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LAMINAR FREE CONVECTION DUE TO A LINE SOURCE OF HEAT

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FOREWORD

This report is No. 7 of a series written for the Diffusion Project presently being conducted by Colorado Agricultural and Mechanical College for the Office of Naval Research under Contract No. N9onr 82401. The experimental phase of this project is being carried out in a wind-tunnel at the Fluid Mechanics Laboratory of the College. The project is under the general supervision of Dr. M. L. Albertson, Head of Fluid Mechanics Research of the Civil Engineering Department.

To Dr. M. L. Albertson, and to Dr. D. F. Peterson, Head of the Civil Engineering Department and Chief of the Civil Engineering Section of the Experiment Station, as well as to Professor T. H. Evans, Dean of the Engineering School and Chairman of the Engineering Division of the Experiment Station, the writer wants to express his appreciation for their kind interest in the present work.

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Laminar Free Convection due to a Line Source of Heat

by

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Abstract

Closed solutions are given in this paper for the velocity and temperature distributions in a fluid which is set in laminar free convection by a line source of heat embedded in an infinite horizontal plane, for Prandtl numbers 2/3 and 7/3. The variations of the kinematic viscosity and of the thermal diffusivity of the fluid with temperature as well as the heat due to viscous dissipation are neglected. Since for heat transfer in air under normal conditions the Prandtl number is 0.73 and for diffusion of water vapor in air it is approximately 0.60, and since the treatment for a mass source is identical, the solution for Prandtl number 2/3 gives an approximate solution for the problem of laminar free convection due to a line source of either heat or water vapor in air.

1. Statement of the Problem

A line source of heat is placed in an infinite horizontal plane above which the fluid was originally isothermal and at rest. Free convection is induced by the difference of specific weight due to the temperature variation in the fluid. The resulting interdependent velocity and temperature distributions are sought.

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2. Solution

From the equation of state

$$\frac{p}{\gamma} = RT \tag{1}$$

where p , γ , R , and T are respectively the pressure, the specific weight, the gas constant, and the absolute temperature, one has, for small variations in T and nearly constant pressure:

$$-\frac{\Delta\gamma}{\gamma_0} = \frac{\Delta T}{T_0} \tag{2}$$

where γ_0 and T_0 are the datum values of γ and T , and $\Delta\gamma$ and ΔT are the deviations of γ and T from these values.

In a vertical plane perpendicular to the line source, the trace of the line source is taken as the origin, and the vertical and horizontal lines intersecting there are taken as the x and y axes respectively. Denoting by u and v the velocity components in the x- and y-directions, and by α the thermal diffusivity, the boundary-layer equation of heat diffusion, being linear and homogeneous, can be written

$$u\frac{\partial \Delta\gamma}{\partial x} + v\frac{\partial \Delta\gamma}{\partial y} = \alpha\frac{\partial^2 \Delta\gamma}{\partial y^2} \tag{3}$$

in virtue of (2).

The boundary-layer equation of motion is

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - g\frac{\Delta\gamma}{\gamma_0} \tag{4}$$

where ν and g are respectively the kinematic viscosity and the gravitational acceleration. Eqs (3) and (4) are to be solved simultaneously with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

with the boundary conditions

$$y = 0: \quad v = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial \Delta\gamma}{\partial y} = 0$$

$$y = \pm\infty: \quad u = 0, \quad \Delta\gamma = 0$$

Since steady conditions are considered, the quantity

$$G = \int_{-\infty}^{\infty} u \Delta\gamma \, dy \tag{6}$$

must be independent of x , and in fact is a measure of the strength of the line source. This integral condition of the continuity of heat can be deduced from (3), integration of which yields in virtue of (5),

$$\int_{-\infty}^{\infty} u \frac{\partial \Delta\gamma}{\partial x} dy + v\Delta\gamma \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Delta\gamma \frac{\partial u}{\partial x} dy = \alpha \frac{\partial \Delta\gamma}{\partial y}\Big|_{-\infty}^{\infty}$$

or, since $\Delta\gamma$ and $\frac{\partial \Delta\gamma}{\partial y}$ both vanish at $y = \pm\infty$, one has

$$\frac{d}{dx}\int_{-\infty}^{\infty} u \Delta\gamma \, dy = 0$$

which gives (6).

