DISSERTATION

THE $\mathcal{D}$-NEIGHBORHOOD COMPLEX OF A GRAPH

Submitted by

Corrine Previte

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2014

Doctoral Committee:

Advisor: Chris Peterson
Co-Advisor: Alexander Hulpke

Dan Bates
Martin Gelfand

ABSTRACT

THE $\mathcal{D}$-NEIGHBORHOOD COMPLEX OF A GRAPH

The Neighborhood complex of a graph, G, is an abstract simplicial complex formed by the subsets of the neighborhoods of all vertices in G. The construction of this simplicial complex can be generalized to use any subset of graph distances as a means to form the simplices in the associated simplicial complex.

Consider a simple graph G with diameter d. Let $\mathcal{D}$ be a subset of $\{0, 1, \ldots, d\}$. For each vertex, u, the $\mathcal{D}$-neighborhood is the simplex consisting of all vertices whose graph distance from u lies in $\mathcal{D}$. The $\mathcal{D}$-neighborhood complex of G, denoted $DN(G, \mathcal{D})$, is the simplicial complex generated by the $\mathcal{D}$-neighborhoods of vertices in G. We relate properties of the graph G with the homology of the chain complex associated to $DN(G, \mathcal{D})$.

ACKNOWLEDGEMENTS

I would like to begin by thanking both of my advisors, Chris Peterson and Alexander Hulpke. I appreciate your endless guidance and support. Because of your various perspectives, this work is a beautiful blend of combinatorics and algebraic topology. I have learned much from both of you.

I would like to thank my parents. I cannot express how much I appreciate your support throughout the years, your continued faith in me, and all of the advice you have offered. I would also like to thank Shawn Johnson. You have been a steady rock and someone on whom I could fully depend. You win an award for having the most patience.

Finally, I would like to thank my fellow graduate students both at CSU and in the EDGE program. There has been a genuine camaraderie in both programs, for which I am truly grateful. In particular, thank you Leif Anderson for your seemingly never ending work on this document class!

TABLE OF CONTENTS

Abstract ii

Acknowledgements iii

Chapter 1. Introduction 1

Chapter 2. Background 3

2.1. Graph Theory 3

2.2. Simplicial Homology 5

2.3. Construction of Chain Complexes 10

Chapter 3. $\mathcal{D}$-Neighborhood Complex 18

3.1. The $\mathcal{D}$-Neighborhood Complex of a Graph 18

3.2. Examples 21

3.3. Propositions 24

Chapter 4. One-Point Union of Graphs 28

4.1. One-Point Union Definition 28

4.2. The Main Theorem 30

4.3. Graph Decomposition 35

4.4. Corollary for Trees 35

Chapter 5. Trees 37

5.1. The $\mathcal{D}$-Neighborhood Complex of Path Graphs 37

5.2. The $\mathcal{D}$-Neighborhood Complex of Unicyclic Graphs 38

Chapter 6. The $\mathcal{D}$-Neighborhood Complex of Cycle Graphs 44

6.1. Cycle Graphs 44

6.2. The $\mathcal{D}$-Neighborhood Complex with Minimal Distance Set 44

6.3. One-Point Unions of Cycles 46

6.4. The $\mathcal{D}$-Neighborhood Complex for $\mathcal{D} = \{1\}$ 47

6.5. The $\mathcal{D}$-Neighborhood Complex with Maximal Distance Set 49

Chapter 7. The $\mathcal{D}$-Neighborhood Complex of Vertex Weighted Trees 51

7.1. The $\mathcal{D}$-Neighborhood Complex of Weighted Graphs 51

7.2. Maximally Weighted Graphs 53

7.3. The Main Theorem 56

Chapter 8. Conclusion 60

8.1. Open Questions 61

Bibliography 63

CHAPTER 1

Introduction

A natural question in mathematics asks "When are objects the same, and when are they different?" Consider the two graphs in Figures 1.1 and 1.2. Each graph contains 12 vertices and the local viewpoint from any vertex is the same. In particular, each vertex has degree two. Globally, one can see that these two graphs are different. Graph G_1 is a disconnected graph consisting of two cycles, while graph G_2 is a disconnected graph consisting of three cycles.

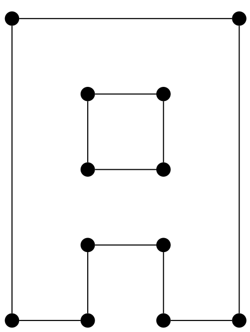

FIGURE 1.1. Graph G_1

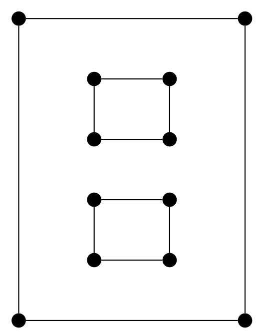

FIGURE 1.2. Graph G_2

It is desirable to find techniques for distinguishing between two graphs. For example, if graphs share the same local information, isomorphism tests can become difficult. Algebraic topology allows one to study connectivity information in a precise manner. In particular, persistent homology is an algebraic method for measuring the topological features of a space. This field allows for the study of various levels of connectivity such as path connectivity, loop connectivity, as well as higher dimensions.

One can use tools from algebraic topology to distinguish between graphs. A topological space can be associated to a graph in many different ways. One such way is to use the information about how the vertices in a graph are connected. Studying the topological features of the space can give insight into the global structure of the graph.

Chapters 2 and 3 will introduce the theory behind building a topological space from a graph. Chapters 4 - 7 will discuss particular types of graphs and what can be said about the features of the spaces associated to these graphs.

CHAPTER 2

Background

In order to construct a topological space from a graph, some background information from both graph theory and simplicial homology is needed. The next three sections will introduce vocabulary, notation, and concepts from these areas.

2.1. Graph Theory

The following definitions from graph theory are consistent with those found in [Die10], [GR01], [Gro08], and [HHM08]. A graph, $G = (V, E)$, consists of a set of vertices, V, together with a set of edges, E, which are two element subsets of V. If $v_i, v_j \in V$, then the edge between these vertices will be denoted as the monomial $v_i v_j = v_j v_i$.

The notion of distance in a graph is measured by the number of edges which must be traversed in order to move from one vertex to another. The following defines three terms relating to graph distance. These terms will be used frequently throughout this work.

Definition 1. *Let $G = (V, E)$ be a graph with $v_i, v_j \in V$.*

(1) The distance *between two vertices, denoted $d(v_i, v_j)$, is the length of the shortest path between v_i and v_j.*

(2) The eccentricity *of a vertex, v_i, is given by $\epsilon(v_i) = max\{d(v_i, v_j) : \forall\, v_j \in V\}$.*

(3) The diameter *of a graph is given by $\text{diam}(G) = max\{\epsilon(v_i) : \forall\, v_i \in V\}$.*

In other words, the *distance* between a pair of vertices is the fewest number of edges needed to travel from vertex v_i to vertex v_j. From here on, when speaking of distance, it will be assumed to be graph distance. The *eccentricity* of vertex v_i is the maximum distance between v_i and every other vertex in the graph, and the *diameter* of a graph is the maximum distance between every pair of vertices. If $\mathrm{diam}(G) = d$, then there exists a path of length d or less between every pair of vertices. Two vertices are said to be *adjacent* if there exists an edge between them. This is to say, the vertices have distance 1 from each other.

DEFINITION 2. *A graph, $G = (V, E)$, is* disconnected *if the vertices can be partitioned into two non-empty sets, G_1 and G_2, such that no vertex in G_1 is adjacent to any vertex in G_2. We say G is the* disjoint union *of the two subgraphs and denote this graph by $G_1 \sqcup G_2$.*

A graph is *connected* if there exists a path between every pair of vertices. A graph is *simple* if there are no loops (edges connecting a vertex to itself) or multiple edges between a pair of vertices. Unless otherwise noted, all graphs are assumed to be simple and connected.

A *cycle* is a path which starts and ends at the same vertex, but otherwise has no repeated edges or vertices. A *cycle graph*, denoted C_n, is a single cycle on n vertices (See Figure 2.1).

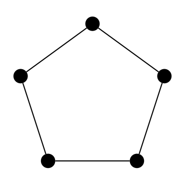

FIGURE 2.1. Cycle graph on 5 vertices, C_5

A *tree* is a simple, connected graph in which there are no non-trivial cycles (See Figure 2.2). In a tree, there is exactly one path between every pair of vertices. A *leaf* of a tree is a vertex of degree 1; that is, there is exactly one edge incident with that vertex.

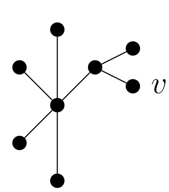

FIGURE 2.2. Tree with leaf v

2.2. Simplicial Homology

The definitions from algebraic topology are consistent with [Car09], [EH10], [Koz08], [Mun84], and [MS05]. An *(abstract) simplicial complex*, Δ, on the vertex set $V(\Delta) = \{1, \ldots, n\}$, is a collection of subsets from the vertex set, called faces or simplices, which is closed under taking subsets. Equivalently, the faces of a simplicial complex can be expressed as monomials. Monomial division acts as a closure operation and it follows that if $\sigma \in \Delta$, then $\{\tau : \tau|\sigma\} \subset \Delta$. A face $\sigma \in \Delta$ of cardinality $|\sigma| = i + 1$ has dimension i and is called an *i-face* of Δ. A face, τ, is a *facet* if there is no distinct $\sigma \in \Delta$ such that $\tau|\sigma$; that is to say, τ is a maximal face. Many texts emphasize the term "abstract" when referring to an abstract simplicial complex. For the purpose of this dissertation, the term "abstract" will be dropped and all simplicial complexes will be assumed to be abstract simplicial complexes.

Given a set of faces, a simplicial complex can be generated by taking the *simplicial closure* of this set. Mathematically, given some vertex set V, if there exists a face $\sigma \subset V$, then the simplicial closure of this face is $\{\tau : \tau|\sigma\}$. Now, given a set of faces, S one can take the set of simplicial closures of each face in S, call this Δ. By definition, Δ is a simplicial complex. Taking the simplicial closure a second time stabilizes Δ; in other words, there will be no new faces.

There is an algebraic structure called a chain complex which is a sequence of abelian groups or modules connected by homomorphisms. From the definition that follows, we will show one way in which a simplicial complex can be associated to a chain complex.

Definition 3. *A* chain complex*, $\mathcal{C} = \{\mathcal{C}_i, \partial_i\}$, is a collection of abelian groups or modules $\mathcal{C}_i$, one for each integer i, and of homomorphisms $\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ such that $\partial_i \circ \partial_{i+1} = 0$, for each i.*

Suppose Δ is a simplicial complex on the vertex set $V(\Delta) = \{1, \ldots, n\}$. For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ. This means that $F_0(\Delta)$ can be thought of as the set of all vertices, $F_1(\Delta)$ as the set of edges, $F_2(\Delta)$ as the set of solid triangles, $F_3(\Delta)$ as the set of solid tetrahedron, etc. Let $\mathbb{K}^{F_i(\Delta)}$ be a vector space over a field $\mathbb{K}$ whose basis elements e_σ correspond to i-faces $\sigma \in F_i(\Delta)$. Suppose $e_\sigma = v_0 v_1 \cdots v_i$.

Definition 4. *The* boundary operator $\delta_i : \mathbb{K}^{F_i(\Delta)} \to \mathbb{K}^{F_{i-1}(\Delta)}$ *is a homomorphism given by*

$$\delta_i(e_\sigma) = \sum_{j=0}^{i} (-1)^j (v_0 \cdots \hat{v}_j \cdots v_i)$$

where the face $v_0 \cdots \hat{v}_j \cdots v_i$ is the j^{th} face of e_σ obtained by removing the j^{th} vertex.

Definition 5. *A* chain complex*, $\mathcal{C} = \{\Delta, \delta\}$, is the sequence of vector spaces:*

$$0 \longrightarrow \mathbb{K}^{F_{n-1}(\Delta)} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow \mathbb{K}^{F_i(\Delta)} \xrightarrow{\delta_i} \mathbb{K}^{F_{i-1}(\Delta)} \longrightarrow \cdots \xrightarrow{\delta_1} \mathbb{K}^{F_0(\Delta)} \xrightarrow{\delta_0} 0$$

where δ_i is the boundary operator defined in Definition 4.

By the definition of the boundary operator, it follows that $\delta_i \circ \delta_{i+1} = 0$ for all i. As a consequence, $\mathrm{Im}(\delta_{i+1}) \subseteq \mathrm{Ker}(\delta_i)$. If $\mathrm{Im}(\delta_{i+1}) = \mathrm{Ker}(\delta_i)$, then the sequence is *exact at i*. A chain complex is *exact* if it is exact for all i.

Definition 6. *The i-*th homology *of a chain complex, $\mathcal{C}$, is defined to be*

$$H_i(\mathcal{C}) = \mathrm{Ker}(\delta_i)/\mathrm{Im}(\delta_{i+1})$$

The i-th homology is said to be trivial if $H_i(\mathcal{C}) \cong \mathbb{K}$ when $i = 0$ or if $H_i(\mathcal{C}) = 0$ for $i > 0$. A chain complex has *trivial homology* if both $H_0(\mathcal{C}) \cong \mathbb{K}$ and if $H_i(\mathcal{C}) = 0$ for all $i > 0$.

Notice that a chain complex has trivial homology if it is exact for each $i > 0$ and if $H_0(\mathcal{C})$ is one-dimensional. The i-th homology measures the failure of a chain complex to be exact. Intuitively, the dimension of the i-th homology can be thought as indicating the number of boundaries of i-dimensional holes in the simplicial complex, where the dimension of $H_0(\mathcal{C})$ indicates the number of connected components. To say the homology of a chain complex is trivial indicates the associated simplicial complex bounds no holes.

2.2.1. Koszul Complex. Consider a simplicial complex consisting of all subsets on the vertex set $V(\Delta) = \{1, 2, \dots, n\}$ defined below and shown in Figure 2.3.

Definition 7. *Let Δ be a simplicial complex. A* complete simplex *of Δ is a face which contains every vertex $v \in V(\Delta)$.*

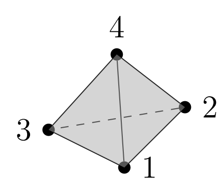

Figure 2.3. Simplicial complex on the set $\{1, 2, 3, 4\}$

By this definition and the definition of a simplicial complex, it follows that $|F_i(\Delta)| = \binom{n}{i+1}$ for each $0 \le i < n$, and the associated chain complex, $\mathcal{C}$, is given by:

$$0 \longrightarrow \mathbb{K}^1 \xrightarrow{\delta_{n-1}} \cdots \longrightarrow \mathbb{K}^{\binom{n}{i+1}} \xrightarrow{\delta_i} \mathbb{K}^{\binom{n}{i}} \longrightarrow \cdots \xrightarrow{\delta_1} \mathbb{K}^n \xrightarrow{\delta_0} 0$$

This chain complex is a Koszul Complex, see [Eis95]. The homology of the Koszul complex can be computed directly to show that $H_0(\mathcal{C}) \cong \mathbb{K}$ and $H_i(\mathcal{C}) = 0$ for all $i > 0$ [MS05]. Furthermore, the homology of the Koszul complex does not depend on the characteristic of the field. Topologically, a complete simplex on n vertices is a solid ball of dimension $n - 1$. As can be seen, Figure 2.3 is a ball of dimension 3.

2.2.2. Trivial Homology. There are different techniques that can be used to determine when the homology of a chain complex is trivial. One such way is by identifying whether or not the chain complex is a copy of the Koszul complex, as in Section 2.2.1. Another is by using the ranks of the boundary operators. This is described in the lemma that follows.

