DISSERTATION

GENERALIZED INFERENCE FOR MIXED LINEAR MODELS

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY PAUL L. PATTERSON III ENTITLED GENERALIZED INFERENCE FOR MIXED LINEAR MODELS BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

GENERALIZED INFERENCE FOR MIXED LINEAR MODELS

Tsui and Weerahandi (1989) introduced the concept of a generalized p-value for a hypothesis test and subsequently Weerahandi (1993) introduced the concept of a generalized pivotal quantity for a scalar parameter. Generalized pivotal quantities allow one to construct confidence intervals for parameters where standard frequentist confidence intervals are not available. Generalized confidence intervals are not guaranteed to be exact in the frequentist sense. Also, Weerahandi did not provide a systematic way for constructing generalized pivotal quantities. In this work we discuss three techniques for constructing generalized pivotal quantities – a simple recipe, a two stage procedure and a general structural method – all motivated by Fraser's structural method. We apply the simple recipe to obtain generalized pivotal quantities for the variance components of a general balanced mixed linear model and, using a theorem in Hannig et al. (JASA, March 2006), verify the asymptotic exact coverage of the associated confidence intervals. In addition we consider several commonly occurring problems and for each one derive a generalized pivotal quantity and evaluate the performance of the associated confidence interval using a Monte Carlo simulation. The list of problems we consider is (a) tolerance intervals for the distributions of population characteristics of the unbalanced one-way random model (b) tolerance intervals for population characteristics of the two-way random effects model (c) the proportion of conformance for population characteristics of a single normal population (d) proportion of conformance for populations characteristics of the one-way random model and (e) the Common Mean problem. We verify that all the generalized confidence intervals in (a)-(e) have asymptotically exact frequentist coverage.

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DEDICATION

To Kathryn Dubiel,

my partner in this journey through life.

For us both, so many careers, so little time.

CONTENTS

1 Introduction 1

1.1 Introduction to Tolerance Intervals and Proportion of Conformance 3

1.1.1 Tolerance Intervals 3

1.1.2 Proportion of Conformance 5

1.2 The Common Mean Problem 6

2 Fiducial Generalized Confidence Intervals 7

2.1 Definitions of Generalized Pivotal Quantities and Fiducial Generalized Pivotal Quantities 8

2.2 Asymptotic Exact Coverage for FGCIs 12

2.3 Structural Method for Constructing FGPQs 15

2.4 Saturated Balanced Factorial Linear Mixed Models with Replication 22

2.5 Approximate FGPQ for Saturated Unbalanced Factorial Linear Mixed Models with Replication 25

2.6 Alternative Representations of FGPQs 27

2.7 Properties of FGQPs and FGCIs for Scalar Parameters 29

3 One-sided Tolerance Intervals for Linear Mixed Model **32**

3.1 One-sided Tolerance Intervals Under a Saturated Balanced Factorial Linear Mixed Model with Replication 33

3.2 One-way Random Effects Model 34

3.3 Two-way Random Effects Model 37

3.4 Balanced Two-way Random Effects Model with Interaction 39

3.4.1 Lower Tolerance Intervals for $Y \sim N(\mu, \sigma_Y^2)$ 40

3.4.2 Lower Tolerance Intervals for $\mu + A \sim N(\mu, \sigma_A^2)$ 45

3.5 Balanced Two-way Random Effects Model Without Interaction 46

3.5.1 Tolerance intervals for $Y \sim N(\mu, \sigma_Y^2)$ 47

3.5.2 Lower Tolerance Intervals for $\mu + A \sim N(\mu, \sigma_A^2)$ 48

3.6 Upper Tolerance Bound for the Balanced Two-way Random Effects Model with Interaction 49

3.7 Simulation Study and Discussion 49

3.7.1 Results for $Y \sim N(\mu, \sigma_Y^2)$ under the Model with Interaction 50

3.7.2 Results for $\mu + A \sim N(\mu, \sigma_A^2)$ under the Model with Interaction 52

3.7.3 Results for $Y \sim N(\mu, \sigma_Y^2)$ and $\mu + A \sim N(\mu, \sigma_A^2)$ under the Model without Interaction 56

4 Proportion of Conformance for a Normal Distribution **59**
4.1 Methods 60
4.1.1 L2 Method 60
4.1.2 Generalized Confidence Interval 62
4.2 Simulation Study and Discussion 67

5 Proportion of Conformance in One-way Random Effects Models **72**
5.1 FGPQs for the Proportion of Conformance for a One-way Random Effects Models 73
5.1.1 Proportion of Conformance for $Y \sim N(\mu, \sigma_Y^2)$ 73
5.1.2 Proportion of Conformance for $\mu + A \sim N(\mu, \sigma_A^2)$ 77
5.2 Simulation Study and Discussion 79

6 Generalized Structural Methods for Fiducial Generalized Pivotal Quantities **82**
6.1 Two Stage Construction of a FGPQ 82
6.2 General Structural Method for Constructing a FGPQ 84

7 Common Mean of Several Populations **89**
7.1 Notation 90
7.2 Confidence Intervals for μ based on a Two-Stage FGPQ 90
7.3 Confidence Intervals for μ based on a General Structural Method FGPQ . . . 95
7.4 Fisher's Method 100
7.5 Simulation Results 102
7.6 Discussion of Simulation Results 104
7.7 Tables of Results 106

8 Conclusions **111**

9 References **115**

A Empirical Coverage Probabilities for Lower Tolerance Intervals for Two-Way Random Effects Models **119**
A.1 With Interaction 120
A.2 Without Interaction 132

B Simulation Results for Confidence Intervals for the Proportion of Conformance of One-way Random Effects Model **138**

C Computer Programs **177**
C.1 FORTRAN Program for Simulation Study of the Imputation Method Applied to Tolerance Intervals for Distributions from Unbalanced One-way Random Effects Model 177
C.2 FORTRAN Programs for Simulation Study for Tolerance Intervals for Distributions Associated with Two-way Random Effects Model with Interaction . 192
C.2.1 Program for $Y = \mu + A + B + AB + e$ 192
C.2.2 Modifications Needed for $\mu + A$ and Example of Input File 198
C.3 FORTRAN Programs for Simulation Study of Tolerance Intervals for Distribution Associated with Two-way Random Effects Models Without Interaction 199
C.3.1 Program for $\mu + A$ 200

C.3.2 Modifications Needed for $Y = \mu + A + B + e$ and example of Input File . . . 205
C.4 FORTRAN Programs for Simulation Study of Lower Confidence Intervals for Proportion of Conformance of IID Sample 206
C.4.1 Program L2 method of Wang and Lam [54] Using Quantiles as Inputs 206
C.4.2 Program Using Percentile Pairs as Inputs 211
C.5 FORTRAN Programs for Simulation Study of Lower Confidence Intervals for Proportion of Conformance of Population Characteristics of the One-way Random Effects Model 216
C.5.1 Program for $Y = \mu + A + e$ 216
C.5.2 Program for $\mu + A$ 221
C.6 FORTRAN Program for the Simulation Study of Three Lower Tolerance Intervals for the Common Mean Problem 227

LIST OF FIGURES

3.1 Empirical coverage probabilities for tolerance intervals of the distribution of Y, model with interaction, all parameter combinations. 51

3.2 Empirical coverage probabilities for tolerance intervals of the distribution of Y, model with interaction over subsets of the parameters settings. The subsets are defined by intersecting over the listed parameters. 53

3.3 Empirical coverage probabilities for tolerance intervals of the distribution of $\mu + A$, model with interaction, all parameter settings. 54

3.4 Empirical coverage probabilities for tolerance intervals of the distribution of $\mu + A$, model with interaction, over subsets of parameter values. 55

3.5 Empirical coverage probabilities for tolerance intervals of the distribution of Y and for tolerance intervals of the distribution of $\mu + A$, model without interaction, all parameter combinations. 57

3.6 Empirical coverage probabilities for tolerance intervals of the distribution of Y, over parameter combinations with $b = 5$, $\sigma_A^2 = 10$, and $\sigma_B^2 \in \{0, 0.1, 0.25\}$. . 58

LIST OF TABLES

3.1 Monte Carlo Estimates of the Coverage Probability and Expected Value of the Absolute Distance of the $(p, 1-\alpha)$ Tolerance Bound from the True Value, $Q = \mu + z_p\sigma_A^2$, for the Random Variable $\mu + A$. Parameter values: $\mu = 0$, $p = .9$ and $\rho = \sigma_A^2/(\sigma_A^2 + \sigma_e^2)$ 38

4.1 Empirical Coverage Probabilities and Estimated Expected Values of Lower Confidence Bounds for Proportion Conformance Based on Two Different Methods – GCI and L2. 69

4.2 Empirical Coverages and "Expected Absolute Differences" of Generalized Lower Confidence Bound for Proportion of Conformance: Percentile Pairs $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$ 69

4.3 Empirical Coverages and "Expected Absolute Differences" of Generalized Lower Confidence Bound for Proportion of Conformance: Other Percentile Pairs . 70

7.1 Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 2$ populations & smaller sample sizes, $\alpha = .05$ 106

7.2 Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 2$ populations & larger sample sizes, $\alpha = .05$ 107

7.3 Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 3$ populations & smaller sample sizes, $\alpha = .05$ 108

7.4 Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 3$ populations & larger sample sizes, $\alpha = .05$ 109

7.5 Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 10$ populations, $\alpha = .05$ 110

A.1 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$. 120

A.2 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$. 121

A.3 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 10$. 122

A.4 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 2$. 123

A.5 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 5$. 124

A.6 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 10$. 125

A.7 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 2$. 126
A.8 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 5$. 127
A.9 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 10$. 128
A.10 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 2$. 129
A.11 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 5$. 130
A.12 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 10$. 131
A.13 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$. 132
A.14 Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$. 133

A.15 Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 10$. 134
A.16 Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$. . . 135
A.17 Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$. . . 136
A.18 Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 10$. . 137

B.1 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$. 139
B.2 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for asymptotics and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$. 142
B.3 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs. 143
B.4 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for Asymptotics and Other Percentile Pairs. 154
B.5 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$. 157
B.6 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$. 160

B.7 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs. 161
B.8 Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and Other Percentile Pairs. 173

Chapter 1

INTRODUCTION

We are interested in methods for constructing confidence intervals for population characteristics in the context of mixed linear models. That is, we have an observable random vector $\mathbb{S} \in \mathbb{R}^k$ whose distribution arises from a mixed linear model and is indexed by a (possibly vector) parameter $\xi \in \mathbb{R}^p$. We are interested in making inference about the population characteristic $\theta = \pi(\xi)$. In some simple applications exact confidence intervals for θ exist. In most general situations only approximate confidence intervals are available. Some general methods for constructing confidence intervals are the pivotal method, inversion of generalized likelihood ratio tests, and Bayesian interval estimators. These methods are discussed in most books on mathematical statistics. See for instance [42]. Some less commonly used methods are the fiducial method of R. A. Fisher [15] and structural inference of Fraser [17] [18]. The last two methods have some similarities to Baysian interval estimators in that they assign a distribution to θ and generate confidence intervals using the quantiles of the distribution.

We illustrate the *fiducial argument* using a simple example. Let $X_1, \ldots, X_n$ be iid with $X_i \sim N(0, \sigma^2)$ and let $SSE = \sum_{i=1}^n (X_i - \bar{X})^2$, where $\bar{X} = \sum_{i=1}^n X_i/n$. Then $\frac{SSE}{\sigma^2} = E \sim \chi^2_{n-1}$. If sse is a realization of SSE, and e is a realization of E, then sse and e are related by the algebraic relation $\frac{sse}{\sigma^2} = e$. Suppose we observe that the value of sse is 1. Then we can measure the "likelihood" of values of σ^2 by the corresponding values of e. For instance, we would say $\sigma^2 \geq 10$ is highly unlikely, since $e \leq 0.1$ is a highly unlikely event. The fiducial argument is to say the probability density function (pdf) of σ^2 is the same as the pdf by $\frac{sse}{E}$. We stress that sse is observed data and E is regarded as having a χ^2_{n-1} distribution independent of the realized value of sse. Confidence intervals for σ^2 are based on the distribution of $\frac{sse}{E}$, which is referred to as the fiducial distribution of σ^2.

For example a lower 95% confidence bound for σ^2 is given by $\frac{sse}{\chi^2_{.95,n-1}}$, where $\chi^2_{.95,n-1}$ is the 0.95^{th} quantile of a chi-squared distribution with $n-1$ degrees of freedom. The reader may note that $\frac{SSE}{\sigma^2}$ is a pivotal quantity; the lower bounds derived from $\frac{SSE}{\sigma^2}$ are the same, in this illustration, as those obtained from the fiducial distribution.

In 1993, Weerahandi [55] generalized the concept of a pivotal quantity by defining a Generalized Pivotal Quantity (GPQ). He then proposed a method for constructing confidence intervals based on GPQs. The quantiles of the distribution of the GPQ, conditional on the data, define the confidence bounds. We will refer to a confidence interval generated using a GPQ as a generalized confidence interval (GCI). In 2002, Iyer and Patterson [29] presented a recipe for constructing GPQs based on Fraser's ideas of structural representations. They also provided certain clarifications of the original definition of GPQs that facilitates a frequentist framework for addressing the properties of GPQs. This framework was further refined in Hannig et al. [25] who also showed that a strong connection exists between Fisher's fiducial argument, structural inference, and a subclass of GPQs. They label the GPQs in this subclass as Fiducial Generalized Pivotal Quantities (FGPQs). A confidence interval derived from a FGPQ is referred to as a fiducial generalized confidence interval (FGCI). In most cases FGCIs are approximate confidence intervals. Hannig et al. gave sufficient conditions for FGCIs to have asymptotically exact frequentist coverage.

In Chapter 2 we present the definitions of generalized pivotal quantities and fiducial generalized pivotal quantities. Chapter 2 also contains a presentation of the structural method for constructing generalized pivotal quantities. For completeness we state the convergence theorem for fiducial generalized confidence intervals from Hannig et al.. When θ is a scalar, we provide necessary and sufficient conditions for exact frequentist coverage of fiducial generalized confidence intervals. Examples are also presented to illustrate the above definitions, methods and theorems.

The structural method for constructing FGPQs is well suited for situations where $\mathbb{S}$ is a complete and sufficient statistic. However, the structural method does apply in more general situations as well. In Chapter 6 we present a two stage method for constructing a FGPQ in certain situations where $\mathbb{S}$ is an incomplete minimal sufficient statistic.

In cases where $\mathbb{S}$ is an incomplete minimal sufficient statistic the structural method does not use all available information to produce a FGPQ. Hannig et al. [25] presented a general structural method for constructing FGPQs that is applicable in very general situations and uses all available information by using a conditional approach. The general structural method does not give unique results. Hannig et al. [25] show that this situation is unavoidable. The general structural method is presented in Chapter 6 along with an example that illustrates the non-uniqueness of the FGPQs.

There are a number of applications of FGPQs that are of interest. Consider the factorial linear mixed model, $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{ZA} + \boldsymbol{e}$, where $\boldsymbol{X}$ and $\boldsymbol{Z}$ are known matrices, $\boldsymbol{\beta}$ is a vector of unknown parameters. Assume $\boldsymbol{e} \sim MVN(\boldsymbol{0}, \phi_e \boldsymbol{I}_N)$, $\boldsymbol{A} \sim MVN(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\Sigma = \text{Diag}(\phi_1 \mathbf{I}_{a_1}, \ldots, \phi_q \mathbf{I}_{a_q})$, with $\phi_1 \geq 0$ for $i = 1, \ldots, q$, and $\boldsymbol{e}$ and $\boldsymbol{A}$ are independent. In Chapter 2, for a saturated model with balanced data, we derive FGPQs for variance components and for linear functions of the fixed effects in the model. We also show that FGCIs for the variance components have asymptotically exact coverage probability. For a saturated model with unbalanced data we propose a method based on data imputation to produce approximate FGPQs for the variance components and linear functions of the variance components. In Chapter 3 we use the imputation method to construct tolerance intervals for the unbalanced one-way random effects model.

The rest of the dissertation is comprised of three applications – tolerance intervals, proportions of conformance, and the common mean problem. Tolerance intervals and proportion of conformance are closely related to each other and are discussed in the next section. The common mean problem is discussed in a later section.

1.1 Introduction to Tolerance Intervals and Proportion of Conformance

At the end of second subsection we will discuss the relationship between tolerance intervals and proportion of conformance.

1.1.1 Tolerance Intervals

Let W be a random variable with cumulative distribution function $F_W(w)$. A two-sided (p, γ) tolerance interval for W is an interval (L, U) such that $P[F_W(U) - F_W(L) \geq p] = \gamma$, where $0 < p, \gamma < 1$ and L and U are statistics. In other words, with probability equal to γ at least p% of the distribution of the realizations of W will lie between L and U. Generally L and U are referred to as *lower* and *upper tolerance bounds* respectively, p is referred to as the *content* and γ as the *coverage* or *confidence* (see for instances [21]). Tolerance bounds are also referred to as tolerance limits.

The definition of a two-sided tolerance interval has the deficiency that it does not specify the size of the tails of the distribution that are not covered by the tolerance interval. An alternative is to use one-sided tolerance bounds. A *lower* (p, γ) tolerance bound for a random variable W is a statistic L such that $P[1 - F_W(L) \geq p] = \gamma$, where, as above, p is referred to as the content and γ is referred to as the confidence. L is referred to as a lower (p, γ) tolerance bound. Observe that a lower (p, γ) tolerance bound is the same as a lower $100\gamma\%$ confidence bound for the $(1-p)^{th}$ quantile of W. Similarly, an upper (p, γ) tolerance bound is a statistic U such that $P[F_W(U) \geq p] = \gamma$ and an upper (p, γ) tolerance bound is the same an upper $100\gamma\%$ confidence bound for the p^{th} quantile of W.

In Chapter 3 we will use FGPQs to construct tolerance bounds for distributions of certain linear combinations of normal random variables associated with the linear mixed model. If $W \sim N(\mu, \sigma_W^2)$, then the $(1-p)^{th}$ quantile of W is $\mu - z_p\sigma_W$ and a lower (p, γ) tolerance bound for W is a statistic L such that

$$P[L \leq \mu - z_p\sigma_W] = \gamma, \tag{1.1}$$

where z_p is the p^{th} quantile of the standard normal distribution.

For a normal random variable with unknown mean and unknown standard deviations Chou and Owen [12] and Papp [44] have proposed methods for two-sided tolerance intervals. Brown et al. [5] proposed tolerance intervals for assessing individual bioequivalence for several higher-order cross-over designs Recently, Liao and Iyer, [36], proposed a two-sided tolerance interval for a normal distribution with several variance components; the interval is based on fiducial generalized pivotal quantities for the parameters of the model.

In Chapter 3 we propose a FGPQ-based method for constructing one-sided tolerance intervals for the distributions of two population characteristics of the two-way random effects model with replication. Also we propose a similar method for constructing one-sided tolerance intervals for the two-way random effects model without replication. A simulation study is conducted to assess the performance of the proposed methods.

1.1.2 Proportion of Conformance

Let W be a random variable and let θ denote the proportion of realizations of W that is contained in the interval (C_1, C_2), i.e., $\theta = P[C_1 < W < C_2]$ where $C_1 < C_2$ are real numbers (C_1 may be negative infinity and C_2 may be positive infinity). The quantities C_1 and C_2 are often referred to as specification limits. If C_1 and C_2 are finite then lower confidence bounds for θ are of interest. If one of C_1 or C_2 is infinite then either lower or upper confidence bounds for θ are of interest.

To illustrate this, consider the following examples. In the manufacturing of machine parts, a *part* is said to meet specifications provided its performance characteristic is contained in a pre-specified interval (C_1, C_2) determined by engineering requirements. One would like to be assured that θ is large, so a lower confidence bound for θ would be of interest. In environmental applications it is often of interest to ensure that pollutant levels in soil, water, or air are below an amount determined to be safe by the US Environmental Protection Agency. Here θ may represent, for example, the proportion of all possible water samples in which the concentration of arsenic exceeds 5 $\mu g/L$. One would like to be assured that this proportion is small and so an upper confidence bound for θ would be of interest. Note that, equivalently, one may be interested in a lower confidence bound for $1-\theta$, the proportion of all water samples in which the concentration is less than 5 $\mu g/L$.

For C_1 and C_2 finite, the relative sizes of the tails is unknown. To address this one can look at confidence bounds of the proportion of conformance for $(-\infty, C_2)$ and (C_1, ∞).

In Chapters 4 and 5 we propose confidence intervals for the proportion of conformance under different model assumptions. Chapter 4 considers the case where $W \sim N(\mu, \sigma^2)$ and Chapter 5 considers the case where the variance of W is the sum of two variance components and sample observations follow a one-way random model.

There is a duality between tolerance intervals and conformance intervals. For tolerance intervals, the percentage of the distribution is fixed while the location of the associated quantiles are estimated using confidence bounds, whereas for proportion of conformance the "quantile" locations are fixed and the associated probability content is estimated using confidence bounds. Two-sided tolerance intervals and the proportion of conformance for C_1 and C_2 finite have the limitation that the relative sizes of the left and right tails of the distribution are unknown.

1.2 The Common Mean Problem

Suppose $X_{i1}, \ldots, X_{in_i}$ are iid with $X_{ij} \sim N(\mu, \sigma_i^2)$, for $i = 1, \ldots, k$. A commonly occurring situation in the field of metrology is to construct a confidence interval for the common mean μ of these k populations. For example, suppose k different laboratories are asked to measure a certain physical or chemical property associated with a certain artefact. Let the *true* value of the property of interest be μ. The i^{th} laboratory reports the results of n_i independent measurements. Then the goal is not only to combine the information from all the laboratories to obtain an efficient point estimate of μ, but also to provide an interval estimate for μ.

The difficulty is that for each of the k populations there is a complete and sufficient statistic for parameter (μ, σ_i^2). The question is how to combine the statistics in an efficient manner to generate confidence intervals for μ. The problem has been addressed from a frequentist perspective by many authors, see for instance, Jordan and Krishnamoorthy [31], Yu et al. [59], and Krishnamoorthy and Lu [32]. All these frequentist methods are based on the construction of pivotal quantities for μ, which combine information about μ from each of the samples. A limitation of many of these methods is there is a positive probability of obtaining a confidence set for μ that is not an interval. In Chapter 7 we will address the common mean problem from a FGPQ perspective. We will propose two FGPQs. One is based on the two stage method and the other is based on the general structural method from Hannig et al. [25]. We compare the performance of the associated FGCIs with the frequentist methods.

We conclude the dissertation with some closing remarks and identify some future directions.

Chapter 2

FIDUCIAL GENERALIZED CONFIDENCE INTERVALS

Since Weerahandi's introduction of Generalized Pivotal Quantities (GPQs) and the associated Generalized Confidence Intervals (GCIs) in 1993, [55], GCIs have been constructed for many problems. See for instance Weerahandi [56], Chang and Huang [9], Hamada and Weerahandi [22], McNally et al. [39], Burdick and Park [7], Krishnamoorthy and Lu [32], Krishnamoorthy and Mathew [33] [34], Iyer et al. [30], Weerahandi [57], Burdick et al. [6] [8], Daniels et al. [14], and Roy and Matthew [45]. While most of the GCIs are only approximate, simulation studies by the various authors showed the coverage probabilities are close enough to the stated coverages for GCIs to be useful in practical problems. Up until recently there were two aspects of GCIs which had not been addressed, methods for constructing GPQs and the asymptotic behavior of the GCIs. Both of these issues are addressed in this chapter using Fiducial Generalized Pivotal Quantities (FGPQs), which are a special subclass of GPQs.

Tsui and Weerahandi [51] introduced the related concept of generalized P-values and generalized test variables prior to the introduction of generalized pivotal quantities. We will not discuss generalized tests.

The chapter is organized as follows. In the first section we define GPQs and FGPQs, and illustrate the definitions with examples. This is followed by a section in which we reproduce a result of Hannig et al. on the asymptotic convergence of FGPQs. In section three we present the structural method for constructing FGPQs and give several examples illustrating the method. These examples include FGPQs for the population parameters associated with the one-way random effects model and the two-way random effects model. Section four covers constructing FGPQs for parameters of the saturated balanced factorial linear mixed model with replication. We verify that the GCIs based on the FGPQs for the variance components of the saturated balance factorial linear mixed model with replication have asymptotic exact coverage. In Section five we propose a method for generating FGPQs for the factorial linear model with unbalanced data. The penultimate section presents two alternative representations of FGPQs. These representations are useful in the calculating GCIs and in showing invariance properties of FGPQs. In the last section we present necessary and sufficient conditions for the coverage probability of a GCI for a scalar parameter to be exact. We also discuss the relationship between FGPQs and pivotal quantities for scalar parameters.

2.1 Definitions of Generalized Pivotal Quantities and Fiducial Generalized Pivotal Quantities

Iyer and Patterson [29] presented the following definition for a GPQ. The definition is superficially different from Weerahandi's [55] definition but is identical in spirit. The differences between the Weerahandi's definition and our definition will be discussed after Definition 2.1.

Let $\mathbb{S} \in \mathbb{R}^k$ denote an observable random vector whose distribution is indexed by a (possibly vector) parameter $\xi \in \mathbb{R}^p$. Suppose we are interested in making inferences about $\theta = \pi(\xi) \in \mathbb{R}^q$ $(q \geq 1)$. Let $\mathbf{s}$ denote the observed value of $\mathbb{S}$. In subsequent discussions, $\mathbf{s}$ will be regarded as a vector of known constants, and any reference to the distribution of $\mathbb{S}$ will be to its unconditional distribution, ignoring the fact that $\mathbf{s}$, a realization of $\mathbb{S}$ is available.

Definition: Let $R = R(\mathbb{S}; \mathbf{s}, \xi)$ be a function of $\mathbb{S}$. If R has the following two properties, then it is said to be a generalized pivotal quantity:

Property A: R has a probability distribution that is free of nuisance parameters.

Property B: The observed pivotal, $R(\mathbf{s}; \mathbf{s}, \xi)$, depends on ξ only through θ.

In the Hannig et al. the the definition was further refined to explicitly state the independence of $\mathbf{s}$ and $\mathbb{S}$. We now present the Hannig et al. version of the definition.

Let $\mathbb{S}^\star$ represent an independent copy of $\mathbb{S}$. We will use $\mathbf{s}^\star$ to denote a realized value of $\mathbb{S}^\star$.

Definition 2.1. A *generalized pivotal quantity* for θ, denoted by $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ (or simply $\mathcal{R}_\theta$ or $\mathcal{R}$, when there is no ambiguity) is a function of $(\mathbb{S}, \mathbb{S}^\star, \xi)$ with the following properties.

(GPQ1) The conditional distribution of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$, conditional on $\mathbb{S} = \mathbf{s}$, is free of ξ.

(GPQ2) For every allowable $\mathbf{s} \in \mathbb{R}^k$, $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi)$ depends on ξ only through θ.

The above form of the definition allows one to more clearly explain the connection between ordinary pivotal quantities and generalized pivotal quantities and also facilitates a mathematically rigorous discussion of the asymptotic behavior of the resulting generalized confidence intervals. We will use Definition 2.1.

The differences between Weerahandi's definition given in [55] and the one we give, are first the use of $\mathbb{S}^\star$ as the second argument in $\mathcal{R}$, making explicit that the second argument is an independent copy of $\mathbb{S}$. The second is, although Weerahandi does not explicitly state it, there is an underlying assumption that θ is a scalar parameter, whereas we allow θ to be a vector which is a function of the population parameters.

Note that the second condition uses $\mathbf{s}$ in both the first and the second argument positions of $\mathcal{R}_\theta(\cdot\,,\cdot\,,\xi)$. This is an important aspect of the definition of generalized pivotal quantity and explains both its similarity to, and its difference from, an ordinary pivotal quantity. If, in condition (GPQ2), we instead had the requirement $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)$ depends on ξ only through θ, then conditional on $\mathbb{S}$, $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ would indeed be an ordinary pivotal quantity based on $\mathbb{S}^\star$. If in addition to requiring $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)$ depend on ξ only through θ, we had instead of condition (GPQ1) the requirement that the distribution of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ be free of ξ, then $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ would indeed be an ordinary pivotal quantity based on $(\mathbb{S}, \mathbb{S}^\star)$.

We do not intend to actually observe a realization of $\mathbb{S}^\star$ and therefore are not seeking a pivotal quantity for θ based on both $\mathbb{S}$ and $\mathbb{S}^\star$. We will observe a realization of $\mathbb{S}$. Heuristic reasoning suggests that, under appropriate conditions $\mathbb{S}$ and $\mathbb{S}^\star$ will be sufficiently close to each other, and hence we can substitute $\mathbb{S}$ in both argument positions in $\mathcal{R}_\theta(\cdot\,,\cdot\,,\xi)$, and use the distribution from condition (GPQ1) to make approximate confidence statements about the quantity $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi)$ and hence about θ. To further illustrate this point, suppose θ is a scalar parameter and we are interested in a $100(1-\alpha)\%$ confidence set for θ. We observe a realization $\mathbf{s}$ of $\mathbb{S}$. By (GCQ1) the probability statement $P_{\mathbb{S}^\star}[\mathcal{R}_{\theta,\alpha}(\mathbf{s}) \leq \mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)] = 1 - \alpha$, where $\mathcal{R}_{\theta,\alpha}$ is the α^{th} quantile of the distribution of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$ is independent of ξ. If we then observed an independent realization $\mathbf{s}^\star$ of $\mathbb{S}^\star$ we have the confidence statement $C[\mathcal{R}_{\theta,\alpha}(\mathbf{s}) \leq \mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)] = 1 - \alpha$. Under appropriate conditions we expect $\mathbf{s}$ to be close to $\mathbf{s}^\star$, so we can substitute $\mathbf{s}$ for $\mathbf{s}^\star$ and obtain the approximate confidence statement $C[\mathcal{R}_{\theta,\alpha}(\mathbf{s}) \leq \mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi)] = 1 - \alpha$. By (GPQ2) there exists a function $f(\theta, \mathbf{s})$ which is equal to $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi)$. Hence we can solve the confidence statement for θ to obtain an approximate $100(1-\alpha)\%$ confidence set. Theorem 2.1 confirms that our heuristic reasoning is indeed valid. For later reference we give a more general statement of a generalized confidence set for a scalar parameter.

Remark 2.1. Property GPQ2 implies that $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi) = f(\mathbf{s}, \theta)$ for some function f. If θ is a scalar parameter and $\mathbf{s}$ is an observed realization of $\mathbb{S}$ then a $100(1-\alpha)\%$ generalized confidence set for θ is given by the $\{\theta : \mathcal{R}_{\theta,\alpha_l} \leq f(\mathbf{s}, \theta) \leq \mathcal{R}_{\theta,1-\alpha_r}\}$, where $\mathcal{R}_{\theta,\gamma} = \mathcal{R}_{\theta,\gamma}(\mathbf{s})$ denotes the $100\gamma^{th}$ percentile of distribution of $\mathcal{R}_\theta$ conditional on $\mathbb{S} = \mathbf{s}$ and $\alpha_l + \alpha_r = \alpha$ (note that one of α_l or α_r can be zero).

It turns out that the subclass of GPQs for which $f(\mathbf{s}, \theta)$ is a function of θ only, say $f(\mathbf{s}, \theta) = f(\theta)$, has a special connection with fiducial inference. Generalized confidence regions obtained using such GPQs are not guaranteed to be intervals unless the function $f(\theta)$ is invertible. In this case, one may assume $f(\theta)$ to be identically equal to θ without loss in generality. Such GPQs exist in practically every application we have considered. This leads us to single out the following subclass of GPQs.

Definition 2.2. A GPQ $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ for a parameter θ is called a *Fiducial Generalized Pivotal Quantity* (FGPQ) if it satisfies the following two conditions.

(FGPQ1) The conditional distribution of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$, conditional on $\mathbb{S} = \mathbf{s}$, is free of ξ.

(FGPQ2) For every allowable $\mathbf{s} \in \mathbb{R}^k$, $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi) = \theta$.

Remark 2.2. Condition (FGPQ1) of Definition 2.2 is the same as condition (GPQ1) of Definition 2.1 but condition (FGPQ2) is a stronger version of condition (GPQ2) in the definition of a GPQ.

Example 2.1. (*The Behrens-Fisher problem*)

Consider m iid observations $X_i, i = 1, \ldots, m$ from $N(\mu_X, \sigma_X^2)$ and n iid observations $Y_j, j = 1, \ldots, n$ from $N(\mu_Y, \sigma_Y^2)$ where $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ are unknown parameters. The problem is to obtain confidence bounds for the difference $\theta = \mu_X - \mu_Y$. Let $\bar{X}$, $\bar{Y}$ denote the sample means and S_X^2, S_Y^2 denote the sample variances for the two samples. Then $\bar{X} \sim N(\mu_X, \sigma_X^2/m)$, $\bar{Y} \sim N(\mu_Y, \sigma_Y^2/n)$, $(m-1)S_X^2/\sigma_X^2 \sim \chi^2(m-1)$, and $(n-1)S_Y^2/\sigma_Y^2 \sim \chi^2(n-1)$. The statistic $\mathbb{S} = (\bar{X}, \bar{Y}, S_X^2, S_Y^2)$ is complete and sufficient for $\xi = (\mu_X, \mu_Y, \sigma_X, \sigma_Y)$.

A nontrivial exact confidence interval, in the frequentist sense, is unavailable for this problem. See, for instance, Linnik [38]. A solution to this problem was put forward by Behrens [2] and later, Fisher [15] showed this solution could be derived very simply using the fiducial argument. The fiducial distribution of $\mu_X - \mu_Y$ is referred to as the Behrens-Fisher distribution. Critical values for this distribution have been tabulated by Sukhathme [49]. The quantiles of this distribution lead to lower and upper confidence bounds for $\mu_X - \mu_Y$.

Many other approximate confidence interval procedures for this problem have been discussed in the literature – see, for instance, Welch [58], Satterthwaite [46] [47], Cochran [13], and Graybill and Wang [20]. Weerahandi [55] derived a generalized confidence interval for $\mu_X - \mu_Y$ and noted that it is the same as the Behrens-Fisher solution. He derived the GCI for $\theta = \mu_X - \mu_Y$ by starting with the statistics $\bar{X} - \bar{Y}$, S_X^2, and S_Y^2. Weerahandi justified this based on invariance arguments. The GPQ proposed by Weerahandi is

$$\mathcal{R} = \mathcal{R}(\mathbb{S}, \mathbb{S}^\star, \xi) = (\bar{X}^\star - \bar{Y}^\star - \theta)\left(\frac{\sigma_X^2 S_X^2/(mS_X^{\star 2}) + \sigma_Y^2 S_Y^2/(nS_Y^{\star 2})}{\sigma_X^2/m + \sigma_Y^2/n}\right)^{1/2}$$

where $(\bar{X}^\star, \bar{Y}^\star, S_X^{\star 2}, S_Y^{\star 2})$ is an independent copy of $(\bar{X}, \bar{Y}, S_X^2, S_Y^2)$. Although $\mathcal{R}(\mathbb{S}, \mathbb{S}^\star, \xi)$ is not itself a FGPQ for θ, $\mathcal{R}_\star(\mathbb{S}, \mathbb{S}^\star, \xi) = (\bar{X} - \bar{Y}) - \mathcal{R}(\mathbb{S}, \mathbb{S}^\star, \xi)$ is a FGPQ for θ. Moreover, it leads to the same confidence interval for θ as does the FGPQ $\mathcal{R}_\theta$ defined by

$$\mathcal{R}_\theta = (\bar{X} - \bar{Y}) - \left[(\bar{X}^\star - \mu_X)\sqrt{\frac{S_X^2}{S_X^{\star 2}}} - (\bar{Y}^\star - \mu_Y)\sqrt{\frac{S_Y^2}{S_Y^{\star 2}}}\right]. \tag{2.1}$$

This latter FGPQ will be derived by a direct application of Theorem 2.2.

Example 2.2. (*Balanced One-way Random Effects Model*)

Consider the one-way random effects model

$$Y_{ij} = \mu + A_i + e_{ij}, \qquad i = 1, \ldots, a; \;\; j = 1, \ldots, n$$

where $A_i \sim N(0, \sigma_A^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$, and $\{A_i\}$ and $\{e_{ij}\}$ are all mutually independent. Let $\mathbb{S} = (\bar{Y}_{..}, S_B^2, S_W^2)$ where

$$S_B^2 = \frac{n \sum_{i=1}^a (\bar{Y}_{i.} - \bar{Y}_{..})^2}{a-1}, \qquad S_W^2 = \frac{\sum_{i=1}^a \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i.})^2}{a(n-1)},$$

are, respectively, the *between groups* and *within groups* mean squares, and

$$\bar{Y}_{i.} = \frac{\sum_{j=1}^n Y_{ij}}{n}, \quad \text{and} \quad \bar{Y}_{..} = \frac{\sum_{i=1}^a \sum_{j=1}^n Y_{ij}}{an}.$$

The statistic $\mathbb{S}$ is complete and sufficient for $\xi = (\mu, \sigma_A^2, \sigma_e^2)$. Suppose one is interested in a confidence interval for $\theta = \sigma_A^2$. Weerahandi [55] derived the FGPQ $\mathcal{R}_{\sigma_A^2}(\mathbb{S}, \mathbb{S}^\star, \xi) = \dfrac{S_B^2}{n {S_B^\star}^2} (\sigma_e^2 + n\sigma_A^2) - \dfrac{S_W^2}{n {S_W^\star}^2} \sigma_e^2$, by inverting a generalized test. A one-sided generalized $100\gamma\%$ confidence interval for σ_A is given by $\sigma_A^2 \leq \mathcal{R}_\gamma$, where $\mathcal{R}_\gamma$ is the $\gamma 100$-percentile of the conditional distribution of $\mathcal{R}(\mathbb{S}, \mathbb{S}^\star, \xi)$, conditional on $\mathbb{S} = \mathbf{s}$. The coverage probability, $P_{\mathbb{S}}[\sigma_A^2 \leq \mathcal{R}_\gamma]$, is only approximate.

Another FGPQ, $\max(0, \mathcal{R})$, is guaranteed to take on only nonnegative values. Asymptotically, this modified FGPQ is equivalent to $\mathcal{R}$ as long as σ_A^2, σ_e^2 are nonzero. In later examples we will see that $\max(0, \mathcal{R})$ is useful when deriving a FGPQ for σ_A.

2.2 Asymptotic Exact Coverage for FGCIs

As mentioned earlier, the literature abounds with examples of generalized confidence intervals, but until recently no theoretical results were available that guaranteed satisfactory performance of these intervals. Only evidence of their acceptability for practical use is through simulation studies which suggest that almost all reported generalized confidence interval methods appear to have coverage probabilities close to their stated values.

Next we state a theorem from Hannig et al. [25] that appears to be the first result that guarantees, under some fairly mild conditions, the asymptotic correctness of the coverage probability of a generalized confidence interval.

Let us consider a parametric statistical problem where we observe $X_1, \ldots, X_n$ whose joint distribution belongs to some family of distributions parameterized by $\xi \in \Xi \subset \mathbb{R}^p$. Let $\mathbb{S} = (S_1, \ldots, S_k)$ denote a statistic based on the X_i's. In theory we can consider an independent copy of $X_1^\star, \ldots, X_n^\star$ and denote the statistic based on $X_i^\star$'s by $\mathbb{S}^\star$. Finally, suppose a function $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ is available that is a FGPQ for a scalar parameter $\theta = \pi(\xi)$. We will consider some notation and assumptions first.

Assumptions 2.1. 1. Assume that there exists $\boldsymbol{t}(\xi) \in \mathbb{R}^k$ such that

$$\sqrt{n}\,(S_1^\star - t_1(\xi), \ldots, S_k^\star - t_k(\xi)) \xrightarrow{\mathcal{D}} \boldsymbol{N} = (N_1, \ldots, N_k),$$

where $\boldsymbol{N}$ has a non-degenerate multivariate normal distribution.

2. Assuming existence and continuity of second partial derivatives with respect to $\mathbf{s}^\star$ of $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)$ we have the following one-term Taylor expansion around $\boldsymbol{t}(\xi)$ with a remainder term:

$$\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi) = g_{0,n}(\mathbf{s}, \xi) + \sum_{j=1}^{k} g_{1,j,n}(\mathbf{s}, \xi)\left(S_j^\star - t_j(\xi)\right) + R_n(\mathbf{s}, \mathbb{S}^\star, \xi).$$

Here

$$g_{0,n}(\mathbf{s}, \xi) = \mathcal{R}_\theta(\mathbf{s}, \boldsymbol{t}(\xi), \xi),$$
$$g_{1,j,n}(\mathbf{s}, \xi) = \frac{\partial}{\partial s_j^\star}\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)\Big|_{\mathbf{s}^\star = \mathbf{t}(\xi)}, \text{ and}$$
$$R_n(\mathbf{s}, \mathbf{s}^\star, \xi) = \sum_{i=1}^{k}\sum_{j=1}^{k}(\mathbf{s}_i^\star - t_i(\xi))(\mathbf{s}_j^\star - t_j(\xi))\frac{1}{2}\frac{\partial^2}{\partial s_i^\star s_j^\star}\mathcal{R}_\theta(\mathbf{s}, \tilde{\mathbf{s}}, \xi),$$

where $\tilde{\mathbf{s}}$ lies on the line segment connecting $\boldsymbol{t}(\xi)$ and $\mathbf{s}^\star$. Suppose $\mathcal{A} \subset \mathbb{R}^k$ is an open set containing $\boldsymbol{t}(\xi)$ with the following properties:

(a) The functions $g_{1,j,n}(\mathbf{s}, \xi)$ converge uniformly in $\mathbf{s} \in \mathcal{A}$ to a function $g_{1,j}(\mathbf{s}, \xi)$ continuous at $\mathbf{s} = \boldsymbol{t}(\xi)$.

(b) For all $\mathbf{s} \in \mathcal{A}$, $\sum_{j=1}^{k} |g_{1,j}(\mathbf{s}, \xi)| > 0$.

(c) There is $M < \infty$ such that

$$\left|\frac{\partial^2}{\partial s_i^\star s_j^\star}\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)\right| < M$$

for all n and all $(\mathbf{s}, \mathbf{s}^\star) \in \mathcal{A} \times \mathcal{A}$.

Theorem 2.1. *Suppose* Assumptions 2.1 *hold and that for each fixed* $\mathbf{s}$*,* n*, and* $\gamma \in (0,1)$*, there exists a real number* $C_n(\mathbf{s},\gamma)$ *such that*

$$\lim_{n\to\infty} P_\xi\left(\mathcal{R}_\theta(\mathbf{s},\mathbb{S}^\star,\xi) \le C_n(\mathbf{s},\gamma)\right) = \gamma.$$

Then

$$\lim_{n\to\infty} P_\xi\left(\mathcal{R}_\theta(\mathbb{S},\mathbb{S},\xi) \le C_n(\mathbb{S},\gamma)\right) = \gamma.$$

In particular, since $\mathcal{R}_\theta(\mathbb{S},\mathbb{S},\xi) = \theta$*, it follows that the interval* $-\infty < \theta \le C_n(\mathbb{S},\gamma)$ *is a one-sided confidence interval for* θ *with asymptotic coverage probability equal to* γ*.*

For proof see Hannig et al. [25].

Example 2.3. (*Verification of Assumptions of Theorem 2.1*)

For Examples 2.1 and 2.2, the Behrens-Fisher problem and the balanced one-way random model, one may show that the GPQs in these two examples satisfy the assumptions of Theorem 2.1. Hence the resulting GCIs will have asymptotically correct coverage probabilities. This is well known for the Behrens-Fisher problem and may be directly verified for the one-way random model. However, our purpose here is to illustrate the application of Theorem 2.1 by considering some familiar examples. In later chapters we will discuss some less familiar examples that satisfy the conditions of Theorem 2.1. We will illustrate by checking the conditions for the Behrens-Fisher problem. For the one-way random model see Hannig et al. [25].

Proposition 2.1. *For the Behrens-Fisher problem, a* $100(1-\alpha)\%$ *GCI for* $\mu_X - \mu_Y$ *based on the FGPQ defined in Equation 2.1 has asymptotically* $100(1-\alpha)\%$ *frequentist coverage as* $n \to \infty$*.*

Proof. By standard results for linear models and Serfling (Theorem 3.3.A [48])

$$\sqrt{n}\left(\bar{X}^\star - \mu_X, \bar{Y}^\star - \mu_Y, {S_X^\star}^{-1} - \sigma_X^{-1}, {S_Y^\star}^{-1} - \sigma_Y^{-1}\right) \xrightarrow{\mathcal{D}} \boldsymbol{N},$$

where $\boldsymbol{N}$ is a non-degenerate multivariate normal random variable. Thus $\boldsymbol{t}(\xi) = (\mu_X, \mu_Y, \sigma_X^{-1}, \sigma_X^{-1})$ satisfies condition 1 of Assumptions 2.1, where $\mathcal{R}_\theta$ is a function of $\mathbb{S} = (\bar{X}, \bar{Y}, {S_X}^{-1}, {S_Y}^{-1})$ and $\mathbb{S}^\star$. Note that

$$R_\theta(\mathbf{s},\mathbf{s}^\star,\xi) = (\bar{x} - \bar{y}) - \left[(\bar{x}^\star - \mu_X)s_X {s_X^\star}^{-1} - (\bar{y}^\star - \mu_Y)s_Y {s_Y^\star}^{-1}\right]$$

does not depend on n, so any conditions involving n do not need to be checked. Let $\mathcal{O} = \mathbb{R}^2 \times \mathbb{R}^{+2}$. Over $\mathcal{O} \times \mathcal{O}$, $\mathcal{R}(\cdot, \cdot, \xi)$ has continuous second order partial derivatives with respect to $\mathbf{s}^\star$. Hence on any bound neighborhood conditions 2a and 2c are satisfied. Note that $g_{1,1}(\mathbf{s}, \xi) = -s_X \sigma_X^{-1}$ is non-zero over $\mathcal{O}$ so condition 2b is satisfied. $\square$

The next example illustrates that the conditions of Theorem 2.1 are only sufficient.

Example 2.4. (An situation with two FGPQs which have different asymptotic properties)

Let $X_{i1}, \ldots, X_{in}$ be an independent i.i.d sample of the i^{th} population of k populations, where $X_{ik} \sim N(\mu_i, \sigma^2)$. Confidence intervals for $\theta = \sum_{i=1}^k \mu_i^2$ are of interest. Let $\bar{X}_i$ and S_i^2 denote the mean and the variance of the i^{th} sample and let $S^2 = (\sum_{i=1}^k S_i^2)/k$ denote the pooled estimator of σ^2. For the statistic $\mathbb{S} = (\bar{X}_i, \ldots, \bar{X}_k, S_\nu)$ and $\xi = (\mu_1, \ldots, \mu_k, \sigma^2)$ Daniels et al.[14] gave the following FGPQ for θ

$$\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \sum_{i=1}^k \left(\bar{X}_i + \frac{S}{S^\star}(\bar{X}_i^\star - \mu_i) \right)^2 ,$$

where $\mathbb{S}^\star$ is an independent copy of $\mathbb{S}$. Fix $\theta = 0$ and let $\mathbf{s}$ be a realization of $\mathbb{S}$, then $\mathcal{R}(\mathbf{s}, \mathbb{S}^\star, \xi) > 0$ almost everywhere, so $\mathcal{R}_{\theta,\gamma}(\mathbf{s}) > \theta = 0$ for all γ. Hence no generalized lower confidence interval for $\theta = 0$ can have asymptotically correct coverage. Hannig et al. [25] show that $\mathcal{R}_\theta$ satisfies the conditions of Theorem 2.1 if and only if $\theta > 0$ (for $\theta = 0$ item 2b of Assumptions 2.1 is violated). Hannig et al. define an alternative FGPQ $\tilde{\mathcal{R}}_\theta$ for θ using the statistic $\tilde{\mathbb{S}} = (S_H^2, S^2)$ for the parameter $\tilde{\xi} = (\theta, \sigma^2)$, where $S_H^2 = \sum_{i=1}^k \bar{X}_i^2$. Note $\tilde{\mathbb{S}}$ is an insufficient statistic for the model. Hannig et al. show that although $\tilde{\mathcal{R}}_\theta$ fails to satisfy the conditions of Theorem 2.1 for $\theta = 0$, $\tilde{\mathcal{R}}_\theta$ has asymptotic exact coverage for $\theta = 0$.

2.3 Structural Method for Constructing FGPQs

As mentioned earlier, during the past few years, the idea of GCIs and generalized tests have been used by many authors to obtain useful inference procedures in nonstandard problems. Although Weerahandi [55] provided a few examples illustrating the application of GPQs, he did not provide a systematic approach for finding them. As a matter of fact, Weerahandi [55] says, "The problem of finding an appropriate pivotal quantity is a nontrivial task..." and "...Further research is necessary to develop simple methods of constructing generalized pivotals for classes of general problems, and this is beyond the scope of this article."

Until recently, a general method for constructing GPQs was not available in the literature and each particular problem appeared to require some ingenuity in constructing an appropriate GPQ or a test variable. Chiang [10] proposed the method of *surrogate variables* for deriving approximate confidence intervals for functions of variance components in balanced mixed linear models. Iyer and Mathew [28] pointed out that his intervals are identical to generalized confidence intervals. Nevertheless, Chiang had indirectly provided a systematic method for computing GCIs for the class of problems he considered. However, he did not extend his method to any other class of problems. He also did not discuss the connection between his method and either Weerahandi's GCIs or Fisher's fiducial intervals.

In an unpublished technical report, Iyer and Patterson [29] formulated a method for constructing GPQs and generalized test variables for a parameter $\theta = \pi(\xi)$ and observed that the method can be applied to the class of problems where there exists a k-dimensional pivotal quantity $\mathbb{S}$ that bears an invertible pivotal relationship with the parameter ξ (see Definition 2.4 below). Nearly every GCI in the published statistical literature may be obtained by the use of this recipe and the construction always yields FGPQs. For instance, Burdick et al. [6] [8] report GCIs for parameters of interest in Gage R&R studies using this recipe. In Theorem 2.2 we give a reformulation of the construction using the notation of this paper.

Since the method underlying Theorem 2.2, as well as its generalizations given in Proposition 6.1 and in Theorem 6.1 are inspired by, and related to, Fraser's [16] [17] [18] development of structural inference, we refer to this method as the *structural method* for constructing FGPQs.

The following definitions will be useful in stating the Theorem and in applying the method given in the Theorem.

Definition 2.3. Let $\mathbb{S} = (S_1, \ldots, S_k) \in \mathcal{S} \subset \mathbb{R}^k$ be a k-dimensional statistic whose distribution depends on a p-dimensional parameter $\xi \in \Xi$. Suppose there exist mappings $f_1, \ldots, f_q$, with $f_j : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}$, such that, if $E_i = f_i(\mathbb{S}; \xi)$, for $i = 1, \ldots, q$, then

$\mathbb{E} = (E_1, \ldots, E_q)$ has a joint distribution that is free of ξ. We say that $\boldsymbol{f}(\mathbb{S}, \xi)$ is a *pivotal quantity* for ξ where $\boldsymbol{f} = (f_1, \ldots, f_q)$.

