Computer Programs of Probability Distribution Functions in Hydrology (*)

by

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PREFACE

A manual of computer programs of probablity distribution functions of common use in hydrology and water resources is presented herein. The functions used in the manual are the Normal. Lognormal. Gamma, Log-Pearson III, Gumbel and the General Extreme Value. The programs cover four main topics: the estimation of parameters by the method of moments and the method of maximum likelihood, the determination of the PDF and CDF given the parameters, the determination of the value X given the CDF (the inverse problem) and the determination of the confidence limits. For each topic, the analytical procedure is described in some detail as well as the programming algorithm. Each program is written in the form of independent subroutines which may be lumped together for solving specific problems.

Should a user detect any inaccuracies or errors, the writers will appreciate the corresponding comments.

ACKNOWLEDGEMENTS

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DISCLAIMER

PROBABILITY DISTRIBUTION FUNCTIONS IN HYDROLOGY

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1.0 NORMAL PROBABILITY DISTRIBUTION FUNCTION

1.1 General Properties

The probability density function (PDF) of the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\} \tag{1}$$

where μ is the location parameter and σ is the scale parameter. It is continuous, $-\infty \leq x \leq +\infty$, and tends to zero as x tends to $-\infty$ or $+\infty$. It has a symmetrical bell shape as shown in Fig. 1 and, as a result, the mean, mode and median are equal. It is a common practice to use the notation $N(\mu,\sigma^2)$ or $N(\mu,\sigma)$ when referring to a normal random variable with mean μ and variance σ^2.

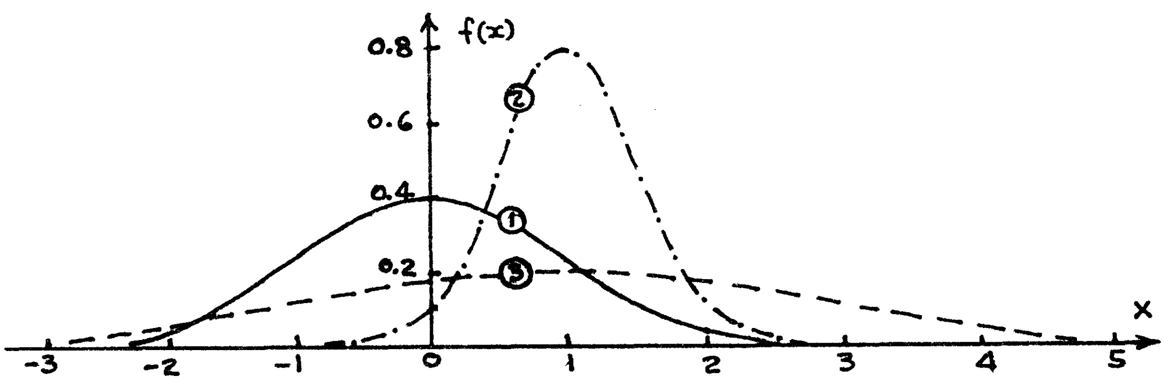

Fig. 1 Three Normal PDF with (1) $\mu = 0$, $\sigma^2 = 1$; (2) $\mu = 1$, $\sigma^2 = 0.25$ and (3) $\mu = 1$, $\sigma^2 = 4$.

Theoretically it may be shown that the mean and variance of the normal distribution function are

$$E(x) = \mu \quad , \tag{1'}$$

and

$$Var(x) = \sigma^2 \quad , \tag{1''}$$

respectively. Similarly the skewness and kurtosis of x are

$$\gamma = 0 \tag{2}$$

and

$$\kappa = 3 \tag{3}$$

In practice these parameters are often used as a criteria for checking the validity of the normal approximation. The criteria is that both the sample skewness coefficient and sample excess coefficient, $E = \kappa - 3$, should be approximately zero (Yevjevich, 1972).

All odd central moments of the normal PDF are zero and all even central moments may be expressed in terms of the second order central moment, μ_2, (Kite, 1977) as:

$$\mu_{2m} = \frac{(2m)!}{2^m \; m!} \mu_2{}^m \quad . \tag{4}$$

For example, if $m = 2$:

$$\mu_4 = 3 \; \mu_2{}^2 = 3 \; \sigma^4 \quad . \tag{5}$$

If the original variable x in Eq. (1) is transformed into a new variable u with mean zero and variance unity, or

$$u = (x - \mu)/\sigma \quad , \tag{6}$$

the PDF of the new variable u takes the form:

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp \left[- \frac{u^2}{2}\right] \quad , \quad -\infty < u < \infty \quad . \tag{7}$$

This is the so-called "standard normal distribution" with mean zero and variance one or u is $N(0,1)$.

The cumulative probability distribution function (CDF) of the normal distribution is the integral of Eq. (1):

$$F(x) = \frac{1}{\sqrt{2\pi} \; \sigma} \int_{-\infty}^{x} \exp \left\{- \frac{1}{2} \left(\frac{y - \mu}{\sigma}\right)^2\right\} dy \quad . \tag{8}$$

The equivalent integral of Eq. (7) is

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp \left[-v^2/2\right] dv \quad . \tag{9}$$

The normal CDF has the properties $F(-\infty) = 0$, $F(\mu) = 0.5$, and $F(+\infty) = 1$.

1.2 Estimation of Parameters

For the particular case of the normal distribution function the moment and maximum likelihood estimates of the parameters, μ and σ are the same. They are

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{10}$$

$$\hat{\sigma} = \left[\frac{\sum_{i=1}^{N} (x_i - \hat{\mu})^2}{N}\right]^{1/2} . \tag{11}$$

The estimate $\hat{\mu}$ is an unbiased estimator of the parameter μ. On the other hand the estimate $\hat{\sigma}$ of Eq. (11) yields a biased estimate of the variance σ^2. But if $\hat{\sigma}$ of Eq. (13) is transformed to

$$\hat{\sigma}' = \hat{\sigma}[N/(N-1)]^{1/2} = \left[\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \hat{\mu})^2\right]^{1/2} \tag{12}$$

then $\hat{\sigma}'^2$ is an unbiased estimator of σ^2.

Subroutine PARNOR

This subroutine estimates the parameters μ and σ of the normal distribution function. The location parameter $\hat{\mu}$ (XLO) is determined by Eq. (10) and the scale parameter $\hat{\sigma}$ (XSC) is determined either by Eq. (11) or by Eq, (12) as desired. The input to PARNOR is the data set X(I), I=1,...,N with N the sample size; the index IEST which indicates the type of estimator for σ, that is if IEST = 0 Eq. (11) is used and if IEST = 1 Eq. (12) is used; and the index IWR = 1 if the printed output is desired or otherwise IWR = 0. The output from PARNOR is the location parameter XLO and the scale parameter (XSC). The listing of PARNOR is given below.

```
      SUBROUTINE PARNOR( N, X, XLO, XSC, IEST, IWR )
C.......................................................................
C.....PARAMETERS OF THE NORMAL DISTRIBUTION FUNCTION
C.....XLO     =LOCATION PARAMETER(MEAN)       XSC     =SCALE PARAMETER(ST. DEV.)
C.....IEST=1  MOMENT OR MAXIMUN LIKELIHOOD ESTIMATORS(BIASED ST. DEV.)
C.....IEST=2  UNBIASED MOMENT ESTIMATORS
C.....IWR=0   DO NOT WRITE RESULTS            IWR=1   WRITE RESULTS
C.....DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C.....HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION  X(N),  EST(2)
      DATA EST /9H(BIASED  ,9H(UNBIASED/
      XM = 0.    $    XS = 0.
      DO 10  I=1,N
      XM = XM + X(I)/N
   10 XS = XS + X(I)**2 / N
      XLO = XM
      XSC = SQRT( XS - XM**2 )
      IF (IEST.GT.1) XSC=XSC*SQRT(FLOAT(N)/(N-1))
      IF (IWR.EQ.0) RETURN
      PRINT 20, EST(IEST), XLO, XSC
   20 FORMAT (1H1///7X,"PARAMETERS OF THE NORMAL DISTRIBUTION"/10XA9"  M
     1OMENT ESTIMATORS)"//5X,"LOCATION PARAMETER="F13.5,"  (MEAN)",/5X,"
     2SCALE     PARAMETER=",F13.5,"  (STD.DEV.)"//)
      RETURN
      END
```

Example 1. The annual maximum streamflow of St. Marys River at Stillwater (data shown in Appendix) is used to compute the unbiased moment estimators of the parameters assuming a normal distribution. The inputs to PARNOR are N = 60 (sample size); X(I), I=1, ..., 60; IEST = 1 (unbiased variance) and IWR = 1 (print results). The main program for calling PARNOR is shown below.

```
      PROGRAM NORPAR(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     PARAMETERS OF THE NORMAL DISTRIBUTION FUNCTION
      DIMENSION  X(60)
      READ 10,N
      READ 20, (X(I),I=1,N)
   10 FORMAT(16I5)
   20 FORMAT (12F6.0)
      CALL PARNOR(N, X, XLO, XSC, 2, 1)
      END
```

The resulting computer output is shown below.

```
        PARAMETERS OF THE NORMAL DISTRIBUTION
           (UNBIASED  MOMENT ESTIMATORS)

      LOCATION PARAMETER=  14554.66667   (MEAN)
      SCALE    PARAMETER=   5226.88988   (STD.DEV.)
```

1.3 Probability Density and Cumulative Distribution Function

Since the PDF of Eq. (1) has a simple functional form, it can easily be evaluated for a given value of x. On the other hand the CDF's of Eqs. (8) or (9) have no explicit forms, and hence, they need to be determined by approximate functions.

Abramowitz and Stegun (1965) gave several approximations for the CDF of the standard normal variate. A polynomial approximation with error less than 10^{-5} is:

$$F(u) \cong 1 - f(u)\ (0.43618\ v - 0.12017\ v^2 + 0.9373\ v^3) \quad , \qquad (13)$$

where $f(u)$ is the density and v is defined for $u \geq 0$ as,

$$v = 1/[1 + 0.33267\ u] \quad . \qquad (14)$$

A similar polynomial approximation was given by Hastings (1955) and has been used by IBM (1968). This approximation is:

$$F(u) \cong 1 - f(u)\ w\ \{0.3193815 + w\ [-0.3565638 + \\ + w\ [1.781478 + w\ (-1.821256 + w\ 1.330274)]]\} \quad , \qquad (15)$$

where w is defined for $u \geq 0$ as:

$$w = 1/(1 + 0.2316419u) \quad . \qquad (16)$$

In both approximations, the cumulative probability is $1 - F(u)$ if $u < 0$. Tables for the cumulative probabilities of the normal distribution are also available.

The relationship between the PDF $f(u)$ of the standard normal variable and PDF $f(x)$ of a normal variable x such that $x = \mu + \sigma u$ is

$$f(x) = f(u)/\sigma \quad . \tag{17}$$

Subroutine DISNOR

This subroutine evaluates the PDF and the CDF of a normal variable with mean μ (XLO) and standard deviation σ (XSC). Equations (6), (7) and (17) are used to determine the PDF and the polynomial approximation of Eqs. (15) and (16) are used to determine the CDF. These probability functions are evaluated for either (a) a given number of points X(I), I=1,...,N which must be supplied to DISNOR or (b) for N equally spaced number of points determined in DISNOR where N is supplied by the user. In the first case, the index IX = 1 and in the second case IX = 0.

For the case (b) DISNOR has a DATA statement specifying the values UMIN = -2.4 and UMAX = 2.4 (corresponding to probabilities of 0.0082 and 0.9918, respectively) which serve to determine N (supplied to DISNOR) equally spaced points beginning at

$$\text{XMIN} = \text{XLO} + \text{UMIN} * \text{XSC} = \text{X}(1) \tag{17'}$$

Subsequent points are determined by

$$\text{X(I)} = \text{XMIN} + (\text{I}-1) * \text{DELX} \quad , \qquad \text{I} = 2,\ldots,\text{N} \tag{18}$$

where the increment DELX is

$$\text{DELX} = (\text{UMAX} - \text{UMIN}) * \text{XSC}/(\text{N}-1) \quad . \tag{19}$$

If the user wants another range of the X values, the UMIN and UMAX of the DATA statement should be modified as desired.

The input to DISNOR is: the total number of points N; the variable X(I); I=1,...,N if IX = 1; the parameters XLO and XSC; the index IX = 0 (if variable X(I) is to be computed in DISNOR) or IX = 1 (if X(I) is given); and the index IWR. The output from DISNOR are the PDF (I) and CDF (I) of X(I), I=1,...,N. The listing of DISNOR is given below.

```
      SUBROUTINE DISNOR( N, X, XLO, XSC, DEN, CDF, IX, IWR )
C...........................................................................
C.....DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS  OF THE NORMAL
C  N = MAXIMUM NUMBER OF DISCRETE POINTS FOR DENSITY AND CUMULATIVE
C  XLO = LOCATION PARAMETER (MEAN), XSC = SCALE PARAMETER (STAND.DEV.)
C  IX = 0  X(I), I=1,2,...,N SET  IS NOT GIVEN AS INPUT
C  IX = 1  X(I), I=1,2,...,N SET  SHOULD BE GIVEN IN ITS ORDER OF MAGNITUDE
C  UMIN, UMAX = ASSUMED MINIMUM AND MAXIMUM VALUES OF THE STANDARD
C  NORMAL DEVIATE LIMITING THE COMPUTATIONS WHEN IX=0 (USER SUPPLIED).
C  IWR = 0  DO NOT WRITE,  IWR = 1  WRITE THE RESULTS
C  DEN = DENSITY,  CDF = CUMULATIVE PROBABILITY OF EXCEEDENCE
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C...........................................................................
      DIMENSION  X(N), DEN(N), CDF(N)
      DATA UMIN, UMAX / -2.4, 2.4/
      IF( IX. GT .0 ) GO TO 20
      XMIN = XLO + UMIN * XSC
      DELX = (UMAX-UMIN)*XSC / (N-1)
      DO 10 I = 1,N
   10 X(I) = XMIN + (I-1) * DELX
   20 CONS = 1. / SQRT(6.2831852)
      DO 30 I = 1,N
      U = ( X(I) - XLO ) / XSC
      DEN(I)= CONS * EXP( -0.5* U**2 )
      V = 1. / (1. + 0.2316419 * ABS(U) )
      CDF(I)= 1. - DEN(I)*V*(0.3193815 + V*(-0.3565638
     1     + V*(1.781478 + V*(-1.821256 + V*1.330274))))
      IF( U. LT .0 )  CDF(I) = 1. - CDF(I)
   30 DEN(I) = DEN(I) / XSC
      IF( IWR. LE .0 ) RETURN
      PRINT 40, XLO, XSC
   40 FORMAT(1H1//5X,"NORMAL DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIO
     1NS",/5X,"XLO=",F12.5,"  (MEAN), XSC=",F12.5," (ST.DEV.)"//7X"ORDER
     2",5X,"VARIABLE",5X,"DENSITY",6X,"CUM.DISTR.",/18X,"VALUE",7X,2("FU
     3NCTION",6X)/)
      PRINT 50, (I, X(I), DEN(I), CDF(I), I=1,N)
   50 FORMAT (8X, I3, F14.5, 2F13.6)
      RETURN
      END
#
```

Example 2. It is desired to compute the standard normal PDF and CDF at 10 equally spaced points. DISNOR is called from a main program with the information N = 10, XLO = 0.0, XSC = 1.0, IX = 0 and IWR = 1. The calling main program is shown below.

```
      PROGRAM NORDIS (INPUT,OUTPUT)
C     DENSITY AND CUMULATIVE PROBABILITY FUNCTIONS OF NORMAL DISTRIBUTION
      DIMENSION  X(60), DEN(60), CDF(60)
      CALL DISNOR(10, X, 0., 1., DEN, CDF, 0, 1)
      END
```

The resulting printed output for this example is shown below.

```
NORMAL DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS
XLO=     0.00000   (MEAN), XSC=      1.00000 (ST.DEV.)

 ORDER        VARIABLE       DENSITY        CUM.DISTR.
               VALUE         FUNCTION        FUNCTION

   1         -2.40000        .022395         .008198
   2         -1.86667        .069867         .030974
   3         -1.33333        .164010         .091211
   4          -.80000        .289692         .211855
   5          -.26667        .385007         .394863
   6           .26667        .385007         .605137
   7           .80000        .289692         .788145
   8          1.33333        .164010         .908789
   9          1.86667        .069867         .969026
  10          2.40000        .022395         .991802
```

1.4 The Inverse of the Cumulative Distribution Function

Consider the variable u to be normal $N(0,1)$ whose CDF is represented by $F(u)$. The problem is to find u for a specified value of $F(u)$. That is, if

$$F(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} y^2\} dy = p \qquad (20)$$

then

$$u = F^{-1}(p) \quad , \tag{21}$$

or u is the inverse of the CDF corresponding to the probability p. For a normal distribution with mean μ and standard deviation σ, the inverse x of F(x) (corresponding to the probability p) may be found by

$$x = \mu + \sigma u \quad . \tag{22}$$

The inverse u is computed herein by the polynomial approximation given by Abramowitz and Stegun (1965) as

$$u = v - \frac{a_0 + a_1 v + a_2 v^2}{1+b_1 v+b_2 v^2+b_3 v^3} \quad , \tag{23}$$

where

$$a_0 = 2.515517 \quad , \quad b_1 = 1.432788 \quad ,$$

$$a_1 = 0.802853 \quad , \quad b_2 = 0.189269 \quad ,$$

$$a_2 = 0.010328 \quad , \quad b_3 = 0.001308 \quad ,$$

and:

$$v = [-2 \ell n F(u)]^{1/2} \quad , \quad 0 < F(u) \leq 0.5. \tag{24}$$

The approximation has an error less than 4.5×10^{-4}. For values of $0.5 < F(u) \leq 1.0$, F(u) of Eq. (20) should be changed to $1 - F(u)$ and the computed u of Eq. (23) becomes $-u$.

Subroutine INVNOR

Subroutine INVNOR determines the inverse x of $F(x)$ of the normal probability function with mean μ and standard deviation σ. The inverse x is represented by X(I) and F(x) by CDF(I). INVNOR has the option to determine X(I) for either (a) N specified values CDF(I), I = 1, ..., N or (b) N equally spaced CDF(I) computed in INVNOR by

CDF(I) = FLOAT(I)/(N+1). In the first case the index ICDF = 1, while in the second case ICDF = 0. Inputs to INVNOR are the mean XLO; the standard deviation XSC; the index ICDF, the number of inverse points N; the set of CDF's CDF(I), I=1,...,N if ICDF=1, and the index IWR. The output of INVNOR is the set X(I), I=1,...,N corresponding to the set CDF(I), I=1,...,N. The program listing of INVNOR is shown below.

```
      SUBROUTINE INVNOR( XLO, XSC, N, CDF, CLAS, ICDF, IWR )
C.......................................................................
C.....INVERSE OF THE NORMAL CDF
C   CDF = CUMULATIVE PROBABILITY AT CLASS LIMIT X .
C   ICDF= 0   CDF(I), I=1,2,..,N SET IS NOT GIVEN AS INPUT
C   ICDF= 1   CDF(I), I=1,2,..,N SET SHOULD BE GIVEN AS AN INPUT VECTOR
C   CLAS= CLASS LIMIT OF X CORRESPONDING TO CDF(X).
C   N = NUMBER OF CLASS LIMITS OF X TO BE COMPUTED
C.....XLO    =LOCATION PARAMETER(MEAN)      XSC     =SCALE PARAMETER(ST. DEV.
C.....IWR=0   DO NOT WRITE RESULTS          IWR=1   WRITE RESULTS
C.....DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C.....HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION  CDF(N), CLAS(N)
      IF( ICDF. GT .0 ) GO TO 20
      DO 10 I = 1,N
   10 CDF(I) = FLOAT(I) / (N+1)
C.....POLYNOMIAL APPROXIMATION FROM ABRAMOWITZ-STEGUN(1965)
   20 DO 30  I=1,N
      CCDF = CDF(I)
      IF( CCDF. GT  .0.5 ) CCDF= 1.-CDF(I)
      V = SQRT(-2.*ALOG(CCDF) )
      CLAS(I)= V - (2.515517 + 0.802853*V + 0.010328* V**2)/
     1    (1. + 1.432788*V + 0.189269*V**2 + 0.001308* V**3)
      IF( CDF(I). LT . 0.5) CLAS(I) = - CLAS(I)
   30 CLAS(I)= XLO + CLAS(I) * XSC
      IF( IWR .EQ. 0 ) RETURN
      PRINT 40, XLO, XSC
   40 FORMAT(1H1//5X,"INVERSE OF THE NORMAL CUMULATIVE DENSITY FUNCTION"
     1/5X,"( XLO =",F12.5,11X,"XSC =",F12.5," )"//13X,"ORDER      CUMULAT
     2IVE      INVERSE"/22X,"DISTRIBTION"/15X,"I",9X,"CDF(X)",9X"X(I)"/)
      PRINT 50, (I, CDF(I), CLAS(I), I=1,N)
   50 FORMAT (13X, I3, F16.6, F14.5)
      RETURN
      END
```

Example 3. Subroutine INVNOR is used to determine the inverse of the CDF of the standard normal for 10 equally spaced values of the CDF. The input to INVNOR is XLO = 0.0, XSC = 1.0, ICDF = 0, N = 10 and IWR = 1. The main program to call INVNOR is shown below.

```
      PROGRAM NORINV (INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
C     INVERSE OF THE NORMAL CDF
      DIMENSION  CDF(100), CLAS(100)
      CALL INVNOR (0., 1., 10, CDF, CLAS, 0, 1)
      END
```

The output from INVNOR is:

```
INVERSE OF THE NORMAL CUMULATIVE DENSITY FUNCTION
( XLO =      0.00000              XSC =      1.00000 )

         ORDER       CUMULATIVE        INVERSE
                     DISTRIBUTION
           I           CDF(X)            X(I)

           1          .090909          -1.33539
           2          .181818           -.90835
           3          .272727           -.60423
           4          .363636           -.34831
           5          .454545           -.11392
           6          .545455            .11392
           7          .636364            .34831
           8          .727273            .60423
           9          .818182            .90835
          10          .909091           1.33539
```

1.5 Confidence Limits

The confidence limits x_ℓ for the T-year event x_T (T represents the return period) is determined assuming a normal standard error s_T so that

$$x_\ell = x_T \pm u_\alpha \, s_T \quad , \tag{25}$$

where u is the standard normal variate corresponding to the confidence level α. The value x_T is determined by

$$x_T = \mu + K \sigma \quad , \tag{26}$$

with K the normal frequency factor which is equal to the standard normal variate u_T (corresponding to T) and μ and σ are the parameters of the normal variable x. The standard error s_T may be shown to be (Kite, 1977)

$$s_T = (1 + U_T^2/2)^{1/2} \, \sigma/\sqrt{N} \quad , \tag{27}$$

with N the sample size.

Subroutine CLINOR

Subroutine CLINOR computes the upper and lower confidence limits of an estimated normal value $x_T(I)$ for a given return period T(I) or a given nonexceedence probabilities PXT(I). The upper and lower confidence limits are represented by XUT(I) and XLT(I) and they are computed for a specified confidence level CONF. The subroutine CLINOR uses the subroutine RETCDF which makes inverse transformations between return periods and nonexceedence probabilities and determines the standard normal deviate UCL corresponding to the specified confidence level CONF. This surbroutine is also used in subsequent programs.

Inputs to CLINOR are the sample size N; the number NT of return periods or nonexceedence probabilities; the confidence level CONF; the mean XLO; the standard deviation XSC; the return period or the nonexceedence probability T(I),I=1,...,NT; and the index IWR. Outputs from CLINOR are XUT(I) and XLT(I) for the corresponding T(I),I=1,...,NT. The program listings of both subroutines CLINOR and RETCDF are given below.

```
      SUBROUTINE CLINOR (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,IWR)
C**CONFIDENCE LIMITS OF DESIGN EVENTS FOR NORMAL DISTRIBUTION
C   XLO = LOCATION PARAMETER( MEAN),   XSC = SCALE PARAMETER(STAND.DEV)
C   N   = SAMPLE SIZE USED FOR ESTIMATION OF THE PARAMETERS
C   NT  = NUMBER OF CONFIDENCE LIMITS TO BE COMPUTED
C   CONF= SELECTED PROBABILITY OF CONFIDENCE(AS 0.90, 0.95, 0.99)
C   T   = AN INPUT VECTOR INDICATING EITHER RETURN PERIODS (IN YEARS)
C   OR NONEXCEEDANCE PROBABILITIES CORRESPONDING TO THE RETURN PERIODS.
C   PXT = PROBABILITY OF NONEXCEEDANCE FOR  T- YEAR  EVENT
C   XT  = ESTIMATE OF THE T-YEAR EVENT FROM FREQUENCY EQUATION
C   UCL = TWO-TAIL STANDARD NORMAL DEVIATE FOR GIVEN CONFIDENCE LEVEL
C   ST  = STANDARD ERROR OF THE ESTIMATED T- YEAR EVENT
C   XUT = UPPER CONFIDENCE LIMIT,   XLT = LOWER CONFIDENCE LIMIT
C** SUBROUTINE REQUREMENT= ONLY THE  SUBROUTINE  RETCDF
C** ABRAMOWITZ-STEGUN POLYNOMIAL APPROXIMATION IS USED TO COMPUTE
C  THE VALUES OF STANDARD NORMAL DEVIATES FOR GIVEN PROBABILITIES.
C     DEVELOPED BY JOSE D. SALAS AND E. HENZLDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION T(NT), XT(NT), XUT(NT), XLT(NT), PXT(NT)
      CALL RETCDF(NT, T, CONF, PXT, UCL)
      DO  10  I= 1,NT
      PROB= PXT(I)
      IF( PXT(I). GT . 0.5 ) PROB = 1.-PXT(I)
      V   = SQRT( -2.*ALOG( PROB ) )
      UXT= V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1        (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      IF( PXT(I). LT .0.5 )  UXT = -UXT
      XT(I) = XLO + UXT * XSC
      ST     = XSC * SQRT( (1.+UXT**2/2.)/N )
      XUT(I)= XT(I) + UCL * ST
   10 XLT(I)= XT(I) - UCL * ST
      IF( IWR. LE .0 ) RETURN
      PRINT 20, CONF, XLO, XSC
      PRINT 30, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   20 FORMAT (1H1///5X,"MOMENT ESTIMATES OF THE ",F4.2,"  CONFIDENCE LIM
     1ITS FOR NORMAL DISTRIBUTION"/10X,"( XLO=",F12.5," (MEAN)      XSC="F
     212.5," (STAND.DEV) )"//10X,"RETURN NONEXCEEDANCE     EVENT",8X,"CON
     3FIDENCE LIMITS"/10X,"PERIOD  PROBABILITY   MAGNITUDE",6X,"UPPER",8
     4X,"LOWER"/)
   30 FORMAT (10X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

```
      SUBROUTINE RETCDF(NT, T, CONF, PXT, UCL)
      DIMENSION  T(NT), PXT(NT)
      IF( T(1). LT .1. ) GO TO 20
      DO  10  I= 1,NT
   10 PXT(I) = 1. - 1./T(I)
      GO TO 40
   20 CONTINUE
      DO  30  I= 1,NT
      PXT(I) = T(I)
   30 T(I) = 1. / (1. - PXT(I) )
   40 PCL = (1. -CONF) / 2.
      V  = SQRT( -2.*ALOG(PCL) )
      UCL= V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1        (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      RETURN
      END
```

Example 4. We wish to determine the values of the variable x for return periods of 2, 5, 10, 20, 50 and 100 years and the corresponding 95% confidence limits of the variable x. The variable x is normal with mean 14554.67 and standard deviation 5226.89 computed from 60 years of record. The subroutine CLINOR is used with input variables N=60; NT=6; CONF=0.95; XLO=14554.67; XSC=5226.89; T(I)=2, 5, 10, 20, 50, 100, for I=1,...,6 respectively; and IWR=1. The main program to input this information is shown below.

```
      PROGRAM NORCON (INPUT, OUTPUT)
C     CONFIDENCE LIMITS OF THE NORMAL DISTRIBUTION
      DIMENSION T(100), XT(100), XUT(100), XLT(100), PXT(100)
      READ 10, N,NT,CONF,XLO,XSC
   10 FORMAT (2I5,3F10.2)
      READ 20, (T(I),I=1,NT)
   20 FORMAT (12F6.0)
      CALL CLINOR (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,1)
      END
```

The output from the program is:

MOMENT ESTIMATES OF THE .95% CONFIDENCE LIMITS FOR NORMAL DISTRIBUTION
(XLO= 14554.67000 (MEAN) XSC= 5226.89000 (STAND.DEV))

RETURN PERIOD	NONEXCEEDANCE PROBABILITY	EVENT MAGNITUDE	CONFIDENCE LIMITS UPPER	LOWER
2.	.50000	14554.66947	15877.52151	13231.81734
5.	.80000	18952.87170	20492.17641	17413.56700
10.	.90000	21254.12522	23039.44359	19468.80685
20.	.95000	23154.00922	25183.35251	21124.66594
50.	.98000	25291.68779	27624.50495	22958.87063
100.	.99000	26716.52099	29253.47173	24169.57025

REFERENCES

Abramowitz, M., and Stegun, I. A., 1965, "Handbook of Mathematical Functions," Dover Publications, New York.

Kite, G. W., 1977, "Frequency and Risk Analysis in Hydrology," Water Resources Publications, Fort Collins, Colorado.

Yevjevich, V., 1972, "Probability and Statistics in Hydrology," Water Resources Publications, Fort Collins, Colorado.

APPENDIX

Flood Data of St. Mary's River at Stillwater
(From Kite, 1977)

10400	16100	11800
10700	10200	13900
20100	12300	12300
8210	19900	15100
14300	19900	11900
8040	18600	11000
8210	18000	16000
13900	13100	11600
8390	29100	14500
18500	10300	13000
13000	12200	17200
16400	15000	13900
23000	12700	11900
6700	13100	13600
7130	19200	12400
14300	19500	18300
20600	9020	12900
25600	11800	18200
8180	16100	9900
34400	16900	10200

2. LOGNORMAL DISTRIBUTION WITH TWO AND THREE PARAMETERS

It has been shown (Sangal and Biswas, 1970; Chow, 1954; Markovic, 1965) that the log probability law can be applied to a wide variety of hydrologic events because most have a physical lower bound; they are a symmetrical and their causative factors are often independent with multiplicative effects (Yevjevich, 1972). Finney (1941) and Chow (1954) attempted to show theoretical justification for the lognormal law. Markovic (1965) studied annual precipitation and runoff series of 2060 stations and found that the two-parameter lognormal distribution gave an excellent fit.

2.1 GENERAL PROPERTIES

Let us consider the transformation $y = \ell n(x)$. It can be shown that if x is lognormal then y is normal. Assume that the parameters of y are the mean μ_y and variance σ_y^2, then the PDF of x can be written as (Finney, 1941; Chow, 1954; 1966; Yevjevich, 1972):

$$f(x) = \frac{1}{\sqrt{2\pi}\ x\ \sigma_y} \exp\left[-\frac{1}{2}\left(\frac{\ell n\ (x)-\mu_y}{\sigma_y}\right)^2\right] \tag{1}$$

which is the two parameter lognormal distribution (lognormal - 2) where μ_y is the location parameter (mean of y) and σ_y is the scale parameter (standard deviation of y) both in the log normal domain. When the parameters are considered in the x domain, μ_y controls the scale while σ_y controls the skewness and hence it may be regarded as a shape parameter (NERC, 1975).

The following relations exist between the parameters of the normal and the lognormal-2 distributions (Yevjevich, 1972):

$$\mu_y = \frac{1}{2} \ln \left[\frac{\mu_x^2}{\eta_x^2 + 1}\right] \quad , \tag{2}$$

and

$$\sigma_y = \ln (1 + \eta_x^2) \quad , \tag{3}$$

where η_x is the coefficient of variation, $\eta_x = \frac{\sigma_x}{\mu_x}$.

The skewness and kurtosis coefficients of the variable x are:

$$\gamma_x = \eta_x^3 + 3 \eta_x \quad , \tag{4}$$

and

$$\kappa_x = \eta_x^8 + 6 \eta_x^6 + 15 \eta_x^4 + 16 \eta_x^2 + 3 \quad , \tag{5}$$

and they are always positive. The mode and median are:

$$m_x = \exp (\mu_y - \sigma_y^2) = \mu_x/(1 + \eta_x^2)^{3/2} \quad , \tag{6}$$

$$M_x = \exp (\mu_y) = \mu_x/(1 + \eta_x^2)^{1/2} \quad , \tag{7}$$

respectively (Yevjevich, 1972).

Examples of lognormal-2 PDF's are shown in Fig. 2.1 below.

If the x-variate has a lower bound, x_o, different from zero, and if the variate, $z = x - x_o$, follows a lognormal distribution with two-parameters, then the x-variate is lognormally distributed with three parameters or simply lognormal-3 (Chow, 1956; Sangal and Biswas, 1970). The lognormal-3 PDF can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}\,(x-x_o)\,\sigma_y} \exp \left\{- \frac{1}{2} \left[\frac{\ln (x-x_o) - \mu_y}{\sigma_y}\right]^2\right\} \quad . \tag{8}$$

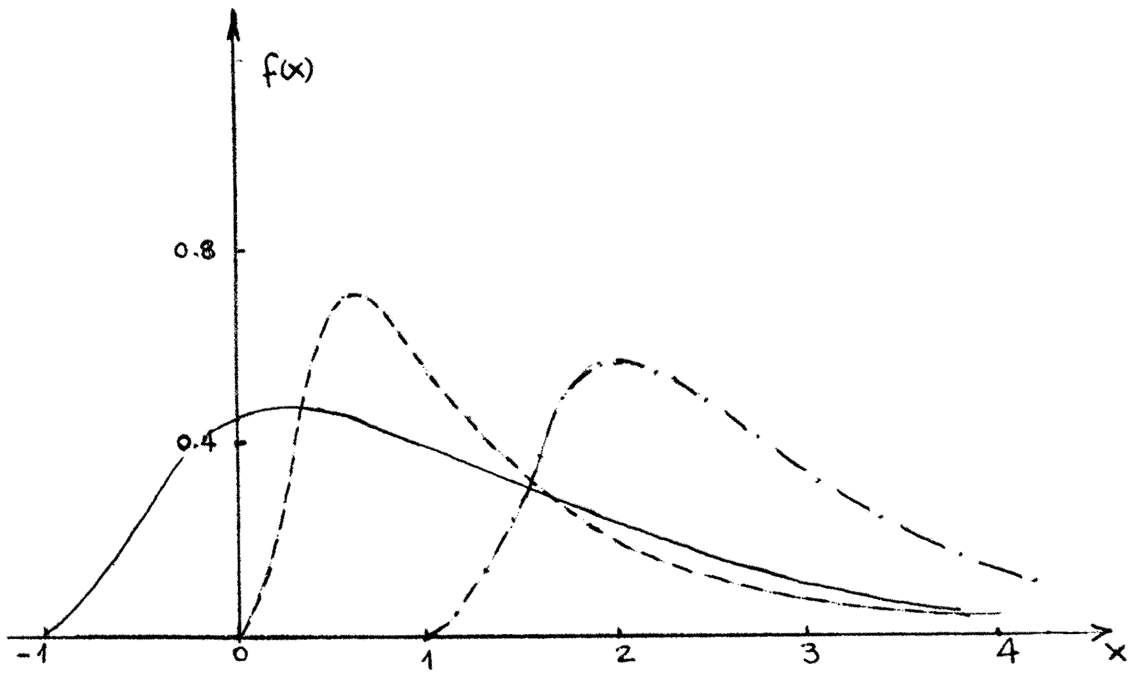

Figure 2.1. Examples of lognormal PDF's with $\mu_y = 0$, $\sigma_y^2 = 1$ and (a) $x_o = 0$ (lognormal-2), (b) $x_o = -1$, and (c) $x_o = 1$.

The parameters μ_y, σ_y and x_o are called the scale (mean of $\ell n(x - x_o)$), the shape (standard deviation of $\ell n(x - x_o)$), and the location parameters, respectively (Yevjevich, 1972; NERC, 1975; Kites, 1977).

The lognormal-3 PDF can be applied to positive or negative valued events provided $x - x_o > 0$, while the lognormal-2 distribution should always be applied to positive valued events. Two examples of the PDF of the lognormal-3 distribution are shown in Fig. 2.1.

2.2 ESTIMATION OF PARAMETERS

The selection of the parameter estimation procedure of lognormal distributions is a matter of reliability. The method of moments is very simple but less accurate since the sample moments may involve some loss of information and bias. Particularly, the estimation of three parameters may result in a lower bound that is greater than the observed minimum value, or in some cases there may not even be a solution. The method of maximum likelihood is more laborious if a computer is not available, but the results are more reliable. Sangal and Biswas (1970) devised a method for estimating the lower bound using the mean, median and standard deviation, but Burges et. al. (1975) concluded that the method of moments was better. The methods of moments and maximum likelihood are the only methods discussed below.

Method of Moments

The moment estimators of the lognormal-2 distribution may be obtained by

$$\hat{\mu}_y = \frac{1}{N} \sum_{i=1}^{N} \ln(x_i) \quad , \tag{9}$$

$$\hat{\sigma}_y = \left\{ \frac{\sum_{i=1}^{N} [\ln(x_i) - \hat{\mu}_y]^2}{N} \right\}^{1/2} . \tag{10}$$

In the case of three parameters, if the lower bound x_o is not known in advance, as most often occurs, it is necessary to estimate x_o either graphically or using the procedure given below provided that the skewness γ_x is positive (Yevjevich, 1972). Since the second and third moments of $z = x-x_o$ do not depend on x_o, and since $\mu_z = \mu_x - x_o$ and $\sigma_z = \sigma_x$, the lower bound may be expressed in terms of μ_x, η_x, and η_z as

$$x_o = \mu_x \left(1 - \frac{\eta_x}{\eta_z}\right) \quad . \tag{11}$$

The skewness coefficient of x and z are the same, i.e.:

$$\gamma_x = \gamma_z = \eta_z^3 + 3\eta_z \quad . \tag{12}$$

Estimating $\hat{\gamma}_x$ from data the solution for Eq. (12) is (Yevjevich, 1972):

$$\hat{\eta}_z = (1 - w^{2/3})/w^{1/3} \tag{13}$$

where w is:

$$w = \frac{1}{2} \left[- \hat{\gamma}_x + (\hat{\gamma}_x^{\ 2} + 4)^{1/2} \right] \quad . \tag{14}$$

Thus, the use of $\hat{\mu}_x$, $\hat{\eta}_x$, and $\hat{\eta}_z$ in Eq. (11) determines the value of $\hat{x}_o$ and $\hat{\mu}_y$, and $\hat{\sigma}_y$ can be estimated from Eqs. (9) and (10) replacing x_i by $(x_i - \hat{x}_o)$. The estimates of σ_y and γ_x may be obtained either from Eqs. ( ) and ( ), or Eqs. ( ) and ( ), respectively.

Subroutine PARLO 1

This subroutine computes the moment estimators of the parameters of the lognormal-2 and lognormal-3 probability functions. The main variables used are the sample size N; the data set X(I), I=1,...,N; the parameters XSC, XSH and XLO; the index IPAR = 2 for the lognormal-2 function or IPAR = 3 for the lognormal-3 function; the index IEST=1 for baised or

IEST=2 for (approximate) unbiased estimators; and the index IWR. When estimating the lower bound parameters XLO for the lognormal-3, it may result in a value greater than the minimum value (XMIN) of the data set X(I); in this case PARLO1 prints out both XLO and XMIN. Also, the coefficient of skewness of X may be negative for which there is no solution for the coefficient of variation η_z of Eq. (12). In such case a message is printed out. The program listing of PARLO1 is given below.