Lemma 1. *Let* $\mathcal{C} = \{\Delta, \delta\}$ *be a chain complex. If* $i > 0$ *and* $\mathrm{rank}(\delta_i) + \mathrm{rank}(\delta_{i+1}) = \dim(\mathbb{K}^{F_i(\Delta)})$*, then* $H_i(\mathcal{C})$ *is trivial.*

Proof. Suppose $i > 0$. It follows from Definition 6 that

$$\begin{aligned}\dim(H_i(\mathcal{C})) &= \mathrm{null}(\delta_i) - \mathrm{rank}(\delta_{i+1}) \\ &= \left(\dim(\mathbb{K}^{F_i(\Delta)}) - \mathrm{rank}(\delta_i)\right) - \mathrm{rank}(\delta_{i+1})\end{aligned}$$

Therefore, if $\mathrm{rank}(\delta_i) + \mathrm{rank}(\delta_{i+1}) = \dim(\mathbb{K}^{F_i(\Delta)})$, then $\dim(H_i(\mathcal{C})) = 0$. □

Consider a simplicial complex on n vertices. The boundary operator, δ_0, will be a zero array of dimension n and therefore, $\mathrm{null}(\delta_0) = n$. If $\mathrm{rank}(\delta_1) = n-1$, then it will follow that $\dim(H_0(\mathcal{C})) = \mathrm{null}(\delta_0) - \mathrm{rank}(\delta_1) = n - (n-1) = 1$. Therefore, in order for $H_0(\mathcal{C})$ to be trivial, $\mathrm{rank}(\delta_1)$ must be $n-1$. Combining this fact with Lemma 1 describes one method for determining when the i-th homology of a chain complex is trivial for all i.

Another technique for identifying when homology is trivial utilizes knowledge of other chain complexes. Since a chain complex is a sequence of vector spaces, it is natural to study homomorphisms between chain complexes. These homomorphisms give insight into the homology of the associated chain complexes.

Definition 8. *Let $\mathcal{A} = \{\Delta_1, \delta\}$ and $\mathcal{B} = \{\Delta_2, \varphi\}$ be two chain complexes. A* chain map *$\boldsymbol{f}\colon \mathcal{A} \to \mathcal{B}$ is a family of homomorphisms*

$$f_i : \mathbb{K}^{F_i(\Delta_1)} \to \mathbb{K}^{F_i(\Delta_2)}$$

such that $\varphi_i \circ f_i = f_{i-1} \circ \delta_i$ for all i.

That is to say that the following diagram commutes for each i:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{K}^{F_{i+1}(\Delta_1)} \arrow[r, "\delta_{i+1}"] \arrow[d, "f_{i+1}"'] & \mathbb{K}^{F_i(\Delta_1)} \arrow[r, "\delta_i"] \arrow[d, "f_i"'] & \mathbb{K}^{F_{i-1}(\Delta_1)} \arrow[r] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r] & \mathbb{K}^{F_{i+1}(\Delta_2)} \arrow[r, "\varphi_{i+1}"] & \mathbb{K}^{F_i(\Delta_2)} \arrow[r, "\varphi_i"] & \mathbb{K}^{F_{i-1}(\Delta_2)} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose $\mathcal{A} = \{\Delta_1, \delta\}$ and $\mathcal{B} = \{\Delta_2, \varphi\}$ are two chain complexes and suppose there is a chain map $\mathbf{f} : \mathcal{A} \to \mathcal{B}$. Let $f_0 : F_0(\Delta_1) \to F_0(\Delta_2)$ map the vertex set $V(\Delta_1)$ to the vertex set $V(\Delta_2)$. This map can be extended to a continuous map $f_i : \mathbb{K}^{F_i(\Delta_1)} \to \mathbb{K}^{F_i(\Delta_2)}$ with $f_i(\sigma) = f_0(v_0)f_0(v_1)\cdots f_0(v_i)$ where $\sigma = v_0v_1\cdots v_i \in F_i(\Delta_1)$ [Arm83]. A chain map induces a homomorphism between the i-th homology groups of the two chain complexes: $f_* : H_i(\mathcal{A}) \to H_i(\mathcal{B})$ [Hat02].

Chain maps can be further extended to a sequence of chain complexes. It is interesting to study when this sequence is exact and how this affects the induced maps on the homology of the chain complexes.

Definition 9. *Suppose $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are chain complexes. Let 0 denote the trivial chain complex. Let $\boldsymbol{f}\colon \mathcal{A} \to \mathcal{B}$ and $\boldsymbol{g}\colon \mathcal{B} \to \mathcal{C}$ be chain maps. The sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{\boldsymbol{f}} \mathcal{B} \xrightarrow{\boldsymbol{g}} \mathcal{C} \longrightarrow 0$$

is a short exact sequence of chain complexes *if in each dimension i, the sequence*

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$$

is an exact sequence.

The following lemma is called the Snake Lemma in references such as [EH10]. However, this name is often used to describe an additional result in homological algebra and to avoid confusion, the name Zig-Zag Lemma found in [Mun84] and [Koz08] will be used.

The Zig-Zag lemma says that a short exact sequence of chain complexes induces a long exact sequence in homology. Given knowledge of the homology of two chain complexes, this lemma allows one to bound the homology of the third chain complex.

Lemma 2. *(Zig-Zag Lemma)*
Assume that

$$0 \longrightarrow \mathcal{A} \xrightarrow{\boldsymbol{f}} \mathcal{B} \xrightarrow{\boldsymbol{g}} \mathcal{C} \longrightarrow 0$$

is a short exact sequence of chain complexes. Then there is a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{f_*} H_n(\mathcal{B}) \xrightarrow{g_*} H_n(\mathcal{C}) \xrightarrow{\partial_*} H_{n-1}(\mathcal{A}) \xrightarrow{f_*} \cdots$$

where f_* and g_* are the maps between the homology groups induced by the maps $\boldsymbol{f}$ and $\boldsymbol{g}$, and ∂_* is induced by the boundary operator in $\mathcal{B}$.

2.3. Construction of Chain Complexes

The Zig-Zag Lemma is a powerful lemma, but requires three chain complexes. The purpose of this section is to identify tools which can be used to construct three chain complexes.

With specific knowledge of the homology of two of these chain complexes, the Zig-Zag Lemma will allow us to deduce the homology of the third chain complex.

Given a simplicial complex, there are natural ways to study a second simplicial complex where one is a subset of the other. Induction on the number of vertices is one such method. There will be two chain complexes associated to these simplicial complexes. These chain complexes will be used to construct a third chain complex. It will then be shown that there exists a short exact sequence between the three chain complexes, thus giving us the opportunity to use the Zig-Zag Lemma.

2.3.1. Injective Chain Maps. The next definition describes when one chain complex is a subcomplex of another. The notation $\partial_i|_{\mathcal{C}'_i}$ indicates restriction of the homomorphism ∂_i to the module $\mathcal{C}'_i$.

Definition 10. *A* subcomplex $\mathcal{C}' = \{\mathcal{C}'_i, \partial'_i\}$ *of a chain complex* $\mathcal{C} = \{\mathcal{C}_i, \partial_i\}$*, denoted* $\mathcal{C}' \subset \mathcal{C}$*, is a chain complex such that* $\mathcal{C}'_i \subset \mathcal{C}_i$ *and* $\partial'_i = \partial_i|_{\mathcal{C}'_i}$ *for all* i*.*

The idea of subcomplexes can be extended for simplicial complexes. Recall, a basis for $\mathbb{K}^{F_i(\Delta)}$ is the set of i-faces $F_i(\Delta)$. Suppose $\Delta_1 \subset \Delta_2$. By the choice of basis, it is easy to see that $\mathbb{K}^{F_i(\Delta_1)} \subset \mathbb{K}^{F_i(\Delta_2)}$ for each i. This implies the chain complex associated to Δ_1 is a subcomplex of the chain complex associated to Δ_2.

The following lemma shows that there exists a natural chain map between a subcomplex and a chain complex. In particular, this chain map is a collection of injective linear transformations.

Lemma 3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two chain complexes such that* $\mathcal{A} \subset \mathcal{B}$*. Then there exists a chain map* $\boldsymbol{f}: \mathcal{A} \to \mathcal{B}$*.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two chain complexes such that $\mathcal{A} \subset \mathcal{B}$. For each i, partition $\mathcal{B}_i = \{\mathcal{A}_i, \mathcal{A}'_i\}$. Suppose $\dim(\mathcal{A}_i) = n$ and $\dim(\mathcal{A}_{i-1}) = m$. Then $\dim(\mathcal{B}_i) = n + n'$, and $\dim(\mathcal{B}_{i-1}) = m + m'$.

There exists an injective linear transformation, $f_i : \mathcal{A}_i \hookrightarrow \mathcal{B}_i$. This map can be represented by the following block matrix

$$f_i = \left(\begin{array}{c} I_n \\ \hline 0 \end{array} \right)$$

where 0 is a $n' \times n$ zero matrix.

Similarly,

$$f_{i-1} = \left(\begin{array}{c} I_m \\ \hline 0 \end{array} \right)$$

where 0 is an $m' \times m$ zero matrix.

Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"'] & \mathcal{A}_{i-1} \arrow[r] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r] & \mathcal{B}_i \arrow[r, "\beta_i"] & \mathcal{B}_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Now, $\alpha_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ can be represented by some $m \times n$ matrix A, and due to the partition of bases, $\beta_i : \mathcal{B}_i \to \mathcal{B}_{i-1}$ can be represented by a block matrix:

$$\beta_i = \left(\begin{array}{c|c} A & * \\ \hline 0 & ** \end{array} \right)$$

It follows that

$$f_{i-1} \circ \alpha_i = \left(\begin{array}{c} A \\ \hline 0 \end{array} \right)$$

and

$$\beta_i \circ f_i = \left(\begin{array}{c} A \\ \hline 0 \end{array} \right)$$

Since the diagram commutes for each i, then $\mathbf{f}$ must be a chain map. $\square$

2.3.2. Surjective Chain Maps. From linear algebra, we know that if W is a vector space and if there exists a subspace $V \subset W$, then there exists a subspace U such that $U \oplus V = W$ [Hal58]. For our purposes, since chain complexes are sequences of vector spaces, then we can extend this property to chain complexes. In particular, if a simplicial complex is a subset of another simplicial complex, $\Delta_1 \subset \Delta_2$, then as vector spaces, $\mathbb{K}^{F_i(\Delta_1)} \subset \mathbb{K}^{F_i(\Delta_2)}$ for each i. The next lemma describes a way to obtain another subspace of $\mathbb{K}^{F_i(\Delta_2)}$.

Lemma 4. *Suppose $\Delta_1 \subset \Delta_2$ are two simplicial complexes. There exists a subspace, call this $\mathbb{K}^{F_i(\Delta_3)}$, such that $\mathbb{K}^{F_i(\Delta_1)} \oplus \mathbb{K}^{F_i(\Delta_3)} = \mathbb{K}^{F_i(\Delta_2)}$ for all i.*

Proof. Suppose $\Delta_1 \subset \Delta_2$ are two simplicial complexes. Then for each i, $\mathbb{K}^{F_i(\Delta_1)}$ is a subspace of $\mathbb{K}^{F_i(\Delta_2)}$. Therefore, the basis elements in $\mathbb{K}^{F_i(\Delta_2)}$ can be partitioned into the set $\{F_i(\Delta_1), F_i(\Delta_3)\}$ where $F_i(\Delta_3) = F_i(\Delta_2) \setminus F_i(\Delta_1)$. Thus, there must exist a subspace over the field $\mathbb{K}$ with basis $F_i(\Delta_3)$, say $\mathbb{K}^{F_i(\Delta_3)}$, where $\mathbb{K}^{F_i(\Delta_1)} \oplus \mathbb{K}^{F_i(\Delta_3)} = \mathbb{K}^{F_i(\Delta_2)}$. $\square$

Let $F_i(\Delta_3)$ be a basis for the subspace $\mathbb{K}^{F_i(\Delta_3)}$, and let Δ_3, be the collection of bases $\{F_i(\Delta_3)\}$.[1] Define homomorphisms, $\delta_i : \mathbb{K}^{F_i(\Delta_3)} \to \mathbb{K}^{F_{i-1}(\Delta_3)}$, to be the boundary operator defined in Definition 4. The following lemma shows that the sequence of homomorphisms between these vector spaces is in fact a chain complex. This implies that given two chain complexes $\mathcal{A} \subset \mathcal{B}$, a third chain complex, $\mathcal{C}$, can be constructed. It will be shown in Lemma 6 that one can also obtain a chain map $\mathbf{g} : \mathcal{B} \to \mathcal{C}$.

[1]Despite the notation, Δ_3 is not necessarily a simplicial complex.

LEMMA 5. *Suppose* Δ_1 *and* Δ_2 *are two simplicial complexes and* $\Delta_1 \subset \Delta_2$*. For each* i*, let* $F_i(\Delta_3)$ *be a basis for* $\mathbb{K}^{F_i(\Delta_3)}$ *where* $\mathbb{K}^{F_i(\Delta_1)} \oplus \mathbb{K}^{F_i(\Delta_3)} = \mathbb{K}^{F_i(\Delta_2)}$*. Then there exists a non-trivial chain complex* $\mathcal{C} = \{\Delta_3, \delta\}$*.*

PROOF. Let Δ_1 and Δ_2 be two simplicial complexes such that $\Delta_1 \subset \Delta_2$. For each i, let $F_i(\Delta_3)$ be a basis for $\mathbb{K}^{F_i(\Delta_3)}$ where $\mathbb{K}^{F_i(\Delta_1)} \oplus \mathbb{K}^{F_i(\Delta_3)} = \mathbb{K}^{F_i(\Delta_2)}$. Using the boundary operator defined in Definition 4, there is a non-trivial sequence:

$$0 \longrightarrow \mathbb{K}^{F_{n-1}(\Delta_3)} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow \mathbb{K}^{F_i(\Delta_3)} \xrightarrow{\delta_i} \mathbb{K}^{F_{i-1}(\Delta_3)} \longrightarrow \cdots \xrightarrow{\delta_1} \mathbb{K}^{F_0(\Delta_3)} \longrightarrow 0$$

Let $\sigma = a_0 a_1 \cdots a_i \in F_i(\Delta_3)$ and let $a_0 a_1 \cdots \hat{a}_j \cdots a_i$ denote the $(i-1)$-dimensional face which excludes vertex a_j. Suppose $j < k$. Then the following holds:

$$\begin{aligned}
\delta_{i-1} \circ \delta_i(\sigma) &= \delta_{i-1}(\ldots + (-1)^j a_0 \cdots \widehat{a_j} \cdots a_i + \ldots \\
&\quad +(-1)^k a_0 \cdots \widehat{a_k} \cdots a_i + \ldots) \\
&= \ldots + (-1)^j (-1)^{k-1} a_0 \cdots \widehat{a_j} \cdots \widehat{a_k} \cdots a_i + \ldots \\
&\quad +(-1)^k (-1)^j a_0 \cdots \widehat{a_j} \cdots \widehat{a_k} \cdots a_i + \ldots \\
&= 0
\end{aligned}$$

Since $\mathbb{K}^{F_i(\Delta_3)}$ is a vector space, then $\tau \in \mathbb{K}^{F_i(\Delta_3)}$ implies that the additive inverse $-\tau \in \mathbb{K}^{F_i(\Delta_3)}$ for all i. This ensures that all terms will sum to 0. Therefore, the definition of a chain complex is satisfied. $\square$

With this construction of the third chain complex, we call $\mathcal{C}$ the *difference complex.*

Definition 11. *Suppose* $\mathcal{A} = \{\Delta_1, \delta\}$ *and* $\mathcal{B} = \{\Delta_2, \varphi\}$ *are chain complexes and* $\mathcal{A} \subset \mathcal{B}$. *The* difference complex, $\mathcal{C} = \{\Delta_3, \psi\}$, *is the sequence of vector spaces together with a boundary operator,* $\psi_i : \mathbb{K}^{F_i(\Delta_3)} \to \mathbb{K}^{F_{i-1}(\Delta_3)}$, *where* $\mathbb{K}^{F_i(\Delta_3)} \oplus \mathbb{K}^{F_i(\Delta_1)} = \mathbb{K}^{F_i(\Delta_2)}$.

It follows that $\mathcal{C} \subset \mathcal{B}$. The following lemma shows that there exists a natural chain map $\mathbf{g} : \mathcal{B} \to \mathcal{C}$.

Lemma 6. *Let* $\mathcal{B}$ *and* $\mathcal{C}$ *be two chain complexes such that* $\mathcal{C} \subset \mathcal{B}$. *Then there exists a chain map* $\boldsymbol{g} : \mathcal{B} \to \mathcal{C}$.