Definition 2.4. Let $\boldsymbol{f}(\mathbb{S}, \xi)$ be a *pivotal quantity* for ξ as described in Definition 2.3 with $q = p$. For each $\mathrm{s} \in \mathcal{S}$, define $\mathcal{E}(\mathrm{s}) = \boldsymbol{f}(\mathrm{s}, \Xi)$. Suppose the mapping $\boldsymbol{f}(\mathrm{s}, \bullet) : \Xi \to \mathcal{E}(\mathrm{s})$ is invertible for every $\mathrm{s} \in \mathcal{S}$. We then say that $\boldsymbol{f}(\mathbb{S}, \xi)$ is an *invertible pivotal quantity* for ξ. In this case we write $\boldsymbol{g}(\mathrm{s}, \bullet) = (g_1(\mathrm{s}, \bullet), \ldots, g_p(\mathrm{s}, \bullet))$ for the inverse mapping so that whenever $\mathbf{e} = \boldsymbol{f}(\mathrm{s}, \xi)$, we have $\boldsymbol{g}(\mathrm{s}, \mathrm{e}) = \xi$.

The following theorem gives a simple recipe for constructing a FGPQ when an invertible pivotal quantity exits.

Theorem 2.2. *Let $\mathbb{S} = (S_1, \ldots, S_k) \in \mathcal{S} \subset \mathbb{R}^k$ be a k-dimensional statistic whose distribution depends on a k-dimensional parameter $\xi \in \Xi$. Suppose there exist mappings $f_1, \ldots, f_k$, with $f_j : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, such that, $\boldsymbol{f} = (f_1, \ldots, f_k)$ is an invertible pivotal quantity with inverse mapping $\boldsymbol{g}(\mathrm{s}, \bullet)$. Define*

$$\mathcal{R}_\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \pi(g_1(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi)), \ldots, g_k(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))) \tag{2.2}$$

$$= \pi(g_1(\mathbb{S}, \mathbb{E}^\star), \ldots, g_k(\mathbb{S}, \mathbb{E}^\star)) \tag{2.3}$$

where $\mathbb{E}^\star = \boldsymbol{f}(\mathbb{S}^\star, \xi)$ is an independent copy of $\mathbb{E}$. Then $\mathcal{R}_\theta$ is a FGPQ for $\theta = \pi(\xi)$. When θ is a scalar parameter, an equal-tailed 2-sided $100(1-\alpha)\%$ GCI for θ is given by $\mathcal{R}_{\theta,\alpha/2} \leq \theta \leq \mathcal{R}_{\theta,1-\alpha/2}$. Here $\mathcal{R}_{\theta,\gamma} = \mathcal{R}_{\theta,\gamma}(\mathrm{s})$ denotes the $100\gamma^{th}$ percentile of the distribution of $\mathcal{R}_\theta$ conditional on $\mathbb{S} = \mathrm{s}$. One-sided generalized confidence bounds are obtained in an obvious manner.

Proof. Notice that, since the distribution of $\mathbb{E}^\star = \boldsymbol{f}(\mathbb{S}^\star, \xi)$ does not depend on ξ, the distribution of $\mathcal{R}(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathrm{s}$ does not depend on ξ. Additionally, $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}, \xi) = \theta$ by definition of $\boldsymbol{g}$ and $\boldsymbol{f}$. Therefore $\mathcal{R}_\theta$ satisfies the requirements for it to be a FGPQ for θ. $\square$

We will now present several examples that illustrate the application of the Theorem 2.2. The first is the Behrens-Fisher problem. The second and third are the balanced one-way random model and balanced two-way random effects model with interaction.

Example 2.5. (*Continuation of Behrens-Fisher, Example 2.1*)

Continuing with the notation of Example 2.1 note that $\mathbb{S} = (\bar{X}, \bar{Y}, S_X^2, S_Y^2)$ and $\xi = (\mu_X, \mu_Y, \sigma_X, \sigma_Y)$ have an inverse pivotal relationship given by

$$f_1(\mathbb{S}, \xi) = \frac{\sqrt{m}(\bar{X} - \mu_X)}{\sigma_X} = E_1 \sim N(0,1) \qquad f_3(\mathbb{S}, \xi) = \frac{(m-1)S_X^2}{\sigma_X^2} = E_3 \sim \chi^2(m-1)$$
$$f_2(\mathbb{S}, \xi) = \frac{\sqrt{n}(\bar{Y} - \mu_Y)}{\sigma_Y} = E_2 \sim N(0,1) \qquad f_4(\mathbb{S}, \xi) = \frac{(n-1)S_Y^2}{\sigma_Y^2} = E_4 \sim \chi^2(n-1)$$

with inverse

$$g_1(\mathbb{S}, \mathbb{E}) = \bar{X} - \frac{1}{\sqrt{m}} E_1 \sqrt{\frac{(m-1)S_X^2}{E_3}} \qquad g_3(\mathbb{S}, \mathbb{E}) = \frac{(m-1)S_X^2}{E_3}$$
$$g_2(\mathbb{S}, \mathbb{E}) = \bar{Y} - \frac{1}{\sqrt{n}} E_2 \sqrt{\frac{(n-1)S_Y^2}{E_4}} \qquad g_4(\mathbb{S}, \mathbb{E}) = \frac{(n-1)S_Y^2}{E_4}$$

Now by the recipe in Theorem 2.2 a FGPQ for $\pi_X(\xi) = \mu_X$ is given by

$$\mathcal{R}_{\mu_X} = \mathcal{R}_{\mu_X}(\mathbb{S}, \mathbb{S}^\star, \xi) = g_1(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi)) = \bar{X} - (\bar{X}^\star - \mu_X)\sqrt{\frac{S_X^2}{S_X^{\star 2}}}$$

There is a similar expression for $\mathcal{R}_{\mu_Y}$ and lastly for $\theta = \pi_X(\xi) - \pi_Y(\xi)$ the recipe produces the following FGPQ

$$\mathcal{R}_\theta = \mathcal{R}_{\mu_X} - \mathcal{R}_{\mu_Y} = (\bar{X} - \bar{Y}) - \left[(\bar{X}^\star - \mu_X)\sqrt{\frac{S_X^2}{S_X^{\star 2}}} - (\bar{Y}^\star - \mu_Y)\sqrt{\frac{S_Y^2}{S_Y^{\star 2}}} \right]$$

as claimed in Example 2.1.

Example 2.6. (*Continuation of Balanced One-Way Random Effects Model, Example 2.2*)

Continuing with the notation of Example 2.2, it is of interest to construct FGPQs for several parameters, namely μ, σ_A^2 σ_e^2, σ_A and σ_Y, where $\sigma_Y^2 = \sigma_A^2 + \sigma_e^2$. The complete and sufficient statistic $\mathbb{S} = (\bar{Y}_{..}, S_B^2, S_W^2)$ for $\xi = (\mu, \sigma_A^2, \sigma_e^2)$ has an invertible pivotal relationship with ξ given by

$$\begin{aligned} f_1(\mathbb{S}, \xi) &= \frac{\bar{Y} - \mu}{\sqrt{\dfrac{n\sigma_A^2 + \sigma_e^2}{an}}} &= E_1 \sim N(0,1) \\ f_2(\mathbb{S}, \xi) &= \frac{(a-1)S_B^2}{n\sigma_A^2 + \sigma_e^2} &= E_2 \sim \chi^2_{a-1}, \\ f_3(\mathbb{S}, \xi) &= \frac{a(n-1)S_W^2}{\sigma_e^2} &= E_3 \sim \chi^2_{a(n-1)}, \end{aligned} \tag{2.4}$$

where E_1, E_2, and E_3 are jointly independent. The inverse mapping $\boldsymbol{g}(\mathbb{S}, \mathbb{E})$ is defined by

$$g_1(\mathbb{S}, \mathbb{E}) = \bar{Y} - E_1\sqrt{\frac{(a-1)S_B^2}{an\,E_2}},$$
$$g_2(\mathbb{S}, \mathbb{E}) = \frac{(a-1)S_B^2}{n\,E_2} - \frac{a(n-1)S_W^2}{n\,E_3},$$
$$g_3(\mathbb{S}, \mathbb{E}) = \frac{a(n-1)S_W^2}{E_3}.$$

By the structural method of Theorem 2.2, a FGPQ for μ is given by $\mathcal{R}_\mu = g_1(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))$, a FGPQ for σ_A^2 is given by $\mathcal{R}_{\sigma_A^2} = g_2(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))$, and a FGPQ for σ_e^2 is given by $\mathcal{R}_{\sigma_e^2} = g_3(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))$. Evaluating these expressions we get

$$\mathcal{R}_\mu = \mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = \bar{Y} - (\bar{Y}^\star - \mu)\sqrt{\frac{S_B^2}{{S_B^\star}^2}} \tag{2.5}$$
$$\mathcal{R}_{\sigma_A^2} = \mathcal{R}_{\sigma_A^2}(\mathbb{S}, \mathbb{S}^\star, \xi) = \frac{(n\sigma_A^2 + \sigma_e^2)S_B^2}{n{S_B^\star}^2} - \frac{\sigma_e^2 S_W^2}{n{S_W^\star}^2}$$
$$\mathcal{R}_{\sigma_e^2} = \mathcal{R}_{\sigma_e^2}(\mathbb{S}, \mathbb{S}^\star, \xi) = \frac{\sigma_e^2 S_W^2}{{S_W^\star}^2}$$

It is worth noting that f_3 is an invertible pivotal quantity for σ_e^2 and that the confidence intervals defined by f_3 agree with the confidence intervals defined by $\mathcal{R}_{\sigma_e^2}$. Later in the chapter we will prove that any invertible pivotal quantity for a scalar parameter produces the same confidence intervals as the FGPQ constructed using the structural method of Theorem 2.2.

For σ_A, the straight forward approach of using $\pi_{\sigma_A}(\xi) = \sqrt{\sigma_A^2}$ in the recipe from Theorem 2.2 yields $\mathcal{R}_{\sigma_A} = \sqrt{g_2(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))}$. But there a non-zero probability that $\mathcal{R}_{\sigma_A}$ will be an imaginary number. A solution is to define $\pi_{\sigma_A}(\xi) = \sqrt{\max(0, \sigma_A^2)}$. Then the recipe produces the following FGPQ for σ_A,

$$\mathcal{R}_{\sigma_A} = \mathcal{R}_{\sigma_A}(\mathbb{S}, \mathbb{S}^\star, \xi) = \sqrt{\max\left(0, \frac{(n\sigma_A^2 + \sigma_e^2)S_B^2}{n S_B^{\star 2}} - \frac{\sigma_e^2 S_W^2}{n S_W^{\star 2}}\right)} \tag{2.6}$$

The recipe of Theorem 2.2 applied to σ_Y, where $\sigma_Y^2 = \sigma_A^2 + \sigma_e^2$, produces the following FGPQ

$$\mathcal{R}_{\sigma_Y} = \sqrt{g_2(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi)) + g_3(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))} = \sqrt{\frac{(n\sigma_A^2 + \sigma_e^2)S_B^2}{n S_B^{\star 2}} + \frac{(n-1)\sigma_e^2 S_W^2}{n S_W^{\star 2}}} \tag{2.7}$$

The last two FGPQ's, (2.7) and (2.6), will be used in later chapters.

In the next example we look at the balanced two-way random effects model with interaction. The main reason for presenting it at this point is to illustrate a technical difficultly that users of the recipe should be aware of. The issue is the extension of the inverse function $\boldsymbol{g}$ when the independent copy $\mathbb{S}^*$ is used in place of $\mathbb{S}$.

Example 2.7. *Balanced Two-Way Random Effects Model with Interaction and Replication)*

Consider the two-way random effects model which is expressed as:

$$Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + e_{ijk}, \qquad i = 1, \ldots, a; \;\; j = 1, \ldots, b; \;\; k = 1, \ldots, n,$$

where $A_i \sim N(0, \sigma_A^2)$, $B_j \sim N(0, \sigma_B^2)$, $(AB)_{ij} \sim N(0, \sigma_{AB}^2)$ and $e_{ijk} \sim N(0, \sigma_e^2)$, and $\{A_i\}$, $\{B_j\}$, $\{(AB)_{ij}\}$ and $\{e_{ijk}\}$ are all mutually independent. Define the following sums of squares.

$$SSA = bn \sum_{i=1}^{a} (\bar{Y}_{i..} - \bar{Y}_{...})^2 \qquad SSB = an \sum_{j=1}^{b} (\bar{Y}_{.j.} - \bar{Y}_{...})^2$$

$$SSAB = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 \qquad SSE = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (Y_{ijk} - \bar{Y}_{ij.})^2$$

where

$$\bar{Y}_{...} = \frac{1}{abn} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} Y_{ijk} \qquad \bar{Y}_{ij.} = \frac{1}{n} \sum_{k=1}^{n} Y_{ijk}$$

$$\bar{Y}_{i..} = \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} Y_{ijk} \qquad \bar{Y}_{.j.} = \frac{1}{an} \sum_{i=1}^{a} \sum_{k=1}^{n} Y_{ijk}$$

To simplify notation let $\bar{Y}$ denote the over all mean $\bar{Y}_{...}$. The complete and sufficient statistic $\mathbb{S} = (\bar{Y}, SSA, SSB, SSAB, SSE)$ and $\xi = (\mu, \sigma_A^2, \sigma_B^2, \sigma_{AB}^2, \sigma_e^2)$ have a pivotal relationship

$$f_0(\mathbb{S}, \xi) = \frac{\bar{Y} - \mu}{\sigma_{\bar{Y}}} = E_0 \sim N(0, 1) \tag{2.8}$$

$$f_1(\mathbb{S}, \xi) = \frac{SSA}{\psi_A} = E_1 \sim \chi^2_{a-1} \qquad f_2(\mathbb{S}, \xi) = \frac{SSB}{\psi_B} = E_2 \sim \chi^2_{b-1}$$

$$f_3(\mathbb{S}, \xi) = \frac{SSAB}{\psi_{AB}} = E_3 \sim \chi^2_{(a-1)(b-1)} \qquad f_4(\mathbb{S}, \xi) = \frac{SSE}{\psi_e} = E_4 \sim \chi^2_{ab(n-1)}$$

with $E_0, \ldots, E_4$ jointly independent and where

$$\sigma_{\bar{Y}}^2 = \frac{bn\sigma_A^2 + an\sigma_B^2 + n\sigma_{AB}^2 + \sigma_e^2}{abn}$$

$$\psi_A = bn\sigma_A^2 + n\sigma_{AB}^2 + \sigma_e^2 \qquad \psi_B = an\sigma_B^2 + n\sigma_{AB}^2 + \sigma_e^2$$

$$\psi_{AB} = n\sigma_{AB}^2 + \sigma_e^2 \qquad \psi_e = \sigma_e^2 \tag{2.9}$$

It is straightforward to show $\boldsymbol{f}(\mathbb{S}, \xi)$ is an invertible pivotal quantity, but a little care is needed in defining the g_0 component of the inverse mapping $\boldsymbol{g}$. We will outline the process. Let $(\mathbf{s}, \xi, \mathbf{e})$ be such that $\boldsymbol{f}(\mathbf{s}, \xi) = \mathbf{e}$. The first step is to solve the system of linear equations defined by $\psi = (\psi_A, \psi_B, \psi_{AB}, \psi_e)$ for $\phi = (\sigma_A^2, \sigma_B^2, \sigma_{AB}^2, \sigma_e^2)$ in terms of *psi*. Then substitute for ψ the appropriate ratios defined by $\mathbf{e}_1 = (f_1(\mathbf{s}, \xi), \ldots f_4(\mathbf{s}, \xi))$, i.e., $\psi_A = ssa/e_1$, $\psi_B = ssb/e_2$, $\psi_{AB} = ssab/e_3$, $\psi_e = sse/e_4$. This defines functions $g_1, \ldots, g_4$ such that $\boldsymbol{g}_1(\mathbf{s}, \mathbf{e}_1) = \phi$, where $\boldsymbol{g}_1 = (g_1, \ldots, g_4)$. This process produces

$$g_1(\mathbb{S}, \mathbb{E}) = \frac{1}{bn}\left(\frac{SSA}{E_1} - \frac{SSAB}{E_3}\right) \qquad g_2(\mathbb{S}, \mathbb{E}) = \frac{1}{an}\left(\frac{SSB}{E_2} - \frac{SSAB}{E_3}\right)$$

$$g_3(\mathbb{S}, \mathbb{E}) = \frac{1}{n}\left(\frac{SSAB}{E_3} - \frac{SSE}{E_4}\right) \qquad g_4(\mathbb{S}, \mathbb{E}) = \frac{SSE}{E_4}$$

Notice that $\boldsymbol{g}_1$ is an inverse for $\boldsymbol{f}_1 = (f_1, \ldots, f_4)$ so that a FGPQ for ϕ is given by

$$\mathcal{R}_\phi = \mathcal{R}_\phi(\mathbb{S}, \mathbb{S}^\star, \phi) = \boldsymbol{g}_1(\mathbb{S}, \boldsymbol{f}_1(\mathbb{S}^\star, \phi))$$

To finish finding the inverse for *bsymf* one solves $f_0(\mathbf{s}, \xi) = e_0$ for μ, i.e., $\mu = \bar{y} - e_0\sigma_{\bar{Y}}$. Then substitute $(g_1, \ldots, g_4)$ for ϕ in $\sigma_{\bar{Y}}$ to obtain $g_0(\mathbb{S}, \mathbb{E}) = \bar{Y} - E_0\sqrt{\left(\frac{SSA}{E_1} + \frac{SSB}{E_2} - \frac{SSAB}{E_3}\right)\frac{1}{abn}}$. The problem arises when one substitutes the independent copy $\mathbb{E}^\star$ for $\mathbb{E}$, since then there is a positive probability of a negative number under the square root. So we define

$$g_0(\mathbb{S}, \mathbb{E}^\star) = \bar{Y} - E_0^\star\sqrt{\max\left[0, \left(\frac{SSA}{E_1^\star} + \frac{SSB}{E_2^\star} - \frac{SSAB}{E_3^\star}\right)\right]\frac{1}{abn}}.$$

Note that g_0 agrees with the inverse of f_0 on the set of $(\mathbf{s}, \mathbf{e})$ such that there exists ξ with $\boldsymbol{f}(\mathbf{s}, \xi) = \mathbf{e}$. The FGPQ for μ given by the recipe in Theorem 2.2 is

$$\mathcal{R}_\mu = \mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = \bar{Y} - \left(\frac{\bar{Y}^\star - \mu}{\sigma_{\bar{Y}}}\right)\sqrt{\max\left(0, \frac{SSA}{SSA^\star}\psi_A + \frac{SSB}{SSB^\star}\psi_B - \frac{SSAB}{SSAB^\star}\psi_{AB}\right)\frac{1}{abn}}. \tag{2.10}$$

The method used in Example 2.6 and 2.7 is in fact generalizable to arbitrary saturated balanced linear factorial models with replication. This is the topic of the next section.

2.4 Saturated Balanced Factorial Linear Mixed Models with Replication

We will use the subset notation of Hocking [27]. Let $F_1, \ldots, F_{k+1}$ denote the factors with a_i the number of levels for F_i, $i = 1, \ldots, k$, and a_{k+1} the number of replication. The total number of treatments is equal $a_1 a_2 \cdots a_k$ and the replicates per treatment is a_{k+1}. Let $\mathcal{T}$ denote the set of all admissible subsets of $\{1, \ldots, k+1\}$, $\mathcal{T}_r$ denote the admissible subsets associated with the random effects, and $\mathcal{T}_{r0}$ denote $\mathcal{T}_r$ minus $\{1, \ldots, k+1\}$. Let $q = |\mathcal{T}_{r0}|$. Order the $q+1$ elements of $\mathcal{T}_r$ by the cardinality of the elements, with some order picked for the elements with the same cardinality; let $k = 1, \ldots, q+1$ denote the ordering. We will use this ordering whenever we list a set that is indexed by $\mathcal{T}_r$. For $\boldsymbol{u} \in \mathcal{T}$ let

$$a_{\boldsymbol{u}} = \prod_{i \in \boldsymbol{u}} a_i \quad \text{and} \quad b_{\boldsymbol{u}} = \begin{cases} \prod_{i \notin \boldsymbol{u}} a_i, & \text{if } \boldsymbol{u} \neq \{1, \ldots, k+1\}; \\ 1, & \text{if } \boldsymbol{u} = \{1, \ldots, k+1\}. \end{cases} \tag{2.11}$$

We consider the saturated balanced factorial mixed linear model with replication, (SBFLMMWR),

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{A} + \boldsymbol{e}$$

where $\boldsymbol{X}$ and $\boldsymbol{Z}$ are known matrices, $\boldsymbol{\beta}$ is a vector of unknown parameters. Assume $\boldsymbol{e} \sim MVN(\boldsymbol{0}, \phi_e \boldsymbol{I}_N)$, $\boldsymbol{A} \sim MVN(\boldsymbol{0}, \Sigma)$, where $\Sigma = \text{Diag}(\phi_{\boldsymbol{u}} \mathrm{I}_{a_{\boldsymbol{u}}} | \boldsymbol{u} \in \mathcal{T}_{r0})$, with $\phi_{\boldsymbol{u}} \geq 0$ for $\boldsymbol{u} \in \mathcal{T}_r$, and $\boldsymbol{e}$ and $\boldsymbol{A}$ are independent. We will derive FGPQs for the $\{\phi_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$ and for any linear form $\boldsymbol{l}^T \boldsymbol{\beta}$. First we present several well known properties of the SBFLMMWR, (see Hocking [27] for proofs).

The expectation is $E[\boldsymbol{Y}] = \boldsymbol{X}\boldsymbol{\beta}$. There exist known pairwise commuting matrices $\{\boldsymbol{V}_{\boldsymbol{s}}\}_{\boldsymbol{s} \in \mathcal{T}_r}$ such that $V[\boldsymbol{Y}] = \boldsymbol{Z}\Sigma\boldsymbol{Z}^T + \phi_e \boldsymbol{I}_N$, can be expressed as

$$V[\boldsymbol{Y}] = \sum_{\boldsymbol{u} \in \mathcal{T}_r} \phi_{\boldsymbol{u}} \boldsymbol{V}_{\boldsymbol{u}},$$

where one term is $\phi_e \boldsymbol{I}_N$. There exist statistics $\boldsymbol{S}_0$, and $\{S_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$, with linear functions $\{\psi_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$ of the $\{\phi_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$, and positive integers $\{r_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$, with the following properties

P1: $\boldsymbol{S}_0 \sim MVN(\boldsymbol{\beta}, \Sigma_{\boldsymbol{\beta}})$, where $\Sigma_{\boldsymbol{\beta}} = \sum_{\boldsymbol{u} \in \mathcal{T}_r} \phi_{\boldsymbol{u}} \boldsymbol{V}^*_{\boldsymbol{u}}$ with
$\boldsymbol{V}^*_{\boldsymbol{u}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{\boldsymbol{u}}\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}$.

P2: $\dfrac{S_{\boldsymbol{u}}}{\psi_{\boldsymbol{u}}} \sim \chi^2_{r_{\boldsymbol{u}}}, \boldsymbol{u} \in \mathcal{T}_r$, with $r_{\boldsymbol{u}} = \prod_{i \in \boldsymbol{u}} a^*_i$ where $a^*_i = \begin{cases} a_i, & \text{if } i(j) \text{ for } j \in s; \\ a_i - 1, & \text{otherwise.} \end{cases}$

P3: $\{S_0,\ S_u | u \in \mathcal{T}_r\}$ are jointly independent.

P4: For $u \in \mathcal{T}_r$, $\psi_u = \sum_{v \supseteq u} b_v \phi_v$, where b_v is defined in (2.11).

The linear functions defined in item P4 are invertible. The statement of the next lemma describes the structure of the inverse functions. While the proof of the lemma describes a method for finding the coefficients of the inverse functions.

Lemma 2.1. *For $u \in \mathcal{T}_r$, there exists constants $\{c_{v,u}\}_{v \supseteq u}$, with $c_{v,u} \in \mathbb{Z}$ and $c_{u,u} = 1$, such that $\phi_u = \frac{1}{b_u}\left[\sum_{v \supseteq u} c_{v,u}\psi_v\right]$.*

Proof. The proof is by finite induction, where the induction step assumes the result to be true for all $v \supset u$ and the base case is $u = \{1, \ldots, k+1\}$. □

We now construct FGPQs for each of component of $\xi = (l^T\beta, \phi_1, \ldots, \phi_{q+1})$ using the recipe in Theorem 2.2. Based on P1-P3 the statistic $\mathbb{S} = (l^T S_0, S_1, \ldots, S_{q+1})$ and ξ have the following invertible pivotal relationship

$$f_0(\mathbb{S}, \xi) = \frac{l^T S_0 - l^T \beta}{\sqrt{\Sigma_l}} = E_0 \sim N(0,1)$$
$$f_i(\mathbb{S}, \xi) = \frac{S_i}{\psi_i} = E_i \sim \chi^2_{r_i}, \text{ for } i = 1, \ldots, q+1, \tag{2.12}$$

where $(E_0, E_1, \ldots, E_{q+1})$ are jointly independent and

$$\Sigma_l = \mathrm{Var}(l^T\Sigma) = \sum_{s \in \mathcal{T}_r} \phi_s l^T V_s^* l.$$

The pivotal quantity is given by $f(\mathbb{S}, \xi) = (f_0, f_1) = \mathbb{E}$, where $\mathbb{E} = (E_0, \ldots, E_{q+1})$ and $f_1 = (f_1, \ldots, f_{q+1})$. We will now find the inverse function $g(\mathbb{S}, \cdot)$. First we show that $f_1(\mathbb{S}, \cdot)$ is invertible. Define

$$g_u(\mathbb{S}, \mathbb{E}) = \frac{1}{b_u}\left[\sum_{v \supseteq u} c_{v,u}\frac{S_v}{E_v}\right] \quad \text{for} \quad u \in \mathcal{T}_r.$$

Lemma 2.1 and (2.12) imply $g_u(\mathbb{S}, f_1(\mathbb{S}, \xi)) = \phi_u$, for $u \in \mathcal{T}_r$. So f_1 is an invertible pivotal quantity for $\mathbb{S}_1 = (S_1, \ldots, S_{q+1})$ and $\phi = (\phi_1, \ldots, \phi_{q+1})$. Hence by Theorem 2.2 a FGPQ for ϕ_u is given by

$$\mathcal{R}_{\phi_u}(\mathbb{S}_1, \mathbb{S}_1^*, \phi) = \frac{1}{b_u}\left[\sum_{v \supseteq u} c_{v,u}\frac{S_v}{S_v^*}\psi_v\right], \text{ where } u \in \mathcal{T}_r \tag{2.13}$$

We will now finish finding the inverse for $\boldsymbol{f}(\mathbb{S},\cdot)$. Let $\Sigma_l(\mathbb{S},\mathbb{E}) = \sum_{\boldsymbol{u}\in\mathcal{T}_r} g_{\boldsymbol{u}}(\mathbb{S},\mathbb{E})\boldsymbol{l}^T\boldsymbol{V}^*_{\boldsymbol{u}}\boldsymbol{l}$ and let $g_0(\mathbb{S},\mathbb{E}) = \boldsymbol{l}^T\boldsymbol{S}_0 - E_0\sqrt{\Sigma_l(\mathbb{S},\mathbb{E})}$. Then $g_0(\mathbb{S},\boldsymbol{f}(\mathbb{S},\xi)) = \boldsymbol{l}^T\boldsymbol{\beta}$. Special care must be used in extending g_0 to the independent copy $\mathbb{E}^\star = \boldsymbol{f}(\mathbb{S}^\star,\xi)$ of $\mathbb{E}$. There is a possibility that

$$\mathcal{R}_{\Sigma_l}(\mathbb{S},\mathbb{S}^\star,\xi) = \Sigma_l(\mathbb{S},\boldsymbol{f}(\mathbb{S}^\star,\xi)) = \sum_{\boldsymbol{u}\in\mathcal{T}_r} \mathcal{R}_{\phi_{\boldsymbol{u}}}(\mathbb{S},\mathbb{S}^\star,\phi)\boldsymbol{l}^T\boldsymbol{V}^*_{\boldsymbol{u}}\boldsymbol{l} \tag{2.14}$$

will be negative. As with the previous examples, the solution is to extend g_0 using $\max(0,\mathcal{R}_{\Sigma_l})$. Thus an FGPQ for $\boldsymbol{l}^T\boldsymbol{\beta}$ is given by

$$\mathcal{R}_l(\mathbb{S},\mathbb{S}^\star,\xi) = \boldsymbol{l}^T\boldsymbol{S}_0 - \frac{\boldsymbol{l}^T\boldsymbol{S}^\star_0 - \boldsymbol{l}^T\boldsymbol{\beta}}{\sqrt{\Sigma_l}}\sqrt{\max(0,\mathcal{R}_{\Sigma_l}(\mathbb{S},\mathbb{S}^\star,\xi))}, \tag{2.15}$$

where R_{Σ_l} is defined in (2.14). We summarize these results in the following proposition.

Proposition 2.2. *For a saturated balanced factorial linear mixed model with replication with parameters $(\boldsymbol{\beta},\phi_{\boldsymbol{u}}|\boldsymbol{u}\in\mathcal{T}_r)$, where $\boldsymbol{\beta}$ are the fixed effects and the $\phi_{\boldsymbol{u}}$ are the variance components of the random effects, let $\boldsymbol{S}_0$ be the sums of squares for the fixed effects and $S_{\boldsymbol{u}}$ be the sums of squares for the $\boldsymbol{u}^{th}$ random effect. A FGPQ for $\phi_{\boldsymbol{u}}$ is defined in Equation 2.13, where the $c_{\boldsymbol{v},\boldsymbol{u}}$ are described in Lemma 2.1. In addition for any fixed vector $\boldsymbol{l}$, a FGPQ for $\boldsymbol{l}^T\boldsymbol{\beta}$ is defined in (2.15).*

We will next show that the above FGPQs for the variance components of the SBFLMMWR satisfy Assumptions 2.1 and hence generalized confidence intervals based on the FGPQs have asymptotically exact frequentist coverage.

Proposition 2.3. *Let $\phi_{\boldsymbol{u}}$ be a variance component of a saturated balance factorial linear mixed model with replication. Then the $100(1-\alpha)\%$ GCIs for the variance components based on the FGPQ from Proposition 2.2 have asymptotically $100(1-\alpha)\%$ frequentist coverage as $a_{\boldsymbol{u}} \to \infty$.*

Proof. We verify that $\mathcal{R}_{\phi_{\boldsymbol{u}}}$ satisfies Assumptions 2.1 of Theorem 2.1. Let $\mathbb{W}_{\boldsymbol{u}} = (S_1,\ldots,S_p)$ be a list of $\{S_{\boldsymbol{v}}|\boldsymbol{v}\supseteq\boldsymbol{u}\}$, and let $\xi_{\boldsymbol{u}} = (\phi_1,\ldots,\phi_p)$ be the corresponding list of the $\{\phi_{\boldsymbol{v}}|\boldsymbol{v}\supseteq\boldsymbol{u}\}$. Note that $\mathcal{R}_{\phi_{\boldsymbol{u}}}$ is a function of $(\mathbb{W}_{\boldsymbol{u}},\mathbb{W}^\star_{\boldsymbol{u}},\xi_{\boldsymbol{u}})$ and not the entire vector $(\mathbb{S},\mathbb{S}^\star,\xi)$. We will consider the factor levels $\{a_i|i\notin\boldsymbol{u}\}$ to be fixed. Note, $b_{\boldsymbol{u}}$ is the product of these factor levels.

If $\boldsymbol{v} \supseteq \boldsymbol{u}$, by standard results from linear models and distribution theory we have

$$\sqrt{a_{\boldsymbol{u}}}\left(\frac{r_{\boldsymbol{v}}}{S_{\boldsymbol{v}}} - \frac{1}{\psi_{\boldsymbol{v}}}\right) \xrightarrow{\mathcal{D}} N(0, 2\sqrt{a_s/a_m}\,\psi_m^{-2}).$$

Since the $\{S_s\}$ are jointly independent,

$$\sqrt{a_{\boldsymbol{u}}}\left(\frac{r_1}{S_1} - \frac{1}{\psi_1}, \ldots, \frac{r_p}{S_p} - \frac{1}{\psi_p}\right) \xrightarrow{\mathcal{D}} \boldsymbol{N},$$

where $\boldsymbol{N}$ is a non-degenerate multivariate normal. Thus $\boldsymbol{t}(\xi_{\boldsymbol{u}}) = (\psi_1^{-1}, \ldots, \psi_p^{-1})$ satisfies condition 1 of Assumptions 2.1.

We will next show that condition 2 of Assumptions 2.1 is satisfied. The only factor levels $\mathcal{R}_{\phi_{\boldsymbol{u}}}$ is a function of are the $\{a_i | i \notin \boldsymbol{u}\}$, which we consider to be fixed. While the convergence is with respect to the product of the factor levels $\{a_i | i \in \boldsymbol{u}\}$. Thus there is nothing to check with respect to the uniform convergence in item 2a. Since $\mathcal{R}_{\phi_{\boldsymbol{u}}}(w_{\boldsymbol{u}}, w_{\boldsymbol{u}}^{\star}, \xi_{\boldsymbol{u}})$ has continuous partial derivatives with respect to both $\boldsymbol{w}_{\boldsymbol{u}}$ and $\boldsymbol{w}_{\boldsymbol{u}}^{\star}$, $\mathcal{R}_{\phi_{\boldsymbol{u}}}$ satisfies rest of condition 2. □

Next we present a method for constructing approximate FGPQ's for the unbalanced saturated linear mixed models with replication.

2.5 Approximate FGPQ for Saturated Unbalanced Factorial Linear Mixed Models with Replication

In the unbalanced setup, one solution is to impute data to make a balanced model and then use the imputed data and Proposition 2.2 to construct FGPQs. In the following discussion of this method we will use the notation from Section 2.4.

Consider the unbalanced factorial mixed linear model, $\boldsymbol{Y}_0 = \boldsymbol{X}_0\boldsymbol{\beta} + \boldsymbol{Z}_0\boldsymbol{A} + \boldsymbol{e}_0$, where $\boldsymbol{X}_0$ and $\boldsymbol{Z}_0$ are known matrices, $\boldsymbol{\beta}$ is a vector of unknown parameters. Assume $\boldsymbol{e} \sim MVN(\boldsymbol{0}, \phi_e \boldsymbol{I}_N)$, $\boldsymbol{A} \sim MVN(\boldsymbol{0}, \Sigma)$, where $\Sigma = \mathrm{Diag}(\phi_{\boldsymbol{u}} \mathbf{I}_{a_{\boldsymbol{u}}} | \boldsymbol{u} \in \mathcal{T}_{r0})$, with $\phi_{\boldsymbol{u}} \geq 0$ for $\boldsymbol{u} \in \mathcal{T}_r$, and $\boldsymbol{e}$ and $\boldsymbol{A}$ are independent. We will derive approximate FGPQs for the $\{\phi_{\boldsymbol{u}}\}_{\boldsymbol{u} \in \mathcal{T}_r}$ and for any linear form $\boldsymbol{l}^T\boldsymbol{\beta}$. By approximate FGPQS we mean using imputed data for the missing data in the FGPQs from Proposition 2.2.

For $\boldsymbol{Y}_0$, the expectation is $E[\boldsymbol{Y}_0] = \boldsymbol{X}_0\boldsymbol{\beta}$, and the variance is $\boldsymbol{V}_0 = V[\boldsymbol{Y}_0] = \boldsymbol{Z}_0\Sigma_0\boldsymbol{Z}_0^T + \phi_e\boldsymbol{I}_{N_0}$, where $\boldsymbol{V}_0$ is a function of $\phi = (\phi_1, \ldots, \phi_{q+1})$, with $\phi_{q+1} = \phi_e$. Given a realization $\boldsymbol{y}_0$ of $\boldsymbol{Y}_0$, denote by $\hat{\phi}$ the REML estimate of ϕ. Let

$$\hat{\boldsymbol{V}}_0 = \boldsymbol{V}_0(\hat{\phi}) \quad \text{and} \quad \hat{\boldsymbol{\beta}} = (\boldsymbol{X}_0^T\hat{\boldsymbol{V}}_0^{-1}\boldsymbol{X}_0)^{-1}\boldsymbol{X}_0^T\hat{\boldsymbol{V}}_0^{-1}\boldsymbol{Y}_0$$

Denote by $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{ZA} + \boldsymbol{e}$, the balanced model such that exists a known matrix $\boldsymbol{D}$ with

$$\boldsymbol{Y}^* = \boldsymbol{D}^T\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_0 \\ \boldsymbol{Y}_m \end{bmatrix},$$

where $\boldsymbol{Y}_m$ is random vector of missing values. By standard results for multivariate normal random variables (see [27]), there exits matrices $\boldsymbol{V}_{m0}$ and $\boldsymbol{V}_m$ which are functions of ϕ such that

$$V[\boldsymbol{Y}^*] = \begin{bmatrix} \boldsymbol{V}_0 & \boldsymbol{V}_{0m}^T \\ \boldsymbol{V}_{0m} & \boldsymbol{V}_m \end{bmatrix},$$

and

$$\boldsymbol{Y}_m|\boldsymbol{Y}_0 \sim MVN(\boldsymbol{X}_m\boldsymbol{\beta} + \boldsymbol{V}_{m0}\boldsymbol{V}_0^{-1}(\boldsymbol{Y}_0 - \boldsymbol{X}_0\boldsymbol{\beta}), \boldsymbol{V}_m).$$

For a realization $\boldsymbol{y}_0$ of $\boldsymbol{Y}_0$ define the imputed missing values, $\tilde{\boldsymbol{y}}_m$, by

$$\tilde{\boldsymbol{y}}_m = \boldsymbol{X}_m\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{V}}_{m0}\hat{\boldsymbol{V}}_0^{-1}(\boldsymbol{y}_0 - \boldsymbol{X}_0\hat{\boldsymbol{\beta}}),$$

i.e., $\tilde{\boldsymbol{y}}_m$ is the EBLUP for $\boldsymbol{y}_m$. The realization of $\boldsymbol{Y}$ corresponding to $\boldsymbol{y}_0$ is defined as

$$\tilde{\boldsymbol{y}} = \boldsymbol{D}^T \begin{bmatrix} \boldsymbol{y}_0 \\ \tilde{\boldsymbol{y}}_m \end{bmatrix},$$

For the balanced model $\boldsymbol{Y}$ let $\mathbb{S}$ be the statistics defined in Section 2.4. Denote by $\tilde{\mathbb{s}}$ the realization of $\mathbb{S}$ corresponding to $\boldsymbol{y}_0$, i.e., $\tilde{\mathbb{s}}$ is calculated using the vector $\tilde{\boldsymbol{y}}$. We will use the symbol $\tilde{\mathbb{S}}$ to emphasize that realizations of $\mathbb{S}$ are calculated using imputed data for the missing values. An approximate FGPQ for the variance component $\phi_{\boldsymbol{v}}$, is given by $\mathcal{R}_{\phi_{\boldsymbol{v}}}(\tilde{\mathbb{S}}, \mathbb{S}^*, \xi)$, where $\mathcal{R}_{\phi_{\boldsymbol{v}}}$ is defined in (2.13). For any fixed vector $\boldsymbol{l}$, a FGPQ for $\boldsymbol{l\beta}$ is given by $R_{\boldsymbol{l}}(\tilde{\mathbb{S}}, \mathbb{S}^*, \xi)$, where $\mathcal{R}_{\boldsymbol{l}}$ is defined in (2.15).

Based on an suggestion by Hocking [27], we used $\dfrac{\tilde{s}_{\boldsymbol{u}}}{\hat{\psi}_{reml,\boldsymbol{u}}}$, $\boldsymbol{u} \in \mathcal{T}_r$, for the degrees of freedom $r_{\boldsymbol{u}}$ in Equation (2.12), where $\hat{\psi}_{reml,\boldsymbol{u}} = \sum_{\boldsymbol{v} \supseteq \boldsymbol{u}} b_{\boldsymbol{v}}\hat{\phi}_{\boldsymbol{v}}$. In simulation studies we came across scenarios where the suggested degree of freedom were less than 1 and we found the performance of the generalized confidence intervals could be improved by using as the degrees of freedom the following:

$$\tilde{r}_{\boldsymbol{u}} = \max\left(1, \frac{\tilde{s}_t}{\hat{\psi}_{reml,\boldsymbol{u}}}\right) \text{ for } \boldsymbol{u} \in \mathcal{T}_r,$$

We will refer to the method given in this section for constructing approximate FGPQs for saturated unbalanced factorial linear mixed models as the imputation method. An illustration of the imputation method is presented in Section 3.2.

Remark 2.3. A natural question is whether conditions exist similar to those for Theorem 2.1 that guarantee the correct asymptotic coverage for the variance components in the unbalanced case. For the variance component $\phi_{\boldsymbol{u}}$, we conjecture that the addition condition needed is that the proportion of missing data goes to zero as $a_{\boldsymbol{u}} \to \infty$; some consideration must be given to relative rate that the proportion of missing values goes to zero as $a_{\boldsymbol{u}} \to \infty$.

2.6 Alternative Representations of FGPQs

The definition of a FGPQ expresses $\mathcal{R}_\theta$ as a function of $(\mathbb{S}, \mathbb{S}^\star, \xi)$. If the FGPQ is constructed using the structural method from Theorem 2.2 then there is at least one and possibly two other representations of $\mathcal{R}_\theta$ as functions of random variables. Both of these representations are useful and the next two remarks discuss the details.

Remark 2.4. If $\mathcal{R}_\theta$ is obtained using the structure method of Theorem 2.2 then by Equation (2.3) we can also express $\mathcal{R}_\theta$ as a function of $(\mathbb{S}, \mathbb{E}^\star)$. We will denote this representation by

$$\mathcal{R}_\theta = \dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star) = \pi(g_1(\mathbb{S}, \mathbb{E}^\star), \ldots, g_k(\mathbb{S}, \mathbb{E}^\star)).$$

It may appear that in this representation the dependence on ξ no longer exists, but dependence on ξ is through the distribution of $\mathbb{S}$, which does depend on ξ. The representation $\dot{\mathcal{R}}_\theta$ is useful in discussions of how to calculate confidence bounds using $\mathcal{R}_\theta$. As noted in the Theorem 2.2, if θ is a scalar parameter, then the quantiles of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathrm{s}$ are used as the confidence bounds. In most cases the intractability of calculating the conditional distribution means that the quantile are estimated using a

Monte Carlo simulation of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathbf{s}$. Both the direction calculation and the Monte Carlo simulation of the conditional distribution use the representation $\dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star)$. In many cases the E_i's are independent and have distributions for which random number generators are readily available. This makes simulating the distribution of $\dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbb{E}^\star)$ straightforward. This will be illustrated with several examples that are contained in the later chapters.

Remark 2.5. If $\boldsymbol{f}(\mathbb{S}, \xi)$ is an invertible pivotal quantity and the mapping $\boldsymbol{f}(\cdot\,, \xi) : \mathbb{S} \to \mathcal{E}(\mathbf{s})$ is invertible for every $\xi \in \Xi$, we denote the inverse mapping by $\boldsymbol{h}(\cdot\,, \xi)$, that is $\boldsymbol{h}(\mathbf{e}, \xi) = \mathbf{s}$, when $\boldsymbol{f}(\mathbf{s}, \xi) = \mathbf{e}$. If $\boldsymbol{h}$ exists then it can be used to express $\mathcal{R}_\theta$ as a function $(\mathbb{E}, \mathbb{E}^\star, \xi)$. That is,

$$\begin{aligned}\mathcal{R}_\theta = \ddot{\mathcal{R}}_\theta(\mathbb{E}, \mathbb{E}^\star, \xi) &= \pi(g_1(\boldsymbol{h}(\boldsymbol{f}(\mathbb{S}, \xi), \xi), \mathbb{E}^\star), \ldots, g_k(\boldsymbol{h}(\boldsymbol{f}(\mathbb{S}, \xi), \xi), \mathbb{E}^\star)) \\ &= \pi(g_1(\boldsymbol{h}(\mathbb{E}, \xi), \mathbb{E}^\star), \ldots, g_k(\boldsymbol{h}(\mathbb{E}, \xi), \mathbb{E}^\star)). \qquad (2.16)\end{aligned}$$

We call this the canonical representation. The canonical representation of $\mathcal{R}_\theta$ is useful in the proofs of invariance properties of the coverage probability under transformations of the parameters.

We will illustrate Remarks 2.4 and 2.5 with the one-way random effects model.

Example 2.8. (*Continuation of Balanced One-Way Random Effects Model, Example 2.2*)

The function $\boldsymbol{h}$ for the one-way random effects model is given by.

$$\bar{Y} = h_1(\mathbb{E}, \xi) = \mu + E_1\sqrt{\frac{n\sigma_A^2 + \sigma_e^2}{an}} \qquad (2.17a)$$

$$S_B^2 = h_2(\mathbb{E}, \xi) = E_2\frac{(n\sigma_A^2 + \sigma_e^2)}{a-1} \qquad (2.17b)$$

$$S_W^2 = h_3(\mathbb{E}, \xi) = E_3\frac{\sigma_e^2}{a(n-1)}. \qquad (2.17c)$$

where these functions follow easily from Equation (2.4). The representations $\dot{\mathcal{R}}_\theta(\mathbb{S}, E^\star)$ and $\ddot{\mathcal{R}}_\theta$ for the three population characteristics μ, σ_A and σ_Y will be used multiple times in following chapters. For convenience we list them here. The FGPQs were derived in Example 2.6.

$$\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) \quad = \bar{Y} - E_1^\star\sqrt{\frac{(a-1)S_B^2}{an\,E_2^\star}} \qquad (2.18a)$$

$$\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi) = \mu + \sqrt{\frac{n\sigma_A^2 + \sigma_e^2}{an}}\left(E_1 - E_1^\star\sqrt{\frac{E_2}{E_2^\star}}\right). \qquad (2.18b)$$

$$\dot{\mathcal{R}}_{\sigma_Y}(\mathbb{S},\mathbb{E}^\star) = \sqrt{\frac{(a-1)S_B^2}{n\,E_2^\star} + \frac{(n-1)(a-1)S_W^2}{E_3^\star}}, \tag{2.19a}$$

$$\ddot{\mathcal{R}}_{\sigma_Y}(\mathbb{E},\mathbb{E}^\star,\xi) = \sqrt{\frac{(n\sigma_A^2+\sigma_e^2)E_2}{nE_2^\star} + \frac{(n-1)\sigma_e^2 E_3}{nE_3^\star}}. \tag{2.19b}$$

$$\dot{\mathcal{R}}_{\sigma_A}(\mathbb{S},\mathbb{E}^\star) = \sqrt{\max\left(0, \frac{(a-1)S_B^2}{n\,E_2^\star} - \frac{a(n-1)S_W^2}{n\,E_3^\star}\right)} \tag{2.20a}$$

$$\ddot{\mathcal{R}}_{\sigma_A}(\mathbb{E},\mathbb{E}^\star,\xi) = \sqrt{\max\left(0, \frac{(n\sigma_A^2+\sigma_e^2)E_2}{n\,E_2^\star} - \frac{\sigma_e^2\,E_3}{n\,E_3^\star}\right)} \tag{2.20b}$$

2.7 Properties of FGQPs and FGCIs for Scalar Parameters

In this sections we a present a necessary and sufficient condition for a fiducial generalized confidence interval of a scalar parameter to have frequentist exact coverage. For scalar parameters we also discuss the relationship between a invertible pivotal quantity for the parameter and the FGPQ derived from the pivotal quantity using the structural method. We start with the discussion of conditions to guarantee exact coverage.

Let $\mathcal{R}_\theta$ be a FGPQ for θ. From Remark 2.1 a $100(1-\alpha)\%$ confidence interval for θ based on $\mathcal{R}_\theta$ is given by $\mathcal{R}_{\theta,\alpha_l} < \theta < \mathcal{R}_{\theta,1-\alpha_r}$, where $\mathcal{R}_{\theta,\gamma} = \mathcal{R}_{\theta,\gamma}(\mathrm{s})$ denotes the $100\gamma^{\text{th}}$ quantile of the distribution of $\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi)$ conditional on $\mathbb{S} = \mathrm{s}$ and $\alpha_l + \alpha_r = \alpha$. The coverage probability is given by $P_{\mathbb{S}}\left[\mathcal{R}_{\theta,\alpha_l}(\mathbb{S}) < \theta < \mathcal{R}_{\theta,1-\alpha_r}(\mathbb{S})\right]$. Notice that $\theta < \mathcal{R}_{\theta,1-\alpha_r}(\mathbb{S})$ if and only if $P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \le \theta \,\middle|\, \mathbb{S}\right] \le 1-\alpha_r$, and $\mathcal{R}_{\theta,\alpha_l}(\mathbb{S}) < \theta$ if and only if $P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \le \theta \,\middle|\, \mathbb{S}\right] \ge \alpha_l$. Thus the coverage probability is given by

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \le \theta \,\middle|\, \mathbb{S}\right] \ge \alpha_l \text{ and } P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \le \theta \,\middle|\, \mathbb{S}\right] \le 1-\alpha_r\right]. \tag{2.21}$$

We now present a proposition which gives a necessary and sufficient condition for exact coverage.

Proposition 2.4. *Let $\mathcal{R}_\theta$ be a FGPQ for a scalar parameter θ and define the random variable $U(\mathbb{S};\theta) = P_{\mathbb{S}^\star}[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \le \theta|\,\mathbb{S}]$. Then the generalized confidence intervals based on $\mathcal{R}_\theta$ are frequentist exact if and only if the distribution of $U(\mathbb{S};\theta)$ is uniform on $[0,1]$ for all θ. In particular $U(\mathbb{S};\theta)$ is an ordinary pivotal quantity for θ.*

Proof. The proof follows from Expression (2.21). □

The following corollary uses the notation of Proposition 2.4 and gives a sufficient condition for the distribution of $U(\mathbb{S};\theta)$ to be uniform [0, 1] and hence by the previous proposition generalized confidence intervals based on $\mathcal{R}_\theta$ are frequentist exact.

Corollary 2.1. *$U(\mathbb{S};\theta)$ will have a uniform* [0, 1] *distribution whenever the inequality $\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi)\leq\theta$ can be put in the form $q(\mathbb{S},\xi)\leq q(\mathbb{S}^\star,\xi)$ for some function $q(\cdot,\cdot)$.*

Proof. This follows from the general fact that if $U(W)=P[W\leq W^\star|\,W]=1-F_{W^\star}(W)$, where W and $W^\star$ are independent identically distributed random variables, then the distribution of $U(W)$ is uniform on $[0,1]$. □

This condition is easy to check in many applications. An interesting example is reported by Roy and Mathew [45] where, for a given constant t, they obtained a generalized confidence bound for $\tau=(t-\mu)/\theta$ based on a right censored sample from a shifted exponential distribution with density $f(x;\theta,\mu)=(1/\theta)exp(-(x-\mu)/\theta)I_{[\mu,\infty)}(x)$. They observed that the resulting GCI had exact frequentist coverage. In their proof of the exact coverage they express $\mathcal{R}_\theta\leq\theta$ in the form $q(\mathbb{S},\xi)\leq q(\mathbb{S}^\star,\xi)$. One might note that an ordinary pivotal quantity is available in their problem and is given by the cdf of $T=(t-\hat{\mu})/\hat{\theta}$ where $\hat{\mu}$ and $\hat{\theta}$ are the MLEs for μ and θ, respectively.