```
      SUBROUTINE PARLO1(N,X,XSC,XSH,XLO,IPAR,IEST,IWR)
C   MOMENT ESTIMATORS OF LOGNORMAL DISTRIBUTIONS WITH 2 AND 3 PARAMETERS
C   IPAR= NUMBER OF PARAMETERS (TAKES ONLY THE VALUES 2 OR 3)
C       IEST=1 BIASED, IEST=2 UNBIASED ESTIMATORS
C       IWR=0 DO NOT WRITE, IWR=1 WRITE THE RESULTS
C   N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C
      DIMENSION X(N), EST(2)
      DATA EST/9H(BIASED   , 9H(UNBIASED/
C     XLO =LOCATION PARAMETER
C     XSC =SCALE     PARAMETER (MEAN OF LN(X-XO))
C     XSH =SHAPE PARAMETER(STD.DEV. OF LN(X-XO))
C   DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C   HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      VN=N
      XM=XS=XG=0.0
      IF (IPAR.GT.2) GO TO 30
      DO 10 I=1,N
      XM=XM+ALOG(X(I))/N
   10 XS=XS+(ALOG(X(I)))**2/N
      XSC=XM
      XSH=SQRT(XS-XM**2)
      IF(IEST.EQ.2) XSH=XSH*SQRT(VN/(VN-1.))
      IF(IWR.EQ.0) RETURN
      PRINT 20, IPAR, EST(IEST), XSC,XSH
   20 FORMAT (1H1///8X,"PARAMETERS  OF  THE  LOGNORMAL-",I2,"  DISTRIBUT
     1ION"/8X,A9," MOMENT OR MAX.LIKELIHOOD ESTIMATORS)"//5X,"SCALE    P
     2ARAMETER=",F12.5,2X,"(MEAN   OF  LN(X))"/5X,"SHAPE    PARAMETER=",
     3F12.5,2X,"(S. DEV. OF LN(X))"//)
      RETURN
   30 XMIN = X(1)
      DO 40  I=1,N
      XM=XM+X(I)/N
      XS=XS+X(I)**2/N
      XG=XG+X(I)**3/N
   40 XMIN = AMIN1(XMIN,X(I))
      XG=XG+2*XM**3-3*XM*XS
      XS=XS-XM**2
      CS= XG/(XS**1.5)
      IF (CS.LT.0.) GO TO 70
      W= (SQRT(CS**2+4.)-CS)/2.
      Z= (1.-W**(2./3))/(W**(1./3))
      XLO= XM-SQRT(XS)/Z
```

```
      IF ((XMIN-XLO).LE.0.) GO TO 90
      XM=XS=0.
      DO 50  I=1,N
      XM= XM + ALOG(X(I)-XLO)/N
   50 XS = XS+ALOG(X(I)-XLO)**2/N
      XSC = XM
      XSH = SQRT(XS-XM**2)
      IF(IEST.EQ.2) XSH= XSH*SQRT(VN/(VN-1.))
      IF(IWR.EQ.0) RETURN
      PRINT 60, IPAR, EST(IEST), XLO, XSC,XSH
   60 FORMAT (1H1///8X,"PARAMETERS  OF  THE  LOGNORMAL-",I2,"  DISTRIBUT
     1ION",/17X,A9," MOMENT ESTIMATORS)".// 5X,"LOCATION PARAMETER=",F12
     2.5,2X,"(LOWER BOUND, XLO)"/ 5X,"SCALE    PARAMETER=",F12.5,2X,"(ME
     3AN   OF  LN(X-XLO))"/ 5X,"SHAPE    PARAMETER=",F12.5,2X,"(S. DEV O
     4F  LN(X-XLO))"//)
      RETURN
   70 PRINT 80, CS
   80 FORMAT (1H1/// 5X,"MOMENT SOLUTION IS IMPOSSIBLE BECAUSE OF THE NE
     1GATIVE SKEW"/5X,"COEF.OF SKEW=",F10.4)
      RETURN
   90 PRINT 100, XMIN, XLO
  100 FORMAT (1H1/// 5X,"LOWER BOUND IS GREATER THAN OBSERVED MINIMUM VA
     1LUE"/5X,"OBSERVED MIN=",F12.5,"  LOWER BOUND=",F12.5)
      RETURN
      END
```

Example 1. Subroutine PARL01 is used to compute the approximate unbiased moment estimators of the parameters for the lognormal-2 and lognormal-3 probability functions corresponding to the annual maximum flows at St. Marys River at Stillwater (Kite, 1977). The length of record is 60 years. The input data to PARL01 is N=60; the data set X(I), I=1,...,60; IPAR=2 (and IPAR=3 for the lognmoral-3); IEST=2 and IWR=1. The main program to call PARL01 is shown below.

```
      PROGRAM PLOG1 (INPUT, OUTPUT)
C
C     EXAMPLES OF LOGNORMAL DISTRIBUTIONS WITH MOMENT ESTIMATORS
      DIMENSION X(60) , TYPE(2)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARL01(N,X,XSC,XSH,XLO,2,2,1)
      CALL PARL01(N,X,XSC,XSH,XLO,3,2,1)
      END
```

The output from the program is:

```
        PARAMETERS  OF  THE  LOGNORMAL- 2  DISTRIBUTION
        (UNBIASED MOMENT OR MAX.LIKELIHOOD ESTIMATORS)

     LOCATION PARAMETER=      9.52847  (MEAN   OF  LN(X))
     SCALE    PARAMETER=       .33740  (S. DEV. OF LN(X))


          PARAMETERS  OF  THE  LOGNORMAL- 3  DISTRIBUTION
                   (UNBIASED MOMENT ESTIMATORS)

     LOCATION PARAMETER=  2108.94705  (LOWER BOUND, XLO)
     SHAPE    PARAMETER=     9.34951  (S.DEV. OF  LN(X-XLO))
     SCALE    PARAMETER=      .40213  ( MEAN  OF  LN(X-XLO))
```

Method of Maximum Likelihood

The maximum likelihood estimators of the parameters of the lognormal-2 distribution are the same as the moment estimators given in the previous section. However the results are not the same for lognormal-3 distribution. Therefore, this section outlines the procedure for estimating the parameters μ_y, σ_y and x_o of the lognormal-3 distribution by the method of maximum likelihood. The log likelihood function of the density function of Eq. (8) is (Yevjevich, 1972):

$$LL\ (x,\mu_y,\sigma_y,x_o) = -\sum_{i=1}^{N} \ln\ (x_i-x_o) - \frac{N}{2} \ln\ (2\pi) - N \ln \sigma_y -$$

$$- \{\sum_{i=1}^{N} [\ln\ (x_i-x_o) - \mu_y]^2\}/(2\sigma_y^2). \tag{15}$$

Differentiating Eq. (15) with respect to μ_y, σ_y and x_o and equating them to zero yields three equations which should be solved simultaneously in order to obtain the estimates $\hat{\mu}_y$, $\hat{\sigma}_y$ and $\hat{x}_o$. The equations are:

$$\partial LL/\partial\mu_y = \{\sum_{i=1}^{N} [\ln(x_i-x_o)-\mu_y]\}/\sigma_y^2 = 0 \tag{16}$$

$$\partial LL/\partial\sigma_y = -N/\sigma_y + \{\sum_{i=1}^{N} [\ln(x_i-x_o)-\mu_y]^2\}/\sigma_y^3 = 0 \tag{17}$$

and

$$\frac{\partial LL}{\partial x_o} = \sum (x_i-x_o)^{-1} (\mu_y - \sigma_y^2) - \sum(x_i-x_o)^{-1} \ln\ (x_i-x_o) = 0\ . \tag{18}$$

Equations (16) and (17) give the following expressions for $\hat{\mu}_y$ and $\hat{\sigma}_y$:

$$\hat{\mu}_y = \frac{1}{N} \sum_{i=1}^{N} \ln\ (x_i-\hat{x}_o) \tag{19}$$

$$\hat{\sigma}_y = \frac{1}{N} \sum_{i=1}^{N} [\ln\ (x_i-\hat{x}_o) - \hat{\mu}_y]^2 \tag{20}$$

where the location parameter, $\hat{x}_o$, must be solved from Eq. 18 or from

$$F(\hat{x}_o) = [\sum_{i=1}^{N} \frac{1}{x_i-\hat{x}_o}] \{\frac{1}{N} \sum_{i=1}^{N} \ell n^2 (x_i-\hat{x}_o) - [\frac{1}{N} \sum_{i=1}^{N} \ell n (x_i-\hat{x}_o)]^2$$

$$- \frac{1}{N} \sum_{i=1}^{N} \ell n (x_i-\hat{x}_o)\} + \sum_{i=1}^{N} \frac{\ell n(x_i-\hat{x}_o)}{x_i-\hat{x}_o} = o \quad . \tag{21}$$

Equation (21) has no explicit solution, therefore, it must be solved by an iterative procedure such as the Newton's tangent method. This method requires the first derivative of Eq. (21) with respect to x_o. That is

$$F'(x_o) = f(b-c^2-c-1) + \frac{2a}{N} [-d + a(c + 0.5)] + e \tag{22}$$

where a,b,c,d, and e are:

$$a = \sum_{i=1}^{N} \frac{1}{x_i-x_o} \quad , \tag{23}$$

$$b = \frac{1}{N} \sum_{i=1}^{N} \ell n^2 (x_i-x_o) \quad , \tag{24}$$

$$c = \frac{1}{N} \sum_{i=1}^{N} \ell n (x_i-x_o) \quad , \tag{25}$$

$$d = \sum_{i=1}^{N} \frac{\ell n(x_i-x_o)}{x_i-x_o} \quad , \text{ and} \tag{26}$$

$$e = \sum_{i=1}^{N} \frac{\ell n(x_i-x_o)}{(x_i-x_o)^2} \tag{27}$$

Subroutine PARLO 2

Since the maximum likelihood estimators of the two-parameter log-normal distribution are the same as the moment estimators, this program only deals with the maximum likelihood estimators of the three-parameter lognormal distribution.

Subroutine PARLO2 solves Eq. (21) for the lower bound XLO, using the Newton's tanget method. Since XLO < XMIN, a simple bisection procedure is used in order to decrease the number of iterations in the tangent procedure. The maximum number of iterations are restricted to LIM = 20 and the maximum error for XLO is restricted to 10^{-6}. The error is the absolute value of the ratio of difference between two successive values of XLO and the last value of XLO.

Subroutine PARLO2 uses subroutine FXLOG for evaluating the function of Eq. (21), its first derivative (Eq. 22), as well as the scale and shape parameters for a given value of XLO.

The input to PARLO2 is the sample size N; the data set X(I), I=1,...,N; and the index IWR. The corresponding output are the parameters XSC, XSH and XLO. The listing of PARLO2 is given below.

```
      SUBROUTINE PARLO2(N,X,XSC,XSH,XLO,IWR)
C     MAX. LIKELIHOOD ESTIMATORS OF THE LOGNORMAL-3 DISTRIBUTION
C     XLO = LOCATION PARAMETER     (LOWER BOUNDARY OF X )
C     XSH = SHAPE     PARAMETER    (S.DEV. OF LN(X-XLO))
C     XSC = SCALE     PARAMETER    (MEAN OF LN(X-XLO))
C     IWR  = 0 DO NOT,     IWR = 1 WRITE THE RESULTS
C  N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C  ERROR= MAXIMUM ABSOLUTE RELATIVE ERROR REQUIREMENT ON XLO.
C  LIM = MAXIMUM NUMBER OF ITERATIONS PERMITTED IN THE PROCEDURE
C* SUBROUTINE REQUIREMENTS= ONLY THE  SUBROUTINE FXLOG.
C  DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C  HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N)
      DATA LIM,ERROR/20,0.0001/
      XM=0.
      XMIN= X(1)
      DO 10 I=1,N
      XM=XM + X(I)/N
   10 XMIN= AMIN1(XMIN,X(I))
      PRINT 20
   20 FORMAT(1H1///8X,"PARAMETERS  OF  THE  LOGNORMAL-3  DISTRIBUTION"/1
     17X,"(MAX. LIKELIHOOD ESTIMATORS)"//5X,"TRIAL  LOWER BOUND   VALUE O
     2F LIKE-  IMPROVEMENT"/15X,"XLO",8X,"LIHOOD EQUATION   ON XLO"/)
      KTR=1
      IF (XMIN) 40,30,40
   30 XI = -0.001*XM
      GO TO 50
```

```
 40 XI = XMIN-0.01*ABS(XMIN)
 50 CALL FXLOG (N,X,XI,FI,DFI,XSC,XSH)
    IF (IWR.EQ.1) PRINT 60, KTR, XI, FI
 60 FORMAT ( 7X,I2,F12.5,3X,E15.8,F13.5)
    DO 105  KTR = 2,LIM
    IF (XMIN) 80,70,80
 70 XO =-0.1*KTR*XM
    GO TO 90
 80 XO= XMIN - 0.25*(KTR-1)*ABS(XMIN)
 90 CALL FXLOG (N,X,XO,FX,DFX,XSC,XSH)
    IF (IWR.EQ.1)  PRINT 60, KTR, XO, FX
    IF ((FI*FX)) 110,150,100
100 FI = FX
    XI=XO
105 DFI = DFX
    IF (KTR.EQ.LIM) GO TO 190
110 IF (ABS(DFI)-ABS(DFX)) 130,130,120
120 XO = XI-FI/DFI
    GO TO 140
130 XO = XO-FX/DFX
140 KTR = KTR+1
    CALL FXLOG(N,X,XO,FX,DFX,XSC,XSH)
    DXO= FX/DFX
    IF (IWR.EQ.1) PRINT 60, KTR, XO, FX, DXO
    XO= XO - DXO
    IF (XO.GE.XMIN) GO TO 170
    IF (KTR.GE.LIM) GO TO 190
    IF (ABS(DXO/XO).GT.ERROR) GO TO 140
    CALL FXLOG(N,X,XO,FX,DFX,XSC,XSH)
150 XLO = XO
    IF(IWR.EQ.0) RETURN
    PRINT 160, XLO, XSH, XSC
160 FORMAT(//5X,"LOCATION PARAMETER=",F12.5,2X,"(LOWER BOUND OF X,XLO)
   1"/ 5X"SHAPE     PARAMETER="F12.5,2X,"(S.DEV.OF LN(X-XLO))"/ 5X,"SCA
   2LE      PARAMETER=",F12.5,2X,"(MEAN OF LN(X-XLO))"//)
    RETURN
170 PRINT 180, XO, XMIN
180 FORMAT(/5X,"LOWER BOUND=",F12.5," .GT. OBSERVED MIN=",F12.5/)
    RETURN
190 PRINT 200, KTR, XO, FX
200 FORMAT(5X,"NO CONVERGENCE AFTER",I3," ITERATIONS"/5X,"LAST VALUES
   1OF THE ITERATION PROCEDURE ARE"/ 5X,"LOWER BOUNDARY =",F12.5/ 5X,"
   2LIKELIHOOD VALUE=",E12.5)
    RETURN
    END


    SUBROUTINE FXLOG(N,X,XO,FX,DFX,XSC,XSH)
    DIMENSION X(N)
    A=B=C=D=AP=DP=0.
    DO 10 I=1,N
    DX= X(I) - XO
    DLX= ALOG(DX)
    A= A + 1./DX
    B= B + DLX*DLX/N
    C= C + DLX/N
    D= D + DLX/DX
    AP= AP + 1./(DX*DX)
 10 DP= DP + (DLX-1.)/(DX*DX)
    FX= A*(B-C*C-C) + D
    DFX = AP*(B-C*C-C)+A*((-D+A*C)*2.0+A)/N+DP
    XSH= SQRT(B-C*C)
    XSC= C
    RETURN
    END
```

<u>Example 2</u>. Subroutine PARLO2 is used to determine the maximum likelihood estimators of the parameters of the lognormal-3 distribution for the annual maximum flow of St. Mary's River at Stillwater. The record length is of 60 years. The input to PARLO2 is N=60; X(I), I=1,...,60; and IWR=1. The main program to call PARLO2 is given below.

```
      PROGRAM PLOG2 (INPUT, OUTPUT)
C     EXAMPLE FOR MAX. LIKELIHOOD ESTIMATORS OF LOGNORMAL-3 DISTRIBUTION
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARLO2(N,X,XSC,XSH,XLO,1)
      END
```

The resulting output from the program is:

```
          PARAMETERS  OF  THE  LOGNORMAL-3  DISTRIBUTION
                   (MAX. LIKELIHOOD ESTIMATORS)

  TRIAL   LOWER BOUND     VALUE F LIKE-   IMPROVEMENT
              XLO         LIHOOD EQUATION    ON XLO

     1   6633.00000      -.51781074E-01
     2   5025.00000      -.39015073E-03
     3   3350.00000      -.38577519E-04
     4   1675.00000       .94657998E-05
     5   2729.86806      -.10573685E-04    333.41411
     6   2396.45395      -.16762043E-05     75.42820
     7   2321.02575      -.65723629E-07      3.20566
     8   2317.82009      -.11253154E-09       .00551

  LOCATION PARAMETER=  2317.81458   (LOWER BOUND OF X,XLO)
  SHAPE    PARAMETER=      .40676   (S.DEV.OF LN(X-XLO))
  SCALE    PARAMETER=     9.32961   (MEAN OF LN(X-XLO))
```

The first four trials in the above result correspond to the bisection iteration procedure. Thereafter the tangent procedure was used.

2.3 PROBABILITY DENSITY AND CUMULATIVE DISTRIBUTION FUNCTION

An indirect way of evaluating the cumulativè probabilities for the lognormal distribution is to transform the original x values into logarithmic values as $y = \ell n\ (x-x_o)$ and use the procedure for the normal distribution (section 4.2.3). Consider the CDF of the lognormal-3 distribution as

$$F(x) = \frac{1}{\sqrt{2\pi}\ \sigma_y} \int_{-\infty}^{x} \frac{1}{x-x_o} \exp\left[-\frac{1}{2}\left(\frac{\ell n(x-x_o) - \mu_y}{\sigma_y}\right)^2\right] dx \quad . \tag{28}$$

Since $dx = (x-x_o)\ dy$, one can express Eq. (28) as:

$$F(x) = \frac{1}{\sqrt{2\pi}\ \sigma_y} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right] dy = F(y) \tag{29}$$

which can be numerically evaluated using the standardized variate:

$$u = \frac{y-\mu_y}{\sigma_y} \tag{30}$$

and using the approximate procedure used in the case of the normal function.

The PDF's of the lognormal distribution functions (Eqs. 1 and 8) have the same explicit form as the normal density function but in the log domain. Thus if one used the density computed or tabulated in the u domain, the values of the density function in the x domain are computed by

$$f_X(x) = f_U(u)/[\sigma_Y\ (x-x_o)] \quad . \tag{31}$$

Equation (30) gives a straight line between the log tansformed variate $y = \ell n(x-x_o)$ and the standard normal variate u. Lognormal probability paper is frequently used in practice for plotting empirical and theoretical probabilities (Eq. 28) where the latter yields a straight line and can be plotted easily using Eq. (30).

Subroutine DISLOG

Subroutine DISLOG evaluates densities (PDF) and cumulative probabilities (CDF) for the lognormal functions with two or three parameters. The program uses the procedures outlined in the section above.

The PDF and CDF functions are determined at N points. These points are eigher given by the user or are determined in the program.

If X(I),I=1,2,...,N is not given as input (IX=0), a maximum value, UMAX, for the standard normal deviate should be specified for a selected maximum probability level, say 0.9918 (UMAX=2.4) in order to restrict computations in the subroutine. In such a case, DISLOG gives densities and cumulative probabilities at N equally spaced values of X(I) within the range, XLO to XMAX, where XMAX is the value corresponding to the selected maximum value for the standard normal deviate, UMAX. If IX=1,X(I),I=1,2,...,N, (not necessarily in order of magnitude), should be input by the user in the calling program.

The location parameter should be specified as XLO=0 if the two-parameter lognormal distribution is used. This is because the parameter XLO is used to identify the type of the distribution, i.e., the value of IPAR, and it is also the first x value X(I) when IX=0. The user must realize the fact that the location parameter XLO is the lower bound in the x domain, while XSC and XSH are the mean and standard deviation in the log domain.

The input to DISLOG is: the number of discrete points; the data set X(I),I=1,...,N if IX=1; the parameters XLO, XSC and XSH; the index IX=0 (if X(I) is to be determined in the program) or IX=1 (if X(I) is given); and the index IWR. The corresponding output is the PDF(I) and CDF(I) of X(I),I=1,...,N. The listing of the program is given below.

```
      SUBROUTINE DISLOG(N, X, XLO, XSC, XSH, DEN, CDF, IX, IWR)
C   N = MAXIMUM NUMBER OF DISCRETE POINTS FOR DENSITY AND CUMULATIVE
C   XLO = LOCATION PARAMETER (LOWER BOUND OF X)
C   XSC = SCALE PARAMETER  (MEAN OF  LN(X-XLO))
C   XSH = SHAPE PARAMETER  (S.DEV.OF LN(X-XLO))
C   UMAX= ASSUMED MAXIMUM VALUE FOR THE STANDARD NORMAL DEVIATE
C   LIMITING THE COMPUTATIONS  WHEN  IX=0 (USER SUPPLIED)
C   IX = 0  X(I), I=1,2,..,N SET  IS NOT GIVEN AS INPUT
C   IWR = 0  DO NOT WRITE,  IWR = 1  WRITE THE RESULTS
C   DEN = DENSITY,  CDF = CUMULATIVE PROBABILITY OF EXCEEDENCE
C*  THE LOWER BOUND SHOULD BE SPECIFIED AS XLO=0,EVEN FOR LOGNORMAL-2
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION  X(N), DEN(N), CDF(N)
      DATA  UMAX / 2.4 /
      CONS= 1./ SQRT(6.2831852)  $  IPAR= 3
      IF (IX.GT.0) GO TO 20
      XMAX = XLO + EXP( XSC + UMAX*XSH )
      DELX = (XMAX - XLO) / (N-1)
      DO 10  I= 1,N
   10 X(I) = XLO + (I-1) * DELX
   20 DEN(1) = 0.   $   CDF(1) = 0.
      DO 30  I= 2,N
      U = (ALOG( X(I)-XLO ) - XSC ) / XSH
      DEN(I)= CONS * EXP( -0.5* U**2 )
      V = 1. / (1. + 0.2316419 * ABS(U) )
      CDF(I) = 1.- DEN(I)*V*(0.3193815 + V*(-0.3565638
     1    + V*(1.781478 + V*(-1.821256 + V*1.330274))))
      IF (U.LT.0) CDF(I) = 1.-CDF(I)
   30 DEN(I) = DEN(I) / ( XSH*( X(I)-XLO ) )
      IF (XLO.EQ.0.) IPAR = IPAR-1
      IF (IWR.LE.0) RETURN
      PRINT 40, IPAR, XLO, XSC, XSH
   40 FORMAT (1H1///5X,"LOGNORMAL-",I2," DENSITY AND CUMULATIVE DISTRIBU
     1TION FUNCTIONS",/7X,"( XLO=",F12.5,"    XSC=",F11.7,"    XSH=",F11.7
     2," )"//11X,"ORDER       VARIABLE        DENSITY         CUM.DISTR.",/22X,"
     3VALUE         FUNCTION        FUNCTION"/)
      PRINT 50, (I, X(I), DEN(I), CDF(I), I=1,N)
   50 FORMAT(13X, I3, F14.5, 2F13.6 )
      RETURN
      END
```

Example 3. Subroutine DISLOG is used to evaluate the densities and cumulative probabilites of the lognormal-3 function with parameters XLO=-18.9, XSC=5.365 and XSH=0.696 at ten equally spaced points. The index IX=0 (since X(I) is not given as input) and IWR=1 to print the results. The main program to call DISLOG is shown below.

```
      PROGRAM LOGDIS (INPUT, OUTPUT)
C     LOGNORMAL DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS
C     EVALUATION OF DENSETY AND CUMULATIVE FOR LOGNORMAL DISTRIVUTION
      DIMENSION  X(60), DEN(60), CDF(60)
      CALL DISLOG( 10, X, -18.9, 5.365, 0.696, DEN, CDF, 0, 1 )
      END
```

The resulting output from the program is:

LOGNORMAL- 3 DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS
(XLO= -18.90000 XSC= 5.3650000 XSH= .6960000)

ORDER	VARIABLE VALUE	DENSITY FUNCTION	CUM.DISTR. FUNCTION
1	-18.90000	0.000000	0.000000
2	107.33881	.003410	.224545
3	233.57762	.002206	.594435
4	359.81643	.001080	.794329
5	486.05524	.000530	.891561
6	612.29405	.000271	.940084
7	738.53285	.000145	.965425
8	864.77166	.000081	.979271
9	991.01047	.000047	.987152
10	1117.24928	.000028	.991802

2.4 THE INVERSE OF THE CUMULATIVE DISTRIBUTION FUNCTION

The inverse of the CDF of the lognormal distribution with 2 or 3 parameters may be determined by

$$x_i = x_o + \exp\{\mu_y + u_i\,\sigma_y\} \tag{32}$$

where x_i is the inverse corresponding to the CDF $F(x_i) = p$ of the lognormal function with parameters x_o, μ_y and σ_y and u_i is the inverse of the standard normal variate corresponding also to the probability p. Therefore the numerical approximation used to determine u_i is as given in section 4.2.4.

Subroutine INVLOG

This subroutine determines the inverse variable x of the lognormal CDF F(x) with parameters x_o, μ_y and σ_y. The inverse variable x is represented by X(I) and the CDF F(x) by CDF(I). The parameters x_o, μ_y and σ_y are represented by XLO, XSC and XSH, respectively. INVLOG has the option to determine X(I) for either N specified values CDF(I), I=1,...,N or for N equally spaced CDF(I) computed in INVLOG by CDF(I) = FLOAT(I)/(N+1). In the first case, the index ICDF=1 while in the second case ICDF=0. Inputs to INVLOG are the parameters XLO, XSC and XSH; the number of inverse points N, the set of CDF's CDF(I), I=1,...,N if ICDF=1; the index ICDF; and the index IWR. The output of INVLOG is the set X(I), I=1,...,N corresponding to CDF(I). The program listing INVLOG is given below.

```
      SUBROUTINE INVLOG (XLO, XSC, XSH, N, CDF, X, ICDF, IWR)
C   INVERSE OF THE LOGNORMAL DISTIBUTION FUNCTION
C   N = NUMBER OF VALUES OF X TO BE DETERMINED
C   XLO = LOCATION PARAMETER (LOWER BOUND OF X)
C   XSC = SCALE PARAMETER   (MEAN OF LN(X-XLO))
C   XSH = SHAPE PARAMETER   (S.DEV.OFLN(X-XLO))
C   ICDF= 0  CDF(I), I=1,2,...,N SET IS NOT GIVEN AS AN INPUT VECTOR
C   ICDF= 1  CDF(I), I=1,2,...,N SET SHOULD BE GIVEN BY THE USER
C   IWR = DO NOT WRITE,  IWR =1 WRITE THE RESULTS
C   THE LOWER BOUND SHOULD BE ASSIGNED AS XLO=0 FOR LOGNORMAL-2
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION CDF(N), X(N)
      IPAR = 3
      IF  (ICDF.GT.0) GO TO 20
      DO 10 I = 1,N
   10 CDF(I) = FLOAT(I) / (N+1)
C* ABROMOWITZ-STEGUN POLYNOMIAL APPROXIMATION
   20 DO 30  I=1,N
      CCDF = CDF(I)
      IF  (CCDF.GT.0.5) CCDF=1.-CDF(I)
      V = SQRT( -2. * ALOG(CCDF) )
      X(I) = V-(2.515517+0.802853*V+0.010328*V**2)/
     1    (1. + 1.432788*V + 0.189269*V**2 + 0.001308* V**3)
      IF  (CDF(I).LT.0.5) X(I) = -X(I)
   30 X(I) = XLO+EXP(XSC+X(I)*XSH)
      IF  (XLO.EQ.0.) IPAR=IPAR-1
      IF  (IWR.LE.0) RETURN
      PRINT 40, IPAR, XLO, XSC, XSH
   40 FORMAT  (1H1///5X,"INVERSE OF THE LOGNORMAL-",I2,"DISTRIBUTION FUN
     1CTION"/5X,"( XLO=",F12.5,"  XSC=",F9.6,"  XSH=",F9.6," )"//15X,"OR
     2DER",5X,"CUMULATIVE",6X,"CLASS"/24X,"DISTRIBUTION"/17X,"I",9X,"CDF
     3(X)"9X,"X(I)"/)
      PRINT 50, (I, CDF(I), X(I), I=1,N)
   50 FORMAT (15X,I3,F16.6,F14.5)
      RETURN
      END
```

Example 3. Subroutine INVLOG is used to determine the inverse of 9 values of the CDF corresponding to a lognormal-3 distribution with parameters XLO=-18.9, XSC=5.365, and XSH=0.696. In addition to the above information the inputs to INVLOG are ICDF=1, IWR=1 and the set CDF(I), I=1,...,9 shown in the program output below. The main program to call INVLOG is shown below.

```
      PROGRAM LOGCLA (INPUT, OUTPUT)
C     INVERSE OF THE LOGNORMAL DISTRIBUTION
      DIMENSION CDF(100), X(100)
      N=9  $  XLO=-18.9  $  XSC=5.365  $  XSH=0.696
      READ 10, (CDF(I), I=1,N)
   10 FORMAT(10F8.6)
      CALL INVLOG (XLO, XSC, XSH, N, CDF, X, 1, 1)
      END
```

The output from INVLOG is:

```
INVERSE OF THE LOGNORMAL- 3DISTRIBUTION FUNCTION
( XLO=   -18.90000  XSC= 5.365000  XSH=  .696000 )
```

ORDER I	CUMULATIVE DISTRIBUTION CDF(X)	CLASS X(I)
1	.224545	107.35938
2	.594435	233.50637
3	.794329	359.76865
4	.891561	486.10736
5	.940084	612.43621
6	.965425	738.74903
7	.979271	865.04400
8	.987152	991.32439
9	.991802	1117.57364

2.5 Confidence Limits

The confidence limits x_ℓ for the T-year event x_T are determined by

$$x_\ell = x_T \pm u_\alpha S_T \tag{33}$$

where u_α is the standard normal variate corresponding to the confidence level α. The value X_T is determined by

$$x_T = \hat{x}_o + \exp\{\hat{\mu}_y + u_T \hat{\sigma}_y\} \quad , \tag{34}$$

where $\hat{x}_o$, $\hat{\mu}_y$ and $\hat{\sigma}_y$ are the parameters of the lognormal function and u_T is the standard normal variate corresponding to the return period T.

The moment estimator of the standard error S_T may be obtained from

$$S_T = (1 + u_T^2/2)^{1/2}\, \hat{\sigma}_y/\sqrt{N} \quad , \tag{35}$$

where N is the sample size and u_T and $\hat{\sigma}_y$ are as defined above.

The maximum likelihood estimator of the standard error S_T may be computed by (Kite, 1977)

$$S_T^2 = \mathrm{Var}\,(\hat{x}_o) + \left(\frac{u_T\ z_T}{2\ \hat{\sigma}_y}\right)^2 \mathrm{Var}\,(\hat{\sigma}_y^2) + z_T^2\ \mathrm{Var}\,(\hat{\mu}_y) + \frac{u_T\ z_T}{\hat{\sigma}_y}\ \mathrm{Cov}\,(\hat{x}_o, \hat{\sigma}_y^2)$$

$$+\ 2\ z_T\ \mathrm{Cov}\,(x_o, \hat{\mu}_y) + \frac{u_T\ z_T}{\hat{\sigma}_y}\ \mathrm{Cov}\,(\hat{\sigma}_y^2, \hat{\mu}_y) \quad , \tag{36}$$

where

$$z_T = \exp\{\hat{\mu}_y + u_T\,\hat{\sigma}_y\} \quad , \tag{37}$$

$$\mathrm{Var}\,(x_o) = 1/(2ND) \quad , \tag{38}$$

$$\mathrm{Var}\,(\hat{\mu}_y) = \hat{\sigma}_y^{\,2}\,[0.5\,(1 + \hat{\sigma}_y^{\,-2})(a/b)^2 - a/b^2]/(ND) \quad , \tag{39}$$

$$\mathrm{Var}\,(\hat{\sigma}_y^{\,2}) = \hat{\sigma}_y^{\,2}\,[(1 + \hat{\sigma}_y^{\,2})(a/b)^2 - a/b^2]/(ND) \quad , \tag{40}$$

$$\mathrm{Cov}\,(\hat{x}_o, \hat{\mu}_y) = -\sqrt{a}\;\; b/(2ND) \quad , \tag{41}$$

$$\mathrm{Cov}\,(\hat{x}_o, \hat{\sigma}_y^{\,2}) = \hat{\sigma}_y^{\,2}\,\sqrt{a}/(bND) \quad , \tag{42}$$

$$\mathrm{Cov}\,(\hat{\mu}_y, \hat{\sigma}_y^{\,2}) = -\,\hat{\sigma}_y^{\,2}\;a/(b^2 ND) \quad , \tag{43}$$

$$D = 0.5\,(1 + \hat{\sigma}_y^{\,-2})(a/b)^2 - (1 + \frac{1}{2\hat{\sigma}_y^2})(a/b^2) \quad , \tag{44}$$

$$a = \exp\{\hat{\sigma}_y^{\,2}\} \quad , \tag{45}$$

and

$$b = \exp\{\hat{\mu}_y\} \quad . \tag{46}$$

For the case of the longnormal-2 Eq. (36) simplifies to

$$S_T = \hat{\sigma}_y\; x_T\;[(1 + u_T^{\,2}/2)/N]^{1/2} \quad . \tag{47}$$

<u>Subroutine CLILO1</u>

This subroutine determines the T-year event magnitude x_T of Eq. (34) and its upper and lower confidence limits x_ℓ of Eq. (33) based on x_T and the moment estimator of the standard error S_T of Eq. (35). The upper and lower confidence limits XUT(I) and XLT(I) are computed for specified return periods T(I),I=1,...,NT or nonexceedence probabilities PXT(I),I=1,...,NT; and given confidence level CONF. The subroutine CLILO1 uses the subroutine RETCDF (given in section 4.2.5) which makes inverse transformations between return periods and nonexceedence probabilities and determines the standard normal deviate UCL corresponding to the confidence level CONF.

Inputs to CLILO1 are the sample size N; the number NT of return periods or nonexceedence probabilities, both represented in the input by T(I),I=1,...,NT, the confidence level CONF; the parameters XLO, XSC and XSH; and the index IWR. Outputs from CLILO1 are the T-year value XT(I) and its confidence limits XUT(I) and XLT(I) for the corresponding T(I), I=1,...,NT. The program listing of CLILO1 is given below. The listing of RETCDF was given before.

```
      SUBROUTINE CLILO1 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IWR)
C   CONFIDENCE LIMITS OF LOGNORMAL DISTRIBUTION BY METHOD OF MOMENTS
C   XLO = LOCATION PARAMETER  (LOWER BOUND OF LN(X-XLO) )
C   XSC = SCALE PARAMETER(MEAN OF LN(X-XLO) )
C   XSH = SHAPE PARAMETER (STANDARD DEVIATION OF LN(X-XLO) )
C**LOCATION PARAMETER SHOULD BE SPECIFIED AS XLO=0 EVEN FOR LOGNORMAL-2
C   N    = SAMPLE SIZE USED FOR ESTIMATION OF THE PARAMETERS
C   NT   = NUMBER OF CONFIDENCE LIMITS TO BE COMPUTED
C   CONF= SELECTED PROBABILITY OF CONFIDENCE(AS 0.90, 0.95, 0.99)
C   T    = AN INPUT VECTOR INDICATING EITHER RETURN PERIODS (IN YEARS)
C   OR NONEXCEEDANCE PROBABILITIES CORRESPONDING TO THE RETURN PERIODS.
C   PXT = PROBABILITY OF NONEXCEEDANCE FOR  T- YEAR  EVENT
C   XT   = ESTIMATE OF THE T-YEAR EVENT FROM FREQUENCY EQUATION
C   UCL = TWO-TAIL STANDARD NORMAL DEVIATE FOR GIVEN CONFIDENCE LEVEL
C   ST   = STANDARD ERROR OF THE ESTIMATED T- YEAR EVENT
C   XUT = UPPER CONFIDENCE LIMIT,  XLT = LOWER CONFIDENCE LIMIT
C** ABRAMOWITZ-STEGUN POLYNOMIAL APPROXIMATION IS USED TO COMPUTE
C  THE VALUES OF STANDARD NORMAL DEVIATES FOR GIVEN PROBABILITIES.
C**SUBROUTINE REQUIREMENTS= ONLY THE  SUBROUTINE RETCDF
C       DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C       HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION T(NT),XT(NT),XUT(NT),XLT(NT),PXT(NT)
      IPAR= 2
      CALL RETCDF(NT, T, CONF, PXT, UCL)
      DO  10  I= 1,NT
      PROB= PXT(I)
      IF  (PXT(I).GT.0.5) PROB = 1.-PXT(I)
      V   = SQRT( -2.*ALOG( PROB ) )
      UXT= V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1       (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      IF  (PXT(I).LT.0.5) UXT  = -UXT
      XT(I)  = XLO + EXP( XSC + UXT*XSH )
      ST  = SQRT(  (1.+ UXT**2/2.)/N )  * XSH
      XUT(I)= XT(I)  * EXP(UCL*ST)
   10 XLT(I)= XT(I)  * EXP(-UCL*ST)
      IF( XLO.LT.0. OR .XLO.GT.0. ) IPAR = IPAR + 1
      IF  (IWR.LE.0) RETURN
      PRINT 20, CONF, IPAR, XLO, XSC, XSH
      PRINT 30,  (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   20 FORMAT (1H1///5X,"MOMENT ESTIMATES OF THE ",F4.2,"  CONFIDENCE LIM
     1ITS FOR LOGNORMAL-"I1" DISTRIBUTION"/12X"( XLO="F12.5"   XSC="F12.
     27"   XSH="F12.7" )"//12X"RETURN NONEXCEEDANCE     EVENT"6X"CONFIDEN
     3CE  LIMITS"/12X"PERIOD  PROBABILITY   MAGNITUDE"6X"UPPER"8X"LOWER"
     4/)
   30 FORMAT (12X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

Example 4. Subroutine CLILO1 is used to determine the x values for return periods of 2, 5, 10, 20, 50 and 100 years and the corresponding 95% confidence limits based on moment estimators. The variable x is lognormal-3 with parameters $\hat{x}_o = 2108.95$, $\hat{\mu}_y = 9.34951$ and $\hat{\sigma} = 0.40213$ computed from 60 years of record. The input variables to CLILO1 are N=60, NT=6; T(I)=2, 5, 10, 20, 50 and 100 for I=1,...,6, respectively; CONF = 0.95; XLO = 2108.95, XSC = 9.34951; XSH = 0.40213; and IWR = 1. The main program to input this information is given below.

```
      PROGRAM LOGCON (INPUT, OUTPUT)
C     CONFIDENCE LIMITS OF THE LOGNORMAL DISTRIBUTION BY MOMENTS
      DIMENSION T(100), XT(100), XUT(100), XLT(100), PXT(100)
      N = 60                  $      NT = 6
      T(1) = 2.               $      T(2) = 5.
      T(3) = 10.              $      T(4) = 20.
      T(5) = 50.              $      T(6) = 100.
      XLO= 2108.95   $   XSC= 9.34951   $   XSH= 0.40213
      CONF = 0.95
      CALL CLILO1 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The resulting computer output is:

```
MOMENT ESTIMATES OF THE  .95  CONFIDENCE LIMITS FOR LOGNORMAL-3 DISTRIBUTION
       ( XLO=  2108.95000   XSC=    9.3495100    XSH=       .4021300 )

       RETURN NONEXCEEDANCE     EVENT            CONFIDENCE  LIMITS
       PERIOD  PROBABILITY    MAGNITUDE          UPPER          LOWER

         2.        .50000     13602.13994   15059.37257   12285.91762
         5.        .80000     18230.06189   20522.01395   16194.08103
        10.        .90000     21352.50705   24496.30674   18612.17538
        20.        .95000     24381.24652   28501.06552   20856.94590
        50.        .98000     28362.61447   33938.39011   23702.88918
       100.        .99000     31404.18594   38202.09924   25815.93457
```

Subroutine CLILO2

This subroutine determines the confidence limits x_ℓ of Eq. (33) of the T-year event x_T of Eq. (34) based on the maximum likelihood estimator of the standard error s_T of Eq. (36). The overall description of this subroutine is the same as the subroutine CLILO1 referred before. The program listing of CLILO2 is given below.