Eq (5) permits the use of the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{7}$$

Before one attempts a solution it is desirable to perform a dimensional analysis which will provide a guide to the most adequate transformations to be made. With ψ and $\Delta\gamma$ as the dependent variables, and ρ , μ , G , x , and y (the thermal diffusivity will appear through the Prandtl number, and for the time being can be omitted) as the independent variables, a dimensional analysis yields the relationships:

$$\frac{\psi}{\nu}=F_1\left[\left(\frac{\rho^2 G}{\mu^3}\right)^{\frac{1}{3}}x,\ \frac{y}{x}\right]$$

$$\frac{\nu\Delta\gamma}{G}=F_2\left[\left(\frac{\rho^2 G}{\mu^3}\right)^{\frac{1}{3}}x,\ \frac{y}{x}\right]$$

For simplicity one writes

$$C=\left(\frac{\rho^2 G}{\mu^3}\right)^{\frac{1}{3}} \tag{8}$$

In search for a similarity-solution (Ähnlichkeitslösung) one tries the following substitutions:

$$\frac{\psi}{\nu}=(Cx)^m f(\eta) \tag{9}$$

$$\frac{\nu\Delta\gamma}{G}=-(Cx)^p h(\eta) \tag{10}$$

where

$$\eta=(Cx)^n \frac{y}{x} \tag{11}$$

Eqs (9) and (10) give

$$-g\frac{\Delta\gamma}{\gamma_0}=\frac{G}{\mu}(Cx)^p h(\eta) \tag{12}$$

and, by virtue of (7)

$$u=\nu C^{m+n} x^{m+n-1} f' \tag{13}$$

$$v=-\nu C^m x^{m-1}[mf+(n-1)\eta f'] \tag{14}$$

where the primes denote differential with respect to η .
The necessary derivatives of u and $\Delta\gamma$ are then calculated. Substituting (10), (13) and (14) into (3) and (4), and

demanding equal powers in C and x for the terms of each of the resulting equations, one has

$$m = n, \quad m + 3n - 3 = p \tag{15}$$

Another condition for the exponents is furnished by the fact that G given by (6) is independent of x , which demands that

$$m + p = 0 \tag{16}$$

Solution of (15) and (16) gives

$$m = n = \frac{3}{5}, \qquad p = -\frac{3}{5}$$

With these exponents, the equations resulting from the substitution of (10), (13) and (14) into (3) and (4) are

$$f'f' - 3ff'' = 5f''' + 5h \tag{17}$$

$$hf' + fh' = -\frac{5}{3\sigma} h'' \tag{18}$$

where $\sigma = \nu/\alpha$ is the Prandtl number. Eq (6) can now be expressed in the dimensionless form

$$\int_{-\infty}^{\infty} f'h\,d\eta = 1 \tag{19}$$

With the following transformations

$$\xi = 5\eta \qquad \theta(\eta) = 5h(\eta)$$

one has, instead of (17) and (19):

$$f'f' - 3ff'' = f''' + \theta \tag{20}$$

$$\theta f' + f\theta' = -\frac{1}{3\sigma}\theta'' \tag{21}$$

$$\int_{-\infty}^{\infty} f'\theta\,d\xi = 1 \tag{22}$$

the primes now denoting differentiation with respect to ξ .
The boundary conditions are

$$f(0) = f''(0) = \theta'(0) = 0 \tag{23}$$

$$f'(\pm\infty) = \theta'(\pm\infty) = 0 \tag{24}$$

The conditions imposed by (23) are automatically satisfied if f is an odd function and θ an even function of ξ .

In virtue of (21),

$$\theta = K e^{-3\sigma \int_0^{\xi} f d\xi} \tag{25}$$

Keeping in mind the boundary conditions, one can try

$$f = A \tanh a\xi \tag{26}$$

which yields, from (25),

$$\theta = K(\operatorname{sech} a\xi)^{\frac{3\sigma A}{a}} \tag{27}$$

which evidently satisfies (23) and (24). Substituting (26) and (27) into (20), one finds that (20) is satisfied if

$$7A = 4a + K, \quad A = a, \quad 3\sigma A = 2a$$

which yield

$$\sigma = 2/3, \quad K = 3A$$

or if

$$7A = 4a, \quad -6A = -6a + K, \quad 3\sigma A = 4a$$

which yield

$$\sigma = 7/3, \qquad K = 9A/2$$

In the case $\sigma = 2/3$, $K = 3A$, and $A = a$, A can be determined from (22) to be $\sqrt{5/2}$, and one has

$$f = \frac{\sqrt{5}}{2} \tanh \frac{\sqrt{5}}{2} \zeta \tag{28}$$

$$\theta = \frac{3\sqrt{5}}{2} \operatorname{sech}^2 \frac{\sqrt{5}}{2} \zeta \tag{29}$$

