SOME REMARKS ON THE HYDRAULICS OF STEADY-STATE WELLS
IN UNCONFINED MEDIA

By

D. F. Peterson, Jr.



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# Some Remarks on the Hydraulics of Steady-State Wells in Unconfined Media[1]

D. F. Peterson, Jr.[2]

## General Case

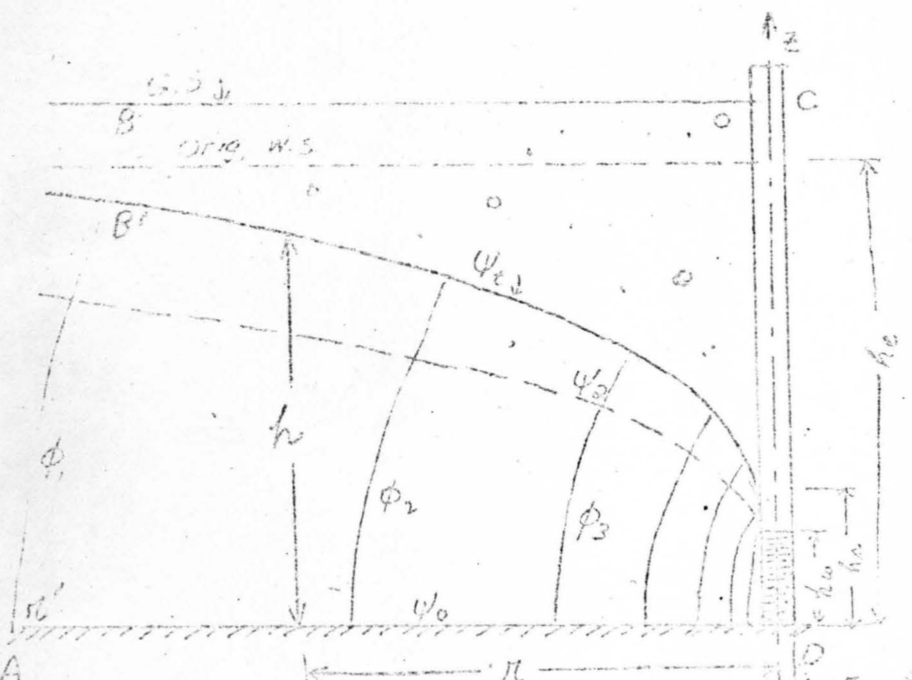

FIG 1

This discussion is concerned with the theoretical aspects of the so-called "gravity" well, - one of the most intriguing problems in natural philosophy which has come to the author's attention. In order to make appreciable progress on this problem one should consider first the simplest case, in which the casing penetrates the full depth of the stratum and is perforated the entire length and in which complete symmetry about the axis of the well exists. If the water surface in the well is lowered to the height $h_w$, flow into the well will result, and after a time, a steady flow condition will be established. For any particular case a definite position of the water surface and a definite flow pattern represented by the flow lines $\psi$ and the iso-head lines $\phi$ will result.

In addition to symmetry the obvious pertinent characteristics of the media are the permeability $k$ and the original water thickness $h_e$; and of the well, the radius $r_w$ and the drawdown, of which $h_w$ is a measure when considered with $h_e$. These four conditions however are insufficient to uniquely determine the pattern and quantity of flow in the region. The flow pattern is described by the 2nd order differential equations of Laplace which require 2 independent boundary conditions for their determination. The foregoing characteristics include only one boundary condition.

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1 Presented to a graduate seminar in irrigation engineering, Colorado A & M College, March 27, 1951

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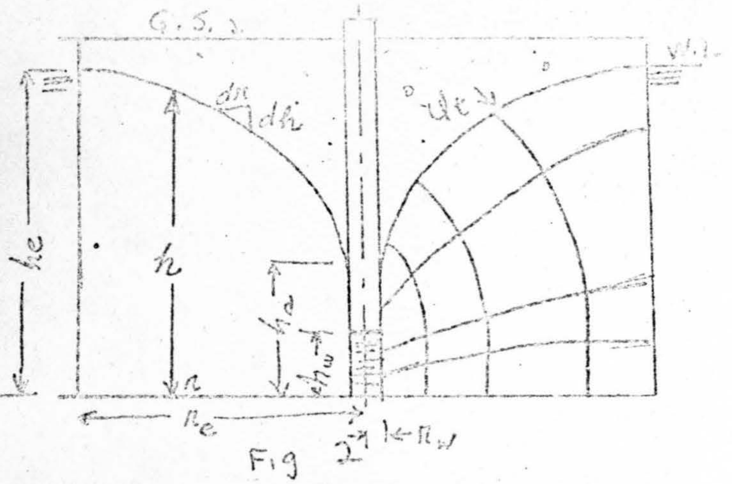
SYMBOLS AND ABBREVIATIONS