Proof. Suppose $\mathcal{B}$ and $\mathcal{C}$ are two chain complexes such that $\mathcal{C} \subset \mathcal{B}$. For each i, patrition $\mathcal{B}_i = \{\mathcal{C}_i, \mathcal{C}'_i\}$. Suppose $\dim(\mathcal{B}_i) = n + n'$, and $\dim(\mathcal{B}_{i-1}) = m + m'$. Then $\dim(\mathcal{C}_i) = n$ and $\dim(\mathcal{C}_{i-1}) = m$.

There exists a surjective linear transformation $g_i : \mathcal{B}_i \twoheadrightarrow \mathcal{C}_i$. This can be represented by the following block matrix

$$g_i = \left(\begin{array}{c|c} I_n & 0 \end{array} \right)$$

where 0 is an $n \times n'$ zero matrix.

Similarly,

$$g_{i-1} = \left(\begin{array}{c|c} I_m & 0 \end{array} \right)$$

where 0 is an $m \times m'$ zero matrix.

Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{B}_i \arrow[r, "\alpha_i"] \arrow[d, "g_i"'] & \mathcal{B}_{i-1} \arrow[r] \arrow[d, "g_{i-1}"'] & \cdots \\
\cdots \arrow[r] & \mathcal{C}_i \arrow[r, "\beta_i"] & \mathcal{C}_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

By the partition of bases, $\alpha_i : \mathcal{B}_i \to \mathcal{B}_{i-1}$ can be represented by the block matrix

$$\alpha_i = \left(\begin{array}{c|c} A & * \\ \hline 0 & ** \end{array}\right)$$

where $\beta_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ is represented by the $m \times n$ matrix A. It follows that

$$g_{i-1} \circ \alpha_i = \left(\begin{array}{c|c} A & 0 \end{array}\right)$$

and

$$\beta_i \circ g_i = \left(\begin{array}{c|c} A & 0 \end{array}\right)$$

Since the diagram commutes for each i, then $\mathbf{g}$ must be a chain map. □

Beginning with two simplicial complexes $\Delta_1 \subset \Delta_2$, we have shown there is a natural injective chain map between the associated chain complexes $\mathcal{A} = \{\Delta_1, \delta\}$ and $\mathcal{B} = \{\Delta_2, \psi\}$. Using the faces in $F_i(\Delta_2) \setminus F_i(\Delta_1) = F_i(\Delta_3)$ as a basis, it is possible to build a sequence of vector spaces which induces a difference complex, $\mathcal{C} = \{\Delta_3, \varphi\}$. Therefore, $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$. There is a natural surjective chain map between $\mathcal{B}$ and $\mathcal{C}$. The following lemma is a result from [DF04] and shows that this construction of chain maps is a short exact sequence of chain complexes. The proof of this lemma follows from the fact that $\mathbf{f}$ is injective, $\mathbf{g}$ is surjective, and it is a simple exercise to show that $\text{Ker}(\mathbf{g}) = \text{Im}(\mathbf{f})$.

Lemma 7. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are chain complexes. Then*

$$0 \longrightarrow \mathcal{A} \xrightarrow{\mathbf{f}} \mathcal{A} \oplus \mathcal{B} \xrightarrow{\mathbf{g}} \mathcal{B} \longrightarrow 0$$

is a short exact sequence of chain complexes.

Suppose we start with a chain complex with known homology. Then we can construct two more chain complexes in such a way to guarantee a short exact sequence. It is then possible to apply the Zig-Zag Lemma. Computing the homology of the remaining two chain complexes becomes a task in record keeping. At times, it will be possible to use some of the techniques from Section 2.2.2 for identifying trivial homology in one chain complex to show the homology of the two remaining chain complexes must be equivalent. Since the homology of the chain complex was known for one chain complex, this gives us a method for determining the homology of the other chain complex.

CHAPTER 3

$\mathcal{D}$-Neighborhood Complex

This chapter describes one way to construct a simplicial complex from a graph. In particular, the distance between vertices will be used as a rule for generating a face in the simplicial complex. To this simplicial complex, one can associate a chain complex. The homology of the chain complex can be computed to identify topological features of the simplicial complex. These features will then be used as a way to compare different classes of graphs.

3.1. The $\mathcal{D}$-Neighborhood Complex of a Graph

Consider a graph, $G = (V, E)$, on n vertices. The next definition describes how to construct a simplex from one vertex in a graph.

Definition 12. *Let $\mathcal{D}$ be a subset of the set of graph distances $\{0, 1, \dots, \mathrm{diam}(G)\}$. The* $\mathcal{D}$-neighborhood *of a vertex, v_i, is given by $N_i = \{v_j \in V : d(v_i, v_j) \in \mathcal{D}\}$.*

In other words, the $\mathcal{D}$-neighborhood of a vertex is a collection of vertices lying at specific distances from that vertex. A natural choice for $\mathcal{D}$ is to choose the set of consecutive distances $\{0, 1, \dots, d\}$, for some number d, as this uses the immediate neighborhood around a vertex. However, there is no requirement for $\mathcal{D}$ to be a set containing consecutive distances or to include 0. In some graphs, it will be interesting to study the case when the set of distances do not contain 0, such as the case when $\mathcal{D} = \{1\}$.

Distance in a graph is measured by the number of edges in the shortest path between two vertices (Definition 1). This value will always be a positive integer or 0. It will be explicitly stated when $0 \in \mathcal{D}$; therefore, without loss of generality, an arbitrary value $d \in \mathcal{D}$ will be assumed to be an integer greater than 0.

Consider the following graph:

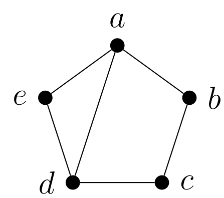

FIGURE 3.1. Graph G

Let $\mathcal{D} = \{0, 2\}$. The $\mathcal{D}$-neighborhood of vertex a is $N_a = \{a, c\}$ since these are the vertices with distance 0 or 2 from vertex a.

The concept of a $\mathcal{D}$-neighborhood can be used to build a simplicial complex from a graph. The following definition describes this process.

DEFINITION 13. *The* $\mathcal{D}$-neighborhood complex *of a graph* $G = (V, E)$ *with distance set* $\mathcal{D}$, *denoted* $DN(G, \mathcal{D})$, *is the simplicial complex with simplex* σ *included whenever* $\sigma \subset N_i$ *for some vertex* v_i.

One can generate the $\mathcal{D}$-neighborhood complex of a graph by taking the simplicial closure of the set of $\mathcal{D}$-neighborhoods. The set of all facets are a subset of the $\mathcal{D}$-neighborhoods of the graph. The $\mathcal{D}$-neighborhood, N_i, will be written as a monomial of vertices from the graph. The operation of monomial division ensures the definition of a simplicial complex is satisfied. That is,

$$\{\tau : \tau | N_i\} \subset DN(G, \mathcal{D}) \tag{1}$$

The $\mathcal{D}$-neighborhood complex is a generalization of a simplicial complex called the Neighborhood complex. The Neighborhood complex is equivalent to considering the case when

$\mathcal{D} = \{1\}$ and the "closed" Neighborhood complex is equivalent to the case when $\mathcal{D} = \{0,1\}$. More information about this simplicial complex can be found in [Cso07] and [Kah07]. It is interesting to study $\mathcal{D} = \{0,1,\dots,d\}$ for increasing choices of d because one can create a nested sequence of the associated chain complexes. The induced maps in homology allow one to look for topological features which persist as d increases. In other words, this allows one to study the persistent homology of the nested sequence of $\mathcal{D}$-neighborhood complexes.[1]

Recall from Figure 3.1, the $\mathcal{D}$-neighborhood on vertex a was $N_a = ac$. Finding the $\mathcal{D}$-neighborhoods of the remaining vertices and taking the simplicial closure yields the following $\mathcal{D}$-neighborhood complex:

$$DN(G,\mathcal{D}) = \{ace,\ bce,\ bde,\ ac,\ ae,\ bc,\ bd,\ be,\ ce,\ de,\ a,\ b,\ c,\ d,\ e\}$$

Using Section 2.2, the chain complex, $\mathcal{C}$, associated to $DN(G,\mathcal{D})$ is the sequence

$$0 \longrightarrow \mathbb{K}^3 \xrightarrow{\delta_2} \mathbb{K}^7 \xrightarrow{\delta_1} \mathbb{K}^5 \xrightarrow{\delta_0} 0$$

Recall the definition of the boundary operator (Definition 4). Using a matrix to represent δ_1, we have

$$\delta_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccc} ac & ae & bc & bd & be & ce & de \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

[1] This is not the purpose of this dissertation, but is one possible direction for future research.

The boundary operator for δ_2 can be represented by

$$\delta_2 = \begin{array}{c} \\ ac \\ ae \\ bc \\ bd \\ be \\ ce \\ de \end{array} \begin{array}{c} \begin{array}{ccc} ace & bce & bde \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

The boundary operator for δ_0 can be represented by a zero array. Using Definition 6, the homology of $\mathcal{C}$ is $H_0(\mathcal{C}) = \mathrm{Ker}(\delta_0)/\mathrm{Im}(\delta_1) \cong \mathbb{K}$, $H_1(\mathcal{C}) = 0$, and $H_2(\mathcal{C}) = 0$. This indicates that the $\mathcal{D}$-neighborhood complex has trivial homology. Using the facets $\{ace, bce, bde\}$, the $\mathcal{D}$-neighborhood complex can be visualized (Figure 3.2).

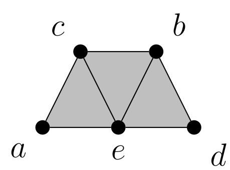

FIGURE 3.2. $\mathcal{D}$-neighborhood complex for graph G

3.2. Examples

In Chapters 4 - 7 we will compute the homology of the chain complex associated to the $\mathcal{D}$-neighborhood complex of general classes of graphs and for various choices of distance sets. This section will study the homology of the $\mathcal{D}$-neighborhood complex of a few special graphs.

Consider the Petersen Graph (See Figure 3.3). The Petersen Graph is a strongly regular graph. Specifically, every vertex has degree 3, every pair of adjacent vertices has 0 common neighbors, and every pair of non-adjacent vertices has 1 common neighbor [GR01]. The symmetry and regularity of this graph makes it interesting to study.

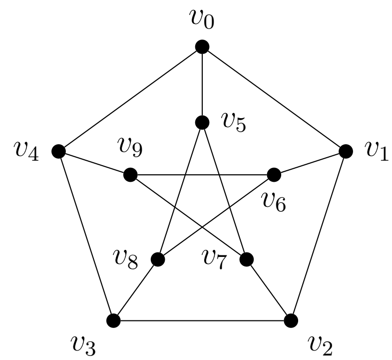

FIGURE 3.3. Petersen graph

Let $\mathcal{D} = \{0, 1\}$ and let $\mathcal{C}$ be the chain complex associated to $DN(G, \mathcal{D})$. There are 10 distinct facets of $DN(G, \mathcal{D})$ which correspond to the $\mathcal{D}$-neighborhoods on each of the 10 vertices in the Petersen Graph. Since each vertex has degree 3, each of these facets is 3-dimensional. The chain complex has the form:

$$0 \longrightarrow \mathbb{K}^{10} \xrightarrow{\delta_3} \mathbb{K}^{40} \xrightarrow{\delta_2} \mathbb{K}^{45} \xrightarrow{\delta_1} \mathbb{K}^{10} \xrightarrow{\delta_0} 0$$

Using Definition 6 to compute the homology of the chain complex, the only non-trivial homology class is $H_1(\mathcal{C}) \cong \mathbb{K}^6$. This means that there are 6 boundaries of one-dimensional "holes" in the $\mathcal{D}$-neighborhood complex of the Petersen Graph. Something to notice is that there are 40 faces of dimension 2. Since there are 10 faces of dimension 3, then a simple counting argument shows that there are no 3-faces "glued" together along the same 2-face; otherwise, there would be fewer than 40 faces of dimension 2. However, there are many 3-faces glued together along the same 1-face. In other words, many tetrahedron share edges. This topology explains why $H_1(\mathcal{C})$ is non-trivial.

Suppose $\mathcal{D} = \{0,1,2\}$. Then since 2 is the diam(G), the homology of the chain complex associated to the $\mathcal{D}$-neighborhood complex would be trivial (See Proposition 3). Extending the radius of the neighborhoods around each vertex from a radius of 1 to radius of 2, in effect, "solidifies" the simplicial complex.

Next, we will look at the homology of the chain complex of the $\mathcal{D}$-neighborhood complex of the skeletons of the Platonic solids. By definition, each of the skeletons of the Platonic solids are regular graphs. This means that all of the facets of the associated $\mathcal{D}$-neighborhood complex will have the same cardinality. It is uninteresting to study the skeleton of the tetrahedron since this is a complete graph.

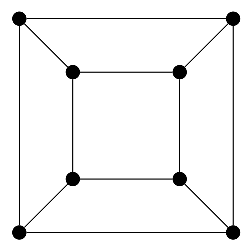

FIGURE 3.4. Skeleton of cube

Consider the graph of the skeleton of a cube (Figure 3.4). Let $\mathcal{D} = \{0,1\}$. Then the chain complex, $\mathcal{C}$, associated to $DN(G,\mathcal{D})$ is given by:

$$0 \longrightarrow \mathbb{K}^8 \xrightarrow{\delta_3} \mathbb{K}^{32} \xrightarrow{\delta_2} \mathbb{K}^{24} \xrightarrow{\delta_1} \mathbb{K}^8 \xrightarrow{\delta_0} 0$$

Once again, the 8 vertices in the graph give 8 distinct facets. In computing the homology of the chain complex, the only non-trivial homology class is $H_2(\mathcal{C}) \cong \mathbb{K}^7$. This means that the $\mathcal{D}$-neighborhood complex of the skeleton of the cube contains 7 boundaries of two-dimensional "holes". In other words, this simplicial complex has the same homology as a wedge sum of 7 hollow tetrahedron. As with the Petersen Graph, since the diameter of the cube is 2, then letting $\mathcal{D} = \{0,1,2\}$ will lead to trivial homology of the associated $\mathcal{D}$-neighborhood complex.

The $\mathcal{D}$-neighborhood complexes of the skeletons of the octahedron, dodecahedron, and icosahedron (Figure 3.5) also have interesting homology for various choices of $\mathcal{D}$.

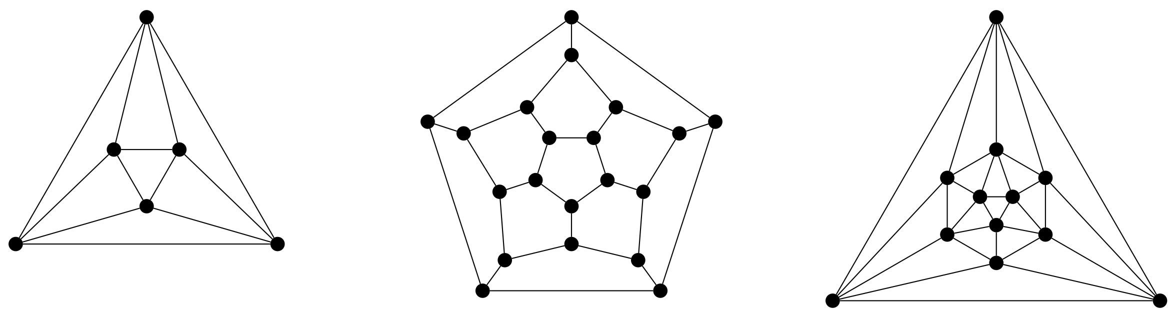

FIGURE 3.5. Skeleton of octahedron (left), dodecahedron (center), icosahedron (right)

Table 3.1 summarizes the results from direct computations. Notice that since the skeleton of the octahedron has diameter 2, then the homology of the chain complex associated to the $\mathcal{D}$-neighborhood complex will be trivial for $\mathcal{D} = \{0,1,2\}$.