From (11), (13) and (14), and remembering $\zeta = 5\eta$ and $\theta(\eta) = 5h(\eta)$, one has

$$\left(\frac{x^3}{\rho^3 \mu^2 G^4}\right)^{1/5} \Delta\gamma = -\frac{3}{2\sqrt{5}} \operatorname{sech}^2 \frac{\sqrt{5}}{2} \zeta \tag{30}$$

$$\left(\frac{\rho\mu}{G^2 x}\right)^{1/5} u = \frac{25}{4} \operatorname{sech}^2 \frac{\sqrt{5}}{2} \zeta \tag{31}$$

$$\left(\frac{\rho^3 x^2}{G \mu^2}\right)^{1/5} v = -\frac{3}{2\sqrt{5}} \tanh \frac{\sqrt{5}}{2} \zeta + \frac{1}{2} \zeta \operatorname{sech}^2 \frac{\sqrt{5}}{2} \zeta \tag{32}$$

where

$$\zeta = 5\eta = 5\left(\frac{\rho^2 G}{\mu^3 x^2}\right)^{\frac{1}{5}} y \tag{33}$$

The main results for the case $\sigma = 2/3$ are thus obtained.

In the case $\sigma = 7/3$, $K = 9A/2$, and $7A = 4a$, A can be determined from (22) to be $5/2\sqrt{6}$, and one has

$$f = \frac{5}{2\sqrt{6}} \tanh \frac{35}{8\sqrt{6}} \zeta \tag{34}$$

$$\theta = \frac{45}{4\sqrt{6}} \operatorname{sech}^4 \frac{35}{8\sqrt{6}} \zeta \tag{35}$$

where ζ is given by (33). Results similar to those contained in (30) to (32) can be easily obtained.

3. Discussion of Results

From (31) and (32), one sees that

$$u \sim x^{\frac{1}{5}}, \quad v \sim x^{-\frac{2}{5}}$$

for any fixed value of ξ. The first proportionality introduces difficulty at $x = \infty$, and the second at $x = 0$. The difficulty for u at $x = \infty$ is perhaps inherent with the problem and can be tolerated when one remembers that in reality the flow will become turbulent at some height and the solution will cease at any rate to apply at higher elevations. The singularity at $x = 0$ is more serious. Perhaps the present solution is valid only in a thin layer directly above the line source, and has to be matched with another solution for the rest of the flow field to constitute a complete solution. If so, it will be valid for a region with which an investigator of the present problem is concerned the most. In this connection it may be pointed out that solutions due to Schlichting and Bickley for laminar jets (reference 1 and) all suffer from difficulty with V at $x = 0$. For the plane jet, the Schlichting-Bickley solution gives

$$V \sim x^{-2/3}$$

which introduced an even worse singularity at $x = 0$ than the one encountered in the present paper. Since the Schlichting-Bickley solution has been experimentally verified and is generally accepted to be valid, there is no reason why the present solution cannot be expected also to be valid.

4. Remark

The problem of free convection due to a line source of mass in air can be treated in an identical manner. Assume the mass to be water vapor for instance. Denoting the difference between the vapor concentration at any point and that in the ambient air far from the source by Δc , then

$$-\Delta \gamma = \beta \Delta c$$

where β is a constant equal to

$$(M_a - 18)/18$$

M_a being the means molecular weight of the ambient air, and 18 being that for water vapor. Thus, in the boundary-layer equation of diffusion $\Delta \gamma$ can replace Δc , and (3) is again obtained. The subsequent treatment is identical to the foregoing.

5. Acknowledgments

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The derivation of (6) from the energy equation was suggested by a communication in 1948 of Professor C. C. Lin of the Massachusetts Institute of Technology, in which he recommended a similar procedure for an analogous problem where a point source of heat was considered.