In Example 2.6 it was noted that the FGPQ for σ_e^2 was derived from an invertible pivotal quantity for σ_e^2. It was claimed that these two quantities produce the same confidence intervals. We now justify that special case with the following proposition.

Proposition 2.5. *Let θ be a scalar parameter and S a statistic which has an invertible pivotal relationship with θ. Denote the pivotal relationship by $f(S,\theta)$, and denote by $\mathcal{R}_\theta$ the FGPQ that the recipe in Theorem 2.2 produces. If for each fixed s, $f(s,\cdot)$ is monotonic in θ then the confidence intervals defined by f are the same as the confidence intervals defined by $\mathcal{R}_\theta$.*

Proof. By the assumption there exists a random variable E, such that $f(S,\theta)=E$, where the distribution of E is free of parameters. There also exists a function $g(S,E)$ such that $g(s, f(s, \theta)) = \theta$, where s is any realization of S. Denote by e_γ, the γ quantile of the distribution of E . Without loss of generality assume $f(s, \cdot)$ is increasing. If s is a realization of S, then the $100(1-\alpha)\%$ confidence interval defined by f with a left tail probability of α_l and right tail probability of α_R is given by $e_{\alpha_l} \leq f(s, \theta) \leq e_{1-\alpha_r}$ where $\alpha_l + \alpha_r = 1 - \alpha$ and one of α_l, α_r can be zero. We now derive the γ^{th} quantile of the distribution of $\mathcal{R}_\theta$ given $S = s$, which we denote by $\mathcal{R}_\gamma(s)$.

$$\begin{aligned}\gamma &= P_{S^\star}[g(s, f(S^\star, \theta)) \leq \mathcal{R}_\gamma(s)] \\ &= P_{S^\star}[f(s, g(s, f(S^\star, \theta))) \leq f(s, \mathcal{R}_\gamma(s))] \\ &= P_{S^\star}[f(S^\star, \theta) \leq f(s, \mathcal{R}_\gamma(s))], \end{aligned} \tag{2.22}$$

where the last equality follows from $g(s, \cdot)$ and $f(s, \cdot)$ being inverses of each other. Equation (2.22) implies $f(s, \mathcal{R}_\gamma(s)) = e_\gamma$. Hence $\mathcal{R}_\gamma(s) = g(s, e_\gamma)$. Then the $100(1-\alpha)\%$ confidence interval based on $\mathcal{R}_\theta$ with a left tail probability of α_l and right tail probability of α_R is given by $\mathcal{R}_{\alpha_l}(s) \leq \theta \leq \mathcal{R}_{1-\alpha_r}(s)$. We have

$$\begin{aligned}\mathcal{R}_{\alpha_l}(s) \leq \theta \leq \mathcal{R}_{1-\alpha_r}(s) &\iff g(s, e_{\alpha_l}) \leq \theta \leq g(s, e_{1-\alpha_r}), \text{ by } \mathcal{R}_\gamma(s) = g(s, e_\gamma); \\ &\iff e_{\alpha_l} \leq f(s, \theta) \leq e_{1-\alpha_r}, \text{ since } f(s, \cdot) \text{ is an increasing function.}\end{aligned}$$

□

Chapter 3

ONE-SIDED TOLERANCE INTERVALS FOR LINEAR MIXED MODEL

A general discussion of tolerance intervals is in Section 1.1.1. Tolerance intervals for linear mixed models are of natural interest. As an example suppose one is interested in percentiles of the distribution of concentrations of pesticide on fruit trees within a grove. Such information is useful in establishing environmental safety standards and dosages for efficacy in killing pests. An experiment is run where several plots are selected from the grove, within each plot n_i trees are selected, and one measurement of the concentration of pesticide is made for each tree. The quantity of interest is the concentration value where $p\%$ of the trees have a concentration above that value (one is interested in determining whether the concentration is sufficient to kill insects). We can model the response from the j^{th} tree of the i^{th} plot as $Y_{ij} = \mu + A_i + e_{ij}$ with the usual normality assumptions. We are interested in a lower tolerance interval for $Y \sim N(\mu, \sigma_A^2 + \sigma_e^2)$. A similar example would be concentrations of nitrogen in lakes in a geographical area. One might be interested in an upper tolerance interval for the nitrogen concentration to determine if fertilizer run off from farms is a problem.

As another example, at an engine factory a number of engines are chosen at random. Each engine is tested a number of times for noise level at a fixed power level. Each time a test is run the noise level is tested a fixed number of times. The measured noise level could be modelled as $Y_{ijk} = \mu + A_i + B_{j(i)} + e_{ijk}$, where A_i is the engine effect, $B_{j(i)}$ is the test effect, which is nested in the engine effect, and e_{ijk} is the measurement effect. If one is interested in the noise threshold that a given percentage of the engines do not exceed when tested, then an upper tolerance interval for $\mu + A_i + B_{j(i)} \sim N(\mu, \sigma_A^2 + \sigma_B^2)$ is of interest.

As the last example, suppose one is interested in a lower threshold for the gas mileage on a Subaru Forester. A test is conducted where a random set of Foresters are selected and a random group of drivers is selected. Each car/driver combination goes over a fixed course several times and the miles per gallon is measured each time the course is driven. One might consider the model $Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + e_{ijk}$ for this scenario, where A_i is the random effect due to car, B_j is the random effect due to the driver, $(AB)_{ij}$ is the interaction of the driver and car, and e_{ijk} is random error. If we assume there is negligible measurement error and the course is constant, then we are interested in a lower tolerance bound for $W \sim N(\mu, \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2)$.

An extensive body of work exists on methods for tolerance intervals for the one-way random effects model and we will discuss some of these methods later in the chapter.

The rest of the chapter is organized as follows. First, we propose an FGPQ that can be used to construct tolerance intervals for distributions associated with a saturated balanced liner mixed model with replication. This will be followed by a description of recent developments for one-sided tolerance intervals for distribution associate with the balanced and unbalanced one-way random effects models. For the unbalanced random effects model we present the results of a simulation study for tolerance intervals constructed using the imputation technique discussed in Section 2.5. In the last section we will present a method for constructing tolerance intervals for the distributions of two population characteristics associated with the two-way random effects model.

3.1 One-sided Tolerance Intervals Under a Saturated Balanced Factorial Linear Mixed Model with Replication

Consider the saturated balanced factorial linear mixed model with replication $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{ZA} + \boldsymbol{e}$ defined in Section 2.4. We will use the notation developed in Section 2.4. It is of interest to construct one-sided $(p, 1-\alpha)$-tolerance intervals for random variables of the form $W = \boldsymbol{l}^T\boldsymbol{\beta} + \sum_{i=1}^{q} b_i C_i + b_{q+1}e$, where $C_i \sim N(0, \phi_i)$, $e \sim N(0, \phi_e)$, $(C_1, \ldots, C_{q+1}, e)$ are jointly independent, $\boldsymbol{l}$ is a vector of real numbers, and $b_1, \ldots, b_{q+1}$ are real numbers. One-sided tolerance bounds for the distribution of a random variable are equivalent to one-sided confidence bounds for the quantiles of the distribution (see Section 1.1.1). Our method for constructing one-sided tolerance bounds for W is based on finding generalized one-sided confidence bounds for the quantiles of the distribution of W. In the next proposition we define a FGPQ for the quantiles of the distribution of W.

Proposition 3.1. *Let θ denote the p^{th} quantile of the distribution of $W \sim N(\boldsymbol{l}^T\boldsymbol{\beta}, \sum_{i=1}^{q+1} b_i^2\phi_i)$. Then an FGPQ for θ is given by*

$$\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \mathcal{R}_l(\mathbb{S}, \mathbb{S}^\star, \xi) + z_p\sqrt{\max(0, \sum_{i=1}^{q+1} b_i^2 \mathcal{R}_{\phi_i}(\mathbb{S}, \mathbb{S}^\star, \xi))} \tag{3.1}$$

where $\mathcal{R}_l, \mathcal{R}_{\phi_1} \ldots \mathcal{R}_{\phi_{q+1}}$ are defined in Proposition 2.2.

Proof. Note $\theta = \boldsymbol{l}^T\boldsymbol{\beta} + z_p\sqrt{\sum_{i=1}^{q+1} b_i^2\phi_i}$. Into this expression one substitutes the FGPQs for the parameters of the saturated balanced factorial linear mixed model from Proposition 2.2. The maximum is needed since the $\mathcal{R}_{\phi_i}$s can be negative. □

We remind the reader that a lower $(p, 1-\alpha)$ tolerance bound for W is equivalent to a lower $100(1-\alpha)\%$ confidence bound for the $(1-p)^{\text{th}}$ quantile of the distribution of W, and an upper $(p, 1-\alpha)$ tolerance bound for W is equivalent to an upper $100(1-\alpha)\%$ confidence bound for the p^{th} quantile of the distribution of W. We now give a construction of lower and upper $(p, 1-\alpha)$ tolerance bounds for W based on FGPQs for quantiles of the distribution of W.

Corollary 3.1. *Let $\theta = \boldsymbol{l}^T\boldsymbol{\beta} - z_p\sqrt{\sum_{i=1}^{q+1} b_i^2\phi_i}$ be the $(1-p)^{th}$ quantile of distribution for the random variable W and let $R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ be the FGPQ for θ defined in Proposition 3.1. Then a lower $(p, 1-\alpha)$ tolerance bound for W is given by $\mathcal{R}_{\theta,\alpha}$, where $\mathcal{R}_{\theta,\alpha}$ is the α^{th} quantile of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathbf{s}$. An upper $(p, 1-\alpha)$ tolerance bound for W is given by $\mathcal{R}_{\eta,1-\alpha}$, where $\mathcal{R}_{\eta,1-\alpha}$ is the $(1-\alpha)^{th}$ quantile of the conditional distribution of $\mathcal{R}_\eta$ given $\mathbb{S} = \mathbf{s}$, with $\eta = \boldsymbol{l}^T\boldsymbol{\beta} + z_p\sqrt{\sum_{i=1}^{q+1} b_i^2\phi_i}$.*

3.2 One-way Random Effects Model

Consider the one-way random effects model introduced in Example 2.2. Tolerance intervals for population characteristics of the balanced and unbalanced one-way random effects models have been studied by many authors. The population characteristics that are of interest are the "measured value" $Y \sim N(\mu, \sigma_A^2 + \sigma_e^2)$, where Y has the same distribution as any of the Y_{ij}'s, and the "true" value $\mu + A \sim N(\mu, \sigma_A^2)$, where A has the same distribution as any of the A_i's. Although this discussion will be restricted to one-sided tolerance bounds, there have been methods proposed for constructing two-sided tolerance intervals, see for example Liman and Thomas [37], Mee [40], Beckman and Tietjen [1], and Wang and Iyer [53]. For simplicity we will restrict the discussion to upper tolerance bounds. As stated previously, finding an upper $(p, 1-\alpha)$ tolerance bound for either population characteristic is equivalent to finding a $100(1-\alpha)\%$ upper confidence bound for the p^{th} quantile of the distribution of the population characteristic.

This can be stated as finding a statistic U with $P[\mu + z_p\sigma_* \leq U] = 1-\alpha$, where $*$ is either Y or A. Several methods suggested are based on finding k and $\hat{\sigma}_*$ such that

$$P[\mu + z_p\sigma_* \leq \bar{Y} + k\hat{\sigma}_*] = 1-\alpha, \tag{3.2}$$

where k is a function of p, α and the data, and again $*$ is either Y or A. All the methods proposed to date are approximate. Initially we consider balanced data and later discuss unbalanced data. For the measured value Y, methods have been suggested by Lemon [35], Mee and Owen [41], Vangel [52] and Mosier [43]. The methods proposed by Lemon, and Mee and Owen are conservative. Vangel's and Moser's methods were independent proposals of the same method. The Vangel-Mosier method is based on iteratively solving an integral equation. This method produces tolerance bounds with close to exact coverage. For the true value $\mu + A$ Mosier [43] proposed a method for constructing tolerance intervals.

The method given in Proposition 3.1 for upper $(p, 1-\alpha)$ tolerance bounds based on FGPQs is applies to both Y and $\mu + A$. Let $\theta = \mu + z_p\sigma_Y$ and $\eta = \mu + z_p\sigma_A$, then FGPQs for θ and η are given by

$$\mathcal{R}_\theta = \mathcal{R}_\mu + z_p\mathcal{R}_{\sigma_Y} \qquad \text{and} \qquad \mathcal{R}_\eta = \mathcal{R}_\mu + z_p\mathcal{R}_{\sigma_A}, \tag{3.3}$$

where $\mathcal{R}_\mu$ is given in Equation (2.5), $\mathcal{R}_{\sigma_Y}$ is given in Equation (2.6), and $\mathcal{R}_{\sigma_A}$ is given in Equation (2.7). As pointed out in Theorem 2.2, given a realization $\mathbf{s}$ of $\mathbb{S} = (\bar{X}, S_B^2, S_W^2)$, an upper $(p, 1-\alpha)$ tolerance bound is given by $\mathcal{R}_{\theta,1-\alpha}(\mathbf{s})$, where $\mathcal{R}_{\theta,1-\alpha}(\mathbf{s})$ is the $(1-\alpha)^{\text{th}}$ quantile of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathbf{s}$. Krisnamoorthy and Mathew [34] used simulation study to compar the performance of $\mathcal{R}_{\theta,1-\alpha}(\mathbf{s})$ with Vangel's method and found that except when n is very big and the intraclass correlation is small Vangel's method appears to perform quite well. While $\mathcal{R}_{\theta,1-\alpha}(\mathbf{s})$ appears to be conservative when the intra-class correlation is small but otherwise performs quite well. Krisnamoorthy and Mathew's simulation study results indicate that, except when σ_A^2 is small, the tolerance intervals based on $\mathcal{R}_\eta$ perform well. When σ_A^2 is small the tolerance bound $\mathcal{R}_{\eta,1-\alpha}(\mathbf{s})$ appears to be conservative.

We now discuss the methods have been proposed by other authors for unbalanced data. If $R = \sigma_A^2/\sigma_e^2$ is known, then the random effects model reduces to a fixed effects model and a solution to Equation (3.2) is known, see for instance Graybill [19]. To obtain a tolerance bound for the random effects model one can use a suitable estimate of R in the solution for the fixed effects model. This method, which we call the "known R" method, follows the spirit of a method Harville [26] proposed for solving prediction and confidence interval problems. Bhaumik and Kulkarni [3] [4] proposed a procedure for constructing tolerance intervals for Y in the unbalanced case. Their proposed method turns out to be the same as the "known R" method. They suggest using the estimates of Thomas and Hultquist [50] to estimate R. In the unbalanced case, Krisnamoorthy and Mathew [34] propose two methods for constructing upper tolerance bounds for Y and two similar methods for constructing upper tolerance bounds for$\mu + A$. We outline their methods for constructing the two upper tolerance bounds for Y. While the outline does not use the same notation they did, it is the same in spirit and uses the notation we have developed.

M1: Using the Thomas and Hultquist estimate for R, Krisnamoorthy and Mathew define an approximate invertible pivotal quantity $\boldsymbol{f}_A(\mathbb{S},\xi)$ for ξ, i.e, the distribution of $\boldsymbol{f}_A(\mathbb{S},\xi)$ is approximated by a distribution F which does not depend on ξ. Let $\mathbb{E}_A$ be a random variable with the distribution F. They use the recipe in Theorem 2.2 to construct an approximate FGPQ, $\mathcal{R}_{A,\theta}$, which is given by

$$\mathcal{R}_{A,\theta} = \mathcal{R}_{A,\theta}(\mathbb{S},\mathbb{E}_A^\star) = \pi_\theta(\boldsymbol{g}_A(\mathbb{S},\mathbb{E}_A^\star)),$$

where $\mathbb{E}_A^\star$ is an independent copy of $\mathbb{E}_A$ and $\boldsymbol{g}_A$ is the inverse function for $\boldsymbol{f}_A$. The first upper $(p, 1-\alpha)$ tolerance bound they propose is $R_{A,\theta,1-\alpha}(\mathbf{s})$, where $R_{A,\theta,\gamma}(\mathbf{s})$ is the γ^{th} quantile of the distribution of $\mathcal{R}_{A,\theta}(\mathbb{S},\mathbb{E}_A^\star)$ conditional on $\mathbb{S} = \mathbf{s}$.

M2: Their second tolerance bound is an approximation to $R_{A,\theta,1-\alpha}(\mathbf{s})$ that does not involve the conditional distribution, i.e., it is only a function of $\mathbf{s}$, a, the n_i's, p, and α.

For A they used $\pi_{\tilde{\theta}}$ in place of π_θ in M1 and in M2 find an approximation to $R_{A,\tilde{\theta},1-\alpha}(\mathbf{s})$ that does not involve the conditional distribution. Krisnamoorthy and Mathew report the results of simulation studies they conducted to assess the performance of their proposed tolerance bounds. For the random variable Y they conclude that method M1 performed satisfactorily and better than M2 for some parameter combinations. For the population characteristic $\mu + A$ they checked the performance of upper tolerance bound based on Method M2 and found it produced satisfactory coverage.

We propose using the imputation method from Section 2.5 in conjunction with Proposition 3.1 to construct one-sided tolerance bounds for $\mu + A$. We conducted a simulation study to compare the performance of the tolerance bounds based the imputation method to the tolerance bounds based on Method M2. The results of the simulation study are in Table 3.1. The parameter combinations used in the simulation study included all the parameter combinations used by Krisnamoorthy and Mathew in [34]. In the simulation study 10,000 realization of the data were generated for each parameter combination. To assess the performance the empirical coverage probability was calculated along with an empirical estimate of the expected absolute distance of the tolerance bound from the true value. The results show the imputation method appears to be quite conservative for smaller values of $\rho = \sigma_A^2/(\sigma_A^2 + \sigma_e^2)$. For larger values of ρ the empirical coverage is close to exact and the empirical estimates of the distance are not statistically different from those reported by Krisnamoorthy and Mathew. Although it is clear that for the unbalanced one-way random effects model the tolerance bounds based on Method M2 are superior to the imputation method, the performance of the imputation method indicates that for more general models the method should be considered. A copy of the computer program used to implement the simulation study is in Section C.1.

3.3 Two-way Random Effects Model

The saturated two-way random effects model was introduced in Example 2.7. We use the notation and results presented in Example 2.7. One sided tolerance bounds for two

Table 3.1: Monte Carlo Estimates of the Coverage Probability and Expected Value of the Absolute Distance of the $(p, 1-\alpha)$ Tolerance Bound from the True Value, $Q = \mu + z_p \sigma_A^2$, for the Random Variable $\mu + A$. Parameter values: $\mu = 0$, $p = .9$ and $\rho = \sigma_A^2/(\sigma_A^2 + \sigma_e^2)$

n_1	n_2	n_3	ρ (Q)	$1-\alpha$ .90	.95	.99
2	3	3	0.00	0.995	0.999	1.000
			0.00	3.57	6.42	25.78
			0.20	0.985	0.996	1.000
			0.64	3.95	6.74	24.20
			0.40	0.984	0.997	1.000
			1.05	4.57	7.41	23.53
			0.50	0.974	0.994	1.000
			1.28	5.05	8.01	24.06
			0.80	0.945	0.987	1.000
			2.56	8.12	12.08	30.12
			0.90	0.921	0.975	0.999
			3.84	11.64	17.03	40.33
			0.95	0.908	0.966	0.998
			5.59	16.58	24.11	55.92
			0.99	0.896	0.951	0.995
			12.75	37.40	54.08	123.69
4	2	5	0.00	0.994	0.998	1.000
			0.00	3.02	5.54	22.99
			0.20	0.980	0.995	1.000
			0.64	3.57	6.13	22.32
			0.40	0.976	0.995	1.000
			1.05	4.27	6.91	21.78
			0.50	0.973	0.995	0.999
			1.28	4.75	7.50	22.15
			0.80	0.939	0.989	1.000
			2.56	8.03	11.94	29.62
			0.90	0.917	0.978	1.000
			3.84	11.62	17.00	40.23
			0.95	0.899	0.961	1.000
			5.59	16.59	24.10	55.83
			0.99	0.895	0.948	0.995
			12.75	37.22	53.85	123.26

n_1	n_2	n_3	n_4	ρ (Q)	$1-\alpha$ .90	.95	.99
2	3	3	2	0.00	1.000	1.000	1.000
				0.00	3.04	5.14	18.48
				0.20	0.996	1.000	1.000
				0.64	3.17	5.01	15.64
				0.40	0.989	0.998	1.000
				1.05	3.52	5.22	13.87
				0.50	0.977	0.997	1.000
				1.28	3.86	5.55	13.55
				0.80	0.927	0.977	0.999
				2.56	6.21	8.33	16.23
				0.90	0.902	0.958	0.997
				3.84	8.97	11.83	21.75
				0.95	0.902	0.956	0.994
				5.59	13.03	17.06	30.65
				0.99	0.898	0.949	0.990
				12.75	29.21	38.17	67.96
5	2	12	5	0.00	0.998	1.000	1.000
				0.00	2.05	3.73	15.68
				0.20	0.990	0.999	1.000
				0.64	2.44	3.90	12.54
				0.40	0.981	0.999	1.000
				1.05	3.02	4.42	11.34
				0.50	0.970	0.998	1.000
				1.28	3.45	4.87	11.17
				0.80	0.913	0.976	1.000
				2.56	6.05	8.05	15.16
				0.90	0.903	0.962	1.000
				3.84	8.89	11.70	21.21
				0.95	0.901	0.954	0.997
				5.59	12.87	16.85	30.19
				0.99	0.899	0.949	0.992
				12.75	29.16	38.08	67.75

population characteristics associated with the balanced two-way random effects model are of interest. The population characteristics of interest are Y_{ijk} and $\mu + A_i$. If Y_{ijk} represents the repeated measurements of multiple copies of an item (factor A_i) by several technicians (factor B_j), then the population characteristics that are of interest are the measured values $Y_{ijk} \sim N(\mu, \sigma_Y^2)$ and true values $\mu + A_i \sim N(\mu, \sigma_A^2)$. Our literature search did not reveal any papers on tolerance intervals for two-way random effects models.

In the next two sections we will present a method for constructing $(p, 1-\alpha)$ lower tolerance bounds for measured value and the true value for the two-way random effects model with an interaction factor and without interaction respectively. Our proposed method is to find a fiducial generalized pivotal quantity for the $(1-p)^{\text{th}}$ quantile and use the associated generalized lower $100(1-\alpha)\%$ confidence bound for the $(1-p)^{\text{th}}$ quantile. The generalized confidence bounds are only approximate and calculating the exact coverage probability is intractable, so a Monte Carlo simulation will be used to check the performance.

The penultimate section covers upper tolerance bounds. The last section contains the results of the simulation study used to evaluate the performance of the lower tolerance bounds and a discussion of the results.

3.4 Balanced Two-way Random Effects Model with Interaction

As stated above there is an interest in finding lower $(p, 1-\alpha)$ tolerance bounds for the distributions of the two population characteristics the measured value Y_{ijk} and the true value $\mu + A_i$. This section begins with notation and results that are common to the proposed methods for constructing lower $(p, 1-\alpha)$ tolerance bounds for the two population characteristics. This is followed by a subsection on the proposed method for finding lower tolerance bounds for distribution of the measured value. The subsection also contains results that are useful in the Monte Carlo simulation that was conducted to measure the performance of the lower tolerance bounds. This is followed by a a subsection on tolerance intervals for the distribution of the true value.

In Example 2.7 it was shown that

$$S = (\bar{Y}_{...}, SSA, SSB, SSAB, SSE) \quad \text{and} \quad \xi = (\mu, \sigma_A^2, \sigma_B^2, \sigma_{AB}^2, \sigma_e^2)$$

have a inverse pivotal relationship and using Theorem 2.2 FGPQs were constructed for each of the components of ξ. Example 2.7 contains the definitions of the sums of squares, the invertible pivotal quantity $\boldsymbol{f}$, the inverse function $\boldsymbol{g}$, and the expected mean squares. It will be useful to express the FGPQs in their canonical representations (see Remark 2.5). One derives the inverse function $\boldsymbol{h}(\mathbb{E}, \xi)$ from (2.8). We use the mean squares, see (2.9), to simplify the notation.

$$\begin{aligned} \bar{Y} &= h_0(\mathbb{E}, \xi) = \mu + E_0\sigma_{\bar{Y}} && && (3.4)\\ SSA &= h_1(\mathbb{E}, \xi) = E_1\psi_A, && SSB = h_2(\mathbb{E}, \xi) = E_2\psi_B \\ SSAB &= h_3(\mathbb{E}, \xi) = E_3\psi_{AB} && SSE = h_4(\mathbb{E}, \xi) = E_4\sigma_e^2. \end{aligned}$$

The FGPQ for μ constructed using the structural method of Theorem 2.2 was presented in Example 2.7 and will be used in this subsection and in next subsection. For convenience we state the needed properties of the FGPQ here. As mentioned in Remark 2.4, we can also express $\mathcal{R}_\mu$ as the representation $\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) = g_0(\mathbb{S}, \mathbb{E}^\star)$. The canonical representation $\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi)$ follows from Equations (3.4). For ease of reference we repeat the Equation (2.10) here and present the other two representations.

$$\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = \bar{Y} - \left(\frac{\bar{Y}^\star - \mu}{\sigma_{\bar{Y}}}\right)\sqrt{\max\left(0, \frac{SSA}{SSA^\star}\psi_A + \frac{SSB}{SSB^\star}\psi_B - \frac{SSAB}{SSAB^\star}\psi_{AB}\right)\frac{1}{abn}} \tag{3.5a}$$

$$\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) = \bar{Y} - E_0^\star\sqrt{\max\left(0, \frac{SSA}{E_1^\star} + \frac{SSB}{E_2^\star} - \frac{SSAB}{E_3^\star}\right)\frac{1}{abn}} \tag{3.5b}$$

$$\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi) = \mu + E_0\sigma_{\bar{Y}} - E_0^\star\sqrt{\max\left(0, \frac{E_1}{E_1^\star}\psi_A + \frac{E_2}{E_2^\star}\psi_B - \frac{E_3}{E_3^\star}\psi_{AB}\right)\frac{1}{abn}} \tag{3.5c}$$

Remark 3.1. Equation (3.5c) will help show the coverage probabilities of our proposed tolerance intervals are only a function of a, b, n and the following ratios:

$$\tau_A = \frac{\sigma_A}{\sigma_e}, \quad \tau_B = \frac{\sigma_B}{\sigma_e}, \quad \tau_{AB} = \frac{\sigma_{AB}}{\sigma_e}.$$

For example see Proposition 3.2. Note that the following expressions involve only a, b, n, and the ratios τ_A, τ_B and τ_{AB}

$$\sigma_e^{-1}\sigma_{\bar{Y}}, \quad \sigma_e^{-2}\psi_A, \quad \sigma_e^{-2}\psi_B, \quad \sigma_e^{-2}\psi_{AB}.$$

3.4.1 Lower Tolerance Intervals for $Y \sim N(\mu, \sigma_Y^2)$

We are interested in lower $(p, 1-\alpha)$ tolerance intervals for the measured value $Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + e_{ijk}$. Note that $Y_{ijk} \sim N(\mu, \sigma_Y^2)$, where $\sigma_Y^2 = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma_e^2$. The $(1-p)^{\text{th}}$ quantile of Y is given by $\theta = \mu - z_p\sigma_Y$. The recipe in Theorem 2.2 implies a FGPQ for θ is the given by $\mathcal{R}_\theta = \mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_Y}$, where $\mathcal{R}_{\sigma_Y} = \sqrt{g_1(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi)) + \cdots + g_4(\mathbb{S}, \boldsymbol{f}(\mathbb{S}^\star, \xi))}$. Evaluating this last expression yields

$$\mathcal{R}_{\sigma_Y} = \sqrt{\frac{1}{bn}\frac{SSA}{SSA^\star}\psi_A + \frac{1}{an}\frac{SSB}{SSB^\star}\psi_B + \frac{ab-(a+b)}{abn}\frac{SSAB}{SSAB^\star}\psi_{AB} + (1-\frac{1}{n})\frac{SSE}{SSE^\star}\sigma_e^2}$$

Note that if $a, b \geq 2$, then $ab > (a+b)$ and the above expression is well defined. So without further comment we will assume a and b are greater than 1. A lower $100(1-\alpha)\%$

generalized confidence bound for θ is given by $\mathcal{R}_{\theta,\alpha}$, where $\mathcal{R}_{\theta,\gamma}$ denotes the 100γ-percentile of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathbf{s}$.

Given a realization $\mathbf{s}$ of $\mathbb{S}$, the distribution of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$ is usually intractable. One can estimate $\mathcal{R}_{\theta,\alpha}$ using the Monte Carlo method. We will outline the process for finding the estimate. The simulation is conducted using the representation $\dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star) = \dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) - z_p \dot{\mathcal{R}}_{\sigma_Y}(\mathbb{S}, \mathbb{E}^\star)$, where $\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)$ is given in (3.5b) and

$$\dot{\mathcal{R}}_{\sigma_Y}(\mathbb{S}, \mathbb{E}^\star) = \sqrt{\frac{1}{bn}\frac{SSA}{E_1^\star} + \frac{1}{an}\frac{SSB}{E_2^\star} + \frac{ab-(a+b)}{abn}\frac{SSAB}{E_3^\star} + (1-\frac{1}{n})\frac{SSE}{E_4^\star}}.$$

We remind the reader that $E_0^\star \sim N(0,1)$, $E_1^\star \sim \chi^2_{a-1}$, $E_2^\star \sim \chi^2_{b-1}$, $E_3^\star \sim \chi^2_{(a-1)(b-1)}$, $E_4^\star \sim \chi^2_{ab(n-1)}$ and $E_0^\star, \ldots, E_4^\star$ are independent. Let $\mathbb{E}^\star$ to denote the vector $(E_0^\star, \ldots, E_4^\star)$. Given a realization $\mathbf{s}$ of $\mathbb{S}$, one generates N independent realizations, $(\mathbf{e}_1^\star, \ldots, \mathbf{e}_N^\star)$, of $\mathbb{E}^\star$ and then calculates $\dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_1^\star), \ldots, \dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_N^\star)$. If these N values are ordered from smallest to largest, then the $[\alpha N]^{th}$ element of the ordered sequence gives an estimate of $\mathcal{R}_{\theta,\alpha}$.

Let $F_\mathbf{s}(t)$ denote the conditional cdf of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. The coverage of the proposed generalized lower confidence bound is given by $P_\mathbb{S}\left[F_\mathbb{S}^{-1}(\alpha) \leq \theta\right]$. This is equal to

$$P_\mathbb{S}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \theta | \mathbb{S}\right] > \alpha\right]. \tag{3.6}$$

The calculation of this probability is intractable, so a Monte Carlo simulation will be used to estimate it.

The following proposition is useful in choosing the parameter combinations for the simulation.

Proposition 3.2. *Expression (3.6) depends only on p, α, a, b, n, and the ratios τ_A, τ_B and τ_{AB} (see Remark 3.1).*

Proof. Note that

$$R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \theta \iff \frac{R_\mu - \mu}{\sigma_e} - z_p\frac{R_{\sigma_Y}}{\sigma_e} \leq -z_p\frac{\sigma_Y}{\sigma_e}. \tag{3.7}$$

The canonical representation of $\mathcal{R}_{\sigma_Y}$ is given by

$$\ddot{\mathcal{R}}_{\sigma_Y}(\mathbb{E}, \mathbb{E}^\star, \xi) = \sqrt{\frac{1}{bn}\frac{E_1}{E_1^\star}\psi_A + \frac{1}{an}\frac{E_2}{E_2^\star}\psi_B + \frac{ab-(a+b)}{abn}\frac{E_3}{E_3^\star}\psi_{AB} + (1-\frac{1}{n})\frac{E_4}{E_4^\star}\sigma_e^2},$$

From the canonical representations $\ddot{\mathcal{R}}_{\sigma_Y}$, $\ddot{\mathcal{R}}_{\mu}$ (Equation 3.5c) ones see that $\frac{R_{\sigma_Y}}{\sigma_e}$ and $\frac{R_{\mu}-\mu}{\sigma_e}$ are a function of $(\mathbb{E}, \mathbb{E}^\star)$, a, b, n, and the ratios τ_A, τ_B and τ_{AB}. Thus by (3.7), $R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) \le \theta$ is a function of $(\mathbb{E}, \mathbb{E}^\star)$, a, b, n, and the ratios τ_A, τ_B and τ_{AB}. Since $E_0, \ldots, E_4$, $E_0^\star, \ldots, E_4^\star$ are jointly independent and their distributions are free of unknown parameters, the joint distribution of $(\mathbb{E}, \mathbb{E}^\star)$ is free of unknown parameters, and the result follows. □

Next we show that lower $(p, 1-\alpha)$ tolerance bounds that are based on $\mathcal{R}_\theta$ have asymptotic correct coverage as the number of levels of the two random factors go to infinity.

Proposition 3.3. *For $Y \sim N(0, \sigma_Y^2)$ from a balanced two-way random effects model with interaction let a and b go to infinity at a rate such that $\lim a/(a+b) = \lambda_A$ and $\lim b/(a+b) = \lambda_B$ exist with $0 < \lambda_A, \lambda_B < 1$. Then a lower $(p, 1-\alpha)$ tolerance interval for Y based on the FGPQ $\mathcal{R}_\theta$ has asymptotically $100(1-\alpha)\%$ frequentist coverage.*

Proof. Let $c = a + b$ and let n be fixed. We will verify the conditions of Theorem 2.1. By standard results in mixed linear models and Serfling (Theorem 3.3.A [48]) we have

$$\sqrt{c}\left(\bar{Y} - \mu, \frac{b(a-1)}{SSA} - \frac{1}{n\sigma_A^2}, \frac{a(b-1)}{SSB} - \frac{1}{\sigma_B^2}, \frac{(a-1)(b-1)}{SSAB} - \frac{1}{\psi_{AB}}, \frac{ab(n-1)}{SSE} - \frac{1}{\sigma_e^2}\right) \xrightarrow{\mathcal{D}} \boldsymbol{N}$$

where

$$\boldsymbol{N} \sim MVN(\boldsymbol{0}, \Sigma), \text{ with } \Sigma = Diag(\frac{\sigma_A^2}{\lambda_A} + \frac{\sigma_A^2}{\lambda_A}, \frac{2}{n\lambda_A\sigma_A^4}, \frac{2}{n\lambda_B\sigma_B^4}, 0, 0).$$

Thus $\boldsymbol{t}(\xi) = (\mu, (n\sigma_A^2)^{-1}, (n\sigma_B^2)^{-1}, \psi_{AB}^{-1}, \sigma_e^{-2})$ satisfies condition 1 of Assumptions 2.1.

We will next verify that condition 2 of Assumptions 2.1 is satisfied. To simplify the notation let $x_1 = \frac{b(a-1)}{ssA}$, $x_2 = \frac{a(b-1)}{ssB}$, $x_3 = \frac{(a-1)(b-1)}{ssAB}$ and $x_4 = \frac{ab(n-1)}{ssE}$. Then $\mathrm{s} = (\bar{y}, x_1, x_2, x_3, x_4)$. Let $\mathrm{s}^\star$ be an independent copy of s. There exists an open set $\mathcal{O}$ containing $\boldsymbol{t}(\xi)$ with the following three properties: (i) for s in $\bar{\mathcal{O}}$ the components x_1, x_2, x_3 and x_4 of s are non-zero, (ii) $\bar{\mathcal{O}}$ is compact, and (iii) $(x_1^{-1}x_1^\star\psi_A + x_2^{-1}x_2^\star\psi_B - x_3^{-1}x_3^\star\psi_{AB})$ is greater than zero for all $(\mathrm{s}, \mathrm{s}^\star) \in \mathcal{O} \times \mathcal{O}$; this condition implies for $(\mathrm{s}, \mathrm{s}^\star) \in \mathcal{O} \times \mathcal{O}$ the max function is not necessary in $\mathcal{R}_\mu(\mathrm{s}, \mathrm{s}^\star, \xi)$.

Since $\mathcal{R}_\theta = \mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_Y}$, to show $\mathcal{R}_\theta$ satisfies items 2a and 2c of Assumptions 2.1, it suffices to show $\mathcal{R}_{\sigma_Y}$ and $\mathcal{R}_\mu$ satisfy items 2a and 2c. The following three limits will be useful

$$\lim_{c\to\infty} \frac{\psi_A}{c} = n\lambda_b\sigma_A^2, \quad \lim_{c\to\infty} \frac{\psi_B}{c} = n\lambda_a\sigma_B^2, \quad \text{and} \quad \lim_{c\to\infty} \frac{\psi_{AB}}{c} = 0.$$

For $(\mathbf{s}, \mathbf{s}^\star) \in \mathcal{O} \times \mathcal{O}$

$$\mathcal{R}_\mu(\mathbf{s}, \mathbf{s}^\star, \xi) = \bar{y} - \left(\frac{\bar{y}^\star - \mu}{\sigma_{\bar{Y}}}\right) \sqrt{(abn)^{-1}(x_1^{-1}x_1^\star\psi_A + x_2^{-1}x_2^\star\psi_B - x_3^{-1}x_3^\star\psi_{AB})}$$
$$= \bar{y} - (\bar{y}^\star - \mu) \sqrt{\frac{x_1^{-1}x_1^\star\psi_A + x_2^{-1}x_2^\star\psi_B - x_3^{-1}x_3^\star\psi_{AB}}{\psi_A + \psi_B - \psi_{AB}}}.$$

To facilitate showing $\mathcal{R}_\mu$ satisfies items 2a and 2c of Assumptions 2.1 let

$$f_\mu(\mathbf{s}, \mathbf{s}^\star) = \frac{x_1^{-1}x_1^\star\psi_A + x_2^{-1}x_2^\star\psi_B - x_3^{-1}x_3^\star\psi_{AB}}{\psi_A + \psi_B - \psi_{AB}}.$$

As $c \to \infty$, $f_\mu(\mathbf{s}, \mathbf{s}^\star)$ converges uniformly on $\mathcal{O} \times \mathcal{O}$ to

$$g_\mu(\mathbf{s}, \mathbf{s}^\star) = \frac{x_1^{-1}x_1^\star n\lambda_b\sigma_A^2 + x_2^{-1}x_2^\star n\lambda_a\sigma_B^2}{n\lambda_b\sigma_A^2 + n\lambda_a\sigma_B^2}.$$

Note that g_μ is continuous and positive at any $(\mathbf{s}, \mathbf{s}^\star) \in \mathcal{O} \times \mathcal{O}$. For $\mathbf{s} \in \mathcal{O}$ as $c \to \infty$ we have

M.1 $\frac{\partial}{\partial \bar{y}^\star}\mathcal{R}_\mu(\mathbf{s}, \mathbf{s}^\star, \xi)\Big|_{\mathbf{s}^\star = t(\xi)} = [f_\mu(\mathbf{s}, \mathbf{t}(\xi))]^{\frac{1}{2}}$ converges uniformly to $[g_\mu(\mathbf{s}, \mathbf{t}(\xi))]^{\frac{1}{2}}$.

M.2 $\frac{\partial}{\partial x_i^\star}\mathcal{R}_\mu(\mathbf{s}, \mathbf{s}^\star, \xi)\Big|_{\mathbf{s}^\star = t(\xi)} = 0$ for $i = 1, 2, 3, 4$.

Thus $\mathcal{R}_\mu$ satisfies item 2a of Assumptions 2.1. When verifying item 2a is satisfied by $\mathcal{R}_\mu$ the following notation and result will be useful. Let $\psi_1 = \psi_A$, $\psi_2 = \psi_B$ and $\psi_3 = \psi_{AB}$; then $\lim_{c\to\infty} \frac{\psi_i}{\psi_A + \psi_B - \psi_{AB}} = L_i$ is finite for $i = 1, 2, 3$. Any second partial derivatives $\frac{\partial^2}{\partial x_4^\star s_j^\star}$ or $\frac{\partial^2}{\partial \bar{y}^\star \bar{y}^\star}$ of $\mathcal{R}_\mu$ are zero. The remaining second order partial derivatives are divided into the following two cases. First, $\frac{\partial^2}{\partial x_i^\star \bar{y}^\star}$ for $i = 1, 2, 3$ and second, $\frac{\partial^2}{\partial x_i^\star x_j^\star}$ with $i, j = 1, 2, 3$. For $(\mathbf{s}, \mathbf{s}^\star) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$ as $c \to \infty$

- For $i = 1, 2, 3$, $\frac{\partial^2}{\partial x_i^\star \bar{y}^\star}\mathcal{R}_\mu(\mathbf{s}, \mathbf{s}^\star, \xi) = \frac{1}{2}\left[f_\mu(\mathbf{s}, \mathbf{s}^\star)\right]^{-\frac{1}{2}} x_i^{-1} \dfrac{\psi_i}{\psi_A + \psi_B - \psi_{AB}}$ converges uniformly to $\frac{1}{2}\left[g_\mu(\mathbf{s}, \mathbf{s}^\star)\right]^{-\frac{1}{2}} x_i^{-1} L_i$.

- For $i, j = 1, 2, 3$, $\frac{\partial^2}{\partial x_i^\star x_j^\star}\mathcal{R}_\mu(\mathbf{s}, \mathbf{s}^\star, \xi) = -\frac{3}{4}(\bar{y}^\star - \mu)\left[f_\mu(\mathbf{s}, \mathbf{s}^\star)\right]^{-\frac{3}{2}} x_i^{-1} x_j^{-1} \frac{\psi_i\psi_j}{(\psi_A + \psi_B - \psi_{AB})^2}$ converges uniformly to $-\frac{3}{4}\left[g_\mu(\mathbf{s}, \mathbf{s}^\star)\right]^{-\frac{3}{2}} x_i^{-1} x_j^{-1} L_i L_j$.

Since $(\mathbf{s}, \mathbf{s}^\star) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$ is a compact set, a uniform bound on the second partial derivatives with respect to $\mathbf{s}^\star$ exists for all a, b, n and all $(\mathbf{s}, \mathbf{s}^\star)$ in $\mathcal{O} \times \mathcal{O}$. Thus $\mathcal{R}_\mu$ satisfies item 2a of Assumptions 2.1.

We will now show $\mathcal{R}_{\sigma_Y}$ satisfies items 2a and 2c of Assumptions 2.1. To facilitate the presentation note

$$\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi) = \sqrt{\frac{1}{bn}x_1^{-1}x_1^\star\psi_A + \frac{1}{an}x_2^{-1}x_2^\star\psi_B + \frac{ab-a-b}{abn}x_3^{-1}x_3^\star\psi_{AB} + \frac{n-1}{n}x_4^{-1}x_4^\star\sigma_e^2}.$$

To simplify evaluating the partial derivatives of $\mathcal{R}_{\sigma_Y}$ with respect to $\mathbf{s}^\star$, let

$$f_\sigma(\mathbf{s}) = \sqrt{\frac{\psi_A}{bn\sigma_A^2}x_1^{-1} + \frac{\psi_B}{an\sigma_B^2}x_2^{-1} + \frac{ab-a-b}{abn}x_3^{-1} + \frac{n-1}{n}x_4^{-1}}.$$

As $c \to \infty$, $f_\sigma(\mathbf{s})$ converges uniformly in $\mathbf{s} \in \mathcal{O}$ to $g_\sigma(\mathbf{s}) = \sqrt{x_1^{-1} + x_2^{-1} + \frac{1}{n}x_3^{-1} + \frac{n-1}{n}x_4^{-1}}$. For $\mathbf{s} \in \mathcal{O}$ as $c \to \infty$ we have

S.1 $\frac{\partial}{\partial \bar{y}^\star}\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} = 0.$

S.2 $\frac{\partial}{\partial x_1^\star}\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} = \frac{1}{2}\left[f_\sigma(\mathbf{s})\right]^{-1}\frac{1}{bn}\psi_A x_1^{-1}$ converges uniformly to $\frac{1}{2}\left[g_\sigma(\mathbf{s})\right]^{-1}x_1^{-1}\sigma_A$.

S.3 $\frac{\partial}{\partial x_2^\star}\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} = \frac{1}{2}\left[f_\sigma(\mathbf{s})\right]^{-1}\frac{1}{an}\psi_B x_2^{-1}$ converges uniformly to $\frac{1}{2}\left[g_\sigma(\mathbf{s})\right]^{-1}x_2^{-1}\sigma_B$.

S.4 $\frac{\partial}{\partial x_3^\star}\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} = \frac{1}{2}\left[f_\sigma(\mathbf{s})\right]^{-1}\frac{ab-a-b}{abn}x_3^{-1}\psi_{AB}$ converges uniformly to $\frac{1}{2}\left[g_\sigma(\mathbf{s})\right]^{-1}\frac{1}{n}x_3^{-1}\psi_{AB}$.

S.5 $\frac{\partial}{\partial x_4^\star}\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} = \frac{1}{2}\left[f_\sigma(\mathbf{s})\right]^{-1}\frac{n-1}{n}x_4^{-1}\psi_{AB}$ converges uniformly to $\frac{1}{2}\left[g_\sigma(\mathbf{s})\right]^{-1}\frac{n-1}{n}x_4^{-1}\sigma_e^2$.

Thus $\mathcal{R}_{\sigma_Y}$ satisfies item 2a of Assumptions 2.1. Any second partial derivative $\frac{\partial^2}{\partial \bar{y}^\star s_j^\star}$ of $\mathcal{R}_{\sigma_Y}$ is 0. All other second partial derivatives $\frac{\partial^2}{\partial x_i^\star x_j^\star}$ are of the form $(\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi))^{-3}\eta\, x_i^{-1}x_j^{-1}$, where η is a function of a, b, ψ_A, ψ_B, n, ψ_{AB} and σ_e^2, and $\lim_{c\to\infty}\eta = L$ exists. Note $\mathcal{R}_{\sigma_Y}(\mathbf{s},\mathbf{s}^\star,\xi)$ converges uniformly in $(\mathbf{s},\mathbf{s}^\star) \in \bar{\mathcal{O}}\times\bar{\mathcal{O}}$ to a continuous function. So the second partial derivatives of $\mathcal{R}_{\sigma_Y}$ also converge uniformly on $\bar{\mathcal{O}}\times\bar{\mathcal{O}}$ to a continuous function. Since $\bar{\mathcal{O}}\times\bar{\mathcal{O}}$ is a compact set, a uniform bound on the second partial derivative with respect to $\mathbf{s}^\star$ exists for all a, b, n and all $(\mathbf{s},\mathbf{s}^\star)$ in $\mathcal{O}\times\mathcal{O}$. Thus $\mathcal{R}_{\sigma_Y}$ satisfies item 2c of Assumptions 2.1.

To show $\mathcal{R}_\theta =$ satisfies item 2b of Assumptions 2.1, note that items S.1 and M.1 imply $\frac{\partial}{\partial \bar{y}^\star}\mathcal{R}_\theta(\mathbf{s},\mathbf{s}^\star,\xi)\Big|_{\mathbf{s}^\star=t(\xi)} > 0$ for $\mathbf{s} \in \mathcal{O}$. □

3.4.2 Lower Tolerance Intervals for $\mu + A \sim N(\mu, \sigma_A^2)$

We will give a method for constructing lower $100(1-\alpha)\%$ confidence intervals for the p^{th} percentile of the distribution of $\mu + A_i$. The p^{th} percentile is given by $\eta = \mu - z_p\sigma_A$. Using the recipe in Theorem 2.2 a FGPQ for η is given by $\mathcal{R}_\eta = \mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_A}$. The three representations of $\mathcal{R}_\mu$ are given in (3.5). If one uses $\pi_{\sigma_A}(\xi) = \sqrt{\max(0, \sigma_A^2)}$, the structural method in Theorem 2.2 produces the following FGPQ for σ_A. Included are the other two representations of $\mathcal{R}_{\sigma_A}$.

$$\mathcal{R}_{\sigma_A}(\mathbb{S}, \mathbb{S}^\star, \xi) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{SSA}{SSA^\star}\psi_A - \frac{SSAB}{SSAB^\star}\psi_{AB}\right)\right]}$$

$$\dot{\mathcal{R}}_{\sigma_A}(\mathbb{S}, \mathbb{E}^\star) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{SSA}{E_1^\star} - \frac{SSAB}{E_3^\star}\right)\right]}$$

$$\ddot{\mathcal{R}}_{\sigma_A}(\mathbb{E}, \mathbb{E}^\star, \xi) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{E_1}{E_1^\star}\psi_A - \frac{E_3}{E_3^\star}\psi_{AB}\right)\right]}$$

The generalized lower $100(1-\alpha)\%$ confidence bound for η is given by $\mathcal{R}_{\eta,\alpha}$, where $\mathcal{R}_{\eta,\alpha}$ is denotes the 100α-percentile of the conditional distribution of $\mathcal{R}_\eta$ given $\mathbb{S} = \mathbf{s}$.

Direct calculation of $\mathcal{R}_{\eta,\alpha}$ is intractable. However one can use a Monte Carlo simulation to obtain an approximation. The previous subsection indicates how this can be done using representation $\dot{\mathcal{R}}_\eta = \dot{\mathcal{R}}_\mu - z_p\dot{\mathcal{R}}_{\sigma_A}$.

The following proposition is useful in a Monte Carlo simulation to evaluate the coverage probabilities of the one-sided confidence interval $\mathcal{R}_{\eta,\alpha} \leq \eta$.

Proposition 3.4. *The expression*

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\eta(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \eta|\mathbb{S}\right] > \alpha\right].$$

depends only on p, α, a, b, n, and the ratios τ_A, τ_B and τ_{AB}.

Proof. With the substitution of the canonical representation of $\mathcal{R}_{\sigma_A}$ for the canonical representation of $\mathcal{R}_{\sigma_A}$, the proof is same as the proof of Proposition 3.2. We omit the details. □

Next we show that lower $(p, 1-\alpha)$ tolerance intervals that are based on $\mathcal{R}_\eta$ have asymptotic correct coverage as the number of levels of the two random factors go to infinity.

Proposition 3.5. *For $\mu + A \sim N(\mu, \sigma_A^2)$ from a balanced two-way random effects model with interaction let a and b go to infinity at rate such that* $\lim a/(a+b) = \lambda_A$ *and* $\lim b/(a+b) = \lambda_B$ *exist with* $0 < \lambda_A, \lambda_B < 1$. *Then the lower* $(p, 1-\alpha)$ *tolerance bounds based on the FGPQ* $\mathcal{R}_\eta$ *has asymptotically* $100(1-\alpha)\%$ *frequentist coverage.*

Proof. The proof is identical in spirit to the proof of Proposition 3.3. We omit the details.