TABLE 3.1. Homology of the $\mathcal{D}$-neighborhood complex of Platonic solids

	$\mathcal{D} = \{0,1\}$	$\mathcal{D} = \{0,1,2\}$
Octahedron	$H_4(\mathcal{C}) \cong \mathbb{K}$	Trivial
Dodecahedron	$H_1(\mathcal{C}) \cong \mathbb{K}^{11}$	$H_2(\mathcal{C}) \cong \mathbb{K}$
Icosahedron	$H_2(\mathcal{C}) \cong \mathbb{K}$	$H_{10}(\mathcal{C}) \cong \mathbb{K}$

3.3. PROPOSITIONS

As mentioned in Section 2.1, unless specified, all graphs will be assumed to be connected and simple. The justification for this follows. Recall the definition of the disjoint union of two graphs (Definition 2). Notice that the definition of graph distance does not allow for simplices in the $\mathcal{D}$-neighborhood complex to form between vertices from two disjoint graphs regardless of the choice of $\mathcal{D}$. As a consequence, the homology of the chain complex associated to the

$\mathcal{D}$-neighborhood complex of a disconnected graph can be treated additively. The following proposition formalizes this scenario, thereby justifying the restriction to connected graphs.

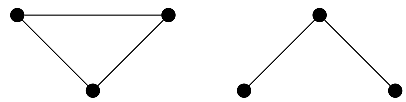

FIGURE 3.6. Disjoint union of two graphs

PROPOSITION 1. *Let $\mathcal{D}$ be fixed. Let $G_1 \amalg G_2$ be the disjoint union of graphs G_1 and G_2. Let $\mathcal{A}$ be the chain complex associated to $DN(G_1, \mathcal{D})$, let $\mathcal{B}$ be the chain complex associated to $DN(G_2, \mathcal{D})$, and let $\mathcal{C}$ be the chain complex associated to $DN(G_1 \amalg G_2, \mathcal{D})$. Then $H_i(\mathcal{C}) = H_i(\mathcal{A}) \oplus H_i(\mathcal{B})$ for all i.*

PROOF. The proof of this proposition is trivial. □

The reason for assuming all graphs to be simple is that loops and multiple edges do not change the distance between vertices and therefore, will not change the $\mathcal{D}$-neighborhoods. If the homology of the $\mathcal{D}$-neighborhood complex of a non-simple graph is needed, one can remove the loops and multiple edges.

Consider a connected graph, $G = (V, E)$, and let $\mathcal{D} = \{0, 1, \ldots, d\}$. Notice that for each vertex $v_j \in V$, there will exist at least one $\mathcal{D}$-neighborhood, N_k, such that $v_j | N_k$. Since $0 \in \mathcal{D}$, then it will also be true that $v_j | N_j$. This means that the associated $\mathcal{D}$-neighborhood complex, $DN(G, \mathcal{D})$, will always be one connected component. In other words, $H_0(\mathcal{C}) \cong \mathbb{K}$. However, this is not necessarily true if $0 \notin \mathcal{D}$.

As implied in Section 3.2, there is a time when there are enough distances included in $\mathcal{D}$ so that the homology of the chain complex associated the $\mathcal{D}$-neighborhood complex becomes trivial. The following expands upon this idea.

Definition 14. *Given a graph $G = (V, E)$, a $\mathcal{D}$-neighborhood, N_i, is said to be* complete *if N_i is a complete simplex on V.*

The first case is applicable to any choice of $\mathcal{D}$. Once there exists a complete $\mathcal{D}$-neighborhood in the $\mathcal{D}$-neighborhood complex, then the associated chain complex is a Koszul complex. From [Mun84], the homology of the chain complex will be trivial.

Proposition 2. *Let $G = (V, E)$ be a graph and let $\mathcal{D}$ be fixed. Let $\mathcal{C}$ be the associated chain complex to $DN(G, \mathcal{D})$. If there exists a complete $\mathcal{D}$-neighborhood in $DN(G, \mathcal{D})$, then $H_i(\mathcal{C})$ is trivial for all i.*

Proof. Let $G = (V, E)$ be a graph and let $\mathcal{D}$ be fixed. Suppose there exists a complete $\mathcal{D}$-neighborhood, $N_i \subset DN(G, \mathcal{D})$. By Definition 13, every monomial which divides N_i will also be a face, and thus, every possible combination of the vertices from V. It follows that the chain complex associated to $DN(G, \mathcal{D})$ is the Koszul complex on the vertex set V. Thus, $H_i(\mathcal{C})$ is trivial for all i. □

The next case is actually a corollary to Proposition 2. This is the case when $\mathcal{D}$ contains every possible distance in a particular graph. At this point in time, every vertex will have a complete $\mathcal{D}$-neighborhood.

Proposition 3. *Let $G = (V, E)$ be a graph and let $\mathcal{D} = \{0, 1, \ldots, \mathrm{diam}(G)\}$. Let $\mathcal{C}$ be the chain complex associated to $DN(G, \mathcal{D})$. Then $H_i(\mathcal{C})$ is trivial for all i.*

Proof. Let $G = (V, E)$ be a graph, let $\mathcal{D} = \{0, 1, \ldots, \mathrm{diam}(G)\}$, and let $\mathcal{C}$ be the chain complex associated to $DN(G, \mathcal{D})$. By definition of the diameter of a graph (Definition 1), $d(v_i, v_j) \leq \mathrm{diam}(G)$ for every pair of vertices $v_i, v_j \in V$. This implies that N_i is a complete $\mathcal{D}$-neighborhood for each vertex $v_i \in V$. By Proposition 2, $H_i(\mathcal{C})$ is trivial for all i. □

In addition to assuming all graphs are simple and connected (unless otherwise noted), then by Proposition 3, we can also assume for distance set $\mathcal{D} = \{0, 1, \ldots, d\}$, that d is an integer and $1 \leq d < \text{diam}(G)$. Otherwise, it will be immediately known that the homology of the chain complex associated to the $\mathcal{D}$-neighborhood complex is trivial.

CHAPTER 4

One-Point Union of Graphs

Beginning with a simple connected graph, we have described one way of building a simplicial complex based on the distance between vertices. The homology of the associated chain complex gives insight into the topological features of the simplicial complex. This chapter begins to draw connections between these features and the graph.

Choosing $\mathcal{D}$ to be the set of consecutive distances $\{0, 1, \ldots, d\}$, this chapter studies how the homology of the $\mathcal{D}$-neighborhood complex changes when two graphs are joined together at a single vertex. Understanding these changes allows one to compute the homology of the $\mathcal{D}$-neighborhood complex of a variety of graphs by decomposing them into smaller pieces.

4.1. One-Point Union Definition

Recall Definition 2 of the disjoint union of graphs. The following definition describes joining two disconnected graphs at a single vertex in order to form the one-point union.

Definition 15. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $v_1 \in V_1$ and $v_2 \in V_2$ be two vertices. The* one-point union *of G_1 and G_2 with respect to v_1 and v_2, denoted $G_1 \tilde{\amalg} G_2 = (V, E)$, is the graph defined by*

(1) the vertex set $V = (\{V_1 \setminus v_1\}) \cup (\{V_2 \setminus v_2\}) \cup \{v\}$

(2) $v_i v_j$ is an edge in $G_1 \tilde{\amalg} G_2$ if and only if either $v_i v_j$ is an edge in $G_1 \cup \{v\}$ and $v_i v_j$ is an edge in G_1 once v is replaced with v_1, or $v_i v_j$ is an edge in $G_2 \cup \{v\}$ and $v_i v_j$ is an edge in G_2 once v is replaced with v_2.

The one-point union of two graphs can be regarded as "gluing" vertices v_1 and v_2 together (See Figures 4.1 and 4.2). The one-point union depends on the choice of vertices, v_1 and v_2, but in practice, these will be suppressed from notation.

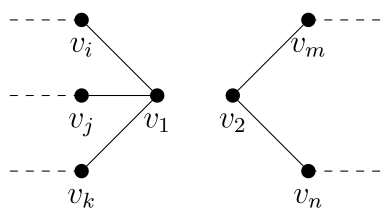

FIGURE 4.1. Disjoint union $G_1 \amalg G_2$

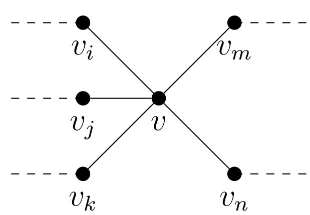

FIGURE 4.2. One-point union $G_1 \widetilde{\amalg} G_2$

The idea of joining objects together at a single point can be extended to simplicial complexes. The following definition is from [Koz08] and will be used in the theorem in Section 4.2.

DEFINITION 16. *Given two simplicial complexes Δ_1 and Δ_2, with vertices $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$, the* wedge *of Δ_1 and Δ_2, with respect to the vertices v_1 and v_2, is the simplicial complex $\Delta_1 \vee \Delta_2$ defined by*

(1) $V(\Delta_1 \vee \Delta_2) = (V(\Delta_1) \setminus \{v_1\}) \cup (V(\Delta_2) \setminus \{v_2\}) \cup \{v\}$;

(2) $\sigma \subset V(\Delta_1 \vee \Delta_2)$ is a simplex of $\Delta_1 \vee \Delta_2$ if and only if either $\sigma \subset V(\Delta_1) \cup \{v\}$ and σ is a simplex of Δ_1 once v is replaced with v_1, or $\sigma \subset V(\Delta_2) \cup \{v\}$ and σ is a simplex of Δ_2 once v is replaced with v_2.

The wedge of two simplicial complexes joins two vertices from each simplicial complex to form one vertex. Notice that the homology of the wedge of two simplicial complexes is equivalent to the sum of the homology of each simplicial complex [Jon08].

4.2. The Main Theorem

The following theorem says that the homology of the $\mathcal{D}$-neighborhood complex of the one-point union of two graphs is equivalent to the homology of the $\mathcal{D}$-neighborhood complex of the disjoint union of the graphs.[1] This is not something that is immediately obvious. The set of facets in $DN(G_1 \widetilde{\amalg} G_2, \mathcal{D})$ is not the union of the set of facets in $DN(G_1, \mathcal{D})$ and $DN(G_2, \mathcal{D})$. In other words, the $\mathcal{D}$-neighborhood complex of $G_1 \widetilde{\amalg} G_2$ is not the wedge sum of $DN(G_1, \mathcal{D}) \vee DN(G_2, \mathcal{D})$. In particular, it is possible that the $\mathcal{D}$-neighborhood of vertex v in Figure 4.2 creates a facet of higher dimension than those in $DN(G_1, \mathcal{D})$ and $DN(G_2, \mathcal{D})$. Despite the changes in facets, it turns out that the topological features of the simplicial complexes for each graph are preserved.

Theorem 1. *Let $\mathcal{D} = \{0, 1, \ldots, d\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $\mathcal{A}$ be the chain complex associated to $DN(G_1 \amalg G_2, \mathcal{D})$ and let $\mathcal{B}$ be the chain complex associated to $DN(G_1 \widetilde{\amalg} G_2, \mathcal{D})$. Then $H_i(\mathcal{B}) = H_i(\mathcal{A})$ for all $i > 0$ and $H_0(\mathcal{B}) \cong \mathbb{K}$.*

Proof. Let $\mathcal{D} = \{0, 1, \ldots, d\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $\mathcal{A} = \{\Delta_1, \delta\}$ be the chain complex associated to the $\mathcal{D}$-neighborhood complex of the disjoint union $G_1 \amalg G_2$. Let $G_1 \widetilde{\amalg} G_2$ be the one-point union of G_1 and G_2 with respect to vertices $a' \in V_1$ and $a'' \in V_2$. Let $\mathcal{B} = \{\Delta_2, \psi\}$ be the chain complex associated to the $\mathcal{D}$-neighborhood complex of $G_1 \widetilde{\amalg} G_2$.

[1] Recall, from Section 3.3 the homology of the $\mathcal{D}$-neighborhood complex of the disjoint union of two graphs is equivalent to the sum of the homology groups of the $\mathcal{D}$-neighborhood complex of each graph.

Let $\mathbf{f} : \mathcal{A} \to \mathcal{B}$ be a chain map. In particular, let $f_0 : a' \mapsto a$, $f_0 : a'' \mapsto a$, and $f_0 : v_i \mapsto v_i$ for all other 0-faces in $DN(G_1 \amalg G_2, \mathcal{D})$. By identifying vertices a' and a'' with vertex a, notice that $DN(G_1 \amalg G_2, \mathcal{D}) \subset DN(G_1 \widetilde{\amalg} G_2, \mathcal{D})$. Let Δ_3 be the collection of faces from $DN(G_1 \widetilde{\amalg} G_2, \mathcal{D}) \smallsetminus DN(G_1 \amalg G_2, \mathcal{D})$. For each $\sigma \in \Delta_3$, there exists at least one vertex $v_1 \in V_1$ and at least one vertex $v_2 \in V_2$ such that $v_1 | \sigma$ and $v_2 | \sigma$. That is, each face in Δ_3 must have at least one vertex from G_1 and one vertex from G_2, otherwise, this face would be in $DN(G_1 \amalg G_2, \mathcal{D})$. Put an ordering on the vertices in Δ_3, such that $a < v_j$ for all other v_j. By Lemma 5, there is an associated chain complex to Δ_3, the difference complex $\mathcal{C} = \{\Delta_3, \varphi\}$.

We first show that $\mathcal{C}$ is exact. By Lemma 1, we want to show that $\text{rank}(\varphi_k)+\text{rank}(\varphi_{k-1}) = \dim(\mathbb{K}^{F_{k-1}(\Delta_3)})$ for all k.

By definition of a chain complex, it follows that $\text{Im}(\varphi_k) \subseteq \text{Ker}(\varphi_{k-1})$. This implies that

$$\text{null}(\varphi_{k-1}) \geq \text{rank}(\varphi_k) \tag{2}$$

$$\dim(\mathbb{K}^{F_{k-1}(\Delta_3)}) - \text{rank}(\varphi_{k-1}) \geq \text{rank}(\varphi_k) \tag{3}$$

$$\dim(\mathbb{K}^{F_{k-1}(\Delta_3)}) \geq \text{rank}(\varphi_{k-1}) + \text{rank}(\varphi_k) \tag{4}$$

Using Equation 4, it is left to show that $\dim(\mathbb{K}^{F_{k-1}(\Delta_3)}) \leq \text{rank}(\varphi_k) + \text{rank}(\varphi_{k-1})$.

Let $k > 0$ and consider $av_0 \cdots v_{k-1} \in F_k(\Delta_3)$. Applying the boundary operator, φ_k, yields the linear combination $\varphi_k(av_0 \cdots v_{k-1}) = v_0 \cdots v_{k-1} - av_1 \cdots v_{k-1} + \ldots$ This is a linear combination of $(k-1)$ dimensional faces where the only subsimplex which is not divisible by vertex a is the face $v_0 \cdots v_{k-1}$ which has coefficient 1. Due to the ordering on the vertices, this will be true for every face, $\sigma \in F_k(\Delta_3)$ where $a|\sigma$. Furthermore, any face $\tau \in F_k(\Delta_3)$ such that $a \nmid \tau$ will be mapped to a linear combination of $(k-1)$-faces which also are not divisible by a.

Recall, the k-faces from $F_k(\Delta_3)$ are a basis for $\mathbb{K}^{F_k(\Delta_3)}$. By the ordering on the elements in $F_k(\Delta_3)$, the matrix representation of φ_k will be a block matrix of the form

$$\varphi_k = \left(\begin{array}{c|c} * & 0 \\ \hline I_t & ** \end{array}\right)$$

Let $A = \{\sigma \in F_k(\Delta_3) : a|\sigma\}$, i.e. the the set of k-faces that contain vertex a. Let $B = \{\tau \in F_{k-1}(\Delta_3) : a \nmid \tau\}$, i.e. the set of $(k-1)$-faces that do not contain vertex a. Notice that the size of A determines the size of the identity block in φ_k and therefore, puts a lower bound on the rank(φ_k) . This implies that $|A| = t \leq \text{rank}(\varphi_k)$. There exists a 1-1 correspondence between A and B. For each $\sigma \in A$, there is exactly one corresponding $(k-1)$-face that is not divisible by a which is obtained from applying the boundary operator, $\varphi_k(\sigma)$. Furthermore, each $\tau \in B$ divides a distinct face in A by construction of Δ_3. Otherwise, if τ is a $(k-1)$-face, if $a \nmid \tau$, and if there does not exist $\sigma \in A$ such that $\tau|\sigma$, then $\tau \in DN(G_1 \amalg G_2, \mathcal{D})$. That is, a $(k-1)$-face which is not divisible by a and which does not divide any face in A would have been a face in $DN(G_1 \amalg G_2, \mathcal{D})$. Therefore, $|A| = |B|$ and $|B| \leq \text{rank}(\varphi_k)$.