```
      SUBROUTINE CLILO2 (N,NI,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IWR)
C  CONFIDENCE LIMITS OF LOGNORMAL DISTRIBUTION BY MAX. LIKELIHOOD
C** PARAMETERS XLO, XSC, XSH SHOULD BE GIVEN AS MAX.LIKELIHOOD ESTIMATES
C  XLO = LOCATION PARAMETER  (LOWER BOUND OF LN(X-XLO) )
C  XSC = SCALE PARAMETER(MEAN OF LN(X-XLO) )
C  XSH = SHAPE PARAMETER (STANDARD DEVIATION OF LN(X-XLO) )
C**LOCATION PARAMETER SHOULD BE SPECIFIED AS XLO=0 EVEN FOR LOGNORMAL-2
C  N   = SAMPLE SIZE USED FOR ESTIMATION OF THE PARAMETERS
C  NT  = NUMBER OF CONFIDENCE LIMITS TO BE COMPUTED
C  CONF= SELECTED PROBABILITY OF CONFIDENCE(AS 0.90, 0.95, 0.99)
C  T   = AN INPUT VECTOR INDICATING EITHER RETURN PERIODS (IN YEARS)
C  OR NONEXCEEDANCE PROBABILITIES CORRESPONDING TO THE RETURN PERIODS.
C  PXT = PROBABILITY OF NONEXCEEDANCE FOR  T- YEAR  EVENT
C  XT  = ESTIMATA OF THE T-YEAR EVENT FROM FREQUENCY EQUATION
C  UCL = TWO-TAIL STANDARD NORMAL DEVIATE FOR GIVEN CONFIDENCE LEVEL
C  ST  = STANDARD ERROR OF THE ESTIMATED T- YEAR EVENT
C  XUT = UPPER CONFIDENCE LIMIT,  XLT = LOWER CONFIDENCE LIMIT
C** ABRAMOWITZ-STEGUN POLYNOMIAL APPROXIMATION IS USED TO COMPUTE
C  THE VALUES OF STANDARD NORMAL DEVIATES FOR GIVEN PROBABILITIES.
C**SUBROUTINE REQUIREMENTS= ONLY THE  SUBROUTINE RETCOF
C  DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C  HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION T(NT),XT(NT),XUT(NT),XLT(NT),PXT(NT)
      IPAR= 3
      IF (XLO.EQ.0.) GO TO 10
      A = EXP( XSH**2)
      B = EXP( XSC )
      C = A / B
      D = 0.5*(1.+1./(XSH**2))*C**2 - (1.+0.5/(XSH**2))*C/B
      E = XSH**2/(N*D)
      VXO = 0.5/(N*D)
      VYM = (0.5*(1.+1./(XSH**2))*C**2 - C/B)*E
      VYS = ((XSH**2+1.)*C**2 - C/B)*E
      CV1 = -0.5*SQRT(A) / (N*B*D)
      CV2 = SQRT(A) * E / B
      CV3 = -C *E / B
   10 CALL RETCOF(NT, T, CONF, PXT, UCL)
      DO  50  I= 1,NT
      PROB= PXT(I)
      IF (PXT(I).GT.0.5) PROB = 1.-PXT(I)
      V  = SQRT( -2.*ALOG( PROB ) )
      UXT= V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1       (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      IF (PXT(I).LT.0.5) UXT = -UXT
      ZT = EXP( XSC + UXT*XSH )
      IF( XLO )  30, 20, 30
   20 XT(I) = ZT
      ST = XSH*XT(I)*SQRT( (1.+UXT**2/2.)/N )
      GO TO 40
   30 F = UXT*ZT/XSH
      XT(I) = XLO + ZT
      ST = SQRT(VXO+0.25*F**2*VYS+ZT**2*VYM+F*CV2+2.*ZT*CV1+ZT*CV3*F)
   40 XUT(I)= XT(I) + UCL*ST
   50 XLT(I)= XT(I) - UCL*ST
      IF (XLO.EQ.0.) IPAR = 2
      IF (IWR.EQ.0) RETURN
      PRINT 60, CONF, IPAR, XLO, XSC, XSH
      PRINT 70, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   60 FORMAT(1H1///5X"MAX.LIKELIHOOD ESTIMATES OF ",F4.2," CONFIDENCE LI
     1MITS FOR LOGNORMAL-"I1" DISTRIBUTION"/13X"( XLO="F12.5"     XSC="F1
     22.7" XSH="F12.7" )"//13X"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDEN
     3CE LIMITS"/13X"PERIOD  PROBABILITY   MAGNITUDE"6X"UPPER"8X"LOWER"/)
     4)
   70 FORMAT (13X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

Example 5. Subroutine CLILO2 is used to determine the 95% confidence limits of event magnitudes for return periods of 2, 5, 10, 20, 50 and 100 years, based on maximum likelihood estimators. The distribution of events is assumed lognormal-3 with parameters $\hat{x}_o = 2317.815$, $\hat{\mu}_y = 9.32961$ and $\hat{\sigma}_y = 0.40676$ obtained from 60 years of record of annual floods. The input variables to CLILO2 are N=60; NT=6; T(I)=2, 5, 10, 20, 50 and 100 for I=1,...,6, respectively; CONF = 0.95; XLO = 2317.815; XSC = 9.32961; XSH = 0.40676; and IWR = 1. The main program to input this information is given below.

```
      PROGRAM LOGCL2 (INPUT, OUTPUT)
C     CONFIDENCE LIMITS ESTIMATES OF LOGNORMAL DISTR.BY MAX.LIKELIHOOD
      DIMENSION T(100), XT(100), XUT(100), XLT(100), PXT(100)
      N = 60                $      NT = 6
      T(1) = 2.             $      T(2) = 5.
      T(3) = 10.            $      T(4) = 20.
      T(5) = 50.            $      T(6) = 100.
      CONF = 0.95
      XLO=2317.815  $  XSC=9.32961  $  XSH=0.40676
      CALL CLILO2(N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The resulting computer output for this example is:

```
MAX.LIKELIHOOD ESTIMATES OF  .95 CONFIDENCE LIMITS FOR LOGNORMAL-3 DISTRIBUTION
   ( XLO=  2317.81500     XSC=    9.3296100 XSH=      .4067600 )

    RETURN NONEXCEEDANCE    EVENT          CONFIDENCE LIMITS
    PERIOD  PROBABILITY   MAGNITUDE       UPPER         LOWER

      2.       .50000     13584.55114  14824.28546  12344.81682
      5.       .80000     18182.97731  20083.70878  16282.24583
     10.       .90000     21294.49221  24005.44690  18583.53753
     20.       .95000     24318.22141  28108.38606  20528.05676
     50.       .98000     28300.13928  33882.31916  22717.95939
    100.       .99000     31346.89026  38539.10753  24154.67299
```

3. GAMMA DISTRIBUTIONS WITH TWO AND THREE PARAMETERS

3.1 General Properties

The gamma probability distribution with two parameters has a density function of the form

$$f(x) = \frac{1}{\alpha^{\beta}\,\Gamma(\beta)}\, x^{\beta-1}\, e^{-x/\alpha} \quad , \qquad 0 \leq x \leq \infty \tag{1}$$

where α and β are the scale and shape parameters, respectively and $\Gamma(\beta)$ is the complete gamma function. The parameters are restricted to $\alpha > o$ and $\beta > o$ and $\Gamma(\beta)$ is the integral

$$\Gamma(\beta) = \int_{o}^{\infty} z^{\beta-1}\, e^{-z}\, dz. \tag{1'}$$

The gamma -2 CDF is:

$$F(x) = \int_{o}^{x} \frac{x^{\beta-1} e^{-x/\alpha}}{\alpha^{\beta}\,\Gamma(\beta)}\, dx \quad , \tag{2}$$

where the integral is known as the incomplete gamma function and must be evaluated by numerical integration (Yevjevich, 1972; NERC, 1975)

The gamma distribution with three parameters (or the so-called Pearson Type III distribution) has a PDF

$$f(x) = \frac{1}{\alpha\Gamma(\beta)} \left[\frac{x-x_o}{\alpha}\right]^{\beta-1} e^{-\left(\frac{x-x_o}{\alpha}\right)} \quad , \qquad x_o \leq x < \infty \tag{3}$$

where α and β are the scale and shape parameters respectively and x_o is the location parameter. As before $\alpha > o$, $\beta > o$ and $\Gamma(\beta)$ is given by Eq. (2).

If the variable $y = x - x_o$ is introduced in the latter expression it can be easily seen that the y variate has a gamma-2 distribution with parameters α and β.

3.2 Estimation of Parameters

Method of Moments

The mean, variance, and skewness of the gamma-2 distribution are:

$$\mu = \alpha\beta \quad , \tag{4}$$

$$\sigma^2 = \alpha^2\beta \quad , \tag{5}$$

and

$$\gamma = 2/\sqrt{\beta} \quad . \tag{6}$$

From the first two equations the parameters α and β can be determined, using the sample estimates, $\hat{\mu}$ and $\hat{\sigma}$, as:

$$\hat{\alpha} = \hat{\sigma}^2/\hat{\mu} \tag{7}$$

$$\hat{\beta} = (\hat{\mu}/\hat{\sigma})^2 \quad . \tag{8}$$

Similarly, for the gamma-3 distribution the following relations exist between the central moments of x and the parameters α, β, and x_o:

$$\mu = \alpha\beta + x_o \quad , \tag{9}$$

$$\sigma^2 = \alpha^2\beta \quad , \tag{10}$$

$$\gamma = 2/\sqrt{\beta} \quad , \tag{11}$$

$$\kappa = 3(1 + \gamma^2/2) \quad . \tag{12}$$

Using the sample mean $\hat{\mu}$, standard deviation $\hat{\sigma}$, and coefficient of skewness $\hat{\gamma}$, the moment estimates of the parameters α, β and x_o are:

$$\hat{\alpha} = \hat{\sigma}/\sqrt{\hat{\beta}} \quad , \tag{13}$$

$$\hat{\beta} = (2/\hat{\gamma})^2 \quad , \tag{14}$$

$$\hat{x}_o = \hat{\mu} - \hat{\sigma}\sqrt{\hat{\beta}} \quad . \tag{15}$$

In the above equations the sample standard deviation $\hat{\sigma}$ may be determined by Eq. ( ) or ( ) (for biased or unbiased estimates of the variances, respectively). Similarly, $\hat{\gamma}$ may be determined by Eq. ( ) or ( ) (for biased or unbiased estimates of the skewness, respectively), (Yevjevich, 1972, Bobee and Robitaille, 1976).

Subroutine PARAGA1

This subroutine computes the moment estimators of the parameters of the gamma-2 and gamma-3 probability functions based on the Eqs. (7) and (8) and Eqs. (13), (14), and (15), respectively. The input variables to PARGA1 are: the sample size N; the data set X(I), I=1,...,N; the index IPAR=2 (for gamma-2) or IPAR=3 (for gamma-3); the index IEST=1 (biased variance and skewness) or IEST=2 (unbiased variance and skewness); and the index IWR. The output variables are the scale parameter XSC, the shape parameter XSH, and the location parameter XLO. The listing of PARGA1 is given below.

```
      SUBROUTINE PARGA1(N,X,XSC,XSH,XLO,IPAR,IEST,IWR)
C     PARAMETERS OF THE GAMMA DISTRIBUTION (MOMENT ESTIMATORS)
C     XLO = LOCATION PARAMETER    (LOWER BOUNDARY OF X)
C     XSC = SCALE     PARAMETER    XSH = SHAPE PARAMETER
C     IEST = 1 BIASED ,     IEST = 2 UNBIASED ESTIMATORS
C     IWR  = 0 DO NOT,      IWR = 1 WRITE THE RESULTS
C     N    = SAMPLE SIZE
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N), EST(2)
      DATA EST/9H(BIASED    , 9H(UNBIASED/
C MOMENT ESTIMATORS OF THE GAMMA DISTRIBUTION WITH 2 PARAMETERS
      VN= N
      XM=XS=XG=0.
      DO 10 I=1,N
      XM= XM + X(I)/N
      XS= XS + X(I)**2/N
   10 XG= XG + X(I)**3/N
      XG= XG + 2*XM**3 - 3*XM*XS
      XS= XS - XM**2
      CS= XG/(XS**1.5)
      IF(IEST.EQ.2) XS= XS*VN/(VN-1.)
      IF(IEST.EQ.2) CS= CS*(1.+8.5/VN)*SQRT(VN*(VN-1.))/(VN-2.)
      IF(IPAR.GT.2) GO TO 40
      XSC= XS/XM
      XSH= XM/XSC
      IF(IWR.EQ.0) RETURN
      PRINT 20, IPAR, EST(IEST), XM, XS, CS
   20 FORMAT (1H1///5X,"PARAMETERS OF THE GAMMA-",I2,"DISTRIBUTION"/10X,
     1A9," MOMENT ESTIMATORS)"//10X"MEAN VALUE OF X =",F12.5,/10X,
     2"VARIANCE  OF  X =",E15.8/10X"COEF.OF SKEWNESS=",F12.5//)
      PRINT 30, XSC, XSH
   30 FORMAT(/10X"SCALE     PARAMETER=",F12.5/10X"SHAPE     PARAMETER="F12
     1.5/)
      RETURN
```

```
C MOMENT ESTIMATORS OF THE GAMMA DISTRIBUTION WITH 3 PARAMETERS
   40 SX = SQRT(XS)
      XSH= 4./(CS**2)
      XSC= SX*CS/2
      XLO= XM - 2.*SX/CS
      IF (IWR.EQ.0) RETURN
      PRINT 20, IPAR, EST(IEST), XM, XS, CS
      PRINT 50, XLO, XSC, XSH
   50 FORMAT (/10X"LOCATION PARAMETER "/10X"(LOWER BOUND OF X)=",F12.5,
     1/10X,"SCALE     PARAMETER=",F12.5/10X"SHAPE     PARAMETER=",F12.5/)
      RETURN
      END
```

Example 1. Subroutine PARGA1 is used to compute the moment estimators of the parameters of the gamma-2 and gamma-3 probability functions for the annual maximum flows at St. Marys River at Stillwater. The unbiased estimates of the variance and skewness are used in this example. The length of record is of 60 years. The input data to PARGA1 is N=60, the data set X(I), I=1,...,60; IPAR=2 (gamma-2) and IPAR=3 (gamma-3); IEST=2 and IWR=1. The main program to call PARGA1 is shown below.

```
      PROGRAM PGAM1 (INPUT,OUTPUT)
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARGA1(N,X,XSC,XSH,XLO,2,2,1)
      CALL PARGA1(N,X,XSC,XSH,XLO,3,2,1)
      END
```

The computer output for this example is:

```
PARAMETERS OF THE GAMMA- 2DISTRIBUTION
     (UNBIASED MOMENT ESTIMATORS)

     MEAN VALUE OF X = 14554.66667
     VARIANCE  OF  X =  .27320378E+08
     COEF.OF SKEWNESS=       1.54781


     SCALE     PARAMETER=  1877.08716
     SHAPE     PARAMETER=     7.75386



PARAMETERS OF THE GAMMA- 3DISTRIBUTION
     (UNBIASED MOMENT ESTIMATORS)

     MEAN VALUE OF X = 14554.66667
     VARIANCE  OF  X =  .27320378E+08
     COEF.OF SKEWNESS=       1.54781


     LOCATION PARAMETER
     (LOWER BOUND OF X)=  7800.73787
     SCALE     PARAMETER=  4045.10896
     SHAPE     PARAMETER=     1.66965
```

Method of Maximum Likelihood

The log-likelihood function of the gamma-2 distribution function of Eq. (1) is

$$LL(x;\alpha,\beta) = -N\beta\ln(\alpha) - N\ln[\Gamma(\beta)] + (\beta-1)\sum_{i=1}^{N}\ln(x_i) - \frac{1}{\alpha}\sum_{i=1}^{N} x_i \quad . \tag{16}$$

Differentiating with respect to β and α yields:

$$\frac{\partial LL}{\partial\beta} = -N\ln(\alpha) - N\frac{\Gamma'(\beta)}{\Gamma(\beta)} + \sum_{i=1}^{N}\ln(x_i) \tag{17}$$

$$\frac{\partial LL}{\partial \alpha} = -N\beta/\alpha + \frac{1}{\alpha^2}\sum_{i=1}^{N} x_i \quad . \tag{18}$$

The maximum likelihood estimates of α and β must satisfy the following equations:

$$\partial LL/\partial \beta = 0 \quad , \quad \partial LL/\partial \alpha = 0 \quad . \tag{19}$$

The latter equation gives $\hat{\alpha}$ in terms of $\hat{\beta}$ and $\hat{\mu}$ as:

$$\hat{\alpha} = \hat{\mu}/\hat{\beta} \quad . \tag{20}$$

Substitution of this expression for α into Eq. (17) gives:

$$F(\hat{\beta}) = \frac{1}{N}\frac{\partial LL}{\partial \beta} = \hat{\mu}_\ell - \ell n(\hat{\mu}) + \ell n(\hat{\beta}) - \psi(\hat{\beta}) = 0 \tag{21}$$

where $\hat{\mu}$ and $\hat{\mu}_\ell$ are the arithmetic and logarithmic means of x, respectively, and $\psi(\hat{\beta})$ is the so-called digamma function which can be computed using the asymptotic expansion as in Condie and Nix (1975),

$$\psi(\hat{\beta}) = \Gamma'(\hat{\beta})/\Gamma(\hat{\beta}) = \ell n\ (\hat{\beta}+2) - \frac{1}{2(\hat{\beta}+2)} - \frac{1}{12(\hat{\beta}+2)^2}$$

$$+ \frac{1}{120(\hat{\beta}+2)^4} - \frac{1}{256(\hat{\beta}+2)^6} - \frac{1}{\hat{\beta}+1} - \frac{1}{\hat{\beta}} \quad . \tag{22}$$

Equation (21) is to be solved by a numerical iterative procedure, such as Newton's tangent method, since it cannot be expressed in an explicit form.

The derivative of Eq. (21) is:

$$F'(\hat{\beta}) = 1/\hat{\beta} - \psi'(\hat{\beta}) \quad , \tag{23}$$

where $\psi'(\hat{\beta})$ is the derivative of $\psi(\hat{\beta})$ and is the so-called trigamma function. It may be approximated by an asymptotic expansion as:

$$\psi'(\hat{\beta}) = \frac{1}{\hat{\beta} + 2} + \frac{1}{2(\hat{\beta} + 2)^2} + \frac{1}{6(\hat{\beta} + 2)^3} - \frac{1}{30(\hat{\beta} + 2)^5}$$

$$+ \frac{1}{42(\hat{\beta} + 2)^7} - \frac{1}{30(\hat{\beta} + 2)^9} + \frac{1}{(\hat{\beta} + 1)^2} + \frac{1}{\hat{\beta}^2} \qquad (24)$$

The log-likelihood function of the gamma-3 distribution function of Eq. (3) takes the form:

$$LL(x; \alpha, \beta, x_o) = -N \,\ell n[\Gamma(\beta)] - \frac{1}{\alpha} \sum_{i=1}^{N} (x_i - x_o) +$$

$$+ (\beta - 1) \sum_{i=1}^{N} \ell n\,(x_i - x_o) - N\,\beta\,\ell n(\alpha) \qquad (25)$$

and its derivatives with respect to α, β, and x_o are:

$$\partial LL/\partial\alpha = \frac{1}{\alpha^2} \sum_{i=1}^{N} (x_i - x_o) - \beta N/\alpha \qquad (26)$$

$$\partial LL/\partial\beta = -N[\Gamma'(\beta)/\Gamma(\beta)] + \sum_{i=1}^{N} \ell n\,(x_i - x_o) - N\,\ell n(\alpha) \qquad (27)$$

$$\partial LL/\partial x_o = N/\alpha - (\beta - 1) \sum_{i=1}^{N} [1/(x_i - x_o)] \quad . \qquad (28)$$

Maximum likelihood solutions for α, β and x_o must satisfy the following equations:

$$\partial LL/\partial\alpha = 0 \quad , \quad \partial LL/\partial\beta = 0 \quad , \quad \partial LL/\partial x_o = 0 \quad , \qquad (29)$$

respectively. Simultaneous solution of these equations yields expressions for $\hat{\beta}$ and $\hat{\alpha}$ in terms of $\hat{x}_o$ as:

$$\hat{\beta} = \left\{1 - N^2 \left[\sum_{i=1}^{N} (x_i - \hat{x}_o) \sum_{i=1}^{N} \frac{1}{x_i - \hat{x}_o}\right]^{-1}\right\}^{-1} \qquad (30)$$

$$\hat{\alpha} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\hat{x}_o) - N\left[\sum_{i=1}^{N}\frac{1}{x_i-\hat{x}_o}\right]^{-1} \qquad (31)$$

and an implicit expression for $\hat{x}_o$ as:

$$F(\hat{x}_o) = -N\,\psi(\beta) + \sum_{i=1}^{N}\ell n\,(x_i-\hat{x}_o) - N\,\ell n(\hat{\alpha}) = 0 \quad . \qquad (32)$$

This is to be solved numerically by the Newton's tangent procedure.

The first derivative of $F(\hat{x}_o)$ of Eq. (32) with respect to $\hat{x}_o$ is:

$$F'(\hat{x}_o) = -N\,\psi'(\beta)\,(f\,d - a\,f')/f^2 - a - N\,h'/h \qquad (33)$$

where $\psi'(\beta)$ is the trigamma function defined before and:

$$a = \sum_{i=1}^{N}\frac{1}{x_1-\hat{x}_o} \quad , \qquad (34)$$

$$d = \sum_{i=1}^{N}\frac{1}{(x_i-\hat{x}_o)^2} \quad , \qquad (35)$$

$$f = a - N^2/b \quad , \qquad (36)$$

$$f' = d - N^3/b^2 \quad , \qquad (37)$$

$$h = b/N - N/a \qquad (38)$$

$$h' = -1 + Nd/a^2 \quad , \qquad (39)$$

and b is:

$$b = \sum_{i=1}^{N}(x_i-\hat{x}_o) \quad . \qquad (40)$$

Subroutine PARGA 2

This subroutine determines the parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{x}_o$ (if the function is gamma-3) by the method of maximum likelihood, outlined above.

The variables XSC, XSH and XLO represent the scale parameter $\hat{\alpha}$, the shape parameter $\hat{\beta}$ and the location parameter $\hat{x}_o$, respectively.

For the case of the two-parameter gamma distribution, the numerical procedure starts using the following value of the shape parameter (Yevjevich, 1972)

$$\hat{\beta} = \left\{1 + [1 + \frac{4}{3} (\ell n\ \hat{\mu} - \hat{\mu}_\ell)]^{1/2}\right\} / 4(\ell n\ \hat{\mu} - \hat{\mu}_\ell) - \Delta\hat{\beta} \qquad (41)$$

where $\Delta\hat{\beta} = 0$ and $\hat{\mu}_\ell$ is the logarithmic mean of the sequence.

A similar procedure as in the case of the lognormal-3 distribution is introduced in PARGA2 for the estimation of the parameters of the gamma-3 function. At first, a coarse bisection procedure is used to find the interval in which Eq. (32) changes sign, and then Newton's procedure is used starting at the steepest tangential point, that is, at the point where the first derivative of Eq. (32) is greatest. The maximum number of iterations and the maximum relative error on the location parameter in this subroutine have been restricted by LIM=20, and ERROR=$\pm 10^{-6}$. Subroutine FXGAM is required in subroutine PARGA2 to compute XSC, XSH, FX (Eq. 32), and DFX (Eq. 33) for a given XLO.

The input variables to PARGA2 are: the sample size N; the data set X(I), I=1,...,N; the index IPAR=2 (for the gamma-2 function) or IPAR=3 (for the gamma-3 function) and the index IWR. The output are the parameters XSC, XSH and XLO. The listing of PARGA2 and FXGAM are given below.

```
      SUBROUTINE PARGA2(N,X,XSC,XSH,XLO,IPAR,IWR)
C     N   = SAMPLE SIZE
C     XLO = LOCATION PARAMETER    (LOWER BOUNDARY OF X)
C     XSC = SCALE    PARAMETER    XSH = SHAPE PARAMETER
C     IPAR = 2  GAMMA - 2    , IPAR = 3  GAMMA - 3 DISTRIBUTION
C     IWR  = 0  DO NOT WRITE, IWR  = 1  WRITE THE RESULTS
C  ERROR= MAXIMUM ABSOLUTE RELATIVE ERROR REQUIREMENT ON XLO.
C  LIM = MAXIMUM NUMBER OF ITERATIONS PERMITTED IN THE PROCEDURE
C* SUBROUTINE REQUIREMENTS= ONLY THE SUBROUTINE FXGAM.
C  DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C  HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N)
      DATA LIM,ERROR/20,0.0001/
C GAMMA DISTRIBUTION WITH TWO PARAMETERS
      XM= XLM = 0.
      IF(IPAR.GT.2) GO TO 70
      DO 10 I=1,N
      XM =  XM + X(I)/N
   10 XLM= XLM + ALOG(X(I))/N
      C = ALOG(XM) - XLM
      XSH = (1.+SQRT(1.+4.*C/3))/(4.*C)
      IF(XSH.GE.5.6) GO TO 30
      DXSH= 0.
      KT= 0
   20 KT= KT + 1
      XSH= XSH - DXSH
      A= XSH + 2.
      DGM=ALOG(A)-1./(2*A)-1./(12*A**2)+1./(120*A**4)
     $-1./(252*A**6)-1./(XSH+1.)-1./XSH
      TRG=1./A+1./(2*A**2)+1./(6*A**3)-1./(30*A**5)+1./(42*A**7)
     $-1./(30*A**9)+1./((XSH+1.)**2)+1./(XSH**2)
      FA= ALOG(XSH) - DGM - C
      DFA= 1./XSH - TRG
      DXSH= FA/DFA
      IF(KT.GT.LIM) GO TO 50
      IF(ABS(DXSH/XSH).GT.ERROR) GO TO 20
      XSH= XSH - DXSH
   30 XSC= XM/XSH
      PRINT 40, XSC, XSH
   40 FORMAT (1H1///12X,"PARAMETERS OF THE GAMMA-2 DISTRIBUTION"/15X"(MA
     1X.LIKELIHOOD ESTIMATORS)"//15X,"SCALE PARAMETER="F12.5,/15X,"SHAPE
     2 PARAMETER=",F12.5//)
      RETURN
   50 PRINT 60, KT, XSH
   60 FORMAT (1H1///5X,"GAMMA-2 DISTRIBUTION (MAX.LIKELIHOOD PROCEDURE)"
     2/5X,"SHAPE PARAMETER AT THE END OF",I3,".TH ITERATION=",F12.5/15X,
     2"N O   C O N V E R G E N C E")
      RETURN
C GAMMA DISTRIBUTION WITH  THREE PARAMETERS
   70 XMIN = X(1)
      DO 80 I=1,N
      XM= XM + X(I)/N
   80 XMIN= AMIN1(XMIN,X(I))
      IF (IWR.EQ.1) PRINT 90
   90 FORMAT (1H1///12X,"PARAMETERS OF THE GAMMA-3 DISTRIBUTION"/15X"(MA
     1X.LIKELIHOOD ESTIMATORS)"//5X,"TRIAL  LOWER BOUND   VALUE OF LIKE-
     2  IMPROVEMENT"/15X,"XLO",8X,"LIHOOD EQUATION   ON XLO"/)
      KTR= 1
      IF(XMIN) 110,100,110
  100 XI= -0.001*XM
      GO TO 120
```

```
110 XI= XMIN - 0.01*ABS(XMIN)
120 CALL FXGAM(N,X,XI,FI,DFI,XSC,XSH,XM)
    IF (IWR.EQ.1) PRINT 130, KTR, XI, FI
130 FORMAT (7X,I2,F14.5,3X,E15.8,2X,F12.5)
    DO 180 KTR= 2,N
    IF(XMIN) 150,140,150
140 XO= 0.1*(KTR-1)*XM
    GO TO 160
150 XO= XMIN - 0.25*(KTR-1)*ABS(XMIN)
160 CALL FXGAM(N,X,XO,FX,DFX,XSC,XSH,XM)
    IF (IWR.EQ.1) PRINT 130, KTR, XO, FX
    IF( (FI*FX) ) 190,230,170
170 FI= FX
    XI= XO
180 DFI=DFX
    IF(KTR.EQ.LIM) GO TO 270
190 IF( ABS(DFI)-ABS(DFX) ) 210,210,200
200 XO= XI - FI/DFI
    GO TO 220
210 XO= XO - FX/DFX
220 KTR= KTR + 1
    CALL FXGAM(N,X,XO,FX,DFX,XSC,XSH,XM)
    DXO= FX/DFX
    IF (IWR.EQ.1) PRINT 130, KTR, XO ,FX, DXO
    XO= XO - DXO
    IF( XO.GE.XMIN ) GO TO 250
    IF( KTR.GE.LIM ) GO TO 270
    IF( ABS(DXO/XO).GT.ERROR ) GO TO 220
    CALL FXGAM(N,X,XO,FX,DFX,XSC,XSH,XM)
230 XLO= XO
    IF( IWR.EQ.0) RETURN
    PRINT 240, XLO, XSC, XSH
240 FORMAT (//15X"LOCATION PARAMETER"/15X"(LOWER BOUND OF X)=",F12.5,/
   115X"SCALE     PARAMETER=",F12.5/15X"SHAPE     PARAMETER=",F12.5//)
    RETURN
250 PRINT 260, XO, XMIN
260 FORMAT (/5X,"LOWER BOUND=",F12.5," .GT. OBSERVED MIN="F12.5)
    RETURN
270 PRINT 280, KTR, XO, FX
280 FORMAT (/5X,"NO CONVERGENCE AFTER",I3," ITERATIONS"/5X,"LAST VALUE
   1S OF THE ITERATION  ARE"/5X,"LOWER BOUNDARY =",F12.5/5X,"LIKELIHOO
   2D VALUE=",E12.5/)
    RETURN
    END


    SUBROUTINE FXGAM (N,X,XO,FX,DFX,XSC,XSH,XM)
    DIMENSION X(N)
    A=B=C=D=0.
    VN = N
    DO 10  I=1,N
    DX = X(I)-XO
    A=A+1./DX
    C=C+ALOG(DX)
 10 D=D+1./(DX*DX)
    B=N*(XM-XO)
    XSH=A/(A-VN*VN/B)
    XSC=B/(XSH*N)
    E=XSH+2.
    DGM=ALOG(E)-1./(2.*E)-1./(12.*E**2)+1./(120.*E**4)-1./(252.*E**6)-
   11./(XSH+1.)-1./XSH
    FX=-N*DGM+C-N*ALOG(XSC)
    TRG=1./E+1./(2.*E**2)+1./(6.*E**3)-1./(30.*E**5)+1./(42.*E**7)-1./
   1(30.*E**9)+1./((XSH+1.)**2)+1./(XSH**2)
    F=A-VN*VN/B
    FD=D-(VN**3)/(B*B)
    HD=-1.+VN*D/(A*A)
    DFX=-VN*TRG*((F*D-A*FD)/(F*F))-A-VN*HD/XSC
    RETURN
    END
```

Example 2. Subroutine PARGA2 is used to estimate the maximum likelihood parameters of the gamma-2 and gamma-3 probability functions for the annual maximum flows at St. Mary's River at Stillwater. The input data to PARGA2 are N = 60; the data set X(I), I = 1,...,60; IPAR = 2 (gamma-2) and IPAR = 3 (gamma-3); and IWR = 1. The main program to call PARGA2 is shown below.

```
      PROGRAM PGAM2 (INPUT,OUTPUT)
C     EXAMPLES FOR THE MAXIMUM LIKELIHOOD ESTIMATORS FO GAMMA DISTRIBUTIONS
      DIMENSION X(60)
      READ 10,N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARGA2(N,X,XSC,XSH,XLO,2,1)
      CALL PARGA2(N,X,XSC,XSH,XLO,3,1)
      END
```

The computer output for this example is:

```
          PARAMETERS OF THE GAMMA-2 DISTRIBUTION
             (MAX.LIKELIHOOD ESTIMATORS)

             SCALE PARAMETER=  1634.28594
             SHAPE PARAMETER=     8.90583
```

PARAMETERS OF THE GAMMA-3 DISTRIBUTION
(MAX.LIKELIHOOD ESTIMATORS)

TRIAL	LOWER BOUND XLO	VALUE OF LIKE-LIHOOD EQUATION	IMPROVEMENT ON XLO
1	6633.00000	.74097456E+01	
2	5025.00000	-.20094077E+00	
3	6510.62173	.34529710E+01	200.56112
4	6310.06061	.15005988E+01	273.60235
5	6036.45826	.57166388E+00	275.86425
6	5760.59401	.17019648E+00	169.27479
7	5591.31922	.29106101E-01	42.39931
8	5548.91991	.12865864E-02	2.05242
9	5546.86748	.27975711E-05	.00448

LOCATION PARAMETER
(LOWER BOUND OF X)= 5546.86300
SCALE PARAMETER= 2946.61799
SHAPE PARAMETER= 3.05700

3.3 Probability Density and Cumulative Distribution Functions

For a given set of parameters the PDF of the gamma-2 may be obtained directly from Eq. (1) and the PDF of the gamma-3 from Eq. (3). In both cases $\Gamma(\beta)$ is determined by a polynomial approximation as in Hastings (1955) (see Appendix). On the other hand, the computation of the corresponding CDF's require a numerical integration procedure since they cannot be expressed explicitly. The IMSL (1975) uses a series expansion for the exponential term $\exp\{-(x-x_0)/\alpha\}$ to integrate numerically the gamma CDF. This text uses the numerical integration procedure suggested by O'Hara and Smith (1969) which is general and applicable for the integration of any continuous function (see Appendix).

Subroutine DISGAM

Subroutine DISGAM computes probability densities (PDF) and cumulative probabilities (CDF) for the gamma distribution with two or three parameters at N discrete points. These points can be either specified by the user or they can be determined in the program assuming they are equally spaced.

If one wishes to determine the PDF and CDF values at N equally spaced points, the index IX must be set to zero and a maximum XMAX value must be specified in addition to the parameters XSC, XSH and XLO. Thus, N values of the densities and cumulative probabilities will be determined beginning at XLO up to the specified XMAX. On the other hand, if one needs the values of the density and cumulative functions for a set of X(I), I=1,...,N values, (such as sample observations) then IX=1 and X(I) and N should be input in addition to the parameters. The set X(I) should be input in increasing order of magnitude. In the case of the gamma-2 distribution it is necessary to set the location parameter as XLO=0. This is because the parameter XLO is used to identify the type of the distribution, i.e., the value of IPAR.

Subroutine DISGAM requires subroutine CDFINT to evaluate the incomplete integral for a given interval (see Appendix); subroutine GAMMA which evaluates the complete gamma function $\Gamma(\beta)$ for a given shape parameter (see Appendix) and a function FGAM to compute densities. This function should be declared external in subroutine DISGAM.

The input variables to DISGAM are: the number of points N; the data set X(I);I=1,...,N if IX=1; the parameters XLO, XSC and XSH; the maximum value XMAX if IX=0; the index IX=0 if the N points are to be determined in the program or IX=1 if the N points are specified by the user; and the index IWR. The output are the PDF(I) and CDF(I) values corresponding to X(I), I=1,...,N. The program listings of DISGAM and function outprogram FGAM are given below.

```
      SUBROUTINE DISGAM(N,X,XLO,XSC,XSH,XMAX,DEN,CDF,IX,IWR)
C...DENSITY AND CUMULATIVE DISTRIBUTION OF GAMMA PROBABILITY FUNC.
C...IPAR = 2  GAMMA-2        IPAR = 3  GAMMA-3 DISTRIBUTION
C...XMIN = OBSERVED MINIMUM, XMAX = OBSERVED MAXIMUM (USER SUPPLIED)
C...XLO = LOCATION PARAMETER OR LOWER BOUND OF X, IT SHOULD BE
C    SPECIFIED AS XLO = 0 WHEN GAMMA-2 DISTRIBUTION IS IN PREFERENCE
C    XSC = SCALE PARAMETER (ALWAYS POSITIVE), XSH = SHAPE PARAMETER
C...DEN = DENSITY, CDF = CUMULATIVE PROBABILITY AT GIVEN X VALUE
C...IF IX = 0  X(I), I=1,2,...,N VALUES ARE INTERNALLY EVALUATED
C    IF IX = 1  X(I) VALUES MUST BE SUPPLIED BY THE USER IN ORDER
C...IF IWR = 1  WRITE, IF IWR = 0 DO NOT WRITE THE RESULTS
C...ACC = THE VALUE OF THE ABSOLUTE ERROR REQUIREMENT DESIRED
C    IN THE EVALUATION OF CUMULATIVE PROBABILITIES(USER SUPPLIED).
C...DX  = THE SMALLEST FRACTION OF THE TOTAL INTERVAL TO BE ALLOWED
C    DURING EVALUATION OF THE CDF ( USER SUPPLIED).
C...SUBPROGRAM REQUIREMENTS= (1) SUBROUTINE CDFINT TO EVALUATE
C    CUMULATIVE PROBABILITY AT ANY X, (2) FUNCTION FGAM TO EVALUATE
C    DENSITY AT GIVEN X, AND (3) SUBROUTINE GAMCOM TO EVALUATE
C    THE VALUE OF COMPLETE GAMMA FUNCTION FOR GIVEN XSH
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      EXTERNAL FGAM
      DIMENSION X(N), DEN(N), CDF(N)
      DATA ACC,DX/1.E-08,1.E-06/
      IPAR = 3
      IF (IX.GT.0) GO TO 20
      DEN(1) = 0.   $   CDF(1) = 0.
      DELX= (XMAX-XLO)/(N-1)
      DO 10 I= 1,N
   10 X(I) = XLO + (I-1)*DELX
      GO TO 30
   20 XMIN = X(1)
      DEN(1) = FGAM(XMIN)
      XMAX = X(N)
      CALL CDFINT(XLO,XMIN,ACC,DX,P,NUP,FGAM,ID)
      CDF(1) = P
   30 IF (N.EQ.1) GO TO 50
      DO 40  I=2,N
      XU   = X(I)
      XL   = X(I-1)
      DEN(I)= FGAM(XU)
      CALL CDFINT(XL,XU,ACC,DX,P,NUP,FGAM,ID)
   40 CDF(I) = CDF(I-1) + P
   50 IF (IWR.EQ.0) RETURN
      IF (XLO.EQ.0) IPAR=IPAR-1
      PRINT 60, IPAR, XLO, XSC, XSH
   60 FORMAT (1H1///5X,"DENSITY AND CUMULATIVE DISTRIBUTION OF GAMMA"I2/
     15X"(XLO="F12.6"XSC="F12.6"XSH="F10.6")"/6X"ORDER      VARIABLE
     2DENSITY       CUM. DISTR."/17X,"VALUE"7X"FUNCTION"6X"FUNCTION"/)
      PRINT 70, (I, X(I), DEN(I), CDF(I), I=1,N)
   70 FORMAT (7X,I3,F14.6,2F13.6)
      RETURN
      END


      FUNCTION FGAM(X)
      COMMON / BLK1/ XLO, XSC, XSH
      CALL GAMMA (XSH,GB,IER)
      CONS = XSC*GB
      Z = (X - XLO)/XSC
      FGAM = Z**(XSH-1.)*EXP(-Z)/CONS
      RETURN
      END
```

Example 3. Subroutine DISGAM is used to determine the PDF and CDF of the gamma-2 probability function with parameters $x_o = 0.$, $\alpha = 0.5$ and $\beta = 2.0$. The PDF and CDF are desired at 10 equally spaced points beginning from $x = x_o = 0$ up to the value of $x = 5$. The input data to DISGAM are N=10, XLO=0.0, XSC=0.5, XSH=2.0, XMAX=5.0, IX=0 and IWR=1. The main program to call DISGAM is shown below.