| Symbol | Definition | Dimensions |
|---|---|---|
| $\gamma_w$ | Unit weight of water | $F/L^3$ |
| $\epsilon$ | Base of Naperian logarithm system | |
| F | Froude number | |
| $F_n$ | Denoting a functional relationship | |
| $\eta$ | Ratio of the discharge originating by vertical replenishment to the total discharge | |
| g | Acceleration of gravity | $\frac{L}{T^2}$ |
| h | Vertical distance to free boundary | L |
| $h_e$ | Depth of water at the radius of influence | L |
| $h_s$ | Height of seepage surface | L |
| $h_w$ | Depth of water in the well | L |
| $i_n$ | Natural slope of groundwater table | |
| k | Permeability | $\frac{L}{T}$ |
| ln | Logarithm to base $\epsilon$. | |
| $\varsigma$ | Unit replenishment (general case) | $L/T$ |
| $\varsigma_1$ | Unit replenishment (vertical case) | $L/T$ |
| $\varsigma_2$ | Unit replenishment (horizontal case) | $L/T$ |
| $\psi$ | The flow function | |
| $\psi_t$ | Flow line described by the free boundary | |
| $\phi$ | The head (potential) function | |
| p | Pressure | $F/L^2$ |
| Q | Total discharge | $L^3/T$ |
| $Q_r$ | Partial discharge at radial distance r | $L^3/T$ |
| r | A radial distance from the well center | L |
| $r_e$ | Radial distance to the edge of the region of influence | L |
| $r_w$ | Radius of well | L |
| $r_w'$ | An equivalent well radius | L |
| v | Velocity | $L/T$ |
| | Vertical distance from impervious boundary to any point | L |

# The Sand Island




A sand island with a well in the exact center has the additional boundary condition required for a solution. The water level is held at the height $h_e$ at the radius, $r_e$, of the island. With the imposition of the well having the characteristics $r_w$ and $h_w$, the steady-state flow pattern is completely established independently of the total discharge Q and the permeability k. An increased value of k simply causes an increase in Q. The position of the water surface and the flow pattern remain unchanged in other words, the flow pattern as represented by $\psi$ and $\phi$ is uniquely determined by the four boundary values $r_w$, $h_w$, $r_e$, $h_e$. If k is added to these four characteristics, then the discharge, Q, as well is uniquely determined. This conclusion may be stated as

$$F_1\,(Q,\ k,\ r_w,\ h_w,\ r_e,\ h_e) = 0 \qquad (1)$$

Choosing k and $r_w$ as repeating variables yields

$$\frac{Q}{kr_w^2} = F_2\left(\frac{h_w}{r_w},\ \frac{r_e}{r_w},\ \frac{h_e}{r_w}\right) \qquad (2)$$

One may logically choose $\psi_t = \dfrac{Q}{kr_w^2}$. In this case the relative flow pattern depends only on the ratios $\dfrac{h_w}{r_w}$, $\dfrac{r_e}{e_w}$, $\dfrac{h_e}{r_w}$ and not on the absolute dimensions.

One might expect that a possible solution to the indeterminate general case may be found by considering the boundary conditions at $r_e = \infty$. The writer has concluded, however that the discharge will be zero for any finite value of $h_e$ and that the discharge and flow pattern will be indeterminate for $h_e = \infty$. (App. A.)

## Replenishment

The sand island, of course is not a practical case. In practice, a steady-state flow pattern for an incompressible fluid cannot exist in a bounded region unless the region is replenished at the boundaries. The total replenishment must, of course equal the discharge of the well. If $S$ is the unit replenishment called "replenishment" in volume per unit of time per unit of area then

$$q = \int_S \mathfrak{S}(dS) \qquad (3)$$

where the integral is taken over boundary of the region. Equation 3 simply states the law of continuity in an integral form. The general case problem may be made determinate by introducing the condition $\mathfrak{S}$ .[1] as a property of the media. Thus

$$F_3\ (Q,\ k,\ r_w,\ h_w,\ h_e,\ \mathfrak{S}\ ) = 0 \qquad (4)$$

and one may write

$$\frac{Q}{kr_w^3} = F_4\left(\frac{h_w}{r_w}\quad \frac{h_e}{r_w}\quad \frac{\mathfrak{S}}{k}\right) = 0 \qquad (5)$$

In this instance the shape of the flow pattern depends on the ratio of replenishment to permeability , $\mathfrak{S}/k$ , as well as on the original and final depth of water in the well expressed as a ratio of the well radius.