Now let $C = \{\sigma \in F_{k-1}(\Delta_3) : a|\sigma\}$, i.e. the set of $(k-1)$-faces that contain vertex a. It follows that $|B| + |C| = \dim(\mathbb{K}^{F_{k-1}(\Delta_3)})$. By the same argument as above, it follows that $|C| \leq \text{rank}(\varphi_{k-1})$. Therefore, $\dim(\mathbb{K}^{F_{k-1}(\Delta_3)}) = |B| + |C| \leq \text{rank}(\varphi_k) + \text{rank}(\varphi_{k-1})$. Combining this result with Equation 4, it has been shown that $\text{rank}(\varphi_k) + \text{rank}(\varphi_{k-1}) = \dim(\mathbb{K}^{F_{k-1}(\Delta_3)})$. By Lemma 1, this means that $H_i(\mathcal{C})$ is trivial for all $i > 0$.

Let $\Delta_1 = DN(G_1 \amalg G_2, \mathcal{D})$ and let $\Delta_2 = DN(G_1 \widetilde{\amalg} G_2, \mathcal{D})$. In order to use the Zig-Zag Lemma, we want a short exact sequence of chain complexes between $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. However, the map $f_0 : \mathbb{K}^{F_0(\Delta_1)} \to \mathbb{K}^{F_0(\Delta_2)}$ produces a non-trivial kernel. In particular, $\text{Ker}(f_0)$ is one-dimensional. In order to remedy this, the chain complex $\mathcal{A}$ can be adjusted.

Since G_1 and G_2 are disjoint, then $\delta_1 : \mathbb{K}^{F_1(\Delta_1)} \to \mathbb{K}^{F_0(\Delta_1)}$ can be represented as the block matrix

$$\delta_1 = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & S \end{array}\right)$$

Take the wedge $DN(G_1, \mathcal{D}) \vee DN(G_2, \mathcal{D})$ at vertices a' and a'' by defining $\widetilde{\delta_1} : \mathbb{K}^{F_1(\Delta_1)} \to \mathbb{K}^{F_0(\Delta_1)}/\mathbb{K}$. This means that $\widetilde{\delta_1}$ can be represented by the matrix

$$\widetilde{\delta_1} = \left(\begin{array}{c|c} \multicolumn{2}{c}{T} \\ \hline \widetilde{R} & 0 \\ \hline 0 & \widetilde{S} \end{array}\right)$$

where T can be thought of as a single row which takes the sum of the row from R which corresponded to vertex a' and the row from S which corresponded to vertex a''.

This modified chain complex, denoted $\widetilde{\mathcal{A}}$, is given below:

$$0 \longrightarrow \cdots \xrightarrow{\delta_i} \mathbb{K}^{F_i(\Delta_1)} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_2} \mathbb{K}^{F_1(\Delta_1)} \xrightarrow{\widetilde{\delta_1}} \mathbb{K}^{F_0(\Delta_1)}/\mathbb{K} \xrightarrow{\widetilde{\delta_0}} 0$$

From the modification on the chain complex, $\mathcal{A}$, the changes in the homology, $H_i(\mathcal{A})$, can be tracked. First, we show that the $\text{rank}(\delta_1) = \text{rank}(\widetilde{\delta_1})$. In [Bap10], it is shown if a graph, G, on x vertices has k connected components, and has incidence matrix $Q(G)$, then $\text{rank}(Q(G)) = x - k$. Since δ_1 is the (oriented) incidence matrix for $G_1 \sqcup G_2$, and there are two connected components, then $\text{rank}(\delta_1) = (|V_1| + |V_2|) - 2$. Since joining $DN(G_1, \mathcal{D})$ with $DN(G_2, \mathcal{D})$ at vertices a' and a'' means there is one less vertex in $DN(G_1, \mathcal{D}) \vee DN(G_2, \mathcal{D})$, then $\text{rank}(\widetilde{\delta_1}) = (|V_1| + |V_2| - 1) - 1 = \text{rank}(\delta_1)$. Since $\dim(\mathbb{K}^{F_1(\Delta_1)})$ is the same in the chain complexes $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, then it follows by the rank-nullity theorem that $\text{null}(\delta_1) = \text{null}(\widetilde{\delta_1})$. Therefore, the homology on $\widetilde{\mathcal{A}}$ is equivalent to $H_i(\mathcal{A})$ for all $i \geq 1$.

Since $\operatorname{null}(\widetilde{\delta_0}) = \operatorname{null}(\delta_0) - 1$, then the dimension of $H_0(\widetilde{\mathcal{A}})$ is one less than the dimension of $H_0(\mathcal{A})$. By assuming that G_1 is connected and G_2 is connected, then $H_0(\mathcal{A}) \cong \mathbb{K}^2$, and therefore, $H_0(\widetilde{\mathcal{A}}) \cong \mathbb{K}$.

Let $\widetilde{\mathbf{f}} : \widetilde{\mathcal{A}} \to \mathcal{B}$ and let $\mathbf{g} : \mathcal{B} \to \mathcal{C}$ be chain maps. Then we have the following sequence of chain complexes:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{K}^{F_2(\Delta_1)} \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & \mathbb{K}^{F_1(\Delta_1)} \arrow[r, "\widetilde{\delta_1}"] \arrow[d, "f_1"'] & \mathbb{K}^{F_0(\Delta_1)}/\mathbb{K} \arrow[r, "\widetilde{\delta_0}"] \arrow[d, "\widetilde{f_0}"'] & 0 \\
\cdots \arrow[r] & \mathbb{K}^{F_2(\Delta_2)} \arrow[r, "\psi_2"] \arrow[d, "g_2"'] & \mathbb{K}^{F_1(\Delta_2)} \arrow[r, "\psi_1"] \arrow[d, "g_1"'] & \mathbb{K}^{F_0(\Delta_2)} \arrow[r, "\psi_0"] \arrow[d, "g_0"'] & 0 \\
\cdots \arrow[r] & \mathbb{K}^{F_2(\Delta_3)} \arrow[r, "\varphi_2"] \arrow[d] & \mathbb{K}^{F_1(\Delta_3)} \arrow[r, "\varphi_1"] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Notice that by construction $\mathcal{B} = \widetilde{\mathcal{A}} \oplus \mathcal{C}$. Therefore, this sequence of chain complexes is of the form

$$0 \longrightarrow \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}} \oplus \mathcal{C} \longrightarrow \mathcal{C} \longrightarrow 0$$

By Lemma 7, this is a short exact sequence of chain complexes. By the Zig-Zag Lemma, there is a long exact sequence in homology.

$$\cdots \longrightarrow H_i(\widetilde{\mathcal{A}}) \longrightarrow H_i(\mathcal{B}) \longrightarrow H_i(\mathcal{C}) \longrightarrow H_{i-1}(\widetilde{\mathcal{A}}) \longrightarrow \cdots$$

It was shown that $H_i(\mathcal{C})$ was trivial for $i > 0$ and since $\mathbb{K}^{F_0(\Delta_3)} = 0$, then $H_0(\mathcal{C}) = 0$. Therefore, the homology of $\widetilde{\mathcal{A}}$ is equivalent to the homology of $\mathcal{B}$ for all i. As shown before,

$H_i(\widetilde{\mathcal{A}}) = H_i(\mathcal{A})$ for $i > 0$ and $H_0(\widetilde{\mathcal{A}}) \cong \mathbb{K}$. Therefore, $H_i(\mathcal{B}) = H_i(\mathcal{A})$ for all $i > 0$ and $H_0(\mathcal{B}) \cong \mathbb{K}$. □

4.3. Graph Decomposition

Theorem 1 allows for a decomposition of graphs into subgraphs that meet at exactly one vertex. For example, let $\mathcal{D} = \{0, 1\}$ and consider the $\mathcal{D}$-neighborhood complex of the graph in Figure 4.3.

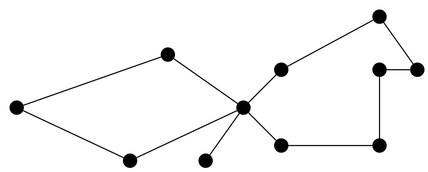

Figure 4.3. Graph G

This graph can be decomposed into 3 one-point unions of subgraphs (See Figure 4.4). By Theorem 1, the homology of the $\mathcal{D}$-neighborhood complex of G is the sum of the homology groups of the $\mathcal{D}$-neighborhood complex of each subgraph. We can compute the homology of the $\mathcal{D}$-neighborhood complex of each subgraph using theorems from Chapters 5 and 6. It follows that $H_2(\mathcal{C}) \cong \mathbb{K}$, $H_1(\mathcal{C}) \cong \mathbb{K}$, and $H_i(\mathcal{C})$ is trivial for $i \neq 1, 2$.

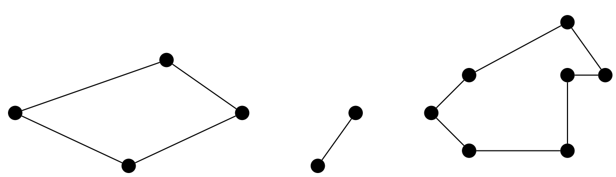

Figure 4.4. Decomposition of graph G

4.4. Corollary for Trees

Edges are the building blocks of a graph. By joining a set of edges as one-point unions, one is able to build any tree. Restricting to one-point unions of edges ensures that a cycle is not formed and thus, protects the definition of a tree. The next result is an immediate consequence of Theorem 1 and looks at the homology of of the $\mathcal{D}$-neighborhood complex of a tree.

COROLLARY 1. *Let $T = (V, E)$ be a tree, let $\mathcal{D} = \{0, 1, \ldots, d\}$, and let $\mathcal{C}$ be the chain complex associated to $DN(T, \mathcal{D})$. Then $H_i(\mathcal{C})$ is trivial for all i.*

PROOF. Consider a tree on two vertices, v_j and v_k. This tree must be the edge $v_j v_k$. It follows that $\sigma = v_j v_k$ is a face in $DN(T, \mathcal{D})$. Since σ is a complete $\mathcal{D}$-neighborhood, then by Proposition 2, $H_i(\mathcal{C})$ must be trivial for all i.

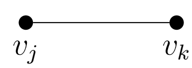

FIGURE 4.5. Tree on two vertices

Consider any tree, T. A tree is a collection of one-point unions of edges. The homology of the $\mathcal{D}$-neighborhood complex of each edge is trivial. Thus, by Theorem 1, if $\mathcal{C}$ is the chain complex associated to $DN(T, \mathcal{D})$, then $H_i(\mathcal{C})$ must be trivial for all i. □

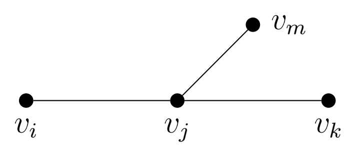

FIGURE 4.6. One-point union of edges

CHAPTER 5

Trees

As mentioned in Chapter 2, there is exactly one path between any pair of vertices in a tree. Equivalently, adding any edge to a pair of existing vertices will form a cycle. For this reason, the connectivity information in a tree seems trivial. This chapter will look at how different choices of $\mathcal{D}$ and how adding edges to a tree changes the connectivity information. First, we study the $\mathcal{D}$-neighborhood complex of path graphs for a specific choice of $\mathcal{D}$. Section 5.2 will look at how the homology of the $\mathcal{D}$-neighborhood complex of a tree changes when one edge is added to a pair of existing vertices.

5.1. The $\mathcal{D}$-Neighborhood Complex of Path Graphs

The most basic type of tree to study is a path graph. A path graph, denoted P_n, is a tree on n vertices with exactly two leaves and where all other vertices have degree 2. This graph can be thought as a straight line through n vertices (See Figure 5.1).

By Corollary 1, the homology of the $\mathcal{D}$-neighborhood complex associated to a path graph is trivial if $\mathcal{D}$ consists of consecutive distances starting with 0. The homology is no longer trivial if 0 is excluded from $\mathcal{D}$. In particular, consider the case when $\mathcal{D} = \{1\}$.

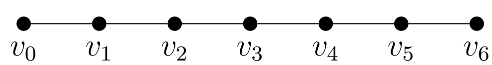

Figure 5.1. Path graph P_7

Theorem 2. *Let P_n be a path graph, and let $\mathcal{D} = \{1\}$. Suppose $\mathcal{C}$ is the chain complex associated to $DN(P_n, \mathcal{D})$. Then $H_0(\mathcal{C}) \cong \mathbb{K}^2$ and $H_i(\mathcal{C})$ is trivial for all $i > 0$.*

Proof. Let P_n be a path graph on the vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and let $\mathcal{D} = \{1\}$. Consider $DN(P_n, \mathcal{D})$. Notice that $N_0 = v_1$ and $N_{n-1} = v_{n-2}$. It follows that $N_0|N_2$ and

$N_{n-1}|N_{n-3}$. Thus, the facets of $DN(P_n, \mathcal{D})$ are 1-faces of the form $v_{j-1}v_{j+1}$ for each integer $j \in [1, n-2]$. Whenever j is even, N_j will be a 1-face of vertices with odd subscripts and when j is odd, N_j will be a 1-face of vertices with even subscripts. This implies $H_i(\mathcal{C})$ is trivial for $i > 1$. Also, this implies the set of facets can be separated into exactly two disjoint components (see Figure 5.2). Since $H_0(\mathcal{C})$ measures the number of connected components, then $H_0(\mathcal{C}) \cong \mathbb{K}^2$.

FIGURE 5.2. $\mathcal{D}$-neighborhood complex of P_7

Now, if $H_1(\mathcal{C})$ were not trivial, then at the minimum, there would exist a set of 1-faces which contains vertices of the form $a_j a_k, a_j a_m, a_k a_m$ which would form the boundary of a one-dimensional hole. If $v_{j-1}v_{j+1}$ is in this set, then this set must also contain a 1-face with vertex v_{j-1} and a 1-face with vertex v_{j+1}. However, by construction of the $\mathcal{D}$-neighborhoods, these faces would have to be $v_{j-3}v_{j-1}$ and $v_{j+1}v_{j+3}$. Since $v_{j-3} \neq v_{j+3}$, then no such set exists. Thus, $H_1(\mathcal{C})$ is trivial. □

5.2. THE $\mathcal{D}$-NEIGHBORHOOD COMPLEX OF UNICYCLIC GRAPHS

A graph is *unicyclic* if it contains exactly one cycle. One such graph is C_n. A more general example is if exactly one edge is added between two existing vertices in a tree. Corollary 1 from Section 4.4 can be extended to compute the homology of the $\mathcal{D}$-neighborhood complex of a unicyclic graph in the case of $\mathcal{D} = \{0, 1\}$. In order to do this, we start with a tree and add exactly one edge which forms one cycle. This new graph is a *copy* of the tree. The homology of the $\mathcal{D}$-neighborhood complex of a unicyclic graph will vary depending on the size of the cycle. An immediate corollary of this theorem classifies the homology of the $\mathcal{D}$-neighborhood complex of C_n for this same choice of distance set.

THEOREM 3. *Let $\mathcal{D} = \{0,1\}$. Consider a tree T with distinct leaves v_1 and v_2. Let $G = (V,E)$ be a copy of T where the edge $v_1v_2 \in E$. Let $\mathcal{B}$ be the chain complex associated to $DN(G,\mathcal{D})$.*

(i) *If $d(v_1,v_2) = 2$ in T, then $H_i(\mathcal{B})$ is trivial for all i.*

(ii) *If $d(v_1,v_2) = 3$ in T, then $H_2(\mathcal{B}) \cong \mathbb{K}$ and $H_i(\mathcal{B})$ is trivial for all $i \neq 2$.*

(iii) *If $d(v_1,v_2) \geq 4$ in T, then $H_1(\mathcal{B}) \cong \mathbb{K}$ and $H_i(\mathcal{B})$ is trivial for all $i \neq 1$.*

PROOF. Let $\mathcal{D} = \{0,1\}$. Suppose T is a tree with distinct leaves v_1 and v_2. Let $\mathcal{A} = \{\Delta_1, \delta\}$ be the associated chain complex to $DN(T,\mathcal{D})$. Let $G = (V,E)$ be a copy of T with edge $v_1v_2 \in E$. Let $\mathcal{B} = \{\Delta_2, \psi\}$ be the chain complex associated to $DN(G,\mathcal{D})$. Let $\mathbf{f} : \mathcal{A} \to \mathcal{B}$ be a chain map. Notice that $\Delta_1 \subset \Delta_2$. Let Δ_3 be a collection of i-dimensional faces from $\Delta_2 \setminus \Delta_1$. By Lemma 5, let $\mathcal{C} = \{\Delta_3, \varphi\}$ be the difference complex.