```
      PROGRAM GAMDIS (INPUT,OUTPUT)
C     EXAMPLE FOR THE EVALUATION OF GAMMA DENSITY AND CUMULATIVE
      COMMON/BLK1/XLO,XSC,XSH
      DIMENSION X(60), DEN(60), CDF(60)
      READ 10, N, XLO, XSC, XSH, XMIN, XMAX
   10 FORMAT (I5,5F12.0)
      CALL DISGAM(N,X,XLO,XSC,XSH,XMAX,DEN,CDF,0,1)
      END
```

The printed program output for this example is:

```
DENSITY AND CUMULATIVE DISTRIBUTION OF GAMMA 2
(XLO=      0.000000XSC=       .500000XSH=   2.000000)
```

ORDER	VARIABLE VALUE	DENSITY FUNCTION	CUM. DISTR. FUNCTION
1	0.000000	0.000000	0.000000
2	.555556	.731540	.305037
3	1.111111	.481636	.650814
4	1.666667	.237827	.845413
5	2.222222	.104388	.936062
6	2.777778	.042955	.974657
7	3.333333	.016968	.990243
8	3.888889	.006517	.996323
9	4.444444	.002452	.998636
10	5.000000	.000908	.999501

3.4 The Inverse of the Cumulative Distribution Function

The inverse of the CDF of the gamma distribution is not simple to obtain because there is neither an explicit solution nor an approximation for the incomplete gamma function. Therefore, a procedure of successive approximations is used in this text to compute the inverse of F(x).

The inverse x of a given F(x) is found by computing the $F(x_i)$, $i=1,2,\ldots$ corresponding to $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$, ..., $x_i = x_{i-1} + \Delta x$ until $F(x_i)$ is within a specified error ε relative to F(x), or

$$\Delta F(x) = |F(x_i) - F(x)| < \varepsilon \quad . \tag{42}$$

Figure 2 shows graphically the above procedure.

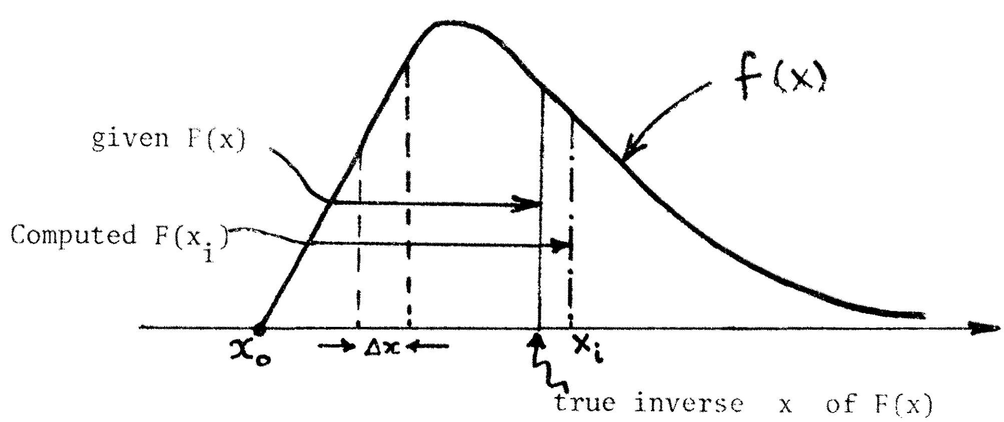

Fig. 2 Successive Approximation for Finding the Inverse x of a given F(x).

Subroutine INVGAM

This subroutine determines the inverse of the CDF of the gamma distribution by the procedure of successive approximations described above. The inverse x is represented by X(I) and the cumulative distribution function F(x) by CDF(I), I=1,...,N. INVGAM determines X(I) either for a set of specified CDF(I), I=1,...,N; or for a set of equally spaced CDF(I) = I/(N+1), I=1,...,N, where N is given by the user. In the first case the index ICDF=1, while in the second case ICDF=0. The subprograms needed for INVGAM are CDFINT and GAMMA (see Appendix) and FGAM (described previously).

In the approximation followed herein the integration increment Δx is represented by DELX which is initially set to

$$\text{DELX} = (\text{XLO} + \text{XSC} * \text{XSH})/(\text{N}+1) \quad . \tag{43}$$

Since the numerator in Eq. (43) represents the mean of the gamma distribution, the computed DELX is not generally a small value which enables to locate rather quickly the interval where the actual inverse X(I) of CDF(I) lies. When this interval is found a new DELX is computed by dividing the previous DELX over M which is specified as 10 in a DATA statement of INVGAM. In addition to M, this data statement specifies ERROR=10^{-6} which is the maximum absolute error of $\Delta F(x)$ of Eq. (42); ACC=10^{-8} which is the accuracy of the computed CDF (represented by FT in INVGAM) in subroutine CDFINT; and DX=10^{-6} which is the increment of integration in this last subroutine. The four values, M, ERROR, ACC, and DX may be changed by the user if desired.

Inputs to INVGAM are the parameters XLO, XSC and XSH; N the number of inverses of $F(x)$ to be computed; the index ICDF and the index IWR. The output of INVGAM is the set of inverses X(I), I=1,...,N corresponding to CDF(I). The program listing of INVGAM is given below.

```
      SUBROUTINE INVGAM (XLO,XSC,XSH,N,CDF,X,ICDF,IWR)
C   INVERSE OF THE GAMMA DISTRIBUTION FUNCTION
C..XLO = LOCATION PARAMETER OR LOWER BOUND OF X, IT SHOULD BE GIVEN
C   AS XLO=0 EVEN IF ONE NEEDS THE GAMMA-2 DISTRIBUTION.
C..XSC = SCALE PARAMETER,  XSH= SHAPE PARAMETER( XSH SHOULD BE .LE.57)
C..CDF = CUMULATIVE PROBABILITY AT X(I)
C   ICDF= 0 CDF(I), I=1,2,...,N SET IS NOT GIVEN BY THE USER.
C   ICDF= 1 CDF(I), I=1,2,...,N SET SHOULD BE GIVEN BY THE USER.
C   N    = SIZE OF X(I)
C   X    = INVERSE CORRESPONDING TO CDF(I)
C..IWR = 0  DO NOT WRITE,  IWR = 1 WRITE THE RESULTS.
C**XLO,XSC,XSH,N,ICDF AND IWR VALUES SHOULD BE SUPPLIED BY THE USER.
C..SUBPROGRAMS NEEDED ARE...CDFINT, FGAM, GAMMA
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
```

```
      EXTERNAL FGAM
      DIMENSION X(N), CDF(N)
      DATA M, ERROR, ACC, DX /10,1.0E-06,1.0E-08,1.0E-06/
      ZL=XLO   $   FT=0.   $   IPAR=3
      DO 60 I=1,N
      DELX=(XLO+XSC*XSH)/(N+1)
      IF(ICDF.GT.0) GO TO 20
      CDF(I)= FLOAT(I)/(N+1)
      GO TO 20
   10 DELX=  (ZU-ZL)/M
   20 DO 30   K=1,M
      ZU = ZL + DELX
      CALL CDFINT(ZL,ZU,ACC,DX,PT,NUP,FGAM,ID)
      FT = FT + PT
      IF(FT-CDF(I)) 30, 50, 40
   30 ZL = ZU
   40 IF(ABS(FT-CDF(I)).LE.ERROR) GO TO 50
      FT = FT-PT
      GO TO 10
   50 X(I) = ZU
   60 ZL = ZU
      IF( XLO.EQ.0.) IPAR = IPAR-1
      IF (IWR.EQ.0) RETURN
      PRINT 70, IPAR, XLO, XSC, XSH
   70 FORMAT (1H1///7X,"INVERSE OF THE GAMMA-",I1" DISTRIBUTION FUNCTION
     1"/5X"( XLO="F10.3"    XSC="F10.4"    XSH="F8.5" )"//10X,"ORDER",7X,"
     2CUMULATIVE"/21X"DISTRIBUTION"/11X"(I)"10X"CDF(X)"13X"X(I)"/)
      PRINT 80, (I, CDF(I), X(I), I=1,N)
   80 FORMAT (10X,I3,2F18.6)
      RETURN
      END
```

Example 4. Subroutine INVGAM is used to determine the inverse x of $F(x)$ for a gamma distribution with parameters XLO = 0.0, XSC = 0.5 and XSH = 2.0. Two cases are considered: (1) the CDF $F(x)$ is given as CDF(I), I=1,...,10 (see the first table of the computer output below), therefore, ICDF=1, and IWR = 1; (2) the inverse is desired for 10 equally spaced values of the CDF or N=10, ICDF=0, and IWR=1. The main program to call INVGAM is shown below.

```
      PROGRAM GAMCLA (INPUT,OUTPUT)
C     INVERSE OF THE GAMMA DISTRIBUTION FUNCTION
      COMMON /BLK1/ XLO, XSC, XSH
      DIMENSION X(100), CDF(100)
      N=10    $    XLO=0.    $    XSC=0.5    $    XSH=2.
      READ 10, (CDF(I), I=1,N)
   10 FORMAT (10F8.6)
      CALL INVGAM (XLO,XSC,XSH,N,CDF,X,1,1)
      CALL INVGAM (XLO,XSC,XSH,N,CDF,X,0,1)
      END
```

The computed output of INVGAM for this example is:

INVERSE OF THE GAMMA-2 DISTRIBUTION FUNCTION
(XLO= 0.000 XSC= .5000 XSH= 2.00000)

ORDER (I)	CUMULATIVE DISTRIBUTION CDF(X)	X(I)
1	0.000000	.000091
2	.305037	.555556
3	.650614	1.111111
4	.845413	1.666668
5	.936062	2.222223
6	.974657	2.777786
7	.990243	3.333332
8	.996323	3.888968
9	.998636	4.444423
10	.999501	5.000786

INVERSE OF THE GAMMA-2 DISTRIBUTION FUNCTION
(XLO= 0.000 XSC= .5000 XSH= 2.00000)

ORDER (I)	CUMULATIVE DISTRIBUTION CDF(X)	X(I)
1	.090909	.251162
2	.181818	.386895
3	.272727	.511535
4	.363636	.636577
5	.454545	.768598
6	.545455	.914022
7	.636364	1.081590
8	.727273	1.286285
9	.818182	1.560397
10	.909091	2.004588

3.5 Confidence Limits

As before, the α confidence limits x_ℓ for the T-year event are determined by

$$x_\ell = x_T \pm u_\alpha s_T \quad , \tag{44}$$

where u_α is the standard normal variate corresponding to the α confidence level. The value of x_T may be determined by

$$x_T = \hat{\mu} + K_T \hat{\sigma} \tag{45}$$

where K_T is the frequency factor, a function of the skewness and T.

The variable x is gamma distributed with parameters $\hat{\beta}$, $\hat{\alpha}$ and $\hat{x}_o$ or x is $G(\hat{\beta}, \hat{\alpha}, \hat{x}_o)$. The standardized variable $(x-\hat{x}_o)/\hat{\alpha}$ is $G(\hat{\beta}, 1, 0)$. Similarly the variable $2(x-\hat{x}_o)/\hat{\alpha}$ is $G(\hat{\beta}, 2, 0)$ or simply $G(\beta, 2)$. But a gamma variable with parameters $(\beta, 2)$ is also χ^2 distributed with $\nu = 2\beta$ degrees of freedom (Mood, et. al, 1974). Therefore, we may also write that

$$\chi_T^{\,2} = \frac{2(x_T - \hat{x}_o)}{\hat{\alpha}} \quad ,$$

or

$$x_T = \frac{1}{2}\hat{\alpha}\,\chi_T^{\,2} + \hat{x}_o \quad . \tag{46}$$

From Eqs. (45) and (46) we have

$$K_T = (\frac{1}{2}\hat{\alpha}\,\chi_T^{\,2} + \hat{x}_o - \hat{\mu})/\hat{\sigma} \quad .$$

Furthermore, since $\hat{\mu} = \hat{\alpha}\hat{\beta} + \hat{x}_o$, $\hat{\sigma} = \hat{\alpha}/\sqrt{\hat{\beta}}$ and $\hat{\gamma} = 2/\sqrt{\hat{\beta}}$, the frequency factor K_T may be expressed as

$$K_T = 0.25\,\chi_T^{\,2}\,\hat{\gamma} - 2(1/\hat{\gamma}) \quad , \tag{47}$$

where $\hat{\gamma}$ is the skewness coefficient of the variable x.

There are two ways to compute χ_T^2 of Eq. (47) for given T and $\nu = 2\beta$. The first way would be to find the inverse of the gamma distribution with parameters (β, 2) by a numerical integration procedure (as shown in previous section); and the second procedure is to use the approximation (Abramowitz and Stegun, 1970)

$$\chi_T^2 \approx \nu[1 - (2/9\nu) + u_T\,(2/9\nu)^{1/2}]^3 \quad , \tag{48}$$

with $\nu = 2\beta$, and it is valid for $\nu > 30$. Since $\nu = 2\beta = 8/\hat{\gamma}^2$, Eqs. (48) and (47) give (Kite, 1977)

$$\begin{aligned} K_T \approx\ & u_T + (u_T^2 - 1)(\hat{\gamma}/6) + (1/3)(u_T^3 - 6u_T)(\hat{\gamma}/6)^2 - (u_T^2 - 1) \\ & (\hat{\gamma}/6)^3 + u_T\,(\hat{\gamma}/6)^4 + (1/3)(\hat{\gamma}/6)^5 \quad . \end{aligned} \tag{49}$$

The moment estimator of the standard error s_T of Eq. (44) may be shown to be (Kite, 1977)

$$\begin{aligned} s_T^2 = \frac{\hat{\mu}_2}{N}\,[1 + K_T\hat{\gamma} + \frac{K_T^2}{2}\,(3\hat{\gamma}^2/4 + 1) + 3\,K_T\,\frac{\partial K_T}{\partial\hat{\gamma}}\,(\gamma + \gamma^3/4) + \\ + 3(\frac{\partial K_T}{\partial\gamma})^2\,(2 + 3\gamma^2 + 5\gamma^4/8)] \quad , \end{aligned} \tag{50}$$

where

$$\begin{aligned} \frac{\partial k_T}{\partial\hat{\gamma}} \approx\ & \{(u_T^2 - 1)[1 - 3(\frac{\hat{\gamma}}{6})^2] + 4\,u_T\,(\frac{\hat{\gamma}}{6})^3\}\,(1/6) + \\ & + (\frac{\hat{\gamma}}{6})\,\{4(u_T^3 - 6u_T) - 10\,(\frac{\hat{\gamma}}{6})^3\}\,(1/6)^2 \quad . \end{aligned} \tag{51}$$

Equation (51) was obtained from Eq. (49), therefore, it is only an approximation.

The maximum likelihood estimator of the standard error s_T may be computed by (Kite, 1977)

$$s_T^2 = \left(\frac{\partial x}{\partial \alpha}\right)^2 \text{Var } \alpha + \left(\frac{\partial x}{\partial \beta}\right) \text{Var } \beta + \left(\frac{\partial x}{\partial x_o}\right) \text{Var } x_o +$$

$$+ 2\left(\frac{\partial x}{\partial \alpha}\right) \left[\left(\frac{\partial x}{\partial \beta}\right) \text{Cov }(\alpha,\beta) + \left(\frac{\partial x}{\partial x_o}\right) \text{Cov }(\alpha, x_o)\right] + 2 \left(\frac{\partial x}{\partial \beta}\right)\left(\frac{\partial x}{\partial x_o}\right)$$

$$\text{Cov }(\beta, x_o) \quad , \tag{52}$$

where

$$\text{Var } \alpha = \frac{1}{N \alpha^2 D} \left\{\frac{\psi'(\beta)}{\beta-2} - \frac{1}{(\beta-1)^2}\right\} \quad , \tag{53}$$

$$\text{Var } \beta = 2/[N \alpha^4 D (\beta-2)] \quad , \tag{54}$$

$$\text{Var } x_o = [\beta \psi'(\beta) - 1]/N \alpha^2 D \quad , \tag{55}$$

$$\text{Cov }(\alpha,\beta) = -\left[\frac{1}{\beta-2} - \frac{1}{\beta-1}\right]/N \alpha^3 D \quad , \tag{56}$$

$$\text{Cov }(\alpha,x_o) = \left[\frac{1}{\beta-1} - \psi'(\beta)\right]/N \alpha^2 D \quad , \tag{57}$$

$$\text{Cov }(\beta,x_o) = -\left[\frac{\beta}{\beta-1} - 1\right]/N \alpha^3 D \quad , \tag{58}$$

where D is the determinant of the matrix of likelihood derivatives;

$$D = \left[2\psi'(\beta) - \frac{2\beta-3}{(\beta-1)^2}\right]/[(\beta-2)\alpha^4] \tag{59}$$

and $\psi'(\beta)$ is the trigamma function. The partial derivatives of x are from Eqs. (46) and (48) approximately

$$\frac{\partial x}{\partial \alpha} \cong \left[\beta^{1/3} - \frac{1}{9\beta^{2/3}} + \frac{u}{3\beta^{1/6}}\right]^3 \quad , \tag{60}$$

$$\frac{\partial x}{\partial \beta} \cong 3\alpha \left(\frac{\partial x}{\partial \alpha}\right)^{2/3} \left[\frac{1}{3\beta^{2/3}} + \frac{2}{27\beta^{5/3}} - \frac{u}{18\beta^{7/6}}\right] \quad , \tag{61}$$

$$\frac{\partial x}{\partial x_o} = 1 \quad . \tag{62}$$

Subroutine CLIGA1

This subroutine determines the approximate moment estimates of the T-year event magnitude x_T of Eq. (45) and the (100α) percent confidence limits x_ℓ of Eq. (44) based on the moment estimator of the standard error s_T of Eq. (50). The event magnitude XT(I) and its upper and lower confidence limits XUT(I) and XLT(I) are computed for specified return periods T(I) or non-exceedence probabilities PXT(I), I=1,...,NT, with NT the number of desired T or PXT values, and for a given confidence level CONF. Subroutine RETCDF makes inverse transformations between T and PXT and determines the standard normal deviate UCL corresponding to the confidence level CONF.

Inputs to CLIGA1 are the sample size N; the number NT of return periods or non-exceedence probabilities, the confidence level CONF; the parameters XLO, XSC, and XSH; the return period or non-exceedence probabilities T(I),I=1,...,NT and the index IWR. Outputs from CLIGA1 are the T-year value XT(I) and its confidence limits XUT(I) and XLT(I) corresponding to the input T(I), I=1,...,NT. The program listing of CLIGA1 is given below. The listing of RETCDF was given before.

```
      SUBROUTINE CLIGA1 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IWR)
C  CONFIDENCE LIMITS OF THE GAMMA DISTRIBUTIONS BY MOMENTS
C  XLO   = LOCATION  PARAMETER ( LOWER  BOUND )
C  XSC   = SCALE  PARAMETER ,  XSH = SHAPE  PARAMETER
C *  ALL PARAMETERS SHOULD BE THE MOMENT ESTIMATORS
C**LOCATION PARAMETER SHOULD BE SPECIFIED AS XLO=0 EVEN FOR GAMMA-2
C   N    = SAMPLE SIZE USED FOR ESTIMATION OF THE PARAMETERS
C   NT   = NUMBER OF CONFIDENCE LIMITS TO BE COMPUTED
C   CONF= SELECTED PROBABILITY OF CONFIDENCE(AS 0.90, 0.95, 0.99)
C   T    = AN INPUT VECTOR INDICATING EITHER RETURN PERIODS (IN YEARS)
C   OR NONEXCEEDANCE PROBABILITIES CORRESPONDING TO THE RETURN PERIODS.
C   PXT = PROBABILITY OF NONEXCEEDANCE FOR  T- YEAR  EVENT
C   XT   = ESTIMATE OF THE T-YEAR EVENT FROM FREQUENCY EQUATION
C   UCL  = TWO-TAIL STANDARD NORMAL DEVIATE FOR GIVEN CONFIDENCE LEVEL
C   ST   = STANDARD ERROR OF THE ESTIMATED T- YEAR EVENT
C   XUT = UPPER CONFIDENCE LIMIT,  XLT = LOWER CONFIDENCE LIMIT
C** ABRAMOWITZ-STEGUN POLYNOMIAL APPROXIMATION IS USED TO COMPUTE
C   THE VALUES OF STANDARD NORMAL DEVIATES FOR GIVEN PROBABILITIES.
C      SUBPROGRAM NEEDED...RETCDF
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
```

```
      DIMENSION  T(NT), XT(NT), XUT(NT), XLT(NT), PXT(NT)
      CALL RETCDF(NT,T,CONF,PXT,UCL)
      B = 1. / (9.*XSH)
      XM2 = XSC * XSH
      IF( XLO.LT.0. OR .XLO.GT.0. )  GO TO 10
      IPAR = 2  $  CVX = 1./SQRT(XSH)  $  W = CVX/3.
      GO TO 20
   10 IPAR = 3   $   G = 2./SQRT(XSH)   $   Z = G/6.
   20 SX = XSC * SQRT(XSH)
      DO  50  I=1,NT
      PROB = PXT(I)
      IF  (PXT(I).GT.0.5) PROB = 1.-PXT(I)
      V = SQRT( -2.*ALOG(PROB) )
      U  = V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1        (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      IF  (PXT(I).LT.0.5) U=-U
      XT(I) = XM2 * ( 1.-B+U*SQRT(B) )**3 + XLO
      IF  (IPAR.EQ.3) GO TO 30
      FK = ( (1.-W**2+U*W)**3-1.) / CVX
      ST = SQRT( (1.+2.*FK*CVX**2 + 0.5*FK*( 1.+3.*CVX**4 ) )/N )*SX
      GO TO 40
   30 FK = U+(U**2-1.)*(Z-Z**3)+(U**3-6.*U)*Z**2/3.+U*Z**4+Z**5/3.
      DFK=( (U**2-1.)*(1.-3.*Z**2) + 4.*U*Z**3 ) / 6.
     1    + Z*( 4.*(U**3-6.*U) - 10.*Z**3 ) / 36.
      ST = ( 1.+ FK*G + (1.+.75*G**2)*FK**2/2 + 3*FK*DFK*G*(1+.25*G**2)
     1    + 3.*( 2.+3.*G**2+.625*G**4 )*DFK**2 ) / N
      ST = SX * SQRT(ST)
   40 XUT(I) = XT(I) + UCL*ST
   50 XLT(I) = XT(I) - UCL*ST
      IF  (IWR.EQ.0) RETURN
      PRINT 60, CONF, IPAR, XLO, XSC, XSH
   60 FORMAT (1H1///5X,"MOMENT ESTIMATES OF THE",F4.2," CONFIDENCE LIMIT
     1S FOR GAMMA-",I1" DISTRIBUTION",/12X,"( XLO=",F12.5,"   XSC=",F12.
     27,"   XSH=",F12.7," )"//12X,"RETURN NONEXCEEDANCE",7X"EVENT"8X,
     3"CONFIDENCE  LIMITS"/12X"PERIOD  PROBABILITY"6X"MAGNITUDE"6X"UPPER
     4"8X"LOWER"/)
      PRINT 70, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   70 FORMAT (14X,F4.0,2X,F10.5,4X,3F13.5)
      RETURN
      END
```

Example 5. Subroutine CLIGA1 is applied to compute moment estimates of design floods and their corresponding 95 percent confidence limits for 2, 5, 10, 20, 50 and 100 years of return periods. The floods are gamma distributed with moment parameter estimates $\hat{\alpha}$ = 4045.11, $\hat{\beta}$ = 1.66965 and $\hat{x}_o$ = 7800.74 obtained from 60 years of flood data of St. Mary's River at Stillwater using subroutine PARGA1. The input variables to CLIGA1 are N=60; NT=6; T(I)=2, 5, 10, 20, 50 and 100 for I=1,...,6; CONF=0.95; XLO=7800.74; XSC=4045.11; XSH=1.66965 and IWR=1. The main program to input this information is given below.

```
      PROGRAM GAMCLA (INPUT,OUTPUT)
C     EXAMPLE OF USE OF SUBPROGRAM ON CONFIDENCE LIMITS FOR THE GAMMA DIST.
      DIMENSION T(60), XT(60), XUT(60), XLT(60), PXT(60)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5,7F8.2)
      XLO=7800.74     $     XSC=4045.11     $     XSH=1.66965
      CALL CLIGA1 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The computer output from CLIGA1 for this example is:

```
MOMENT ESTIMATES OF THE .95 CONFIDENCE LIMITS FOR GAMMA-3 DISTRIBUTION
       ( XLO=   7800.74000    XSC=4045.1100000    XSH=    1.6696500 )

       RETURN  NONEXCEEDANCE         EVENT          CONFIDENCE   LIMITS
       PERIOD   PROBABILITY        MAGNITUDE        UPPER          LOWER

           2.        .50000        13294.02771    14989.93442   11598.12101
           5.        .80000        18086.58420    20320.36870   15852.79970
          10.        .90000        21443.35422    24406.89011   18479.81833
          20.        .95000        24710.02019    29077.37996   20342.66043
          50.        .98000        28965.63349    35917.97284   22013.29414
         100.        .99000        32165.80514    41441.17426   22890.43603
```

Subroutine CLIGA2

This subroutine determines the maximum likelihood estimates of the T-year event magnitude x_T of Eq. (45) and the 100α percent confidence limits x_ℓ of Eq. (44) based on the maximum likelihood estimator of the standard error s_T of Eq. (52).

The overall description of this subroutine is the same as the subrountine CLIGA1 referred before. The program listing of CLIGA2 is given below.

```
      SUBROUTINE CLIGA2 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IWR)
C   MAX.LIKELIHOOD ESTIMATES OF THE CONFIDENCE LIMITS FOR GAMMA DISTR.
C   XLO  = LOCATION  PARAMETER ( LOWER  BOUND )
C   XSC  = SCALE  PARAMETER ,  XSH = SHAPE  PARAMETER
C** PARAMETERS SHOULD BE GIVEN AS MAXIMUM LIKELIHOOD ESTIMATES
C**LOCATION PARAMETER SHOULD BE SPECIEFIED AS XLO=0 WHEN GAMMA-2
C   N    = SAMPLE SIZE USED FOR ESTIMATION OF THE PARAMETERS
C   NT   = NUMBER OF CONFIDENCE LIMITS TO BE COMPUTED
C   CONF= SELECTED PROBABILITY OF CONFIDENCE(AS 0.90, 0.95, 0.99)
C   T    = AN INPUT VECTOR INDICATING EITHER RETURN PERIODS (IN YEARS)
C   OR NONEXCEEDANCE PROBABILITIES CORRESPONDING TO THE RETURN PERIODS.
C   PXT = PROBABILITY OF NONEXCEEDANCE FOR  T- YEAR  EVENT
C   XT  = ESTIMATE OF THE T-YEAR EVENT FROM FREQUENCY EQUATION
C   UCL = TWO-TAIL STANDARD NORMAL DEVIATE FOR GIVEN CONFIDENCE LEVEL
C   ST  = STANDARD ERROR OF THE ESTIMATED T- YEAR EVENT
C   XUT = UPPER CONFIDENCE LIMIT,  XLT = LOWER CONFIDENCE LIMIT
C   TRG = THE TRIGAMMA FUNCTION
C** ABRAMOWITZ-STEGUN POLYNOMIAL APPROXIMATION IS USED TO COMPUTE
C   THE VALUES OF STANDARD NORMAL DEVIATES FOR GIVEN PROBABILITIES.
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION  T(NT), XT(NT), XUT(NT), XLT(NT), PXT(NT)
      CALL RETCDF(NT,T,CONF,PXT,UCL)
      B = 1. / (9.*XSH)
      XM2 = XSC * XSH
      Z = XSH**(1./3)
      A = XSH + 2.
      TRG = 1./A + 0.5/(A**2) + 1./(6.*A**3) -1./(30.*A**5)
     1      +1./(42.*A**7) - 1./(30.*A**9) + 1./((XSH+1.)**2) +1./(XSH**2)
      E = 1. / (XSH-1.)
      F = 1. / (XSH-2.)
      D = (2.*TRG - (2.*XSH-3.)*E**2)*F / (XSC**4)
      C = 1. /(N*D*XSC**2)
      VA = C*( TRG*F -E**2 )
      VB = 2.*C*F / (XSC**2)
      CV1= -C*( F-E ) / XSC
      IF( XLO)  10, 20, 10
   10 IPAR = 3
      VLB=  C*( XSH*TRG - 1. )
      CV2=  C*( E - TRG )
      CV3= -C*( E*XSH - 1. ) / XSC
      GO TO 30
   20 IPAR = 2
      VLB = CV2 = CV3 = 0.0
   30 DO 40  I=1,NT
      PROB = PXT(I)
      IF (PXT(I).GT.0.5) PROB = 1.-PXT(I)
      V = SQRT( -2.*ALOG(PROB) )
      U  = V - (2.515517 + 0.802853*V + 0.010328*V**2) /
     1      (1.+ 1.432788*V + 0.189269*V**2 + 0.001308*V**3)
      IF (PXT(I).LT.0.5) U=-U
      W = Z - 1./(9.*Z**2) + U/( 3.*SQRT(Z) )
      DXA= W**3
         FF = 1./(3.*Z**2) + 2./(27.*Z**5) - U/(18.*Z**3.5)
         DXB = 3.* XSC *FF * W**2
      ST = SQRT( VA*DXA**2 + VB*DXB**2 + VLB + 2.*( DXA*DXB*CV1
     1                      + DXA*CV2 + DXB*CV3 ) )
      XT(I) = XM2 * ( 1.-B+U*SQRT(B) )**3 + XLO
      XUT(I) = XT(I) + UCL*ST
   40 XLT(I) = XT(I) - UCL*ST
      IF (IWR.EQ.0) RETURN
      PRINT 50, CONF, IPAR, XLO, XSC, XSH
   50 FORMAT(1H1///5X"MAX.LIKELIHOOD ESTIMATES OF "F4.2" CONFIDENCE LIMI
     1TS FOR  GAMMA-"I1"  DISTRIBUTION"/13X"( XLO="F12.5"   XSC="F12.7"
     2  XSH="F12.7" )"//13X"RETURN NONEXCEEDANCE"6X"EVENT"8X"CONFIDENCE
     3LIMITS"/13X"PERIOD PROBABILITY"6X"MAGNITUDE"6X"UPPER"8X"LOWER"/)
      PRINT 60, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   60 FORMAT (14X,F4.0,2X,F10.5,4X,3F13.5)
      RETURN
      END
```

<u>Example 6</u>. Subroutine CLIGA2 is applied to compute the maximum likelihood estimates of design floods and their corresponding 95 percent confidence limits for 0.5, 0.8, 0.9, 0.95, 0.98 and 0.99 non-exceedence probabilities. The floods are gamma distributed with maximum likelihood parameter estimates of $\hat{\alpha}$ = 2946.618, $\hat{\beta}$ = 3.057 and $\hat{x}_o$ = 5546.863 obtained from 60 years of flood data of St. Mary's River at Stillwater using subroutine PARGA2. The input variables to CLIGA2 are N=60; NT=6; T(I)=0.5, 0.8, 0.9, 0.95, 0.98, and 0.99 for I=1,...,6; CONF=0.95; XLO=5546.863; XSC=2946.618; XSH=3.057 and IWR=1. The main program to input this information is given below.

```
      PROGRAM GAMCL2 (INPUT,OUTPUT)
C     EXAMPLE OF USE OF SUBPROGRAM ON CONFIDENCE LIMITS FOR THE GAMMA
      DIMENSION  T(60), XT(60), XUT(60), XLT(60), PXT(60)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5, 7F8.2)
      XLO=5546.863  $  XSC=2946.618  $  XSH=3.05700
      CALL CLIGA2 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The computer output from CLIGA2 for this example is:

```
MAX.LIKELIHOOD ESTIMATES OF  .95 CONFIDENCE LIMITS FOR  GAMMA-3  DISTRIBUTION
      ( XLO=   5546.86300    XSC=2946.6180000    XSH=    3.0570000 )

      RETURN NONEXCEEDANCE        EVENT          CONFIDENCE LIMITS
      PERIOD PROBABILITY        MAGNITUDE        UPPER         LOWER

        2.       .50000        13607.73493   14848.03852   12367.43134
        5.       .80000        18340.83207   20262.10190   16419.56225
       10.       .90000        21426.21920   24031.01803   18821.42037
       20.       .95000        24318.71498   27691.00497   20946.42499
       50.       .98000        27970.48547   32440.54043   23500.43051
      100.       .99000        30650.25250   35996.53575   25303.96926
```

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APPENDIX

1. Program Listing of Subroutine GAMMA

```
      SUBROUTINE GAMMA (Z,G,IER)
C.....FROM IBM/360 SCIENT. SUB. PACKAGE,1968
      IF (Z.LE.57.) GO TO 10
      IER = 2
      G = 1.E75
      RETURN
   10 X = Z
      ERR = 1.0E-06
      IER = 0
      G = 1.0
      IF (X-2.) 40,40,30
   20 IF (X.LE.2.) GO TO 80
   30 X = X-1.
      G = G*X
      GO TO 20
   40 IF (X-1.) 50,90,80
   50 IF (X.GE.ERR) GO TO 70
      Y = FLOAT(INT(X))-X
      IF (ABS(Y).LE.ERR) GO TO 100
      IF ((1.-Y-ERR).LE.0.) GO TO 100
   60 IF (X.GT.1.) GO TO 80
   70 G = G/X
      X = X+1.
      GO TO 60
   80 Y = X-1.
      GY = 1.+Y*(-.57710166+Y*(.98585399+Y*(-.87642182+Y*(.8328212+Y*(-.
     15684729+Y*(.25482049+Y*(-.0514993)))))))
      G = G*GY
   90 RETURN
  100 IER = 1
      RETURN
      END
```

2. Program Listing of Subroutine CDFINT

```
      SUBROUTINE CDFINT (AA,BB,ACC,SML,ANS,NUP,F,ID)
C     THE SUBROUTINE CDFINT USES THE O"HARA-SMITH ALGORITHM TO EVALUATE
C     THE DEFINITE INTEGRAL OF A GIVEN FUNCTION BETWEEN THE LIMITS A $ B.
C     FOR REFERENCE, SEE H. O"HARA AND F. J. SMITH--THE EVALUATION OF
C     THE DEFINITE INTEGRALS BY INTERVAL SUBDIVISION, COMPUTER JOURNAL,
C     VOL. 12, 1969, P. 179.
C     AA IS THE LOWER LIMIT OF INTEGRATION--USER SUPPLIED.
C     BB IS THE UPPER LIMIT OF INTEGRATION--USER SUPPLIED.
C     ACC IS THE VALUE OF THE ABSOLUTE ERROR REQUIREMENT DESIRED IN THE
C     RESULT. EPS IS USER SUPPLIED.
C     SML IS THE VALUE OF THE SMALLEST FRACTION OF THE TOTAL INTERVAL TO
C     BE ALLOWED DURING EVALUATION OF THE INTEGRAL--USER SUPPLIED.
C     ANS IS THE VARIABLE CONTAINING THE VALUE OF THE INTEGRAL AS
C     EVALUATED IN THE SUBROUTINE.
C     NUP IS AN INTEGER VARIABLE CONTAINING THE NUMVER OF INTEGRAND
C     EVALUATIONS THAT HAVE TAKEN PLACE IN THE SUBROUTINE.
C     F IS THE NAME OF THE USER SUPPLIED FUNCTION SUBPROGRAM DEFINING
C     THE INTEGRAND.  F MUST BE DECLARED EXTERNAL IN THE CALLING PROGRAM.
C     ID IS AN INTEGER VARIABLE INDICATING WHETHER CONVERGENCE HAS
C     OCCURRED IN THE SUBROUTINE.  ID=1 IF CONVERGES AND ID=-1 OTHERWISE.
C     USER SHOULD CHECK VALUE OF ID ON RETURN TO ASSURE CORRECT RESULTS.
```