## Characteristics of the Gravity Well

General.- Combination of the law of continuity (in differential form) with that of Darcy leads to Laplace's equation for describing the flow and head pattern in a permeable isotropic media. The effect of gravity for the unconfined case is that the location of the upper boundary of the flow region is not known in advance. Solution to Laplace's equation however consists principally in discovering mathematical functions which satisfy the boundary conditions. For the case of a well in unconfined media, Laplace's equation must be expressed in cylindrical co-ordinates rather than polar co-ordinates because there exists a flow component in the $z$ direction. This requirement as well as the unknown boundary adds immeasurably to the mathematical difficulties of obtaining a theoretical solution.

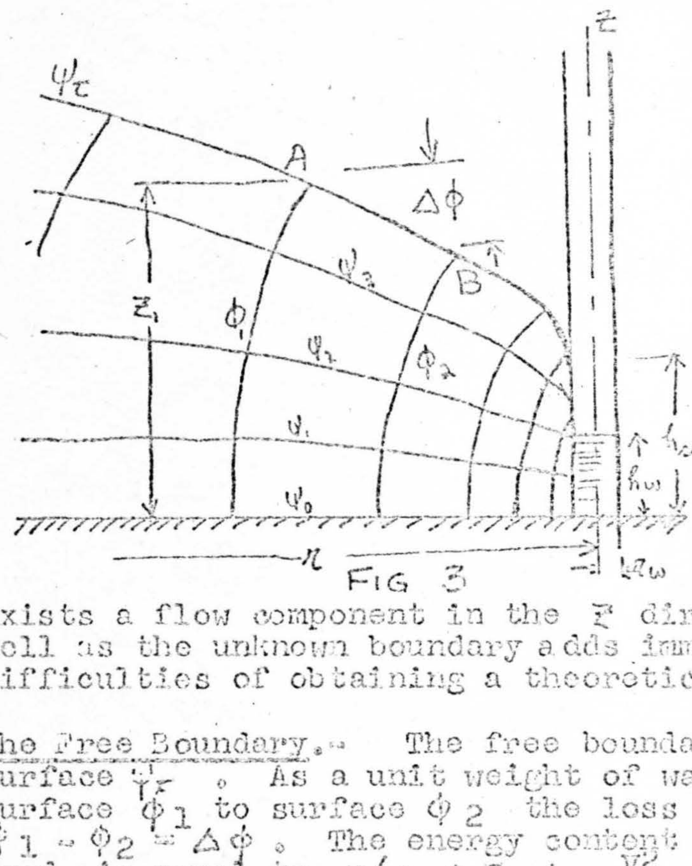


FIG 3

The Free Boundary.- The free boundary is represented by the flow surface $\psi_t$ . As a unit weight of water moves from the iso-head surface $\phi_1$ to surface $\phi_2$ the loss of energy is equal to $\phi_1 - \phi_2 = \Delta\phi$ . The energy content of the unit weight, called head, is equal to $p/\gamma_w + z + \frac{V^2}{2g} =$ where p is the pressure,

[1] Hydraulics of steady-flow well systems, D. F. Peterson and O. W. Israelsen, Utah St. Agric. College, Dept. of Irrig. Engr. 1951, 43 pp. mimeo.

$z$ the elevation and $\frac{V^2}{2g}$ is the energy of motion. In ground-water hydrodynamics the energy of motion is ignored because of relative insignificance. Along the surface $\psi_t$, $P/\gamma_w = 0$ using atmospheric pressure as datum, thus $\Delta\phi = \Delta z$ and $\phi_1 = z_1$, so that the condition for a free surface is established. If the free boundary is at some other position $\psi_t'$ less than $\psi_t$ the internal piezometric pattern would exceed $z$ along $\psi_t$, however, since the media is not confined, water will raise in the pores, just as in a piezometer in a confined media, until the free condition is satisfied. If a new value of $\psi_t''$ greater than $\psi_t$ is assumed, water would drain from the pores between $\psi_t$ and $\psi_t''$ until the gravity condition is satisfied.