(i) Suppose $d(v_1,v_2) = 2$ in T, and suppose v_3 is the parent of v_1 and v_2 (see Figure 5.3). That is, $d(v_1,v_3) = 1$ and $d(v_2,v_3) = 1$. In Δ_1, $N_1 = v_1v_3$ and $N_2 = v_2v_3$. In Δ_2, these $\mathcal{D}$-neighborhoods change to $N_1 = v_1v_2v_3 = N_2$. However, since the face $v_1v_2v_3|N_3$ in both Δ_1 and Δ_2, it follows that $\Delta_3 = \varnothing$. That is, $\Delta_1 = \Delta_2$, so $H_i(\mathcal{A}) = H_i(\mathcal{B})$ for each i. Therefore, by Corollary 1, $H_i(\mathcal{B})$ is trivial for all i.

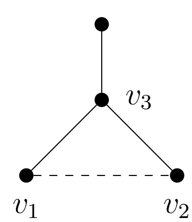

FIGURE 5.3. Tree with $d(v_1,v_2) = 2$

(ii) Suppose $d(v_1,v_2) = 3$ in T. Suppose v_3 is the parent of vertex v_1 and that v_4 is the parent of vertex v_2 (see Figure 5.4). That is, $d(v_1,v_3) = 1$ and $d(v_2,v_4) = 1$. The facets that are in Δ_3 come from the $\mathcal{D}$-neighborhoods on v_1 and v_2 in Δ_2 which are given by $N_1 = v_1v_2v_3$ and $N_2 = v_1v_2v_4$.

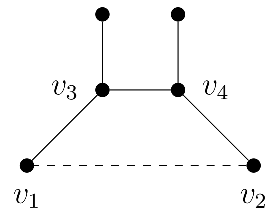

FIGURE 5.4. Tree with $d(v_1, v_2) = 3$

All of the faces in Δ_3 can be given explicitly, $\Delta_3 = \{v_1v_2v_3,\ v_1v_2v_4,\ v_1v_2\}$. This can be used to write the chain complex $\mathcal{C}$:

$$0 \longrightarrow \mathbb{K}^2 \xrightarrow{\varphi_2} \mathbb{K}^1 \xrightarrow{\varphi_1} 0$$

where $\varphi_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\varphi_1 = (0)$.

There are chain maps between the chain complexes $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ which are demonstrated below:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{K}^{F_3(\Delta_1)} \arrow[r, "\delta_3"] \arrow[d, "f_3"'] & \mathbb{K}^{F_2(\Delta_1)} \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & \mathbb{K}^{F_1(\Delta_1)} \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & \mathbb{K}^{F_0(\Delta_1)} \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r] & \mathbb{K}^{F_3(\Delta_2)} \arrow[r, "\psi_3"] \arrow[d, "g_3"'] & \mathbb{K}^{F_2(\Delta_2)} \arrow[r, "\psi_2"] \arrow[d, "g_2"'] & \mathbb{K}^{F_1(\Delta_2)} \arrow[r, "\psi_1"] \arrow[d, "g_1"'] & \mathbb{K}^{F_0(\Delta_2)} \arrow[r, "\psi_0"] \arrow[d, "g_0"'] & 0 \\
 & 0 \arrow[r] & \mathbb{K}^2 \arrow[r, "\varphi_2"] \arrow[d] & \mathbb{K}^1 \arrow[r, "\varphi_1"] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

By construction, $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$. Therefore, this sequence of chain complexes is of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{C} \longrightarrow \mathcal{C} \longrightarrow 0$$

By Lemma 7, this is a short exact sequence of chain complexes, and by the Zig-Zag Lemma, there is a long exact sequence in homology:

$$\cdots \longrightarrow H_n(\mathcal{A}) \longrightarrow H_n(\mathcal{B}) \longrightarrow H_n(\mathcal{C}) \longrightarrow H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

Recall, by Corollary 1, $H_i(\mathcal{A})$ is trivial for all i. Since φ_k is the zero map for $k \neq 2$, then $H_i(\mathcal{C})$ is trivial for $i = 0$ and for all $i \geq 3$.

Using φ_2 and φ_1, explicit computations show that $H_2(\mathcal{C}) \cong \mathbb{K}$ and $H_1(\mathcal{C}) = 0$. Filling in known homology groups, the long exact sequence becomes:

$$\cdots \longrightarrow 0 \longrightarrow H_2(\mathcal{B}) \longrightarrow \mathbb{K} \longrightarrow 0 \longrightarrow H_1(\mathcal{B}) \longrightarrow 0 \longrightarrow \cdots$$

Since this sequence is exact, then $H_2(\mathcal{B}) \cong \mathbb{K}$, and $H_i(\mathcal{B})$ is trivial for all $i \neq 2$.

(iii) Suppose $d(v_1, v_2) \geq 4$ in T. Suppose v_3 is the parent of vertex v_1 and that v_4 is the parent of vertex v_2 (see Figure 5.5). That is, $d(v_1, v_3) = 1$ and $d(v_2, v_4) = 1$.

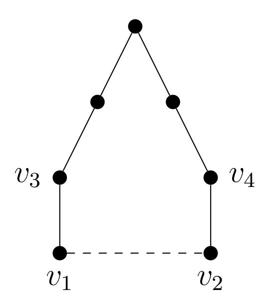

FIGURE 5.5. Tree with $d(v_1, v_2) \geq 4$

The facets that are in Δ_3 come from the $\mathcal{D}$-neighborhoods on v_1 and v_2 in Δ_2 which are given by $N_1 = v_1v_2v_3$ and $N_2 = v_1v_2v_4$. It follows that the faces in Δ_3 can be given explicitly, $\Delta_3 = \{v_1v_2v_3,\ v_1v_2v_4,\ v_1v_2,\ v_1v_4, v_2v_3\}$. This can be used to write the chain complex $\mathcal{C}$:

$$0 \longrightarrow \mathbb{K}^2 \xrightarrow{\varphi_2} \mathbb{K}^3 \xrightarrow{\varphi_1} 0$$

where φ_1 is a zero array and

$$\varphi_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$$

There are chain maps between $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ which are demonstrated below:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{K}^{F_3(\Delta_1)} \arrow[r, "\delta_3"] \arrow[d, "f_3"'] & \mathbb{K}^{F_2(\Delta_1)} \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & \mathbb{K}^{F_1(\Delta_1)} \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & \mathbb{K}^{F_0(\Delta_1)} \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r] & \mathbb{K}^{F_3(\Delta_2)} \arrow[r, "\psi_3"] \arrow[d, "g_3"'] & \mathbb{K}^{F_2(\Delta_2)} \arrow[r, "\psi_2"] \arrow[d, "g_2"'] & \mathbb{K}^{F_1(\Delta_2)} \arrow[r, "\psi_1"] \arrow[d, "g_1"'] & \mathbb{K}^{F_0(\Delta_2)} \arrow[r, "\psi_0"] \arrow[d, "g_0"'] & 0 \\
 & 0 \arrow[r] & \mathbb{K}^2 \arrow[r, "\varphi_2"] \arrow[d] & \mathbb{K}^3 \arrow[r, "\varphi_1"] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

By construction, $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$. Therefore, this sequence of chain complexes is of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{C} \longrightarrow \mathcal{C} \longrightarrow 0$$

By Lemma 7, this is a short exact sequence of chain complexes, and by the Zig-Zag Lemma, there is a long exact sequence in homology:

$$\cdots \longrightarrow H_n(\mathcal{A}) \longrightarrow H_n(\mathcal{B}) \longrightarrow H_n(\mathcal{C}) \longrightarrow H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

Recall, by Corollary 1, $H_i(\mathcal{A})$ is trivial for all i. Since φ_k is the zero map for $k \neq 2$, then $H_i(\mathcal{C})$ is trivial for $i = 0$ and for all $i \geq 3$.

The homology for $H_2(\mathcal{C})$ and $H_1(\mathcal{C})$ can be computed explicitly using φ_2 and φ_1. It follows that $H_2(\mathcal{C}) = 0$ and $H_1(\mathcal{C}) \cong \mathbb{K}$. Filling in known homology groups, the long exact sequence becomes:

$$\cdots \longrightarrow 0 \longrightarrow H_2(\mathcal{B}) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_1(\mathcal{B}) \longrightarrow \mathbb{K} \longrightarrow 0 \longrightarrow \cdots$$

Since this sequence is exact, then it follows that $H_1(\mathcal{B}) \cong \mathbb{K}$, and $H_i(\mathcal{B})$ is trivial for $i \neq 1$.

$\square$

Theorem 3 looks at the $\mathcal{D}$-neighborhood complex of a graph when an edge has been added between the leaves of a tree. One can add edges to these same vertices as one-point unions to obtain any tree in which an edge has been added between any pair of existing vertices. Therefore, by Theorem 3, the homology of the $\mathcal{D}$-neighborhood complex of any unicyclic graph is known.

CHAPTER 6

The $\mathcal{D}$-Neighborhood Complex of Cycle Graphs

6.1. Cycle Graphs

As indicated in Section 5.2, the graph of C_n can be formed from a tree. Begin with the path graph P_n and add an edge between the two leaves. The graph is now a cycle on n vertices. The homology of the $\mathcal{D}$-neighborhood complex of cycle graphs is non-trivial for several choices of $\mathcal{D}$. Section 6.2 uses the "smallest" set of consecutive distances, while subsequent sections in this chapter will look at other choices for $\mathcal{D}$.

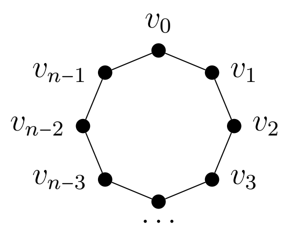

Figure 6.1. Cycle graph C_n

6.2. The $\mathcal{D}$-Neighborhood Complex with Minimal Distance Set

The "smallest" set of consecutive distances is $\mathcal{D} = \{0,1\}$. For this choice of $\mathcal{D}$, the homology of the $\mathcal{D}$-neighborhood complex of a cycle graph is a corollary to Theorem 3.

Corollary 2. *Let C_n be a cycle graph and let $\mathcal{D} = \{0,1\}$. Let $\mathcal{C}$ be the chain complex associated to $DN(C_n, \mathcal{D})$.*

(i) *If $n = 3$, then $H_i(\mathcal{C})$ is trivial for all $i > 0$.*

(ii) *If $n = 4$, then $H_2(\mathcal{C}) \cong \mathbb{K}$, and $H_i(\mathcal{C})$ is trivial for $i \neq 2$.*

(iii) *If $n > 4$, then $H_1(\mathcal{C}) \cong \mathbb{K}$, and $H_i(\mathcal{C})$ is trivial for $i \neq 1$.*

Proof. Consider a path graph P_n. Add an edge between the two leaves to form the cycle graph C_n.

(i) If $n = 3$, then the leaves of P_3, say v_0 and v_2, are of distance $d(v_0, v_2) = 2$, so this is case (i) from Theorem 3.

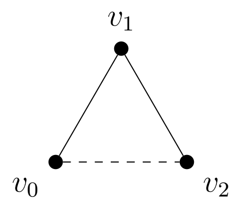

FIGURE 6.2. P_3 with edge added to form C_3

(ii) If $n = 4$, then the leaves of P_4, say v_0 and v_3, are of distance $d(v_0, v_3) = 3$, so this is case (ii) from Theorem 3.

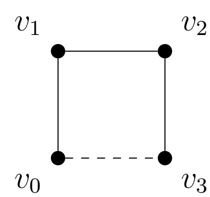

FIGURE 6.3. P_4 with edge added to form C_4

(iii) If $n > 4$, then the leaves of P_n, say v_0 and v_{n-1}, are of distance $d(v_0, v_{n-1}) = n - 1 \geq 4$, so this is case (iii) from Theorem 3.

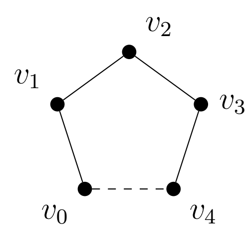

FIGURE 6.4. P_5 with edge added to form C_5

The homology of the $\mathcal{D}$-neighborhood complex of C_n follows from Theorem 3. $\square$

By Corollary 2, the $\mathcal{D}$-neighborhood complex of C_4 has one boundary of a 2-dimensional "hole". The facets of $DN(C_4, \mathcal{D})$ are $\{v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3\}$. One can see that these facets form a 2-sphere, or a hollow tetrahedron.

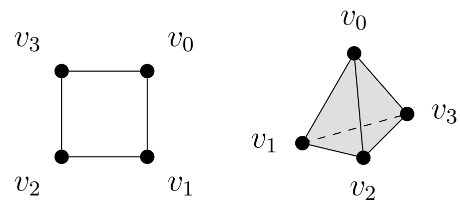

FIGURE 6.5. Graph C_4 (left) and $DN(C_4, \{0, 1\})$ (right)

One way to think about the $\mathcal{D}$-neighborhood complex of C_n when $n > 4$, is to suppose the vertices of C_n are labeled consecutively $v_0, v_1, \ldots v_{n-1}$. The $\mathcal{D}$-neighborhoods are of the form $v_{i-1}v_iv_{i+1}$ (mod n) for all $i \in [0, n-1]$. Therefore, there are n facets of dimension 2 which can be thought of as solid triangles. Each triangle, $v_{i-1}v_iv_{i+1}$, shares an edge with two other triangles, $v_{i-2}v_{i-1}v_i$ and $v_iv_{i+1}v_{i+2}$. Depending on the parity of n, the $\mathcal{D}$-neighborhood complex of C_n will either be a cylinder or a Möbius band. However, the homology of the chain complex for both simplicial complexes will still be one-dimensional at $H_1(\mathcal{C})$ [Mun84]. See Figures 6.6 and 6.7

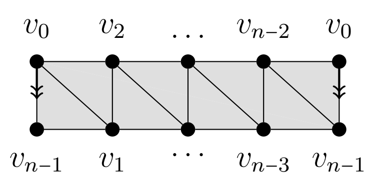

FIGURE 6.6. $\mathcal{D}$-neighborhood complex of C_n when n is even

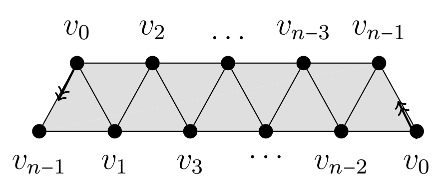

FIGURE 6.7. $\mathcal{D}$-neighborhood complex of C_n when n is odd

6.3. One-Point Unions of Cycles

There is a class of graphs called *Dutch windmill* graphs, denoted $D_3^{(m)}$, which consists of m copies of C_3 joined at a single vertex. These graphs can be extended to $D_n^{(m)}$, which consists of m copies of C_n joined at a single vertex. Theorem 1 and Corollary 2 can be used to compute the homology of the $\mathcal{D}$-neighborhood complex of $D_n^{(m)}$ when $\mathcal{D} = \{0, 1\}$.

FIGURE 6.8. Dutch windmill graphs D_3^3 and D_4^3

COROLLARY 3. *Let* $\mathcal{D} = \{0, 1\}$ *and let* $D_n^{(m)}$ *be a Dutch windmill graph. Let* $\mathcal{C}$ *be the chain complex associated to* $DN(D_n^{(m)}, \mathcal{D})$.

(i) *If* $n = 3$, *then* $H_i(\mathcal{C})$ *is trivial for all* $i > 0$.

(ii) *If* $n = 4$, *then* $H_2(\mathcal{C}) \cong \mathbb{K}^m$, *and* $H_i(\mathcal{C})$ *is trivial for* $i \neq 2$.

(iii) *If* $n > 4$, *then* $H_1(\mathcal{C}) \cong \mathbb{K}^m$, *and* $H_i(\mathcal{C})$ *is trivial for* $i \neq 1$.