□

3.5 Balanced Two-way Random Effects Model Without Interaction

We assume $\sigma_{AB}^2 = 0$. The minimal sufficient statistic for $\xi = (\mu, \sigma_A^2, \sigma_B^2, \sigma_e^2)$ is $\mathbb{S} = (\bar{Y}_{...}, SSA, SSB, SSEP)$, where $SSEP = SSAB + SSE$ is the pooled sums of squares for the error and the definition of the sums of squares in given in Example 2.7. The results in this section mirrors the results for the previous section on the model with interaction, the differences are only in the form of expressions and equations. The inverse pivotal quantity for ξ is given by

$$\begin{aligned}
f_0(\mathbb{S}, \xi) &= \frac{\bar{Y} - \mu}{\sqrt{\frac{bn\sigma_A^2 + an\sigma_B^2 + \sigma_e^2}{abn}}} = E_0 \sim N(0,1) \\
f_1(\mathbb{S}, \xi) &= \frac{SSA}{bn\sigma_A^2 + \sigma_e^2} = E_1 \sim \chi^2_{a-1} \\
f_2(\mathbb{S}, \xi) &= \frac{SSB}{an\sigma_B^2 + \sigma_e^2} = E_2 \sim \chi^2_{b-1} \\
f_3(\mathbb{S}, \xi) &= \frac{SSEP}{\sigma_e^2} = E_43 \sim \chi^2_{(a-1)(b-1)+ab(n-1)}
\end{aligned}$$

In the interest of simplifying the presentation we will use the following notation for the expected mean squares and the variance of $\bar{Y}$:

$$\begin{aligned}
\sigma_{\bar{Y}}^2 &= \frac{bn\sigma_A^2 + an\sigma_B^2 + \sigma_e^2}{abn} \\
\psi_A &= bn\sigma_A^2 + \sigma_e^2 \\
\psi_B &= an\sigma_B^2 + \sigma_e^2
\end{aligned}$$

The three representations of the FGPQ for μ are given by

$$\mathcal{R}_\mu(\mathbb{S},\mathbb{S}^\star,\xi) = \bar{Y} - \left(\frac{\bar{Y}^\star-\mu}{\sigma_{\bar{Y}}}\right)\sqrt{\max\left(0,\frac{SSA}{SSA^\star}\psi_A + \frac{SSB}{SSB^\star}\psi_B - \frac{SSEP}{SSEP^\star}\sigma_e^2\right)\frac{1}{abn}}$$

$$\dot{\mathcal{R}}_\mu(\mathbb{S},\mathbb{E}^\star) = \bar{Y} - E_0^\star\sqrt{\max\left(0,\frac{SSA}{E_1^\star} + \frac{SSB}{E_2^\star} - \frac{SSEP}{E_3^\star}\right)\frac{1}{abn}}$$

$$\ddot{\mathcal{R}}_\mu(\mathbb{E},\mathbb{E}^\star,\xi) = \mu + E_0\sigma_{\bar{Y}} - E_0^\star\sqrt{\max\left(0,\frac{E_1}{E_1^\star}\psi_A + \frac{E_2}{E_2^\star}\psi_B - \frac{E_3}{E_3^\star}\sigma_e^2\right)\frac{1}{abn}}$$

3.5.1 Tolerance intervals for $Y \sim N(\mu,\sigma_Y^2)$

In the interest of brevity we will not introduce new notation for $(1-p)^{\text{th}}$ percentile $\theta = \mu - z_p\sigma_Y$ or for $\mathcal{R}_\theta = \mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_Y}$, but let the context determine if the model is with or without interaction.

The the FGPQ $\mathcal{R}_{\sigma_Y}$.

$$\mathcal{R}_{\sigma_Y} = \sqrt{\frac{1}{bn}\frac{SSA}{SSA^\star}\psi_A + \frac{1}{an}\frac{SSB}{SSB^\star}\psi_B + \frac{abn-(a+b)}{abn}\frac{SSEP}{SSEP^\star}\sigma_e^2}$$

Note that if $a, b, n \geq 2$ then $abn > (a+b)$ and the above expression is well defined. The other two representations are

$$\dot{\mathcal{R}}_{\sigma_Y} = \sqrt{\frac{1}{bn}\frac{SSA}{E_1^\star} + \frac{1}{an}\frac{SSB}{E_2^\star} + \frac{abn-(a+b)}{abn}\frac{SSEP}{E_3^\star}\sigma_e^2}$$

$$\ddot{\mathcal{R}}_{\sigma_Y} = \sqrt{\frac{1}{bn}\frac{E_1}{E_1^\star}\psi_A + \frac{1}{an}\frac{E_2}{E_2^\star}\psi_B + \frac{abn-(a+b)}{abn}\frac{E_3}{E_3^\star}\sigma_e^2}$$

For a realization s of the data $\mathbb{S}$ the lower $(p, 1-\alpha)$ tolerance bound is equal to the α^{th} quantile of distribution of $\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi)$ conditional on $\mathbb{S}$ equal s.

Proposition 3.6. *The coverage probability*

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\xi) \leq \theta|\mathbb{S}\right] > \alpha\right]$$

depends only on p, α, a, b, n, and the ratios τ_A and τ_B.

Proof. The proof is the same as the proof of Proposition 3.2 and the details are omitted. □

Next we show that lower tolerance intervals that are based on $\mathcal{R}_\theta$ have asymptotic correct coverage as the number of levels of the two random factors go to infinity.

Proposition 3.7. *For $Y \sim N(0, \sigma_Y^2)$ from a balanced two-way random effects model without interaction let a and b go to infinity at rate such that $\lim a/(a+b) = \lambda_A$ and $\lim b/(a+b) = \lambda_B$ exist with $0 < \lambda_A, \lambda_B < 1$. Then a lower $(p, 1-\alpha)$ tolerance interval for Y based on the FGPQ $\mathcal{R}_\theta$ has asymptotically $100(1-\alpha)\%$ frequentist coverage.*

Proof. The proof is identical in spirit to the proof of Proposition 3.3. We omit the details.

□

3.5.2 Lower Tolerance Intervals for $\mu + A \sim N(\mu, \sigma_A^2)$

In the interest of brevity we will not introduce new notation for $(1-p)^{\text{th}}$ percentile $\eta = \mu - z_p\sigma_A$ or for $\mathcal{R}_\theta = \mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_A}$, but let the context determine if the model is with or without interaction.

The three representations of $\mathcal{R}_{\sigma_A}$.

$$\mathcal{R}_{\sigma_A}(\mathbb{S}, \mathbb{S}^\star, \xi) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{SSA}{SSA^\star}\psi_A - \frac{SSEP}{SSEP^\star}\sigma_e^2\right)\right]}$$
$$\dot{\mathcal{R}}_{\sigma_A}(\mathbb{S}, \mathbb{E}^\star) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{SSA}{E_1^\star} - \frac{SSEP}{E_3^\star}\sigma_e^2\right)\right]}$$
$$\ddot{\mathcal{R}}_{\sigma_A}(\mathbb{E}, \mathbb{E}^\star, \xi) = \sqrt{\max\left[0, \frac{1}{bn}\left(\frac{E_1}{E_1^\star}\psi_A - \frac{E_3}{E_3^\star}\sigma_e^2\right)\right]}$$

For a realization $\mathbf{s}$ of the data $\mathbb{S}$ a lower $(p, 1-\alpha)$ tolerance bound is equal to the α^{th} quantile of distribution of $\mathcal{R}_\eta(\mathbb{S}, \mathbb{S}^\star, \xi)$ conditional on $\mathbb{S}$ equal $\mathbf{s}$.

The coverage probability is a function of less than the full set of parameters.

Proposition 3.8. *The coverage probability*

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\eta(\mathbb{S}, \mathbb{S}^\star, \xi) \le \eta|\mathbb{S}\right] > \alpha\right].$$

depends only on p, α, a, b, n, and the ratios τ_A and τ_B.

Proof. The proof is similar to the proof of Proposition 3.2 and the details are omitted. □

The following proposition states that asymptotically the tolerance intervals based on $\mathcal{R}_\eta$ have exact frequentist coverage.

Proposition 3.9. *For $\mu + A \sim N(\mu, \sigma_A^2)$ from a balanced two-way random effects model without interaction let a and b go to infinity at rate such that $\lim a/(a+b) = \lambda_A$ and $\lim b/(a+b) = \lambda_B$ exist with $0 < \lambda_A, \lambda_B < 1$. Then the lower $(p, 1-\alpha)$ tolerance bound based on the FGPQ $\mathcal{R}_\eta$ has asymptotically $100(1-\alpha)\%$ frequentist coverage.*

Proof. The proof is identical in spirit to the proof of Proposition 3.3. We omit the details.

□

3.6 Upper Tolerance Bound for the Balanced Two-way Random Effects Model with Interaction

As pointed out in Section 1.1.1 an upper $(p, 1-\alpha)$ tolerance bound for a random variable $X \sim N(\mu, \sigma_X)$ is equivalent to an upper $100(1-\alpha)\%$ confidence bound for $\zeta = \mu + z_p\sigma_X$, the $100p^{\text{th}}$-percentile of the distribution of X. From this statement, one can see there are two differences between upper $(p, 1-\alpha)$ tolerance bounds and the lower $(p, 1-\alpha)$ tolerance bounds that are based on FGPQs constructed using the method of Theorem 2.2. First, one constructs the FGPQ $\mathcal{R}_\zeta = \mathcal{R}_\mu + z_p\mathcal{R}_{\sigma_X}$ instead of $\mathcal{R}_\mu - z_p\mathcal{R}_{\sigma_X}$. Second, one finds the $100(1-\alpha)$-percentile of the conditional distribution of $\mathcal{R}_\zeta$ given $\mathbb{S} = \mathrm{s}$.

3.7 Simulation Study and Discussion

A simulation study was used to assess the performance of the proposed lower $(p, 1-\alpha)$ tolerance bounds for the two population characteristics $Y \sim N(\mu, \sigma_Y^2)$ and $\mu + A \sim N(\mu, \sigma_A^2)$ of the balanced two-way random effects model with and without interaction, i.e., four simulation studies were conducted. The performance was assessed by calculating the empirical coverage probability. By the Propositions 3.2, 3.4, 3.6, and 3.8 without loss of generality we assume $\mu = 0$ and $\sigma_e^2 = 1$. For all four simulation studies we used $p = 0.9$ and $\alpha = 0.05$. The other parameters the model depends on are a, b, n, σ_A^2, σ_B^2 and for the model with interaction, σ_{AB}^2. The values used for a, b and n were $\{2, 5, 10\}$, for a total 27 combinations. Due to symmetry for σ_A^2 and σ_B^2 in σ_Y we can assume without loss of generality that $\sigma_B^2 \leq \sigma_A^2$ for Y. The values for σ_A^2 and the σ_B^2 were $\{0, 0.1, 0.25, 0.5, 1, 2, 4, 10, 100\}$, for a total of 81 combinations for $\mu + A$ and 45 combinations for Y. Finally, for the model with interaction the values used for σ_{AB}^2 were $\{0.25, 0.5, 1, 2, 4\}$. Thus the total parameter combinations for Y with interaction was 6075, for $\mu + A$ with interaction was 10,935, for Y without interaction was 1215 and for $\mu + A$ without interaction was 2187.

For each parameter combination 4000 realizations of $\mathbb{S}$ were generated. For each realization $\mathbf{s}$ of $\mathbb{S}$ an empirical distribution of $\dot{\mathcal{R}}_*(\mathbf{s}, \mathbb{E}^\star)$ was generated using $N = 10,000$ realizations of $\mathbb{E}^\star$, where $*$ stands for one of the four random variable and model combinations. For details on generating the empirical distribution see Section 3.4.1. The 0.05^{th} percentile of the empirical distribution was used for the lower tolerance bound. The empirical coverage probability was calculated from the 4000 lower tolerance bounds.

A FORTRAN program was written to perform the simulation. The program was implemented using Digital Fortran Professional Edition 6.0.A. The realizations of $\mathbb{S}$ were generated using the DRNNOR random number generator for $\bar{Y}$ and DRNCHI random number generator for the sums of squares, SS*. The same random number generators were used in the simulation of the conditional distribution of $\mathcal{R}_*$ given $\mathbb{S} = \mathbf{s}$, where $\mathbf{s}$ is the realization of $\mathbb{S}$ and $*$ stand for one of the four random variable model combinations. Copies of the programs and input files are contained in Sections C.2 and C.3.

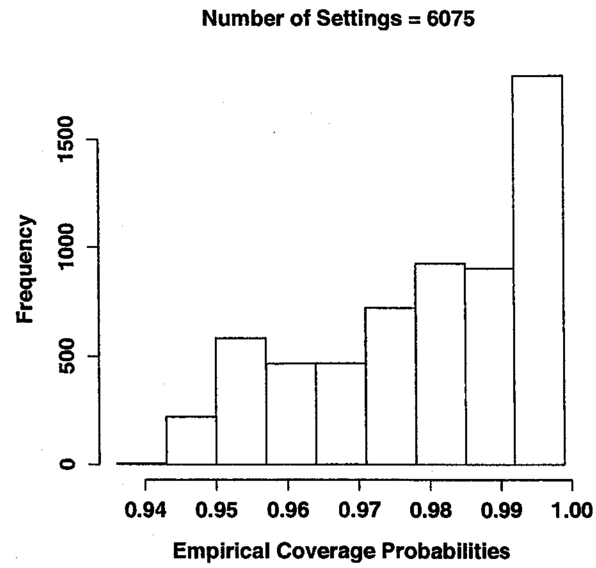
The acceptance region of the empirical coverage probability based on 4000 observations for a confidence level of 95% is $0.95 \pm .007$ or stated as an interval $[0.943, 0.957]$. For the histograms used to illustrate the results, the empirical coverage probabilities were divided into the categories $.95 \pm k0.007$ for $k \in \mathbb{N}$, so the categories immediately adjacent to the point 0.95 contain empirical coverage probabilities which are in the acceptance region.

The results of the simulation study are organized into three subsections (1) Y under the model with interaction, (2) $\mu + A$ under the model with interaction and (3) Y and $\mu + A$ under the model without interaction.

3.7.1 Results for $Y \sim N(\mu, \sigma_Y^2)$ under the Model with Interaction

Out of the 6075 parameter combinations tested in the simulation study 819 of the empirical coverage probabilities were within the acceptance region of $0.95 \pm .007$. The histogram of the empirical coverage probabilities in Figure 3.1 indicates the empirical coverage probabilities for a majority of parameter settings are conservative to very conservative.

Figure 3.1: Empirical coverage probabilities for tolerance intervals of the distribution of Y, model with interaction, all parameter combinations.



There are five parameter combinations for which the empirical coverage probability was liberal, with a minimum empirical coverage probability of 0.940.

A classification tree was built using the the binary response variable Z, with $Z = 1$ if the empirical coverage probability was in the interval $[0.943, 0.957]$ and $Z = 0$ otherwise. The factors used to construct the tree were a, b, n, σ_A^2, σ_B^2 and σ_{AB}^2. The classification tree had 10 leaves with a maximum length of five branches. None of the splits involved a or n and only one split involved σ_{AB}^2. The split involving σ_{AB}^2 was the node of maximum depth. Based on this we concluded the empirical coverage probabilities depend on b, σ_A^2 and σ_B^2 with $\sigma_B^2 \leq \sigma_A^2$. For each combination of the $(b, \sigma_A^2, \sigma_B^2)$ the mean, standard deviation, minimum and maximum of the empirical coverage probabilities for the 45 parameter settings over a, n and σ_{AB}^2 we computed. The results are contained in Tables A.1-A.3. Table A.1 contains the results for $b = 2$, Table A.2 results for $b = 5$ and Table A.3 for $b = 10$.

Also of interest is whether there are any subsets of parameter settings for which the empirical coverage probabilities are within or close to the acceptance region and also subsets where all of the empirical coverage probabilities are very conservative. Figure 3.2 contains histograms illustrating four subsets of the parameter settings, two for which the empirical coverage probabilities are close to the nominal coverage and two where the empirical coverage probabilities appear to be mostly very conservative. The subsets are defined by listing the restrictions on parameters (i.e., the parameters setting are found by intersecting the conditions for the listed parameters and the other parameters are unrestricted). The two "good" subsets share the pattern that the levels of one factor are high, along with corresponding variance component low, and the other variance component high. For one subset of parameter setting, where the empirical coverage probability is mostly very conservative, the levels of one factor are low, while the variance component of the other factor is high. The other subset of parameter combinations where the empirical coverage appears to be conservative is when the variance components of both factors are less than four times the error variance component.

3.7.2 Results for $\mu + A \sim N(\mu, \sigma_A^2)$ under the Model with Interaction

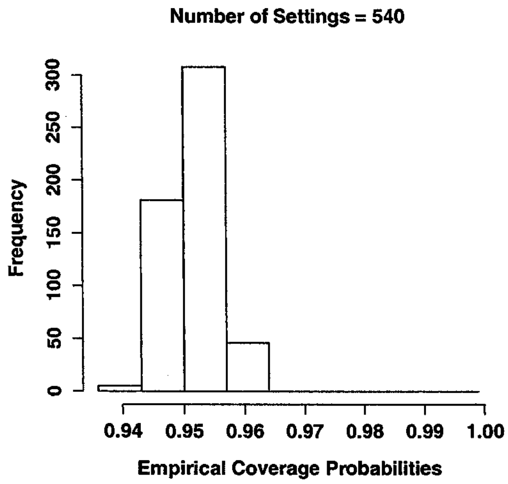
Out of the 10,935 parameter combinations tested in the simulation study, the empirical coverage probabilities for 3249 parameter combinations were within the acceptance region of $0.95 \pm .007$. For the histogram of the empirical coverage probabilities see Figure 3.3. Note the bimodal nature of the histogram. There are 33 parameter combinations for which the empirical coverage probability was liberal. The minimum empirical coverage probability was 0.938.

A classification tree was built using the the binary response variable Z, with $Z = 1$ if the empirical coverage probability was in the interval $[0.943, 0.957]$ and $Z = 0$ otherwise. The factors used to construct the tree were a, b, n, σ_A^2, σ_B^2 and σ_{AB}^2. Based on the splits of the classification tree, the empirical coverage probabilities depend on the values of a, b, σ_A^2, σ_B^2. The mean, standard deviation, minimum and maximum of the empirical coverage probability over the 15 values of σ_{AB}^2 and n for each parameter combination of a, b, σ_A^2 and σ_B^2 are in Tables A.4-A.12. Each table is a 9×9 grid of $\sigma_A^2 \times \sigma_B^2$ values for fixed levels of a and b.

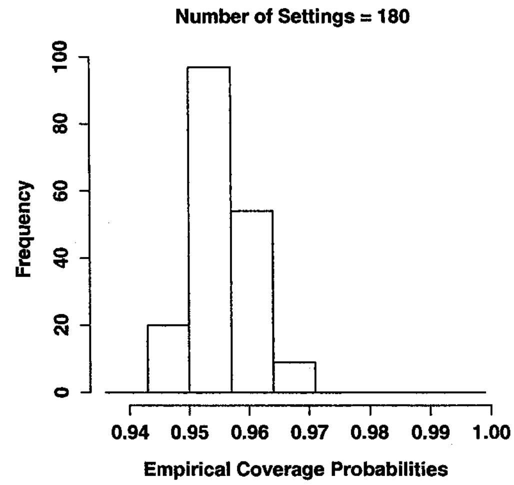
Figure 3.4 contains histograms for five subsets of the parameter combinations. For four subsets the empirical coverage probability are either within acceptance interval $[0.943, 0.957]$ or close to the acceptance interval. For the other subset of the parameter

Figure 3.2: Empirical coverage probabilities for tolerance intervals of the distribution of Y, model with interaction over subsets of the parameters settings. The subsets are defined by intersecting over the listed parameters.

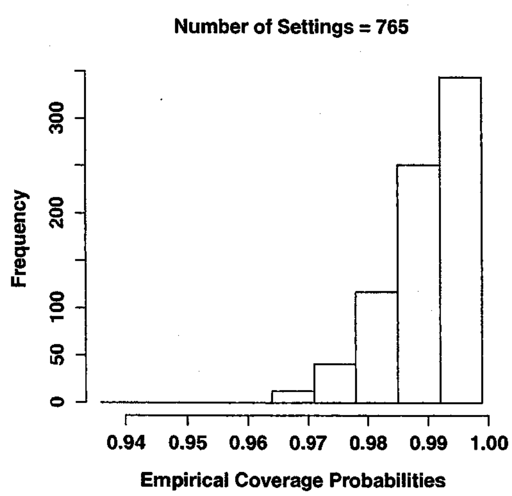
(A) $b \in \{5, 10\}$, $\sigma_A^2 = 100$ and $\sigma_B^2 \in \{0, 0.1, 0.25, 0.5, 1, 2\}$



(B) $b = 10$, $\sigma_A^2 = 10$ and $\sigma_B^2 \in \{0, 0.1, 0.25, 0.5\}$



(C) $b = 2$ and $\sigma_A^2 \in \{10, 100\}$



(D) $\sigma_A^2, \sigma_B^2 \in \{0, 0.1, 0.25, 0.5, 1, 2, 4\}$, with $\sigma_B^2 \leq \sigma_A^2$

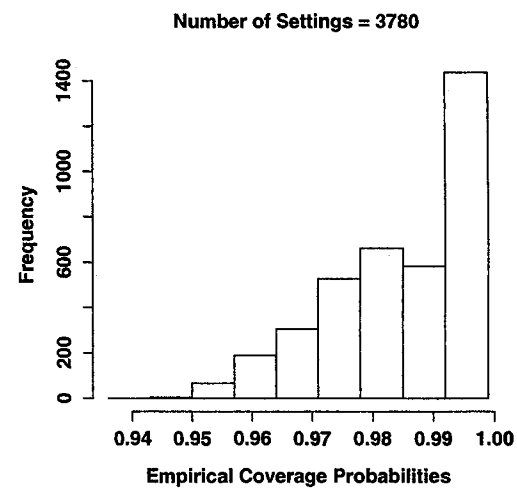
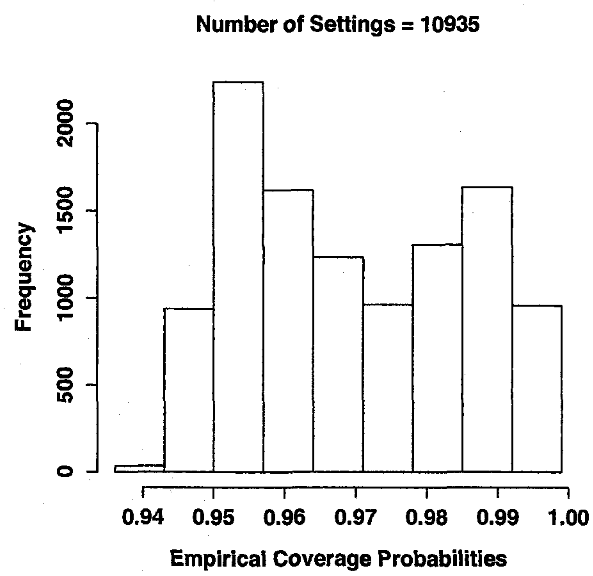


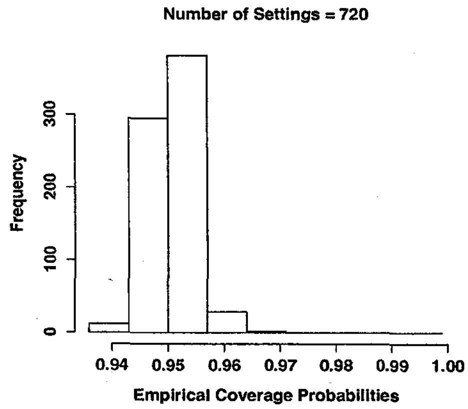
Figure 3.3: Empirical coverage probabilities for tolerance intervals of the distribution of $\mu+A$, model with interaction, all parameter settings.



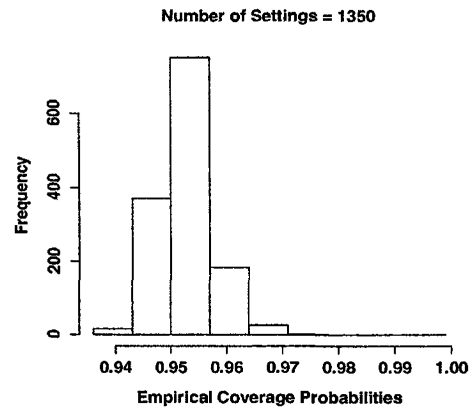
combinations the empirical coverage probabilities are conservative. The sets of parameters are defined by listing the restrictions on the parameter combinations, where the restrictions are defined by intersecting over all the listed parameters. For any parameter not list, the values are unrestricted over the values used in the simulation. The number of parameter combinations in each subset in referred to as the number of settings and is at the top of each histogram. We note that the restrictions that define the parameter settings used in Figures 3.4 (A), (B), and (C) do involve the number of levels of A. It appears that the number of levels of B should be 5 or larger, σ_B^2 should be smaller than σ_A^2, and σ_A^2 should be much larger than the error variance. Figure 3.4 (D) illustrates another situation where the coverage probability appears to be close to nominal, e.g., when the variance component of factor B is much larger than the variance component of factor A and the number of levels of factor A are greater than the number of levels of factor B, with both greater than 5. Lastly Figure 3.4 (E) indicates if the number of replicates of B is small, then the coverage probability appears to be conservative and maybe very conservative; note the large number parameter settings for this case.

Figure 3.4: Empirical coverage probabilities for tolerance intervals of the distribution of $\mu + A$, model with interaction, over subsets of parameter values.

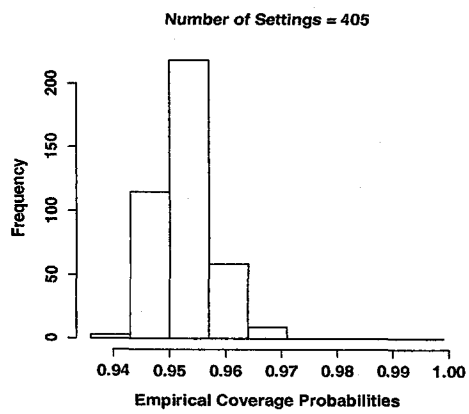
(A) $b = 5, 10$, $\sigma_A^2 = 100$ and $\sigma_B^2 \in \{0, 0.1, 0.25, 0.5, 1, 2, 4, 10\}$



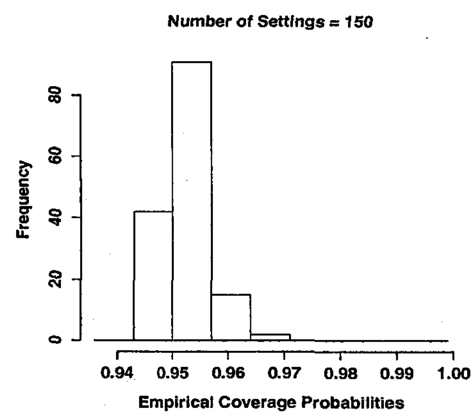
(B) $b = 5, 10$, $\sigma_A^2 \in \{2, 4, 10\}$ and $\sigma_B^2 \in \{0, 0.1, 0.25, 0.5, 1\}$



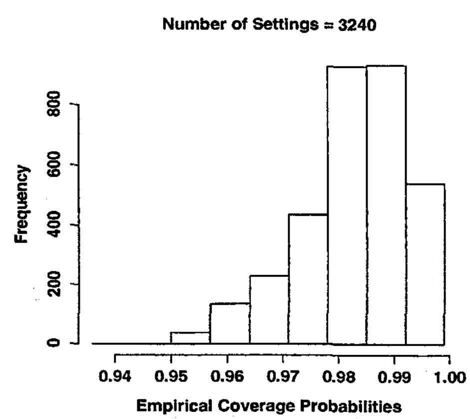
(C) $b = 10$, $\sigma_A^2 \in \{0.25, 0.5, 1\}$ and $\sigma_B^2 \in \{0, 0.1, 0.25\}$



(D) $a = 10$, $b \in \{5, 10\}$
$\sigma_A^2 \in \{0, 0.1, 0.25, 0.5, 1\}$ and $\sigma_B^2 = 100$



(E) $b = 2$ and $\sigma_B^2 \in \{0, 0.1, 0.25, 0.5, 1, 2, 4, 10\}$



3.7.3 Results for $Y \sim N(\mu, \sigma_Y^2)$ and $\mu + A \sim N(\mu, \sigma_A^2)$ under the Model without Interaction

The histograms of the empirical coverage probabilities for all parameter combinations for Y and $\mu + A$ are contained in Figures 3.5 (A) and (B) respectively. The shape is similar to that of the histogram for the corresponding random variable under the model with interaction. The minimum empirical coverage probability for Y was 0.940 and the minimum empirical coverage probability $\mu + A$ was 0.943. For Y and for $\mu + A$ a classification tree was built based on the binary variable Z, with $Z = 1$ if the empirical coverage probability was in the interval $[0.943, 0.957]$ and $Z = 0$ otherwise. For both Y and $\mu + A$ there was little or no involvement of n and a in the splits. The data was collapsed over the nine values of a and n for each fixed parameter values of b, σ_A^2 and σ_B^2, with the data being represented by the mean, standard deviation, minimum and maximum. The results for Y are given in Tables A.13-A.15, with the value of b being fixed for each table. While the results for $\mu + A$ are in Tables A.16-A.18.

The subsets of the parameter combinations for which the empirical coverage probability is close to the nominal coverage probability were identical to subsets of parameter combination for the corresponding random variable under the model with interaction. There was a slight difference in the shape of the histograms. The same was true for the subsets of the parameter combinations where the empirical coverage probability appears to be always conservative. One interesting aspect is for certain subsets, the value of the interaction variance component has a profound effect on the shape of the histogram. It appears that as the interaction variance component converges to zero the histogram "converges" to the histogram for the model without interaction. We have illustrated this in Figure 3.6, which is for the random variable Y and parameter combinations with $b = 5$, $\sigma_A^2 = 10$, and $\sigma_B^2 \in \{0, 0.1, 0.25\}$. Figure 3.6 (A) is the histogram of the empirical coverage probabilities for the model with interaction, while Figure 3.6 (B) is the histogram for the model without interaction. Figures 3.6 (C)-(F) are for the model with interaction, with a restriction on the value of σ_{AB}^2. Notice that for $\sigma_{AB}^2 = 4$ the shape of the histogram is far different than the histograms in Figure 3.6 (A) and (B). As σ_{AB}^2 decreases the shape of the histogram approaches both Figures 3.6 (A) and (B). We stress that this does not appear to be true for all subsets of the parameter combinations and further investigation is needed.

Figure 3.5: Empirical coverage probabilities for tolerance intervals of the distribution of Y and for tolerance intervals of the distribution of $\mu + A$, model without interaction, all parameter combinations.

(A) Y (B) $\mu + A$

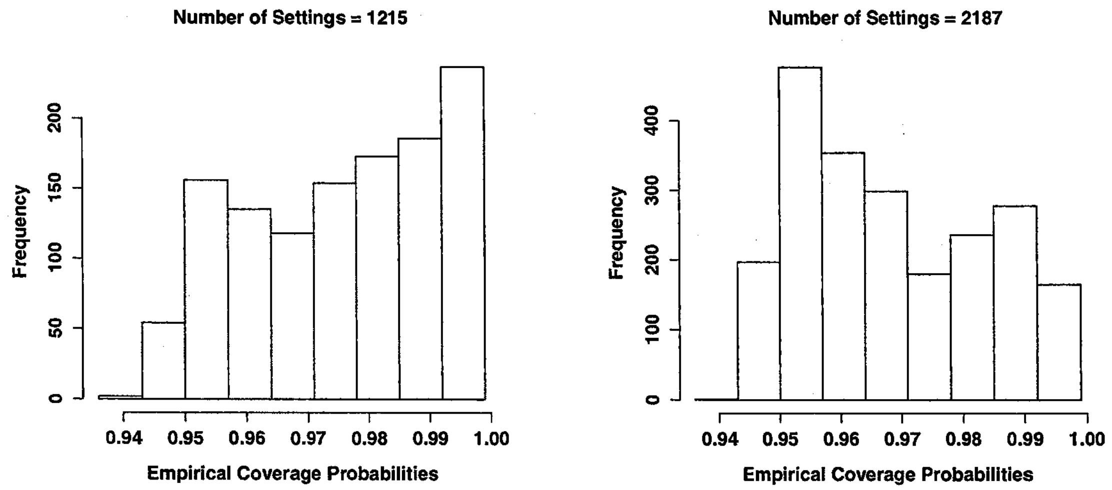
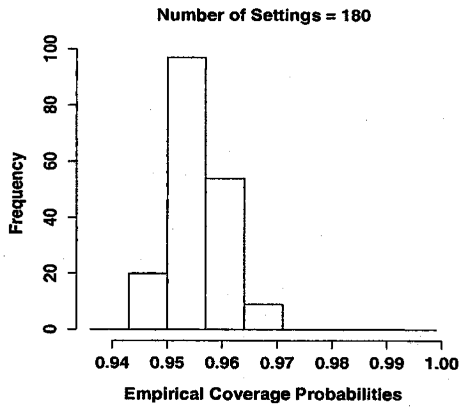
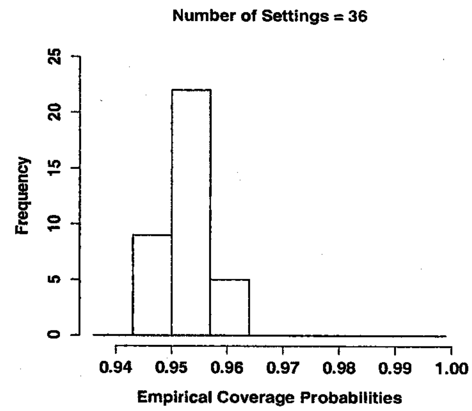


Figure 3.6: Empirical coverage probabilities for tolerance intervals of the distribution of Y, over parameter combinations with $b = 5$, $\sigma_A^2 = 10$, and $\sigma_B^2 \in \{0, 0.1, 0.25\}$.

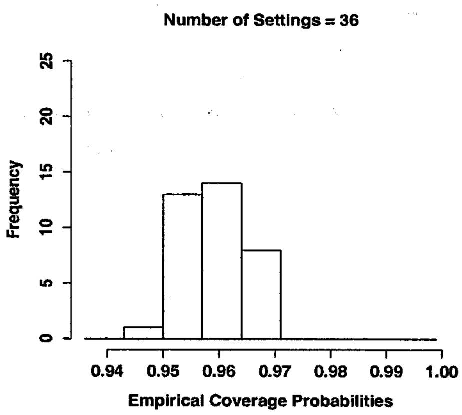
(A) with interaction



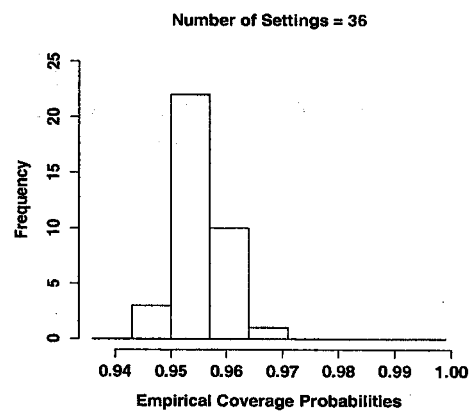
(B) without interaction



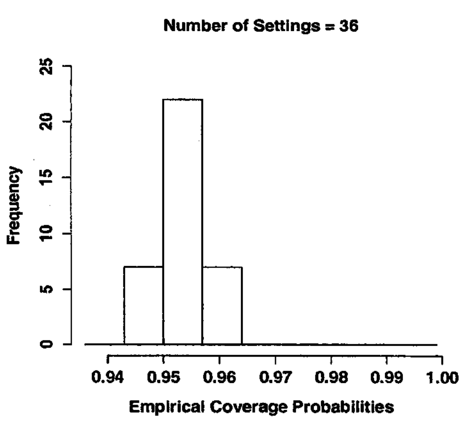
(C) with interaction, $\sigma_{AB}^2 = 4$



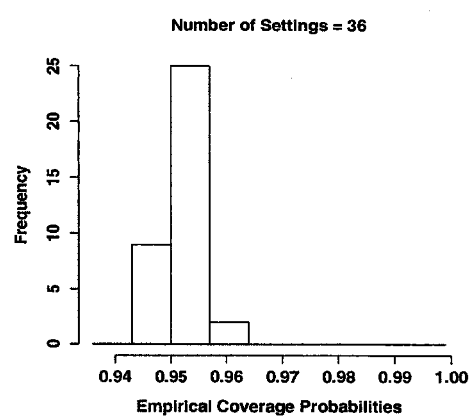
(D) with interaction, $\sigma_{AB}^2 = 1$



(E) with interaction, $\sigma_{AB}^2 = 0.5$



3.6.F: with interaction, $\sigma_{AB}^2 = 0.25$



Chapter 4

PROPORTION OF CONFORMANCE FOR A NORMAL DISTRIBUTION

In this chapter and Chapter 5 we are interested in confidence intervals for the proportion of conformance for situations where the population characteristics are distributed $N(\mu, \sigma^2)$. For a general discussion of proportion of conformance see Section 1.1.2. We repeat the definition of proportion of conformance. The *proportion of conformance* is the proportion θ of the population that is contained in the interval (C_1, C_2), where $C_1 < C_2$ are real numbers chosen to meet some performance requirements (C_1 may be negative infinity and C_2 may be positive infinity). The quantities C_1 and C_2 are often referred to as specification limits. When the population characteristic is distributed $N(\mu, \sigma^2)$, θ can be expressed as,

$$\theta = \Phi\left(\frac{C_2 - \mu}{\sigma}\right) - \Phi\left(\frac{C_1 - \mu}{\sigma}\right). \tag{4.1}$$

In this chapter we will consider constructing confidence intervals for the proportion of conformance for the distribution of $X \sim N(\mu, \sigma^2)$ based on an iid sample $X_1, \ldots, X_n$ with $X_i \sim N(\mu, \sigma^2)$. In Chapter 5 we consider confidence intervals for the proportion of conformance for population characteristics associated with the one-way random effects model.

To construct confidence intervals for the proportion of conformance for the distribution of $X \sim N(\mu, \sigma^2)$ based on a iid sample Chou and Owen [11] proposed combining simultaneous one-side confidence limits for the right and left tails of the distribution. Wang and Lam [54] proposed two methods which they labelled 'L1' and 'L2' for constructing lower confidence bounds. In a simulation study Wang and Lam found that both the L1 and L2 methods performed better than the method of Chou and Owen. We will derive a FGPQ for θ using the structural method and will compare the performance of the lower generalized confidence bounds obtained using the FGPQ with one of the methods proposed by Wang and Lam. The simulation study of Wang and Lam [54] found that the L2 method performed better than L1 in some situations and was nearly as good as L1 in other situations. It was also the simpler method to compute. For these reasons, we chose to compare the lower generalized confidence bounds with the lower confidence bounds produced using the L2 method. Both of these methods are only approximate. Analytical derivations of the actual coverage probabilities are intractable, so a Monte Carlo evaluation is used to assess the relative performance of the two methods.

The chapter is organized into a methods section, a section describing the simulation study and a discussion section.

4.1 Methods

This section consists of a two subsections, one for the L2 method and one GCI method. The subsections will consists of a derivation of the lower bounds for θ and a presentation of some properties of the lower bounds that will be useful in reducing the number of parameter combinations needed in the simulation study.

For a iid sample $X_1, \ldots, X_n$ with $X_i \sim N(\mu, \sigma^2)$ let $\bar{X}$ denote the sample mean and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ the sample variance. Let $\mathbb{S} = (\bar{X}, S^2)$.

4.1.1 L2 Method

The L2 method of Wang and Lam is based on

$$K_1 = \frac{\bar{X} - C_1}{S} \quad \text{and} \quad K_2 = \frac{C_2 - \bar{X}}{S}.$$

The L2 $100(1-\alpha)\%$ lower confidence bound for θ denoted by $\hat{\theta}$ is given by

$$\hat{\theta} = \Phi\left(\frac{1}{\sqrt{n}} + \max(K_1, K_2)\sqrt{\frac{\chi^2_{\alpha,n-1}}{n-1}}\right) - \Phi\left(\frac{1}{\sqrt{n}} - \min(K_1, K_2)\sqrt{\frac{\chi^2_{\alpha,n-1}}{n-1}}\right), \tag{4.2}$$

where $\chi^2_{\alpha,n-1}$ is the α percentile of the chi–squared distribution with $n-1$ degrees of freedom. The comparison between this method and the GCI will be based on the coverage probability, $P_S[\hat{\theta} \le \theta]$, and the expected value of $\hat{\theta}$. The analytical derivation of these two quantities is intractable, so a Monte Carlo simulation will be used to assess the performance. The two propositions which follow the next lemma will be useful in reducing the number of parameter combinations needed for the simulation study.

Lemma 4.1. *The distribution of $\hat{\theta}$ depends only on n, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$.*

Proof. For the iid sample $X_1, \ldots, X_n$ with $X_i \sim N(\mu, \sigma^2)$, define the transformed random variables $\check{X}_i = (X_i - \mu)/(\sigma)$ for $i = 1, \ldots, n$. Then $\check{X}_1, \ldots, \check{X}_n$ are iid $N(0,1)$. The sample mean $\check{\bar{X}}$ is distributed $N(0, n^{-1})$, while the sample variance $(n-1)\check{S}^2$ is distributed $\sim \chi^2_{n-1}$. The following relationship exists between the statistic $\check{\mathbb{S}} = (\check{\bar{X}}, \check{S}^2)$ and $\mathbb{S}$

$$\check{\bar{X}} = \frac{\bar{X} - \mu}{\sigma}, \quad \text{and} \quad \check{S}^2 = \frac{S^2}{\sigma^2}.$$

We can express K_1 as follows

$$K_1 = \frac{(\bar{X} - \mu)/\sigma - (C_1 - \mu)/\sigma}{S/\sigma} = \frac{\check{\bar{X}} - C_1^\star}{\check{S}}.$$

By a similar calculation, $K_2 = (C_2^\star - \check{\bar{X}})/\check{S}$. The distributions of both $\check{\bar{X}}$ and $\check{S}$ depend only on n and since $\check{\bar{X}}$ and $\check{S}$ are independent their joint distribution depends only on n. The result now follows from (4.2). □

The following two propositions will allows us to greatly reduce the number of parameter combinations needed in the simulation study.

Proposition 4.1. *The coverage probability, $P_{\mathbb{S}}[\hat{\theta} \leq \theta]$, depends only on n, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$.*

Proof. It is clear from (4.1) that θ depends only on $C_1^\star$ and $C_2^\star$. The result now follows from Lemma 4.1. □

Proposition 4.2. *The expected value of $\hat{\theta}$, $E_{\mathbb{S}}[\hat{\theta}]$, depends only on n, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$.*

Proof. The proposition is an immediate consequence of Lemma 4.1. □

4.1.2 Generalized Confidence Interval

We now derive a FGPQ for θ using the structural method of Theorem 2.2. This FGPQ can be used to obtain lower confidence bounds, upper confidence bounds, or two-sided confidence intervals for θ.

Let $\mathbb{S} = (\bar{X}, S^2)$ and $\xi = (\mu, \sigma^2)$. The following is an invertible pivotal relationship for $\mathbb{S}$ and ξ

$$\begin{aligned} f_1(\mathbb{S}, \xi) &= \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} &&= E_1 \sim N(0,1), \\ f_2(\mathbb{S}, \xi) &= \frac{(n-1)S^2}{\sigma^2} &&= E_2 \sim \chi^2_{n-1}, \end{aligned}$$

with E_1 and E_2 are independent. The inverse mapping $\mathbf{g}(\mathbb{S}, \mathbb{E})$ is given by

$$\begin{aligned} g_1(\mathbb{S}, \mathbb{E}) &= \bar{X} - \frac{E_1}{\sqrt{n}} \sqrt{\frac{(n-1)S^2}{E_2}}, \\ g_2(\mathbb{S}, \mathbb{E}) &= \frac{(n-1)S^2}{E_2}. \end{aligned}$$

It is convenient to derive the FGPQ for θ in two stages. Let $\pi_1(\xi) = \mu$ and $\pi_2(\xi) = \sqrt{\sigma^2} = \sigma$. By structural method of Theorem 2.2 we have the following two FGPQ's for σ and μ respectively:

$$\begin{aligned} \mathcal{R}_\sigma &= \mathcal{R}_\sigma(\mathbb{S}, \mathbb{S}^\star, \xi) = \sqrt{g_2(\mathbb{S}, \mathbf{f}(\mathbb{S}^\star, \xi))} = \frac{\sigma S}{S^\star}, \quad \text{and} \\ \mathcal{R}_\mu &= \mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = g_1(\mathbb{S}, \mathbf{f}(\mathbb{S}^\star, \xi)) = \bar{X} - (\bar{X}^\star - \mu)\frac{S}{S^\star}. \end{aligned}$$

Observe that

$$\theta = \Phi\left(\frac{C_2 - \pi_1(\xi)}{\pi_2(\xi)}\right) - \Phi\left(\frac{C_1 - \pi_1(\xi)}{\pi_2(\xi)}\right),$$

so from the the recipe given in Theorem 2.2 we get the following FGPQ for θ,

$$\mathcal{R}_\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \Phi\left(\frac{C_2 - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{C_1 - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) \tag{4.3}$$

A $100(1-\alpha)\%$ lower generalized confidence interval for θ is given by $\mathcal{R}_{\theta,\alpha} \leq \theta$, where $\mathcal{R}_{\mu,\gamma}$ denotes the 100γ-percentile of the conditional distribution of $\mathcal{R}_\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Given a realization $\mathbf{s}$ of $\mathbb{S}$, the distribution of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$ is usually intractable. One can estimate $\mathcal{R}_{\theta,\alpha}$ using a Monte Carlo simulation of the empirical density of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$. We will give an outline of process for estimating $\mathcal{R}_{\theta,\alpha}$. One uses the representation

$$\dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star) = \Phi\left(\frac{C_2 - \dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)}{\dot{\mathcal{R}}_\sigma(\mathbb{S}, \mathbb{E}^\star)}\right) - \Phi\left(\frac{C_1 - \dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)}{\dot{\mathcal{R}}_\sigma(\mathbb{S}, \mathbb{E}^\star)}\right), \tag{4.4}$$

where

$$\dot{\mathcal{R}}_\sigma(\mathbb{S}, \mathbb{E}^\star) = \sqrt{g_2(\mathbb{S}, \mathbb{E}^\star)} = S\sqrt{\frac{n-1}{E_2^\star}}, \quad \text{and}$$
$$\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) = g_1(\mathbb{S}, \mathbb{E}^\star) = \bar{X} - S\sqrt{\frac{n-1}{n}}\frac{E_1^\star}{\sqrt{E_2^\star}}.$$

We remind the reader that $E_1^\star \sim N(0,1)$, $E_2^\star \sim \chi^2_{n-1}$ and $E_1^\star$ and $E_2^\star$ are independent. Let $\mathbb{E}^\star$ to denote the vector $(E_1^\star, E_2^\star)$. Given a realization $\mathbf{s}$ of $\mathbb{S}$, one generates N independent realizations $(\mathbf{e}_1^\star, \ldots, \mathbf{e}_N^\star)$ of $\mathbb{E}^\star$ and calculates $\dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_1^\star), \ldots, \dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_N^\star)$, which gives an empirical distribution of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$. One orders these values from smallest to largest and $[\alpha N]^{th}$ element of the ordered sequence is used as the estimate of $\mathcal{R}_{\theta,\alpha}$.

The performance of the GCI will be evaluated using the coverage probability, the expected value of $\mathcal{R}_{\theta,\alpha}$, and the expected value of $|\mathcal{R}_{\theta,\alpha} - \theta|$. The analytical derivation of these quantities is intractable, so a Monte Carlo simulation will be used to find estimates of them.

The following propositions will allow a reduction in the number of parameter combinations needed in the simulation study.

Let $F_{\mathbf{s}}(t)$ denote the cdf of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Then the coverage probability of the proposed generalized lower confidence interval is given by $P_{\mathbb{S}}\left[F_{\mathbb{S}}^{-1}(\alpha) \leq \theta\right]$. This is equal to

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \theta | \mathbb{S}\right] > \alpha\right]. \tag{4.5}$$

The calculation of this probability is intractable, so a Monte Carlo simulation will be used to estimate it. The following proposition will be useful in allowing us to reduce the number of parameter combinations for the simulation study.

Proposition 4.3. *Expression (4.5) depends only on n, α, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$.*

Proof. From Equation (4.1) ones see that θ only depends on $C_1^\star$ and $C_2^\star$. So is suffices to show the distribution of $\mathcal{R}_\theta$ depends only on n, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$. We start by deriving the canonical representations of $\mathcal{R}_\mu$ and $\mathcal{R}_\sigma$, see Remark 2.5. The function $\boldsymbol{h}(\mathbb{E}, \xi)$ can be derived by inverting the expression for $\boldsymbol{f}(\mathbb{S}, \xi)$ and $\boldsymbol{h}$ is given by

$$\bar{X} = h_1(\mathbb{E}, \xi) = \mu - \frac{\sigma}{\sqrt{n}} E_1$$
$$S^2 = h_2(\mathbb{E}, \xi) = \frac{\sigma^2}{n-1} E_2$$

Substituting these expression into Equation (2.16) we get the canonical representations

$$\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi) = \mu - \frac{\sigma}{\sqrt{n}} \left(E_1 - E_1^\star \sqrt{\frac{E_2}{E_2^\star}} \right)$$
$$\ddot{\mathcal{R}}_\sigma(\mathbb{E}, \mathbb{E}^\star, \xi) = \sigma \sqrt{\frac{E_2}{E_2^\star}}$$

From the canonical representations one sees that $\mathcal{R}_\sigma/\sigma$ and $(\mathcal{R}_\mu - \mu)/\sigma$ only depend on $(\mathbb{E}, \mathbb{E}^\star)$ and n. Thus

$$\mathcal{R}_\theta = \Phi\left(\frac{\frac{C_2 - \mu}{\sigma} - \frac{\mathcal{R}_\mu - \mu}{\sigma}}{\mathcal{R}_\sigma/\sigma} \right) - \Phi\left(\frac{\frac{C_1 - \mu}{\sigma} - \frac{\mathcal{R}_\mu - \mu}{\sigma}}{\mathcal{R}_\sigma/\sigma} \right),$$

only depends on $C_1^\star$, $C_2^\star$, n and $(\mathbb{E}, \mathbb{E}^\star)$. Since E_1, E_2, $E_1^\star$ and $E_2^\star$ are jointly independent and their distributions are free of unknown parameters, the joint distribution of $(\mathbb{E}, \mathbb{E}^\star)$ is free of unknown parameters and the result follows. □

We will now prove $E_\mathbb{S}[\mathcal{R}_{\theta,\alpha}(\mathbb{S})]$ and $E_\mathbb{S}[|\mathcal{R}_{\theta,\alpha}(\mathbb{S}) - \theta|]$ also only depend on $C_1^\star$ and $C_2^\star$.