## The Seepage Surface

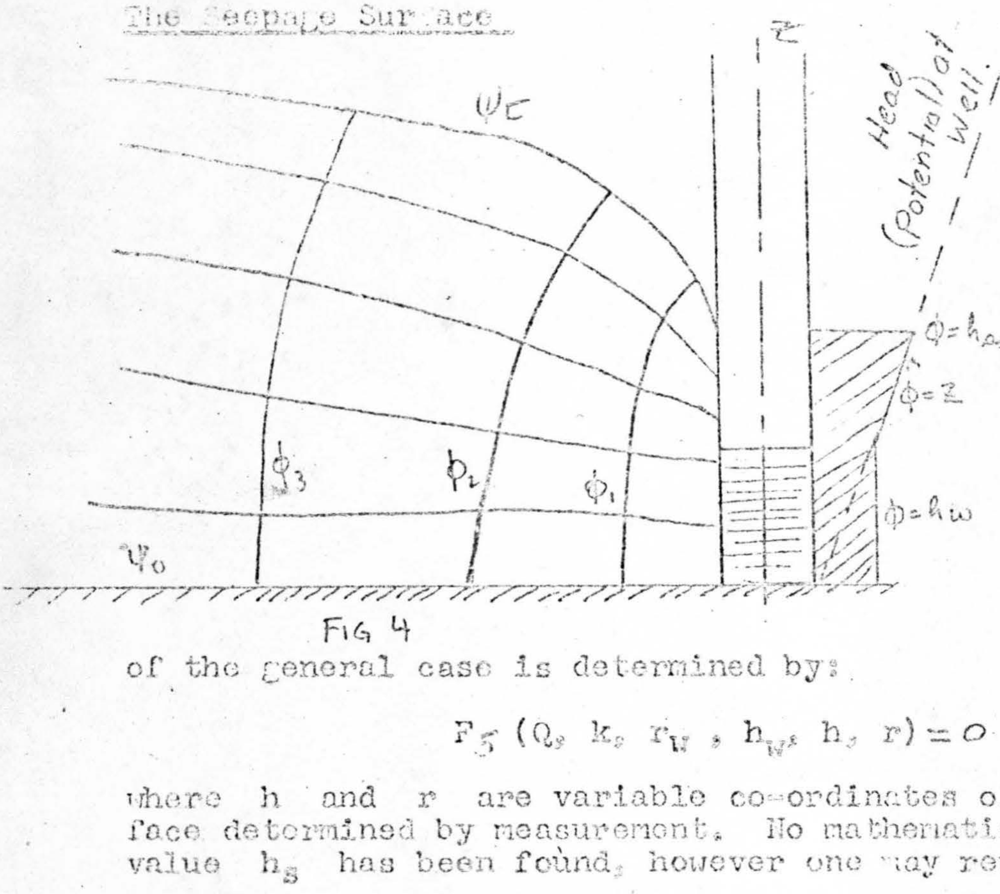



FIG 4

The sketch represents the flow in the region of the well. The surface at the radius of the well between $h_s$ and $h_w$ is called the seepage surface. A finite seepage surface always exists when flow in an unconfined media occurs. This apparently occurs as a manifestation of the principle of least action. The value of $\phi$ at the well radius for $z \leq h_w$ is $h_w$ but is $z$ for $h_w < z < h_s$. A solution of the general case is determined by:

$$F_5 (Q, k, r_w, h_w, h, r) = 0 \qquad (6)$$

where $h$ and $r$ are variable co-ordinates of the free water surface determined by measurement. No mathematical solution for the value $h_s$ has been found, however one may rewrite equation 6

$$\frac{Q}{kr_w^2} = F_6 \left( \frac{h_w}{r_w}, \frac{h}{r_w}, \frac{r}{r_w} \right) \qquad (7)$$

and for the case where $r = r_w$; $h = h_s$, so that

$$\frac{Q}{kr_w^2} = F_7 \left( \frac{h_w}{r_w}, \frac{h_s}{r_w} \right) \qquad (8)$$

as proposed by Vaughn Hansen in his doctoral thesis. Equation 8 was plotted by Hansen on the basis of a number of experimental solutions. Peterson and Israelsen added the six solutions of Yang made by relaxation to Hansen's curves to obtain Fig 5.

Another interesting theoretical fact about the seepage surface is that the slope of $\psi = \psi_t$ is tangent to the seepage surface and that flow lines of lesser values of $\psi$ intersect the seepage surface at decreasing values of slope until $z = h_w$ where they intersect the well at 90° to the well boundary. Proof is not given. The reader is referred to the works of Muskat[1] and Hubbert[2].

Some interesting consequences are implied by Equation 8. Let $\psi_t = \frac{Q}{k r_w^2}$ represent the free surface and hold the value of $\frac{h_w}{r_w}$ constant. An infinite number of free surfaces are possible. As the value of $\psi_t$ increases, the corresponding curve becomes steeper. The greater the replenishment factor $S/k$ the steeper is the free-surface curve. All of these curves theoretically extend to $r = \infty$. The values of $r_e$ and $h_e$ for the sand island are simply particular values of $r$ and $h$ with the exception that the inflow surface for the island is a vertical cylinder rather than the curved iso-head surface of the general case. The greater is $r$, the closer the approximation of the sand island becomes.