```
      DIMENSION B1(30), B2(30), B3(30), B4(30), P(30)
      DATA CNC1,CNC2,CNC3,CNC4/ .3111111111111111, .6222222222222222,
     1 .5333333333333333, 1.422222222222222/, C8R1, C8R2, C8R3, C8R4/
     2.3061728395061728, .6151675485008818, .4966490299823633,
     3 1.444797178130511/, C7L1, C7L2, C7L3, C7L4/ .1142857142857143,
     4 2.082539682539683, 1.828571428571429, 1.015873015873016/,
     5 RT3/ 1.732050807568877/
      ID = 1
      X = AA       $     Z = BB     $     Y = (BB+AA)/2     $    H = (BB-AA)/4
      U = X+H      $     V = Y+H    $     T0 = F(AA)        $     T4 = F(Y)
      T8 = F(BB)   $     T2 = F(U)  $     T6 = F(V)
      NUP = 5      $     ERR = ACC  $     ANS = 0.          $     K = 0
      HMIN = SML*H
   10 H = .5*H
      IF (ABS(H).LE.ABS(HMIN)) GO TO 50
      W1 = X+H     $     W2 = U+H    $     W3 = Y+H          $     W4 = V+H
      T1 = F(W1)   $     T3 = F(W2)  $     T5 = F(W3)        $     T7 = F(W4)
      NUP = NUP+4
      SA = T0+T8 $     SC = T2+T6   $     SD = T1+T3+T5+T7
      ROMBRG = (C8R1*SA+C8R2*T4+C8R3*SC+C8R4*SD)*H
      AISNC = (CNC1*SA+CNC2*T4+CNC3*SC+CNC4*SD)*H
      EST = ABS(ROMBRG-AISNC)
      UV = 10.
      IF (K.EQ.0) UV = 1.
      ZUB = ERR/UV
      IF (EST.GT.ZUB) GO TO 20
      XX = RT3*H
      T9 = F(U-XX)+F(U+XX)
      T10 = F(V-XX)+F(V+XX)
      NUP = NUP+4
      ACL7 = (C7L1*(SA+2.*T4)+C7L2*SC+C7L3*SD+C7L4*(T9+T10))*H*.5
      EST = ABS(ACL7-ROMBRG)
      EMAX = ABS(.5*(T0+T4)+T1+T3-T2-T9)
     1       +(ABS(.5*(T4+T8)+T5+T7-T6-T10))*H*64./945.
      IF (EMAX.GT.EST) EST=EMAX
      IF (EST.LE.ZUB) GO TO 40
   20 K = K+1
      IF (K.LE.30) GO TO 30
      ID = -1
      GO TO 60
   30 B1(K)=T5 $  B2(K)=T6 $  B3(K)=T7 $  B4(K)=T8 $  P(K)=H
      T8=T4    $  T6=T3    $  T4=T2    $  T2=T1
      Z=Y      $  V=W2     $  Y=U      $  U=W1
      GO TO 10
   40 ANS = ANS+ACL7
      ERR = ERR+EST
   50 IF (K.LE.0) RETURN
      H = P(K)          $     X = Z         $     T0 = T8
      T2 = B1(K)        $     T4 = B2(K)    $     T6 = B3(K)      $    T8 = B4(K)
      U = X+H           $     Y = U+H       $     V = Y+H         $    Z = V+H
      K = K-1
      GO TO 10
   60 PRINT 70
   70 FORMAT (//20X,"ID = -1 INDICATES THAT CONVERGENCE WAS NOT ATTAINED
     1"/)
      STOP
      END
```

4. LOG-PEARSON TYPE III DISTRIBUTION

4.1 General Properties

The probability density function of the Log-Pearson III distribution is

$$f(x) = \frac{1}{\alpha\Gamma(\beta)x} \left\{\frac{\ell n(x)-y_o}{\alpha}\right\}^{\beta-1} \exp\left\{-\frac{\ell n(x)-y_o}{\alpha}\right\} \quad 0 < x < \infty \tag{1}$$

where α, β and $y_o = (\ell n\ x)_o$ are the scale, shape and location parameters in the log domain respectively. Actually the Log-Pearson III distribution is a gamma (Pearson III) distribution in the log domain. That is, if x is Log-Pearson III distributed, then $y = \ell n(x)$ is gamma distributed.

4.2 Estimation of Parameters

There are three methods for estimating the parameters of the Log-Pearson III distribution: (1) method of moments applied to the original x data; (2) method of moments applied to the log transformed data y; and (3) the method of maximum likelihood. Bobee (1976) has compared these three methods on annual flood peaks and concluded that the first gives the best results.

Method of Moments

The scale parameter α may be obtained following Bobee (1976) and Kite (1977) by

$$\hat{\alpha} = 1/(A+3) \quad , \tag{2}$$

in which

$$A = -\ 0.23019 + 1.65262C + 0.20911C^2 - 0.04557C^3 \quad , \tag{3}$$

for $3.5 < B \leq 6$, and

$$A = -0.45157 + 1.99955\ C \quad , \tag{4}$$

for $3 < B \leq 3.5$. In these equations C is computed by

$$C = 1/(B-3) \quad , \tag{5}$$

and

$$B = \frac{\ell n(\hat{\mu}_3) - 3\ \ell n(\hat{\mu}_1)}{\ell n(\hat{\mu}_2) - 2\ \ell n(\hat{\mu}_1)} \quad , \tag{6}$$

where $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$ are the sample first three moments of x about the origin. Similarly, the shape parameter β and the location parameter y_o are estimated by

$$\hat{\beta} = \frac{\ell n(\hat{\mu}_2) - 2\ \ell n(\hat{\mu}_1)}{\ell n(1-\alpha)^2 - \ell n(1-2\alpha)} \quad , \tag{7}$$

and

$$\hat{y}_o = \ell n(\hat{\mu}_1) + \hat{\beta}\ell n(1-\hat{\alpha}) \quad . \tag{8}$$

The relationships between the sample statistics of $y = \ell n(x)$ such as the mean $\hat{\mu}_y$, standard deviation $\hat{\sigma}_y$ and skewness coefficient $\hat{\gamma}_y$, and the estimated parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{y}_o$ are

$$\hat{\mu}_y = \hat{\alpha}\ \hat{\beta} + \hat{y}_o \quad , \tag{9}$$

$$\hat{\sigma}_y = \hat{\alpha}\ \sqrt{\hat{\beta}} \quad , \tag{10}$$

and

$$\hat{\gamma}_y = 2/\sqrt{\hat{\beta}} \ . \tag{11}$$

The moment estimators $\hat{\alpha}$ and $\hat{\beta}$ of Eqs. (2), (7) were obtained directly in terms of the original x values. However, since $y = \ell n(x)$ is gamma distributed, the moment estimates of the parameters may be obtained indirectly by first estimating the mean $\hat{\mu}_y$, the standard deviation $\hat{\sigma}_y$ and the unbiased skewness coefficient $\hat{\gamma}_y$ and then finding the parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{y}_o$ from Eqs. (9), (10) and (11) as

$$\hat{\alpha} = 0.5\ \ \hat{\sigma}_y\ \hat{\gamma}_y \quad , \tag{12}$$

$$\hat{\beta} = (2/\hat{\gamma}_y)^2 \quad , \tag{13}$$

and

$$\hat{y}_o = \hat{\mu}_y - 2\,\hat{\sigma}_y/\hat{\gamma}_y \quad . \tag{14}$$

Subroutine PARLP1

This subroutine computes the moment estimators of the parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{y}_o$ of the Log-Pearson III distribution by either the direct approach from Eqs. (2), (7) and (8) or by the indirect approach from Eqs. (12), (13) and (14). In the first case the input index METH = 1 while in the second case METH = 2. The input variables to PARLP1 are the sample size N; the data set X(I), I = 1,...,N; the index METH; the index IEST = 1 for biased or IEST = 2 for unbiased estimates and the index IWR. The output variables are the parameters XSC($\hat{\alpha}$); XSH($\hat{\beta}$) and XLO($\hat{y}_o$). The program listing of PARLP1 is given below.

```
      SUBROUTINE PARLP1(N,X,XSC,XSH,XLO,METH,IEST,IWR)
C   MOMENT ESTIMATORS OF LOG-PEARSON TYPE-3 DISTRIBUTION
C   N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C   XLO = LOCATION PARAMETER (LOWER BOUND IN THE LOG DOMAIN)
C   XSC = SCALE     PARAMETER, XSH = SHAPE PARAMETER
C   METH= 1 DIRECT,  METH= 2 INDIRECT METHOD OF MOMENTS
C   IWR = 0 DO NOT WRITE, IWR = 1  WRITE THE RESULTS
C   IEST= 1 BIASED, IEST= 2 UNBIASED MOMENT ESTIMATORS
C*  SUBROUTINE REQUIREMENTS= ONLY  THE  SUBROUTINE  PARGA1
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N), EST(2)
      DATA EST/9H(BIASED   , 9H(UNBIASED/
      IF( METH.EQ.2 ) GO TO 60
C DIRECT METHOD OF MOMENTS
      XM=XS=XG=0.
      DO 10 I= 1,N
      XM= XM + X(I)/N
      XS= XS + X(I)**2/N
   10 XG= XG + X(I)**3/N
      B =(ALOG(XG)-3*ALOG(XM))/(ALOG(XS)-2*ALOG(XM))
      C = 1./(B-3.)
      IF( B.LE.3..OR.B.GT.6.) GO TO 40
      IF( B.LE.3.5) A= -0.47157+1.99955*C
      IF( B.GT.3.5 ) A=-0.23019+1.65262*C+.20911*C**2-.04557*C**3
      XSC= 1./(A+3.)
      XSH= ALOG(XS/(XM**2))/ALOG((1.-XSC)**2/(1.-2*XSC))
      XLO= ALOG(XM)+XSH*ALOG(1.-XSC)
      IF( IWR.EQ.0 ) RETURN
      PRINT 20
   20 FORMAT (1H1///5X,"PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION",/
     115X,"(DIRECT METHOD OF MOMENTS)"//)
      PRINT 30, XLO, XSC, XSH
      RETURN
```

```
   30 FORMAT (10X,"LOCATION      PARAMETER"/10X,"(LOWER BOUND OF LN(X))="
     1F12.6,/10X"SCALE"8X"PARAMETER=",F12.6,/10X"SHAPE"8X"PARAMETER=",
     2F12.6/)
   40 PRINT 50, B
   50 FORMAT (1H1///5X,"LOG-PEARSON TYPE-3 DISTRIBUTION"/8X,"(DIRECT MET
     1HOD OF MOMENTS)"/5X,"NO SOLUTION BECAUSE B=",F12.6/5X,"FALLS OUTSI
     2DE THE RANGE B(3.0,6.0)"//)
      RETURN
C INDIRECT  METHOD  OF  MOMENTS
   60 DO 70  I=1,N
   70 X(I)= ALOG(X(I))
      CALL PARGA1(N,X,XSC,XSH,XLO,3,IEST,0)
      IF( IWR.EQ.0 ) RETURN
      PRINT 80, EST(IEST)
   80 FORMAT (1H1///5X,"PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION",
     1/13X,"(INDIRECT METHOD OF MOMENTS)"/13X,A9," MOMENT ESTIMATORS)"//
     2)
      PRINT 30, XLO, XSC, XSH
      RETURN
      END
```

<u>Example 1</u>. Subroutine PARLP1 is applied to compute the moment estimators of the parameters of the Log-Pearson III distribution for the annual flood peaks of St. Mary's River at Stillwater. The direct and indirect approaches are used and for the second case the unbiased estimators are desired. The length of record is of 60 years. The input data to PARLP1 are N=60; the data set X(I), I=1,...,60; METH=1 (direct approach) or METH=2 (indirect approach), IEST=2 and IWR=1. The main program to call PARLP1 is shown below.

```
      PROGRAM PLOGP1 (INPUT, OUTPUT)
C     EXAMPLES FOR THE MOMENT ESTIMATORS OF LOG-PEARSON-3 DISTRIBUTION
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARLP1(N,X,XSC,XSH,XLO,1,2,1)
      CALL PARLP1(N,X,XSC,XSH,XLO,2,2,1)
      END
```

The computer output for this example is:

```
PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION
          (DIRECT METHOD OF MOMENTS)

     LOCATION        PARAMETER
     (LOWER BOUND    OF LN(X))=      5.109656
     SCALE           PARAMETER=       .025646
     SHAPE           PARAMETER=    172.282998


PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION
          (INDIRECT METHOD OF MOMENTS)
          (UNBIASED MOMENT ESTIMATORS)

     LOCATION        PARAMETER
     (LOWER BOUND    OF LN(X))=      6.515795
     SCALE           PARAMETER=       .037788
     SHAPE           PARAMETER=     79.726449
```

Method of Maximum Likelihood

The log likelihood function for the PDF in Eq. (1) is

$$LL(x,\alpha,\beta,y_o) = -\sum_{i=1}^{N} \ell n\ (x_i) - N\ \ell n[(\beta)] - \frac{1}{a} \sum_{i=1}^{N} [\ell n\ (x_i) - y_o]$$

$$+ (\beta-1) \sum_{i=1}^{N} \ell n[\ell n\ (x_i) - y_o] - N\ \beta\ \ell n(\alpha) \qquad (15)$$

Differentiating with respect to the parameters and equating this to zero yields three equations which need to be solved simultaneously to determine the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{y}_o$. These equations are

$$\sum_{i=1}^{N} (y_i - \hat{y}_o) = N\ \hat{\alpha}\ \hat{\beta} \qquad (16)$$

$$N\,\psi(\hat{\beta}) = \sum_{i=1}^{N} \ell n\left(\frac{y_i - \hat{y}_o}{\alpha}\right) \tag{17}$$

$$N = \hat{\alpha}(\hat{\beta}-1) \sum_{i=1}^{N} \frac{1}{(y_i - y_o)} \tag{18}$$

where $y_i = \ell n(x_i)$ and $\psi(\beta)$ is the digamma function which can be calculated by the asymptotic expansion as mentioned before.

Equations (16), (17) and (18) give

$$\hat{\alpha} = \frac{1}{N} \sum_{i=1}^{N} (y_i - \hat{y}_o) - N\left[\sum_{i=1}^{N} 1/(y_i - \hat{y}_o)\right]^{-1} \quad , \tag{19}$$

$$\hat{\beta} = \left\{1 - N^2\left[\sum_{i=1}^{N} (y_i - \hat{y}_o) \sum_{i=1}^{N} 1/(y_i - \hat{y}_o)\right]^{-1}\right\}^{-1} \quad , \tag{20}$$

and

$$F(\hat{y}_o) = -N\psi(\hat{\beta}) + \sum_{i=1}^{N} \ell n(y_i - \hat{y}_o) - N\,\ell n(\hat{\alpha}) = 0 \quad , \tag{21}$$

which are solved by a numerical iterative procedure. It is noted that Eqs. (19), (20) and (21) are the same as Eqs. (29), (30) and (31) corresponding to the gamma distribution. Therefore, the solution in this case parallels that of the gamma PDF described before.

Subroutine PARLP2

This subroutine determines the maximum likelihood estimates of the parameters of the Log-Pearson III distribution. The numerical procedure is the same as that applied in subroutine PARGA2 for the gamma-3 distribution. In PARLP2 the moment estimates of the parameters are used as initial estimates to start the iterative procedure for the maximum likelihood estimates. If B of Eq. (6) is outside the range $3 < B \leq 6$, the moment estimates are not computed using Eqs. (2), (7) and (8) but by setting an initial value for the location parameter $\hat{y}_o$(XLO) less than the minimum observed x value in the log domain XMIN. If B of Eq. (6) is within the range $3 < B \leq 6$, the moment initial estimate of $\hat{y}_o$ may be greater than XMIN. Also, within the iteration, $\hat{y}_o$ may assume a value x greater than $\hat{y}_o$ > XMIN. In these cases, a second approximation is set for $\hat{y}_o$ equal to the arithmetic mean $\hat{y}_o = 0.5\ (\hat{y}_o + \text{XMIN})$. The maximum number of iterations denoted by LIM has been set to 20, and the relative maximum error denoted by ERROR has been set to 10^{-8} in the DATA statement. These two values may be changed by the user if desired.

The input variables to PARLP2 are the sample size N; the data set X(I), I=1,...,N; and the index IWR. The output variables are the scale parameter XSC, the shape parameter XSH and the location parameter XLO. The program listing of PARLP2 is given below.

```
      SUBROUTINE PARLP2(N,X,XSC,XSH,XLO,IWR)
C MAX.LIKELIHOOD ESTIMATORS OF LOG-PEARSON TYPE-3 DISTRIBUTION
C   N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C   XLO = LOCATION PARAMETER (LOWER BOUND IN THE LOG DOMAIN)
C   XSC = SCALE     PARAMETER, XSH = SHAPE PARAMETER
C   IWR = 0 DO NOT WRITE, IWR = 1  WRITE THE RESULTS
C   LIM = MAXIMUM NUMBER OF ITERATIONS PERMITTED IN THE PROCEDURE
C   ERROR= MAXIMUM ABSOLUTE RELATIVE ERROR REQUIREMENT ON XLO.
C* SUBROUTINE REQUIREMENTS= ONLY THE SUBROUTINE FXGAM.
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N)
      DATA LIM,ERROR/20,0.00000001/
      VN= N
C DIRECT MOMENT ESTIMATES OF THE PARAMETERS(AS A FIRST APPROXIMATION)
      XM=XS=XG=0.
      DO 10  I=1,N
      XM= XM + X(I)/N
      XS= XS + X(I)**2/N
   10 XG=XG+X(I)**3/N
      B =(ALOG(XG)-3*ALOG(XM))/(ALOG(XS)-2*ALOG(XM))
      C = 1./(B-3.)
      IF( B.LE.3.OR.B.GT.6. ) GO TO 130
      IF( B.LE.3.5 ) A= -0.47157+1.99955*C
      IF( B.GT.3.5 ) A= -.23019+1.65262*C+.20911*C**2-.04557*C**3
      SCA= 1./(A+3.)
      SHA= ALOG(XS/(XM**2))/ALOG((1.-SCA)**2/(1.-2*SCA))
      XOM= ALOG(XM)+SHA*ALOG(1.-SCA)
      XM= 0.
      XMIN=ALOG(X(1))
      DO 20  I=1,N
      X(I)= ALOG(X(I))
```

```
      XM= XM+X(I)/N
   20 XMIN = AMIN1(XMIN,X(I))
      IF (IWR.GT.0) PRINT 30
   30 FORMAT (1H1///5X,"PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION",
     1/14X"(MAX.LIKELIHOOD ESTIMATORS)"//5X"TRIAL  LOWER BOUND   VALUE OF
     2F LIKE-  IMPROVEMENT"/15X"XLO"8X"LIHOOD EQUATION   ON XLO"/)
      IF (XOM.GE.XMIN) GO TO 130
      KTR=0
   40 KTR = KTR+1
      CALL FXGAM(N,X,XOM,FM,DFM,XSC,XSH,XM)
      DXO= FM/DFM
      IF (IWR.GT.0) PRINT 70, KTR, XOM, FM, DXO
      XO= XOM-DXO
      IF (XO-XMIN) 60,50,50
   50 XOM = (XOM+XMIN)/2.
      GO TO 40
   60 KTR = KTR+1
      CALL FXGAM(N,X,XO,FX,DFX,XSC,XSH,XM)
      DXO= FX/DFX
      IF (IWR.GT.0) PRINT 70, KTR, XO, FX, DXO
   70 FORMAT (7X,I2,F14.6,2X,E15.8,F12.6)
      XO= XO-DXO
      IF (XO.GE.XMIN) GO TO 90
      IF (KTR.GE.LIM) GO TO 110
      IF (ABS(DXO/XO).GT.ERROR) GO TO 60
      XLO=XO
      IF (IWR.GT.0) PRINT 80, XLO, XSC, XSH
   80 FORMAT (10X,"LOCATION     PARAMETER"/10X,"(LOWER BOUND OF LN(X))="
     1F12.6,/10X,"SCALE"8X"PARAMETER=",F12.6/10X"SHAPE"8X"PARAMETER=",
     2F12.6/)
      RETURN
   90 PRINT 100, XO, XMIN
  100 FORMAT (/5X,"LOWER BOUND ="F12.6," .GT.OBSERVED MIN=",F12.6)
      RETURN
  110 PRINT 120, KTR, XO, FX
  120 FORMAT (/5X,"NO CONVERGENCE AFTER",I3,"ITERATIONS"/5X,"LAST VALUES
     1 OF THE ITERATION ARE "/5X,"LOWER BOUNDARY =",F12.6/5X"LIKELIHOOD
     2VALUE=",F12.5/)
      RETURN
  130 KTR = 0
      XO = XMIN - 0.01*ABS(XMIN)
      GO TO 60
      END
```

Example 2. Subroutine PARLP2 is applied to compute the maximum likelihood estimators of the parameters of the Log-Pearson III distribution for the flood data of Example 1. The input data to PARLP2 are the sample size N=60, the data set X(I), I=1,...,60, and the index IWR=1. The main program to call PARLP2 is shown below.

```
      PROGRAM PLOGP2 (INPUT, OUTPUT)
C     EXAMPLE FOR THE MAX.LIKELIHOOD ESTIMATORS OF LOG-PEARSON-3 DISTRIBUTION
      DIMENSION X(60)
      READ 10, N
      READ 20, (X(I), I=1,N)
   10 FORMAT (16I5)
   20 FORMAT (12F6.0)
      CALL PARLP2(N,X,XSC,XSH,XLO,1)
      END
```

The computer output for this example is:

```
PARAMETERS OF LOG-PEARSON TYPE-3 DISTRIBUTION
         (MAX.LIKELIHOOD ESTIMATORS)

TRIAL   LOWER BOUND    VALUE OF LIKE-   IMPROVEMENT
            XLO        LIHOOD EQUATION    ON XLO

  1       5.109656    -.19908979E-03   -94.778258
  2       6.959759     .12446181E-02      .343356
  3       6.616403     .40881530E-03      .263404
  4       6.352999     .10583123E-03      .126219
  5       6.224781     .14740685E-04      .024311
  6       6.200470     .42558804E-06      .000744
  7       6.199726     .38016879E-09      .000001
  8       6.199725     .45474735E-11      .000000
     LOCATION        PARAMETER
     (LOWER BOUND OF LN(X))=     6.199725
     SCALE           PARAMETER=     .033645
     SHAPE           PARAMETER=   98.938852
```

In this output the moment initial estimate of the lower bound is XLO = 5.109656 (first trial). However, the derivative of Eq. (21) at this point would give an unappropriate second approximation of XLO = =5.109656 - (-94.778947) = 99.89, which is greater than the observed minimum value in the log domain XMIN. Therefore, a new value of XLO is set equal to the arithmetic mean of 5.109656 and $\ell n(\text{XMIN})$ for the second approximation.

4.3 Probability Density and Cumulative Distribution Function

If x is Log-Pearson III then $y = \ell n(x)$ is gamma distributed. Hence, the PDF of x is related to the PDF of y by

$$f(x) = \frac{1}{x} f(y) \quad . \tag{22}$$

Equation (22) is used to find the Log-Pearson III density $f(x)$ based on the gamma density of $y = \ell n(x)$. As for the CDF of x, the following relation holds

$$F(x) = \int_0^x f(x)dx = F(y) = \int_0^{\ell n(x)} f(y)dy \quad , \tag{23}$$

where $F(x)$ and $F(y)$ are the CDF of x and y, respectively. That is, $F(x)$ may be obtained by integrating the gamma density in the log domain.

Subroutine DISLP3

This subroutine determines the PDF and CDF of the Log-Pearson III distribution of parameters $\hat{y}_o$(XLO), $\hat{\alpha}$(XSC) and $\hat{\beta}$(XSH) at specified points x represented by X(I). Both the PDF and CDF of x are obtained by using the corresponding functions for $y = \ell n(x)$ as in Eqs. (22) and (23).

The input variables to DISLP3 are the number of points N at which the PDF and CDF are to be computed; the parameters XLO, XSC and XSH; the index IX, the index IWR and either the data set X(I), I=1,...,N if IX=1 or the maximum value of x, represented by XMAX, if IX=0. In this last case, the PDF and CDF are determined at N equally spaced points (in the log domain) between XMIN and XMAX, where XMIN = EXP (XLO). The output variables from DISLP3 are the PDF denoted by DEN(I), the cumulative function CDF(I), I = 1,...,N, and the X(I)'s if IX=0. The program listing of DISLP3 is given below.

```
      SUBROUTINE DISLP3(N,X,XLO,XSC,XSH,XMAX,DEN,CDF,IX,IWR)
C..DENSITY AND CUMULATIVE OF LOG-PEARSON TYPE 3 DISTRIBUTION
C**X(I) AND XMAX VALUES SHOULD BE GIVEN IN THE REAL DOMAIN
C**SUBROUTINE REQUIREMENTS= SUBROUTINE DISGAM AND RELATED SUBPROGRAMS
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N), DEN(N), CDF(N)
      IF( IX.LE.0 ) GO TO 20
      DO 10 I= 1,N
   10 X(I) = ALOG(X(I))
      XMAX = X(N)
      CALL DISGAM(N,X,XLO,XSC,XSH,XMAX,DEN,CDF,1,0)
      GO TO 30
   20 ZMAX= ALOG(XMAX)
      CALL DISGAM(N,X,XLO,XSC,XSH,ZMAX,DEN,CDF,0,0)
   30 DO 40  I=1,N
      X(I) = EXP(X(I))
   40 DEN(I) = DEN(I)/X(I)
      IF (IWR.LE.0) RETURN
      PRINT 50, XLO, XSC, XSH
   50 FORMAT (1H1///5X,"DENSITY AND CUMULATIVE DISTRIBUTION OF LOG-PEARS
     1ON TYPE 3"//8X,"( XLO="F10.5"  XSC="F10.3"  XSH="F8.4" )"/9X,"ORDE
     2R      VARIABLE       DENSITY       CUM. DISTR."/20X,"VALUE"/X"FUNCTIO
     3N       FUNCTION"/)
      PRINT 70, (I,X(I),DEN(I),CDF(I), I=1,N)
   70 FORMAT (10X,I3,F14.6,2F13.6)
      RETURN
      END
```

Example 3. Subroutine DISLP3 is used to determine the PDF and CDF of the Log-Pearson III distribution with parameters $\hat{y}_o = 4.0723$, $\hat{\alpha} = 0.0863$ and $\hat{\beta} = 10.75$ (Natural Environment Research Council, 1975) at 10 points. The inputs to DISLP3 are N=10, XLO=4.0723, XSC=0.0863, XSH=10.75, IX=0 and IWR=1. The main program to call DISLP3 in this example is:

```
      PROGRAM LPDIST (INPUT,OUTPUT)
C     EXAMPLE FOR THE DENSITY AND CUMULATIVE OF LOG-PEARSON-3
      COMMON/BLK1/XLO,XSC,XSH
      DIMENSION X(60), DEN(60), CDF(60)
      N=10 $ XLO=4.0723 $ XSC=0.0863 $ XSH=10.75 $ XMAX=235.7
      CALL DISLP3(N,X,XLO,XSC,XSH,XMAX,DEN,CDF,0,1)
      END
```

The computer output for this example is:

```
DENSITY AND CUMULATIVE DISTRIBUTION OF LOG-PEARSON TYPE 3

  ( XLO=      4.07230    XSC=        .086    XSH=  10.7500  )
  ORDER       VARIABLE         DENSITY         CUM. DISTR.
                VALUE          FUNCTION          FUNCTION

     1        58.691799        0.000000          0.000000
     2        68.495860         .000004           .000005
     3        79.937624         .000502           .001638
     4        93.290654         .003745           .026593
     5       108.874218         .008854           .126509
     6       127.060908         .011158           .315707
     7       148.285560         .009444           .539336
     8       173.055644         .006073           .731055
     9       201.963401         .003194           .861393
    10       235.700000         .001441           .935780
```

4.4 The Inverse of the Cumulative Distribution Function

Since the Log-Pearson III distribution is a gamma-3 in the log domain, its inverse x for given probabilities can be determined using the same procedure as for the gamma-3. That is, for a given set of parameters $\hat{y}_o$, $\hat{\alpha}$ and $\hat{\beta}$, of the Log-Pearson III and for a given CDF, say p, the inverse y of the gamma-3 is obtained from which $x = \exp\{y\}$ is the inverse corresponding to the Log-Pearson III.

<u>Subroutine INVLP3</u>

This subroutine determines the inverse of the CDF of the Log-Pearson III distribution by using the procedure outlined above and the subroutine INVGAM. The reader should be referred to the description of INVGAM. Inputs to INVLP3 are the parameters XLO, XSC and XSH; the number of inverses N to be computed; the index ICDF; and the index IWR. If ICDF=1, the set CDF(I), I=1,...,N is given as input but if ICDF=0, the set CDF(I) is not given as input, but it is computed as CDF(I)=I/(N+1) in the program. The program listing of INVLP3 is given below.

```
      SUBROUTINE INVLP3 (XLO,XSC,XSH,N,CDF,X,ICDF,IWR)
C..INVERSE OF THE LOG-PEARSON TYPE 3 DISTRIBUTION
C..XLO = LOCATION PARAMETER (LOWER BOUND IN THE LOG DOMAIN)
C..XSC = SCALE PARAMETER,  XSH= SHAPE PARAMETER( XSH SHOULD BE .LE.57)
C..CDF = CUMULATIVE PROBABILITY CORRESPONDING TO X(I)
C  ICDF= 0 CDF(I), I=1,2,..,N SET IS NOT GIVEN BY THE USER.
C  ICDF= 1 CDF(I), I=1,2,..,N SET SHOULD BE GIVEN BY THE USER.
C  N = NUMBER OF CLASS LIMITS OF X DESIRED TO BE COMPUTED.
C  X(I) IS THE INVERSE CORRESPONDING TO CDF(I)
C..IWR = 0  DO NOT WRITE,  IWR = 1 WRITE THE RESULTS.
C**XLO,XSC,XSH,N,ICDF AND IWR VALUES SHOULD BE SUPPLIED BY THE USER.
C**SUBPROGRAM REQUIREMENTS ARE INVGAM AND RELATED SUBPROGRAMS
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N), CDF(N)
      CALL INVGAM (XLO,XSC,XSH,N,CDF,X,ICDF,0)
      DO 10 I=1,N
   10 X(I) = EXP(X(I))
      IF( IWR.LE.0 ) RETURN
      PRINT 20, XLO, XSC, XSH
   20 FORMAT (1H1///7X,"CLASS LIMITS OF THE LOG-PEARSON-3 DISTRIBUTION F
     1UNCTION"/10X"( XLO="F10.5"    XSC="F10.4"    XSH="F8.5" )"//15X"ORDE
     2R"7X"CUMULATIVE"/26X"DISTRIBUTION"/16X"(I)"10X"CDF(X)"12X"X(I)"/)
      PRINT 30, (I, CDF(I), X(I), I=1,N)
   30 FORMAT (15X,I3,2F18.6)
      RETURN
      END
```

<u>Example 4</u>. Subroutine INVLP3 is used to determine the inverse of F(x) for x Log-Pearson III distributed with parameters $\hat{y}_o = 4.0723$, $\hat{\alpha} = 0.0863$ and $\hat{\beta} = 10.75$. Nine inverse values are desired for the CDF's of Example 3. The inputs to INVLP3 are N=9, XLO=4.0723, XSC= =0.0863, XSH=10.75, ICDF=1, IWR=1 and the set CDF(I), I=1,...,9 shown in Example 3. The main program to call INVLP3 is:

```
      PROGRAM LP3CLA (INPUT,OUTPUT)
C     INVERSE OF THE LOG-PEARSON TYPE 3 DISTRIBUTION
      COMMON /BLK1/ XLO,XSC,XSH
      DIMENSION CLAS(100), CDF(100)
      N=9  $ XLO=4.0723 $ XSC=0.0863 $ XSH=10.75 $ ICDF=1 $ IWR=1
      READ 10, (CDF(I), I=1,N)
   10 FORMAT (10F8.6)
      CALL INVLP3 (XLO,XSC,XSH,N,CDF,CLAS,ICDF,IWR)
      END
```

The computer output of INVLP3 for this example is:

```
CLASS LIMITS OF THE LOG-PEARSON-3 DISTRIBUTION FUNCTION
   ( XLO=    4.07230    XSC=      .0863    XSH=10.75000 )

          ORDER          CUMULATIVE
                        DISTRIBUTION
           (I)             CDF(X)               X(I)

            1              .000006            68.875519
            2              .001638            79.938017
            3              .026593            93.290877
            4              .126509           108.874255
            5              .315707           127.060946
            6              .539336           148.285599
            7              .731055           173.055682
            8              .861393           201.963337
            9              .935780           235.699798
```

4.5 Confidence Limits

Considering that the Log-Pearson III distribution is a gamma-3 in the log domain, the T-year event magnitudes and 100α percent confidence limits of the Log-Pearson III are determined by first computing the corresponding magnitudes in the y (log) domain and transferring them back to the x domain by

$$x_T = \exp\{y_T\} \tag{24}$$

$$X_\ell = \exp\{y_\ell\} \tag{25}$$

where x_T and y_T represent the T-year event magnitudes and x_ℓ and y_ℓ represent the 100α percent confidence limits for the Log-Pearson

III and gamma-3, respectively. Both moment and maximum likelihood estimators are determined by the above transformations.

Subroutine CLILP3

This subroutine determines the approximate moment or maximum likelihood estimates of the T-year event magnitudes and the 100α percent confidence limits for the Log-Pearson III. The event magnitude X(I) and its upper and lower confidence limits XUT(I) and XLT(I), respectively, are computed for specified return periods T(I) or non-exceedence probabilities PXT(I), I=1,...,NT, with NT the number of desired T or PXT values, and for a given confidence level CONF. Subroutine CLILP3 needs subroutines CLIGA1, CLIGA2 and RETCDF, described previously.

Inputs to CLILP3 are the sample size N; the number NT of return periods or non-exceedence probabilities both represented by the input variable T(I), I=1,...,NT; the confidence level CONF; the parameters XLO, XSC and XSH; the index IEST=1 (moment estimates) or IEST=2 (maximum likelihood estimates); and the index IWR. Outputs from CLILP3 are the T-year event magnitude XT(I) and its XUT(I) and XLT(I) corresponding to the input T(I), I=1,...,NT. The program listing of CLILP3 is given below. Other needed subroutines CLIGA1, CLIGA2 and RETCDF were given previously.

```
      SUBROUTINE CLILP3(N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IEST,IWR)
C** CONFIDENCE LIMITS FOR LOG-PEARSON TYPE 3 DISTRIBUTION
C   IEST = 1 ESTIMATION BY MOMENTS,   IEST = 2 ESTIMATION BY MAX.LIKELIHOOD
C   XLO  = LOCATION  PARAMETER ( LOWER  BOUND )
C   XSC  = SCALE  PARAMETER ,  XSH = SHAPE  PARAMETER
C**SUBPROGRAMS NEEDED ARE CLIGA1 AND CLIGA2
      DIMENSION T(NT), XT(NT),XUT(NT), XLT(NT), PXT(NT)
      IF( IEST. GT .1 ) GO TO 20
```

```
      CALL CLIGA1(N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,0)
      IF (IWR.GT.0) PRINT 10, CONF, XLO, XSC, XSH
   10 FORMAT(1H1///5X"MOMENT ESTIMATES OF THE "F4.2"  CONFIDENCE LIMITS
     1 FOR LOG-PEARSON-3  DISTRIBUTION"/12X"( XLO="F12.5"   XSC="F12.7"
     2  XSH="F12.7" )"//12X"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDENCE
     3 LIMITS"/12X"PERIOD  PROBABILITY    MAGNITUDE"6X"UPPER"8X"LOWER"/)
      GO TO 40
   20 CALL CLIGA2(N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,0)
      IF (IWR.GT.0) PRINT 30, CONF, XLO, XSC, XSH
   30 FORMAT (1H1///5X"MAX.LIKELIHOOD ESTIMATES OF "F4.2" CONFIDENCE LIM
     1ITS FOR LOG-PEARSON-3 DISTRIBUTION"/13X"( XLO="F12.5"    XSC="F12.7
     2"    XSH="F12.7" )"//13X"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDENC
     3E LIMITS"/13X"PERIOD  PROBABILITY    MAGNITUDE"6X"UPPER"8X"LOWER"/)
   40 DO 50  I=1,NT
      XT(I) =  EXP( XT(I) )
      XUT(I)=  EXP( XUT(I)  )
   50 XLT(I)=  EXP( XLT(I)  )
      IF (IWR.EQ.0) RETURN
      PRINT 60, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   60 FORMAT (13X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

Example 5. Subroutine CLILP3 is applied to compute the moment and maximum likelihood estimates of design floods and 95 percent confidence limits for 2, 5, 10, 20, 50 and 100 years of return periods. The floods are assumed to be Log-Pearson III distributed with moment parameter estimates of $\hat{y}_o = 6.51579$, $\hat{\alpha} = 0.037788$ and $\hat{\beta} = 79.726449$ and maximum likelihood parameter estimates of $\hat{y}_o = 6.19973$, $\hat{\alpha} = 0.033645$ and $\hat{\beta} = 98.938852$ obtained from subroutines PARLP1 and PARLP2, respectively, for 60 years of annual floods at St. Mary's River at Stillwater. The input variables to CLILP3 are N=60; NT=6; CONF=0.95; XLO=6.51579, XSC=0.037788 and XSH=79.726449 for moment estimators or XLO=6.19973, XSC=0.033645 and XSH=98.938852 for maximum likelihood estimators; T(I)=2, 5, 10, 20, 50 and 100 for I=1,...,6; IEST=1 (moment estimators) and IEST=2 (maximum likelihood estimators); and IWR=1. The main program to input this information is given below.

```
      PROGRAM PEACON (INPUT,OUTPUT)
C     MOMENT AND MAX. LIKELIHOOD ESTIMATES OF CONF. LIMITS FOR LOG-PEARSON-3
      DIMENSION T(10), XT(10),XUT(10),XLT(10),PXT(10)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5,7F8.2)
      XLO=6.515795   $   XSC=.037788   $ XSH=79.726449
      CALL CLILP3 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1,1)
      XLO=6.199725   $   XSC=.033645   $ XSH=98.938852
      CALL CLILP3 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,2,1)
      END
```

The computer output for this example is:

```
OMENT ESTIMATES OF THE  .95  CONFIDENCE LIMITS  FOR LOG-PEARSON-3  DISTRIBUTION
      ( XLO=      6.51580    XSC=      .0377880    XSH=  79.7264490 )

      RETURN  NONEXCEEDANCE     EVENT           CONFIDENCE  LIMITS
      PERIOD   PROBABILITY    MAGNITUDE          UPPER        LOWER

        2.        .50000      13574.10981   14890.23527   12374.31470
        5.        .80000      18179.36435   20206.65819   16355.46488
       10.        .90000      21336.50179   24281.15228   18748.95819
       20.        .95000      24446.07456   28768.31741   20773.21912
       50.        .98000      28610.35872   35535.66117   23034.68120
      100.        .99000      31852.83677   41397.74602   24508.65827


AX.LIKELIHOOD ESTIMATES OF   .95 CONFIDENCE LIMITS FOR LOG-PEARSON-3 DISTRIBUTION
        ( XLO=      6.19973    XSC=     .0336450    XSH=  98.9388520 )

        RETURN  NONEXCEEDANCE     EVENT           CONFIDENCE LIMITS
        PERIOD   PROBABILITY    MAGNITUDE          UPPER          LOWER

          2.        .50000      13593.13591   14890.11071   12409.13164
          5.        .80000      18147.65501   20141.28263   16351.36096
         10.        .90000      21246.94294   24120.32443   18715.85871
         20.        .95000      24283.45742   28452.37248   20725.38256
         50.        .98000      28327.50932   34909.98796   22986.19482
        100.        .99000      31460.30006   40446.75622   24470.45381
```

5. GUMBEL AND LOG-GUMBEL DISTRIBUTIONS

5.1 General Properties

The cumulative distribution function of the Gumbel (double exponential or Type I general extreme value) distribution has the form (Gumbel, 1958)

$$F(x) = \exp\{-\exp[-(x-x_o)/\alpha]\} \quad , \quad -\infty < x < \infty \quad , \tag{1}$$

where $\alpha > 0$ is the scale parameter and x_o is the location parameter. The parameter x_o is also called the central value or mode for which $F(x_o) = 0.368$.

Differentiating Eq. (1) with respect to x yields the PDF

$$f(x) = \exp\{-(x-x_o)/\alpha - \exp[-(x-x_o)/\alpha]\} \quad , \quad -\infty < x < \infty \quad . \tag{2}$$

The reduced Gumbel variate commonly used in practice is defined as:

$$z = (x-x_o)/\alpha \quad . \tag{3}$$

Substituting Eq. (3) into Eq. (1) and taking twice the logarithms of both sides gives

$$-\ln[-\ln F(x)] = z = (x-x_o)/\alpha \quad . \tag{4}$$

Hence the Gumbel CDF plots as a straight line on a paper with double logarithmic scale for $F(x)$ and the cartesian scale for x. This probability paper is called the "Gumbel probability paper".