Proposition 4.4. *The expectation, $E_\mathbb{S}[\mathcal{R}_{\theta,\alpha}(\mathbb{S})]$, depends only on n, α, $C_1^\star = (C_1 - \mu)/\sigma$ and $C_2^\star = (C_2 - \mu)/\sigma$*

Proof. Using the representation $\dot{\mathcal{R}}_\theta$, (4.4), one easily sees that the distribution of $\mathcal{R}_\theta$ conditional on $\mathbb{S} = \mathrm{s} = (\bar{x}, s)$ is a function of $\frac{C_2 - \bar{x}}{s}$, $\frac{C_1 - \bar{x}}{s}$ and $\mathbb{E}^\star$. Since the distribution of $\mathbb{E}^\star$ is free of unknown parameters, the quantile $\mathcal{R}_{\theta,\alpha}(\mathrm{s})$ only depends on $\frac{C_2 - \bar{x}}{s}$ and $\frac{C_1 - \bar{x}}{s}$. Note that

$$\frac{C_i - \bar{X}}{S} = \left(\frac{\frac{C_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma}}{S/\sigma} \right) = \left(\frac{C_i^\star - E_1}{\sqrt{E_2/(n-1)}} \right), \text{ for } i = 1, 2.$$

Since the distribution of $\mathbb{E}$ is free of unknown parameters the joint distribution of $\frac{C_2 - \bar{X}}{S}$ and $\frac{C_1 - \bar{X}}{S}$ only depends on C_2^* and C_1^* and the result follows. □

Proposition 4.5. *The expectation, $E_{\mathbb{S}}[|\mathcal{R}_{\theta,\alpha}(\mathbb{S}) - \theta|]$, depends only on n, α, $C_1^* = (C_1 - \mu)/\sigma$ and $C_2^* = (C_2 - \mu)/\sigma$*

Proof. The proof is similar to the proof of Proposition 4.4. □

For $i = 1, 2$, let $p_i = \Phi(C_i^*)$. Using standard notation for the quantiles of the standard normal distribution we have $z_{p_i} = C_i^*$, where $p_i = \Phi(C_i^*)$. In Propositions 4.3, 4.4 and 4.5 we can replace the phase "depends only on $C_1^* = (C_1 - \mu)/\sigma$ and $C_2^* = (C_2 - \mu)/\sigma$" with "depends only on z_{p_1} and z_{p_2}, where $p_i = \Phi((C_i - \mu)/\sigma)$ for $i = 1, 2$". We will refer to (p_1, p_2) with $0 \le p_1 < p_2 \le 1$ as a *percentile pair.* We will use percentile pairs in parts of the simulation study.

Besides comparing the performance of L2 method lower confidence bound and lower generalized confidence bound we will also simulate other scenarios for the lower generalized confidence bound. These scenarios will be expressed in terms of percentile pairs. The following proposition greatly reduces the number of percentile pairs (p_1, p_2) needed in the simulation study.

Proposition 4.6. *Assume $\mu = 0$. If (p_1, p_2) is a percentile pair and $z_{p_i} = \Phi^{-1}(p_i)$, $i = 1, 2$, then*

$$\Phi\left(\frac{z_{p_2} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{z_{p_1} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right)$$

and

$$\Phi\left(\frac{z_{1-p_1} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{z_{1-p_2} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right).$$

have the same distribution.

Proof. Note that

$$\begin{aligned}
\Phi\left(\frac{z_{p_2} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{z_{p_1} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) &= 1 - \Phi\left(-\frac{z_{p_2} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \left(1 - \Phi\left(-\frac{z_{p_1} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right)\right) \\
&= \Phi\left(\frac{-z_{p_1} + \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{-z_{p_2} + \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) \\
&= \Phi\left(\frac{z_{1-p_1} + \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{z_{1-p_2} + \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) \\
&\sim \Phi\left(\frac{z_{1-p_1} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) - \Phi\left(\frac{z_{1-p_2} - \mathcal{R}_\mu}{\mathcal{R}_\sigma}\right),
\end{aligned}$$

where the last line follows from the following three items.

1. Let $\mathbb{F} = (-E_1, E_2)$. Since E_1 and E_2 are jointly independent and $E_1 \sim N(0,1)$, $\mathbb{F}$ and $\mathbb{E}$ have the same distribution;

2. If $\mu = 1$, then $\ddot{\mathcal{R}}_\mu(\mathbb{F}, \mathbb{F}^\star, \xi) = -\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi)$, where $\mathbb{F}^\star$ is an independent copy of $\mathbb{F}$;

3. $\ddot{\mathcal{R}}_\sigma(\mathbb{F}, \mathbb{F}^\star, \xi) = \ddot{\mathcal{R}}_\sigma(\mathbb{E}, \mathbb{E}^\star, \xi)$.

□

We will now verify that $\mathcal{R}_\theta$ satisfies the conditions of Assumptions 2.1 and hence confidence intervals based on $\mathcal{R}_\theta$ have asymptotical exact frequentist coverage.

Proposition 4.7. *For a normal population, the* $100(1-\alpha)\%$ *GCIs for the proportion of conformance based on the FGPQ defined in Equation (4.3) has asymptotically* $100(1-\alpha)\%$ *frequentist coverage as* $n \to \infty$.

Proof. By standard results and Serfling (Theorem 3.3.A [48]) we have

$$\sqrt{n}(\bar{X}^\star - \mu, S^\star - \sigma) \xrightarrow{\mathcal{D}} \boldsymbol{N}$$

where $\boldsymbol{N}$ is a non-degenerate multivariate normal random variable. Thus $\boldsymbol{t}(\xi) = (\mu, \sigma)$ satisfies condition 1 of Assumptions 2.1, with $\mathbb{S} = (\bar{X}, S)$. For item 2 of Assumptions 2.1 $\mathbf{s}$ is considered an element of $\mathcal{O} = \mathbb{R} \times \mathbb{R}^+$. To simplify the notation for $i = 1, 2$ let

$$w_i(\mathbf{s}, \mathbf{s}^\star, \xi) = s^\star(C_i - \bar{x})(\sigma\, s)^{-1} + (\bar{x}^\star - \mu)\sigma^{-1}.$$

Note that w_i does not depend on n and has continuous second order partial derivatives with respect to $\mathbf{s}^\star$ for $(\mathbf{s}, \mathbf{s}^\star) \in \mathcal{O} \times \mathcal{O}$. Thus

$$\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi) = \Phi\left(w_2(\mathbf{s}, \mathbf{s}^\star, \xi)\right) - \Phi\left(w_1(\mathbf{s}, \mathbf{s}^\star, \xi)\right)$$

satisfies conditions 2a and 2c, where $\mathcal{A} \subset \mathcal{O}$ is any bounded open neighborhood of $\boldsymbol{t}(\xi)$. For $i = 1, 2$ let $w_i = w_i(\mathbf{s}, \boldsymbol{t}(\xi), \xi)$. Then for $\mathbf{s} \in \mathcal{O}$

$$\left.\frac{\partial}{\partial \bar{x}^\star}\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi)\right|_{\mathbf{s}^\star = \boldsymbol{t}(\xi)} = \sigma^{-1}\left(\phi(w_2) - \phi(w_1)\right) > 0,$$

since $\phi(w_2) = \phi(w_1)$ if and only if $C_2 = C_1$; thus condition 2c is satisfied. □

4.2 Simulation Study and Discussion

The simulation study was conducted to assess and compare the performances of the L2 method and the GCI for constructing lower confidence bounds for the proportion of conformance. The performance was assessed by calculating the empirical coverage probability and the empirical mean the lower confidence bound. By Propositions 4.1, 4.2, 4.3, and 4.4 with out loss of generality we can assume $\mu = 0$ and $\sigma = 1$. For the simulation study we used $\alpha = 0.05$. To compare the L2 method and the GCI we used the same parameter combinations of $(n, (C_1^\star, C_2^\star))$ that were used in Wang and Lam [54]. In addition, we investigated several other scenario for the GCI by using parameter combinations $(n, (p_1, p_2))$, where (p_1, p_2) is a percentile pair. For each parameter combination 4000 independent realizations of $\mathbb{S}$ were calculated. For each realization $\mathbf{s}$ of $\mathbb{S}$ the L2 lower bound was calculated. The generalized lower bound was calculated by simulating the 0.05^{th} quantile of the conditional distribution of $\dot{\mathcal{R}}(\mathbf{s}, \mathbb{E})$, as detailed in Section 4.1.2, using $N = 10,000$ realization of $\mathbb{E}^\star$. We will next discuss the computer program used to implement the simulation study and then discuss the results of the simulation study.

A FORTRAN program was written to perform the simulation. The realizations of $\mathbb{S}$ were generated using the IMSL DRNNOR to generate the values for $\bar{X}$ and the IMSL DRNCHI to generate the values for S^2. The same random number generators were used in the simulation of the conditional distribution of $\mathcal{R}_\theta$ given $\mathbb{S} = \mathbf{s}$. A copy of the program is in Section C.4. The program for comparing the L2 method and the GCI method is in Section C.4.1. The program for assessing the GCI method using percentile pairs is in Section C.4.2.

Table 4.1 displays the results of the simulation study comparing the performance of the GCI with the L2 method proposed by Wang and Lam [54]. In Table 4.1, the column labelled 'GCI' refers to the lower generalized confidence bound and the column labelled L2 refers to the L2 method of Wang and Lam. For each scenario, the first row gives the empirical coverages and the second row gives the empirical expected value of the lower confidence bound. For a given coverage, a higher empirical expected value for the lower confidence bound implies better performance and is the analog of expected length of two-sided intervals. The results demonstrate that the empirical coverage for the lower generalized confidence bound is comparable to that of the L2 method in some cases, but is much closer to the stated confidence level in other cases. The generalized lower bound is also seen to have a higher empirical expected value. Overall, GCI appears to perform better than the L2 method.

In addition to comparing the generalized lower confidence bound with the L2 method, several other scenarios were simulated for the lower generalized confidence bound. In Tables 4.2–4.3 the empirical coverage probabilities and empirical expected absolute difference are reported, where by the expected absolute difference we mean the expected value of the absolute distance of the lower confidence bound from the true value. The results in the methods sections indicate that it sufficient to check the coverage and lengths for a variety of sample sizes and pairs (p_1, p_2) where p_1 is the lower percentile and p_2 is the upper percentile. Proposition 4.6 allows extension of the results of the simulations to the reflected percentile pairs. Table 4.2 contains the results for percentile pairs which are of usual interest for both small sample sizes and larger sample sizes: $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$. The larger sample sizes are used for judging rate of coverage of the coverage probability to state coverage. Table 4.3 contains the results for a variety of other percentile pairs and sample sizes. The results indicate the empirical coverage probability range from near-exact to conservative. The results also indicate that as the sample size increases the expected absolute differences appears to decrease.

Table 4.1: Empirical Coverage Probabilities and Estimated Expected Values of Lower Confidence Bounds for Proportion Conformance Based on Two Different Methods – GCI and L2.

n	C_1^*	C_2^*	GCI	L2
10	-1	1	0.9760	0.9790
			0.4440	0.4398
10	-1	2	0.9748	0.9795
			0.5756	0.5658
10	-1	3	0.9597	0.9760
			0.6233	0.6107
10	-1	4	0.9535	0.9722
			0.6375	0.6249
10	-1	5	0.9507	0.9712
			0.6392	0.6272
10	-1	6	0.9485	0.9720
			0.6432	0.6306
10	-2	2	0.9732	0.9792
			0.7519	0.7415
10	-2	3	0.9635	0.9770
			0.8243	0.8066
10	-2	4	0.9595	0.9772
			0.8439	0.8181
10	-2	5	0.9613	0.9792
			0.8524	0.8238
10	-2	6	0.9505	0.9722
			0.8570	0.8276
30	-1	1	0.9755	0.9788
			0.5578	0.5526
30	-1	2	0.9647	0.9762
			0.6991	0.6877
30	-1	3	0.9525	0.9730
			0.7311	0.7222
30	-1	4	0.9543	0.9710
			0.7337	0.7263
30	-1	5	0.9490	0.9673
			0.7362	0.7289
30	-1	6	0.9530	0.9710
			0.7369	0.7295
30	-2	2	0.9720	0.9758
			0.8721	0.8671
30	-2	3	0.9577	0.9725
			0.9189	0.9092
30	-2	4	0.9480	0.9712
			0.9256	0.9144
30	-2	5	0.9500	0.9710
			0.9271	0.9157
30	-2	6	0.9445	0.9712
			0.9273	0.9158
50	-1	1	0.9667	0.9730
			0.5913	0.5873
50	-1	2	0.9623	0.9762
			0.7311	0.7213
50	-1	3	0.9557	0.9722
			0.7580	0.7512
50	-1	4	0.9510	0.9700
			0.7619	0.7561
50	-1	5	0.9517	0.9698
			0.7615	0.7559
50	-1	6	0.9503	0.9655
			0.7633	0.7576
50	-2	2	0.9613	0.9675
			0.8984	0.8954
50	-2	3	0.9600	0.9770
			0.9375	0.9306
50	-2	4	0.9457	0.9722
			0.9419	0.9344
50	-2	5	0.9533	0.9732
			0.9427	0.9352
50	-2	6	0.9477	0.9738
			0.9434	0.9359

Table 4.2: Empirical Coverages and "Expected Absolute Differences" of Generalized Lower Confidence Bound for Proportion of Conformance: Percentile Pairs $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$

n	p_1	p_2	coverage	length
2	0.000	0.950	0.9460	0.5207
2	0.000	0.990	0.9540	0.4935
2	0.025	0.975	0.9657	0.7577
2	0.005	0.995	0.9587	0.7457
5	0.000	0.950	0.9480	0.2612
5	0.000	0.990	0.9503	0.1916
5	0.025	0.975	0.9755	0.3728
5	0.005	0.995	0.9680	0.2895
10	0.000	0.950	0.9477	0.1565
10	0.000	0.990	0.9530	0.0928
10	0.025	0.975	0.9728	0.2060
10	0.005	0.995	0.9690	0.1331
20	0.000	0.950	0.9497	0.0938
20	0.000	0.990	0.9533	0.0488
20	0.025	0.975	0.9720	0.1186
20	0.005	0.995	0.9735	0.0646
50	0.000	0.950	0.9440	0.0507
50	0.000	0.990	0.9520	0.0226
50	0.025	0.975	0.9643	0.0603
50	0.005	0.995	0.9657	0.0265
100	0.000	0.950	0.9503	0.0334
100	0.000	0.990	0.9557	0.0134
100	0.025	0.975	0.9613	0.0374
100	0.005	0.995	0.9660	0.0155
500	0.000	0.950	0.9460	0.0132
500	0.000	0.990	0.9435	0.0047
500	0.025	0.975	0.9555	0.0140
500	0.005	0.995	0.9607	0.0051

Table 4.3: Empirical Coverages and "Expected Absolute Differences" of Generalized Lower Confidence Bound for Proportion of Conformance: Other Percentile Pairs

n	p_1	p_2	coverage	length
2	0.000	0.100	0.9513	0.0901
2	0.000	0.300	0.9557	0.2401
2	0.000	0.500	0.9547	0.3606
2	0.000	0.700	0.9470	0.4527
2	0.000	0.900	0.9493	0.5178
2	0.050	0.300	0.9895	0.2362
2	0.050	0.500	0.9858	0.4086
2	0.050	0.700	0.9760	0.5669
2	0.050	0.900	0.9718	0.7115
2	0.050	0.950	0.9690	0.7349
2	0.100	0.300	0.9915	0.1901
2	0.100	0.500	0.9820	0.3684
2	0.100	0.700	0.9738	0.5312
2	0.100	0.900	0.9702	0.6770
2	0.300	0.500	0.9915	0.1893
2	0.300	0.700	0.9848	0.3677
5	0.000	0.100	0.9530	0.0795
5	0.000	0.300	0.9513	0.1941
5	0.000	0.500	0.9520	0.2666
5	0.000	0.700	0.9543	0.2997
5	0.000	0.900	0.9527	0.2897
5	0.050	0.300	0.9905	0.1736
5	0.050	0.500	0.9838	0.2733
5	0.050	0.700	0.9775	0.3527
5	0.050	0.900	0.9802	0.4085
5	0.050	0.950	0.9768	0.3991
5	0.100	0.300	0.9915	0.1375
5	0.100	0.500	0.9845	0.2463
5	0.100	0.700	0.9795	0.3373
5	0.100	0.900	0.9798	0.3978
5	0.300	0.500	0.9890	0.1272
5	0.300	0.700	0.9868	0.2407
10	0.000	0.100	0.9505	0.0710
10	0.000	0.300	0.9505	0.1569
10	0.000	0.500	0.9545	0.2022
10	0.000	0.700	0.9495	0.2126
10	0.000	0.900	0.9555	0.1854
10	0.050	0.300	0.9788	0.1283
10	0.050	0.500	0.9798	0.1916
10	0.050	0.700	0.9742	0.2350
10	0.050	0.900	0.9755	0.2487
10	0.050	0.950	0.9795	0.2407
10	0.100	0.300	0.9852	0.1021
10	0.100	0.500	0.9795	0.1714
10	0.100	0.700	0.9795	0.2265
10	0.100	0.900	0.9755	0.2549
10	0.300	0.500	0.9852	0.0857
10	0.300	0.700	0.9820	0.1611
20	0.000	0.100	0.9505	0.0593
20	0.000	0.300	0.9440	0.1191
20	0.000	0.500	0.9493	0.1451
20	0.000	0.700	0.9520	0.1494
20	0.000	0.900	0.9497	0.1215
20	0.050	0.300	0.9683	0.0935
20	0.050	0.500	0.9745	0.1325
20	0.050	0.700	0.9750	0.1568
20	0.050	0.900	0.9685	0.1560
20	0.050	0.950	0.9768	0.1475
20	0.100	0.300	0.9782	0.0725
20	0.100	0.500	0.9730	0.1170
20	0.100	0.700	0.9698	0.1510
20	0.100	0.900	0.9758	0.1624
20	0.300	0.500	0.9748	0.0574
20	0.300	0.700	0.9768	0.1094
50	0.000	0.100	0.9507	0.0438
50	0.000	0.300	0.9470	0.0816
50	0.000	0.500	0.9463	0.0948
50	0.000	0.700	0.9547	0.0932
50	0.000	0.900	0.9505	0.0699
50	0.050	0.300	0.9635	0.0601
50	0.050	0.500	0.9660	0.0817
50	0.050	0.700	0.9683	0.0936
50	0.050	0.900	0.9647	0.0881
50	0.050	0.950	0.9677	0.0782
50	0.100	0.300	0.9663	0.0444
50	0.100	0.500	0.9640	0.0711
50	0.100	0.700	0.9688	0.0899
50	0.100	0.900	0.9712	0.0934
50	0.300	0.500	0.9685	0.0350
50	0.300	0.700	0.9683	0.0649
100	0.000	0.100	0.9500	0.0335
100	0.000	0.300	0.9583	0.0607
100	0.000	0.500	0.9513	0.0670
100	0.000	0.700	0.9490	0.0650
100	0.000	0.900	0.9453	0.0456
100	0.050	0.300	0.9607	0.0425
100	0.050	0.500	0.9590	0.0564
100	0.050	0.700	0.9627	0.0646
100	0.050	0.900	0.9655	0.0587
100	0.050	0.950	0.9650	0.0505
100	0.100	0.300	0.9570	0.0307
100	0.100	0.500	0.9613	0.0501
100	0.100	0.700	0.9560	0.0611
100	0.100	0.900	0.9643	0.0619
100	0.300	0.500	0.9708	0.0246
100	0.300	0.700	0.9585	0.0453

continued on next page

Table 4.3: Empirical Coverages and "Expected Absolute Differences" of Generalized Lower Confidence Bound for Proportion of Conformance: Other Percentile Pairs.

continued from previous page				
n	p_1	p_2	coverage	length
500	0.000	0.100	0.9543	0.0164
500	0.000	0.300	0.9477	0.0272
500	0.000	0.500	0.9543	0.0299
500	0.000	0.700	0.9485	0.0284
500	0.000	0.900	0.9497	0.0191
500	0.050	0.300	0.9575	0.0189
500	0.050	0.500	0.9565	0.0248
500	0.050	0.700	0.9570	0.0270
500	0.050	0.900	0.9585	0.0237
500	0.050	0.950	0.9583	0.0203
500	0.100	0.300	0.9560	0.0135
500	0.100	0.500	0.9547	0.0214
500	0.100	0.700	0.9605	0.0263
500	0.100	0.900	0.9583	0.0258
500	0.300	0.500	0.9620	0.0107
500	0.300	0.700	0.9615	0.0199

Chapter 5

PROPORTION OF CONFORMANCE IN ONE-WAY RANDOM EFFECTS MODELS

Consider the balanced one-way random effects model that was introduced in Example 2.2. It is of interest to calculate confidence intervals for proportion of conformance for two population characteristics of the one-way random effects model. For a general overview of proportion of conformance see Section 1.1.2 and the first part of Chapter 4.

For the sake of this discussion we assume the one-way random effects model is for repeated measurements, so that the e_{ij}'s represent measurement error. For instance, a machine part is being manufactured. Several copies of the part are chosen at random and each copy is measured multiple times. One might consider the model $Y_{ij} = \mu + A_i + e_{ij}$, where A_i is the random effect due to the part, e_{ij} is the measurement error and we make the usual distributional assumptions. The first population characteristic that is of interest is the measured value Y_i, which is distributed $N(\mu, \sigma_A^2 + \sigma_e^2)$. The second population characteristic of interest is the true value in the presence of measurement error, $\mu + A_i$, which is distributed $N(\mu, \sigma_A^2)$. It is of interest in practical applications to find either an upper or a lower confidence bound for for the proportion of conformance for these two distributions. The purpose of this chapter is to present for both the measured value and the true value a FGPQ for the proportion of conformance of their distributions. This FGPQ can be used to obtain 2-sided confidence intervals along with upper or lower confidence bounds for the proportion of conformance. The coverage probabilities are approximate and the analytical derivation of the coverage probability is intractable. The performance of the 95% lower confidence bound will be assessed using a Monte Carlo simulation.

We will use the notation and results from Examples 2.2, 2.6 and 2.8. We remind the reader that $\mathbb{S} = (\bar{Y}_{..}, S_B^2, S_W^2)$ and $\xi = (\mu, \sigma_A^2, \sigma_e^2)$. A FGPQ for μ, $\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi)$, was derived in Example 2.6 and other representations for $\mathcal{R}_\mu$ were presented in Example 2.8. Both $\mathcal{R}_\mu$ and its representations will be used in the following sections.

5.1 FGPQs for the Proportion of Conformance for a One-way Random Effects Models

In this section FGPQs are derived for the proportion of conformance for the distribution of the measured value and the true value. We will also present results that reduce the number of parameter combination needed in the simulation study that evaluates the performance of the GCIs that are based on the FGPQ.

5.1.1 Proportion of Conformance for $Y \sim N(\mu, \sigma_Y^2)$

Let Y be a random variable with distribution $N(\mu, \sigma_A^2 + \sigma_e^2)$ and let $\sigma_Y^2 = \sigma_A^2 + \sigma_e^2$. We are interested in lower confidence bounds for θ, the proportion of conformance of the distribution $N(\mu, \sigma_Y^2)$. In Example 2.6 a FGPQ for σ_Y was derived using the structural method. Based on (4.1), the structural method of Theorem 2.2 yields the following FGPQ for the proportion of conformance θ:

$$\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \Phi\left(\frac{C_2 - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right) - \Phi\left(\frac{C_1 - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right) \tag{5.1}$$

A $100(1-\alpha)\%$ lower generalized confidence interval for θ is given by $\mathcal{R}_{\theta,\alpha} < \theta$, where $\mathcal{R}_{\theta,\gamma}$ denotes the 100γ-percentile of the conditional distribution of $\mathcal{R}_\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathrm{s}$.

Given a realization s of $\mathbb{S}$, the distribution of $\mathcal{R}_\theta(\mathrm{s}, \mathbb{S}^\star, \xi)$ is usually intractable, but one can calculate $\mathcal{R}_{\theta,\alpha}$ using Monte Carlo simulation. The representation $\dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star)$ is used to generate an empirical distribution of $\mathcal{R}_\theta(\mathrm{s}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathrm{s}$. The simulation is conducted as follows:

$$\dot{\mathcal{R}}_\theta(\mathbb{S}, \mathbb{E}^\star) = \Phi\left(\frac{C_2 - \dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)}{\dot{\mathcal{R}}_\sigma(\mathbb{S}, \mathbb{E}^\star)}\right) - \Phi\left(\frac{C_1 - \dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)}{\dot{\mathcal{R}}_\sigma(\mathbb{S}, \mathbb{E}^\star)}\right), \tag{5.2}$$

where $\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star)$ is defined in (2.18a) and $\dot{\mathcal{R}}_{\sigma_Y}(\mathbb{S}, \mathbb{E}^\star)$ is defined in (2.19a). We remind the reader that $E_1^\star \sim N(0,1)$, $E_2^\star \sim \chi^2_{a-1}$, $E_3^\star \sim \chi^2_{a(n-1)}$ and $E_1^\star$, $E_2^\star$, $E_3^\star$ are independent. Let $\mathbb{E}^\star$ denote the vector $(E_1^\star, E_2^\star, E_3^\star)$. Given a realization s of $\mathbb{S}$, we generate N independent realizations, $(\mathbf{e}_1^\star, \ldots, \mathbf{e}_N^\star)$, of $\mathbb{E}^\star$ and calculate $\dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_1^\star), \ldots, \dot{\mathcal{R}}_\theta(\mathbf{s}, \mathbf{e}_N^\star)$, which gives an empirical distribution of $\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi)$. These values are ordered from smallest to largest and the $[\alpha N]^{th}$ element of the ordered sequence is used as the estimate of $\mathcal{R}_{\theta,\alpha}$.

Let $F_{\mathbf{s}}(t)$ denote the conditional cdf of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Then the coverage of the proposed generalized lower confidence bound is given by $P_{\mathbb{S}}\left[F_{\mathbb{S}}^{-1}(\alpha) \leq \theta\right]$. This is equal to

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \theta | \mathbb{S}\right] > \alpha\right]. \tag{5.3}$$

The calculation of this probability is intractable, so a Monte Carlo simulation will be used to estimate the coverage probability.

Next we will determine the parameters we will use in the simulation study. It is more illustrative to express specification limits, C_1 and C_2, in terms of percentiles of the standard normal distribution. Let (p_1, p_2) be such that $0 \leq p_1 < p_2 \leq 1$, we shall refer to such pairs as *percentile pairs.* There is a one-to-one correspondence between specification limits, (C_1, C_2), and percentile pairs (p_1, p_2), where the correspondence is given by $C_i = \mu + z_{p_i}\sigma_Y$, where $p_i = \Phi\left(\dfrac{C_i - \mu}{\sigma_Y}\right)$ for $i = 1, 2$. In terms of percentile pairs, $\mathcal{R}_\theta$ is given by

$$\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \Phi\left(\frac{\mu + z_{p_2}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right) - \Phi\left(\frac{\mu + z_{p_1}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right). \tag{5.4}$$

The following material allow one to greatly reduce the number of parameter combinations needed in the simulation study. We start with a lemma that shows the distribution of $\mathcal{R}_\theta$ depends on a reduced set of parameters.

Lemma 5.1. *The distribution of $\mathcal{R}_\theta$ depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$.*

Proof. From the canonical form of $\mathcal{R}_{\sigma_Y}$

$$\ddot{\mathcal{R}}_{\sigma_Y}(\mathbb{E}, \mathbb{E}^\star, \xi) = \sigma_e\sqrt{\frac{(n\tau^2 + 1)E_2}{nE_2^\star} + \frac{(n-1)E_3}{nE_3^\star}},$$

one sees that $\mathcal{R}_{\sigma_Y}/\sigma_e$ is a function of n, the parameters a and τ, and $(\mathbb{E}, \mathbb{E}^\star)$. Similarly, the canonical form $\ddot{\mathcal{R}}_\mu$, (2.18b) one sees that $(\mathcal{R}_\mu - \mu)/\sigma_e$ is a function of n, parameters a

and τ, and $(\mathbb{E}, \mathbb{E}^\star)$. Hence

$$\mathcal{R}_\theta = \Phi\left(\frac{z_{p_2}\frac{\sigma_Y}{\sigma_e} - \frac{\mathcal{R}_\mu - \mu}{\sigma_e}}{\mathcal{R}_{\sigma_Y}/\sigma_e}\right) - \Phi\left(\frac{z_{p_1}\frac{\sigma_Y}{\sigma_e} - \frac{\mathcal{R}_\mu - \mu}{\sigma_e}}{\mathcal{R}_{\sigma_Y}/\sigma_e}\right).$$

depends only on n, the percentile pair (p_1, p_2), the parameters a and τ, and $(\mathbb{E}, \mathbb{E}^\star)$. Since the $(E_1, E_2, E_3, E_1^\star, E_2^\star, E_3^\star)$ are jointly independent and their distributions do not depend on unknown parameters, the joint distribution of $(\mathbb{E}, \mathbb{E}^\star)$ is free of unknown parameters and the results follows. $\square$

To assess the performance of $\mathcal{R}_\theta$ one use a Monte Carlo simulation to estimate, for selected parameter combinations, the coverage probability and the expectation of the absolute distance of the lower generalized confidence bound from θ. The following two propositions allow a reduction in the number parameters combination needed in the simulation study.

Proposition 5.1. *The Expression (5.3) depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$.*

Proof. By definition $\theta = p_2 - p_1$ and by Lemma 5.1 the distribution of $\mathcal{R}_\theta$ depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$. The result now follows. $\square$

Proposition 5.2. *The expectation, $E_{\mathbb{S}}\left[|\mathcal{R}_{\theta,\alpha}(\mathbb{S}) - \theta|\right]$, depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$.*

Proof. Using the representation $\mathcal{R}_\theta$, (5.2), one easily sees that the distribution of $\mathcal{R}_\theta$ conditional on $\mathbb{S}$ equal $\mathbf{s} = (\bar{y}, s_B^2, s_W^2)$ is a function of $\frac{\mu + z_{p_2}\sigma_Y - \bar{y}}{s_B}$, $\frac{\mu + z_{p_1}\sigma_Y - \bar{y}}{s_B}$, $\frac{s_W^2}{s_B^2}$ and $(\mathbb{E}^\star$. Since the distribution of $\mathbb{E}^\star$ is free of unknown parameters, the quantile $\mathcal{R}_{\theta,\alpha}(\mathbf{s})$ only depends on $\frac{\mu + z_{p_2}\sigma_Y - \bar{y}}{s_B}$, $\frac{\mu + z_{p_1}\sigma_Y - \bar{y}}{s_B}$ and $\frac{s_W^2}{s_B^2}$. Note

$$\frac{\mu + z_{p_i}\sigma_Y - \bar{Y}}{S_B} = \frac{z_{p_i}}{\sqrt{E_2/(a-1)}} - \frac{E_1}{E_2}\sqrt{\frac{a-1}{an}}, \quad \text{for} \quad i = 1, 2 \quad \text{and}$$

$$\frac{S_W^2}{S_B^2} = \frac{(a-1)E_3}{a(n-1)(n\tau+1)E_2}.$$

Since the distribution of $\mathbb{E}$ is free of unknown parameters the joint distribution of $\frac{\mu + z_{p_2}\sigma_Y - \bar{Y}}{S_B}$, $\frac{\mu + z_{p_1}\sigma_Y - \bar{Y}}{S_B}$ and $\frac{S_W^2}{S_B^2}$ depends only on the percentile pair (p_1, p_2), and the ratio τ and the result follows. $\square$

The following proposition allow us to greatly reduce the number of percentile pairs needed in the simulation study.

Proposition 5.3. *Assume $\mu = 0$. Let (p_1, p_2) be a percentile pair. Then the distribution of*

$$\Phi\left(\frac{z_{p_2}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right) - \Phi\left(\frac{z_{p_1}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right)$$

is equal to the distribution of

$$\Phi\left(\frac{z_{(1-p_1)}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right) - \Phi\left(\frac{z_{(1-p_2)}\sigma_Y - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_Y}}\right).$$

Proof. The proof is the same as the proof of Proposition 4.6 and it is omitted. □

We now show that generalized confidence intervals based on $\mathcal{R}_\theta$ have asymptotically exact coverage as the number of factor levels goes to infinity.

Proposition 5.4. *For measured value Y from a one-way random model, the $100(1-\alpha)\%$ GCIs for the proportion of conformance based on the FGPQ defined in Equation (5.1) has asymptotically $100(1-\alpha)\%$ frequentist coverage as $a \to \infty$.*

Proof. By standard results in mixed linear models and Serfling (Theorem 3.3.A [48]) we have

$$\sqrt{a}\left(\bar{X}^\star - \mu,\, S_B^\star - \sqrt{n\sigma_A^2 + \sigma_e^2},\, \frac{S_B^{\star\,2}}{S_w^{\star\,2}} - \frac{n\sigma_A^2 + \sigma_e^2}{\sigma_e^2/\sqrt{n-1}}\right) \xrightarrow{\mathcal{D}} \boldsymbol{N},$$

where $\boldsymbol{N}$ is a non-degenerate multivariate normal. To simplify the notation let $V = S_B^{\star\,2}/S_w^{\star\,2}$. We have $\boldsymbol{t}(\xi) = (\mu, \sqrt{n\sigma_A^2 + \sigma_e^2}, \frac{n\sigma_A^2+\sigma_e^2}{\sigma_e^2/\sqrt{n-1}})$ satisfies condition 1 of Assumptions 2.1, where $\mathbb{S} = (\bar{X}, S_B, V)$ and $\mathbb{S}^\star$ is an independent copy of $\mathbb{S}$. For the rest of the proof let n be fixed. For item 2 of Assumptions 2.1 $\mathbf{s}$ is an element of $\mathcal{O} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$. To simplify the notation, for $i = 1, 2$, let

$$w_i(\mathbf{s}, \mathbf{s}^\star, \xi) = \frac{s_B^\star(C_i - \bar{x}) + s_B(\bar{x}^\star - \mu)}{s_B\sqrt{\dfrac{n\sigma_A^2 + \sigma_e^2}{n} + \dfrac{n-1}{n}\sigma_e^2 v^{-1} v^\star}}.$$

Note that w_i does not depend on a and has continuous second order partial derivatives with respect to $\mathbf{s}^\star$ $(\mathbf{s}, \mathbf{s}^\star)$ in $\mathcal{O} \times \mathcal{O}$. Thus

$$\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}^\star, \xi) = \Phi\left(w_2(\mathbf{s}, \mathbf{s}^\star, \xi)\right) - \Phi\left(w_1(\mathbf{s}, \mathbf{s}^\star, \xi)\right)$$

satisfies conditions 2a and 2c, where $\mathcal{A} \subset \mathcal{O}$ is any bounded open neighborhood of $\boldsymbol{t}(\xi)$. For $i = 1, 2$, let $w_i = w_i(\mathrm{s}, \boldsymbol{t}(\xi), \xi)$. Then for $\mathrm{s} \in \mathcal{O}$

$$\frac{\partial}{\partial \bar{x}^*}\mathcal{R}_\theta(\mathrm{s}, \mathrm{s}^*, \xi)\bigg|_{\mathrm{s}^*=t(\xi)} = \left((n\sigma_A^2 + \sigma_e^2)\left(\frac{1}{n} + \frac{n-1}{n}v^{-1}\right)\right)^{-1/2} \left(\phi(w_2) - \phi(w_1)\right) > 0,$$

since $\phi(w_2) = \phi(w_1)$ if and only if $C_2 = C_1$. Thus condition 2c is satisfied. □

5.1.2 Proportion of Conformance for $\mu + A \sim N(\mu, \sigma_A^2)$

We are interested in lower confidence bounds for the proportion of conformance $\tilde{\theta}$ for the distribution $N(\mu, \sigma_A^2)$. Here we assume that $\sigma_A^2 > 0$. If σ_A^2 is zero, then $\mu + A_i$ is the point mass μ and either $\tilde{\theta} = 1$ or $\tilde{\theta} = 0$ and it is clearly impossible to find $100(1-\alpha)\%$ lower or upper confidence bounds.

In Example 2.6 a FGPQ for σ_A was derived using $\pi_{\sigma_A}(\xi) = \sqrt{\max(0, \sigma_A^2)}$. A similar modification of $\pi_{\tilde{\theta}}$ is needed for when σ_A is zero. If $\sigma_A = 0$, then we define $(C_i - \mu)/\sigma_A$ is $\pm\infty$, with the sign determined by the sign of $C_i - \mu$. This leads to the following definition for $\pi_{\tilde{\theta}}$

$$\pi_{\tilde{\theta}}(\xi) = \begin{cases} \Phi\left(\dfrac{C_2 - \pi_\mu(\xi)}{\pi_{\sigma_A}(\xi)}\right) - \Phi\left(\dfrac{C_1 - \pi_\mu(\xi)}{\pi_{\sigma_A}(\xi)}\right), & \text{if } \pi_{\sigma_A}(\xi) > 0; \\ 1, & \text{if } \pi_{\sigma_A}(\xi) = 0 \text{ and} \\ & \quad (C_2 - \pi_\mu(\xi))(C_1 - \pi_\mu(\xi)) < 0; \\ 0, & \text{if } \pi_{\sigma_A}(\xi) = 0 \text{ and} \\ & \quad (C_2 - \pi_\mu(\xi))(C_1 - \pi_\mu(\xi)) \geq 0. \end{cases}$$

Using $\pi_{\tilde{\theta}}$, the recipe of Theorem 2.2 produces the following FGPQ for $\tilde{\theta}$,

$$\mathcal{R}_{\tilde{\theta}}(\mathbb{S}, \mathbb{S}^*, \xi) = \begin{cases} \Phi\left(\dfrac{C_2 - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right) - \Phi\left(\dfrac{C_1 - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right), & \text{if } \mathcal{R}_{\sigma_A} > 0; \\ 1, & \text{if } \mathcal{R}_{\sigma_A} = 0 \text{ and} \\ & \quad (C_2 - \mathcal{R}_\mu)(C_1 - \mathcal{R}_\mu) < 0; \\ 0, & \text{if } \mathcal{R}_{\sigma_A} = 0 \text{ and} \\ & \quad (C_2 - \mathcal{R}_\mu)(C_1 - \mathcal{R}_\mu) \geq 0. \end{cases}$$

The realization $\dot{\mathcal{R}}\theta$ is obtained by replacing $\mathcal{R}_\mu$ and $\mathcal{R}_{\sigma_A}$ with $\dot{\mathcal{R}}_\mu$ and $\dot{\mathcal{R}}_{\sigma_A}$ in the definition of $\mathcal{R}_{\tilde{\theta}}$.

Given a realization s of $\mathbb{S}$, the distribution of $\mathcal{R}_{\tilde{\theta}}(\mathrm{s}, \mathbb{S}^*, \xi)$ is usually intractable, but one can calculate $\mathcal{R}_{\tilde{\theta},\alpha}$ using Monte Carlo simulation. This is done using the representation $\dot{\mathcal{R}}_{\tilde{\theta}}(\mathbb{S}, \mathbb{E}^*)$. For details see Section 5.1.1.

A $100(1-\alpha)\%$ lower generalized confidence interval for $\tilde{\theta}$ is given by $\mathcal{R}_{\tilde{\theta},\alpha} < \theta$, where $\mathcal{R}_{\tilde{\theta},\gamma}$ denotes the 100γ-percentile of the conditional distribution of $\mathcal{R}_{\tilde{\theta}} = \mathcal{R}_{\tilde{\theta}}(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$.

Let $F_{\mathbf{s}}(t)$ denote the conditional cdf of $\mathcal{R}_{\tilde{\theta}}(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Then the coverage of the proposed generalized lower confidence bound is given by $P_{\mathbb{S}}\left[F_{\mathbb{S}}^{-1}(\alpha) \leq \tilde{\theta}\right]$. This is equal to

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_{\tilde{\theta}}(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \tilde{\theta}|\mathbb{S}\right] > \alpha\right]. \tag{5.5}$$

The following three propositions will allows us to reduce the number of parameters used in the simulation study for assessing the performance of lower generalized confidence bounds that based on $\mathcal{R}_{\tilde{\theta}}$

Proposition 5.5. *The Expression (5.5) depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$.*

Proof. The proof is the same as the proof of Proposition 5.1. We omit the details. □

Proposition 5.6. *The expectation, $E_{\mathbb{S}}[|\mathcal{R}_{\tilde{\theta},\alpha}(\mathbb{S}) - \theta|]$, depends only on a, n, the percentile pair (p_1, p_2), and the ratio $\tau = \sigma_A/\sigma_e$.*

Proof. The proof is the same as the proof of Proposition 5.2. We omit the details. □

Proposition 5.7. *Assume $\mu = 0$ and $\sigma_e = 1$ If (p_1, p_2) is a percentile pair then the distribution of*

$$\Phi\left(\frac{z_{p_2}\sigma_A - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right) - \Phi\left(\frac{z_{p_1}\sigma_A - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right)$$

is equal to the distribution of

$$\Phi\left(\frac{z_{(1-p_1)}\sigma_A - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right) - \Phi\left(\frac{z_{(1-p_2)}\sigma_A - \mathcal{R}_\mu}{\mathcal{R}_{\sigma_A}}\right).$$

Proof. The proof is the same as the proof of Proposition 4.6 and the details are omitted. □

We now show that generalized confidence intervals based on $\mathcal{R}_{\tilde{\theta}}$ have asymptotically exact coverage as the number of factor levels goes to infinity.

Proposition 5.8. *For true value $\mu + A$ from a one-way random model, the* $100(1-\alpha)\%$ *GCIs for the proportion of conformance based on the FGPQ* $\mathcal{R}_{\tilde{\theta}}$ *has asymptotically* $100(1-\alpha)\%$ *frequentist coverage as* $a \to \infty$.

Proof. We will use the same $\mathbb{S}^\star$ and $\boldsymbol{t}(\xi)$ as in Proposition 5.4. Since $\mathcal{R}_{\sigma_A}(\boldsymbol{t}(\xi), \boldsymbol{t}(\xi), \xi) = \sigma_A > 0$ and $\mathcal{R}_{\sigma_A}(\cdot, \cdot, \xi)$ is continuous in a neighborhood of $(\boldsymbol{t}(\xi), \boldsymbol{t}(\xi))$ there exists a bounded open set $\mathcal{A}'$ of $\boldsymbol{t}(\xi)$ such that $\mathcal{R}_{\sigma_A}(\mathrm{s}, \mathrm{s}^\star)$ is greater than zero for all $(\mathrm{s}, \mathrm{s}^\star)$ in $\mathcal{A}' \times \mathcal{A}'$. Let

$$\tilde{w}_i(\mathrm{s}, \mathrm{s}^\star, \xi) = \frac{s_B^\star(C_i - \bar{x}) + s_B(\bar{x}^\star - \mu)}{s_B\sqrt{\dfrac{n\sigma_A^2 + \sigma_e^2}{n} - \dfrac{1}{n}\sigma_e^2 v^{-1} v^\star}}$$

for $i = 1, 2$ and note that for $(\mathrm{s}, \mathrm{s}^\star)$ in $\mathcal{A}' \times \mathcal{A}'$

$$\mathcal{R}_{\tilde{\theta}}(\mathrm{s}, \mathrm{s}^\star, \xi) = \Phi\left(\tilde{w}_2(\mathrm{s}, \mathrm{s}^\star, \xi)\right) - \Phi\left(\tilde{w}_1(\mathrm{s}, \mathrm{s}^\star, \xi)\right)$$

exits. The proof now follows using the arguments from the proof of Proposition 5.4, where we use $\mathcal{A}'$ and $\mathcal{R}_{\tilde{\theta}}$ (restricted to $\mathcal{A}' \times \mathcal{A}'$) in place of $\mathcal{O}$ and $\mathcal{R}_\theta$. □

5.2 Simulation Study and Discussion

A simulation study was conducted assess the performance of the proposed lower confidence bounds for the proportion of conformance for the population characteristics $Y \sim N(\mu, \sigma_A^2 + \sigma_e^2)$ and $\mu + A \sim N(\mu, \sigma_A^2)$. We refer to the former as the measured value and the later as true value. The performance was assessed by calculating the empirical coverage probability and the empirical expected value of absolute distance of the lower confidence bound from the true proportion of conformance. The later we refer to as the "Expected Distance". By Propositions 5.1, 5.2 5.5 and 5.6, without lost of generality we can assume $\mu = 0$ and $\sigma_e^2 = 1$. We used the value of $\alpha = 0.05$ for the simulation. The other four parameters used in the simulation were – (1) a, the number of levels, (2) n, the number of replicates, (3) σ_A, the standard deviation of the random effect, and (4) the percentile pairs (p_1, p_2). Two types of behavior were explored, small sample and the asymptotics, i.e., the convergence of the coverage probability to the stated coverage.

The same set of percentile pairs was used to explore both the small sample behavior and the asymptotic behavior. Propositions 5.3 and 5.7 allow a reduction in the number of percentile pairs needed in the simulation, e.g., by the propositions the coverage probability for $(0, 0.95)$ are the same as for $(0.05, 1)$. The percentile pairs were divided into two sets, first a set of four which are percentile pairs that are usually of interest: $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$. Second, seventeen other percentile pairs that cover a wide range of other situations. These pairs are denoted *Other Percentile Pairs* in the tables of results.

The parameters values used for the small sample behavior were – (1) $a = 2, 5,$ and 10; (2) $n = 2, 5,$ and 10; and (3) $\sigma_A = 0.01$, 0.25, 0.5, 1, 2, 4, and 10. Additionally, for the measured value Y, $\sigma_A = 0$ was included.

The asymptotic behavior was explored using the parameter values – (1) $a = 20$, 50, 100 and 500; (2) $n = 10$; and (3) $\sigma_A = 0.25$, 1, 2, 4, and 10.

For each parameter combination $(a, n, \sigma_A, (p_1, p_2))$, 4000 realization of $\mathbb{S}$ were generated. For each realization $\mathbf{s}$ of $\mathbb{S}$ an empirical distribution of $\dot{\mathcal{R}}_*(\mathbf{s}, \mathbb{E}^*)$ was generated using $N = 10,000$ realizations of $\mathbb{E}^*$, where $*$ denotes either θ or $\tilde{\theta}$. The details on how to generate the empirical distribution are in Section 5.1.1. The 0.05^{th} percentile of the empirical distribution was used for the lower tolerance bound.

The FORTRAN programs that were written to implement the simulation study are contained in Sections C.5.1 and C.5.2. The realizations of $\mathbb{S}$ and $\mathbb{E}^*$ were generated using the random number generators DRNNOR and DRNCHI.

The results of the simulations are organized into four tables for the true value $\mu + A$ and four tables for the measured value Y. For the true value $\mu + A$ Tables B.1 and B.2 contain the results for the percentile pairs $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$, with Table B.1 dedicated to the small values for a and n and Table B.2 for the Asymptotics. Tables B.3-B.4 contain the results for the Other Percentile Pairs, with Table B.3 for the small values of a and n, and Table B.4 for the Asymptotics. Tables B.5-B.8 contain the results of the simulations for the measured value Y. The tables are organized in same fashion as the tables for the true value.

We will first discuss the simulation results for the true value and start with the percentile pairs of interest, i.e., Tables B.1 and B.2. The results in Table B.1 indicates that for $a = 2$ the coverage is conservative and the expected distances are large. As a increases, and to a lesser degree as n increases, it appears that the coverage becomes less conservative and the expected distance decreases. The rates are slower for values of σ_A less than one. In fact when $a = n = 10$ for both σ_A equal 0.01 and 0.25 the empirical coverage probabilities are conservative but the expected distances are quite different. It is noteworthy that, for a fixed a, n and a percentile pair, the expected length stabilizes for σ_A greater than one. Table B.2 indicates the rate of convergence to stated coverage for each percentile pair along with a corresponding reduction in the expected distance. The results for the Other Percentile Pairs in Tables B.3 and B.4 follow a similar pattern, with one notable exception. For $a = 2$ and $\sigma_A \leq 1$ and the percentile pairs concentrated in the tail of the distribution, e.g., (0,0.10), the empirical coverage probabilities are liberal. This is not surprising considering that a lower bound for the proportion of the distribution is being estimated. Also note that the percentile pairs with $p_U - p_L$ small and $p_L \neq 0$ have a more conservative empirical coverage probability than a percentile pair with $p_U - p_L$ large. For example, observe the rate of convergence of empirical coverage probability for the percentile pair $(0.3, 0.5)$.

The results of the simulation study for the measured value in Tables B.5-B.8 show the same patterns as those described above. It is worth noting that for the percentile pairs $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, and $(0.005, 0.995)$, even when the model is incorrectly specified, i.e., when $\sigma_A = 0$, the coverage probabilities appear to be satisfactory when the number of level is as low as 20 (see Table B.6).

Chapter 6

GENERALIZED STRUCTURAL METHODS FOR FIDUCIAL GENERALIZED PIVOTAL QUANTITIES

Let $\mathbb{S}$ be a k dimensional statistic whose distribution is indexed by a p-dimensional parameter ξ. In the examples up until now k has been equal to p and $\mathbb{S}$ has been a complete sufficient statistic for ξ. There are examples, with $k > p$, where confidence intervals for a parameter $\pi(\xi)$ are of interest. For example, consider iid samples from m normal populations which have a common mean but heterogeneous variances. It is of interest to construct confidence intervals for the common mean. The minimal sufficient statistic has dimension $2m$, while the number of parameters is $m + 1$.

In this chapter we will present two methods that are potentially applicable in problems where $k > p$. The first method, the two stage method is a special case of the structural construction method of Theorem 2.2 . The second method is the general structural method from Hannig et al. [25]. In the Chapter 7 we will use the two methods to generate FGPQs for the "common mean" problem.

6.1 Two Stage Construction of a FGPQ

In the two stage construction method the invertible pivotal quantity is constructed in two stages. The first stage is existence of an invertible pivotal quantity for a subset ξ_1 of ξ and the second stage is the existence of a invertible pivotal quantity for fixed values of ξ_1.

Proposition 6.1. *Let $\xi = (\xi_1, \xi_2)$ and suppose the conditions (a)–(c) hold.*

(a) There is a statistic $\mathbb{S}_1$ such that $\mathbb{S}_1$ and ξ_1 have an invertible pivotal relationship, $\boldsymbol{f}_1(\mathbb{S}_1, \xi_1) = \mathbb{E}_1$ with inverse $\boldsymbol{g}_1(\mathbb{S}_1, \cdot\,)$. See Definition 2.4. By Theorem 2.2 there exists a FGPQ for ξ_1, given by $\mathcal{R}_{\xi_1} = \boldsymbol{g}_1(\mathbb{S}_1, \boldsymbol{f}_1(\mathbb{S}_1^\star, \xi_1))$

(b) There exists a statistic $\mathbb{S}_2$*, and a pivotal quantity* $\mathbb{E}_2 = \boldsymbol{f}_2(\mathbb{S}_2, \xi_1, \xi_2)$ *such that for each fixed* ξ_1*,* $\boldsymbol{f}_2(\mathbb{S}_2, \xi_1, \cdot\,)$ *is an invertible pivotal quantity for* ξ_2*. Denote the inverse by* $\boldsymbol{g}_2(\mathbb{S}_2, \xi_1, \cdot\,)$*, that is* $\boldsymbol{g}_2(\mathbb{S}_2, \xi_1, \boldsymbol{f}_2(\mathbb{S}_2, \xi_1, \xi_2)) = \xi_2$*.*

(c) The joint distribution of $\mathbb{E}_1$ *and* $\mathbb{E}_2$ *is parameter free.*

Let $\mathbb{S} = (\mathbb{S}_1, \mathbb{S}_2)$ *and* $\mathbb{E} = (\mathbb{E}_1, \mathbb{E}_2)$*, then*

$$\mathcal{R}_\xi = \mathcal{R}_\xi(\mathbb{S}, \mathbb{S}^\star, \xi) = [\mathcal{R}_{\xi_1}, \boldsymbol{g}_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, \boldsymbol{f}_2(\mathbb{S}_2^\star, \xi_1, \xi_2))]$$

is a FGPQ for ξ*, where* $\mathbb{S}_1^\star$ *and* $\mathbb{S}_2^\star$ *are independent copies of* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *respectively, and* $\mathbb{E}_1^\star = \boldsymbol{f}_1(\mathbb{S}_1^\star, \xi_1)$ *and* $\mathbb{E}_2^\star = \boldsymbol{f}_2(\mathbb{S}_2^\star, \xi_1, \xi_2)$*.*

Proof. Since $\boldsymbol{f}(\mathbb{S}, \xi) = (\boldsymbol{f}_1(\mathbb{S}_1, \xi_1), \boldsymbol{f}_2(\mathbb{S}_2, \xi_1, \xi_2))$ is an invertible pivotal quantity for ξ, with inverse $\boldsymbol{g}(\mathbb{S}, \mathbb{E}) = (\boldsymbol{g}_1(\mathbb{S}_1, \mathbb{E}_1), \boldsymbol{g}_2(\mathbb{S}_2, \boldsymbol{g}_1(\mathbb{S}_1, \mathbb{E}_1), \mathbb{E}_2))$, the proposition follows from Theorem 2.2. □

Remark 6.1. It is worth noting that $\boldsymbol{g}_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, \boldsymbol{f}_2(\mathbb{S}_2^\star, \xi_1, \xi_2))$ is a FGPQ for ξ_2.