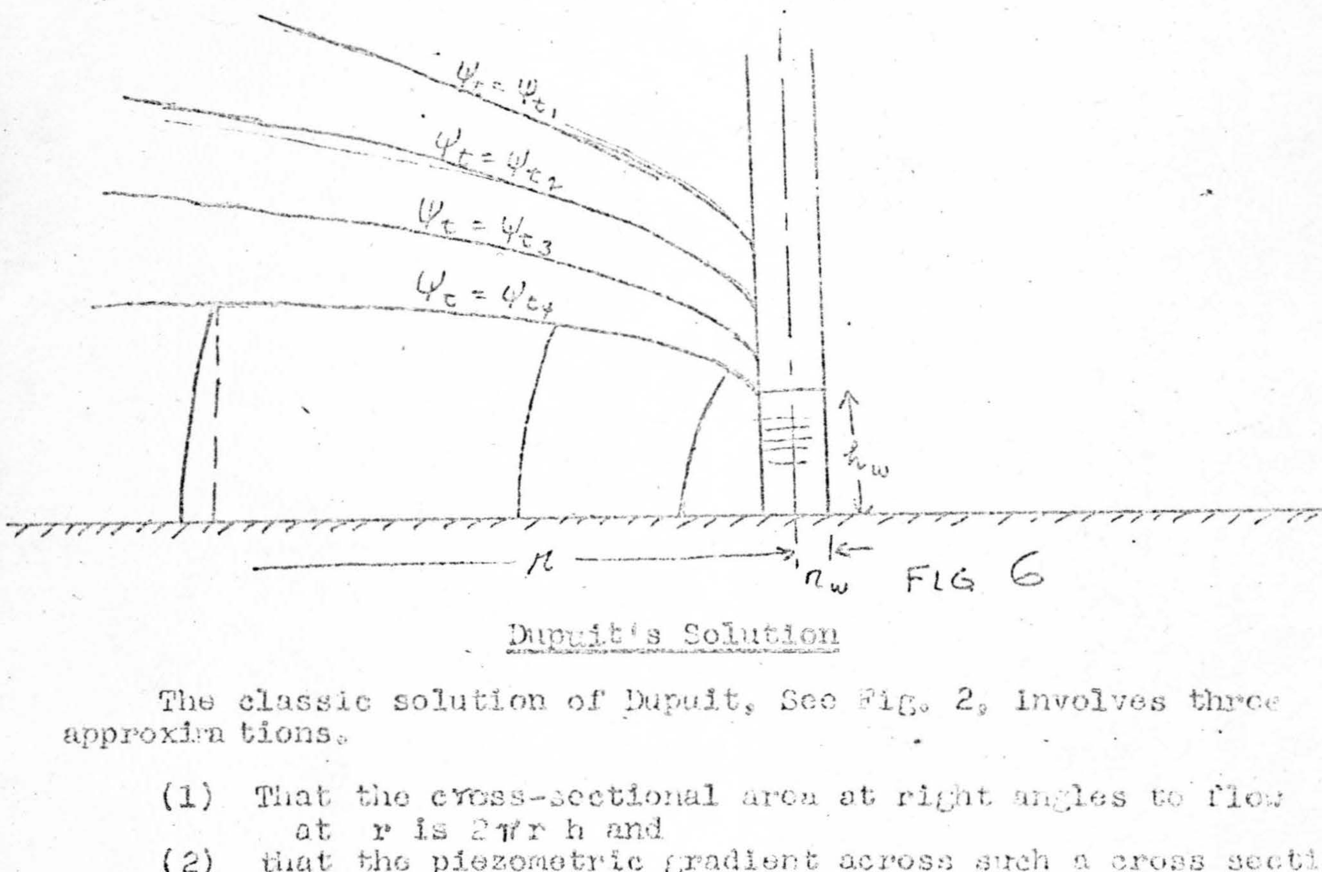



FIG 6

Dupuit's Solution

The classic solution of Dupuit, See Fig. 2, involves three approximations.

(1) That the cross-sectional area at right angles to flow at $r$ is $2\pi r\, h$ and
(2) that the piezometric gradient across such a cross section is everywhere $dr/dh$
(3) that $h$ for $r_w$ is $h_w$ rather than $h_s$

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1 The flow of homogeneous fluids through porous media. New York, McGraw Hill 1937
2 The theory of groundwater motion. Jor. of Geology. Nov-Dec, 1940.