PROOF. Let $\mathcal{D} = \{0, 1\}$ and let $D_n^{(m)}$ be a Dutch windmill graph. Let $\mathcal{C}$ be the chain complex associated to $DN(D_n^{(m)}, \mathcal{D})$. Using the same choice of $\mathcal{D}$, let $\mathcal{A}$ be the chain complex associated to $DN(C_n, \mathcal{D})$.

Since $D_n^{(m)}$ is a one-point union of m copies of C_n, then by Theorem 1, $H_i(\mathcal{C}) = \bigoplus_m H_i(\mathcal{A})$. The homology groups $H_i(\mathcal{A})$ are given in Corollary 2. $\square$

6.4. THE $\mathcal{D}$-NEIGHBORHOOD COMPLEX FOR $\mathcal{D} = \{1\}$

Just as Section 5.1 computes the homology of the $\mathcal{D}$-neighborhood complex of P_n when $\mathcal{D} = \{1\}$, this section will compute the homology of the $\mathcal{D}$-neighborhood complex of the cycle graph, C_n, for the same choice of $\mathcal{D}$. Recall, the $\mathcal{D}$-neighborhood complex of a path graph is two disjoint sequences of 1-faces. However, for C_n, the $\mathcal{D}$-neighborhood complex is either homotopy equivalent to a circle or homotopy equivalent to two circles. The $\mathcal{D}$-neighborhood complex depends on the parity of n.

THEOREM 4. *Let C_n be a cycle graph and let $\mathcal{D} = \{1\}$. Let $\mathcal{C}$ be the chain complex associated to $DN(C_n, \mathcal{D})$.*

(i) *If n is even, then $H_0(\mathcal{C}) \cong \mathbb{K}^2$, $H_1(\mathcal{C}) \cong \mathbb{K}^2$, and $H_i(\mathcal{C})$ is trivial for $i > 1$.*

(ii) *If n is odd, then $H_1(\mathcal{C}) \cong \mathbb{K}$, and $H_i(\mathcal{C})$ is trivial for $i \neq 1$.*

PROOF. Let C_n be a cycle graph and suppose $\mathcal{D} = \{1\}$. Suppose the vertices are labeled in consecutive order, $v_0, v_1, \ldots, v_{n-1}$. Then the $\mathcal{D}$-neighborhood on vertex v_j is of the form $N_j = v_{j-1}v_{j+1} \pmod{n}$, for each $j \in [0, n-1]$. It follows that the facets of $DN(C_n, \mathcal{D})$ are $\{v_0v_2,\ v_1v_3,\ v_2v_4, v_3v_5, \ldots, v_{n-2}v_0,\ v_{n-1}v_1\}$. Since the facets are all of dimension 1, this implies that the only non-trivial homology can come from $H_0(\mathcal{C})$ and $H_1(\mathcal{C})$.

(i) If n is even, then each edge pair $v_{j-1}v_{j+1}$ will either contain only even numbered subscripts or odd numbered subscripts. This divides $DN(C_n, \mathcal{D})$ into exactly two disjoint components: $\{v_0v_2,\ v_2v_4,\ \ldots, v_{n-2}v_0\}$ and $\{v_1v_3,\ v_3v_5,\ \ldots, v_{n-1}v_1\}$, (See Figure 6.9). This implies $H_0(\mathcal{C}) \cong \mathbb{K}^2$. Furthermore, each component forms the boundary of a circle. Together, these two circles give $H_1(\mathcal{C}) \cong \mathbb{K}^2$.

FIGURE 6.9. $\mathcal{D}$-neighborhood complex of C_n, n even

(ii) Suppose n is odd. Then the $\mathcal{D}$-neighborhood on vertex v_{n-1} is $N_{n-1} = v_{n-2}v_0$, which is an edge consisting of a vertex with an odd numbered subscript and a vertex with an even numbered subscript. Rather than have two disjoint cycles consisting of vertices of strictly even numbered subscripts or odd numbered subscripts, this edge connects the two circles from case (i) to create one component which is the boundary of a circle (See Figure 6.10). Therefore $H_1(\mathcal{C}) \cong \mathbb{K}$ and $H_i(\mathcal{C})$ is trivial for $i \neq 1$.

□

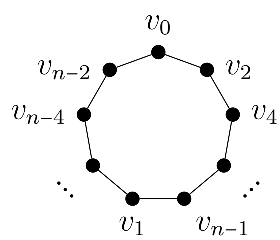

FIGURE 6.10. $\mathcal{D}$-neighborhood complex of C_n, n odd

6.5. The $\mathcal{D}$-Neighborhood Complex with Maximal Distance Set

Corollary 2 looked at the case where $\mathcal{D}$ is the "minimum" set of consecutive distances, namely $\mathcal{D} = \{0, 1\}$. Proposition 3 showed that once $\mathcal{D} = \{0, 1, \ldots, \operatorname{diam}(G)\}$, then the homology of the $\mathcal{D}$-neighborhood complex would always be trivial. For this reason, we regard $\mathcal{D} = \{0, 1, \ldots, \operatorname{diam}(C_n) - 1\}$ as the "maximum" choice of distance set. By construction of the $\mathcal{D}$-neighborhood on a vertex, the following case only applies when n is even. When n is even, then $\operatorname{diam}(G) = \frac{n}{2}$.

THEOREM 5. *Let n be even and let $\mathcal{D} = \left\{0, 1, \ldots, \frac{n}{2} - 1\right\}$. Let $\mathcal{C}$ be the chain complex associated to $DN(C_n, \mathcal{D})$. Then $H_{n-2}(\mathcal{C}) \cong \mathbb{K}$ and $H_i(\mathcal{C})$ is trivial for all other i.*

PROOF. Let n be even and let $\mathcal{D} = \left\{0, 1, \ldots, \frac{n}{2} - 1\right\}$. Consider $DN(C_n, \mathcal{D})$. For each $j \in [0, n-1]$, $N_j = v_0 \cdots \widehat{v_k} \cdots v_{n-1}$ where vertex v_k is excluded and $k = j + \frac{n}{2} \pmod{n}$. These

n distinct $\mathcal{D}$-neighborhoods are the facets of $DN(C_n, \mathcal{D})$. Notice all facets have dimension $n-2$.

Now consider an $(n-1)$-ball which can be represented as the facet $v_0 \cdots v_{n-1}$. Notice that applying the boundary operator yields

$$\delta(v_0 \cdots v_{n-1}) = \sum_{j=1}^{n-1} (-1)^j N_j$$

This is to say, the facets of $DN(C_n, \mathcal{D})$ form the boundary of the $(n-1)$-ball. Therefore, $DN(C_n, \mathcal{D})$ can be represented as an $(n-2)$-sphere and thus, $H_{n-2}(\mathcal{C}) \cong \mathbb{K}$ and $H_i(\mathcal{C})$ is trivial for all other i [Mun84]. $\square$

If n were odd. Then $\operatorname{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$. This means that each $\mathcal{D}$-neighborhood is of the form $N_i = v_0 \cdots \widehat{v_j} \widehat{v}_{j+1} \cdots v_{n-1}$, where $j = i + \lfloor \frac{n}{2} \rfloor \pmod{n}$. This is no longer the boundary of an $(n-1)$-ball. As a result, the homology of $DN(C_n, \mathcal{D})$ is more difficult to predict for various choices of n when n is odd.

CHAPTER 7

The $\mathcal{D}$-Neighborhood Complex of Vertex Weighted Trees

Up to this point, the $\mathcal{D}$-neighborhoods of the vertices of a graph have been formed using the same distance set, $\mathcal{D}$. This chapter studies the case when $\mathcal{D}$ varies for each vertex. We assign weights to the vertices of the graph. These weights are used to determine the distance set that applies to the $\mathcal{D}$-neighborhood on each vertex. In Section 7.3, we compute the homology of the $\mathcal{D}$-neighborhood complex of vertex weighted trees.

7.1. The $\mathcal{D}$-Neighborhood Complex of Weighted Graphs

The following definitions formalize the process for constructing the $\mathcal{D}$-neighborhood complex of a graph where $\mathcal{D}$ varies for each vertex.

Definition 17. *A* vertex weighted graph, *$G_w = (V, E)$, is a graph in which each vertex $v_i \in V$ is assigned a weight $w_i \in \mathbb{Z}^+$.*

Definition 18. *Suppose $G_w = (V, E)$ is a vertex weighted graph. Define $\mathcal{D}_i = \{0, \ldots, w_i\}$. The $\mathcal{D}$-neighborhood of vertex v_i will be given by $N_i = \{v_j : d(v_i, v_j) \in \mathcal{D}_i\}$. The* $\mathcal{D}$-neighborhood complex of a vertex weighted graph*, denoted $DN(G_w, \mathcal{D})$, is the simplicial complex with simplex σ included whenever $\sigma | N_j$ for some N_j.*

Just as before with the value of $d \in \mathcal{D}$, without loss of generality, we assume each weight, w_i, is a positive integer less than the diameter of the graph, i.e. $1 \le w_i < \text{diam}(G_w)$. If $w_i \ge \text{diam}(G_w)$, then N_i is a complete $\mathcal{D}$-neighborhood. By Proposition 3, the homology of the associated chain complex will be trivial.

Allowing different distance sets for the $\mathcal{D}$-neighborhoods of the vertices in a graph can create interesting connectivity information. The homology of the $\mathcal{D}$-neighborhood complex will depend heavily on the various choices for weights on the vertices.

For example, consider the vertex weighted graph of C_8 in Figure 7.1.

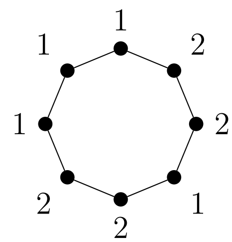

FIGURE 7.1. Vertex weighted C_8, version 1

The chain complex associated to the $\mathcal{D}$-neighborhood complex of this weighted graph is given below.

$$0 \longrightarrow \mathbb{K}^4 \longrightarrow \mathbb{K}^{18} \longrightarrow \mathbb{K}^{32} \longrightarrow \mathbb{K}^{25} \longrightarrow \mathbb{K}^8 \longrightarrow 0$$

Explicit computations show the homology of the $\mathcal{D}$-neighborhood complex of this vertex weighted graph is trivial for all i. In Chapter 6, we showed that $H_1(\mathcal{C})$ would have been one dimensional in the case of $\mathcal{D} = \{0,1\}$. Notice that by increasing this distance set for four vertices, the homology of the $\mathcal{D}$-neighborhood complex becomes trivial.

It will not always be the case that the $\mathcal{D}$-neighborhood complex of a weighted C_8 has trivial homology. From Figure 7.1, we can increase one vertex of weight 1 to weight 2 and rearrange these weights to obtain Figure 7.2.

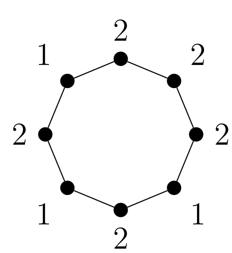

FIGURE 7.2. Vertex weighted C_8, version 2

In version 1, there were four facets of dimension 4. In version 2, there are five facets of dimension 4 which is a result of increasing the weight of one of the vertices. The chain complex associated to the $\mathcal{D}$-neighborhood complex of the graph in Figure 7.2 follows.

$$0 \longrightarrow \mathbb{K}^5 \longrightarrow \mathbb{K}^{23} \longrightarrow \mathbb{K}^{39} \longrightarrow \mathbb{K}^{27} \longrightarrow \mathbb{K}^8 \longrightarrow 0$$

In this case, $H_2(\mathcal{C}) \cong \mathbb{K}$. Again, this differs from the case of $\mathcal{D} = \{0,1\}$ on an unweighted C_8. It can be shown that when $\mathcal{D} = \{0,1,2\}$ in an unweighted C_8, then the homology of the chain complex associated to $DN(C_8,\mathcal{D})$ is $H_2(\mathcal{C}) \cong \mathbb{K}^3$. Intuitively, one would assume that as the weights of more of the vertices are increased to 2, the homology of the $\mathcal{D}$-neighborhood complex of a weighted C_8 will approach the homology of the $\mathcal{D}$-neighborhood complex of an unweighted C_8 with $\mathcal{D} = \{0,1,2\}$.

7.2. Maximally Weighted Graphs

Consider the case when vertex v_i is assigned weight w_i and its corresponding $\mathcal{D}$-neighborhood $N_i|N_j$ for some N_j. There are times when changing w_i will not change the fact that $N_i|N_j$. It is possible to increase the weight, w_i, until it is "maximal". That is, the weight can be increased until it changes the corresponding simplicial complex, so that $N_i \nmid N_j$. In other words, the $\mathcal{D}$-neighborhood, N_i, picks up extra 0-faces, or vertices, which changes the faces in the $\mathcal{D}$-neighborhood complex. The following definition describes the notion of when a vertex weighted graph has maximal weight.

Definition 19. *A vertex weighted graph is* maximally weighted *if there are no complete $\mathcal{D}$-neighborhoods and if increasing any of the weights of the vertices changes the associated $\mathcal{D}$-neighborhood complex.*

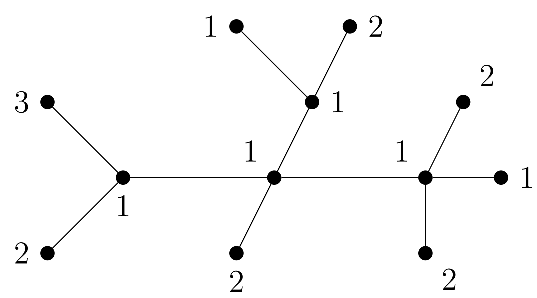

FIGURE 7.3. A vertex weighted tree

Let T_w be a vertex weighted tree (See Figure 7.3). The weights of the vertices in this tree can be increased until the tree is maximally weighted (See Figure 7.4). Notice with these changes in weights, the facets of the associated $\mathcal{D}$-neighborhood complex do not change.

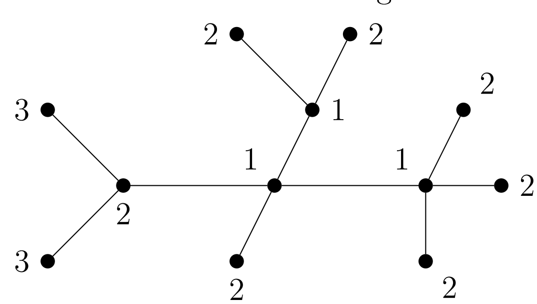

FIGURE 7.4. A maximally weighted tree

The weights on the leaves of the tree determine the weights of the vertices adjacent to each leaf. In some cases, the weight of a vertex adjacent to a leaf might already be maximal. In other cases, this weight will need to be increased in order for the tree to be maximally weighted. The following Lemma describes the weight of such vertices.

LEMMA 8. *Let T_w be a maximally weighted tree and let v_0 be a leaf. Suppose $d(v_0, v_1) = 1$. Then the weight of v_1 is $w_1 = w_0 - 1$.*

PROOF. Suppose T_w is a vertex weighted tree. Let v_0 be a leaf with weight w_0 and suppose vertex v_1 is adjacent to v_0; that is, $d(v_0, v_1) = 1$. Suppose the weight of vertex v_1 is $w_1 < w_0 - 1$. Then $N_1 | N_0$. It will be shown that the weight associated to v_1 can be increased to $w_1 = w_0 - 1$ and this will not change the $\mathcal{D}$-neighborhood complex of T_w.

Let v_j be an arbitrary vertex such that $v_j|N_1$ in $DN(T_w, \mathcal{D})$. Then it follows that

$$\begin{aligned} d(v_1, v_j) &\leq w_1 \\ 1 + d(v_1, v_j) &\leq w_1 + 1 \\ d(v_0, v_1) + d(v_1, v_j) &\leq w_1 + 1 \\ d(v_0, v_j) &\leq w_0 \end{aligned}$$

This implies that $v_j|N_0$, which means that $N_1|N_0$. Therefore, increasing the weight of w_1 does not add any new simplices to $DN(T_w, \mathcal{D})$.

Next, suppose $w_1 > w_0 - 1$, or equivalently, $w_1 + 1 > w_0$. Then $N_0|N_1$. It will be shown that the weight associated to v_0 can be increased to $w_0 = w_1 + 1$ without changing $DN(T_w, \mathcal{D})$.