Remark 6.2. A FGPQ constructed using the recipe in Proposition 6.1 has two other representations. The first representation is as function of $(\mathbb{S}, \mathbb{E}^\star)$, that is

$$\mathcal{R}_\xi = \dot{\mathcal{R}}_\xi(\mathbb{S}, \mathbb{E}^\star) = [\boldsymbol{g}_1(\mathbb{S}_1, \mathbb{E}_1^\star), \boldsymbol{g}_2(\mathbb{S}_2, \boldsymbol{g}_1(\mathbb{S}_1, \mathbb{E}_1^\star), \mathbb{E}_2^\star)]$$

The representation $\dot{\mathcal{R}}_\xi$ facilitates the derivation of the conditional distribution of $\mathcal{R}_\xi$ given $\mathbb{S} = \mathrm{s}$ in either a closed form or Monte Carlo empirical form. The second is the canonical representation $\ddot{\mathcal{R}}_\xi(\mathbb{E}, \mathbb{E}^\star)$ (see Remark 2.5). While it is possible to express a sufficient condition for the existence of the inverse mapping $\boldsymbol{h}(\mathbb{E}, \xi)$, it is tedious to do such. It is more enlightening to note that one uses the invertible pivotal quantities $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ to find an expression for $\mathbb{S}$ as a function of $(\mathbb{E}, \xi)$ and then substitute this function for $\mathbb{S}$ into the expression $\dot{\mathcal{R}}_\xi(\mathbb{S}, \mathbb{E}^\star)$ to obtain the representation $\ddot{\mathcal{R}}_\xi(\mathbb{E}, \mathbb{E}^\star)$. An example of this process will be provided in the next chapter. The canonical representation is useful in proving various invariance properties of the FGPQ.

The two-stage construction is a systematic approach for obtaining a FGPQ for a number of problems where the minimal sufficient statistic is not complete. In many situations it is fairly straightforward to verify that the conditions of Theorem 2.1 hold for the two-stage FGPQ so that the resulting interval will have, at least asymptotically, the correct frequentist coverage.

The two stage construction is also discussed in Hannig et al. [25].

6.2 General Structural Method for Constructing a FGPQ

Hannig et al. [25] presented the general structural method for constructing FGPQs. The method reduces to the structural method of Theorem 2.2 when $k = p$. Of interest to the practitioner is a corollary which gives an alternative construction. Most FGPQs derived using the general structural method are constructed using the alternative construction given in the corollary. We will now state the general construction method.

Let $\mathbb{S} = (S_1, \ldots, S_k) \in \mathcal{S} \subset \mathbb{R}^k$ denote a statistic whose distribution is indexed by $\xi \in \Xi \subset \mathbb{R}^p$. Let $\theta = \pi(\xi)$ be the parameter of interest.

We make the following assumptions.

Assumptions (B)

(a) There exists a mapping $\boldsymbol{f} : \mathcal{S} \times \Xi \longrightarrow \mathbb{R}^k$ such that $\mathbb{E} = \boldsymbol{f}(\mathbb{S}, \xi)$ has a continuous cdf which does not depend on ξ. Let $\mathcal{E} = \boldsymbol{f}(\mathcal{S} \times \Xi)$. Write $\boldsymbol{f} = (f_1, \ldots, f_k)$ so that $E_i = f_i(\mathbb{S}, \xi)$ for $i = 1, \ldots, k$.

(b) Let $\mathbb{E}_0 = (E_1, \ldots, E_p)$ and $\boldsymbol{f}_0 = (f_1, \ldots, f_p)$. We assume that, for each fixed $\mathrm{s} \in \mathcal{S}$, the mapping $\boldsymbol{f}_0(\mathrm{s}, \cdot\,) : \Xi \rightarrow \mathcal{E}$ defined by $\mathrm{e}_0 = \boldsymbol{f}_0(\mathrm{s}, \xi)$ is invertible. Write $\mathcal{E}_0 = \boldsymbol{f}_0(\mathrm{s}, \Xi)$. We denote this inverse mapping from $\mathcal{E}_0$ to Ξ by $\boldsymbol{g}_0(\mathrm{s}, \cdot\,)$ $= (g_1(\mathrm{s}, \cdot\,), \ldots, g_p(\mathrm{s}, \cdot\,))$. Thus we have $\boldsymbol{g}_0(\mathrm{s}, \boldsymbol{f}_0(\mathrm{s}, \xi)) = \xi$ for each $\mathrm{s} \in \mathcal{S}$.

Let $\mathbb{E}_c = (E_{p+1}, \ldots, E_k)$ and $\boldsymbol{f}_c = (f_{p+1}, \ldots, f_k)$. Substituting $\xi = \boldsymbol{g}_0(\mathbb{S}, \mathbb{E}_0)$ in the equations $E_j = f_j(\mathbb{S}, \xi)$, $j = p+1, \ldots, k$, we get the identity

$$\mathbb{E}_c = \boldsymbol{f}_c(\mathbb{S}, \boldsymbol{g}_0(\mathbb{S}, \mathbb{E}_0)). \tag{6.1}$$

For any fixed $\mathrm{s} \in \mathcal{S}$, let $\mathcal{M}(\mathrm{s})$ denote the set of $\mathbf{e} = (e_1, \ldots, e_k) \in \mathcal{E}$ satisfying the following equations:

$$e_j = f_j(\mathrm{s}, \boldsymbol{g}_0(\mathrm{s}, \mathbf{e}_0)), \quad j = p+1, \ldots, k, \tag{6.2}$$

where $\mathbf{e}_0 = (e_1, \ldots, e_p)$, the first p coordinates of $\mathbf{e}$. Thus $\mathcal{M}(\mathrm{s})$ is a manifold in $\mathbb{R}^k$.

Given $\mathbb{S}$, by virtue of (6.1), it follows that $\mathbb{E}$ must lie on the manifold $\mathcal{M}(\mathbb{S})$. Note that the same manifold may be definable using a different set of equations. We will use the notation $\mathbb{B}(\mathrm{s}, \mathrm{e}) = \mathbf{0}$ for the equations chosen to define $\mathcal{M}(\mathrm{s})$.

We now make the following assumption concerning the random vector $\mathbb{E}_0$.

Assumption (C): Conditional on $\mathbb{B}(\mathrm{s}, \mathbb{E})$, $\mathbb{E}_0$ has a jointly continuous distribution.

The following lemma states the multivariate version of the probability integral transform.

Lemma 6.1. *Let $F_1(\cdot \,|\mathbb{B}(\mathrm{s}, \mathbb{E}))$ denote the cdf of E_1 given $\mathbb{B}(\mathrm{s}, \mathbb{E})$ and*

$F_j(\cdot \,|E_1, \ldots, E_{j-1}, \mathbb{B}(\mathrm{s}, \mathbb{E}))$ denote the cdf of E_j given $E_1, \ldots, E_{j-1}, \mathbb{B}(\mathrm{s}, \mathbb{E})$, $j = 2, \ldots, p$.

$F_j(\cdot \,|E_1, \ldots, E_{j-1}, \mathbb{B}(\mathrm{s}, \mathbb{E}))$ denote the cdf of E_j given $E_1, \ldots, E_{j-1}, \mathbb{B}(\mathrm{s}, \mathbb{E})$, $j = 2, \ldots, p$.

The conditional joint distribution of $\mathbb{E}_0$ given $\mathbb{B}(\mathrm{s}, \mathbb{E})$ is completely determined by the univariate cdfs $F_1, \ldots, F_p$. Let the random vector $\mathbb{U}$ be defined by

$$\mathbb{U} = \boldsymbol{F}(\mathbb{E}_0|\mathbb{B}(\mathrm{s}, \mathbb{E})) = (F_1(E_1|\mathbb{B}(\mathrm{s}, \mathbb{E})), F_2(E_2|E_1, \mathbb{B}(\mathrm{s}, \mathbb{E})), \ldots, F_p(E_p|E_1, \ldots, E_{p-1}, \mathbb{B}(\mathrm{s}, \mathbb{E})))$$

The distribution of $\mathbb{U}$, conditional on $\mathbb{B}(\mathrm{s}, \mathbb{E})$, is uniform on $[0, 1]^p$. Hence this is also its unconditional distribution.

We denote the inverse of $\boldsymbol{F}\left(\cdot \;\middle|\mathbb{B}(\mathrm{s}, \mathbb{E})\right)$ by $\boldsymbol{G}\left(\cdot \;\middle|\mathbb{B}(\mathrm{s}, \mathbb{E})\right)$. Both $\boldsymbol{F}$ and $\boldsymbol{G}$ depend on ξ but for clarity of notation this dependence is not explicitly displayed.

We now state the general construction for FGPQs.

Theorem 6.1. *Suppose Assumptions B and Assumption C are satisfied. Let $\theta = \pi(\xi)$ be a scalar parameter. Define*

$$\mathcal{R}_\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = \pi\left(\boldsymbol{g}_0\left(\mathbb{S}, \boldsymbol{G}\left(\boldsymbol{F}\left(\boldsymbol{f}_0(\mathbb{S}^\star, \xi) \;\middle|\; \mathbb{B}(\mathbb{S}, \mathbb{E}^\star)\right) \;\middle|\; \mathbf{0}\right)\right)\right)$$

Then $\mathcal{R}_\theta$ is an FGPQ for θ.

The following corollary from Hannig et al. [25] presents a method for calculating the conditional pdf of $\mathcal{R}_\theta(\mathbf{S},\mathbb{S}^\star,\xi)$ given $\mathbb{S}=\mathbf{s}$. In many cases the conditional pdf is sufficient for determining general confidence bounds.

Corollary 6.1. *The pdf of*

$$\pi\left(\boldsymbol{g}_0\left(\mathbf{s},\boldsymbol{G}\left(\boldsymbol{F}\left(\boldsymbol{f}_0(\mathbb{S}^\star,\xi)\;\middle|\;\mathbb{B}(\mathbf{s},\mathbb{E}^\star)\right)\;\middle|\;\mathbf{0}\right)\right)\right)$$

is the same as the conditional pdf of $\pi\left(\boldsymbol{g}_0\left(\mathbf{s},\mathbb{E}_0^\star\right)\right)$ *given* $\mathbb{B}(\mathbf{s},\mathbb{E}^\star)=\mathbf{0}$.

Remark 6.3. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are jointly continuous random variables and the conditional pdf of $\boldsymbol{X}$ given $\boldsymbol{Y}=\boldsymbol{y}_0$ is of interest, then from the definition of conditional pdf we have $f(\boldsymbol{x}|\boldsymbol{y}_0)=Cf_{\boldsymbol{Y},\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{y}_0)$, where $C^{-1}=\int_{-\infty}^{\infty}f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y}_0)\,d\boldsymbol{x}$. This observation is useful when one is only interested in the conditional pdf at a specific value of $\boldsymbol{Y}$ and calculating $f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y})$ for a generic $\boldsymbol{y}$ is non trivial.

In the description of the general structural method it was noted that same manifold can be defined using different sets of equations. The following example illustrate that different choices for the equations that defining $\mathcal{M}(\mathbb{S})$ may result in nonequivalent FGPQs. The example used is simply for illustrative purposes, the reader will notice that we are using an non-minimal sufficient statistic when a complete sufficient statistic exists.

Example 6.1. Let $X_1,\ldots,X_n$ be a sequence of independent identically distributed random variables, with $X_i\sim N(\mu,1)$. Let $\mathbb{S}=(X_1,\ldots,X_n)$ and $\xi=\mu$. Define $f_i(\mathbb{S},\mu)=X_i-\mu=E_i\sim N(0,1)$, for $i=1,\ldots,n$. Since the E_i's are mutually independent, $\mathbb{E}=(E_1,\ldots,E_n)$ is distributed $MVN(\mathbf{0},\mathbf{I}_n)$ and hence $\mathbb{E}$ satisfies item (a) of Assumptions (B). Let $\mathbb{E}_0=E_1$, so $\boldsymbol{f}_0$ consists of the single element f_1. Then $\boldsymbol{g}_0(\mathbb{S},\mathbb{E}_0)=g_1(\mathbb{S},\mathbb{E})=X_1-E_1$ defines an invertible mapping for $\boldsymbol{f}_0(\mathbf{s},\cdot)$; hence item (b) of Assumptions (B) is satisfied. Let $\mathbb{E}_c=(E_2,\ldots,E_n)$ and $\boldsymbol{f}_c=(f_2,\ldots,f_n)$. The set of equations (6.2) that define the manifold $\mathcal{M}(\mathbf{s})$ are $e_j=f_j(\mathbf{s},g_1(\mathbf{s},e_0))=x_j-(x_1-e_1)$ for $j=2,\ldots,n$. We will now express this set of equations in two equivalent forms. The manifold $\mathcal{M}(\mathbf{s})$ may be described by the set of equations

$$(e_1-e_j)-(x_1-x_j)=0\qquad\text{for } j=2,\ldots,n,$$

which we will denote by $\mathbb{B}(\mathbf{s},\mathbf{e}) = 0$. The manifold can also be described by the set of equations

$$\frac{x_1 - e_1}{x_j - e_j} - 1 = 0 \qquad \text{for } j = 2,\ldots,n,$$

which we will denote by $\tilde{\mathbb{B}}(\mathbf{s},\mathbf{e}) = 0$. Since $\mathbb{E}_0$ is one-dimensional, Assumption (C) is satisfied for both $\mathbb{B}(\mathbf{s},\mathbb{E})$ and $\tilde{\mathbb{B}}(\mathbf{s},\mathbb{E})$. Hence, using either $\mathbb{B}(\mathbb{S},\mathbb{E})$ or $\tilde{\mathbb{B}}(\mathbb{S},\mathbb{E})$, the recipe given in Theorem 6.1 defines a FGPQ for μ. We will denote the FGPQ constructed by conditioning on $\mathbb{B}$ by $\mathcal{R}_\mu$ and the FGPQ constructed by conditioning on $\tilde{\mathbb{B}}$ by $\tilde{\mathcal{R}}_\mu$. By Corollary 6.1 the condition pdf $\mathcal{R}_\mu(\mathbb{S},\mathbb{S}^\star,\xi)$ given $\mathbb{S}=\mathbf{s}$ is the same as the conditional pdf of $g_0(\mathbf{s},\mathbb{E}_0^\star)$ given $\mathbb{B}(\mathbf{s},\mathbb{E}^\star) = \mathbf{0}$. Similarly, the the condition pdf of $\tilde{\mathcal{R}}_\mu(\mathbb{S},\mathbb{S}^\star,\xi)$ is the same as the conditional pdf of $g_0(\mathbf{s},\mathbb{E}_0^\star)$ given $\tilde{\mathbb{B}}(\mathbf{s},\mathbb{E}^\star) = \mathbf{0}$. We will calculate the two conditional pdfs of $g_0(\mathbf{s},\mathbb{E}_0^\star)$.

Let $\mathbf{s} = (x_1,\ldots,x_n)$, so $g_0(\mathbf{s},\mathbb{E}^\star) = x_1 - E_1^\star$.

The equation $\mathbb{B}_1(\mathbf{s},\mathbb{E}^\star) = \mathbf{0}$ is equivalent to $E_1^\star - E_j^\star = x_1 - x_j$ for $j = 2,\ldots,n$. We first find the conditional pdf of $E_1^\star$ given $E_1^\star - E_j^\star = x_1 - x_j$ for $j = 2,\ldots,n$. Note that $(E_1^\star, E_1^\star - E_2^\star,\ldots,E_1^\star - E_n^\star)$ is distributed $MVN(\mathbf{0},\Sigma)$, where $\Sigma_{i,j} = 2$ for $i = j = 2,\ldots,n$ and $\Sigma_{i,j} = 1$ elsewhere. By standard results for conditional distributions of a Multivariate Normal distribution, $(E_1^\star|E_1^\star - E_2^\star,\ldots,E_1^\star - E_n^\star)$ is distributed $N(x_1 - \bar{x}, 1/n)$. Thus $(x_1 - E_1^\star|E_1^\star - E_2^\star,\ldots,E_1^\star - E_n^\star)$ is distributed $N(\bar{x},1/n)$. Let $f_\mathbf{s}$ denote the pdf of $\mathcal{R}_\mu$ given $\mathbf{s}$, then

$$f_\mathbf{s}(y) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{n}{2}(y-\bar{x})^2}.$$

The pdf $\tilde{\mathcal{R}}_\mu$ is the same as the pdf of Y given $\boldsymbol{B} = \mathbf{1}$, where $Y = x_1 - E_1^\star$ and $\boldsymbol{B} = (B_2,\ldots,B_n)$, with $B_j = \frac{x_1 - E_1}{x_j - E_j}$. We use the Jacobian technique to find the joint distribution of $(Y,\boldsymbol{B})$ and then use Remark 6.3 to find the pdf $f_{Y|\boldsymbol{B}=\mathbf{1}}(y)$. The random variable $(Y,\boldsymbol{B})$ is a function of $\mathbb{E}$ and the inverse transform is given by the n equations $E_1^\star = x_1 - Y$ and $E_j^\star = x_j - Y/B_j$ for $j = 2,\ldots,n$. The Jacobian of the transform is

$$J = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{B_2} & \frac{Y}{B_2^2} & 0 & \cdots & 0 \\ -\frac{1}{B_3} & 0 & \frac{Y}{B_3^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{B_n} & 0 & \cdots & \cdots & \frac{Y}{B_n^2} \end{pmatrix} \quad \text{and} \quad |J| = \frac{|y|^{n-1}}{\prod_{j=2}^n B_j^2}.$$

Thus $f_{Y|\mathbf{B}=\mathbf{1}}(y)$ is equal to $C|y|^{n-1}\, e^{-\frac{1}{2}[(x_1-y)^2+\sum_{j=2}^{n}(x_j-y_j)^2]}$, where C is a constant such that $\int_{-\infty}^{\infty} f_{Y|\mathbf{B}=\mathbf{1}}(y)\,dy = 1$. This can be rewritten as

$$\tilde{f}_{\mathbf{s}}(y) = \tilde{K}|y|^{n-1}\, e^{-\frac{n}{2}(y-\bar{x})^2}, \text{ with } \tilde{K}^{-1} = \int_{-\infty}^{\infty} |y|^{n-1}\, e^{-\frac{n}{2}(y-\bar{x})^2}\, dx.$$

It is clear $f_{\mathbf{s}}$ is not equal to $\tilde{f}_{\mathbf{s}}$.

Chapter 7

COMMON MEAN OF SEVERAL POPULATIONS

The common mean problem was discussed in Section 1.2. We will remind the reader of the main points. Suppose $X_{i1}, \ldots, X_{in_i}$ are iid with $X_{ij} \sim N(\mu, \sigma_i^2)$, for $i = 1, \ldots, k$. It is of interest to provide an efficient interval estimate of μ. The question is how to combine the k complete sufficient statistics that exist for the k parameters (μ, σ_i^2). Several frequentist methods have been suggested for the construction of pivotal quantities for μ, based on combining the data from each of the populations. Yu et al. [59] concluded after conducting a study of several of the these interval estimators that using R. A. Fisher's method of combining P-values from the individual populations performed as well as or better than the other frequentist methods.

The purpose of this chapter is to compare the performance of three methods – (1) A generalized confidence interval based on a FGPQ constructed using the two-stage method of Proposition 6.1, (2) a generalized confidence interval based on a FGPQ constructed using the general structural method of Theorem 6.1, and (3) an exact method based on Fisher's proposal for combining P-values from independent tests. The methods will be compared based on upper confidence bounds for μ. While the Fisher-method is exact, the two GCIs are only approximate. It will be shown that the two GCIs have asymptotical exact coverage. Analytical derivation of the actual coverage probabilities for the GCIs is intractable, so a Monte Carlo simulation is used to assess their performance for small sample sizes.

The chapter is organized as follows. The first section contains notation that is common to all sections. Sections 2-4 contain a detailed descriptions of the three completing three confidence interval procedures along with several invariance properties that greatly reduce the number of parameter combinations needed in the simulation study. Description of the simulation study and the results of the simulation study are given in Section 5. Section 6 is a discussion of the results of the simulation study.

7.1 Notation

Let

$$\bar{X}_i = \frac{\sum_{j=1}^{n_i} X_{ij}}{n_i}, \text{ and } S_i^2 = \frac{\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2}{n_i - 1}, \text{ for } i = 1, \dots, k.$$

All three methods are based on these statistics. The statistic $(\bar{X}_1, \dots, \bar{X}_k, S_1^2, \dots, S_k^2)$ is a minimal sufficient statistic for $(\mu, \sigma_1^2, \dots, \sigma_k^2)$, but it is not complete.

Remark 7.1. The following random variables will be used in proving several invariance statements. For $i = 1, \dots, k$, let $\tau_i = \sigma_i/\sigma_1$. Note $\tau_1 = 1$. Define

$$\check{X}_{ij} = \frac{X_{ij} - \mu}{\sigma_1} \quad \text{for } i = 1, \dots, k, \ j = 1, \dots, n_i.$$

Then $\check{X}_{i1}, \dots, \check{X}_{in_i}$ are iid $N(0, \tau_i^2)$, $i = 1, \dots, k$ and there are corresponding statistics $\check{\bar{X}}_i$ and $\check{S}_i^2$ for $i = 1, \dots, k$, with the relations

$$\check{\bar{X}}_i = \frac{\bar{X}_i - \mu}{\sigma_1}, \text{ and } \check{S}_i^2 = \frac{S_i^2}{\sigma_1^2}, \text{ for } i = 1, \dots, k. \tag{7.1}$$

The statistic $(\check{\bar{X}}_1, \dots, \check{\bar{X}}_k, \check{S}_1^2, \dots, \check{S}_k^2)$ is a minimal sufficient statistic for $(0, 1, \tau_2^2, \dots, \tau_k^2)$. We will refer to the $\check{X}_{ij}$'s as the (μ, σ_1)-*transformed populations.*

7.2 Confidence Intervals for μ based on a Two-Stage FGPQ

We will now derive a two-stage FGPQ for μ. In order to match the notation of the Proposition 6.1 we rearrange the order of the parameters above and let $\xi_1 = (\sigma_1^2, \dots, \sigma_k^2)$ and $\xi_2 = \mu$, so that $\xi = (\sigma_1^2, \dots, \sigma_k^2, \mu)$. Additionally, let $\mathbb{S}_1 = (S_1^2, \dots, S_k^2)$ and $\mathbb{S}_2 = (\bar{X}_1, \dots, \bar{X}_k)$, so $\mathbb{S} = (\mathbb{S}_1, \mathbb{S}_2)$.

First we construct a FGPQ for ξ_1. For $i = 1, \dots, k$

$$f_{1,i}(S_i^2, \sigma_i^2) = E_i = \frac{(n_i - 1)S_i^2}{\sigma_i^2},$$

where $E_i \sim \chi^2_{n_i - 1}$, defines an invertible pivotal quantity for σ_i^2, with the inverse mapping given by $g_{1,i}(S_i^2, E_i) = \frac{(n_i-1)S_i^2}{E_i}$. Hence by the structural method of Theorem 2.2 a FGPQ for σ_i^2 is given by,

$$\mathcal{R}_{\sigma_i^2} = \mathcal{R}_{\sigma_i^2}(S_i^2, S_i^{\star^2}, \sigma_i^2) = \frac{S_i^2 \sigma_i^2}{S_i^{\star^2}}$$

The following two representations of $\mathcal{R}_{\sigma_i^2}$ will be used several places in this chapter:

$$\dot{\mathcal{R}}_{\sigma_i^2}(S_i^2, E_i^\star) = \frac{(n_i - 1)S_i^2}{E_i^\star} \tag{7.2}$$

$$\ddot{\mathcal{R}}_{\sigma_i^2}(E_i, E_i^\star, \sigma_i^2) = \frac{\sigma_i^2 E_i}{E_i^\star} \tag{7.3}$$

Since the $\{E_i\}_{i=1}^k$ are independent, $\boldsymbol{f}_1(\mathbb{S}_1, \xi_1) = \mathbb{E}_1 = (E_1, \ldots, E_k)$ is an invertible pivotal quantity for ξ_1 and by Theorem 2.2 an FGPQ for ξ_1 is given by

$$\mathcal{R}_{\xi_1}(\mathbb{S}_1, \mathbb{S}_1^\star, \xi_1) = \boldsymbol{g}_1(\mathbb{S}_1, \boldsymbol{f}_1(\mathbb{S}_1^\star, \xi_1)) = (\mathcal{R}_{\sigma_1^2}, \ldots, \mathcal{R}_{\sigma_k^2})$$

Next we will construct a pivotal quantity $f_2(\mathbb{S}_2, \xi_1, \xi_2)$, which for fixed ξ_1 is an invertible pivotal quantity for μ. Note that

$$\frac{\dfrac{n_1\bar{X}_1}{\sigma_1^2} + \cdots + \dfrac{n_k\bar{X}_k}{\sigma_k^2}}{\dfrac{n_1}{\sigma_1^2} + \cdots + \dfrac{n_k}{\sigma_k^2}} \sim N\left(\mu, \left(\frac{n_1}{\sigma_1^2} + \cdots + \frac{n_k}{\sigma_k^2}\right)^{-1}\right).$$

So, for a fixed ξ_1, an invertible pivotal quantity for μ is given by

$$f_2(\mathbb{S}_2, \xi_1, \mu) = E_{k+1} = \left(\frac{\dfrac{n_1\bar{X}_1}{\sigma_1^2} + \cdots + \dfrac{n_k\bar{X}_k}{\sigma_k^2}}{\dfrac{n_1}{\sigma_1^2} + \cdots + \dfrac{n_k}{\sigma_k^2}} - \mu\right)\left(\frac{n_1}{\sigma_1^2} + \cdots + \frac{n_k}{\sigma_k^2}\right)^{1/2} \tag{7.4}$$

where $E_{k+1} \sim N(0,1)$, with the inverse function for a fixed ξ_1 is given by

$$g_2(\mathbb{S}_2, \xi_1, E_{k+1}) = \frac{\dfrac{n_1\bar{X}_1}{\sigma_1^2} + \cdots + \dfrac{n_k\bar{X}_k}{\sigma_k^2}}{\dfrac{n_1}{\sigma_1^2} + \cdots + \dfrac{n_k}{\sigma_k^2}} - E_{k+1}\left(\frac{n_1}{\sigma_1^2} + \cdots + \frac{n_k}{\sigma_k^2}\right)^{-1/2}. \tag{7.5}$$

Since $\mathbb{E}_1$ and E_{k+1} are independent, item (c) of Proposition 6.1 is satisfied. We have shown all the conditions of Proposition 6.1 are satisfied.

By the two-stage construction of Proposition 6.1 a FGPQ for μ is given by $\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = g_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, f_2(\mathbb{S}_2^\star, \xi_1, \mu))$, where $\mathcal{R}_{\xi_1} = \boldsymbol{g}_1(\mathbb{S}_1, \boldsymbol{f}_1(\mathbb{S}_1^\star, \xi_1))$, and $\mathbb{S}_1^\star$ and $\mathbb{S}_2^\star$ are independent copies of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively (see Remark 6.1). That is, in (7.5), substitute the right hand side of (7.4) for E_{k+1}, and also substitute $\mathcal{R}_{\sigma_i}$ for σ_i. This yields:

$$\begin{aligned}\mathcal{R}_\mu &= \mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) \\ &= \frac{\dfrac{n_1\bar{X}_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k\bar{X}_k}{\mathcal{R}_{\sigma_k^2}}}{\dfrac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k}{\mathcal{R}_{\sigma_k^2}}} - \left(\frac{\dfrac{n_1\bar{X}_1^\star}{\sigma_1^2} + \cdots + \dfrac{n_k\bar{X}_k^\star}{\sigma_k^2}}{\dfrac{n_1}{\sigma_1^2} + \cdots + \dfrac{n_k}{\sigma_k^2}} - \mu\right)\frac{\sqrt{\dfrac{n_1}{\sigma_1^2} + \cdots + \dfrac{n_k}{\sigma_k^2}}}{\sqrt{\dfrac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k}{\mathcal{R}_{\sigma_k^2}}}}\end{aligned}$$

A $100(1-\alpha)\%$ upper generalized confidence bound for μ is given by $\mathcal{R}_{\mu,1-\alpha}$, where $\mathcal{R}_{\mu,\gamma}(\mathbf{s}) = \mathcal{R}_{\mu,\gamma}$ denotes the 100γ-percentile of the conditional distribution of $\mathcal{R}_\mu = \mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Given a realization $\mathbf{s}$ of $\mathbb{S}$, the distribution of $\mathcal{R}_\mu(\mathbf{s}, \mathbb{S}^\star, \xi)$ is usually intractable, but one can estimate $\mathcal{R}_{\mu,1-\alpha}$ using a Monte Carlo simulation.

The Monte Carlo estimate is constructed using the representation $\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) = g_2(\mathbb{S}_2, \boldsymbol{g}_1(\mathbb{S}_1, \mathbb{E}_1^\star), E_{k+1}^\star)$. That is

$$\dot{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{E}^\star) = \frac{\dfrac{n_1\bar{X}_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k\bar{X}_k}{\mathcal{R}_{\sigma_k^2}}}{\dfrac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k}{\mathcal{R}_{\sigma_k^2}}} - \frac{E_{k+1}^\star}{\sqrt{\dfrac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k}{\mathcal{R}_{\sigma_k^2}}}}, \tag{7.6}$$

where $\mathcal{R}_{\sigma_i^2}$ is expressed as the representation $\dot{\mathcal{R}}_{\sigma_i^2}(S_i^2, E_i^\star) = \frac{(n_i-1)S_i^2}{E_i^\star}$, see (7.2). We remind the reader that $E_i^\star \sim \chi^2_{n_i-1}$, for $i = 1, \ldots, k$, $E_{k+1}^\star \sim N(0,1)$ and $E_1^\star, \ldots, E_{k+1}^\star$ are jointly independent. Let $\mathbb{E}^\star$ denote the vector $(E_1^\star, \ldots, E_{k+1}^\star)$. Given a realization $\mathbf{s}$ of $\mathbb{S}$, one generates N independent realizations, $(\mathbf{e}_1^\star, \ldots, \mathbf{e}_N^\star)$, of $\mathbb{E}^\star$ and then calculates $\dot{\mathcal{R}}_\mu(\mathbf{s}, \mathbf{e}_1^\star), \ldots, \dot{\mathcal{R}}_\mu(\mathbf{s}, \mathbf{e}_N^\star)$, which gives an empirical distribution of $\mathcal{R}_\mu(\mathbf{s}, \mathbb{S}^\star, \xi)$. These values are ordered from smallest to largest and the $N(1-\alpha)^{\text{th}}$ element of the ordered sequence is used as an estimate of $\mathcal{R}_{\mu,1-\alpha}$.

The next lemma is used in the proof of two invariance properties of $\mathcal{R}_\mu$. The invariance properties are useful in reducing the parameter combinations needed for the simulation study.

Lemma 7.1. *Let $\{\check{X}_{ij}\}$ be the (μ, σ_1)-transformed populations, see Remark 7.1. The parameters for the (μ, σ_1)-transformed populations are $\check{\xi} = (1, \tau_2^2, \ldots, \tau_k^2, 0)$, where $\tau_i = \sigma_i/\sigma_1$ for $i = 1, \ldots, k$ (note: $\tau_1 = 1$). The statistic corresponding $\mathbb{S}$ is given by $\check{\mathbb{S}} = (\check{S}_1^2, \ldots, \check{S}_k^2, \bar{\check{X}}_1, \ldots, \bar{\check{X}}_k)$. The following relationship exists*

$$\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) = \mu + \sigma_1 \mathcal{R}_\mu(\check{\mathbb{S}}, \check{\mathbb{S}}^\star, \check{\xi}),$$

where $\check{\mathbb{S}}^\star$ is the independent copy of $\check{\mathbb{S}}$ corresponding to $\mathbb{S}^\star$.

Proof. We first derive the canonical representation $\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi)$, where $\mathbb{E}^\star = (\mathbb{E}_1^\star, E_{k+1}^\star)$ is an independent copy of $\mathbb{E} = (\mathbb{E}_1, E_{k+1})$. Since $\mathcal{R}_{\xi_1}$ is independent of $\mathbb{S}_2$, conditional on R_{ξ_1},

$f_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, \mu)$ is distributed as a $N(0,1)$, see (7.4). Hence unconditionally, $f_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, \mu) = E_{k+1} \sim N(0,1)$. The equation $f_2(\mathbb{S}_2, \mathcal{R}_{\xi_1}, \mu) = E_{k+1}$ can be rewritten as

$$\frac{\dfrac{n_1 \bar{X}_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k \bar{X}_k}{\mathcal{R}_{\sigma_k^2}}}{\dfrac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \dfrac{n_k}{\mathcal{R}_{\sigma_k^2}}} = \mu + E_{k+1} \left(\frac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \frac{n_k}{\mathcal{R}_{\sigma_k^2}} \right)^{-1/2}.$$

Using this equality in (7.6) gives the canonical representation of $\mathcal{R}_\mu$,

$$\ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \xi) = \mu + (E_{k+1} - E_{k+1}^\star) \left(\frac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \frac{n_k}{\mathcal{R}_{\sigma_k^2}} \right)^{-1/2}, \tag{7.7}$$

where $\mathcal{R}_{\sigma_i^2}$ is expressed as the canonical representation $\ddot{\mathcal{R}}_{\sigma_i^2}(E_i, E_i^\star, \sigma_i^2) = \frac{\sigma_i^2 E_i}{E_i^\star}$ for $i = 1, \ldots, k$, see (7.3).

Note that $\mathbb{E} = \check{\mathbb{E}}$, so for $\check{\xi}$ Equation (7.7) becomes

$$\ddot{R}_\mu(\mathbb{E}, \mathbb{E}^\star, \check{\xi}) = (E_{k+1} - E_{k+1}^\star) \left(\frac{n_1}{\mathcal{R}_{\tau_1^2}} + \cdots + \frac{n_k}{\mathcal{R}_{\tau_k^2}} \right)^{-1/2} \tag{7.8}$$

Next, for $i = 1, \ldots, k$

$$\begin{aligned} \mathcal{R}_{\sigma_i^2} = \ddot{\mathcal{R}}_{\sigma_i^2}(E_i, E_i, \sigma_i^2) &= \frac{\sigma_i^2 E_i}{E_i^\star} \\ &= \sigma_1^2 \frac{\tau_i^2 E_i}{E_i^\star} = \sigma_1^2 \ddot{\mathcal{R}}_{\tau_i^2}(E_i, E_i^\star, \tau_i^2) = \sigma_1^2 \mathcal{R}_{\tau_i^2}, \end{aligned} \tag{7.9}$$

with $\tau_1 = 1$. From (7.8) and (7.9) we have the following

$$\begin{aligned} (E_{k+1} - E_{k+1}^\star) \left(\frac{n_1}{\mathcal{R}_{\sigma_1^2}} + \cdots + \frac{n_k}{\mathcal{R}_{\sigma_k^2}} \right)^{-1/2} &= \sigma_1 (E_{k+1} - E_{k+1}^\star) \left(\frac{n_1}{\mathcal{R}_{\tau_1^2}} + \cdots + \frac{n_k}{\mathcal{R}_{\tau_k^2}} \right)^{-1/2} \\ &= \sigma_1 \ddot{\mathcal{R}}_\mu(\mathbb{E}, \mathbb{E}^\star, \check{\xi}) \\ &= \sigma_1 \mathcal{R}_\mu(\check{\mathbb{S}}, \check{\mathbb{S}}^\star, \check{\xi}), \end{aligned}$$

Substituting this equality into (7.7) gives the result. □

Let $F_{\mathbf{s}}(t)$ denote the conditional cdf of $\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi)$ given $\mathbb{S} = \mathbf{s}$. Then the coverage of the proposed generalized upper confidence bound is given by $P_{\mathbb{S}}\left[\mu \le F_{\mathbb{S}}^{-1}(1-\alpha)\right]$. This is equal to

$$P_{\mathbb{S}}\left[P_{\mathbb{S}^\star}\left[\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) \le \mu | \mathbb{S}\right] < 1 - \alpha\right]. \tag{7.10}$$

A Monte Carlo simulation will be used to estimate the above coverage probability. The following proposition will be useful in reducing the number of parameter combinations needed in the simulation study.

Proposition 7.1. *Expression (7.10) is independent of μ and depends only the $k-1$ ratios $\tau_2 = \sigma_2/\sigma_1, \ldots, \tau_k = \sigma_k/\sigma_1$.*

Proof. We use the notation developed in the statement of Lemma 7.1. By Lemma 7.1 we have

$$\mathcal{R}_\mu(\mathbb{S}, \mathbb{S}^\star, \xi) \leq \mu \iff \mathcal{R}_\mu(\check{\mathbb{S}}, \check{\mathbb{S}}^\star, \check{\xi}) \leq 0$$

Which shows that (7.10) is independent of μ and depends only on the $k-1$ ratios $\tau_2 = \sigma_2/\sigma_1, \ldots, \tau_k = \sigma_k/\sigma_1$. □

In the simulation study the expected value of the absolute distance of the generalized upper bound $\mathcal{R}_{\mu,1-\alpha}$ from the true value μ will be of interest. The following proposition will be useful in reducing the number of parameter combinations needed in the simulation study.

Proposition 7.2. *We use the notation of Lemma 7.1. Let $\check{\mathcal{R}}_{\mu,\gamma}(\check{\mathbf{s}})$ be the 100γ-percentile of the conditional distribution of $\mathcal{R}_\mu(\check{\mathbb{S}}, \check{\mathbb{S}}^\star, \check{\xi})$ given $\check{\mathbb{S}} = \check{\mathbf{s}}$. Then for any $\gamma \in [0,1]$*

$$E\left[|\,\mu - \mathcal{R}_{\mu,\gamma}(\mathbb{S})|\,\right] = \sigma_1 E\left[|\check{\mathcal{R}}_{\mu,\gamma}(\check{\mathbb{S}})|\,\right].$$

Proof. The result follows from Lemma 7.1. □

Hannig et al. [25] verified the conditions of Theorem 2.1 for $\mathcal{R}_\mu$. Hence the confidence intervals based on $\mathcal{R}_\mu$ have asymptotically exact coverage.

Proposition 7.3. *(Hannig et al. [25]) Let all $n_1, \ldots, n_k$ approach infinity in such a way that $r_j = \lim n_j/(n_1 + \cdots + n_k)$ exists and $0 < r_j < 1$. Then a $100(1-\alpha)\%$ generalized confidence interval for μ based on the FGPQ $\mathcal{R}_\mu$ has asymptotically $100(1-\alpha)\%$ frequentist coverage.*

We will be use the simulation study to check the rate of convergence of the coverage probabilities as the sample sizes become large.

7.3 Confidence Intervals for μ based on a General Structural Method FGPQ

The general structural method for constructing for FGPQs was given in Theorem 6.1. To facilitate the explanation of construction the FGPQ we define ξ and $\mathbb{S}$ as follows, $\xi = (\mu, \sigma_1^2, \ldots, \sigma_k^2)$ and $\mathbb{S} = (\bar{X}_1, \ldots, \bar{X}_k, S_1^2, \ldots, S_k^2)$.

There are following the $2k$ pivotal relationships

$$\left.\begin{aligned} f_i(\mathbb{S}, \xi) &= E_i = \frac{\bar{X}_i - \mu}{\sqrt{\sigma_i^2/n_i}} \\ f_{i+k}(\mathbb{S}, \xi) &= E_{k+i} = \frac{(n_i - 1)S_i^2}{\sigma_i^2} \end{aligned}\right\} \quad i = 1, \ldots, k\ , \tag{7.11}$$

where for $i = 1, \ldots, k$, $E_i \sim N(0,1)$ and $E_{k+i} \sim \chi^2_{n_i-1}$. The $E_1, \ldots, E_{2k}$ are jointly independent. Hence the distribution of $\mathbb{E} = (E_1, \ldots, E_{2k})$ does not depend on ξ and item (a) of Assumptions (B) from Theorem 6.1 is satisfied.

Let $\mathbf{s} = (\bar{x}_1, \ldots, \bar{x}_k, s_1^2, \ldots, s_k^2)$ denote a realization of $\mathbb{S}$ and let $\mathbf{e}$ be the corresponding realization of $\mathbb{E}$. Define the following $k+1$ dimensional vectors: $\mathbb{E}_0 = (E_k, \ldots, E_{2k})$, $\mathbf{e}_0 = (e_k, \ldots, e_{2k})$, and $\boldsymbol{f}_0 = (f_k, \ldots, f_{2k})$. For a fixed $\mathbf{s}$, the function defined by $\mathbf{e}_0 = \boldsymbol{f}_0(\mathbf{s}, \xi)$ is invertible. The inverse $\boldsymbol{g}_0$ is defined by the $k+1$ equations

$$g_k(\mathbf{s}, \mathbf{e}_0) = \bar{x}_k - e_k\sqrt{\frac{(n_k - 1)s_k^2}{n_k e_{2k}}} \tag{7.12}$$

$$g_{k+i}(\mathbf{s}, \mathbf{e}_0) = \frac{(n_i - 1)s_i^2}{e_{k+i}} \qquad i = 1, \ldots, k, \tag{7.13}$$

with $g_k(\mathbf{s}, (f_k(\mathbf{s}, \xi), f_{2k}(\mathbf{s}, \xi))) = \mu$ and $g_{k+i}(\mathbf{s}, f_{k+i}(\mathbf{s}, \xi)) = \sigma_i^2$ for $i = 1, \ldots, k$. So item (b) of Assumptions (B) of Theorem 6.1 is satisfied by $\mathbb{E}_0$, $\boldsymbol{f}_0$ and $\boldsymbol{g}_0$.

We next determine the equations that will define the manifold $\mathcal{M}(\mathbf{s})$. By definition, $\mathcal{M}(\mathbf{s})$ is the set of $\mathbf{e}$ that satisfy $e_i = f_i(\mathbf{s}, \boldsymbol{g}_0(\mathbf{s}, \mathbf{e}_0))$ for $i = 1, \ldots, k-1$. Using (7.11) to evaluate the functions f_i we obtain the set of equations $e_i = (\bar{x}_i - g_k(\mathbf{s}, \mathbf{e}_0))/\sqrt{g_{k+i}(\mathbf{s}, \mathbf{e}_0)/n_i}$, for $i = 1, \ldots, k-1$. We choose to express this set of equation as follows:

$$0 = e_i\sqrt{g_{k+i}(\mathbf{s}, \mathbf{e}_0)/n_i} - (\bar{x}_i - g_k(\mathbf{s}, \mathbf{e}_0)) \qquad i = 1, \ldots, k-1$$

Using (7.12) and (7.13) to evaluate the functions g_k and g_{k+i} in terms of $\mathbf{s}$ and $\mathbf{e}_0$ produces the set of equations

$$\begin{aligned} 0 &= \frac{e_i}{\sqrt{e_{k+i}}}\sqrt{\frac{(n_i-1)s_i^2}{n_i}} - \frac{e_k}{\sqrt{e_{2k}}}\sqrt{\frac{(n_k-1)s_k^2}{n_k}} - (\bar{x}_i - \bar{x}_k), \\ &= \frac{e_i}{\sqrt{e_{k+i}/(n_i-1)}}\sqrt{\frac{s_i^2}{n_i}} - \frac{e_k}{\sqrt{e_{2k}/(n_k-1)}}\sqrt{\frac{s_k^2}{n_k}} - (\bar{x}_i - \bar{x}_k) \quad \text{for } i = 1, \ldots, k-1. \end{aligned}$$

For $i = 1, \ldots, k$, we have $T_i = \frac{E_i}{\sqrt{E_{k+i}/(n_i-1)}} \sim t(n_i - 1)$, where $t(n)$ is a Student t distribution with n degrees of freedom. So if t_i, $i = 1, \ldots, k$ are realizations of k independent random variables which have Student's t distributions with $n_i - 1$ degrees of freedom, then the $k-1$ equations that define the manifold $\mathcal{M}(\mathbf{s})$ can be expressed as

$$0 = t_i\sqrt{\frac{s_i^2}{n_i}} - t_k\sqrt{\frac{s_k^2}{n_k}} - (\bar{x}_i - \bar{x}_k), \qquad i = 1, \ldots, k-1.$$

For $i = 1, \ldots, k-1$, define $B_i = B_i(\mathbf{s}, \mathbb{E}) = T_i\sqrt{\frac{s_i^2}{n_i}} - T_k\sqrt{\frac{s_k^2}{n_k}} - (\bar{x}_i - \bar{x}_k)$. Let $\mathbb{B} = \mathbb{B}(\mathbf{s}, \mathbb{E}) = (B_1, \ldots, B_{k-1})$.

We next check Condition (C) of Theorem 6.1, i.e., conditional on $\mathbb{B}$, $\mathbb{E}_0$ has a continuous distribution. It suffices to show the joint distribution of $(\mathbb{B}, \mathbb{E}_0)$ is continuous. The joint distribution of $(\mathbb{B}, \mathbb{E}_0)$ can be calculated using the Jacobian technique with the mapping $\boldsymbol{c}(\mathbb{E}) = (\mathbb{B}, \mathbb{E}_0)$, as we now outline. The mapping $\boldsymbol{c}$ is invertible with inverse $\boldsymbol{d}$, where

$$\begin{aligned} d_i(\mathbb{B}, \mathbb{E}_0) &= \sqrt{\frac{n_i E_{k+i}}{(n_i-1)s_j^2}}\left(B_i + \frac{E_k}{\sqrt{E_{2k}}}\sqrt{\frac{(n_k-1)s_k^2}{n_k}} + (\bar{x}_i - \bar{x}_k)\right) && i = 1, \ldots, k-1 \\ d_i(\mathbb{B}, \mathbb{E}_0) &= E_i && i = k, \ldots, 2k. \end{aligned}$$

The Jacobian of $\boldsymbol{d}$ is equal to $\prod_{i=1}^{k-1}\sqrt{(n_i E_{k+i})/((n_i-1)s_i^2)}$. The Jacobian of $\boldsymbol{d}$ is continuous and hence by the Jacobian transform technique it follows that the joint distribution of $(\mathbb{B}, \mathbb{E}_0)$ is continuous.

All the conditions of Theorem 6.1 have been shown to hold and hence we could construct the FGPQ $\tilde{\mathcal{R}}_\theta$ using the formula in the statement of the theorem. However, it much easier to calculate the pdf of $\tilde{\mathcal{R}}_\theta$ using Corollary 6.1. By Corollary 6.1 the pdf of $\tilde{\mathcal{R}}_\theta$ is same as the conditional pdf of $\mu = g_k(\mathbf{s}, \mathbb{E}^\star) = \bar{x}_k - T_k\sqrt{s_k^2/n_k}$ conditional on $\mathbb{B}(\mathbf{s}, E^\star) = \mathbf{0}$.

We should substitute $T_i^\star$ for T_i in the equations that define $\mathbb{B}$, for the sake simplifying the notation we don't. The reader should note that in the following the small letters indicate the observed data, while capitol letters indicate the unobserved independent "fiducial" copy of the data. By Remark 6.3 it suffices to calculate the joint pdf of $\mathbb{B}$ and μ at the point $(\mathbf{b}, \mu) = (\mathbf{0}, t)$. The joint pdf of $\mathbb{B}$ and μ will be calculated using the Jacobian technique applied to the map $(T_1, \dots, T_k) \mapsto (B_1, \dots, B_{k-1}, \mu)$. The inverse of this map is given by the following k equations

$$
\begin{aligned}
T_i &= \sqrt{\frac{n_i}{s_i^2}}(B_i + \bar{x}_i - \mu) \qquad\qquad i = 1, \dots, k-1 \\
T_k &= \sqrt{\frac{n_k}{s_k^2}}(\bar{x}_k - \mu)
\end{aligned}
\tag{7.14}
$$

Since the above are all linear as functions of $(\mathbb{B}, \mu)$, the Jacobian is a constant. The joint pdf of $(T_1, \dots, T_k)$ is given by

$$
f_{T_1,\dots,T_k}(t_1, \dots, t_k) = \frac{K}{(1 + \frac{t_1^2}{n_1 - 1})^{n_1/2} \cdots (1 + \frac{t_k^2}{n_k - 1})^{n_k/2}},
$$

where K is a constant. The joint pdf of $\mathbb{B}$ and μ is obtained by substituting the functional relationship given in (7.14) in the above pdf and multiplying the pdf by the Jacobian. Since we are only interested in finding the joint pdf of $\mathbb{B}$ and μ at the point $(\mathbb{B}, \mu) = (\mathbf{0}, t)$ we have

$$
f_{\mathbb{B},\mu}(\mathbf{0}, t) = \frac{K|J|}{\left(1 + \frac{(\bar{x}_1 - t)^2}{(n_1 - 1)s_1^2/n_1}\right)^{n_1/2} \cdots \left(1 + \frac{(\bar{x}_k - t)^2}{(n_k - 1)s_k^2/n_k}\right)^{n_k/2}},
$$

where $|J|$ is the absolute value of Jacobian. Let $\hat{\sigma}_i^2 = (n_i - 1)s_i^2/n_i$ denote the ML estimate of σ_i^2, then by Remark 6.3 the conditional pdf of μ given $\mathbb{B}(\mathbf{s}, \mathbb{E}) = \mathbf{0}$, is given by

$$
f_{\mathbf{S}}(t) = \frac{C}{\left(1 + \frac{(\bar{x}_1 - t)^2}{\hat{\sigma}_1^2}\right)^{n_1/2} \cdots \left(1 + \frac{(\bar{x}_k - t)^2}{\hat{\sigma}_k^2}\right)^{n_k/2}}. \tag{7.15}
$$

where

$$
C^{-1} = \int_{-\infty}^{\infty} \frac{dt}{\left(1 + \frac{(\bar{x}_1 - t)^2}{\hat{\sigma}_1^2}\right)^{n_1/2} \cdots \left(1 + \frac{(\bar{x}_k - t)^2}{\hat{\sigma}_k^2}\right)^{n_k/2}}. \tag{7.16}
$$

Summarizing, we have shown that Equation (7.15) is the conditional pdf of the FGPQ $\tilde{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{S}^*, \xi)$ given $\mathbb{S} = \mathrm{s}$.