[illegible]
are closely [illegible] from Darcy's law

$$Q = kaI = k\,(2\pi rh)\,\frac{dh}{dr} \tag{9}$$

which integrated gives

$$Q \ln r = \pi k h^2 + c \tag{10}$$

If the boundary conditions of the sand island are introduced

$$Q = \frac{\pi k\,(h_e^2 - h_w^2)}{\ln \frac{r_e}{r_w}} \tag{11}$$

In spite of its theoretical shortcomings, Dupuits equation has been found by experimenters to give correct values of discharge within normal error of measurement. If one chooses the variable limits r and h,

$$Q = \frac{\pi k\,(h^2 - h_w^2)}{\ln r/r_w} \tag{12}$$

Substituting the values of Q from equation 12 into equation 11 yields the Dupuit drawdown equation

$$h^2 = \frac{\ln \frac{r}{r_w}}{\ln \frac{r_e}{r_w}}\,(h_e^2 - h_w^2) + h_w^2 \tag{13}$$

Equation 13 gives accurate results for the position of the free water surface at fairly large distances from the well but is inaccurate near the well. It has been said to be accurate for the piezometric head along the botton impermeable boundary and Hansen has advanced theoretical reasons for this.

## Solutions Considering Replenishment

Peterson and Israelsen 1950 propsed solutions of equation 5 for three cases of replenished wells systems. Two of these solutions involve a degree of approximation.

The Vertically replenished well.- For the symmetrical well replenished only by flow vertically into the region the following solution was proposed. Equation 3 yields

$$Q = \pi \varepsilon_1 (r_e^2 - r_w^2) \text{ or approximately } Q = \pi \varepsilon_1 \,(r_e)^2 \tag{14}$$

if $\varepsilon_1$ is constant and is the unit replenishment for the case under discussion. Q is the total discharge. Proceeding as for the Dupuit solution one obtains

$$Q_r = 2\pi r k h \frac{dh}{dr} \tag{15}$$

where $Q_r$ is the flow at the cylinder of radius $r$

but

$$Q_r = Q - \pi \zeta_1 (r^2 - r_w^2) = Q - \zeta r^2 \quad \text{approximately} \qquad (16)$$

Substituting the value of $Q_r$ from equation 14 into equation 15 and integrating between the limits $r_w$, $r_e$ and $h_w$, $h_e$ yields

$$\frac{Q}{2\pi k} \ln \frac{r_e}{r_w} - \zeta_1 \frac{(r_e^2 - r_w^2)}{4k} = 1/2\,(h_e^2 - h_w^2) \qquad (17)$$

By introducing the approximate relationship $Q = \zeta_1 r_e^2 \pi$ and making the approximation

$$r_e^2 - r_w^2 = r_e^2$$

one may rewrite equation 17

$$Q = \frac{\pi k (h_e^2 - h_w^2)}{\ln \frac{r_e}{r_w} - \frac{1}{2}} \qquad (18)$$

or

$$\zeta_1 = \frac{k}{r_e^2} \frac{(h_e^2 - h_w^2)}{\ln \frac{r_e}{r_w} - \frac{1}{2}} \qquad (19)$$

But $1/2 = \ln \epsilon^{1/2}$

If the substitution $r_w = \frac{r'_w}{\epsilon^{1/2}}$ is made one obtains from equation 16

$$Q = \frac{\pi k (h_e^2 - h_w^2)}{\ln \frac{r_e}{r'_w}} \qquad (20)$$

The discharge for the vertically replenished well is therefore the same as for a conventional Dupuit well with radius 1.643 ($\epsilon^{1/2}$) times the radius of the actual well.

By using the varisbles $r$ and $h$ as limits for equation 15. and substituting the resulting value of $Q$ in equation 17 the equation

$$h^2 = (h_e^2 - h_w^2) \frac{\left[\ln \frac{r}{r_w} - \frac{1}{2} \left(\frac{r}{r_e}\right)^2\right]}{\left[\ln \frac{r_e}{r_w} - \frac{1}{2}\right]} + h_w^2 \qquad (21)$$

results. By substituting $r'_w = \epsilon^{1/2} r_w$, equation 21 may be transformed to