Let v_j be an arbitrary vertex such that $v_j|N_0$. Then it follows that

$$\begin{aligned} d(v_0, v_j) &\leq w_0 \\ d(v_0, v_j) &\leq w_1 + 1 \\ d(v_0, v_1) + d(v_1, v_j) &\leq w_1 + 1 \\ d(v_1, v_j) &\leq w_1 \end{aligned}$$

This implies that $v_j|N_1$, which means that $N_0|N_1$. Therefore, increasing the weight of w_0 does not add any new simplices to $DN(T_w, \mathcal{D})$. Therefore, $w_1 = w_0 - 1$. $\square$

As a consequence of this lemma, in a maximally weighted tree, if v_0 is a leaf and v_1 is an adjacent vertex, then it will be true that $N_0 = N_1$.

7.3. The Main Theorem

The following theorem shows that for any vertex weighted tree, the homology of the $\mathcal{D}$-neighborhood complex will be trivial regardless of the weights assigned to the vertices.

To prove this theorem, we compare the $\mathcal{D}$-neighborhood complex of one vertex weighted tree with the $\mathcal{D}$-neighborhood complex of the same vertex weighted tree with one more leaf. Careful bookkeeping allows one to track the differences between these simplicial complexes. In order to record the simplices that are contained in the $\mathcal{D}$-neighborhood complex of the tree with the extra leaf, we assume the trees are maximally weighted. By definition of maximally weighted graphs, we will not lose the information contained in the facets of the $\mathcal{D}$-neighborhood complex. We are then able to use the Zig-Zag Lemma by constructing a short exact sequence of chain complexes. Knowledge of the homology groups on two of the chain complexes allows us to find the homology of the third chain complex.

Theorem 6. *Let T_w be a vertex weighted tree. Let $\mathcal{C}$ be the chain complex associated to $DN(T_w, \mathcal{D})$. Then $H_i(\mathcal{C})$ is trivial for all i.*

Proof. We use induction on the number of vertices on T_w to prove our result.

Suppose T_w is a vertex weighted tree on 3 vertices (see Figure 7.5). Then T_w must be a vertex weighted path graph.

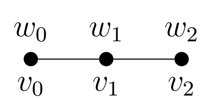

Figure 7.5. Weighted tree on 3 vertices

Since $w_j \geq 1$ for all j, then $N_1 = v_0v_1v_2$. Since N_1 is a complete $\mathcal{D}$-neighborhood, then by Proposition 2, $H_i(\mathcal{C})$ is trivial for all i.

Let T_2 be a vertex weighted tree on $n+1$ vertices, where vertex v_j has corresponding weight $w_j \geq 1$. Without loss of generality, assume T_2 is maximally weighted. Let $DN(T_2, \mathcal{D}) = \Delta_2$ be the $\mathcal{D}$-neighborhood complex of the vertex weighted tree T_2. Let T_1 be the vertex weighted tree obtained by removing a leaf, v_0, from T_2. Now T_1 has n vertices[1]. Let $DN(T_1, \mathcal{D}) = \Delta_1$ be the $\mathcal{D}$-neighborhood complex of the vertex weighted tree T_1. By this construction, $\Delta_1 \subset \Delta_2$. Assume that the $\mathcal{D}$-neighborhood complex of a vertex weighted tree with n vertices has trivial homology; that is, if $\mathcal{A}$ is the chain complex associated to Δ_1, then $H_i(\mathcal{A})$ is trivial for all i. Suppose $\mathcal{B}$ is the chain complex associated to Δ_2, it suffices to show that $H_i(\mathcal{B})$ is trivial.

Suppose $d(v_0, v_1) = 1$, i.e. v_1 is adjacent to v_0. Since T_2 is maximally weighted, then by Lemma 8, $w_1 = w_0 - 1$. It follows that $N_0 = N_1$ in Δ_2.

Let $\mathbf{f}$ be a chain map $\mathbf{f} : \mathcal{A} \to \mathcal{B}$. Since $\Delta_1 \subset \Delta_2$, then let Δ_3 be the collection of faces from $\Delta_2 \setminus \Delta_1$. Consider $\sigma \in \Delta_2$ where $\sigma | N_0$ and $v_0 | \sigma$. Then it must also be true that $\sigma | N_1$. Suppose $\tau | \sigma$, but $v_0 \nmid \tau$. Since $\tau | N_1$ by Definition 13, then $\tau \in \Delta_1$. However, since $v_0 | \sigma$, then by construction of T_1, $\sigma \notin \Delta_1$. Therefore, for every $\alpha \in \Delta_3$, it follows that $v_0 | \alpha$. This is to say, all faces in Δ_3 are divisible by v_0. All faces which are not divisible by v_0 are in Δ_1 from the fact that T_2 is maximally weighted. Notice that Δ_3 contains exactly one facet and this is N_0 from Δ_2. Let $\dim(N_0) = d$; recall this means that the cardinality $|N_0| = d + 1$. Since each face in Δ_3 is divisible by v_0, then fixing v_0 leaves $(d+1) - 1 = d$ vertices and choosing j of these yields a face of dimension j in Δ_3. Thus, there are $\binom{d}{j}$ faces of dimension j in Δ_3, where $1 \leq j \leq d$. In other words, $|F_j(\Delta_3)| = \binom{d}{j}$.

[1]It is important to note the subscripts on the vertex weighted trees here are to distinguish the trees and are not an indication of the weights assigned to the vertices.

By Lemma 5, there is a difference complex, $\mathcal{C}$, associated to Δ_3.

$$0 \longrightarrow \mathbb{K}^{\binom{d}{d}} \xrightarrow{\varphi_d} \cdots \longrightarrow \mathbb{K}^{\binom{d}{i}} \xrightarrow{\varphi_i} \mathbb{K}^{\binom{d}{i-1}} \longrightarrow \cdots \xrightarrow{\varphi_1} \mathbb{K}^{\binom{d}{0}} \longrightarrow 0$$

Notice this chain complex is a copy of the (augmented) Koszul Complex and therefore, $H_i(\mathcal{C}) = 0$ for all i [MS05].

Construct chain maps between these three chain complexes as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{K}^{F_i(\Delta_1)} \arrow[r, "\delta_i"] \arrow[d, "f_i"'] & \mathbb{K}^{F_{i-1}(\Delta_1)} \arrow[r] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r] & \mathbb{K}^{F_i(\Delta_2)} \arrow[r, "\psi_i"] \arrow[d, "g_i"'] & \mathbb{K}^{F_{i-1}(\Delta_2)} \arrow[r] \arrow[d, "g_{i-1}"'] & \cdots \\
\cdots \arrow[r] & \mathbb{K}^{F_i(\Delta_3)} \arrow[r, "\varphi_i"] \arrow[d] & \mathbb{K}^{F_{i-1}(\Delta_3)} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

The structure of these three chain complexes is such that $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$. Therefore, this sequence is of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{C} \longrightarrow \mathcal{C} \longrightarrow 0$$

By Lemma 7 there is a short exact sequence of chain complexes. By the Zig-Zag Lemma, this induces a long exact sequence on homology

$$\cdots \longrightarrow H_n(\mathcal{A}) \longrightarrow H_n(\mathcal{B}) \longrightarrow H_n(\mathcal{C}) \longrightarrow H_{n-1}(\mathcal{A}) \longrightarrow \cdots$$

Recall, $H_i(\mathcal{A})$ is trivial for all i, by assumption. Since $H_i(\mathcal{C}) = 0$ for all i, then the long exact sequence implies that $H_i(\mathcal{B})$ is trivial. $\square$

The homology of the Koszul complex is characteristic free which implies that the proof for Theorem 6 shows that the homology of the $\mathcal{D}$-neighborhood complex of a vertex weighted tree is also characteristic free. Thus, Theorem 6 applies to a very general class of $\mathcal{D}$-neighborhood complexes of trees.

In Chapter 4, it was shown that the homology of the chain complex of the $\mathcal{D}$-neighborhood complex of a tree with distance set $\mathcal{D} = \{0, 1, \dots, d\}$ must be trivial. There is another method for showing this to be true using Theorem 6 where the weight of each vertex is the same.

COROLLARY 4. *Let $T = (V, E)$ be a tree. Let $\mathcal{D} = \{0, 1, \dots, d\}$, and let $\mathcal{C}$ be the associated chain complex to $DN(T, \mathcal{D})$. Then $H_i(\mathcal{C})$ is trivial for all i.*

PROOF. Consider a vertex weighted tree, $T_w = (V, E)$, where $w_j = d$ for each vertex $v_j \in V$. By Theorem 6, $H_i(\mathcal{C})$ mus be trivial for all i. $\square$

In Section 7.1, we showed that adjusting the weights of the vertices of C_8 will impact the homology of the $\mathcal{D}$-neighborhood complex. This should not be surprising since each $\mathcal{D}$-neighborhood is gathering a different amount of connectivity information in the graph. It is interesting that regardless of the weights assigned to the vertices in a tree, the homology of the $\mathcal{D}$-neighborhood complex will still be trivial.

CHAPTER 8

Conclusion

Beginning with a graph, one can build a simplicial complex based on the connectivity of the vertices. Specifically, given a subset of graph distances, $\mathcal{D}$, the $\mathcal{D}$-neighborhood of a vertex, v, is the set of all vertices in the graph with distance in $\mathcal{D}$ from v. The collection of $\mathcal{D}$-neighborhoods on all of the vertices in the graph generate the $\mathcal{D}$-neighborhood complex. It is possible to study the topological features of this simplicial complex.

Much is known about the way in which vertices are connected in a tree since there is exactly one path between each pair of vertices. When $\mathcal{D}$ is a set of consecutive distances beginning with 0, the $\mathcal{D}$-neighborhood complex of a tree has trivial homology. This means the simplicial complex bounds no holes. What may not be intuitive is that if $\mathcal{D}$ is a set of consecutive distances which varies for each vertex in a tree, the homology of the $\mathcal{D}$-neighborhood complex is still trivial.

The connectivity information of a cycle graph is also well known. There are exactly two paths between each pair of vertices; however, there may (or may not) be only one path of shortest distance. The homology of the $\mathcal{D}$-neighborhood complex is non-trivial for many choices of $\mathcal{D}$. In the case when $\mathcal{D} = \{0, 1\}$, once there are more than 5 vertices in the graph, the $\mathcal{D}$-neighborhood complex is equivalent to the boundary of a circle. This agrees with the fact that the graph is also a circle. Once we look at how the vertices in the graph are connected to all but one vertex, i.e. when $\mathcal{D} = \{0, 1, \ldots, \operatorname{diam}(C_n) - 1\}$ for n even, then we find that the $\mathcal{D}$-neighborhood complex is a (hollow) sphere of dimension $n - 2$. Preliminary findings suggest that when $\mathcal{D}$ is between these two distance sets, the $\mathcal{D}$-neighborhood complex has the same homology as the wedge sum of spheres. This means that if we think of looking at the filtration of the $\mathcal{D}$-neighborhood complex of C_n when $\mathcal{D} = \{0, 1, \ldots, d\}$ and we increment d by 1, then the $\mathcal{D}$-neighborhood complex begins as a circle, changes into some number of hyper-spheres, these turn into an $(n-2)$-sphere, which then fills into a solid n-ball.

When $\mathcal{D}$ is a set of consecutive distances which begins with 0, the homology of the $\mathcal{D}$-neighborhood complex of any two graphs joined at a vertex will be equivalent to the sum of the homology groups of the $\mathcal{D}$-neighborhood complex of each graph. This means that the topological features present in each of the $\mathcal{D}$-neighborhood complexes of the individual graphs are preserved. Furthermore, this result provides a method for decomposing graphs into subgraphs in order to detect the structure of the associated $\mathcal{D}$-neighborhood complex.

One method for looking at the global structure of a graph is to look at the topological structure of the associated $\mathcal{D}$-neighborhood complex for a particular choice of $\mathcal{D}$. In the case of two graphs with similar local structure, this can be one way to differentiate the two graphs rather than using an isomorphism test. At times, it may be preferable to categorize two graphs as being "similar". For example, a graph with a few edges joined at a vertex will have a $\mathcal{D}$-neighborhood complex with the same topological features as a graph without these edges. As a result, we could say that although these graphs are not isomorphic, they are similar.

8.1. Open Questions

There are several possible directions one can take with this research. One such direction would be to continue to explore the $\mathcal{D}$-neighborhood complex of other classes of graphs. For example, we have a conjecture for the homology of the $\mathcal{D}$-neighborhood complex of bipartite graphs when $\mathcal{D} = \{0, 1\}$. Since the connectivity information in these graphs is predictable, generalizing to $\mathcal{D} = \{0, 1, \ldots, d\}$ will be a task in extending the current conjecture. Furthermore, using a conjecture for the homology of the $\mathcal{D}$-neighborhood complex of multipartite graphs for $\mathcal{D} = \{0, 1\}$, one could expect to generalize this case to $\mathcal{D} = \{0, 1, \ldots, d\}$ as well.

Another direction for research is to explore connections between the $\mathcal{D}$-neighborhood complex and other simplicial complexes. The Vietoris-Rips complex of a set of points in a metric space is related to the clique complex of a graph. Currently, it is predicted that while the $\mathcal{D}$-neighborhood complex is distinct from the Vietoris-Rips Complex, the homology of the $\mathcal{D}$-neighborhood complex of a cycle graph for sets of consecutive distances beginning with 0 is the same as the Vietoris-Rips complex of a circle of evenly spaced points. This is ongoing work joint with Henry Adams and Michał Adamaszek. The homotopy types of the clique complex of powers of cycle graphs was proved by Adamaszek [Ada13]. By relating the homology of the $\mathcal{D}$-neighborhood complex of a cycle graph with the homology of the clique complex, we hope to solidify the connection with the Vietoris-Rips complex.

BIBLIOGRAPHY

[Ada13] Michał Adamaszek. Clique Complexes and Graph Powers. *Israel J. Math.*, 196(1):295–319, 2013.

[Arm83] Mark Anthony Armstrong. *Basic Topology*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1979 original.

[Bap10] R. B. Bapat. *Graphs and Matrices*. Universitext. Springer, London, 2010.

[Car09] Gunnar Carlsson. Topology and Data. *Bull. Amer. Math. Soc. (N.S.)*, 46(2):255–308, 2009.

[Cso07] Péter Csorba. Homotopy Types of Box Complexes. *Combinatorica*, 27(6):669–682, 2007.

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[Die10] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[EH10] Herbert Edelsbrunner and John L. Harer. *Computational Topology*. American Mathematical Society, Providence, RI, 2010. An introduction.

[Eis95] David Eisenbud. *Commutative Algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[GR01] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[Gro08] Jonathan L. Gross. *Combinatorial Methods with Computer Applications*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2008.

[Hal58] Paul R. Halmos. *Finite-Dimensional Vector Spaces.* The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1958. 2nd ed.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[HHM08] John M. Harris, Jeffry L. Hirst, and Michael J. Mossinghoff. *Combinatorics and Graph Theory.* Undergraduate Texts in Mathematics. Springer, New York, second edition, 2008.

[HW60] P. J. Hilton and S. Wylie. *Homology Theory: An Introduction to Algebraic Topology.* Cambridge University Press, New York, 1960.

[Jon08] Jakob Jonsson. *Simplicial Complexes of Graphs*, volume 1928 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2008.

[Kah07] Matthew Kahle. The Neighborhood Complex of a Random Graph. *J. Combin. Theory Ser. A*, 114(2):380–387, 2007.

[Kin93] L. Christine Kinsey. *Topology of Surfaces.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1993.

[Koz06] Dmitry N. Kozlov. Simple Homotopy Types of Hom-Complexes, Neighborhood Complexes, Lovász Complexes, and Atom Crosscut Complexes. *Topology Appl.*, 153(14):2445–2454, 2006.

[Koz08] Dmitry Kozlov. *Combinatorial Algebraic Topology*, volume 21 of *Algorithms and Computation in Mathematics.* Springer, Berlin, 2008.

[Mas67] William S. Massey. *Algebraic Topology: An Introduction.* Harcourt, Brace & World, Inc., New York, 1967.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*, volume 227 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2005.

[Mun84] James R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.