An $100(1-\alpha)\%$ upper confidence bound for μ is given by $\tilde{\mathcal{R}}_{\mu,1-\alpha}$ where $\tilde{\mathcal{R}}_{\mu,\gamma}(\mathrm{s}) = \tilde{\mathcal{R}}_{\mu,\gamma}$ is the 100γ-percentile of the conditional distribution of $\tilde{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{S}^*, \xi)$ given $\mathbb{S} = \mathrm{s}$. Note that $\tilde{\mathcal{R}}_{\mu,\gamma}$ is the solution to the integral equation

$$1 - \alpha = \int_{-\infty}^{\tilde{\mathcal{R}}_{\mu,\gamma}} f_{\mathbf{S}}(t)\, dt \tag{7.17}$$

In practice, $\tilde{\mathcal{R}}_{\mu,\gamma}$ may be determined using numerical techniques.

The next lemma is used in the proof of two invariance statements, both of which are useful reducing the number of parameter combinations needed for the simulation study.

Lemma 7.2. *Let $\{\check{X}_{ij}\}$ be the (μ, σ_1)-transformed populations, see Remark 7.1. The parameters for the (μ, σ_1)-transformed populations are $\check{\xi} = (0, 1, \tau_2^2, \ldots, \tau_k^2)$, where $\tau_i = \sigma_i/\sigma_1$ for $i = 1, \ldots, k$ (note: $\tau_1 = 1$). The statistic corresponding $\mathbb{S}$ is given by $\check{\mathbb{S}} = (\check{\bar{X}}_1, \ldots, \check{\bar{X}}_k, \check{S}_1^2, \ldots, \check{S}_k^2)$. Let $F_{\mathbf{S}}(x)$ denote the conditional cdf of $\tilde{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{S}^*, \xi)$, given $\mathbb{S} = \mathrm{s}$ and let $F_{\check{\mathbf{S}}}(x)$ denote the conditional cdf of $\tilde{\mathcal{R}}_\mu(\check{\mathbb{S}}, \check{\mathbb{S}}^*, \check{\xi})$, given $\check{\mathbb{S}} = \check{\mathrm{s}}$, then*

$$F_{\mathbf{S}}(x) = F_{\check{\mathbf{S}}}\left(\frac{x - \mu}{\sigma_1}\right).$$

Proof. For realization s and $\check{\mathrm{s}}$ of $\mathbb{S}$ and $\check{\mathbb{S}}$ respectively, the following relations exits:

$$\left.\begin{aligned} \frac{\bar{x}_i - \mu}{\sigma_1} &= \check{\bar{x}}_i \\ \frac{\hat{\sigma}_i^2}{\sigma_1^2} &= \check{\hat{\sigma}}_i^2 \end{aligned}\right. \qquad i = 1, \ldots, k\ ,$$

For $i = 1, \cdots, k$

$$\begin{aligned} \left(1 + \frac{(\bar{x}_i - t)^2}{\hat{\sigma}_i^2}\right) &= \left(1 + \frac{((\bar{x}_i - \mu)/\sigma_1 - (t - \mu)/\sigma_1)^2}{\hat{\sigma}_i^2/\sigma_1^2}\right) \\ &= \left(1 + \frac{(\check{\bar{x}}_i - (t - \mu)/\sigma_1)^2}{\check{\hat{\sigma}}_i^2}\right). \end{aligned}$$

Using this equality followed by the change of variable $s = (t - \mu)/\sigma_1$, the integral

$$\int_{-\infty}^{x} \frac{dt}{\left(1 + \frac{(\bar{x}_1 - t)^2}{\hat{\sigma}_1^2}\right)^{n_1/2} \cdots \left(1 + \frac{(\bar{x}_k - t)^2}{\hat{\sigma}_k^2}\right)^{n_k/2}}$$

can be expressed in the equivalent form

$$\int_{-\infty}^{\frac{x-\mu}{\sigma_1}} \frac{\sigma_1\, ds}{\left(1 + \frac{(\check{\bar{x}}_1 - s)^2}{\check{\hat{\sigma}}_1^2}\right)^{n_1/2} \cdots \left(1 + \frac{\check{\bar{x}}_k - s)^2}{\check{\hat{\sigma}}_k^2}\right)^{n_k/2}}.$$

Similarly we can express C as $\sigma_1^{-1}C'$, where C'^{-1} is given by

$$\int_{-\infty}^{\infty} \frac{ds}{\left(1+\frac{(\check{\bar{x}}_1-s)^2}{\check{\hat{\sigma}}_1^2}\right)^{n_1/2} \cdots \left(1+\frac{\check{\bar{x}}_k-s)^2}{\check{\hat{\sigma}}_k^2}\right)^{n_k/2}}.$$

Hence

$$\begin{aligned} F_{\mathbb{S}}(x) = \int_{-\infty}^{x} f_{\mathbb{S}}(t)\,dt &= \int_{-\infty}^{\mu} \frac{C\,dt}{\left(1+\frac{(\bar{x}_1-t)^2}{\hat{\sigma}_1^2}\right)^{n_1/2} \cdots \left(1+\frac{(\bar{x}_k-t)^2}{\hat{\sigma}_k^2}\right)^{n_k/2}} \\ &= \int_{-\infty}^{\frac{x-\mu}{\sigma_1}} \frac{\sigma_1^{-1}C'\,\sigma_1 ds}{\left(1+\frac{(\check{\bar{x}}_1-s)^2}{\check{\hat{\sigma}}_1^2}\right)^{n_1/2} \cdots \left(1+\frac{\check{\bar{x}}_k-s)^2}{\check{\hat{\sigma}}_k^2}\right)^{n_k/2}} \\ &= \int_{-\infty}^{\frac{x-\mu}{\sigma_1}} f_{\check{\mathbb{S}}}(s)\,ds = F_{\check{\mathbb{S}}}\left(\frac{x-\mu}{\sigma_1}\right). \end{aligned}$$

□

If $F_{\mathbb{S}}(x) = \int_{-\infty}^{x} f_{\mathbb{S}}(t)\,dt$, denotes the conditional cdf of $\tilde{\mathcal{R}}_\mu(\mathbb{S}, \mathbb{S}^*, \xi)$, given $\mathbb{S} = \mathrm{s}$, then $\tilde{\mathcal{R}}_{\mu,1-\alpha}$ is equal to $F_{\mathbb{S}}^{-1}(1-\alpha)$. The coverage probability is equal to

$$P_{\mathbb{S}}\left[\mu \leq F_{\mathbb{S}}^{-1}(1-\alpha)\right]. \tag{7.18}$$

A Monte Carlo simulation will be used to estimate the above coverage probability. The following proposition will be useful in reducing the number of the parameter combinations needed in the simulation study.

Proposition 7.4. *Expression (7.18) is independent of μ and depends only on the $k-1$ ratios $\tau_1 = \sigma_2/\sigma_1, \ldots, \tau_{k-1} = \sigma_k/\sigma_1$.*

Proof. By Lemma 7.2, $P_{\mathbb{S}}\left[F_{\mathbb{S}}(\mu) \leq 1-\alpha\right] = P_{\check{\mathbb{S}}}\left[F_{\check{\mathbb{S}}}(0) \leq 1-\alpha\right]$. Hence,

$$\begin{aligned} P_{\mathbb{S}}\left[\mu \leq F_{\mathbb{S}}^{-1}(1-\alpha)\right] &= P_{\mathbb{S}}\left[F_{\mathbb{S}}(\mu) \leq 1-\alpha\right] \\ &= P_{\check{\mathbb{S}}}\left[F_{\check{\mathbb{S}}}(\mu) \leq 1-\alpha\right] \\ &= P_{\check{\mathbb{S}}}\left[\mu \leq F_{\check{\mathbb{S}}}^{-1}(0)\right] \end{aligned}$$

The last expression independent of μ and depends only on the $k-1$ ratios $\tau_1 = \sigma_2/\sigma_1, \ldots,$ $\tau_{k-1} = \sigma_k/\sigma_1$. □

In the simulation study the expected value of the absolute distance of $\tilde{\mathcal{R}}_{\mu,1-\alpha}$ from the true value, μ, will be useful. The following proposition in conjunction with Proposition 7.4 will be useful in reducing the number parameter combinations needed in the simulation study.

Proposition 7.5. *We will use the notation of Lemma 7.2. Let $\check{\tilde{\mathcal{R}}}_{\mu,\gamma}(\check{\mathrm{s}})$ be the 100γ-percentile of the conditional distribution of $\tilde{\mathcal{R}}_{\mu}(\check{\mathbb{S}},\check{\mathbb{S}}^{\star},\check{\xi})$ given $\check{\mathbb{S}}=\check{\mathrm{s}}$. Then for any $\gamma\in[0,1]$*

$$E\left[\left|\mu-\tilde{\mathcal{R}}_{\mu,\gamma}(\mathbb{S})\right|\right]=\sigma_1 E\left[\left|\check{\tilde{\mathcal{R}}}_{\mu,\gamma}(\check{\mathbb{S}})\right|\right].$$

Proof. By Lemma 7.2 we have

$$\begin{aligned}1-\alpha&=F_{\mathbb{S}}\left(\tilde{\mathcal{R}}_{\mu,\gamma}\right)\\&=F_{\check{\mathbb{S}}}\left(\frac{\tilde{\mathcal{R}}_{\mu,\gamma}-\mu}{\sigma_1}\right).\end{aligned}$$

Which implies

$$\check{\tilde{\mathcal{R}}}_{\mu,\gamma}=\frac{\tilde{\mathcal{R}}_{\mu,\gamma}-\mu}{\sigma_1}\qquad\text{or}\qquad\tilde{\mathcal{R}}_{\mu,\gamma}=\mu+\sigma_1\check{\tilde{\mathcal{R}}}_{\mu,\gamma},$$

from which the result follows. □

Hannig et al. [25] have studied the large sample behavior of the GCIs based on $\tilde{\mathcal{R}}_{\mu}$. Specifically, they proveed the following proposition.

Proposition 7.6. *(Hannig et al. [25]) Let all $n_1,\ldots,n_k$ approach infinity in such a way that $r_j=\lim n_j/(n_1+\cdots+n_k)$ exists and $0<r_j<1$. Then a $100(1-\alpha)\%$ generalized confidence interval for μ, based on the FGPQ $\tilde{\mathcal{R}}_{\mu}$ whose pdf is given in Equation (7.15), has asymptotically $100(1-\alpha)\%$ frequentist coverage.*

We will use a simulation study to check the rate of convergence of the coverage probabilities as the sample sizes become large.

7.4 Fisher's Method

Let $\mathbb{S}=(\bar{X}_1,\ldots,\bar{X}_k,S_1^2,\ldots,S_k^2)$. Fisher's method is based on the k pivotal statistics $T_i(\mathbb{S},\mu_0)=\frac{\sqrt{n_i}(\bar{X}_i-\mu)}{S_i}$, $i=1,\ldots,k$, where $T_i(\mathbb{S},\mu)$ has a student t distribution with n_i-1

degrees of freedom. The statistic $T_i(\mathbb{S}, \mu)$ is the pivotal quantity that would be used to construct confidence intervals based on data from group i only. These k pivotal quantities can be combined using Fisher's method for combining P-values. We will outline Fisher's method. Define the following k integral transforms

$$F_{T_i}(T_i) = P_i(\mathbb{S}, \mu_0) = \int_{-\infty}^{\frac{\sqrt{n_i}(\bar{X}_i-\mu)}{S_i}} f_i(t)\, dt, \quad \text{for } i = 1, \ldots, k,$$

where $f_i(t)$ is the density function for a t distribution with $n_i - 1$ degrees of freedom. The $P_i(\mathbb{S}, \mu)$ are distributed as a Uniform on $[0, 1]$ and $-2\ln(P_i(\mathbb{S}, \mu)) \sim \chi^2_2$. Note that the $T_i(\mathbb{S}, \mu)$, for $i = 1\ldots, k$, are mutually independent, so the k random variables $-2\ln(P_i(\mathbb{S}, \mu))$ are mutually independent and hence $Q(\mathbb{S}, \mu) = -2\sum \ln(P_i(\mathbb{S}, \mu))$ is distributed as χ^2_{2k}. Note that $Q(\mathbb{S}, \mu)$ is a pivotal quantity for μ. Which completes Fisher's method for combining P-values. Since $Q(\mathrm{s}, \cdot)$ is an increasing function of μ an exact $100(1-\alpha)\%$ upper confidence interval for μ is given by $\mu \le \mathcal{U}_{1-\alpha}(\mathbb{S})$, where the upper confidence bound $\mathcal{U}_\gamma = \mathcal{U}_\gamma(\mathbb{S})$ is defined by

$$\mathcal{U}_\gamma(\mathbb{S}) = \sup_\mu \left\{ -2\sum_{i=1}^{k} \ln(P_i(\mathbb{S}, \mu)) \le \chi^2_{2k,\gamma} \right\} \tag{7.19}$$

where $\chi^2_{2k,\gamma}$ is the γ-percentile of the chi-square distribution with $2k$ degrees of freedom.

For the simulation study the proposition will be useful in reducing the number parameter combinations needed in the simulation study. The proposition is similar to Propositions 7.2 and 7.5.

Proposition 7.7. *Let $\{\check{X}_{ij}\}$ be the (μ, σ_1)-transformed populations, see Remark 7.1. The statistic corresponding to $\mathbb{S}$ is given by $\check{\mathbb{S}} = (\check{\bar{X}}_1, \ldots, \check{\bar{X}}_k, \check{S}_1^2, \ldots, \check{S}_k^2)$, then*

$$E[|\mu - \mathcal{U}_\gamma(\mathbb{S})|] = \sigma_1 E[|\mathcal{U}_\gamma(\check{\mathbb{S}})|].$$

Proof. Using the relationships

$$\left.\begin{array}{rcl} \dfrac{\bar{X}_i - \mu}{\sigma_1} & = & \check{\bar{X}}_i \\ \dfrac{S_i^2}{\sigma_1^2} & = & \check{S}_i^2 \end{array}\right\} \quad i = 1, \ldots, k\ ,$$

we have

$$\frac{\sqrt{n_i}(\check{\bar{X}}_i - \eta)}{\check{S}_i} = \frac{\sqrt{n_i}(\bar{X}_i - (\mu + \sigma_1\eta))}{S_i}.$$

Which implies

$$\begin{aligned}\mathcal{U}_\gamma(\mathbb{S}) &= \sup_\eta \left\{ -2\sum_{i=1}^{k} \ln(P_i(\mathbb{S},\eta) \leq \chi^2_{2k,\gamma} \right\} \\ &= \mu + \sigma_1 \sup_\eta \left\{ -2\sum_{i=1}^{k} \ln(P_i(\check{\mathbb{S}},\eta) \leq \chi^2_{2k,\gamma} \right\} \\ &= \mu + \sigma_1 \mathcal{U}_\gamma(\check{\mathbb{S}}).\end{aligned}$$

From this equality the result follows. □

Using the fact that $-2\sum \ln(P_i)$ is an increasing function of μ, Equation (7.19) can be easily solved using numerical procedures to produce an $100(1-\alpha)\%$ upper confidence bound for μ.

7.5 Simulation Results

A simulation study was conducted to compare the performance of the three methods of constructing upper confidence bounds for μ. The performance was measured using two attributes, the coverage probability and the expected lengths for the upper confidence bounds. By "expected length" for an upper confidence bound we mean the expected value of the absolute distance of the upper confidence bound from the true value. Fisher's method produces exact confidence bounds with exact coverage. For the simulation study we used $\alpha = 0.5$. By Propositions 7.1, 7.2 and 7.4 we assume without loss of generality that $\mu = 0$ and $\sigma_e = 1$. For a fixed k the model is determined by $2k-1$ dimensional parameter combinations, which consist of the sample sizes $(n_1, \ldots, n_k)$ and the standard deviations $(\sigma_2, \ldots, \sigma_k)$. We used three values for k; they were $k=$ 2 labs, 3 labs and 10 labs. For each value of k a representative set of parameter combinations was used. For the rest of the discussion consider k to be fixed. For each parameter combination, 4000 independent realizations of $\mathbb{S}$ were generated. For each realization of $\mathbb{S}$ an upper $100(1-\alpha)\%$ confidence bound was computed using of each of the three methods. A FORTRAN program was written to perform the simulation. We will next discuss the technical details of the how the confidence bounds were calculated. This will be followed by a description of how the simulation study results are organized.

The simulation was conducted using Digital Fortran Professional Edition 6.0.A. The IMSL's referred to (names with be in capitol letters) come from the IMSL library attache to the program. The realizations of $\mathbb{S}$ were generated using the DRNNOR random number generator for the $\bar{X}_i$ and the DRNCHI random number generator for the S_i^2. For the remainder of the discussion let s denote one of the 4000 realizations of $\mathbb{S}$. We will now outline how each of the three confidence bounds were calculated for the realization s.

The process for calculating the upper confidence bound for the two-stage method is outlined in Subsection 7.2 right before Lemma 7.1. We used $N = 10,000$ realizations of $\mathbb{E}^\star$, where the realizations of $E_i^\star$ for $i = 1, \ldots, k$ were generated using the DRNCHI random number generator and the realization of $E_{k+1}^\star$ were generated using the DRNNOR random number generator.

The upper confidence bound for the general structural method FGPQ was calculated by solving the integral equation (7.17). A closed form solution is intractable, so numerical integration techniques were used. To simplify the explanation we introduce the following notation. Let

$$\begin{aligned}
max_{\hat{\sigma}^2}(\mathrm{s}) &= \max(\hat{\sigma}_1^2, \ldots, \hat{\sigma}_k^2), \\
min_{\bar{x}}(\mathrm{s}) &= \min(\bar{x}_1, \ldots, \bar{x}_k), \\
max_{\bar{x}}(\mathrm{s}) &= \max(\bar{x}_1, \ldots, \bar{x}_k), \text{ and} \\
\tilde{f}_{\mathrm{s}}(t) &= \frac{1}{\left(1 + \frac{(\bar{x}_1 - t)^2}{\hat{\sigma}_1^2}\right)^{n_1/2} \cdots \left(1 + \frac{(\bar{x}_k - t)^2}{\hat{\sigma}_k^2}\right)^{n_k/2}}
\end{aligned}$$

If $t < min_{\bar{x}}(\mathrm{s})$ and $2 \leq n_i$, then $\tilde{f}_{\mathbf{s}}(t) \leq \left(\frac{min_{\bar{x}}(\mathrm{s}) - t}{max_{\hat{\sigma}^2}(\mathrm{s})}\right)^{-2k}$ and similarly if $t > max_{\bar{x}}(\mathrm{s})$ and $2 \leq n_i$, then $\tilde{f}_{\mathbf{s}}(t) \leq \left(\frac{max_{\bar{x}}(\mathrm{s}) - t}{max_{\hat{\sigma}^2}(\mathrm{s})}\right)^{-2k}$. From which follows

$$\int_{-\infty}^{L(\eta)} \tilde{f}_{\mathbf{s}}(t) \leq \eta \quad \text{if} \quad L(\eta) = min_{\bar{x}}(\mathrm{s}) - max_{\hat{\sigma}}(\mathrm{s})\, \eta^{-(2k-1)} \qquad \text{and}$$

$$\int_{M(\eta)}^{\infty} \tilde{f}_{\mathbf{s}}(t) \leq \eta \quad \text{if} \quad M(\eta) = max_{\bar{x}}(\mathrm{s}) - max_{\hat{\sigma}}(\mathrm{s})\, \eta^{-(2k-1)}.$$

We approximated C^{-1} (7.16) by

$$\hat{C}^{-1} = \int_{L(\eta)}^{M(\eta)} \tilde{f}_{\mathbb{S}}(t)\, dt, \text{ with } \eta = 0.00001.$$

This integral was numerical approximated using Simpson'a rule with $h = .0005$. Equation (7.17) was then approximated by

$$1 - \alpha = \int_{L(\hat{C}^{-1}\eta)}^{\mathcal{R}_{\mu,1-\alpha}} f_{\mathbb{S}}(t)\, dt, \text{ with } \eta = .00001.$$

This integral equation was solved using Simpson's rule with $h = .0005$.

For Fisher's method, the bi-section method was used to solve Equation (7.19). The starting points being integers m_l and m_u such that $m_u = m_l + 1$ and

$$-2\sum_{i=1}^{k} \ln(P_i(\mathbf{s}, m_l)) \leq \chi^2_{2k,1-\alpha} \leq -2\sum_{i=1}^{k} \ln(P_i(\mathbf{s}, m_u)),$$

where $\mathbf{s}$ is the realization of $\mathbb{S}$. We used 20 iterations of the bi-section method.

A copy of the Fortran program is in Section C.6. The Fortran program is for the case $k = 3$. As indicated in the comments section of the program, by modifying the input and output files, and a couple values within the program, the program can produce simulation study for any k.

Tables 7.1–7.5 report the empirical coverage probabilities and empirical expected lengths for the upper $100(1-\alpha)\%$ confidence bounds obtained using each of the three methods. A number of different scenarios were considered with varying sample sizes and variance ratios. Several scenarios with larger sample sizes were included in the study to judge the rate of asymptotic convergence of coverage probability. Results for the case of $k = 2$ populations are given in Table 7.1 (smaller sample sizes) and Table 7.2 (larger sample sizes). For $k = 3$ the results are given in Table 7.3 (smaller sample sizes) and Table 7.2 (larger sample sizes). Results for the case of $k = 10$ populations are given in Table 7.5.

7.6 Discussion of Simulation Results

Fisher's method for obtaining a confidence bound for the common mean μ is an exact method but the two proposed GCI methods are only asymptotically exact. For very small sizes, $n_i = 3$ for $i = 1, \ldots, k$, the coverage appears to be liberal for the two GCI methods.

Nevertheless, at small samples of $n_i = 10$, the two GCI methods are quite satisfactory and they appear to be more efficient than Fisher's method. In fact, even when the GCI are liberal, they appear to be more efficient than Fisher's method. We also note that the GCI based on the general construction performs slightly better than the two-stage GCI in terms of confidence interval expected length. It also appears that increases in sample size for more efficient "labs" (i.e., σ_i less than one) have a greater effect on the coverage and efficiency than an increase in small size where the σ_i are all of similar size. The larger sample sizes give an indication of the rate of coverage probabilities convergence to the stated value of 0.95. For a fixed level of replications per lab, as the number of labs increase the coverage appears to become more liberal. We conclude that either of the two GCI methods may be recommended for practical applications.

The work detailed in this chapter is part of Hannig et al. [25]

7.7 Tables of Results

Table 7.1: Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 2$ populations & smaller sample sizes, $\alpha = .05$

n_1	n_2	σ_1	σ_2	Two-Stage	General FGPQ	Fisher
3	3	1.00	0.10	0.9377	0.9445	0.9507
				0.1638	0.1660	0.2152
3	3	1.00	0.25	0.9305	0.9340	0.9505
				0.3439	0.3398	0.4287
3	3	1.00	0.50	0.9280	0.9213	0.9485
				0.5452	0.5212	0.6660
3	3	1.00	1.00	0.9227	0.9137	0.9475
				0.7788	0.7346	0.9528
3	3	1.00	10.00	0.9307	0.9375	0.9457
				1.6246	1.6445	2.1390
3	10	1.00	0.10	0.9445	0.9450	0.9527
				0.0630	0.0580	0.0738
3	10	1.00	0.25	0.9370	0.9415	0.9517
				0.1542	0.1436	0.1696
3	10	1.00	1.00	0.9323	0.9337	0.9483
				0.4955	0.4840	0.5470
3	10	1.00	4.00	0.9365	0.9305	0.9553
				1.1148	1.0867	1.3120
3	10	1.00	10.00	0.9470	0.9465	0.9557
				1.4431	1.4574	1.8428
10	10	1.00	0.10	0.9500	0.9500	0.9510
				0.0580	0.0574	0.0694
10	10	1.00	0.25	0.9477	0.9460	0.9495
				0.1391	0.1384	0.1574
10	10	1.00	0.50	0.9475	0.9465	0.9557
				0.2531	0.2525	0.2748
10	10	1.00	1.00	0.9280	0.9263	0.9450
				0.3882	0.3875	0.4132
10	10	1.00	10.00	0.9467	0.9467	0.9495
				0.5828	0.5813	0.6973

Table 7.2: Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 2$ populations & larger sample sizes, $\alpha = .05$

n_1	n_2	σ_1	σ_2	Two-Stage	General FGPQ	Fisher
3	50	1.00	0.10	0.9450	0.9447	0.9473
				0.0256	0.0238	0.0322
3	50	1.00	0.25	0.9373	0.9365	0.9397
				0.0634	0.0598	0.0722
3	50	1.00	0.50	0.9400	0.9457	0.9517
				0.1264	0.1185	0.1387
3	50	1.00	1.00	0.9393	0.9443	0.9517
				0.2476	0.2343	0.2654
3	50	1.00	4.00	0.9375	0.9325	0.9517
				0.7243	0.7079	0.7975
3	50	1.00	10.00	0.9430	0.9410	0.9467
				1.1505	1.1267	1.3489
10	50	1.00	0.25	0.9470	0.9457	0.9520
				0.0593	0.0588	0.0689
10	50	1.00	1.00	0.9430	0.9437	0.9455
				0.2228	0.2216	0.2389
10	50	1.00	4.00	0.9457	0.9457	0.9477
				0.4971	0.4973	0.5306
10	50	1.00	10.00	0.9455	0.9457	0.9510
				0.5562	0.5564	0.6341
50	50	1.00	0.10	0.9450	0.9410	0.9483
				0.0243	0.0239	0.0283
50	50	1.00	0.25	0.9417	0.9405	0.9430
				0.0580	0.0576	0.0644
50	50	1.00	1.00	0.9470	0.9460	0.9490
				0.1701	0.1697	0.1767

Table 7.3: Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 3$ populations & smaller sample sizes, $\alpha = .05$

n_1	n_2	n_3	σ_1	σ_2	σ_3	Two-Stage	General FGPQ	Fisher
3	3	3	1.00	1.00	1.00	0.9055	0.8962	0.9493
						0.6172	0.5902	0.7640
3	3	3	1.00	0.50	0.50	0.9075	0.9002	0.9505
						0.3603	0.3424	0.4546
3	3	3	1.00	0.50	2.00	0.9190	0.9160	0.9530
						0.5472	0.5156	0.6948
3	3	3	1.00	0.25	4.00	0.9243	0.9300	0.9545
						0.3768	0.3441	0.5138
3	3	3	1.00	0.50	10.00	0.9303	0.9293	0.9543
						0.5773	0.5246	0.8051
3	3	3	1.00	0.10	10.00	0.9407	0.9493	0.9553
						0.1781	0.1702	0.2949
3	10	3	1.00	0.50	2.00	0.9285	0.9345	0.9513
						0.3018	0.2755	0.3432
3	10	3	1.00	0.25	4.00	0.9430	0.9447	0.9540
						0.1660	0.1466	0.1970
3	10	3	1.00	0.50	10.00	0.9333	0.9415	0.9463
						0.2991	0.2786	0.3685
3	10	3	1.00	0.10	10.00	0.9455	0.9450	0.9490
						0.0625	0.0567	0.0841
3	10	10	1.00	0.50	0.50	0.9323	0.9365	0.9477
						0.2045	0.1928	0.2211
7	20	8	1.00	0.25	1.00	0.9553	0.9577	0.9497
						0.0960	0.0949	0.1183
7	20	12	1.00	1.00	1.00	0.9375	0.9390	0.9497
						0.2791	0.2769	0.3020
7	20	12	1.00	0.10	16.00	0.9465	0.9443	0.9465
						0.0397	0.0392	0.0568
10	10	10	1.00	1.00	1.00	0.9380	0.9345	0.9523
						0.3106	0.3096	0.3380
10	10	10	1.00	0.50	2.00	0.9350	0.9357	0.9447
						0.2464	0.2454	0.2815
10	10	10	1.00	0.25	4.00	0.9537	0.9547	0.9480
						0.1393	0.1385	0.1774
10	10	10	1.00	0.50	10.00	0.9453	0.9457	0.9500
						0.2539	0.2535	0.3092
10	10	10	1.00	0.10	10.00	0.9523	0.9500	0.9480
						0.0570	0.0565	0.0806

Table 7.4: Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 3$ populations & larger sample sizes, $\alpha = .05$

n_1	n_2	n_3	σ_1	σ_2	σ_3	Two-Stage	General FGPQ	Fisher
10	50	10	1.00	0.25	4.00	0.9477	0.9470	0.9467
						0.0602	0.0596	0.0788
10	50	50	1.00	1.00	1.00	0.9430	0.9430	0.9513
						0.1635	0.1625	0.1761
10	50	50	1.00	0.50	0.50	0.9470	0.9470	0.9533
						0.0843	0.0838	0.0934
40	30	50	1.00	0.10	10.00	0.9493	0.9470	0.9487
						0.0313	0.0309	0.0418
40	30	50	1.00	10.00	0.10	0.9495	0.9467	0.9473
						0.0241	0.0237	0.0329
50	50	50	1.00	1.00	1.00	0.9473	0.9470	0.9510
						0.1392	0.1388	0.1461
50	50	50	1.00	0.50	2.00	0.9500	0.9490	0.9527
						0.1046	0.1042	0.1175
50	50	50	1.00	0.25	4.00	0.9485	0.9473	0.9493
						0.0588	0.0583	0.0717
50	50	50	1.00	0.10	10.00	0.9497	0.9463	0.9463
						0.0241	0.0236	0.0321
100	80	90	1.00	0.25	0.25	0.9497	0.9483	0.9523
						0.0319	0.0314	0.0350
100	80	90	1.00	16.00	0.20	0.9483	0.9470	0.9480
						0.0341	0.0337	0.0439
100	100	100	1.00	1.00	1.00	0.9505	0.9497	0.9510
						0.0985	0.0981	0.1039

Table 7.5: Empirical Coverages and "Expected Lengths" for Fisher's Exact Upper Confidence Bound and Two Generalized Upper Bounds: $k = 10$ populations, $\alpha = .05$

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	2-Stage	Gen FGPQ	Fisher
σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}			
3	3	3	3	3	3	3	3	3	3			
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.8672	0.8580	0.9407
										0.3749	0.3258	0.3970
3	3	3	3	3	3	3	3	3	3			
1.00	0.25	0.25	0.50	0.50	10.00	10.00	10.00	10.00	10.00	0.8925	0.8928	0.9540
										0.1974	0.1626	0.2980
7	3	3	3	3	3	10	10	10	10			
1.00	10.00	10.00	10.00	10.00	10.00	0.25	0.25	0.50	0.50	0.9413	0.9410	0.9527
										0.0936	0.0860	0.1273
7	10	10	10	10	10	10	10	10	10			
1.00	0.25	0.25	0.50	0.50	10.00	10.00	10.00	10.00	10.00	0.9413	0.9410	0.9547
										0.0875	0.08562	0.1257
10	10	10	10	10	10	10	10	10	10			
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.9350	0.9357	0.9545
										0.1752	0.1719	0.1911
10	10	10	10	10	50	50	50	50	50			
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.9430	0.9427	0.9525
										0.0983	0.0972	0.1107
50	50	50	50	50	50	50	50	50	50			
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.9477	0.9470	0.9550
										0.0749	0.0744	0.0812
50	50	50	50	50	50	50	50	50	50			
1.00	0.25	0.25	0.50	0.50	10.00	10.00	10.00	10.00	10.00	0.9467	0.9455	0.94527
										0.0377	0.0373	0.0523

Chapter 8

CONCLUSIONS

The pivotal method is perhaps the most well known classical approach for constructing confidence intervals. Unfortunately, for many problems of interest pivotal quantities are unavailable. In this dissertation we show how confidence intervals and tolerance intervals can be constructed in standard as well as nonstandard situations using Weerahandi's idea of GPQs ([55]). Our formulation of FGPQs, although conceptually the same as that of Weerihandi's original definition, facilitates the rigorous study of frequentist properties of GCIs. We also provided a simple recipe for constructing FGPQs in situations where complete, sufficient statistics are readily available for the parameters of the underlying statistical model. This simple construction, inspired by Fraser's ideas of structural inference, provides a systematic approach for developing FGPQs. We also developed a two-stage approach for constructing FGPQs in situations where the minimal sufficient statistics are not complete. Prior to this work, most authors, including Weerahandi, constructed GPQs using either ingenuity or invariance and sufficiency arguments or both. Hannig et al. [25] provide a very general construction method for FGPQs. Their extension bypasses consideration of minimal sufficient statistics and is applicable directly to raw observations whenever a suitable structural representation for the data is available.

In most situations, FGPQ lead to confidence intervals that achieve the stated coverage probabilities only approximately. In the case the population characteristic of interest is a scaler we gave necessary and sufficient conditions for a GCI to be exact. It turns out that, if a GCI is exact, then there is a pivotal quantity that gives rise to the same confidence interval. Hannig et al. [25] gave sufficient conditions so that the confidence intervals based on an FGPQ are asymptotically exact.

For the saturated balanced factorial mixed linear model with replication we constructed FGPQs for the variance components and any linear function of the fixed effects. We verified that FGPQs of the variance components satisfied the sufficient conditions of Hannig et al. and hence the FGCIs that are based on the FGPQs are asymptotically exact. For the saturated factorial mixed linear model with unbalanced data we proposed the imputation method for constructing approximate FGPQs. The imputation method was used to construct approximate FGPQs for a population characteristic of the one-way random effects model with unbalanced data. A simulation study was conducted to compare the imputation method's performance with methods proposed by Krisnamoorthy and Mathew [34]. The performance of the imputation method was not as good as the performance of the method of Krishnamoorthy and Mathew, but the results were sufficiently encouraging that we feel it is a candidate approach for more general models with unbalanced data.

In Chapter 3 we used the FGPQs for parameters from a saturated balanced factorial mixed linear model with replication to construct one sided tolerance intervals for the distribution of normal random variables associated with the linear mixed model. We illustrated the general result by constructing one-sided tolerance intervals for two population characteristics associated with a 2-way random effects model. In addition we constructed one-sided tolerance intervals using FGPQs for two population characteristics associated with a 2-way random effects model without replication. We verified that all the FGPQs satisfied the sufficient conditions of Hannig et al. and hence all the tolerance intervals are asymptotically exact. We used a simulation study to evaluate the small sample behavior of the proposed tolerance intervals. Our simulation study showed that their empirical coverage probabilities ranged from near-exact to very conservative. In our opinion, the performance of these FGCI's is adequate for practical use.

FGPQs were constructed for the proportion of conformance of the distribution of a normal random variable. A simulation study was conducted to the measure the performance of lower generalized confidence bounds (lower-GCBs) against lower confidence bounds suggested by Wang and Lam [54]. The empirical coverage probabilities for the two methods were similar, but the lower-GCB was more efficient as it resulted in tighter bounds. We verified that the FGPQ satisfied the sufficient conditions of Hannig et al. and hence the GCIs are asymptotically exact.

We constructed FGPQs for the proportion of conformance for the distributions of two population characteristics of the one-way random effects model . We verified the FGPQs satisfied the conditions of Hannig et al. and hence the FGCIs have asymptotically exact coverage. We conducted a simulation study to assess the performance of lower-GCBs for small sample sizes. We also evaluated their large sample behavior to estimate when the asymptotics took effect. Our simulation study showed the FGCIs performed adequately for practical use.

The last problem we addressed was finding an FGPQ for the common mean of several populations with possibly different variances. We constructed two FGPQs, one using the two-stage method and the other using the general structural method of Hannig et al. [25]. We conducted a simulation study to assess the performance of upper-GCBs based on the two methods. In the simulation study we also compared the upper-GCBs to Fisher's method of combining P-values. Our simulation study showed the two GCI methods had equivalent performance and appear to be more efficient than Fisher's method.

Several questions remain that need further investigation. The imputation method for constructing approximate FGPQs for the population characteristics associated with the factorial linear mixed model need to be tested under more complicated models. The natural model to explore next is the two-way factorial linear mixed model with unbalanced data.

There is one specific application that may be able to benefit from our study of the common mean problem. The Interior West Forest Inventory and Analysis program of the US Forest Service has fixed sample locations (plots). The sample within each state is divided into 10 panels. Each panel of plots is geographically distributed so it is a sub-sample of the state. One panel is measured each year, so in one cycle of ten years all the plots are measured. Each panel can be viewed as a sample from a population. Combining the estimates from each of the panels can be modelled using the common mean framework, but with the possibility the population distribution is a non-normal distribution. We propose investigating the use of FGPQs to produce confidence intervals for the common mean of the ten panels.

In many of our examples we faced a situation where the natural FGPQ did not have a distribution whose support coincided with the parameter space. For instance, in certain applications, the FGPQ for a variance could take negative values with positive probability. This problem was circumvented by redefining the FGPQ to be zero whenever the original definition resulted in a negative value. We propose investigating other techniquss for handling this issue. This problem becomes particularly interesting in multivariate situations where a FGPQ for a covariance matrix may have a positive probability of not being positive definite.

Fisher and Fraser have provided us with a method for associating a probability distribution with the parameters of a statistical model. These fiducial/structural distributions behave like posterior distributions but do not require explicit specification of prior distributions. The fiducial/structural approach was never accepted by mainstream statisticians. Our investigations and those of Hannig et al. [25], Hannig [23], and Hannig [24] suggest that, at the very least, it would be wise to reevaluate the role of fiducial/structural inference in statistical inference. We believe they have much to offer.

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Appendix A

EMPIRICAL COVERAGE PROBABILITIES FOR LOWER TOLERANCE INTERVALS FOR TWO-WAY RANDOM EFFECTS MODELS

For discussion of results see Section 3.7.

A.1 With Interaction

Table A.1: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.997 0.001 0.992 1				
0.1	0.997 0.002 0.992 1	0.996 0.003 0.987 0.999			
0.25	0.997 0.001 0.995 1	0.996 0.002 0.991 0.999	0.995 0.003 0.987 1		
0.5	0.997 0.002 0.992 1	0.996 0.002 0.992 1	0.996 0.003 0.988 0.999	0.994 0.004 0.983 0.999	
1	0.996 0.002 0.987 0.999	0.996 0.002 0.991 0.999	0.996 0.002 0.991 0.999	0.995 0.003 0.986 1	0.993 0.005 0.981 0.998
2	0.995 0.002 0.989 0.999	0.996 0.002 0.993 0.998	0.996 0.002 0.991 0.999	0.995 0.002 0.991 0.999	0.994 0.003 0.987 0.998
4	0.993 0.003 0.987 0.998	0.995 0.002 0.991 0.998	0.995 0.002 0.992 0.998	0.995 0.002 0.992 0.998	0.995 0.002 0.989 0.999
10	0.991 0.004 0.981 0.997	0.992 0.002 0.987 0.997	0.993 0.002 0.989 0.997	0.994 0.002 0.989 0.997	0.994 0.002 0.990 0.997
100	0.977 0.007 0.965 0.990	0.979 0.005 0.968 0.990	0.982 0.004 0.975 0.990	0.985 0.003 0.978 0.991	0.987 0.002 0.981 0.992
	2	4	10	100	
2	0.992 0.005 0.980 0.998				
4	0.993 0.004 0.987 0.998	0.992 0.005 0.982 0.998			
10	0.994 0.002 0.990 0.998	0.994 0.003 0.986 0.998	0.991 0.006 0.979 0.998		
100	0.990 0.002 0.986 0.994	0.992 0.001 0.989 0.995	0.994 0.001 0.990 0.996	0.990 0.006 0.978 0.998	

Table A.2: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.991 0.006 0.979 1				
0.1	0.990 0.007 0.975 0.999	0.990 0.007 0.980 0.999			
0.25	0.988 0.008 0.972 0.998	0.989 0.006 0.980 0.999	0.988 0.007 0.977 0.999		
0.5	0.985 0.009 0.966 0.999	0.987 0.006 0.977 0.999	0.988 0.006 0.977 0.999	0.988 0.007 0.976 0.998	
1	0.981 0.010 0.962 0.998	0.983 0.008 0.970 0.997	0.986 0.006 0.976 0.998	0.988 0.005 0.979 0.998	0.987 0.006 0.976 0.999
2	0.975 0.011 0.953 0.996	0.979 0.009 0.962 0.996	0.982 0.006 0.971 0.996	0.984 0.005 0.978 0.995	0.986 0.005 0.979 0.996
4	0.968 0.011 0.951 0.991	0.970 0.009 0.956 0.990	0.975 0.007 0.961 0.991	0.978 0.006 0.969 0.992	0.983 0.004 0.978 0.994
10	0.960 0.008 0.948 0.983	0.962 0.008 0.946 0.981	0.964 0.007 0.952 0.982	0.967 0.006 0.957 0.985	0.973 0.005 0.963 0.986
100	0.951 0.004 0.940 0.958	0.951 0.004 0.943 0.959	0.952 0.004 0.945 0.962	0.951 0.004 0.942 0.962	0.953 0.004 0.946 0.964
	2	4	10	100	
2	0.987 0.006 0.976 0.997				
4	0.986 0.004 0.978 0.996	0.986 0.006 0.976 0.995			
10	0.979 0.004 0.971 0.987	0.984 0.003 0.979 0.990	0.986 0.006 0.975 0.995		
100	0.956 0.003 0.948 0.962	0.960 0.003 0.951 0.968	0.969 0.003 0.963 0.977	0.985 0.006 0.974 0.995	

Table A.3: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $a, n \in (2, 5, 10)$, and $b = 10$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.985 0.009				
	0.971 0.998				
0.1	0.982 0.009	0.983 0.008			
	0.965 0.996	0.972 0.996			
0.25	0.978 0.011	0.981 0.008	0.982 0.008		
	0.955 0.996	0.968 0.996	0.967 0.996		
0.5	0.974 0.010	0.977 0.008	0.980 0.007	0.982 0.006	
	0.958 0.992	0.961 0.993	0.969 0.994	0.973 0.995	
1	0.968 0.010	0.972 0.008	0.975 0.007	0.977 0.006	0.981 0.006
	0.955 0.991	0.960 0.993	0.965 0.991	0.968 0.990	0.972 0.993
2	0.963 0.009	0.965 0.008	0.968 0.007	0.972 0.005	0.977 0.004
	0.947 0.985	0.952 0.984	0.958 0.987	0.963 0.985	0.969 0.988
4	0.958 0.007	0.961 0.006	0.963 0.006	0.965 0.005	0.969 0.004
	0.947 0.977	0.953 0.978	0.953 0.979	0.954 0.977	0.963 0.981
10	0.954 0.005	0.955 0.004	0.956 0.005	0.958 0.004	0.959 0.004
	0.945 0.965	0.948 0.967	0.948 0.968	0.948 0.969	0.951 0.975
100	0.950 0.004	0.951 0.004	0.950 0.004	0.951 0.003	0.950 0.003
	0.942 0.959	0.943 0.957	0.943 0.961	0.946 0.958	0.944 0.958
	2	4	10	100	
2	0.980 0.005				
	0.971 0.991				
4	0.974 0.004	0.979 0.005			
	0.968 0.985	0.972 0.988			
10	0.965 0.003	0.972 0.003	0.979 0.004		
	0.961 0.974	0.965 0.979	0.972 0.989		
100	0.950 0.004	0.952 0.003	0.957 0.004	0.978 0.004	
	0.941 0.959	0.944 0.961	0.949 0.967	0.971 0.985	

Table A.4: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 2$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.995	0.001	0.996	0.001	0.996	0.001	0.996	0.001	0.996	0.001
	0.994	0.997	0.994	0.998	0.993	0.998	0.995	0.997	0.993	0.998
0.1	0.992	0.002	0.992	0.001	0.992	0.001	0.993	0.001	0.993	0.001
	0.989	0.995	0.989	0.994	0.990	0.995	0.991	0.994	0.991	0.995
0.25	0.989	0.003	0.990	0.002	0.991	0.002	0.991	0.002	0.992	0.002
	0.982	0.993	0.986	0.993	0.988	0.994	0.986	0.994	0.989	0.994
0.5	0.987	0.003	0.988	0.003	0.989	0.002	0.990	0.002	0.991	0.002
	0.982	0.992	0.982	0.993	0.984	0.992	0.986	0.994	0.988	0.994
1	0.985	0.004	0.986	0.002	0.986	0.003	0.988	0.002	0.989	0.002
	0.979	0.991	0.981	0.989	0.981	0.990	0.984	0.991	0.987	0.992
2	0.982	0.005	0.982	0.004	0.985	0.002	0.985	0.003	0.988	0.002
	0.974	0.990	0.973	0.988	0.981	0.989	0.980	0.989	0.983	0.992
4	0.978	0.006	0.978	0.005	0.980	0.004	0.982	0.002	0.984	0.003
	0.966	0.986	0.970	0.986	0.974	0.989	0.978	0.986	0.980	0.990
10	0.971	0.006	0.973	0.005	0.975	0.005	0.977	0.004	0.981	0.003
	0.956	0.980	0.964	0.980	0.965	0.982	0.970	0.983	0.976	0.984
100	0.959	0.005	0.959	0.004	0.960	0.005	0.961	0.004	0.965	0.003
	0.951	0.967	0.951	0.965	0.953	0.967	0.953	0.968	0.961	0.971
	2		4		10		100			
0	0.996	0.001	0.995	0.002	0.994	0.002	0.988	0.004		
	0.994	0.998	0.992	0.997	0.991	0.997	0.982	0.994		
0.1	0.993	0.001	0.993	0.001	0.993	0.001	0.988	0.003		
	0.990	0.994	0.990	0.996	0.991	0.995	0.982	0.992		
0.25	0.993	0.001	0.993	0.001	0.993	0.002	0.988	0.003		
	0.991	0.995	0.991	0.995	0.990	0.996	0.985	0.993		
0.5	0.992	0.001	0.993	0.001	0.993	0.001	0.989	0.002		
	0.988	0.993	0.990	0.994	0.990	0.995	0.985	0.993		
1	0.991	0.001	0.992	0.002	0.992	0.002	0.990	0.002		
	0.990	0.994	0.990	0.995	0.989	0.996	0.987	0.993		
2	0.988	0.002	0.990	0.002	0.993	0.001	0.991	0.002		
	0.986	0.991	0.986	0.994	0.991	0.994	0.989	0.994		
4	0.987	0.001	0.989	0.002	0.991	0.001	0.992	0.001		
	0.985	0.989	0.985	0.991	0.989	0.993	0.989	0.994		
10	0.983	0.002	0.986	0.002	0.990	0.002	0.994	0.001		
	0.980	0.988	0.984	0.988	0.986	0.993	0.992	0.996		
100	0.968	0.004	0.973	0.003	0.978	0.002	0.990	0.001		
	0.962	0.975	0.968	0.980	0.974	0.981	0.988	0.993		

Table A.5: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 5$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.993	0.001	0.993	0.001	0.993	0.002	0.994	0.002	0.994	0.001
	0.992	0.994	0.990	0.996	0.991	0.996	0.990	0.997	0.992	0.996
0.1	0.976	0.007	0.979	0.005	0.981	0.003	0.984	0.003	0.988	0.002
	0.965	0.986	0.970	0.987	0.976	0.987	0.979	0.989	0.984	0.991
0.25	0.968	0.008	0.972	0.006	0.973	0.006	0.978	0.004	0.981	0.002
	0.955	0.980	0.959	0.986	0.964	0.982	0.972	0.987	0.978	0.987
0.5	0.961	0.007	0.965	0.006	0.967	0.005	0.970	0.005	0.976	0.003
	0.952	0.976	0.956	0.976	0.962	0.978	0.964	0.980	0.970	0.982
1	0.958	0.006	0.958	0.006	0.960	0.007	0.964	0.004	0.969	0.004
	0.948	0.970	0.947	0.968	0.948	0.972	0.956	0.971	0.964	0.976
2	0.953	0.004	0.956	0.005	0.955	0.004	0.960	0.005	0.962	0.004
	0.946	0.960	0.950	0.965	0.946	0.962	0.952	0.973	0.955	0.970
4	0.949	0.005	0.952	0.002	0.953	0.004	0.954	0.004	0.957	0.004
	0.941	0.957	0.948	0.956	0.945	0.959	0.944	0.960	0.952	0.964
10	0.950	0.003	0.950	0.004	0.950	0.003	0.951	0.003	0.954	0.003
	0.943	0.955	0.942	0.956	0.946	0.954	0.946	0.955	0.948	0.958
100	0.950	0.003	0.951	0.004	0.949	0.003	0.952	0.004	0.950	0.004
	0.946	0.954	0.943	0.955	0.943	0.954	0.944	0.960	0.943	0.958
	2		4		10		100			
0	0.994	0.001	0.994	0.001	0.994	0.002	0.986	0.005		
	0.992	0.996	0.992	0.996	0.988	0.996	0.980	0.993		
0.1	0.989	0.002	0.990	0.002	0.991	0.002	0.987	0.003		
	0.986	0.992	0.987	0.992	0.988	0.994	0.982	0.992		
0.25	0.986	0.002	0.988	0.002	0.991	0.002	0.988	0.003		
	0.983	0.990	0.986	0.992	0.988	0.994	0.983	0.992		
0.5	0.981	0.003	0.986	0.002	0.989	0.001	0.989	0.002		
	0.977	0.984	0.981	0.987	0.986	0.991	0.986	0.991		
1	0.974	0.003	0.981	0.002	0.987	0.002	0.989	0.002		
	0.968	0.978	0.978	0.984	0.984	0.989	0.986	0.992		
2	0.968	0.004	0.973	0.002	0.982	0.002	0.990	0.002		
	0.962	0.972	0.970	0.977	0.978	0.986	0.986	0.992		
4	0.962	0.002	0.966	0.003	0.977	0.002	0.989	0.001		
	0.956	0.965	0.962	0.971	0.974	0.981	0.987	0.992		
10	0.956	0.003	0.957	0.002	0.966	0.003	0.987	0.002		
	0.951	0.961	0.954	0.960	0.961	0.974	0.984	0.990		
100	0.949	0.002	0.952	0.003	0.953	0.003	0.967	0.003		
	0.945	0.952	0.949	0.959	0.948	0.956	0.960	0.972		