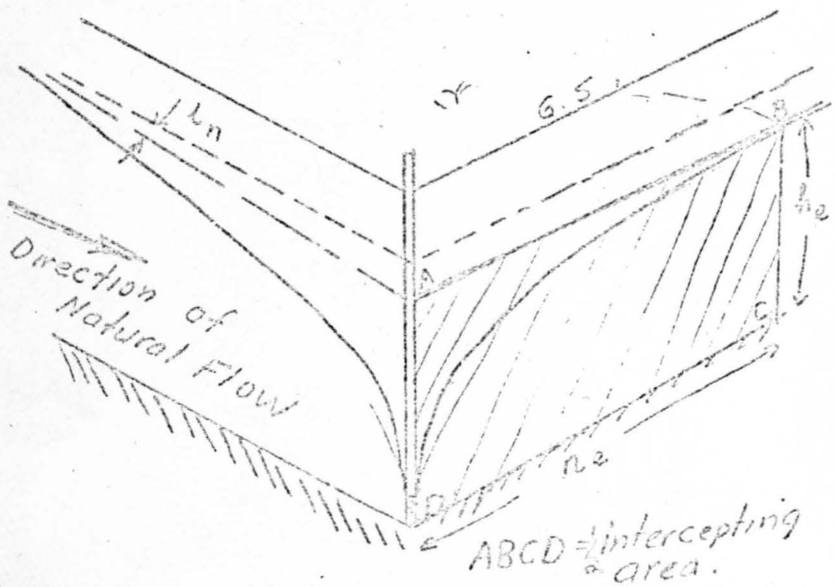
$$h^2 = (h_e^2 - h_w^2) \frac{\left[\ln \frac{r}{r'_w} - \left(\frac{r}{r_e}\right)^2\right]}{\ln \frac{r_e}{r'_w}} + h_w^2 \qquad (22)$$

The transformed equation 22 approaches the Dupuit equation in the neighborhood of the well as one would expect. The inference may be drawn that flow pattern near the well closely approximates that for a Dupuit well in a space in which the radial dimensions relative to the well radius are decreased by $\epsilon^{1/2}$. By means of this transformation the curves of Fig. 5 may be utilized for solution of the seepage face for the vertically replenished well.

## The Horizontally Replenished Well

If the water flows through the unconsolidated sediments of a valley with a slope of $i_n$ the discharge of the well at steady flow equals the water intercepted by the region of influence

$$Q = 2\,k\,i_n\,r_e\,h_e \qquad (23)$$




The symmetry of the system is somewhat distorted; however this will be small because $i_n$ is normally small. A functional relationship of the type of equation 5 may be found by solving equation 23 simultaneously with the Dupuit equation to obtain

$$\frac{Q}{k\,h_e^2}\left[\ln\left(\frac{Q}{k\,h_e^2}\right) - \ln\left(\frac{2\,i_n}{h_e/r_w}\right)\right] = \pi\left[1-\left(\frac{h_w}{h_e}\right)^2\right] \qquad (24)$$

This functional relationship is plotted in Fig 6, $i_n$ corresponds to the dimensionless fraction $\frac{\varsigma_2}{k}$ in equation 5, $\varsigma_2$ represents the normal unit ground water flow through the valley sediments, $\varsigma_2 = k\,i$[1].

Well In System Replenished Partly Horizontally and Partly Vertically.- Let $Q_v$ equal the discharge originating from vertical replenishment and $Q_h$ that originating from horizontal replenishment. Then $Q_t = Q_v + Q_h$ for a steady-state well. Let $\frac{Q_v}{Q_t} = \eta$. For this

1 A similar method of treatment considering replenishment has also been suggested for wells in confined media.

case the following formulas may be derived by a procedure similar to that for the well system recharged vertically

$$Q = \frac{\pi k\,(h_e^2 - h_w^2)}{\ln \frac{r_e}{r_w} - \frac{\eta}{2}} \tag{25}$$

and

$$h^2 = (h_e^2 - h^2)\,\frac{\left[\ln \frac{r}{r_w} - \frac{\eta}{2}\left(\frac{r}{r_e}\right)^2\right]}{\left[\ln \frac{r_e}{r_w} - \frac{\eta}{2}\right]} + h_w^2 \tag{26}$$

This system may be transformed to the Dupuit system in the neighborhood of the well by means of the transformation

$r_w' = e^{\frac{\eta}{2}}\, r_w$.

<u>Some Remarks on the Concept of Replenishment</u>.- The principle of replenishment is, simply a statement of the law of conservation of matter. Either the inflow to a region equals the outflow or the amount of matter in the region is changing. The latter case is not steady flow. The shape of the drawdown curve and the pattern of flow, if steady, depend on the nature and magnitude of replenishment. It is a characteristic of the media in which the system exists.