The mean, standard deviation and skewness of the Gumbel distribution are (Yevjevich, 1972)

$$\mu = x_o + 0.5772\,\alpha \quad , \tag{5}$$

$$\sigma = \pi\,\alpha/\sqrt{6} \quad , \tag{6}$$

and

$$\gamma = 1.14 \quad . \tag{7}$$

When $y = \ell n(x)$ is a Gumbel distributed variable, then x has a log-Gumbel (Frechet or Type II general extreme value) distribution (NERC, 1975).

5.2 Estimation of Parameters

Method of Moments

Moment estimates of the parameters α and x_o may be obtained from Eqs. (5) and (6) as (Yevjevich, 1972)

$$\hat{\alpha} = 0.7806\ \hat{\sigma} \quad , \tag{8}$$

and

$$\hat{x}_o = \hat{\mu} - 0.45\ \hat{\sigma} \tag{9}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the sample estimates of the mean and standard deviation, respectively. In the case of the log-Gumbel distribution $\hat{\mu}$ and $\hat{\sigma}$ should be computed for the transformed variable $y = \ell n(x)$ before using Eqs. (8) and (9) to estimate $\hat{\alpha}$ and $\hat{x}_o$. As in previous cases the sample standard deviation $\hat{\sigma}$ may be determined by Eq. ( ) or ( ).

Subroutine PARGU1

This subroutine computes the moment estimators of the parameters α and x_o of the Gumbel or log-Gumbel distributions by Eqs. (8) and (9) above. The variable XSC and XLO represent the parameters α and x_o, respectively. The index IDIS = 1 for the Gumbel and IDIS = 2 for the log-Gumbel distribution. The index IEST = 1 when $\hat{\sigma}$ of Eq. (8) is determined by Eq. ( ) and IEST = 2 when $\hat{\sigma}$ is determined by Eq. ( ). The input variables to PARGU1 are the sample size N; the data set X(I), I = 1,...,N; and the indeces IEST, IDIS and IWR. The output variables are the scale parameter XSC and the location parameter XLO. The program listing of PARGU1 is shown below.

```
      SUBROUTINE PARGU1 (N,X,XSC,XLO,IEST,IDIS,IWR)
C..........................................................................
C.....PARAMETERS OF THE GUMBEL AND LOG-GUMBEL DISTRIB.(MOMENT ESTIMATORS)
C   N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C   XLO = LOCATION PARAMETER (MODE),   XSC = SCALE PARAMETER
C   IDIS= 1 GUMBEL,    IDIS= 2  LOG-GUMBEL  DISTRIBUTION
C   IEST= 1 BIASED, IEST= 2 UNBIASED MOMENT ESTIMATORS
C   IWR = 0 DO NOT WRITE, IWR = 1  WRITE THE RESULTS
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C..........................................................................
      DIMENSION X(N),EST(2),TRANS(2)
      DATA EST/9H( BIASED  ,9H(UNBIASED/
      DATA TRANS/10H  GUMBEL    ,10HLOG-GUMBEL/
      VN= N
      IF (IDIS.EQ.1) GO TO 20
      DO 10 I=1,N
   10 X(I)= ALOG(X(I))
   20 XM=XS=0.
      DO 30 I=1,N
      XM=XM+X(I)/N
   30 XS=XS+X(I)**2/N
      XS= XS - XM**2
      IF( IEST.EQ.2 )   XS= XS*VN/(VN-1.)
      SX= SQRT(XS)
      XSC= 0.7806 * SX
      XLO= XM- 0.45*SX
      IF( IWR.EQ.0 ) RETURN
      PRINT 40, TRANS(IDIS), EST(IEST), XLO, XSC
   40 FORMAT (1H1///5X"PARAMETERS OF THE EXTREME VALUE DISTRIBUTION"/5X"
     1(TYPE-1 EXTREMAL(",A10," ) DISTRIBUTION)"/13X,A9" MOMENT ESTIMATOR
     2S)"//7X"LOCATION PARAMETER=",F12.5,3X"(MODE)"/7X"SCALE    PARAMETE
     3R="F12.5/)
      RETURN
      END
```

Example 1. Subroutine PARGU1 is used to compute the moment estimators of the parameters of the Gumbel and log-Gumbel distributions for the 60 years of annual flood data of St. Mary's River at Stillwater. Equation ( ) is used for the sample standard derivation $\hat{\sigma}$. The input data to PARGU1 are N = 60; the data set X(I), I = 1,...,60; IEST = 2; IDIS = 1 (Gumbel), and IDIS = 2 (log-Gumbel); and IWR = 1. The main program to call PARGU1 is shown below.

```
      PROGRAM PEV1 (INPUT,OUTPUT)
C     EXAMPLE FOR THE PARAMETERS OF THE GUMBEL AND LOG-GUMBEL
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARGU1 (N,X,XSC,XLO,2,1,1)
      CALL PARGU1 (N,X,XSC2,XLO2,2,2,1)
      END
```

The computer output for this example is:

```
PARAMETERS OF THE EXTREME VALUE DISTRIBUTION
(TYPE-1 EXTREMAL(   GUMBEL    ) DISTRIBUTION)
       (UNBIASED MOMENT ESTIMATORS)

  LOCATION PARAMETER= 12202.56622    (MODE)
  SCALE    PARAMETER=  4080.11024


PARAMETERS OF THE EXTREME VALUE DISTRIBUTION
(TYPE-1 EXTREMAL(LOG-GUMBEL  ) DISTRIBUTION)
       (UNBIASED MOMENT ESTIMATORS)

  LOCATION PARAMETER=      9.37664    (MODE)
  SCALE    PARAMETER=       .26338
```

Method of Maximum Likelihood

The log likelihood function of the Gumbel distribution is (NERC, 1975)

$$LL(\underline{X}\ |\alpha, x_o) = -N\ell n\alpha - \frac{1}{\alpha}\sum_{i=1}^{N}(x_i - x_o) - \sum_{i=1}^{N} e^{-(x_i - x_o)/\alpha} \quad , \tag{10}$$

or

$$LL(\underline{Y}\ |\alpha, x_o) = -N\ell n\alpha - \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} e^{-y_i} \quad , \tag{11}$$

where

$$y_i = (x_i - x_o)/\alpha \quad . \tag{12}$$

The partial derivatives of Eq. (11) with respect to x_o and α are:

$$-\partial LL/\partial x_o = -P/\alpha \quad , \tag{13}$$

and

$$-\partial LL/\partial \alpha = R/\alpha \quad , \tag{14}$$

where P and R are:

$$P = N - \sum_{i=1}^{N} e^{-y_i} \tag{15}$$

$$R = N - \sum_{i=1}^{N} y_i + \sum_{i=1}^{N} y_i e^{-y_i} \quad , \tag{16}$$

respectively. Maximum likelihood estimates $\hat{x}_o$ and $\hat{\alpha}$ must satisfy

$$\frac{\partial LL(\underline{X} \,|\hat{x}_o,\hat{\alpha})}{\partial x_o} = 0 \quad , \quad \frac{\partial LL(\underline{X} \,|\hat{x}_o,\hat{\alpha})}{\partial \alpha} = 0 \tag{17}$$

Jenkinson (1969) gives an iterative procedure for the solution of Eq. (17). The Taylor series expansion of $-\partial LL/\partial x_o$ and $-\partial LL/\partial\alpha$ about the true maximum likelihood estimates $\hat{\alpha}$ and $\hat{x}_o$ omitting the cross products and second order terms yields (NERC, 1975)

$$\begin{bmatrix} -\Delta x_o^{(k)} \\ \\ -\Delta\alpha^{(k)} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial^2 LL(\underline{X}|\hat{x}_o,\hat{\alpha})}{\partial x_o^{\,2}} , & -\dfrac{\partial^2 LL(\underline{X}|\hat{x}_o,\hat{\alpha})}{\partial x_o \partial\alpha} \\ -\dfrac{\partial^2 LL(\underline{X}|\hat{x}_o,\hat{\alpha})}{\partial x_o \partial\alpha} , & -\dfrac{\partial^2 LL(\underline{X}|\hat{x}_o,\hat{\alpha})}{\partial\alpha^2} \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial LL(\underline{X}|x_o^{(k)},\alpha^{(k)})}{\partial x_o} \\ -\dfrac{\partial LL(\underline{X}|x_o^{(k)},\alpha^{(k)})}{\partial\alpha} \end{bmatrix} \tag{18}$$

where $\Delta x_o^{(k)}$ and $\Delta\alpha^{(k)}$ are the differences at the k-th iteration between the true estimates $\hat{x}_o$ and $\hat{\alpha}$ and the actual or k-th estimates $x_o^{(k)}$ and $\alpha^{(k)}$, respectively. That is $\hat{x}_o = x_o^{(k)} + \Delta x_o^{(k)}$ and $\hat{\alpha} = \alpha^{(k)} + \Delta\alpha^{(k)}$.

The elements $-\partial LL(\underline{X}|x_o^{(k)},\alpha^{(k)})/\partial x_o$ and $-\partial LL(\underline{X}|x_o^{(k)},\alpha^{(k)})/\partial\alpha$ of the right-hand side column matrix of Eq. (18) may be determined by Eqs. (13) and (14). As for the inverse matrix of Eq. (18), Jenkinson (1969) suggests to use its expected value which is the large sample maximum likelihood variance covariance matrix (Kendall and Stuart, 1961; (NERC, 1975) as

$$\begin{bmatrix} \mathrm{Var}\ \hat{x}_o & , \mathrm{Cov}\ (\hat{x}_o, \hat{\alpha}) \\ \\ \mathrm{Cov}\ (\hat{x}_o, \hat{\alpha}) & , \mathrm{Var}\ \hat{x}_o \end{bmatrix} = \frac{\alpha^2}{N} \begin{bmatrix} 1.11, & 0.26 \\ 0.26, & 0.61 \end{bmatrix} \tag{19}$$

Therefore, inserting the matrix of Eq. 19 into Eq. (17) and using the notation P and R as in Eqs. (13) through (16) yields

$$\Delta x_o^{(k)} = (1.11\ P^{(k)} - 0.26\ R^{(k)})\ \alpha^{(k)}/N \quad , \tag{20}$$

and

$$\Delta\alpha^{(k)} = (0.26\ P^{(k)} - 0.61\ R^{(k)})\ \alpha^{(k)}/N \quad . \tag{21}$$

The iteration should start with an appropriate set of parameters $x_o^{(1)}$ and $\alpha^{(1)}$, say, the moment estimates. Then, the $P^{(1)}$ and $R^{(1)}$ are computed from Eqs. (15) and (16) which substituted into Eqs. (20) and (21) give the increments $\Delta x_o^{(1)}$ and $\Delta\alpha^{(1)}$, respectively. The new values $x_o^{(2)}$ and $\alpha^{(2)}$ are estimated by

$$x_o^{(k+1)} = x_o^{(k)} + \Delta x_o^{(k)} \quad , \tag{22}$$

and

$$\alpha^{(k+1)} = \alpha^{(k)} + \Delta\alpha^{(k)} \quad . \tag{23}$$

The iteration then proceeds until the two equations in (17) are satisfied within a specified error.

Subroutine PARGU2

This subroutine computes the maximum likelihood estimators of the parameters α and x_o of the Gumbel or log-Gumbel distributions by the numerical procedure described above. The variables XSC and XLO represent the parameters α and x_o, respectively. The index IDIS = 1 for the Gumbel and IDIS = 2 for the log-Gumbel distribution. The maximum number of iterations and the maximum error are specified as LIM = 30 and

ERROR = 0.000001, respectively in a DATA statement of the program (the user can change these values if desired). The input variables for PARGU2 are the sample size N; the data set X(I), I=1,...,N; and the indeces IDIS and IWR. The output variables are the scale parameter XSC and the location parameter XLO. The program listing of PARGU2 is shown below.

```
      SUBROUTINE PARGU2 (N,X,XSC,XLO,IDIS,IWR)
C.......................................................................
C.....PARAMETERS OF THE GUMBEL AND LOG-GUMBEL DISTRIB.(MAX.LIKELIHOOD EST.)
C   N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C   XLO = LOCATION PARAMETER (MODE),   XSC = SCALE PARAMETER
C   IDIS= 1 GUMBEL,    IDIS= 2  LOG-GUMBEL  DISTRIBUTION
C   IWR = 0 DO NOT WRITE, IWR = 1  WRITE THE RESULTS
C   ERROR= MAXIMUM ACCEPTABLE RELATIVE ERROR ON EACH OF THE PARAMETERS
C   LIM = MAXIMUM NUMBER OF ITERATIONS PERMITTED IN THE PROCEDURE
C       DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C       HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION X(N), TRANS(2)
      DATA LIM,ERROR/30,0.000001/
      DATA TRANS/10H  GUMBEL    ,10HLOG-GUMBEL/
      VN= N
C MOMENT ESTIMATORS  AS INITIAL VALUES OF THE PROCEDURE
      IF (IDIS.EQ.1) GO TO 20
      DO 10 I=1,N
   10 X(I)= ALOG(X(I))
   20 XM=XS=0.
      DO 30 I=1,N
      XM= XM+X(I)/N
   30 XS= XS+X(I)**2/N
      XSC= 0.7806*SQRT(XS-XM**2)
      XLO= XM-0.5772*XSC
C JENKINSON"S INVERSE MATRIX SOLUTION
      KIR=0
      IF (IWR.GT.0) PRINT 40, TRANS(IDIS)
   40 FORMAT (1H1///13X"PARAMETERS OF ",A10," DISTRIBUTION"/16X"(MAXIMUM
     1 LIKELIHOOD ESTIMATORS)"//5X"TRIAL  LOCATION"6X"SCALE"5X"LIKELIHOO
     2D EQUATIONS"/12X"PARAMETER   PARAMETER"5X"DLL1"8X"DLL2"//)
   50 KIR= KIR+1
      P= VN
      R= VN
      DO 60 I=1,N
      Y=(X(I)-XLO)/XSC
      P= P-EXP(-Y)
   60 R= R-Y+Y*EXP(-Y)
      DXLO=XSC*(1.11*P-0.26*R)/VN
      DXSC=XSC*(0.26*P-0.61*R)/VN
      F1 = -P/XSC
      F2 =  R/XSC
      IF (IWR.GT.0) PRINT 70, KIR, XLO, XSC, F1, F2
   70 FORMAT (7X,I2,F13.6,F12.6,2E12.5)
      XLO= XLO+DXLO
      XSC= XSC+DXSC
      IF( XSC.LE.0. ) GO TO 90
      IF( KIR.GE.LIM ) GO TO 110
      IF(ABS(DXLO/XLO).GT.ERROR.OR.ABS(DXSC/XSC).GT.ERROR) GO TO 50
      IF (IWR.EQ.0) RETURN
      PRINT 80, XLO, XSC
   80 FORMAT (    ///10X"LOCATION PARAMETER=",F14.8,"  (MODE)"/10X"SCALE
     1    PARAMETER=",F14.8/)
      RETURN
   90 PRINT 100, XSC
  100 FORMAT (1H1///5X"INVALID SOLUTION BECAUSE OF THE NEGATIVE"/10X,"SC
     1ALE PARAMETER=",F14.8/)
      RETURN
  110 PRINT 40, TRANS(IDIS)
      PRINT 120, KIR, XLO, XSC, F1, F2
  120 FORMAT (/7X,I2,F13.6,F12.6,2E12.5//15X"NO CONVERGENCE"//)
      RETURN
      END
```

Example 2. Subroutine PARGU2 is used to compute the maximum likelihood estimators of the parameters of the Gumbel and log-Gumbel distributions for the same data of Example 1. The input data to PARGU2 are the sample size N = 60; the data set X(I), I = 1,...,60; IDIS = 1 (for the Gumbel) and IDIS = 2 (for the log-Gumbel); and IWR = 1. The main program to call PARGU2 is shown below.

```
      PROGRAM PEV2 (INPUT,OUTPUT)
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.0)
      CALL PARGU2 (N,X,XSC,XLO,1,1)
      CALL PARGU2 (N,X,XSC2,XLO2,2,1)
      END
```

The computer output for this example is:

```
          PARAMETERS OF    GUMBEL    DISTRIBUTION
             (MAXIMUM LIKELIHOOD ESTIMATORS)

TRIAL   LOCATION        SCALE       LIKELIHOOD EQUATIONS
        PARAMETER     PARAMETER        DLL1           DLL2

    1 12219.334826 4045.966461  -.42457E-03   .11085E-02
    2 12269.279538 3891.600465  -.23924E-04   .71703E-04
    3 12271.276752 3882.130430  -.61213E-06   .19761E-05
    4 12271.318366 3881.867622  -.15646E-07   .50501E-07
    5 12271.319430 3881.860907  -.39891E-09   .12881E-08

      LOCATION PARAMETER=12271.31945708   (MODE)
      SCALE    PARAMETER= 3881.86073575


          PARAMETERS OF LOG-GUMBEL DISTRIBUTION
             (MAXIMUM LIKELIHOOD ESTIMATORS)

TRIAL   LOCATION        SCALE       LIKELIHOOD EQUATIONS
        PARAMETER     PARAMETER        DLL1           DLL2

    1       9.377723      .261174   .42557E+02  -.12826E+03
    2       9.361932      .337542  -.60450E+01   .23591E+02
    3       9.363026      .313200   .23601E+00  -.22330E+01
    4       9.363547      .315327  -.67302E-01   .30707E+00
    5       9.363538      .315045   .84422E-02  -.41425E-01
    6       9.363541      .315083  -.11546E-02   .56083E-02
    7       9.363540      .315078   .15604E-03  -.75894E-03
    8       9.363540      .315079  -.21120E-04   .10271E-03

      LOCATION PARAMETER=     9.36354041   (MODE)
      SCALE    PARAMETER=      .31507883
```

5.3 Probability Density and Cumulative Distribution Functions

Since the CDF and PDF of the Gumbel distribution are given explicitly by Eqs. (1) and (2), respectively, they can be easily determined for a given set of parameters. For the case that x is log-Gumbel, $y = \ln(x)$ is Gumbel distributed, hence the PDF of x is related to the PDF of y by

$$f(x) = \frac{1}{x} f(y) \quad . \tag{24}$$

Therefore, the PDF of the log-Gumbel can be determined as a function of the PDF of the Gumbel.

The CDF of the Gumbel variable y is related to the CDF of the log-Gumbel variable x as

$$F(x) = \int_0^x f(x)\, dx = F(y) = \int_{-\infty}^{\ln(x)} f(y)\, dy \quad . \tag{25}$$

Therefore, to find the CDF $F(x)$ of the log-Gumbel, Eq. (1) can be used replacing x by $\ln(x)$.

Subroutine DISGUM

This subroutine determines the PDF and CDF of the Gumbel and log-Gumbel distributions for specified parameters α and x_o following the procedures described above. Both PDF and CDF functions are determined either for a specified set of points x represented by X(I), I = 1,...,N or for N equally spaced points between a minimum and a maximum points determined in the program for probabilities of 0.001 and 0.999. These two probabilities are represented by CDFMIN and CDFMAX, respectively and are specified in a DATA statement. They may be modified if desired.

The input variables to DISGUM are the number of points N at which the PDF and CDF are to be computed; the parameters XLO (representing x_o) and XSC (representing α); the data set X(I), I = 1,...,N if the index IX =1, otherwise if IX = 0, the set X(I) is not given; the index IDIS = 1 (Gumbel) or IDIS = 2 (log-Gumbel); and the index IWR. The output variables are the density PDF(I), the cumulative function CDF(I), I = 1,...,N; and the given or computed X(I), I = 1,...,N. The program listing of DISGUM is given below.

```
      SUBROUTINE DISGUM(N, X, XLO, XSC, DEN, CDF, IX, IDIS, IWR)
C.........................................................................
C.....DENSITY AND CUMULATIVE DISTRIB. FUNCTIONS OF THE GUMBEL AND LOG-GUMBEL
C  XLO=LOCATION PARAMETER(MODE),XSC=SCALE PARAMETER (POSITIVE)
C  IDIS=1 GUMBEL, IDIS=2 LOG-GUMBEL DISTRIBUTION
C  DEN = DENSITY FUNCTION, CDF = CUMULATIVE PROBABILITY OF EXCEEDANCE
C  CDFMIN, CDFMAX= DESIRED MINIMUM AND MAXIMUM LIMITS FOR CDF WHICH
C  ARE USED TO SET A SUITABLE RANGE ,(XMIN,XMAX)  WHEN IX=0.
C  N = NUMBER OF DISCRETE POINTS DESIRED TO BE COMPUTED
C  IWR = 1 WRITE,  IWR = 0 DO NOT WRITE THE RESULTS
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.........................................................................
      DIMENSION X(N),DEN(N),CDF(N),TRANS(2)
      DATA TRANS/10H   GUMBEL  ,10HLOG-GUMBEL/
      DATA CDFMIN, CDFMAX/ 0.001, 0.999/
      IF(IX.GT.0) GO TO 20
      XMAX = XLO - XSC * ALOG(-ALOG(CDFMAX))
      XMIN = XLO - XSC * ALOG(-ALOG(CDFMIN))
      IF (IDIS.GT.1) XMAX=EXP(XMAX)
      IF (IDIS.GT.1) XMIN=EXP(XMIN)
      DELX=(XMAX-XMIN)/(N-1)
      DO 10 I=1,N
   10 X(I)=XMIN+(I-1)*DELX
   20 IF(IDIS.LE.1) GO TO 40
      DO 30 I=1,N
   30 X(I)=ALOG(X(I))
   40 DO 50  I=1,N
      CDF(I)=EXP(-EXP(-(X(I)-XLO)/XSC))
   50 DEN(I)=EXP(-(X(I)-XLO)/XSC)*CDF(I)/XSC
      IF (IDIS.EQ.1) GO TO 70
      DO 60 I=1,N
      X(I)=EXP(X(I))
   60 DEN(I)=DEN(I)/X(I)
   70 IF (IWR.EQ.0) RETURN
      PRINT 80, TRANS(IDIS), XLO, XSC
   80 FORMAT (1H1//5X"DENSITY AND CUMULATIVE DISTRIBUTION OF "A10,//11X,
     1"(XLO= "F12.5"   XSC= "F12.5" )"//6X,"ORDER     VARIABLE       DENS
     2ITY        CUM. DISTR."/17X"VALUE"7X"FUNCTION      FUNCTION"/)
      PRINT 90, (I, X(I), DEN(I), CDF(I), I=1,N)
   90 FORMAT (7X,I3,F14.5,F13.6,F14.6)
      RETURN
      END
```

Example 3. Subroutine DISGUM is used to determine the PDF and CDF of the Gumbel distribution with parameters $\hat{x}_o = 77.5$ and $\hat{\alpha} = 39.1$ at eight given points. The inputs to DISGUM are N = 8; the data set

X(I), I = 1,...,8; the parameters XLO = 77.5 and XSC = 39.1; IDIS = 1 and IWR = 1. The main program to call DISGUM in this example is given below.

```
      PROGRAM GUMDIS (INPUT,OUTPUT)
      DIMENSION X(60), DEN(60), CDF(60)
      N=8 $ XLO=77.5 $ XSC=39.1
      READ 10, (X(I), I=1,N)
   10 FORMAT (8F10.2)
      PRINT 20, N, XLO, XSC
   20 FORMAT (1H1//5X,"INPUT DATA..."/5X"N=",I5,6X"XLO=",F10.4,6X,"XSC="
     1,F10.4/)
      PRINT 30, (X(I), I=1,N)
   30 FORMAT (5X,10F8.2)
      CALL DISGUM(N, X, XLO, XSC, DEN, CDF, 1, 1, 1)
      END
```

The computer output for this example is:

```
DENSITY AND CUMULATIVE DISTRIBUTION OF   GUMBEL
        (XLO=       77.50000    XSC=       39.10000 )
 ORDER        VARIABLE        DENSITY         CUM. DISTR.
               VALUE          FUNCTION        FUNCTION

   1           -.70000        .000117         .000618
   2          38.40000        .004588         .065988
   3          57.95000        .008108         .192296
   4         116.60000        .006513         .692201
   5         136.15000        .004565         .800011
   6         155.70000        .003023         .873423
   7         175.25000        .001934         .921194
   8         194.80000        .001211         .951432
```

5.4 The Inverse of the Cumulative Distribution Function

The inverse of the CDF of the Gumbel distribution with parameters y_o and α may be determined from Eq. (1) as

$$y = \hat{y}_o - \alpha \, \ell n[-\ell n \, F(y)] \quad . \tag{26}$$

Since a log-Gumbel variable is Gumbel in the log domain, the inverse of the log-Gumbel CDF may be determined by

$$x = \exp \{x_o - \alpha \ell n[-\ell n F(x)]\} \quad , \tag{27}$$

where in this case x_o and α are the parameters of the log-Gumbel distribution.

Subroutine INVGUM

This subroutine determines the inverse of the CDF of the Gumbel or log-Gumbel distributions by using the Eqs. (26) and (27), respectively. The inverse is determined either for a given set of CDF(I), I = 1,...,N or for N equally spaced CDF's computed in the program by CDF(I) = = I/(N+1).

The input variables to INVGUM are: the parameters XLO and XSC; the number N of inverses to be computed; the set CDF(I), I = 1,...,N if the index ICDF = 1, otherwise when ICDF = 0 the set CDF(I) is not given; the index IDIS = 1 for Gumbel or IDIS = 2 for log-Gumbel distribution; and the index IWR. The output variables are the inverses X(I), I = 1,...,N corresponding to the given or computed CDF(I)'s. The program listing of INVGUM is given below.

```
      SUBROUTINE INVGUM (XLO,XSC,N,X,CDF,ICDF,IDIS,IWR)
C.......................................................................
C.....INVERSE OF THE CDF OF THE GUMBEL AND LOG-GUMBEL DISTRIB.
C   XLO  LOCATION PARAMETER(MODE), XSC= SCALE PARAMETER
C   IDIS=1 GUMBEL , IDIS=2 LOG-GUMBEL DISTRIBUTION
C   ICDF=0 CDF(I),I=1,2,...,N SET IS NOT GIVEN
C   ICDF=1 CDF(I),I=1,2,...,N SET IS GIVEN BY THE USER
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION TRANS(2), X(N), CDF(N)
      DATA TRANS/10H  GUMBEL   ,10HLOG-GUMBEL/
      IF (ICDF.GT.0) GO TO 20
      DO 10 I=1,N
   10 CDF(I)= FLOAT(I)/(N+1)
   20 DO 30  I=1,N
   30 X(I) = XLO-XSC*ALOG(-ALOG(CDF(I)))
      IF (IDIS.EQ.1) GO TO 50
      DO 40 I=1,N
   40 X(I) = EXP(X(I))
   50 IF (IWR.EQ.0) RETURN
      PRINT 60, TRANS(IDIS), XLO, XSC
   60 FORMAT (1H1//6X,"INVERSE OF THE "A10" DISTRIBUTION FUNCTION"/5X"(P
     1ARAMETERS  XLO=",F13.6,"  XSC="F13.6")"//12X"ORDER"5X"CUMULATIVE"/
     221X"DISTRIBUTION"/13X"(I)"8X"CDF(X)"12X"X(I)"/)
      PRINT 70, (I, CDF(I), X(I), I=1,N)
   70 FORMAT (12X,I3,F15.6,4X,F15.6)
      RETURN
      END
```

Example 4. Subroutine INVGUM is used to determine the inverse of the CDF of the Gumbel distribution of parameters $x_o = 10$ and $\alpha = 3$. Seven inverse values are desired for the CDF's shown in the computer output below. The inputs to INVGUM are XLO = 10; XSC = 3; N = 7; CDF(I), I = 1,...,7; ICDF = 1; IDIS = 1 and IWR = 1. The main program to call INVGUM for this example is shown below.

```
      PROGRAM GUMCLA (INPUT,OUTPUT)
      INVERSE OF THE GUMBEL (OR LOG-GUMBEL) DISTRIBUTION
      DIMENSION X(100), CDF(100)
      XLO = 10.    $    XSC = 3.
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (CDF(I), I=1,N)
   20 FORMAT (7F10.3)
      PRINT 30, (CDF(I), I=1,N)
   30 FORMAT (//5X,10F8.4)
      CALL INVGUM (XLO,XSC,N,X,CDF,1,1,1)
      END
```

The computed output for this example is:

```
 INVERSE OF THE   GUMBEL    DISTRIBUTION FUNCTION
(PARAMETERS  XLO=     10.000000   XSC=       3.000000)

        ORDER       CUMULATIVE
                   DISTRIBUTION
         (I)          CDF(X)               X(I)

          1          .143000             8.004352
          2          .598000            11.995636
          3          .873000            15.989281
          4          .965000            20.003940
          5          .991000            24.118041
          6          .998000            28.640822
          7          .999000            30.721765
```

5.5 Confidence Limits

The 100α percent confidence limits x_ℓ for the T-year event are determined by

$$x_\ell = x_T \pm u_\alpha s_T \quad , \tag{28}$$

where u_α is the standard normal variate corresponding to the α confidence level. The value x_T may be determined by

$$x_T = \hat{\mu} + K_T \hat{\sigma} \quad , \tag{29}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the mean and standard deviation of x and K_T is the frequency factor which is a function of the skewness and T.

From Eq. (2) the reduced Gumbel variable $z = (x-x_o)/\alpha$ is

$$z = -\ell n[-\ell n\ F(x)] = -\ell n\ \{-\ell n[1 - Q(x)]\}$$

where $Q(x)$ is the exceedence probability. For a given sample of size N ordered from the largest down, the $Q(x)$ is computed by the Weibul's plotting position formula as $Q(x) = m/N+1$, hence the reduced variable z may be written as

$$z_m = -\ell n\ \{-\ell n[1 - \frac{m}{N+1}]\} \quad ,$$

or

$$z_m = -\ell n\ \{-\ell n[(N+1-m)/(N+1)]\} \quad . \tag{30}$$

The mean μ_z and variance σ_z^2 of z_m of Eq. (30) is

$$\mu_z = (1/N) \sum_{m=1}^{N} z_m \quad , \tag{31}$$

and

$$\sigma_z^2 = (1/N) \sum_{m=1}^{N} (z_m - \mu_z)^2 \quad . \tag{32}$$

Kendall (1959) gives the relation

$$\alpha = \sigma/\sigma_z \quad , \tag{33}$$

and

$$x_o = \mu - \alpha\ \mu_z \quad , \tag{34}$$

where μ and σ are the parameters of x. Replacing these two equations in $z = (x-x_o)/\alpha$ it follows

$$x_m = \mu + (\frac{z_m - \mu_z}{\sigma_z})\sigma \quad , \tag{35}$$

from which the frequency factor K_T is

$$K_T = \frac{z_m - \mu_z}{\sigma_z} \quad . \tag{36}$$

Notice that T and m are related by $T = \frac{N+1}{m}$.

The standard error of estimate by the method of moments for a two-parameter distribution is given by Kite (1977) as

$$s_T^{\;2} = \frac{\mu_2}{N}\,[1 + K_T\gamma + \frac{K_T^{\;2}}{4}(\kappa - 1)] \quad , \tag{37}$$

with μ_2, γ and κ the variance, skewness coefficient and kurtosis coefficient of x. For the Gumbel distribution $\gamma = 1.1396$ and $\kappa = 5.4002$ so that

$$s_T^{\;2} = (\sigma^2/N)(1 + 1.1396\ K_T + 1.1\ K_T^{\;2}) \quad . \tag{38}$$

Kimbal (1949) and Kite (1977) give the maximum likelihood estimate of the standard error as

$$s_T^{\;2} = (\hat{\alpha}^2/N)(1.1086 + 0.514\ y_T + 0.6079\ y_T^{\;2}) \quad , \tag{39}$$

where the $z_T = (x_T - \hat{x}_o)/\hat{\alpha}$ and $\hat{x}_o$ and $\hat{\alpha}$ are the maximum likelihood estimates of the parameters x_o and α.

The event magnitudes x_T and its confidence limits x_ℓ for the log-Gumbel distribution is obtained by

$$x_T = \exp\ \{y_T\} \quad , \tag{40}$$

and

$$x_\ell = \exp\ \{y_\ell\} \quad , \tag{41}$$

where y_T and z_ℓ are the T-year event magnitudes and confidence limits, respectively for the Gumbel distribution $y = \ell n(x)$.

Subroutine CLIGU1

This subroutine determines the moment estimators of the event magnitude x_T of Eq. (29) and the 100α percent confidence limits x_ℓ of Eq. (28) based on the moment estimator of the standard error s_T of Eq. (38). The event magnitude x_T, represented by X(I), and its upper and lower confidence limits XUT(I) and XLT(I) are computed for specified return periods T(I) or non-exceedence probabilities PXT(F), I = 1,...NT, (NT = the number of desired T or PXT values) and for a given confidence level CONF. Subroutine RETCDF is called by CLIGU1 to transform T to PXT or reverse and to determine the standardized normal variate UCL (u_α of Eq. (28)) for the given CONF.

Inputs to CLIGU1 are: the sample size N; the number NT of return periods or non-exceedence probabilities, both represented by the input variable T(I), I = 1,...,N; the confidence level CONF; the parameters XLO and XSC; the index IDIS = 1 (Gumbel) or IDIS = 2 (log-Gumbel); and the index IWR. Outputs from CLIGU1 are: the T-year event magnitudes XT(I) and it confidence limits XUT(I) and XLT(I) corresponding to the input T(I), I = 1,...,NT. The program listing of CLIGU1 is shown below. The listing of RETCDF was given before.

```
      SUBROUTINE CLIGU1 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,IDIS,IWR)
C.......................................................................
C.....MOMENT ESTIMATES OF THE CONFIDENCE LIMITS OF GUMBEL AND LOG-GUMBEL
C   IDIS=1 GUMBEL,   IDIS=2 LOG-GUMBEL DISTRIBUTION
C   XLO = LOCATION PARAMETER (MODE),   XSC = SCALE PARAMETER
C*  XLO PARAMETER SHOULD BE GIVEN IN LOG DOMAIN FOR LOG-GUMBEL DISTRIBUTION
C   IWR = 1  WRITE,    IWR = 0  DO NOT WRITE THE RESULTS
C   SUBPROGRAM NEEDED IS RETCDF
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION  T(NT), XT(NT), XUT(NT), XLT(NT), PXT(NT),TRANS(2)
      DATA  TRANS/ 10H  GUMBEL    ,10HLOG-GUMBEL/
      CALL RETCDF( NT, T, CONF, PXT, UCL )
      SA = XSC / 0.7806
      XM = XLO + 0.5772*XSC
      AN = N+1
      YM = YS = 0.0
      DO  10  M=1, N
      BN = AN - M
      Y = - ALOG( - ALOG(BN/AN) )
      YM = YM + Y / N
```

```
10 YS = YS + Y*Y / N
   SY = SQRT( YS-YM*YM )
   DO  20  I=1,NT
   YT = - ALOG( - ALOG(PXT(I)) )
   FT = (YT - YM ) / SY
   D  = 1. + 1.139547093*FT + 1.1*FT*FT
   ST = SQRT(D/N) * SX
   XT(I) = XM + FT*SX
   XUT(I) = XT(I) + UCL*ST
20 XLT(I) = XT(I) - UCL*ST
   IF (IWR.GT.0) PRINT 30, CONF, TRANS(IDIS), XLO, XSC
30 FORMAT (1H1//5X"MOMENT ESTIMATES OF "F4.2" CONFIDENCE LIMITS FOR "
  1A10" DISTRIBUTION"/12X"( XLO="F12.5"   (MODE)    XSC="F12.5" )",//
  212X,"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDENCE  LIMITS"/12X,
  3"PERIOD  PROBABILITY    MAGNITUDE"6X"UPPER"8X"LOWER"/)
   IF (IDIS.LT.2) GO TO 50
   DO  40  I=1,NT
   XT(I) = EXP( XT(I) )
   XUT(I) = EXP( XUT(I) )
40 XLT(I) = EXP( XLT(I) )
50 IF (IWR.EQ.0) RETURN
   PRINT 60, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
60 FORMAT (12X,F4.0,2X,F10.5,3X,3F13.5)
   RETURN
   END
```

Example 5. Subroutine CLIGU1 is applied to compute moment estimates of design floods and their corresponding 95 percent confidence limits for 2, 5, 10, 20, 50 and 100 years of return periods. The floods are assumed Gumbel distributed with moment parameter estimates of $\hat{x}_o$ = = 12202.566 and $\hat{\alpha}$ = 4080.1 or log-Gumbel with moment parameter estimates of $\hat{x}_o$ = 9.37664 and $\hat{\alpha}$ = 0.26308. These values are obtained from 60 years of data of St. Mary's River at Stillwater. The input variables to CLIGU1 are: N = 60; NT = 6; T(I) = 2, 5, 10, 20, 50 and 100 for I = 1,...,6; CONF = 0.95; XLO = 12202.566 and XSC = 4080.1 if IDIS = 1 (Gumbel), or XLO = 9.37664 and XSC = 0.26308 if IDIS = 2 (log-Gumbel); and IWR = 1. The main program to input this information is shown below.

```
      PROGRAM GUMCON2 (INPUT,OUTPUT)
      DIMENSION  T(60), XT(60), XUT(60), XLT(60), PXT(60)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5,/F7.2)
      XLO=12202.566   $   XSC=4080.1
      CALL CLIGU1 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,1,1)
      XLO= 9.37664    $    XSC= 0.263084
      CALL CLIGU1 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,2,1)
      END
```

The computer output for this example is:

```
MOMENT ESTIMATES OF  .95 CONFIDENCE LIMITS FOR   GUMBEL   DISTRIBUTION
      ( XLO= 12202.56600   (MODE)    XSC=  4080.10000 )

      RETURN NONEXCEEDANCE     EVENT         CONFIDENCE  LIMITS
      PERIOD  PROBABILITY    MAGNITUDE        UPPER        LOWER

        2.       .50000     13731.87060   14949.63158   12514.10961
        5.       .80000     18775.25130   20922.89300   16627.60959
       10.       .90000     22114.40816   25056.59606   19172.22026
       20.       .95000     25317.40759   29060.67886   21574.13631
       50.       .98000     29463.36181   34268.68215   24658.04146
      100.       .99000     32570.17107   38181.06892   26959.27321
```

```
MOMENT ESTIMATES OF  .95 CONFIDENCE LIMITS FOR LOG-GUMBEL DISTRIBUTION
      ( XLO=       9.37664   (MODE)    XSC=        .26308 )

      RETURN NONEXCEEDANCE     EVENT         CONFIDENCE  LIMITS
      PERIOD  PROBABILITY    MAGNITUDE        UPPER        LOWER

        2.       .50000     13033.12200   14097.74617   12048.89541
        5.       .80000     18041.77862   20721.45411   15708.63580
       10.       .90000     22376.20782   27050.61951   18509.54563
       20.       .95000     27509.37844   35019.05475   21610.11792
       50.       .98000     35940.22879   48994.26763   26364.30970
      100.       .99000     43911.82573   63052.84537   30581.46588
```

Subroutine CLIGU2

This subroutine determines the maximum likelihood estimates of the T-year event magnitude x_T of Eq. (29) and the 100α percent confidence limits x_ℓ of Eq. (28) based on the maximum likelihood estimator of the standard error s_T of Eq. (39). The overall description of this subroutine is the same as the subroutine CLIGU1 referred before. The program listing of CLIGU2 is given below.

```
      SUBROUTINE CLIGU2 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,IDIS,IWR)
C.......................................................................
C.....MAX. LIKELIHOOD ESTIMATES OF THE CONF. LIMITS OF GUMBEL AND LOG-GUMBEL
C  IDIS=1 GUMBEL,   IDIS=2 LOG-GUMBEL DISTRIBUTION
C  XLO = LOCATION PARAMETER (MODE),   XSC = SCALE PARAMETER
C* XLO PARAMETER SHOULD BE GIVEN IN LOG DOMAIN FOR LOG-GUMBEL DISTRIBUTION
C  IWR = 1  WRITE,    IWR = 0  DO NOT WRITE THE RESULTS
C  SUBPROGRAM NEEDED IS RETCDF
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
C.......................................................................
      DIMENSION  T(NT), XT(NT), XUT(NT), XLT(NT), PXT(NT), TRANS(2)
      DATA  TRANS/ 10H  GUMBEL    ,10HLOG-GUMBEL/
      CALL RETCDF( NT, T, CONF, PXT, UCL )
      SX = XSC / 0.7806
      XM = XLO + 0.5772*XSC
      DO  10  I=1,NT
```

```
      YT = - ALOG( - ALOG(PXT(I)) )
      ST = SQRT( (1.1086 + 0.514*YT + 0.6079*YT*YT)/N ) * XSC
      XT(I) = XLO + YT*XSC
      XUT(I) = XT(I) + UCL*ST
   10 XLT(I) = XT(I) - UCL*ST
      IF (IWR.GT.0) PRINT 20, CONF, TRANS(IDIS), XLO, XSC
   20 FORMAT (1H1///5X"LIKELIHOOD ESTIMATES OF"F4.2" CONFIDENCE LIMITS F
     1OR "A10" DISTRIBUTION"/12X"( XLO="F12.5"   (MODE)    XSC="F12.5" )"//
     2/12X"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDENCE  LIMITS"//12X"PER
     3IOD  PROBABILITY   MAGNITUDE"6X"UPPER"8X"LOWER"/)
      IF (IDIS.LT.2) GO TO 40
      DO  30  I=1,NT
      XT(I) = EXP( XT(I) )
      XUT(I) = EXP( XUT(I) )
   30 XLT(I) = EXP( XLT(I) )
   40 IF (IWR.EQ.0) RETURN
      PRINT 50, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   50 FORMAT (12X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

Example 6. Subroutine CLIGU2 is applied to compute the maximum likelihood estimates of design floods and corresponding confidence limits for the Gumbel and log-Gumbel distributions for the same data of Example 5. The maximum likelihood parameter estimates are $\hat{x}_o$ = 12271.319 and $\hat{\alpha}$ = 3881.8607 for the Gumbel and $\hat{x}_o$ = 9.36354 and $\hat{\alpha}$ = 0.31508 for the log-Gumbel distributions. The input data variables are similar to those of Example 5. The main program to input data and call CLIGU2 is shown below.