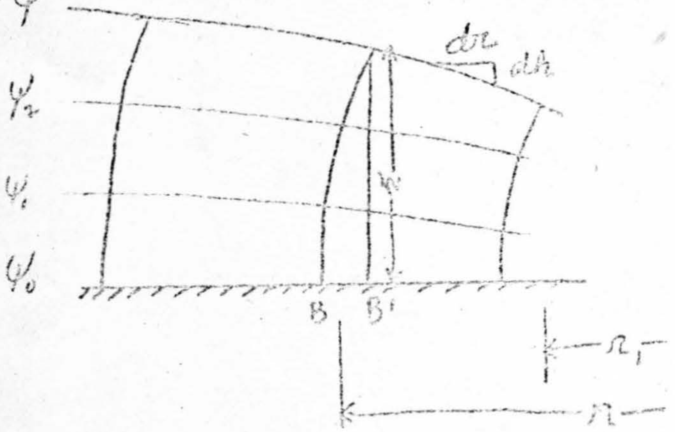
While having an important mathematical significance, replenishment is also quantitatively of major importance in groundwater administration problems. A steady groundwater supply can last indefinitely only if replenishment is considered in planning. True many water supplies now pumped are largely drawn from storage in vast underground reservoirs. The amount pumped may be quite insignificant and the supply may last a long time, but not indefinitely. This fact is recognized and accounted for by groundwater experts.

The mathematical cases herein reported are recognized to be highly idealized. Natural cases are not thus. In many instances, however, the ideal cases may be a reasonably good approximation of the actual case.

<u>The Well Parameter</u>.- The ratio $Q/k\,r_w^2$ seems to have particular significance in well flow. For any particular well at a particular drawdown an infinite number of values of $Q/k\,r_w^2$ are possible depending upon the manner of replenishment or the boundary conditions. The greater the value of $Q/k\,r_w^2$, the steeper is the cone of influence. Israelsen and Peterson have suggested that this parameter measures the response of a well to its hydrologic environment and have suggested that it be designated the "Well-Hydrologic" number or simply "Hydrologic" number. Recently Hansen has demonstrated that this parameter is simply the ratio of the Froude number to the Reynolds number and is accordingly a measure of the relative importance of the inertia forces in relation to the gravity forces. Hansen has suggested that it be designated simply F/R.

## Appendix A

Hypothesis: For a sand island of infinite radius the discharge will be zero for any finite value of $h$ at the boundary and the discharge and the flow pattern will be indeterminate if $h_e$ is infinite at the infinite radius.




The following is not a proof but is helpful in drawing the inference. As the distance from the well increases the Dupuit assumptions become more nearly correct. Assume $r_1$ as a distance such that the effect of the Dupuit approximations are negligible.

From Darcy's equation

$Q = k\,2\pi\,r\,h\,\frac{dh}{dr}$ and considering the case for a finite value of $h_e$ at $r_o = \infty$

$$Q\int_{r_1}^{\infty}\frac{dr}{r} = 2\pi k\int_{h}^{h_e} h\,dh \qquad Q = \lim_{r \to \infty} \frac{\pi k\,(h_e^2 - h_1^2)}{\ln \frac{r}{r_1}} = 0$$

and for an infinite value of $h_e$ at $r_e = \infty$

$$Q = \frac{2\pi k\int_{h_1}^{\infty} h\,dh}{\int_{r_1}^{\infty}\frac{dr}{r}} \quad \frac{\infty}{\infty}$$

however the inference that the flow pattern is indeterminate may be drawn from Fig 1 since the point $B^1$ at radius $r^1$ may have any elevation greater than $h_w$ for any particular $h_w$ and still be part of an infinite system satisfying Laplace's equation and the gravity boundary condition.

# CORRECTIONS

"Some Remarks on the Hydraulics of Steady-State Wells in Unconfined Media", by D. F. Peterson, Jr.

Please make the following corrections:

Fig. 6 and Fig. 7 Appendix

In Figs. 6 and 7 Appendix the symbols $H_y$ and $H_u$ signify the parameter $\frac{Q}{k\ r_w^2}$ in both cases so that the "well factor" (the ordinate on these two figures) is simply $\frac{Q}{k\ h_e^2}$. In Fig 6, Appendix, the symbol q corresponds to $\xi_1$ in equation 14.

Page 5

Insert the word "Appendix" after "Fig. 5" on the last line of page 5.

Page 6

Add the following sentence at the end of the first paragraph: "Using equation 14, $r_e$ may be eliminated from equation 20 to yield the fundamental relationship plotted in Fig. 6, Appendix.

Page 9

First line below equation 24. Change Fig. 6 to Fig. 7 Appendix.