```
      PROGRAM GUMCON2 (INPUT,OUTPUT)
      DIMENSION  T(60), XT(60), XUT(60), XLT(60), PXT(60)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5,7F7.2)
      XLO = 12271.319    $    XSC = 3881.8607
      CALL CLIGU2 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,1,1)
      XLO = 9.36354    $    XSC = 0.3150788
      CALL CLIGU2 (N,NT,CONF,XLO,XSC,T,XT,XUT,XLT,PXT,2,1)
      END
```

The computer output for this example is:

```
LIKELIHOOD ESTIMATES OF .95 CONFIDENCE LIMITS FOR   GUMBEL   DISTRIBUTION
     ( XLO= 12271.31900  (MODE)   XSC=  3881.86070 )
```

RETURN PERIOD	NONEXCEEDANCE PROBABILITY	EVENT MAGNITUDE	CONFIDENCE LIMITS UPPER	LOWER
2.	.50000	13694.07110	14847.61618	12540.52603
5.	.80000	18093.87709	19864.24989	16323.50428
10.	.90000	21006.93149	23278.01173	18735.85125
20.	.95000	23801.20321	26579.66421	21022.74221
50.	.98000	27418.10133	30873.39402	23962.80864
100.	.99000	30128.45750	34099.41322	26157.50177

```
LIKELIHOOD ESTIMATES OF .95 CONFIDENCE LIMITS FOR LOG-GUMBEL DISTRIBUTION
     ( XLO=      9.36354  (MODE)   XSC=       .31508 )
```

RETURN PERIOD	NONEXCEEDANCE PROBABILITY	EVENT MAGNITUDE	CONFIDENCE LIMITS UPPER	LOWER
2.	.50000	13082.36542	14366.43969	11913.06187
5.	.80000	18697.39131	21586.75163	16194.76834
10.	.90000	23684.66419	28478.92722	19697.48767
20.	.95000	29714.35540	37231.28809	23715.07842
50.	.98000	39853.20669	52754.99429	30106.68668
100.	.99000	49659.72385	68545.96833	35977.14399

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6. GENERAL EXTREME VALUE DISTRIBUTIONS

The three extreme value distributions are the Fisher-Tipet Type 1 (EV1 or Gumbel distribution), Type 2 (EV2 or Frechet) and Type 3. They only differ from each other by the value of the shape parameter β. The EV1 or Gumbel distribution is a two parameter unbounded distribution with $\beta = 0$; the EV2 or Frechet distribution is a three parameter distribution with a lower bound and $\beta < 0$; and the EV3 distribution is a three parameter distribution with an upper bound and $\beta > 0$. The cumulative distribution function (CDF) and probability distribution (PDF) of the GEV distribution, which includes both the EV2 and EV3 distribution are:

$$F(x) = \exp\left\{- [1 - \beta(x-x_o)/\alpha]^{1/\beta}\right\} \tag{1}$$

$$f(x) = \frac{1}{\alpha}[1 - \beta(x-x_o)/\alpha]^{1/\beta-1} \cdot F(x) \tag{2}$$

where

$\alpha > 0$ is the scale parameter

β is the shape parameter

x_o is the location parameter.

The Gumbel distribution, as was mentioned, is a special case of the GEV distributions, when $\beta = 0$. So, the limit of the equation number (1) when $\beta \to 0$ should converge to the Gumbel distribution, as shown here. Substituting $u = (x-x_o)/\alpha$, equation No. 1 reduces to

$F(x) = \exp\left\{- [1-\beta u]^{1/\beta}\right\}$, and we can use logarithmic (3)

Taylor series as an approximation to $\ln[1 - \beta u]^{1/\beta}$ as follows

$$\ln[1 - \beta u]^{1/\beta} = 1/\beta \ln(1 - \beta u) = \frac{1}{\beta}\left[- \beta u - \frac{(\beta u)^2}{2} - \frac{(\beta u)^3}{3} - \ldots\right]$$

$$= - u - \frac{\beta u^2}{2} - \frac{\beta^2 u^3}{3} - \ldots$$

Now because $\ln(1-\beta u)^{1/\beta} = -u - \frac{\beta u^2}{2} - \frac{\beta^2 u^3}{3} - \ldots$

$$(1-\beta u)^{1/\beta} = e^{-u - \frac{\beta u^2}{2} - \ldots}$$

and $\lim_{\beta \to 0} e^{[-u - \frac{\beta u^2}{2} - \frac{\beta^2 u^3}{3} - \ldots]} = e^{-u}$

So $\lim_{\beta \to 0} (1 - \beta[x-x_o]/\alpha)^{1/\beta} = e^{-[x-x_o]/\alpha}$

So the general value distribution reduces to the Gumbel distribution which equation is written as

$$F(x) = \exp\{-\exp[-(x-x_o)/\alpha]\} \quad (3a)$$

This chapter is mainly concerned with the EV2 and EV3 distributions, because the EV1 (or Gumbel) distribution will be presented separately in the following chapter, as it is a special case of EV distributions.

The reduced Gumbel variate y and can be written as

$$y = (x-x_o)/\alpha$$

Therefore the Gumbel distribution reduces to

$$F(x) = \exp\{-\exp[-y]\} \quad (4)$$

and we can relate the reduced Gumbel variate with the original extreme value variate as follows

$$[1 - \beta(x-x_o)/\alpha]^{1/\beta} = e^{-y}$$

So $\quad x = x_o + \alpha(1-e^{-\beta y})/\beta \quad (5)$

is the relation between the reduced Gumbel variate y, and the original variate x.

Equation (5) is a straight line when $\beta \to 0$ as will be shown

$$\lim_{\beta \to 0} [x - x_o = \alpha(1-e^{-\beta y})/\beta] = \lim_{\beta \to 0} \alpha(1-e^{-\beta y})/\beta = \alpha y.$$

Therefore $x-x_o = \alpha y$ is a linear relation as shown in figure 6.1. The relation between Gumbel variate and EV2 and EV3 variate is an exponential curve as can be seen in figure 6.1.

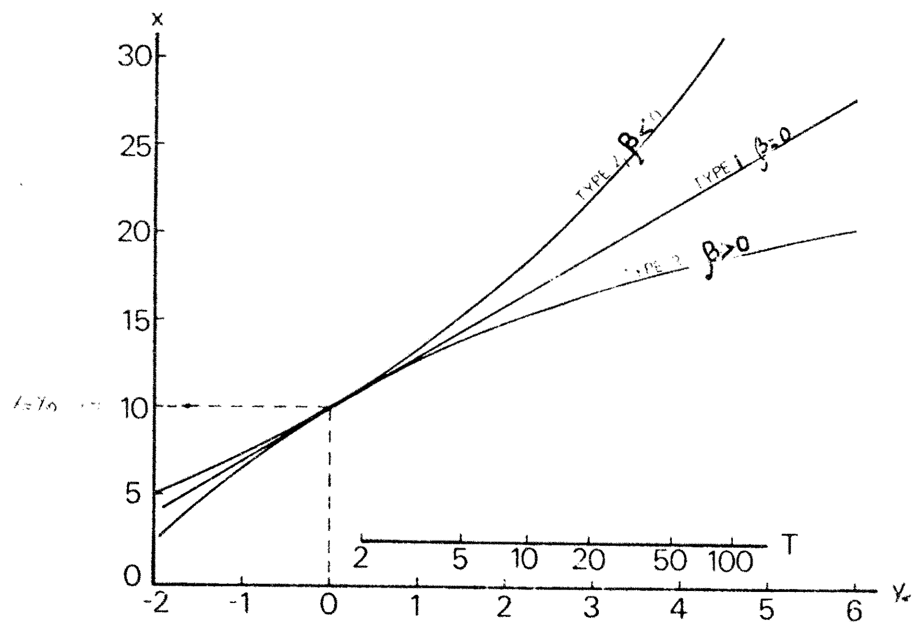

Figure 6.1 The three types of extreme variate shown as functions of type 1 reducted variate by the relation $x = x_o + \alpha(1-e^{-\beta y})/\beta$

There is a practical way of determining the type of the GEV distribution for a given sample of x. Since the theoretical skewness of the EV1 variate is 1.14, for $\beta < 0$ the skewness is greater than 1.14 while for $\beta > 0$, the skewness is less than 1.14. Furthermore if the X variate is EV2 (Frechet) distributed, then log X is Gumbel distributed; and if X is EV3 distributed then - X is said to have a Weibull distribution. (NERC, 1975)

6.1 PROPERTIES OF EV2 AND EV3 DISTRIBUTIONS

Examples of the EV2 and EV3 with parameters $x_o = 10$, $\alpha = 4$, and $\beta = -0.1$ and $x_o = 20$, $\alpha = 4$, $\beta = 0.35$ respectively, are shown in figure 6.2.

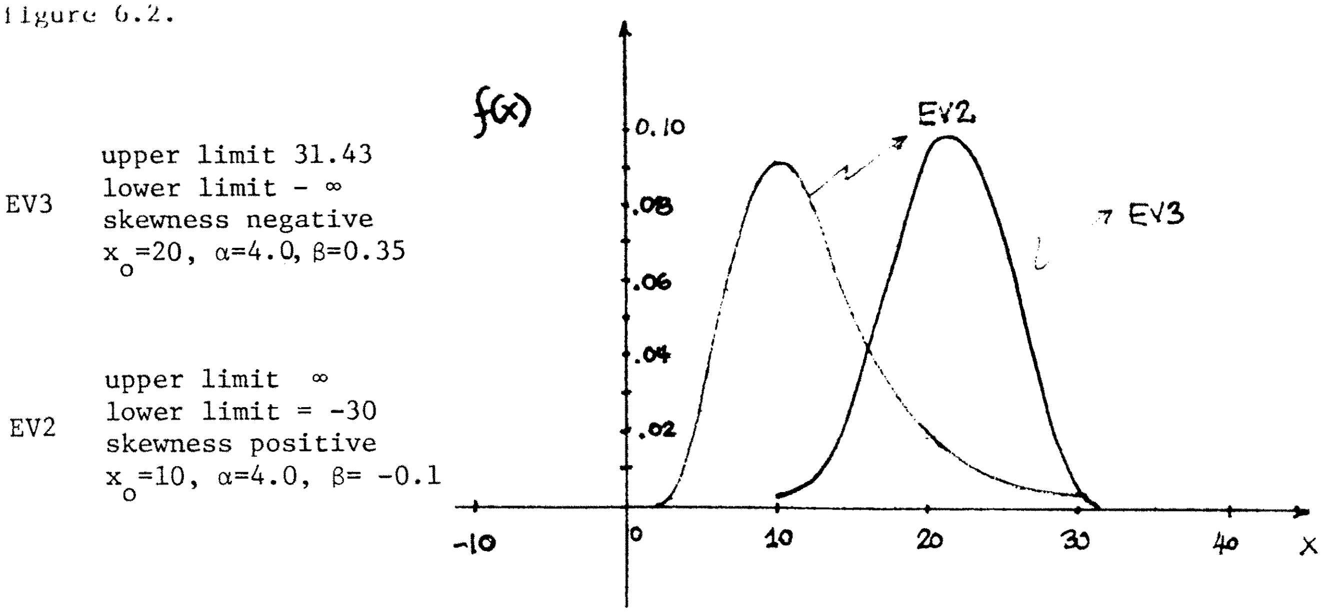

Figure 6.2. Examples of EV2 and EV3 Distributions

Denoting the reduced variates of EV2 and EV3 by z_2 and z_3, one can write:

$$z_2 = 1 - \beta(x-x_o)/\alpha \text{ ; with } 0 \le z_2 \le \infty, \text{ and } \beta < 0 \tag{4}$$

and

$$z_3 = \beta(x-x_o)/\alpha - 1 \text{ ; with } -\infty \le z_3 \le 0, \text{ and } \beta > 0 \tag{5}$$

or the original variate in terms of z_2 or z_3, may be expressed as:

$$x = (x_o + \alpha/\beta) + (-\alpha/\beta)z_2, \text{ with } (x_o + \alpha/\beta) \le x \le \infty \tag{6}$$

and

$$x = (x_o + \alpha/\beta) + (\alpha/\beta)\, z_3, \text{ with } -\infty \le x \le (x_o + \alpha/\beta) \tag{7}$$

where $A = (x_o + \alpha/\beta)$ may be considered as a new location parameter (lower bound of x when $\beta < 0$, and upper bound of x when $\beta > 0$), and

$$B = (-1)^j \ (\alpha/\beta) \qquad \begin{array}{l} j = 1 \text{ for EV2} \\ j = 2 \text{ for EV3} \end{array} \tag{7a}$$

such that $B > 0$ is a new scale parameter. Therefore, equations (6) and (7) can be reduced to one as follows

$$x = A + Bz \tag{8}$$

where z should be considered as the reduced variate either for the EV2 or EV3 distribution.

Replacing the reduced variates z_2 and z_3 in Eqs. 1 and 2, one can obtain the CDF and PDF in terms of the reduced variates:

$$F(z_2) = \exp \left\{ - (z_2)^{1/\beta} \right\} \tag{9}$$

$$f(z_2) = \frac{1}{\beta} \left\{ - (z_2)^{1/\beta - 1} \right\} \ . \ F(z_2) \tag{10}$$

and

$$F(z_3) = \exp \left\{ - [-(z_3)]^{1/\beta} \right\} \tag{11}$$

$$f(z_3) = 1/\beta \left\{ [-z_3]^{1/\beta - 1} \right\} \ . \ F(z_3) \tag{12}$$

One can see in the above relations that the only parameter is β. This is the reason why β determines the type of the distribution. Furthermore, the following simple exponential relations are valid between the reduced Gumbel variate (EV1) y, and the EV2 reduced variate z_2 and the EV3 reduced variate z_3.

$$z_2 = \exp \ [- \beta y] \ , \quad \beta < 0 \tag{13}$$

$$z_3 = - \exp \ [- \beta y] \ , \text{ with } \beta > 0 \tag{14}$$

Two illustrative examples of the PDF of the GEV distributions are given in Figure 6.2.

6.2 ESTIMATION OF PARAMETERS

As was mentioned in the preceding section, the type of the EV distribution strictly depends on the value of the shape parameter, β. Consequently it depends on the sample skewness of the data used in the analysis. Hence, one can make a rather coarse decision of the type of distribution that might be in preference with the aid of the given sample skewness (Figure 6.3).

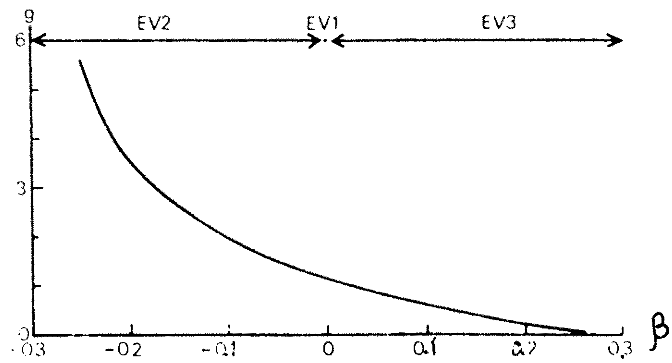

Figure 6.3 Skewness of EV Variates as Function of the Shape Parameter β.

In practice, values of β usually lie in the following range: $-0.4 < \beta < 1.0$. It will be shown later that the central moments of the PDF, i.e. Equation 10, depends only on β in the form of the gamma function, $\Gamma(z)$. Since the values of the gamma function for the arguments 0,-1,-2,... are infinite, the corresponding values of $\beta = -1/3,-1/2,-1,-2...$ and $\beta = 0$ may not permit computation of the central moments analytically.

Obviously the moment estimates of the parameters are less accurate as compared with the maximum likelihood estimates, because of the considerably large skewness of the EV variates. As a result one may find highly different values for the parameters, or even different types of the EV distribution, depending on the estimation procedure used in the analysis.

Since the EV2 and EV3 distributions may be regarded as a GEV distribution differing only by the value of β, only the two basic methods (MONENTS and MAX LIKELIHOOD) have been introduced in the subsequent sections, although there are some graphical procedures and the so-called "Estimation by Sextiles" (Jenkinson, 1969; quoted in NERC).

Method of Moments

Applying the moment generating function of the PDF of z_2 and z_3, one can find the rth moment about $z_2 = 0$ and /or $z_3 = 0$.

$$\mu_r'(z_2) = \Gamma(1 + r\beta), \text{ with } \beta < 0 \tag{15}$$

and

$$\mu_r'(z_3) = (-1)^r \Gamma(1 + r\beta), \text{ with } \beta > 0 \tag{16}$$

The first three moments yield the mean, variance, and skewness coefficients:

$$\mu_1(z_2) = \Gamma(1 + \beta) \tag{17}$$

$$\mu_2(z_2) = \mu_2'(z_2) - [\mu_1'(z_2)]^2 = \Gamma(1 + 2\beta) - \Gamma^2(1 + \beta) \tag{18}$$

$$\gamma_1(z_2) = \mu_3(z_2)/[\mu_2(z_2)]^{3/2} \tag{19}$$

where

$$\mu_3(z_2) = \Gamma(1 + 3\beta) - 3\Gamma(1 + 2\beta)\Gamma(1 + \beta) + 2\Gamma^3(1 + \beta) \tag{20}$$

for z_2, with $\beta < 0$; and:

$$\mu_1(z_3) = -\Gamma(1 + \beta) \tag{21}$$

$$\mu_2(z_3) = \Gamma(1 + 2\beta) - \Gamma^2(1 + \beta) \tag{22}$$

$$\mu_3(z_3) = -\Gamma(1 + 3\beta) + 3\Gamma(1 + 2\beta)\Gamma(1 + \beta) - 2\Gamma^3(1 + \beta) \tag{23}$$

for z_3, with $\beta > 0$; and $\gamma_1(z_3)$ will have the same form as Equation (19).

Since the moments of the reduced variates do not contain the other parameters, and since x is linearly related to z_2 and/or z_3, their skewness will be equal, so they both will have the same value of β. A unique solution of β can be found to equation (19) with a given $\gamma_{1,x} = \gamma_1(z_2)$ or $\gamma_{1,x} = \gamma_1(z_3)$.

Therefore you can find β parameter from the following equation:

$$\gamma = \frac{(-1)^j [\Gamma(1+3\beta) - 3\Gamma(1+2\beta)\Gamma(1+\beta) + 2\Gamma^3(1+\beta)]}{[\Gamma(1+2\beta) - \Gamma^2(1+\beta)]^{3/2}} \qquad (24)$$

if EV2 $j = 2$ and $\beta < 0$

if EV3 $j = 1$ and $\beta > 0$

one can give values to β and find γ values. With this procedure we derived two fifth order polynomal regression equations (β regressed on $\gamma, \gamma^2, \gamma^3, \gamma^4, \gamma^5$)

1.- $$\hat{\beta} = 0.2792377 - 0.3398360\ \hat{\gamma}_{1,x} + 0.1008508\ \hat{\gamma}_{1,x}^2 - 0.0165458\ \hat{\gamma}_{1,x}^3 + 0.0014037\ \hat{\gamma}_{1,x}^4 - 0.0000479\ \hat{\gamma}_{1,x}^5 \qquad (25)$$

$$0.008 < \hat{\gamma}_{1,x} \leq 8.000 \qquad R^2 = 99.97\%$$

2.- $$\hat{\beta} = 0.277439 - 0.322359\ \hat{\gamma}_{1,x} + 0.065677\ \hat{\gamma}_{1,x}^2 + 0.029405\ \hat{\gamma}_{1,x}^3 + 0.0031760\ \hat{\gamma}_{1,x}^4 \qquad (26)$$

$$-3.80 \leq \hat{\gamma}_{1,x} \leq 0.0872 \qquad R^2 = 99.99\%$$

where $\gamma_{1,x}$ is the sample skewness, and we can use the above equations for a range from -3.8 to 8.00, to get the β parameter for either EV2 or EV3 distributions.

Having the computed $\hat{\beta}$, one can estimate the mean and variance of the reduced variate using equation (17) and (18) and/or equations (21) and (22).

Due to the fact that the variate x is linearly related to the reduced variates by equation 8, the mean and variance of x are:

$$\mu_{1,x} = A + B\,\mu_{1,z} \tag{27}$$

$$\mu_{2,x} = B^2\,\mu_{2,z} \tag{28}$$

where $\mu_{1,z}$ and $\mu_{2,z}$ represent the mean and the variance of z_2 and z_3 respectively, as explained before. Then:

$$\hat{B} = (\hat{\mu}_{2,x}/\hat{\mu}_{2,z})^{1/2} \tag{29}$$

and

$$\hat{A} = \hat{\mu}_{1,x} - \hat{B}\,\hat{\mu}_{1,z}$$

As we stated in equations (7a)

$$B = (-1)^j(\alpha/\beta) \qquad \begin{array}{l} j=1 \text{ for EV2} \\ j=2 \text{ for EV3} \end{array} \tag{30}$$

Then since β is known, $\hat{\alpha}$ is calculated with:

$$\hat{\alpha} = (-1)^j\,\hat{B}\,\hat{\beta} \qquad \begin{array}{ll} j=1 \text{ for EV2} & \beta < 0 \\ j=2 \text{ for EV3} & \beta > 0 \end{array} \tag{31}$$

Thus the remaining parameter, x_o, can be obtained from (NERC, 1975) as:

$$\hat{x}_o = \hat{A} - \hat{\alpha}/\hat{\beta} \tag{32}$$

<u>Subroutine PARGEV</u>

This subroutine computes the moment estimation of the GEV distribution including the type I or Gumble distribution, although the occurrence of $\beta=0$ or $\gamma_{1,x} = 1.14$ is almost impossible.

The input variables to the main program PGEV are the sample size N; the data x(I), I = 1,2,...,N; the estimator required, biased (IEST=1) or unbiased (IEST=2); and the treatment of output, do not write results (IWR=0) or write results (IWR=1). The output variables are the type of GEV distribution, ITYP=1,2, or 3; the location parameter or mode x_o (XLO); scale parameter α(XSC), and shape parameter β(XSH).

Subroutine PARGEV requires subroutine GAMMA to compute values of the gamma function, $\Gamma(Z)$ for a given Z. The main program PGEV, and subroutine PARGEV are presented below. GAMMA subroutine will be at appendix.

```
      PROGRAM PGEV (INPUT,OUTPUT)
C     EXAMPLE OF PARAMETER ESTIMATION FOR THE GEV DIST. (MOMENT ESTIMATORS)
      DIMENSION X(60), EST(2)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
20    FORMAT(12F6.1)
      CALL PARGEV (N,X,XSC,XSH,XLO,1,ITYP,1)
       ND
```

```
      SUBROUTINE PARGEV(N,X,XSC,XSH,XLO,IEST,ITYP,IWR)
C     MOMENT ESTIMATORS OF GENERAL EXTREME VALUE DISTRIBUTIONS
C     ITYP = 1, TYPE-1 (GUMBEL) UNBOUNDED AND XSH=0
C     ITYP = 2, TYPE-2 (FRECHET) HAS A LOWER LIMIT, XSH IS NEGATIVE
C     ITYP = 3, TYPE-2 (FRECHET) HAS AN UPPER LIMIT, XSH IS POSITIVE
C     XLO  = LOCATION PARAMETER (CENTRAL VALUE OR MODE)
C     XSC  = SCALE PARAMETER (SHOULD ALWAYS BE POSITIVE)
C     XSH  = SHAPE PARAMETER (DETERMINES THE TYPE OF THE EV DISTR.)
C     N    = SAMPLE SIZE
C     IEST = 1, BIASED,                IEST = 2, UNBIASED MOMENT ESTIMATORS
C     IWR  = 0, DO NOT WRITE RESULTS,  IWR = 1, WRITE RESULTS
C     SUBPROGRAM NEEDED...GAMMA
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
      DIMENSION X(N), EST(2)
      DATA EST/10H ( BIASED  ,10H(UNBIASED /
      VN = N
      XM = XS = XG = ZA = ZB = GA = GB = 0.0
C...  MOMENTS, STAND,DEV.AND SKEWNESS OF THE ORIGINAL DATA
      DO 10 I= 1,N
      XM= XM + X(I)/N
      XS= XS + X(I)**2/N
   10 XG= XG + X(I)**3/N
      XG= XG + 2*XM**3 - 3*XM*XS
      XS= XS - XM**2
      SX = SQRT(XS)
      CS = XG/(XS**1.5)
      IF( IEST.EQ.1 )GO TO 20
      SX =SX*SQRT(VN/(VN-1.))
      CS = CS*SQRT(VN*(VN-1.))/(VN-2.)
20    IF( CS.LE.8.0.OR.CS.GE.-3.8 ) GO TO 30
```

```
      PRINT 90,CS
   90 FORMAT(///20X,"WARNING---THE SAMPLE SKEWNESS IS OUT OF THE PERMISS
     *IBLE RANGE"//20X,"SKEWNESS = ",F12.5//)
      RETURN
30    IF( CS.LT.0.0872.AND.CS.GE.-3.8) GO TO 35
      XSH=  .2792377 -.3398360*CS     +.1008508*CS**2
     1     -.0165458*CS**3 +.0014037*CS**4 -.0000479*CS**5
      GO TO 36
35         XSH=0.277439-0.322359*CS+0.065677*CS**2
     2     +0.029405*CS**3+0.0031760*CS**4
36    ZA = 1. + XSH
      CALL GAMMA (ZA,GA,IER)
      ZB = 1. + 2*XSH
      CALL GAMMA (ZB,GB,JER)
      B = SX/SQRT(GB - GA**2)
      IF( XSH) 50, 80,40
   40 ITYP= 3
      A = XM+B*GA
      XSC =XSH*B
      XLO = A-B
      GO TO 60
   50 ITYP= 2
      A = XM-B*GA
      XSC = -XSH*B
      XLO = A+B
   60 IF (IWR.GT.0) PRINT 70, ITYP, EST(IEST), XLO, XSC, XSH
   70 FORMAT (1H1///5X,"PARAMETERS OF THE EXTREME VALUE TYPE-"I1," DISTR
     1IBUTION"/16X,A10," MOMENT ESTIMATORS)"//15X,"LOCATION PARAMETER="F
     212.6/15X"SCALE    PARAMETER="F12.6/15X"SHAPE    PARAMETER="F12.6
     3/)
      RETURN
   80 ITYP= 1
      XSC = 0.7806*SX
      XLO = XM - 0.5772157*XSC
      IF (IWR.GT.0) PRINT 70, ITYP, EST(IEST), XLO, XSC, XSH
      RETURN
      END
```

Example 1 Subroutine PARGEV is used to compute the moment estimators of the parameters of the annual maximum flows at St. Mary's River at Stillwater. The biased estimates of the variance and skewness are used in this example. The length of the record was 60 years. The input data to PARGEV is N=60, the data set X(I), I=1,...,60; IEST=1, and IWR=1. The computer output for this example is:

```
PARAMETERS OF THE EXTREME VALUE TYPE-2 DISTRIBUTION
            ( BIASED  MOMENT ESTIMATORS)

          LOCATION PARAMETER=    47.235319
          SCALE    PARAMETER=    45.338829
          SHAPE    PARAMETER=     -.049613
```

Method of Maximum Likelihood

Using the Jenkinson approach (1969), the log likelihood of the GEV distribution takes the form

$$LL(x;\ x_o,\ \alpha,\ \beta) = -N\ \ln(\alpha) - (1-\beta)\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} e^{-y_i} \tag{33}$$

where

$$y_i = -\frac{1}{\beta}\ln\left(1 - \frac{x_i - x_o}{\alpha}\ \beta\right) \tag{34}$$

Differentiating with respect to x_o, α, and β yields:

$$-\frac{\partial LL}{\partial x_o} = \frac{Q}{\alpha} \tag{35}$$

$$-\frac{\partial LL}{\partial \alpha} = \frac{(P+Q)}{(\alpha\beta)} \tag{36}$$

$$-\frac{\partial LL}{\partial \beta} = [R-(P+Q)/\beta]/\beta \tag{37}$$

where

$$P = N - \sum_{i=1}^{N} e^{-y_i} \tag{38}$$

$$Q = \sum_{i=1}^{N} e^{(\beta-1)y_i} - (1-\beta) \sum_{i=1}^{N} e^{\beta y_i} \tag{39}$$

$$R = N - \sum_{i=1}^{N} y_i + \sum_{i=1}^{N} y_i\, e^{-y_i} \tag{40}$$

The maximum likelihood estimates of $\hat{x}_o$, $\hat{\alpha}$, and $\hat{\beta}$ must satisfy the equations

$$-\frac{\partial LL}{\partial x_o} = 0 \;, \quad -\frac{\partial LL}{\partial \alpha} = 0 \;, \quad -\frac{\partial LL}{\partial \beta} = 0 \tag{41}$$

These equations must be solved iteratively as there is no explicit solution. It is required to begin the process with initial values x_o, α, β, as for example moment estimates. Let $\Delta x_{o,k}$, $\Delta\alpha_k$, $\Delta\beta_k$ be the differences between the maximum likelihood estimates and the current estimates $x_{o,k}, \alpha_k$, and β_k at the k^{th} step of iteration. That is

$$\begin{aligned} \hat{x}_o &= x_{o,k} + \Delta x_{o,k} \\ \hat{\alpha} &= \alpha_k + \Delta\alpha_k \\ \hat{\beta} &= \beta_k + \Delta\beta_k \end{aligned} \tag{42}$$

Expanding Equation (41) in a Taylor series about the maximum likelihood values and omitting the terms containing higher powers and cross products of $\Delta x_{o,k}$, α_k, and β_k, yields the matrix equation:

$$\begin{bmatrix} \Delta x_{o,k} \\ \Delta\alpha_k \\ \Delta\beta_k \end{bmatrix} = \begin{bmatrix} -\partial^2 LL/\partial x^2_{ok} & -\partial^2 LL/\partial x_{ok}\,\partial\alpha_k & -\partial^2 LL/\partial x_{o,k}\partial\beta_k \\ -\partial^2 LL/\partial\alpha_k\partial x_{ok} & -\partial^2 LL/\partial\alpha_k^{\,2} & -\partial^2 LL/\partial\alpha_k\partial\beta_k \\ -\partial^2 LL/\partial\beta_k\partial x_{ok} & -\partial^2 LL/\partial\beta_k\partial\alpha_k & -\partial^2 LL/\partial\beta_k^{\,2} \end{bmatrix}^{-1} \begin{bmatrix} \partial LL/\partial x_{ok} \\ \partial LL/\partial\alpha_k \\ \partial LL/\partial\beta_k \end{bmatrix} \tag{43}$$

Jenkinson replaces the elements of the matrix to be inverted by their expected values which results in the large sample maximum likelihood variance - covariance matrix of estimators $\hat{x}_o$, $\hat{\alpha}$, $\hat{\beta}$, as:

$$\begin{bmatrix} \text{Var } x_o & \text{Cov }(\alpha, x_o) & \text{Cov }(x_o,\beta) \\ \text{Cov }(\alpha,x_o) & \text{Var }\alpha & \text{Cov }(\alpha,\beta) \\ \text{Cov }(x_o,\beta) & \text{Cov }(\alpha,\beta) & \text{Var }\beta \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \alpha^2 b & \alpha^2 h & \alpha f \\ \alpha^2 h & \alpha^2 a & \alpha g \\ \alpha f & \alpha g & c \end{bmatrix} \qquad (44)$$

where a, b, c, f, g, h are functions of β only and are given in Table 6.3 for various values of β

β	a	b	c	f	g	h
−0.4	1.05	1.29	0.84	0.26	0.09	0.80
−0.3	0.92	1.29	0.73	0.26	0.03	0.69
−0.2	0.81	1.28	0.64	0.26	0.04	0.57
−0.1	0.72	1.27	0.55	0.26	0.10	0.46
0.0	0.65	1.25	0.48	0.26	0.15	0.34
0.1	0.61	1.22	0.39	0.24	0.18	0.21
0.2	0.58	1.20	0.33	0.22	0.21	0.09
0.3	0.58	1.17	0.27	0.19	0.23	−0.03
0.4	0.60	1.14	0.21	0.16	0.24	−0.16
0.5	0.63	1.11	0.15	0.13	0.24	−0.30
0.6	0.68	1.08	0.10	0.09	0.22	−0.43
0.8	0.82	1.02	0.03	0.03	0.15	−0.71
1.0	1.00	1.00	0.00	0.00	0.00	−1.00

Table 6.3

Inserting the expressions for the derivatives $\frac{\partial LL}{\partial x_{o,k}}, \frac{\partial LL}{\partial \alpha_k}, \frac{\partial LL}{\partial \beta_k}$ in P, Q and R, as in equations (35), (36) and (37), with equation (43) and using equation (44) for the inverse of the matrix, yields

$$\begin{bmatrix} \Delta x_{o,k} \\ \Delta\alpha_k \\ \Delta\beta_k \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \alpha_k^2 b & \alpha_k^2 h & \alpha_k f \\ \alpha_k^2 h & \alpha_k^2 a & \alpha_k g \\ \alpha_k f & \alpha_k g & c \end{bmatrix} \begin{bmatrix} -Q_k/\alpha_k \\ -(P_k+Q_k/\alpha_k\beta_k \\ -[R_k - (P_k+Q_k)/\beta_k]/\beta_k \end{bmatrix} \qquad (45)$$

This expression gives the following three equations which can be solved iteratively by varying $x_{o,k}, \alpha_k$ and β_k

$$\begin{aligned} \Delta x_{o,k} &= -\alpha_k\,(b_k A_k + h_k B_k + f_k C_k)/N \\ \Delta\alpha_k &= -\alpha_k\,(h_k A_k + a_k B_k + g_k C_k)/N \qquad (46) \\ \Delta\beta_k &= -\,(f_k A_k + g_k B_k + c_k C_k)/N \end{aligned}$$

where a_k, b_k, c_k, f_k, g_k, and h_k are the current estimates of a, b, c, f, g and h for a given β_k, and A_k, B_k and C_k are:

$$
\begin{aligned}
A_k &= Q_k \\
B_k &= (P_k + Q_k)/\beta_k \\
C_k &= (R_k - B_k)/\beta_k
\end{aligned}
\tag{47}
$$

and P_k, Q_k and R_k are the current estimates, of P, Q and R for a given α_k, β_k and $x_{o,k}$. The new estimates of the parameters are

$$
\begin{aligned}
x_{o,k+1} &= x_{o,k} + \Delta x_{o,k} \\
\alpha_{k+1} &= \alpha_k + \Delta\alpha_k \\
\beta_{k+1} &= \beta_k + \Delta\beta_k
\end{aligned}
\tag{48}
$$

The coefficients a_k, b_k, c_k, f_k, g_k, and h_k used in the variance-covariance matrix are estimated for a given current value of β_k. Then, using the values of table 6.3, the following regression equations, valid for a range of β from -0.4 to 1.0, have been derived

$$\hat{a} = 0.6528060 - 0.5598783\,\beta + 1.0876209\,\beta^2 - 0.054024\,\beta^3 - 0.1270214\,\beta^4$$

with R = 0.9999 and Maximum absolute error (MAE) = 0.0038

$$\hat{b} = 1.2488727 - 0.205244\,\beta - 0.2225715\,\beta^2 + 0.0962481\,\beta^3 + 0.0813214\,\beta^4$$

with R = 0.9996, and MAE = 0.0062

$$\hat{c} = 0.4725506 - 0.7603083\,\beta + 0.2836171\,\beta^2 - 0.1886466\,\beta^3 + 0.1931182\,\beta^4$$

with R = 0.9999, and MAE = 0.0092

$$\hat{f} = 0.2597848 - 0.1727130\,\beta - 0.1370156\,\beta^2 - 0.2437380\,\beta^3 + 0.2937678\,\beta^4$$

with R = 0.9998, and MAE = 0.0029

$$\hat{g} = 0.1432656 + 0.4419219\,\beta - 0.4261602\,\beta^2 - 0.1562171\,\beta^3$$

with R = 0.9993, and MAE = 0.0067

$$\hat{h} = 0.3386324 - 1.2041691\,\beta - 0.1333794\,\beta^2$$

with R $\approx$ 1.000, and MAE = 0.0069

The numerical procedure may be finished when all of the increments of Δx_{ok}, $\Delta\alpha_k$, and $\Delta\beta_k$ become less than a selected percent of the current values of $x_{o,k}$, α_k, and β_k, say 1×10^{-6}.

In summary, in order to estimate the maximum likelihood estimates of the parameters, one needs the initial values of x_o, α, and β (for example, the moment estimates of the parameters), and must establish the computational procedure explained.

Subroutine PARBE2

This subroutine estimates the parameters α, β, and x_o by the Method of Maximum Likelihood, and determines the type of the distribution, taking into account the value of the shape parameter β. The variables XSC, XSH and XLO, represent the scale parameter α, the shape parameter β and the location parameter x_o, respectively.

The computation procedure is explained in the preceding chapter. The input variables to PARBE2 are: the sample size N, the data set X(I), I=1,...N, and the index IWR. Since the moment estimators are used as the initial values of the parameters, subroutine PARGEV is required in addition to subroutine ELEMEN which evaluates the elements a, b, c, f, g, and h for a given β, using the regression equations derived in the previous chapter.

If the shape parameter (XSH = β), lies outside its permissible range $-0.4 \leq \beta \leq 1.0$, subroutine ELEMEN prints out a warning message. If scale parameter (XSC = α), takes a negative value, subroutine PARBE2 warns the user. The maximum number of iterations is restricted by LIM=30 and the maximum relative error on the parameters by ERROR = 10^{-6}. This subroutine doesn't obtain the maximum likelihood estimators of the EV1 (or Gumbel) distribution, but gives a message for the use of subroutine PAREV2.

The listing of PARBE2 and ELEMEN are given below:

```
      SUBROUTINE PARBE2(N,X,XSC,XSH,XLO,ITYP,IWR)
C MAX.LIKELIHOOD ESTIMATORS OF BOUNDED EXPONENTIAL DISTRIBUTIONS
C  N =  MAXIMUM NUMBER OF DATA ANALYZED IN THE PROGRAM.
C  XLO = LOCATION PARAMETER (CENTRAL VALUE OR, MODE)
C  XSC = SCALE PARAMETER (ALWAYS SHOULD BE POSITIVE)
C  XSH = SHAPE PARAMETER  (DETERMINES THE TYPE OF THE DISTRIBUTION)
C  IWR = 0 DO NOT WRITE,    IWR = 1 WRITE THE RESULTS
C  ERROR= MAXIMUM ACCEPTABLE RELATIVE ERROR ON EACH OF THE PARAMETERS
C       ITYP=1  EV1 (OR GUMBEL) DISTRIBUTION
C       ITYP=2  EV2 OR FRECHET DISTR. WITH A LOWER BOUND
C       ITYP=3  EV3 DISTRIBUTION  WITH  AN  UPPER   BOUND
C       SUBPROGRAMS NEEDED...PARGEV, GAMMA, AND ELEMEN
C       DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C       HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N)
      DATA LIM,ERROR/30,0.000001/
      KTR= 0
C MOMENT ESTIMATORS TO INITIALIZE THE PROCEDURE
      CALL PARGEV(N,X,XSC,XSH,XLO,1,ITYP,0)
      IF( ABS(XSH).LT.0.0001 ) GO TO 130
      IF (IWR.GT.0) PRINT 10, ITYP
   10 FORMAT (1H1///14X,"PARAMETERS OF THE EXTREME VALUE TYPE-"I1" DISTR
     1IBUTION"//5X"TRIAL  LOCATION"5X"SCALE"7X"SHAPE  VALUES OF LIKELIHO
     2OD EQUATIONS"//11X"PARAMETER   PARAMETER   PARAMETER   DLL1"6X"DLL2
     32"6X"DLL3"//)
   20 P = R = N    $    Q = 0.0
      KTR= KTR+1
      DO 30 I=1,N
      Y= -ALOG(1.-XSH*(X(I)-XLO)/XSC)/XSH
      YE= EXP(-Y)
      P = P - YE
      Q = Q + (YE-1.+XSH)*EXP(XSH*Y)
   30 R = R - Y + Y*YE
      BB=(P+Q)/XSH
      CC=(R-BB)/XSH
      CALL ELEMEN(XSH,A,B,C,F,G,H)
      DXLO = -XSC*(B*Q+H*BB+F*CC)/N
      DXSC = -XSC*(H*Q+A*BB+G*CC)/N
      DXSH =      -(F*Q+G*BB+C*CC)/N
      F1=  Q/XSC
      F2= BB/XSC
      F3= CC
      IF (IWR.GT.0) PRINT 50, KTR, XLO, XSC, XSH, F1, F2, F3
   50 FORMAT (6X,I2,2X,2F12.6,F10.6,3E10.4)
      XLO= XLO + DXLO
      XSC= XSC + DXSC
      XSH= XSH + DXSH
      IF( XSC.LE.0. ) GO TO 90
      IF( KTR.GE.LIM ) GO TO 110
      IF( ABS(DXLO/XLO).GT.ERROR.OR.ABS(DXSC/XSC).GT.ERROR ) GO TO 20
      IF( ABS(DXSH/XSH).GT.ERROR ) GO TO 20
      IF( XSH.LT.0. ) ITYP=2
      IF( XSH.GT.0. ) ITYP=3
      IF( IWR.GT.0 ) PRINT 70, XLO, XSC, XSH, F1, F2, F3, ITYP
   70 FORMAT (/3X"RESULT"2F12.6, F10.6,3E10.4/3X"FINAL TYPE OF THE DISTR
     1IBUTION IS EV "I1/)
      RETURN
   90 PRINT 100, XSC
  100 FORMAT (//5X,"SCALE PARAMETER="F12.6" MAY NEVER BE NEGATIVE"/)
      RETURN
```

```
110 PRINT 120, KTR, XLO, XSC, XSH, F1, F2, F3
120 FORMAT (//5X,"NO CONVERGENCE AFTER"I3" ITERATIONS.  FINAL VALUES O
   1F THE PROCEDURE"/5X"LOCATION="F12.6"  SCALE="F12.6"  SHAPE="F12.6/
   25X,"DLL1="E14.8,4X,"DLL2="E14.8,4X"DLL3="E14.8/)
    RETURN
130 ITYP= 1
    PRINT 140, ITYP, XLO, XSC, XSH
140 FORMAT (1H1///5X"MOMENT ESTIMATORS OF THE EXTREME VALUE TYPE-"I1"
   1(GUMBEL) DISTRIBUTION"//5X"NOTE...THIS SUBROUTINE IS NOT AVAILABLE
   2 FOR THE MAX. LIKELIHOOD SOLUTION"/5X"USE SUBROUTINE PAREV2, INITI
   3TIALIZING FOLLOWING MOMENT ESTIMATORS"//5X"LOCATION PARAMETER=",
   4F14.6,6X,"CENTRAL VALUE OR MODE)"/5X"SCALE    PARAMETER="F14.6/5X,
   5"SHAPE    PARAMETER="F14.6"(THEORETICALLY SHOULD BE ZERO)"/)
    RETURN
    END
```

```
      SUBROUTINE ELEMEN(S,A,B,C,F,G,H)
C...  ELEMENTS OF VAR-COVAR MATRIX FOR GEV DISTRIBUTIONS
      A= B= C= F= G= H= 0.
      IF( S.LT.-0.4.OR.S.GT.1. ) GO TO 10
      A=  .6528060-.5598783*S+1.0876209*S**2-.054024*S**3-.1270214*S**4
      B=1.2488727-.2053244*S-.2225715*S**2+.0962841*S**3+.0813214*S**4
      C= .4725506-.7603083*S+.2836171*S**2-.1886466*S**3+.1931182*S**4
      F= .2597848-.1727130*S-.1370156*S**2-.2437380*S**3+.2937678*S**4
      IF( S.LE.0. )  F= 0.260
      G= .1432656+.4419219*S-.4261602*S**2-.1562171*S**3
      H= .3386324-1.2041691*S-.1333794*S**2
      RETURN
   10 PRINT 20, S
   20 FORMAT (//5X,"SCALE PARAMETER="F12.6," IS NOT WITHIN THE RANGE (-0
     1.4, 1.0)"/5X"WHICH POLYNOMIAL APPROXIMATIONS ARE VALID"//)
      STOP
      END
```

Example No. 1 Subroutine PARBE2 is used to estimate the maximum likelihood parameters of the EV2 and EV3 probability distribution function for the annual maximum flows at St. Mary's River at Stillwater. The input data to PARBE2 are N=60; the data set X(I), I=1,...,60; and IWR=1. The main program to call PARBE2 is shown below.

```
      PROGRAM PBEXP2 (INPUT,OUTPUT)
C     EXAMPLE FOR THE MAX. LIKELIHOOD ESTIMATORS OF BOUNDED EXP. DISTRIBUTION
      DIMENSION X(60)
      READ 10, N
   10 FORMAT (16I5)
      READ 20, (X(I), I=1,N)
   20 FORMAT (12F6.1)
      CALL PARBE2(N,X,XSC,XSH,XLO,ITYP,1)
      END
```

The computer output for this example is:

PARAMETERS OF THE EXTREME VALUE TYPE-2 DISTRIBUTION

TRIAL	LOCATION PARAMETER	SCALE PARAMETER	SHAPE PAKAMETER	VALUES OF LIKELIHOOD EQUATIONS DLL1	DLL2	DLL3
1	47.235319	45.338829	-.049613	-.1127E+00	.2967E+00	.1268E+02
2	45.559552	38.778996	-.162396	-.1054E+00	.2201E+00	.9434E+01
3	44.419097	35.548372	-.248380	-.6574E-01	.1370E+00	.5905E+01
4	43.472022	33.906521	-.306206	-.3366E-01	.7082E-01	.3234E+01
5	42.885127	33.147515	-.339998	-.1494E-01	.3235E-01	.1582E+01
6	42.576762	32.817540	-.357366	-.6272E-02	.1398E-01	.7223E+00
7	42.430646	32.677672	-.365538	-.2627E-02	.5966E-02	.3184E+00
8	42.365101	32.618444	-.369197	-.1110E-02	.2547E-02	.1381E+00
9	42.336444	32.593243	-.370796	-.4728E-03	.1089E-02	.5952E-01
10	42.324057	32.582480	-.371488	-.2020E-03	.4664E-03	.2557E-01
11	42.318728	32.577874	-.371785	-.8647E-04	.1998E-03	.1097E-01
12	42.316441	32.575902	-.371913	-.3704E-04	.8561E-04	.4703E-02
13	42.315460	32.575057	-.371968	-.1587E-04	.3669E-04	.2016E-02
14	42.315039	32.574695	-.371991	-.6801E-05	.1572E-04	.8641E-03
15	42.314859	32.574540	-.372001	-.2915E-05	.6739E-05	.3703E-03
16	42.314782	32.574473	-.372005	-.1249E-05	.2888E-05	.1587E-03
17	42.314749	32.574445	-.372007	-.5354E-06	.1238E-05	.6803E-04
18	42.314734	32.574432	-.372008	-.2295E-06	.5305E-06	.2916E-04
RESULT	42.314728	32.574427	-.372008	-.2295E-06	.5305E-06	.2916E-04

FINAL TYPE OF THE DISTRIBUTION IS EV 2

6.3 PROBABILITY DENSITY AND CUMULATIVE DISTRIBUTION FUNCTION

The general extreme value probability density and cumulative distribution functions have explicit exponential functional forms. The difference between the GEV type 2 distribution and GEV type 3 distribution is that the shape parameter (β) has different signs. Type 2 has a lower limit to the left as:

$$XMIN = A = x_o + \alpha/\beta \ , \ \beta < 0$$

with f(XMIN) = 0 and F(XMIN) = 0 and unbounded to the right; XMIN $< x < +\infty$, while Type 3 has an upper limit to the right as:

$$XMAX = A = x_o + \alpha/\beta \ , \ \beta > 0$$

with f(XMAX) = 0 and F(XMAX) = 1 and unbounded to the left, $-\infty < x <$ XMAX

For a given set of parameters the Probability Distribution Function PDF of the EV2 and EV3 may be obtained directly from eqs. (9), (10) and (11), (12) respectively.

Subroutine DISGEV

Subroutine DISGEV computes probability densities (PDF) and cumulative probabilities (CDF) for the extreme value distributions type 2 and 3, at N discrete points. These points can be either specified by the user or they can be determined in the program, assuming they are equally spaced. If one wishes to determine the PDF and CDF values at N equally spaced points, the index IX must be set to zero. The parameters of the distribution must be specified and the Maximum (XMAX) and Minimum (XMIN) values are determined analytically. On the other hand, if one needs the values of the density and cumulative functions for a set of X(I), I=1,...N values (such as sample observations), then IX=1 and X(I) and N should be the input in addition to the parameters. The set X(I) should be input in increasing order of magnitude.

The type of the GEV distribution is internally identified depending on the sign of the shape parameter. The input variables to DISGEV are: the number of points N; the data set X(I); I=1,...,N if IX=1; the parameters XLO, XSC, and XSH; the index IX=0 if the N points are to be determined in the program or IX=1 if the N points are specified by the user; and the index IWR. The output are the PDF(I) and CDF(I) values corresponding to X(I), I=1,...,N. The program listings are given below.

```
      SUBROUTINE DISGEV(N,X,XLO,XSC,XSH,DEN,CDF,IX,IWR)
C..DENSITY AND CUMULATIVE OF EXTREME VALUE TYPE 2 AND 3 DISTRIBUTIONS
C  XLO = LOCATION PARAMETER,  XSC = SCALE PARAMETER (POSITIVE)
C  XSH = SHAPE PARAMETER (NEGATIVE IF TYPE 2, POSITIVE IF TYPE 3 )
C  ITYP= 2 EV TYPE 2 DISTRIBUTION WITH A  LOWER BOUND
C  ITYP= 3 EV TYPE 3 DISTRIBUTION WITH AN UPPER BOUND
C  IX = 0  X(I), I=1,2,..,N SET IS NOT GIVEN(IT IS INTERNALLY EVALUATED)
C  IX = 1  X(I), I=1,2,..,N SET SHOULD BE GIVEN IN INCREASING ORDER
C  IWR = 0 DO NOT WRITE,  IWR = 1 WRITE THE RESULTS
C**XMIN AND XMAX VALUES ARE INTERNALLY COMPUTED AS FOLLOWING
C  XMIN= LOWER BOUND OF X, XMAX= X VALUE CORRESPONDIN A PROBABILITY=0.999
C  XMAX= UPPER BOUND OF X, XMIN= X VALUE CORRESPONDIN A PROBABILITY=0.001
C      DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C      HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION X(N), DEN(N), CDF(N)
      B = XSC/XSH  $  C = 1./XSH
      IF( IX.GT.0 ) GO TO 40
      IF( XSH.GT.0.) GO TO 10
      XMIN = XLO + B
      XMAX = B*(1.-(-ALOG(0.999))**XSH) + XLO
      GO TO 20
   10 XMAX = XLO + B
      XMIN = B*(1.-(-ALOG(0.001))**XSH) + XLO
   20 DELX = (XMAX-XMIN)/(N-1)
      DO 30 I= 1,N
   30 X(I) = XMIN + (I-1)*DELX
   40 D = C - 1
      DO 50 I= 1,N
      CDF(I) = EXP(- ((1.-(X(I)-XLO)/B)**C))
   50 DEN(I) = (1. - (X(I)-XLO)/B)**D*CDF(I)/XSC
      IF ( XSH.LT.0.. AND. IX.LE.0 )  CDF(1) = 0.
      IF( XSH.LT.0.) ITYP=2
      IF( XSH.GT.0.) ITYP=3
      IF (IWR.EQ.0) RETURN
      PRINT 60, ITYP, XLO, XSC, XSH
   60 FORMAT (1H1//5X"DENSITY AND CUMULATIVE DISTRIBUTION OF EXTREME VAL
     1UE TYPE-"I1//11X"XLO="F10.3"  XSC="F10.3"  XSH="F8.5/12X"ORDER
     2 VARIABLE      DENSITY       CUM. DISTR."/23X,"VALUE"7X"FUNCTION
     3  FUNCTION"/)
      PRINT 70, (I, X(I), DEN(I), CDF(I), I=1,N)
   70 FORMAT (13X,I3,F14.6,2F13.6)
      RETURN
      END
```

Example Subroutine DISGEV is used to determine the PDF and CDF of the Type 2 and Type 3 extreme value distribution, with parameters XLO=10., and XSC=4., but with the shape parameters of different signs, XSH = -0.1 for type 2, and XSH = 0.1 for type 3 distribution. The maximum and minimum values of cumulative probabilties are calculated assuming CDFMAX = 0.999 and CDFMIN = 0.001. The main program to call DISGAM is shown below.

```
      PROGRAM GEVDIS (INPUT,OUTPUT)
C     EXAMPLE FOR THE DENSITY AND CUMULATIVE OF THE EV TYPE 2 AND TYPE 3
      DIMENSION X(60), DEN(60), CDF(60)
      N=10   $   XLO=10.   $   XSC=4.
      CALL DISGEV(N,X,XLO,XSC, -0.1, DEN,CDF, 0, 1)
      CALL DISGEV(N,X,XLO,XSC,  0.1, DEN,CDF, 0, 1)
      END
```

The printed program output for this example is:

```
DENSITY AND CUMULATIVE DISTRIBUTION OF EXTREME VALUE TYPE-2

        XLO=     10.000   XSC=     4.000   XSH=  -.10000
         ORDER      VARIABLE       DENSITY        CUM. DISTR.
                     VALUE         FUNCTION        FUNCTION

           1      -30.000000      0.000000        0.000000
           2      -21.132611      0.000000        0.000000
           3      -12.265222      0.000000        0.000000
           4       -3.397833       .000000         .000000
           5        5.469556       .033676         .035904
           6       14.336945       .056368         .699615
           7       23.204334       .010236         .943939
           8       32.071723       .001965         .987726
           9       40.939112       .000457         .996756
          10       49.806501       .000125         .999000
DENSITY AND CUMULATIVE DISTRIBUTION OF EXTREME VALUE TYPE-3

        XLO=     10.000   XSC=     4.000   XSH=   .10000
         ORDER      VARIABLE       DENSITY        CUM. DISTR.
                     VALUE         FUNCTION        FUNCTION

           1        1.471856       .001423         .001000
           2        6.863872       .058767         .119169
           3       12.255888       .084723         .571427
           4       17.647904       .032847         .887106
           5       23.039920       .007039         .980839
           6       28.431936       .000961         .997925
           7       33.823952       .000072         .999883
           8       39.215968       .000002         .999998
           9       44.607984       .000000        1.000000
          10       50.000000       .000000        1.000000
```

6.4 THE INVERSE OF THE CUMULATIVE DISTRIBUTION FUNCTION

The inverse of the CDF of the general extreme value distribution function has an explicit exponential function. As we pointed out, the difference between the GEV type 2 distribution and GEV type 3 distribution is that the shape parameter has different signs. The inverse of the CDF of the extreme distributions type 2 and 3, is calculated by the following expression

$$x = x_o + \frac{\alpha}{\beta} - \frac{\alpha}{\beta} (- \ln(F(x)))^{\beta} \tag{49}$$

which was derived from equation (1).

Subroutine INVGEV

This subroutine determines the inverse of the CDF of the extreme value distributions type 2 and 3, using equation (49). The inverse of X is represented by X(I) and the cumulative distribution function by F(X) by CDF(I), I=1,...,N. INVGEV determines X(I) either for a set of specified CDF(I), I=1,...,N; or for a set of equally spaced CDF(I) = I/(N+1), I=1,...,N where N is given by the user. In the first case the index ICDF=1, while in the second case ICDF=0. Inputs to INVGEV are the parameters XLO, XSC and XSH; N is the number of inverses of F(X) to be computed; the index ICDF and the index IWR. The output of INVGAM is the set of inverses, X(I), I=1,...,N corresponding to CDF(I). The program listing of INVGEV is given below.

```
      SUBROUTINE INVGEV (XLO, XSC, XSH, N, X, CDF, ICDF, IWR)
C     INVERSE OF THE GENERAL EXTREME VALUE DISTRIBUTION TYPES 2 AND 3
      DIMENSION X(N), CDF(N)
      ITYP=3  $  B=XSC/XSH  $  A= XLO+B
      IF( ICDF.GT.0 ) GO TO 20
      DO 10 I= 1,N
   10 CDF(I)= FLOAT(I)/(N+1)
   20 DO 30  I=1,N
   30 X(I) = A-B*(-ALOG(CDF(I)))**XSH
      IF( XSH.LT.0. ) ITYP=ITYP-1
      IF (IWR.EQ.0) RETURN
      PRINT 40, ITYP, XLO, XSC, XSH
   40 FORMAT (1H1///7X"INVERSE OF THE EXTREME VALUE TYPE"I2" DISTRIBUTIO
     1N FUNCTION"/5X"(PARAMETERS  XLO="F12.5"  XSC="F12.5"  XSH="F8.5")"
     2//18X"ORDER"5X"CUMULATIVE"/27X"DISTRIBUTION"/19X"(I)",8X,"CDF(X)"1
     32X"X(I)"/)
      PRINT 50, (I, CDF(I), X(I), I=1,N)
   50 FORMAT (18X,I3,F15.6,4X,F15.6)
      RETURN
      END
```

<u>Example</u> Subroutine INVGEV is used to determine the inverse X of F(X) for the extreme value distriubtions type 2 and type 3, with parameters XLO = 10., XSC = 4.0 and XSH = 0.1. The inverse is desired for 10 equally spaced values of the CDF or N=10, ICDF=0, and IWR=1. The main program to call INVGEV is shown below.

```
      PROGRAM GEVCLA (INPUT,OUTPUT)
C     INVERSE OF EXTREME VALUE TYPE 2 AND TYPE 3 DISTRIBUTIONS
      DIMENSION X(100), CDF(100)
      READ 10,N
   10 FORMAT (16I5)
      XLO = 10.    $    XSC = 4.0    $    XSH = 0.1
      CALL INVGEV (XLO,XSC,XSH,N,X,CDF,0,1)
      XLO = 10.    $    XSC = 4.0    $    XSH = -0.1
      CALL INVGEV (XLO,XSC,XSH,N,X,CDF,0,1)
      END
```

The computed output of INVGAM for this example is:

INVERSE OF THE EXTRLME VALUE TYPE 3 DISTRIBUTION FUNCTICN
(PARAMETERS XLO= 10.00000 XSC= 4.00000 XSH= .10000)

ORDER (I)	CUMULATIVE DISTRIBLTION CDF(X)	X(I)
1	.090909	6.344093
2	.181818	7.808378
3	.272727	8.938520
4	.363636	9.953837
5	.454545	10.939499
6	.545455	11.953301
7	.636364	13.053576
8	.727273	14.324947
9	.818182	15.935000
10	.909091	18.379122

INVERSE OF THE EXTREME VALUE TYPE 2 DISTRIBUTION FUNCTION
(PARAMETERS XLO= 10.00000 XSC= 4.00000 XSH= -.10000)

ORDER (I)	CUMULATIVE DISTRIBUTION CDF(X)	X(I)
1	.090909	6.650252
2	.181818	7.922239
3	.272727	8.966340
4	.363636	9.953830
5	.454545	10.962096
6	.545455	12.053522
7	.636364	13.305951
8	.727273	14.849268
9	.818182	16.969030
10	.909091	20.599491

6.5 CONFIDENCE LIMITS

The α confidence limits x_ℓ for the T year event are determined by Kite (1977)

$$x_\ell = x_T \pm u_\alpha s_T \tag{50}$$

where u_α is the standard normal deviate corresponding to the α confidence level. The value of x_T can be determined by

$$x_T = \hat{\mu} + k_T \hat{\sigma} \tag{51}$$

where k_T is the frequency factor, a function of the skewness and T.

$$k_T = A_\beta + B_\beta[(-\ln(1-1/T))^\beta - 1] \tag{52}$$

where

$$B_\beta = \{\Gamma(1+2\beta) - \Gamma^2(1+\beta)\}^{-1/2} \tag{53}$$

and

$$A_\beta = (1-\Gamma(1+\beta))B_\beta \qquad (54)$$

$$\mu = x_o + \alpha/\beta(1 - \Gamma(1+\beta)) \qquad (55)$$

$$\sigma = \left\{(-\alpha/\beta)^2[\Gamma(1+2\beta) - \Gamma^2(1+\beta)]\right\}^{1/2} \qquad (56)$$

From Kite (1977), but keeping the same notation

$$X_T = \mu+\sigma \left\{ (A_\beta + B_\beta[(-\ln(1-1/T))^\beta-1] \right\} \qquad (57)$$

where the frequency factor is

$$k = A_\beta + B_\beta[(-\ln(1-1/T))^\beta-1] \qquad (58)$$

Method of Moments

The general equation for the standard error of estimate for a 3 parameter distribution can be written as:

$$s_T^2 = \frac{\mu_2}{n} \left\{ 1 + k\gamma_1 + \frac{k^2}{4}[\gamma2-1] + \frac{\partial k}{\partial \gamma_1}[\partial\gamma_{2-3}\gamma_1{}^2 - 6 + k(\gamma_3 - 6\gamma_1\gamma_2/4 - 10\gamma_1/4)] + (\frac{\partial k}{\partial \gamma_1})^2[\gamma_4 - 3\gamma_3\gamma_1 - 6\gamma_2 + 9\,\gamma_1{}^2\gamma_2/4 + 35\gamma_1{}^2/4+9] \right\} \qquad (59)$$

where γ_1 = skewness coefficient

γ_2 = kurtosis coefficient

$\gamma_3 = \mu_5/\mu_2{}^{2.5}$

$\gamma_4 = \mu_6/\mu_2{}^3$

k = frequency factor

The derivatives can be found analytically as:

$$\frac{\partial k}{\partial \gamma_1} = \frac{\partial k}{\partial \beta}\frac{\partial \beta}{\partial \gamma_1} \qquad (60)$$

if $y = -\ln(1 - 1/T)$; $Gr = \Gamma(1 + r\beta)$; $Pr = \psi\ (1 + r\beta)$ (61)

---

* G. W. Kite, "Confidence Limits for Design Events", Water Resources Research, February 1975, Vol. 11 No. 1 p 50.

where ψ is the digamma function which can be evaluated from the general approximation

$$\psi(z) = \ln(z+2) - \frac{1}{2(z+2)} - \frac{1}{12(z+2)^2} + \frac{1}{120(z+2)^4} - \frac{1}{252(z+2)^6} - \frac{1}{(z+1)} - \frac{1}{z} \tag{62}$$

Then the two partial derivatives are given by:

$$\frac{\partial\gamma}{\partial\beta} = 3\left\{[G_2-G_1^2].[G_3P_3 - G_1G_2(P_1 + 2P_2) + 2G_1^3P_1] - [G_3 - 3G_2G_1 + 2G_1^3]\cdot[G_2P_2 - G_1^2P_1]\right\}/[G_2 - G_1^2]^{5/2} \tag{63}$$

$$\frac{\partial k}{\partial\beta} = \frac{[\ln Y\cdot Y^\beta - G_1P_1]-[Y^\beta - G_1][G_2 - G_1^2]^{-1}\cdot[G_2P_2 - G_1^2P_1]}{[G_2 - G_1^2]^{1/2}}$$

So:

$$\frac{\partial k}{\partial\gamma_1} = \left\{[\ln Y\cdot Y^\beta - G_1P_1] - [Y^\beta - G_1]\cdot[G_2 - G_1^2]^{-1}\cdot[G_2P_2 - G_1^2P_1]\right\}\cdot [G_2 - G_1^2]^2/\left\{3([G_2 - G_1^2]\cdot[G_3P_3 - G_1G_2(P_1 + 2P_2) + 2G_1^3P_1] - [G_3 - 3G_2G_1 + 2G_1^3]\cdot[G_2P_2 - G_1^2P_1])\right\} \tag{64}$$

With this expression we can calculate the standard error of estimate and find confidence limits of the T year event.

Subroutine CLIEVM

This subroutine determines the approximate moment estimates of the T year event magnitude XT of eq. (57) and the (100X) percent confidence limits X of Eq. (50) based on the moment estimator of the standard error ST of eq. (59). The event magnitude XT(I) and its upper and lower confidence limits XUT(I) and XLT(I) are computed for specific return periods T(I) or non exceedence probabilities PXT(I), I=1,...,NT, where NT is the number of confidence limits to be computed, for a given confidence level CONF. Subroutine RETCDF makes inverse transformations between T and PXT and determines the standard normal deviate UCL corresponding to the confidence level CONF.

Inputs to CLIEVM are the sample size N, the number NT of return periods or non-exceedance probabilities, the confidence level CONF; the parameters XLO, XSC, and XSH (which should be the moment estimators); the return period or non exceedance probabilities (T(I) or PXT) and the index IWR. Outputs from CLIEVM are the T-year value XT(I) and its confidence limits XUT(I) and XLT(I) corresponding to the input T(I), I=1,...,NT. The program listing of CLIEVM is given below.

(Subroutines needed: DIGAMA, GAMMA, RETCDF)

Example: Subroutine CLIEVM is applied to compute moment estimates of design floods and their corresponding 95% confidence limits for 2, 5, 10, 20, 50, 100 years of return periods. The floods are extreme value distributed Type 3 with moment parameter estimates $\hat{\alpha} = 45.3388$, $x_o = 47.2353$, $\beta = -0.0496$ obtained from 60 years of flood data of St. Mary's River at Stillwater, using subroutine PARGEV. The input variables to subroutine CLIEVM are N=60; NT=6; T(I)=2,5,10,20,50 and 100 for I=1,...,6; CONF = 0.95; XLO = 47.2353, XSC = 45.3388, XSH = -0.0496, and IWR = 1. The main program to input this information is given below.

```
      PROGRAM EXTCLM (INPUT,OUTPUT)
      DIMENSION T(60),XT(60),XUT(60),XLT(60),PXT(60)
      READ 10,N,NT,CONF,(T(I),I=1,6)
 10   FORMAT (2I5,7F8.2)
      XLO = 47.235315   $  XSC = 45.338829   $  XSH=-.04 613
      CALL CLIEVM(N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The computer output from CLIEVM for the example is:

```
MOMENT ESTIMATES OF THE .95 CONFIDENCE LIMITS FOR GEV TYPE-2 DISTRIBUTION
     ( XLO=    47.23532    XSC=   45.3388298    XSH=    -.0496135  )

     RETURN  NONEXCEEDANCE        EVENT         CONFIDENCE   LIMITS
     PERIOD   PROBABILITY       MAGNITUDE       UPPER         LOWER

       2.        .50000         64.00458        81.46983      46.54134
       5.        .80000        117.83516       142.96514      92.72518
      10.        .90000        155.17759       186.46362     123.99237
      20.        .95000        192.32851       239.04374     145.61329
      50.        .98000        242.42890       327.87146     156.99583
     100.        .99000        281.51966       409.73158     153.30774
```

Method of Maximum Likelihood

The quantile values of return period T may be written as

$$X(T) = A + By(T) \tag{65}$$

One approximate technique which is widely used is to expand the expression for X(T) considered as a function of estimated parameters in a Taylor series about the population parameter values and on squaring to disregard all terms above the second order, and take expectations of all the remaining terms. This gives an expression for the variance which involves the sampling variance of each parameter and all possible sampling covariances between the parameters.

When estimation is by maximum likelihood (Jenkinson, 1969), the quantile is

$$\hat{x}(T) = \hat{x}_o + \frac{\hat{\alpha}}{\hat{\beta}} [1 - \exp(-\hat{\beta}y)] \tag{66}$$

where y is the EV1 reduced variate of return period T and a constant in this expression. This expression may be written as:

$$\hat{x}(T) = \hat{x}_o + \hat{\alpha}\, W \quad \text{where} \tag{67}$$

$$W = [1 - \exp(-\hat{\beta}y)]/\hat{\beta} \tag{68}$$

Now we can use equations (4) and (5) and the relation of the reduced Gumbel variate with z_2 and z_3 which are given by Eqs. (13) and (14). The expression for the ML sampling variance is: (Jenkinson, 1969)

$$\mathrm{Var}\ \hat{x}(T) = \frac{\alpha^2 w^2}{N} \left\{ a + \frac{b}{w^2} + \frac{c}{w^2} \left(\frac{dw}{d\beta}\right)^2 + 2\,\frac{h}{w} + 2\left(\frac{g}{w} + 2\frac{f}{w^2}\right) \frac{dw}{d\beta} \right\} \tag{69}$$

where the constants a, b, c, f, g, h depend on β and are tabulated in table 6.3 and,

$$\frac{dw}{d\beta} = (ye^{-\beta y} - w)/\beta \tag{70}$$

Using these expressions we can apply equation (50) and find the confidence limits.

Subroutine CLIGEV2

This subroutine determines the maximum likelihood estimates of the T-year event of magnitude XT of equation (57) and the 100 α percent confidence limits x_1 of eq. (50) based on the maximum likelihood sampling variance of eq. (69). The overall description of this subroutine is the same as the subroutine CLIEVM referred before. The program listing of CLIEVM. (Subroutines needed are INVGEV, ELEMEN, RETCDF)

```
      SUBROUTINE CLIGEV2 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,IWR)
C  CONFIDENCE LIMITS OF  GEV DISTRIBUTIONS BY MAX. LIKELIHOOD
C* JENKINSON'S (1969) APPROXIMATION TO THE MAX. LIKELIHOOD SAMPLING
C  VARIANCE IS USED TO ESTIMATE STANDARD ERROR OF T-YEAR EVENT.
C     SUBPROGRAMS NEEDED...INVGEV, RETCDF, AND ELEMEN
C     DEVELOPED BY JOSE D. SALAS AND E. BENZEDEN
C     HYDROLOGY AND WATER RESOURCES PROGRAM, COLORADO STATE UNIVERSITY
      DIMENSION T(NT),XT(NT),XUT(NT),XLT(NT),PXT(NT)
      ITYP = 3
      IF( XSH. LT .0 )  ITYP = 2
      CALL ELEMEN( XSH, A, B, C, F, G, H )
      CALL RETCDF( NT, T, CONF, PXT, UCL )
         CALL INVGEV (XLO,XSC,XSH,NT,XT,PXT,1,0)
      DO  30  I=1,NT
      IF( XSH. GT .0 )  GO TO 10
      Y3 = -1. + XSH*( XT(I) - XLO ) / XSC
      YT = - ALOG( - Y3 ) / XSH
      GO TO 20
   10 Y2 =  1. - XSH*( XT(I) - XLO ) / XSC
      YT = - ALOG( + Y2 ) / XSH
   20 WT = ( 1. - EXP(- XSH*YT ) ) / XSH
      DW = ( YT*EXP( -XSH*YT ) - WT ) / XSH
      WW = WT * WT
      AN = N
      AWN = XSC*WT/ SQRT(AN)
      ST = AWN* SQRT( A + ( B+C*DW*DW)/WW + 2.*H/WT
     1               + 2.*( G/WT + 2.*F/WW )*DW )
      XUT(I) = XT(I) + UCL * ST
   30 XLT(I) = XT(I) - UCL * ST
      IF (IWR.EQ.0) RETURN
      PRINT 40, CONF, ITYP, XLO, XSC, XSH
   40 FORMAT (1H1///5X"LIKELIHOOD ESTIMATES OF THE"F4.2" CONFIDENCE LIMIT
     1TS FOR GEV TYPE-"I1" DISTRIBUTION"/13X"( XLO="F12.5"   XSC="F12.5"
     2   XSH="F12.7" )"//13X"RETURN NONEXCEEDANCE     EVENT"8X"CONFIDENCE
     3  LIMITS"/13X"PERIOD  PROBABILITY   MAGNITUDE"6X"UPPER"8X"LOWER"/)
      PRINT 50, (T(I), PXT(I), XT(I), XUT(I), XLT(I), I=1,NT)
   50 FORMAT (12X,F4.0,2X,F10.5,3X,3F13.5)
      RETURN
      END
```

Example: Subroutine CLIGEV2 is applied to compute maximum likelihood estimates of design floods and their corresponding 95% confidence limits for 2, 5, 10, 20, 50, 100 years of return periods. The floods are extreme value distributed type 2 with maximum likelihood parameter estimates $\hat{\alpha} = 31.7066$, $x_o = 41.3035$, and $\beta = 0.392293$ obtained from 60 years of flood data of St. Mary's River at Stillwater. NT = 6; T(I) = 2,5,10,20, 50 and 100 for I = 1,...,6; WNF = 0.95; XLO = 41.3035, XSC = 31.7066, XSH = 0.392293, and IWR = 1. The main program to input this information is given below.

```
GEVCON      73/73    OPT=1 TRACE                              FTN 4.6+439

      PROGRAM GEVCON (INPUT,OUTPUT)
      DIMENSION  T(10),XT(10),XUT(10),XLT(10),PXT(10)
      READ 10, N, NT, CONF, (T(I), I=1,6)
   10 FORMAT (2I5,7F7.2)
      XLO= 41.3035   $   XSC= 31.7066   $   XSH= - 0.392293
      CALL CLIGEV2 (N,NT,CONF,XLO,XSC,XSH,T,XT,XUT,XLT,PXT,1)
      END
```

The computer output from CLIGEV2 for this example is:

```
LIKELIHOOD  STIMATES OF THE .95 CONFIDENCE LIMITS FOR GEV TYPE-2 DISTRIBUTION
     ( XLO=      41.30350   XSC=      31.70660   XSH=    -.3922930 )

      RETURN  NONEXCEEDANCE     EVENT           CONFIDENCE   LIMITS
      PERIOD   PROBABILITY    MAGNITUDE          UPPER        LOWER

        2.        .50000       53.80137         65.17435     42.42831
        5.        .80000      106.05410        131.49048     80.61772
       10.        .90000      155.83476        214.46842    107.30107
       20.        .95000      219.64292        309.15967    130.12716
       50.        .98000      334.30077        519.58674    148.41201
      100.        .99000      451.69255        758.10895    145.27624
```

APPENDIX

Subroutine to evaluate the gamma function.

```
: GAMMA       73/73    OPT=1 TRACE                              FTN 4.6+452          80/

      SUBROUTINE GAMMA (Z,G,IER)
      IF(Z.LE.57.) GO TO 10
      IER = 2
      G = 1.E75
      RETURN
 10   X = Z
      ERR = 1.0E-06
      IER = 0
      G = 1.0
      IF (X-2.) 40,40,30
 20   IF (X.LE.2.) GO TO 80
 30   X = X-1.
      G = G*X
      GO TO 20
 40   IF (X-1.) 50,90,80
 50   IF (X.GE.ERR) GO TO 70
      Y = FLOAT(INT(X))-X
      IF (ABS(Y).LE.ERR) GO TO 100
      IF ((1.-Y-ERR).LE.0.) GO TO 100
 60   IF (X.GT.1.) GO TO 80
 70   G =G/X
      X = X+1.
      GO TO 60
 80   Y = X-1.
      GY = 1.+Y*(-.57710166+Y*(.98585399+Y*(-.87642182+Y*(.8328212+Y*(-.
     *5684729+Y*(.25482049+Y*(-.0514993)))))))
      G = G*GY
 90   RETURN
 100  IER = 1
      RETURN
      END
```

Subroutine to evaluate the Digamma function.

```
      SUBROUTINE DIGAMA(Z,PSI)
C     RECURRENCE EQUATION FOR THE DIGAMMA FUNCTION
      A=Z+2
      AA=(Z+2)**2
      AC=(Z+2)**4
      AD=(Z+2)**6
      PSI=ALOG(A) - 1/(2*A) - 1/(12*AA) + 1/(120*AC) - 1/(252*AD)
     1 -1/(Z+1) - 1/Z
      RETURN
      END
```

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