Table A.6: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 2$, $\boldsymbol{b} = 10$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.990	0.002	0.991	0.001	0.992	0.002	0.993	0.001	0.994	0.002
	0.988	0.993	0.988	0.993	0.989	0.995	0.990	0.994	0.991	0.997
0.1	0.964	0.007	0.967	0.007	0.969	0.005	0.975	0.003	0.979	0.003
	0.948	0.974	0.957	0.977	0.964	0.980	0.969	0.980	0.974	0.984
0.25	0.957	0.005	0.957	0.006	0.960	0.004	0.964	0.004	0.970	0.003
	0.946	0.964	0.948	0.967	0.955	0.969	0.959	0.972	0.964	0.976
0.5	0.954	0.004	0.955	0.005	0.954	0.004	0.959	0.003	0.963	0.004
	0.946	0.963	0.946	0.964	0.948	0.965	0.955	0.963	0.954	0.967
1	0.953	0.004	0.951	0.004	0.951	0.003	0.954	0.004	0.957	0.003
	0.946	0.960	0.944	0.958	0.946	0.958	0.946	0.961	0.952	0.962
2	0.949	0.004	0.950	0.004	0.952	0.004	0.953	0.004	0.953	0.003
	0.943	0.960	0.944	0.958	0.944	0.957	0.944	0.960	0.947	0.958
4	0.951	0.004	0.950	0.003	0.951	0.002	0.951	0.003	0.951	0.004
	0.943	0.956	0.944	0.957	0.947	0.954	0.944	0.956	0.942	0.958
10	0.951	0.002	0.951	0.004	0.949	0.003	0.950	0.005	0.951	0.003
	0.946	0.955	0.943	0.958	0.940	0.954	0.938	0.958	0.946	0.957
100	0.951	0.003	0.949	0.003	0.951	0.003	0.949	0.004	0.951	0.004
	0.945	0.956	0.945	0.956	0.945	0.955	0.941	0.956	0.942	0.956
	2		4		10		100			
0	0.994	0.001	0.994	0.001	0.994	0.002	0.986	0.005		
	0.992	0.998	0.992	0.996	0.990	0.997	0.978	0.992		
0.1	0.984	0.002	0.988	0.002	0.989	0.002	0.987	0.003		
	0.982	0.988	0.983	0.990	0.986	0.992	0.981	0.992		
0.25	0.978	0.002	0.982	0.002	0.987	0.002	0.987	0.003		
	0.974	0.982	0.978	0.985	0.983	0.989	0.983	0.990		
0.5	0.970	0.002	0.977	0.002	0.984	0.002	0.987	0.002		
	0.965	0.974	0.974	0.982	0.980	0.988	0.982	0.992		
1	0.964	0.003	0.968	0.002	0.979	0.002	0.988	0.001		
	0.958	0.970	0.963	0.972	0.975	0.982	0.986	0.990		
2	0.957	0.003	0.962	0.002	0.971	0.002	0.988	0.002		
	0.953	0.963	0.959	0.965	0.967	0.975	0.985	0.990		
4	0.953	0.004	0.957	0.003	0.965	0.003	0.986	0.002		
	0.946	0.960	0.950	0.963	0.959	0.971	0.981	0.989		
10	0.950	0.004	0.954	0.004	0.956	0.003	0.980	0.002		
	0.945	0.957	0.947	0.958	0.951	0.961	0.976	0.984		
100	0.949	0.004	0.951	0.003	0.949	0.004	0.956	0.004		
	0.944	0.955	0.946	0.956	0.944	0.956	0.951	0.966		

Table A.7: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 2$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.996	0.001	0.996	0.001	0.996	0.001	0.995	0.001	0.993	0.003
	0.995	0.997	0.994	0.998	0.992	0.998	0.992	0.998	0.988	0.997
0.1	0.990	0.003	0.991	0.003	0.990	0.002	0.990	0.003	0.988	0.004
	0.985	0.994	0.984	0.994	0.986	0.994	0.985	0.995	0.982	0.993
0.25	0.986	0.004	0.988	0.002	0.988	0.002	0.988	0.003	0.987	0.004
	0.979	0.992	0.985	0.991	0.986	0.992	0.984	0.991	0.982	0.994
0.5	0.983	0.005	0.985	0.003	0.986	0.002	0.987	0.002	0.987	0.003
	0.974	0.990	0.980	0.990	0.983	0.991	0.984	0.990	0.982	0.993
1	0.980	0.004	0.984	0.002	0.984	0.003	0.986	0.002	0.985	0.002
	0.972	0.985	0.980	0.987	0.978	0.989	0.982	0.989	0.983	0.988
2	0.979	0.004	0.981	0.002	0.982	0.002	0.983	0.003	0.984	0.002
	0.972	0.984	0.977	0.984	0.979	0.986	0.977	0.988	0.982	0.989
4	0.973	0.005	0.976	0.003	0.980	0.003	0.982	0.002	0.984	0.002
	0.965	0.981	0.972	0.982	0.975	0.984	0.979	0.986	0.981	0.987
10	0.971	0.005	0.974	0.003	0.976	0.003	0.979	0.002	0.982	0.002
	0.963	0.979	0.970	0.982	0.972	0.981	0.975	0.982	0.979	0.985
100	0.961	0.004	0.962	0.003	0.962	0.003	0.967	0.003	0.969	0.004
	0.953	0.966	0.957	0.970	0.957	0.968	0.962	0.973	0.964	0.977
	2		4		10		100			
0	0.991	0.004	0.986	0.006	0.978	0.008	0.960	0.006		
	0.984	0.997	0.974	0.995	0.966	0.990	0.949	0.970		
0.1	0.986	0.005	0.983	0.007	0.975	0.008	0.957	0.005		
	0.976	0.992	0.974	0.992	0.964	0.987	0.950	0.966		
0.25	0.986	0.004	0.982	0.006	0.976	0.007	0.957	0.006		
	0.977	0.992	0.973	0.993	0.967	0.986	0.946	0.967		
0.5	0.985	0.004	0.983	0.004	0.977	0.006	0.959	0.005		
	0.978	0.990	0.978	0.993	0.966	0.987	0.954	0.972		
1	0.987	0.002	0.983	0.003	0.979	0.004	0.960	0.005		
	0.984	0.991	0.978	0.988	0.972	0.986	0.953	0.968		
2	0.986	0.002	0.986	0.002	0.981	0.003	0.962	0.003		
	0.983	0.988	0.983	0.988	0.976	0.986	0.956	0.968		
4	0.985	0.002	0.985	0.002	0.984	0.003	0.968	0.002		
	0.982	0.988	0.981	0.990	0.977	0.988	0.965	0.972		
10	0.984	0.002	0.987	0.002	0.987	0.001	0.975	0.003		
	0.978	0.988	0.983	0.990	0.985	0.989	0.968	0.979		
100	0.974	0.002	0.978	0.002	0.982	0.003	0.987	0.002		
	0.970	0.977	0.974	0.980	0.977	0.985	0.983	0.990		

Table A.8: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 5$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.993	0.001	0.994	0.001	0.994	0.001	0.993	0.001	0.991	0.003
	0.991	0.995	0.991	0.996	0.992	0.995	0.990	0.995	0.986	0.995
0.1	0.967	0.007	0.972	0.006	0.975	0.004	0.977	0.004	0.978	0.005
	0.958	0.981	0.964	0.984	0.968	0.980	0.970	0.982	0.967	0.985
0.25	0.958	0.008	0.962	0.006	0.966	0.004	0.969	0.004	0.973	0.005
	0.945	0.972	0.955	0.974	0.959	0.974	0.964	0.976	0.966	0.982
0.5	0.954	0.005	0.959	0.004	0.960	0.002	0.963	0.004	0.970	0.003
	0.946	0.966	0.952	0.967	0.958	0.965	0.957	0.971	0.965	0.974
1	0.952	0.005	0.954	0.004	0.956	0.002	0.959	0.004	0.966	0.004
	0.946	0.960	0.948	0.962	0.953	0.960	0.952	0.964	0.958	0.972
2	0.952	0.003	0.952	0.004	0.953	0.004	0.958	0.003	0.960	0.004
	0.946	0.956	0.945	0.958	0.948	0.961	0.955	0.964	0.950	0.965
4	0.951	0.004	0.952	0.004	0.952	0.003	0.954	0.003	0.958	0.003
	0.942	0.958	0.946	0.961	0.949	0.957	0.950	0.959	0.953	0.964
10	0.949	0.004	0.951	0.003	0.950	0.004	0.951	0.003	0.955	0.003
	0.942	0.953	0.947	0.956	0.941	0.954	0.947	0.956	0.949	0.958
100	0.951	0.003	0.949	0.003	0.950	0.003	0.948	0.004	0.952	0.003
	0.944	0.958	0.945	0.954	0.946	0.956	0.942	0.954	0.946	0.958
	2		4		10		100			
0	0.988	0.004	0.984	0.006	0.977	0.008	0.960	0.004		
	0.981	0.994	0.974	0.993	0.963	0.988	0.951	0.965		
0.1	0.977	0.005	0.974	0.006	0.970	0.006	0.957	0.005		
	0.967	0.985	0.964	0.984	0.959	0.980	0.949	0.966		
0.25	0.974	0.004	0.973	0.004	0.968	0.006	0.956	0.005		
	0.970	0.983	0.968	0.981	0.957	0.978	0.948	0.966		
0.5	0.972	0.003	0.972	0.003	0.969	0.004	0.957	0.003		
	0.968	0.978	0.963	0.976	0.963	0.977	0.950	0.962		
1	0.968	0.003	0.972	0.003	0.972	0.002	0.958	0.002		
	0.964	0.975	0.968	0.977	0.968	0.976	0.954	0.962		
2	0.967	0.004	0.969	0.003	0.972	0.002	0.962	0.003		
	0.959	0.973	0.964	0.973	0.966	0.975	0.955	0.966		
4	0.962	0.004	0.968	0.003	0.972	0.004	0.967	0.003		
	0.957	0.969	0.963	0.975	0.963	0.979	0.962	0.974		
10	0.957	0.003	0.963	0.003	0.968	0.003	0.973	0.003		
	0.952	0.962	0.956	0.968	0.962	0.974	0.968	0.977		
100	0.952	0.002	0.952	0.003	0.953	0.004	0.969	0.004		
	0.948	0.955	0.946	0.957	0.945	0.960	0.962	0.976		

Table A.9: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 5$, $\boldsymbol{b} = 10$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.990	0.002	0.992	0.002	0.992	0.002	0.992	0.002	0.990	0.003
	0.988	0.994	0.989	0.996	0.988	0.994	0.987	0.994	0.983	0.995
0.1	0.953	0.008	0.956	0.005	0.962	0.006	0.966	0.003	0.970	0.003
	0.942	0.966	0.949	0.965	0.951	0.976	0.959	0.972	0.965	0.976
0.25	0.950	0.005	0.951	0.003	0.955	0.003	0.958	0.003	0.963	0.003
	0.938	0.960	0.947	0.956	0.948	0.960	0.952	0.966	0.958	0.969
0.5	0.951	0.003	0.952	0.003	0.952	0.004	0.955	0.003	0.959	0.003
	0.944	0.956	0.948	0.956	0.946	0.958	0.951	0.960	0.952	0.965
1	0.950	0.003	0.951	0.003	0.952	0.004	0.951	0.005	0.955	0.004
	0.944	0.954	0.946	0.960	0.944	0.957	0.942	0.962	0.949	0.964
2	0.951	0.003	0.951	0.003	0.950	0.004	0.951	0.003	0.953	0.003
	0.946	0.956	0.945	0.956	0.942	0.956	0.944	0.957	0.948	0.959
4	0.948	0.003	0.951	0.004	0.950	0.003	0.950	0.004	0.952	0.003
	0.943	0.953	0.945	0.962	0.943	0.954	0.942	0.955	0.947	0.958
10	0.949	0.003	0.948	0.003	0.950	0.004	0.949	0.003	0.952	0.003
	0.944	0.955	0.943	0.953	0.946	0.958	0.945	0.953	0.946	0.958
100	0.951	0.003	0.949	0.004	0.950	0.003	0.951	0.003	0.950	0.002
	0.944	0.955	0.941	0.956	0.943	0.957	0.944	0.956	0.946	0.954

σ_A^2	2		4		10		100	
0	0.988	0.004	0.983	0.006	0.977	0.007	0.960	0.003
	0.977	0.995	0.974	0.990	0.966	0.989	0.955	0.966
0.1	0.972	0.002	0.969	0.005	0.965	0.006	0.957	0.003
	0.967	0.976	0.963	0.977	0.958	0.980	0.952	0.961
0.25	0.966	0.004	0.967	0.003	0.966	0.003	0.956	0.004
	0.959	0.972	0.962	0.973	0.961	0.972	0.948	0.960
0.5	0.962	0.003	0.966	0.004	0.968	0.003	0.957	0.003
	0.957	0.969	0.959	0.973	0.963	0.972	0.952	0.967
1	0.961	0.004	0.964	0.003	0.968	0.003	0.961	0.004
	0.957	0.968	0.960	0.970	0.963	0.974	0.955	0.966
2	0.956	0.003	0.962	0.003	0.966	0.003	0.965	0.002
	0.949	0.961	0.956	0.966	0.960	0.972	0.961	0.969
4	0.953	0.003	0.959	0.004	0.963	0.003	0.969	0.003
	0.947	0.957	0.953	0.966	0.958	0.971	0.964	0.974
10	0.952	0.004	0.953	0.003	0.958	0.002	0.971	0.003
	0.946	0.957	0.950	0.961	0.953	0.962	0.967	0.978
100	0.951	0.004	0.950	0.003	0.951	0.003	0.958	0.002
	0.946	0.959	0.943	0.955	0.947	0.955	0.956	0.962

Table A.10: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 2$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.997	0.001	0.996	0.001	0.995	0.002	0.993	0.004	0.990	0.005
	0.995	0.998	0.994	0.998	0.992	0.997	0.985	0.997	0.981	0.995
0.1	0.988	0.004	0.988	0.003	0.988	0.004	0.986	0.006	0.982	0.008
	0.982	0.994	0.985	0.994	0.981	0.994	0.976	0.994	0.967	0.992
0.25	0.985	0.005	0.985	0.003	0.985	0.004	0.984	0.006	0.981	0.006
	0.978	0.991	0.980	0.990	0.978	0.991	0.972	0.992	0.970	0.991
0.5	0.981	0.005	0.983	0.002	0.984	0.003	0.982	0.004	0.980	0.005
	0.974	0.991	0.978	0.986	0.978	0.990	0.976	0.987	0.971	0.988
1	0.979	0.004	0.982	0.002	0.982	0.003	0.983	0.002	0.980	0.003
	0.972	0.987	0.979	0.986	0.976	0.986	0.980	0.988	0.975	0.986
2	0.976	0.003	0.981	0.002	0.981	0.004	0.983	0.002	0.982	0.002
	0.970	0.980	0.976	0.984	0.972	0.987	0.979	0.986	0.979	0.984
4	0.975	0.003	0.979	0.002	0.982	0.002	0.983	0.002	0.984	0.002
	0.970	0.979	0.976	0.984	0.978	0.985	0.978	0.986	0.981	0.986
10	0.972	0.005	0.976	0.003	0.979	0.002	0.981	0.002	0.983	0.003
	0.965	0.981	0.971	0.982	0.974	0.982	0.977	0.983	0.976	0.987
100	0.960	0.004	0.963	0.005	0.965	0.003	0.969	0.003	0.973	0.003
	0.955	0.969	0.955	0.971	0.960	0.970	0.964	0.974	0.968	0.977
	2		4		10		100			
0	0.985	0.008	0.979	0.009	0.970	0.007	0.958	0.005		
	0.968	0.996	0.967	0.992	0.958	0.983	0.949	0.967		
0.1	0.978	0.010	0.971	0.009	0.964	0.008	0.954	0.004		
	0.958	0.991	0.958	0.985	0.951	0.977	0.946	0.959		
0.25	0.977	0.008	0.970	0.009	0.963	0.007	0.954	0.005		
	0.967	0.989	0.957	0.983	0.954	0.976	0.947	0.961		
0.5	0.976	0.006	0.970	0.006	0.963	0.006	0.952	0.004		
	0.964	0.986	0.962	0.981	0.954	0.973	0.946	0.957		
1	0.977	0.005	0.973	0.006	0.965	0.006	0.954	0.003		
	0.971	0.985	0.962	0.982	0.959	0.977	0.946	0.960		
2	0.980	0.002	0.974	0.003	0.970	0.003	0.954	0.003		
	0.976	0.985	0.967	0.978	0.965	0.975	0.950	0.959		
4	0.983	0.002	0.980	0.003	0.973	0.003	0.957	0.003		
	0.979	0.986	0.974	0.985	0.968	0.979	0.953	0.962		
10	0.985	0.002	0.983	0.002	0.980	0.002	0.960	0.003		
	0.980	0.987	0.980	0.988	0.978	0.984	0.954	0.964		
100	0.978	0.003	0.981	0.002	0.985	0.002	0.981	0.003		
	0.973	0.984	0.978	0.984	0.980	0.988	0.978	0.985		

Table A.11: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 5$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.994	0.001	0.994	0.002	0.993	0.002	0.990	0.003	0.987	0.005
	0.992	0.995	0.990	0.997	0.990	0.996	0.982	0.995	0.976	0.994
0.1	0.962	0.009	0.967	0.007	0.969	0.006	0.970	0.007	0.970	0.008
	0.949	0.976	0.959	0.982	0.956	0.982	0.958	0.980	0.959	0.982
0.25	0.955	0.004	0.958	0.003	0.962	0.003	0.965	0.004	0.966	0.005
	0.951	0.964	0.953	0.964	0.959	0.967	0.958	0.970	0.959	0.974
0.5	0.953	0.003	0.956	0.004	0.959	0.004	0.962	0.004	0.965	0.003
	0.947	0.958	0.949	0.964	0.952	0.966	0.952	0.966	0.959	0.972
1	0.951	0.003	0.954	0.004	0.958	0.004	0.961	0.003	0.964	0.006
	0.947	0.957	0.948	0.961	0.952	0.965	0.956	0.966	0.954	0.970
2	0.952	0.004	0.954	0.003	0.957	0.004	0.960	0.004	0.963	0.003
	0.946	0.959	0.950	0.958	0.950	0.968	0.953	0.965	0.957	0.969
4	0.951	0.004	0.953	0.004	0.954	0.004	0.958	0.004	0.961	0.003
	0.944	0.958	0.946	0.959	0.947	0.961	0.951	0.962	0.956	0.968
10	0.951	0.003	0.951	0.003	0.952	0.004	0.953	0.003	0.956	0.004
	0.946	0.956	0.943	0.955	0.946	0.958	0.948	0.958	0.950	0.962
100	0.950	0.005	0.951	0.004	0.950	0.004	0.949	0.004	0.949	0.005
	0.943	0.958	0.944	0.958	0.942	0.958	0.942	0.958	0.939	0.956
	2		4		10		100			
0	0.982	0.007	0.976	0.007	0.968	0.006	0.956	0.004		
	0.971	0.992	0.966	0.986	0.958	0.979	0.950	0.962		
0.1	0.967	0.009	0.963	0.008	0.958	0.007	0.951	0.003		
	0.954	0.980	0.950	0.976	0.946	0.972	0.947	0.960		
0.25	0.964	0.004	0.960	0.005	0.957	0.005	0.950	0.004		
	0.959	0.972	0.954	0.973	0.948	0.966	0.944	0.956		
0.5	0.964	0.002	0.963	0.003	0.959	0.004	0.952	0.003		
	0.961	0.968	0.955	0.967	0.952	0.964	0.946	0.957		
1	0.965	0.004	0.964	0.003	0.961	0.002	0.952	0.004		
	0.957	0.971	0.958	0.967	0.957	0.964	0.945	0.958		
2	0.965	0.003	0.967	0.003	0.963	0.003	0.955	0.003		
	0.960	0.971	0.962	0.971	0.958	0.968	0.950	0.960		
4	0.964	0.003	0.968	0.003	0.968	0.004	0.957	0.003		
	0.960	0.971	0.964	0.975	0.960	0.973	0.951	0.963		
10	0.961	0.003	0.964	0.003	0.968	0.003	0.961	0.003		
	0.957	0.966	0.960	0.970	0.964	0.972	0.954	0.966		
100	0.952	0.004	0.952	0.003	0.957	0.004	0.969	0.003		
	0.946	0.961	0.946	0.958	0.950	0.964	0.964	0.973		

Table A.12: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $\sigma_{AB}^2 \in (0.25, 0.5, 1, 2, 4)$, $n \in (2, 5, 10)$, and $\boldsymbol{a} = 10$, $\boldsymbol{b} = 10$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.992	0.001	0.992	0.002	0.992	0.002	0.989	0.004	0.986	0.006
	0.990	0.994	0.989	0.996	0.988	0.994	0.981	0.994	0.976	0.994
0.1	0.949	0.005	0.953	0.004	0.957	0.005	0.960	0.005	0.961	0.006
	0.944	0.961	0.946	0.960	0.946	0.967	0.948	0.965	0.951	0.973
0.25	0.949	0.004	0.953	0.004	0.952	0.004	0.956	0.005	0.959	0.002
	0.944	0.956	0.943	0.960	0.943	0.957	0.947	0.964	0.955	0.963
0.5	0.949	0.004	0.950	0.005	0.954	0.003	0.954	0.004	0.959	0.006
	0.944	0.958	0.939	0.957	0.947	0.959	0.947	0.962	0.947	0.968
1	0.950	0.003	0.951	0.004	0.952	0.004	0.955	0.003	0.958	0.005
	0.944	0.954	0.945	0.958	0.945	0.957	0.952	0.962	0.950	0.966
2	0.952	0.003	0.950	0.002	0.952	0.004	0.954	0.002	0.955	0.004
	0.943	0.956	0.947	0.953	0.945	0.956	0.952	0.958	0.949	0.962
4	0.948	0.003	0.951	0.005	0.951	0.003	0.953	0.004	0.952	0.003
	0.941	0.952	0.943	0.962	0.945	0.957	0.948	0.960	0.948	0.956
10	0.948	0.003	0.951	0.004	0.950	0.003	0.950	0.004	0.952	0.003
	0.943	0.954	0.945	0.956	0.944	0.954	0.944	0.957	0.946	0.960
100	0.949	0.004	0.950	0.003	0.949	0.003	0.951	0.004	0.950	0.004
	0.942	0.957	0.946	0.954	0.943	0.954	0.944	0.956	0.943	0.956
	2		4		10		100			
0	0.982	0.008	0.977	0.006	0.969	0.008	0.955	0.003		
	0.964	0.994	0.966	0.986	0.952	0.982	0.951	0.964		
0.1	0.962	0.006	0.958	0.006	0.955	0.005	0.949	0.004		
	0.954	0.974	0.950	0.968	0.949	0.967	0.944	0.959		
0.25	0.959	0.003	0.959	0.004	0.956	0.003	0.952	0.004		
	0.954	0.963	0.952	0.965	0.949	0.962	0.945	0.957		
0.5	0.960	0.004	0.959	0.004	0.956	0.004	0.952	0.003		
	0.952	0.969	0.949	0.963	0.950	0.962	0.947	0.958		
1	0.960	0.004	0.962	0.004	0.962	0.002	0.953	0.004		
	0.954	0.966	0.957	0.968	0.958	0.965	0.948	0.965		
2	0.960	0.004	0.963	0.002	0.962	0.003	0.956	0.002		
	0.954	0.966	0.960	0.967	0.958	0.967	0.952	0.960		
4	0.957	0.004	0.960	0.003	0.963	0.003	0.957	0.004		
	0.949	0.961	0.954	0.962	0.958	0.970	0.950	0.963		
10	0.955	0.002	0.956	0.002	0.961	0.003	0.962	0.003		
	0.952	0.960	0.952	0.960	0.958	0.966	0.957	0.966		
100	0.949	0.004	0.950	0.003	0.952	0.003	0.962	0.002		
	0.944	0.959	0.946	0.954	0.946	0.958	0.958	0.966		

A.2 Without Interaction

Table A.13: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.996 0.003 0.992 0.999				
0.1	0.995 0.003 0.991 0.999	0.993 0.005 0.985 0.998			
0.25	0.995 0.003 0.990 0.999	0.994 0.003 0.990 0.999	0.991 0.007 0.981 0.999		
0.5	0.994 0.003 0.988 0.998	0.995 0.003 0.992 0.998	0.994 0.004 0.987 0.999	0.992 0.005 0.984 0.999	
1	0.992 0.004 0.986 0.999	0.995 0.002 0.991 0.998	0.994 0.003 0.991 0.997	0.994 0.004 0.989 0.999	0.991 0.006 0.983 0.998
2	0.990 0.004 0.983 0.996	0.992 0.002 0.989 0.997	0.995 0.002 0.990 0.996	0.994 0.002 0.991 0.998	0.993 0.005 0.985 0.998
4	0.986 0.006 0.977 0.994	0.993 0.002 0.990 0.996	0.994 0.001 0.993 0.996	0.994 0.001 0.993 0.997	0.994 0.003 0.991 0.998
10	0.980 0.006 0.971 0.988	0.988 0.001 0.986 0.991	0.990 0.001 0.989 0.993	0.992 0.001 0.991 0.994	0.994 0.001 0.993 0.997
100	0.964 0.006 0.956 0.976	0.971 0.003 0.966 0.976	0.976 0.001 0.974 0.978	0.982 0.002 0.979 0.985	0.987 0.002 0.984 0.989
	2	4	10	100	
2	0.990 0.006 0.982 0.998				
4	0.992 0.005 0.985 0.997	0.990 0.006 0.981 0.998			
10	0.995 0.002 0.992 0.997	0.993 0.004 0.987 0.997	0.990 0.006 0.983 0.997		
100	0.989 0.002 0.987 0.994	0.992 0.002 0.990 0.995	0.994 0.001 0.992 0.995	0.990 0.006 0.979 0.998	

Table A.14: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.987 0.008 0.977 0.999				
0.1	0.981 0.009 0.968 0.996	0.985 0.007 0.976 0.997			
0.25	0.975 0.010 0.961 0.991	0.985 0.005 0.980 0.996	0.986 0.005 0.979 0.992		
0.5	0.971 0.009 0.960 0.988	0.983 0.004 0.977 0.991	0.986 0.004 0.981 0.992	0.985 0.007 0.977 0.993	
1	0.962 0.009 0.953 0.982	0.974 0.004 0.969 0.982	0.981 0.003 0.978 0.986	0.984 0.004 0.979 0.991	0.986 0.006 0.977 0.995
2	0.959 0.006 0.949 0.970	0.966 0.005 0.957 0.974	0.975 0.003 0.972 0.980	0.980 0.002 0.978 0.985	0.984 0.004 0.976 0.988
4	0.957 0.005 0.950 0.968	0.961 0.004 0.957 0.970	0.966 0.002 0.962 0.969	0.973 0.003 0.969 0.977	0.981 0.003 0.976 0.987
10	0.953 0.003 0.948 0.960	0.954 0.002 0.950 0.957	0.959 0.003 0.952 0.963	0.961 0.002 0.958 0.965	0.969 0.003 0.964 0.974
100	0.951 0.004 0.943 0.958	0.950 0.002 0.946 0.953	0.953 0.003 0.948 0.959	0.951 0.004 0.945 0.962	0.954 0.003 0.949 0.959
	2	4	10	100	
2	0.986 0.005 0.980 0.993				
4	0.984 0.004 0.979 0.990	0.986 0.005 0.978 0.992			
10	0.977 0.002 0.974 0.980	0.982 0.002 0.979 0.985	0.985 0.006 0.979 0.995		
100	0.957 0.002 0.952 0.960	0.958 0.002 0.953 0.960	0.969 0.001 0.968 0.971	0.985 0.005 0.975 0.990	

Table A.15: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of Y; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 10$.

	σ_B^2				
σ_A^2	0	0.1	0.25	0.5	1
0	0.979 0.011				
	0.964 0.996				
0.1	0.971 0.009	0.979 0.007			
	0.961 0.989	0.970 0.990			
0.25	0.966 0.008	0.973 0.006	0.979 0.006		
	0.956 0.981	0.965 0.986	0.969 0.986		
0.5	0.962 0.006	0.969 0.004	0.975 0.005	0.978 0.005	
	0.953 0.973	0.965 0.977	0.966 0.981	0.972 0.985	
1	0.959 0.007	0.964 0.005	0.967 0.004	0.975 0.003	0.980 0.005
	0.947 0.971	0.957 0.973	0.963 0.977	0.972 0.979	0.975 0.988
2	0.955 0.003	0.957 0.003	0.962 0.003	0.967 0.002	0.973 0.002
	0.950 0.959	0.952 0.962	0.958 0.970	0.964 0.972	0.970 0.977
4	0.952 0.002	0.954 0.005	0.957 0.003	0.961 0.004	0.966 0.003
	0.948 0.956	0.948 0.962	0.950 0.963	0.955 0.967	0.962 0.971
10	0.952 0.003	0.951 0.002	0.953 0.003	0.955 0.004	0.958 0.004
	0.947 0.957	0.948 0.955	0.949 0.959	0.950 0.961	0.952 0.964
100	0.950 0.005	0.950 0.004	0.950 0.002	0.949 0.003	0.950 0.003
	0.944 0.958	0.940 0.954	0.947 0.955	0.943 0.953	0.944 0.957
	2	4	10	100	
2	0.979 0.004				
	0.975 0.986				
4	0.971 0.003	0.979 0.004			
	0.969 0.977	0.973 0.986			
10	0.962 0.002	0.972 0.003	0.978 0.004		
	0.959 0.966	0.967 0.976	0.973 0.984		
100	0.952 0.004	0.954 0.004	0.957 0.004	0.980 0.003	
	0.946 0.956	0.948 0.959	0.953 0.962	0.977 0.984	

Table A.16: Empirical coverage probabilities for lower $(0.9, 0.95)$ tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 2$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.997	0.001	0.995	0.002	0.993	0.005	0.989	0.008	0.984	0.012
	0.996	0.999	0.992	0.998	0.982	0.999	0.975	0.998	0.965	0.998
0.1	0.987	0.004	0.988	0.005	0.987	0.007	0.985	0.009	0.980	0.013
	0.983	0.993	0.980	0.995	0.976	0.995	0.972	0.996	0.960	0.995
0.25	0.981	0.005	0.986	0.003	0.987	0.006	0.987	0.006	0.984	0.010
	0.975	0.990	0.981	0.991	0.979	0.995	0.977	0.994	0.971	0.996
0.5	0.981	0.006	0.985	0.002	0.986	0.003	0.987	0.004	0.984	0.007
	0.971	0.990	0.982	0.988	0.981	0.989	0.981	0.993	0.972	0.993
1	0.977	0.005	0.984	0.002	0.986	0.002	0.987	0.003	0.986	0.005
	0.970	0.984	0.982	0.987	0.983	0.988	0.983	0.992	0.977	0.992
2	0.972	0.006	0.982	0.002	0.984	0.002	0.986	0.002	0.986	0.002
	0.964	0.980	0.977	0.985	0.982	0.988	0.982	0.989	0.982	0.988
4	0.967	0.005	0.976	0.004	0.981	0.003	0.984	0.002	0.986	0.002
	0.961	0.977	0.970	0.982	0.977	0.985	0.980	0.987	0.984	0.989
10	0.965	0.003	0.970	0.004	0.976	0.005	0.980	0.003	0.983	0.003
	0.960	0.969	0.962	0.976	0.969	0.983	0.975	0.983	0.979	0.990
100	0.953	0.004	0.959	0.003	0.962	0.004	0.963	0.004	0.969	0.006
	0.947	0.959	0.955	0.963	0.955	0.967	0.956	0.971	0.962	0.978
	2		4		10		100			
0	0.979	0.014	0.975	0.015	0.970	0.016	0.962	0.014		
	0.961	0.997	0.957	0.996	0.952	0.993	0.949	0.989		
0.1	0.978	0.014	0.975	0.016	0.969	0.019	0.962	0.016		
	0.959	0.997	0.958	0.997	0.947	0.994	0.948	0.986		
0.25	0.979	0.014	0.976	0.014	0.972	0.017	0.962	0.016		
	0.960	0.996	0.958	0.995	0.949	0.995	0.943	0.986		
0.5	0.982	0.011	0.980	0.014	0.971	0.018	0.965	0.019		
	0.967	0.994	0.959	0.995	0.950	0.994	0.951	0.991		
1	0.985	0.008	0.982	0.010	0.977	0.014	0.968	0.018		
	0.974	0.995	0.968	0.995	0.962	0.995	0.951	0.992		
2	0.987	0.005	0.985	0.007	0.980	0.012	0.969	0.018		
	0.979	0.993	0.974	0.994	0.961	0.994	0.950	0.992		
4	0.986	0.003	0.986	0.005	0.983	0.010	0.971	0.017		
	0.981	0.989	0.978	0.992	0.971	0.994	0.954	0.994		
10	0.986	0.002	0.988	0.003	0.987	0.005	0.977	0.013		
	0.984	0.990	0.983	0.994	0.979	0.992	0.961	0.995		
100	0.973	0.004	0.979	0.005	0.982	0.002	0.987	0.006		
	0.967	0.981	0.970	0.985	0.980	0.985	0.978	0.995		

Table A.17: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 5$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.994	0.001	0.993	0.004	0.990	0.006	0.987	0.009	0.982	0.012
	0.993	0.996	0.985	0.997	0.978	0.996	0.971	0.996	0.967	0.997
0.1	0.958	0.006	0.966	0.005	0.971	0.007	0.973	0.009	0.973	0.013
	0.951	0.972	0.961	0.974	0.963	0.982	0.962	0.986	0.954	0.990
0.25	0.955	0.004	0.962	0.003	0.966	0.003	0.972	0.004	0.971	0.009
	0.950	0.964	0.956	0.966	0.959	0.969	0.966	0.978	0.957	0.982
0.5	0.952	0.003	0.959	0.003	0.964	0.004	0.969	0.003	0.972	0.004
	0.947	0.957	0.953	0.963	0.958	0.971	0.965	0.974	0.967	0.980
1	0.949	0.002	0.955	0.004	0.959	0.004	0.962	0.004	0.969	0.003
	0.946	0.951	0.950	0.961	0.955	0.966	0.955	0.970	0.964	0.972
2	0.952	0.002	0.953	0.004	0.956	0.004	0.960	0.005	0.963	0.004
	0.948	0.956	0.945	0.957	0.949	0.962	0.953	0.968	0.959	0.970
4	0.953	0.003	0.953	0.004	0.952	0.003	0.954	0.003	0.959	0.004
	0.948	0.957	0.948	0.959	0.949	0.958	0.949	0.959	0.955	0.967
10	0.951	0.003	0.951	0.004	0.951	0.003	0.952	0.003	0.952	0.005
	0.947	0.956	0.944	0.957	0.948	0.958	0.949	0.957	0.943	0.959
100	0.951	0.003	0.950	0.003	0.951	0.004	0.953	0.003	0.949	0.003
	0.944	0.953	0.946	0.956	0.946	0.957	0.949	0.956	0.945	0.956
	2		4		10		100			
0	0.979	0.012	0.975	0.014	0.969	0.015	0.962	0.011		
	0.964	0.996	0.956	0.994	0.956	0.991	0.953	0.980		
0.1	0.973	0.013	0.969	0.017	0.970	0.016	0.960	0.016		
	0.955	0.990	0.945	0.993	0.949	0.991	0.945	0.984		
0.25	0.972	0.011	0.972	0.012	0.969	0.017	0.964	0.017		
	0.959	0.987	0.958	0.990	0.952	0.993	0.949	0.989		
0.5	0.974	0.008	0.974	0.010	0.972	0.014	0.965	0.018		
	0.962	0.982	0.959	0.987	0.955	0.991	0.946	0.990		
1	0.972	0.004	0.974	0.008	0.972	0.013	0.968	0.016		
	0.964	0.980	0.964	0.985	0.957	0.991	0.953	0.990		
2	0.968	0.002	0.973	0.004	0.974	0.009	0.969	0.015		
	0.965	0.971	0.968	0.978	0.962	0.988	0.950	0.990		
4	0.964	0.004	0.970	0.004	0.972	0.006	0.970	0.014		
	0.958	0.969	0.959	0.975	0.961	0.978	0.955	0.990		
10	0.956	0.005	0.963	0.004	0.967	0.003	0.973	0.012		
	0.949	0.961	0.957	0.968	0.963	0.973	0.960	0.987		
100	0.952	0.003	0.951	0.005	0.956	0.002	0.970	0.002		
	0.947	0.956	0.943	0.957	0.953	0.959	0.968	0.974		

Table A.18: Empirical coverage probabilities for lower (0.9, 0.95) tolerance intervals for the distribution of $\mu + A$; the Mean, Standard deviation, Minimum and Maximum over the parameter sets $\mu = 0$, $\sigma_e^2 = 1$, $a, n \in (2, 5, 10)$, and $\boldsymbol{b} = 10$.

	σ_B^2									
σ_A^2	0		0.1		0.25		0.5		1	
0	0.992	0.001	0.992	0.003	0.989	0.007	0.986	0.009	0.982	0.012
	0.991	0.994	0.987	0.996	0.975	0.996	0.968	0.996	0.962	0.996
0.1	0.950	0.004	0.959	0.003	0.962	0.004	0.966	0.006	0.969	0.008
	0.943	0.958	0.956	0.964	0.955	0.969	0.959	0.974	0.960	0.981
0.25	0.950	0.001	0.953	0.004	0.956	0.004	0.960	0.002	0.967	0.004
	0.947	0.952	0.946	0.958	0.948	0.963	0.957	0.965	0.959	0.972
0.5	0.950	0.003	0.952	0.003	0.955	0.002	0.960	0.004	0.964	0.003
	0.945	0.954	0.947	0.956	0.951	0.958	0.954	0.966	0.960	0.968
1	0.951	0.002	0.952	0.003	0.952	0.004	0.955	0.004	0.960	0.004
	0.949	0.954	0.947	0.957	0.945	0.956	0.951	0.964	0.954	0.967
2	0.949	0.003	0.951	0.004	0.950	0.003	0.953	0.004	0.957	0.003
	0.943	0.954	0.947	0.959	0.945	0.956	0.947	0.959	0.953	0.962
4	0.949	0.004	0.953	0.003	0.953	0.002	0.952	0.004	0.955	0.003
	0.943	0.955	0.947	0.957	0.950	0.956	0.947	0.958	0.951	0.960
10	0.950	0.003	0.950	0.003	0.951	0.003	0.949	0.002	0.951	0.003
	0.946	0.954	0.944	0.955	0.948	0.958	0.945	0.952	0.946	0.954
100	0.950	0.003	0.951	0.001	0.949	0.003	0.950	0.003	0.949	0.002
	0.946	0.953	0.948	0.952	0.945	0.956	0.946	0.954	0.945	0.952
	2		4		10		100			
0	0.978	0.013	0.975	0.014	0.969	0.015	0.960	0.013		
	0.960	0.995	0.959	0.995	0.955	0.991	0.948	0.984		
0.1	0.971	0.012	0.970	0.015	0.967	0.016	0.964	0.016		
	0.955	0.986	0.955	0.990	0.949	0.990	0.952	0.986		
0.25	0.968	0.009	0.970	0.011	0.970	0.014	0.965	0.016		
	0.955	0.981	0.957	0.985	0.953	0.990	0.953	0.987		
0.5	0.966	0.004	0.970	0.007	0.969	0.012	0.966	0.016		
	0.960	0.972	0.961	0.978	0.955	0.986	0.951	0.988		
1	0.963	0.002	0.967	0.004	0.971	0.009	0.969	0.016		
	0.959	0.968	0.961	0.972	0.958	0.984	0.953	0.991		
2	0.959	0.004	0.963	0.003	0.969	0.005	0.969	0.014		
	0.953	0.965	0.959	0.967	0.962	0.975	0.954	0.989		
4	0.956	0.004	0.958	0.002	0.965	0.002	0.968	0.014		
	0.950	0.962	0.955	0.961	0.963	0.969	0.953	0.987		
10	0.952	0.004	0.955	0.004	0.959	0.004	0.971	0.008		
	0.947	0.957	0.950	0.964	0.953	0.966	0.962	0.982		
100	0.950	0.004	0.951	0.004	0.950	0.003	0.959	0.002		
	0.944	0.956	0.943	0.956	0.945	0.954	0.953	0.962		

Appendix B

SIMULATION RESULTS FOR CONFIDENCE INTERVALS FOR THE PROPORTION OF CONFORMANCE OF ONE-WAY RANDOM EFFECTS MODEL

For a discussion of results see Section 5.2.

Table B.1: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

a	n	σ_A	p_L	p_U	coverage	dist
2	2	0.1	0.000	0.950	0.9698	0.8326
2	2	0.1	0.000	0.990	0.9750	0.8547
2	2	0.1	0.025	0.975	0.9945	0.9437
2	2	0.1	0.005	0.995	0.9918	0.9799
2	2	0.25	0.000	0.950	0.9650	0.7699
2	2	0.25	0.000	0.990	0.9620	0.7612
2	2	0.25	0.025	0.975	0.9800	0.9211
2	2	0.25	0.005	0.995	0.9740	0.9431
2	2	0.5	0.000	0.950	0.9475	0.6727
2	2	0.5	0.000	0.990	0.9493	0.6297
2	2	0.5	0.025	0.975	0.9667	0.8719
2	2	0.5	0.005	0.995	0.9600	0.8642
2	2	1	0.000	0.950	0.9457	0.5872
2	2	1	0.000	0.990	0.9465	0.5385
2	2	1	0.025	0.975	0.9590	0.8049
2	2	1	0.005	0.995	0.9580	0.7934
2	2	2	0.000	0.950	0.9470	0.5405
2	2	2	0.000	0.990	0.9460	0.5013
2	2	2	0.025	0.975	0.9603	0.7707
2	2	2	0.005	0.995	0.9545	0.7568
2	2	4	0.000	0.950	0.9427	0.5192
2	2	4	0.000	0.990	0.9505	0.4924
2	2	4	0.025	0.975	0.9617	0.7535
2	2	4	0.005	0.995	0.9617	0.7492
2	2	10	0.000	0.950	0.9573	0.5230
2	2	10	0.000	0.990	0.9447	0.4838
2	2	10	0.025	0.975	0.9615	0.7560
2	2	10	0.005	0.995	0.9610	0.7450
2	5	0.1	0.000	0.950	0.9653	0.7976
2	5	0.1	0.000	0.990	0.9633	0.8082
2	5	0.1	0.025	0.975	0.9920	0.9405
2	5	0.1	0.005	0.995	0.9870	0.9722
2	5	0.25	0.000	0.950	0.9543	0.6975
2	5	0.25	0.000	0.990	0.9487	0.6588
2	5	0.25	0.025	0.975	0.9692	0.8901
2	5	0.25	0.005	0.995	0.9645	0.8850
2	5	0.5	0.000	0.950	0.9457	0.5920
2	5	0.5	0.000	0.990	0.9505	0.5602
2	5	0.5	0.025	0.975	0.9613	0.8125
2	5	0.5	0.005	0.995	0.9520	0.7939
2	5	1	0.000	0.950	0.9460	0.5386
2	5	1	0.000	0.990	0.9430	0.5072
2	5	1	0.025	0.975	0.9623	0.7722
2	5	1	0.005	0.995	0.9595	0.7638
2	5	2	0.000	0.950	0.9520	0.5259
2	5	2	0.000	0.990	0.9463	0.4950
2	5	2	0.025	0.975	0.9610	0.7557
2	5	2	0.005	0.995	0.9597	0.7499
2	5	4	0.000	0.950	0.9500	0.5162
2	5	4	0.000	0.990	0.9483	0.4805
2	5	4	0.025	0.975	0.9593	0.7536
2	5	4	0.005	0.995	0.9597	0.7439

a	n	σ_A	p_L	p_U	coverage	dist
2	5	10	0.000	0.950	0.9480	0.5138
2	5	10	0.000	0.990	0.9473	0.4882
2	5	10	0.025	0.975	0.9752	0.7627
2	5	10	0.005	0.995	0.9633	0.7429
2	10	0.1	0.000	0.950	0.9610	0.7638
2	10	0.1	0.000	0.990	0.9600	0.7610
2	10	0.1	0.025	0.975	0.9840	0.9299
2	10	0.1	0.005	0.995	0.9800	0.9550
2	10	0.25	0.000	0.950	0.9515	0.6474
2	10	0.25	0.000	0.990	0.9480	0.5943
2	10	0.25	0.025	0.975	0.9667	0.8486
2	10	0.25	0.005	0.995	0.9603	0.8361
2	10	0.5	0.000	0.950	0.9477	0.5570
2	10	0.5	0.000	0.990	0.9495	0.5187
2	10	0.5	0.025	0.975	0.9563	0.7860
2	10	0.5	0.005	0.995	0.9600	0.7792
2	10	1	0.000	0.950	0.9455	0.5297
2	10	1	0.000	0.990	0.9530	0.4927
2	10	1	0.025	0.975	0.9655	0.7682
2	10	1	0.005	0.995	0.9575	0.7471
2	10	2	0.000	0.950	0.9510	0.5200
2	10	2	0.000	0.990	0.9523	0.4934
2	10	2	0.025	0.975	0.9647	0.7586
2	10	2	0.005	0.995	0.9600	0.7407
2	10	4	0.000	0.950	0.9560	0.5244
2	10	4	0.000	0.990	0.9540	0.4906
2	10	4	0.025	0.975	0.9655	0.7568
2	10	4	0.005	0.995	0.9565	0.7391
2	10	10	0.000	0.950	0.9450	0.5120
2	10	10	0.000	0.990	0.9485	0.4877
2	10	10	0.025	0.975	0.9615	0.7563
2	10	10	0.005	0.995	0.9585	0.7393
5	2	0.1	0.000	0.950	0.9765	0.7900
5	2	0.1	0.000	0.990	0.9785	0.7931
5	2	0.1	0.025	0.975	0.9992	0.9454
5	2	0.1	0.005	0.995	0.9972	0.9803
5	2	0.25	0.000	0.950	0.9605	0.6478
5	2	0.25	0.000	0.990	0.9540	0.5592
5	2	0.25	0.025	0.975	0.9852	0.8689
5	2	0.25	0.005	0.995	0.9745	0.8061
5	2	0.5	0.000	0.950	0.9383	0.4360
5	2	0.5	0.000	0.990	0.9423	0.3373
5	2	0.5	0.025	0.975	0.9700	0.6093
5	2	0.5	0.005	0.995	0.9627	0.5020
5	2	1	0.000	0.950	0.9345	0.3099
5	2	1	0.000	0.990	0.9440	0.2382
5	2	1	0.025	0.975	0.9647	0.4412
5	2	1	0.005	0.995	0.9570	0.3597
5	2	2	0.000	0.950	0.9477	0.2748
5	2	2	0.000	0.990	0.9460	0.2021
5	2	2	0.025	0.975	0.9660	0.3886
5	2	2	0.005	0.995	0.9657	0.3086

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Table B.1: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

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a	n	σ_A	p_L	p_U	coverage	dist
5	2	4	0.000	0.950	0.9455	0.2591
5	2	4	0.000	0.990	0.9583	0.1949
5	2	4	0.025	0.975	0.9718	0.3761
5	2	4	0.005	0.995	0.9627	0.2939
5	2	10	0.000	0.950	0.9507	0.2626
5	2	10	0.000	0.990	0.9565	0.1953
5	2	10	0.025	0.975	0.9718	0.3695
5	2	10	0.005	0.995	0.9695	0.2891
5	5	0.1	0.000	0.950	0.9692	0.7314
5	5	0.1	0.000	0.990	0.9685	0.7004
5	5	0.1	0.025	0.975	0.9962	0.9384
5	5	0.1	0.005	0.995	0.9870	0.9456
5	5	0.25	0.000	0.950	0.9533	0.5026
5	5	0.25	0.000	0.990	0.9447	0.3994
5	5	0.25	0.025	0.975	0.9698	0.6858
5	5	0.25	0.005	0.995	0.9580	0.5785
5	5	0.5	0.000	0.950	0.9460	0.3368
5	5	0.5	0.000	0.990	0.9467	0.2616
5	5	0.5	0.025	0.975	0.9673	0.4801
5	5	0.5	0.005	0.995	0.9657	0.3944
5	5	1	0.000	0.950	0.9473	0.2804
5	5	1	0.000	0.990	0.9473	0.2093
5	5	1	0.025	0.975	0.9673	0.4025
5	5	1	0.005	0.995	0.9667	0.3199
5	5	2	0.000	0.950	0.9423	0.2602
5	5	2	0.000	0.990	0.9445	0.1931
5	5	2	0.025	0.975	0.9708	0.3792
5	5	2	0.005	0.995	0.9637	0.2942
5	5	4	0.000	0.950	0.9475	0.2585
5	5	4	0.000	0.990	0.9490	0.1906
5	5	4	0.025	0.975	0.9708	0.3750
5	5	4	0.005	0.995	0.9683	0.2858
5	5	10	0.000	0.950	0.9535	0.2603
5	5	10	0.000	0.990	0.9497	0.1910
5	5	10	0.025	0.975	0.9728	0.3705
5	5	10	0.005	0.995	0.9702	0.2912
5	10	0.1	0.000	0.950	0.9550	0.6564
5	10	0.1	0.000	0.990	0.9657	0.5922
5	10	0.1	0.025	0.975	0.9902	0.8955
5	10	0.1	0.005	0.995	0.9762	0.8297
5	10	0.25	0.000	0.950	0.9383	0.3974
5	10	0.25	0.000	0.990	0.9490	0.3161
5	10	0.25	0.025	0.975	0.9657	0.5588
5	10	0.25	0.005	0.995	0.9613	0.4659
5	10	0.5	0.000	0.950	0.9390	0.3021
5	10	0.5	0.000	0.990	0.9507	0.2314
5	10	0.5	0.025	0.975	0.9673	0.4294
5	10	0.5	0.005	0.995	0.9665	0.3484
5	10	1	0.000	0.950	0.9500	0.2718
5	10	1	0.000	0.990	0.9455	0.2062
5	10	1	0.025	0.975	0.9718	0.3875
5	10	1	0.005	0.995	0.9705	0.3061
5	10	2	0.000	0.950	0.9570	0.2648
5	10	2	0.000	0.990	0.9483	0.1890
5	10	2	0.025	0.975	0.9758	0.3754
5	10	2	0.005	0.995	0.9730	0.2934
5	10	4	0.000	0.950	0.9513	0.2624
5	10	4	0.000	0.990	0.9455	0.1887
5	10	4	0.025	0.975	0.9765	0.3761
5	10	4	0.005	0.995	0.9683	0.2875
5	10	10	0.000	0.950	0.9527	0.2583
5	10	10	0.000	0.990	0.9577	0.1935
5	10	10	0.025	0.975	0.9742	0.3730
5	10	10	0.005	0.995	0.9688	0.2868
10	2	0.1	0.000	0.950	0.9792	0.7505
10	2	0.1	0.000	0.990	0.9745	0.7156
10	2	0.1	0.025	0.975	0.9998	0.9403
10	2	0.1	0.005	0.995	0.9988	0.9680
10	2	0.25	0.000	0.950	0.9560	0.5029
10	2	0.25	0.000	0.990	0.9475	0.3911
10	2	0.25	0.025	0.975	0.9778	0.7086
10	2	0.25	0.005	0.995	0.9650	0.5677
10	2	0.5	0.000	0.950	0.9453	0.2818
10	2	0.5	0.000	0.990	0.9415	0.2001
10	2	0.5	0.025	0.975	0.9643	0.3970
10	2	0.5	0.005	0.995	0.9620	0.3064
10	2	1	0.000	0.950	0.9400	0.1926
10	2	1	0.000	0.990	0.9430	0.1269
10	2	1	0.025	0.975	0.9690	0.2666
10	2	1	0.005	0.995	0.9650	0.1853
10	2	2	0.000	0.950	0.9445	0.1644
10	2	2	0.000	0.990	0.9475	0.1014
10	2	2	0.025	0.975	0.9667	0.2212
10	2	2	0.005	0.995	0.9680	0.1456
10	2	4	0.000	0.950	0.9507	0.1562
10	2	4	0.000	0.990	0.9510	0.0945
10	2	4	0.025	0.975	0.9705	0.2092
10	2	4	0.005	0.995	0.9708	0.1353
10	2	10	0.000	0.950	0.9425	0.1533
10	2	10	0.000	0.990	0.9447	0.0953
10	2	10	0.025	0.975	0.9760	0.2066
10	2	10	0.005	0.995	0.9735	0.1344
10	5	0.1	0.000	0.950	0.9585	0.6440
10	5	0.1	0.000	0.990	0.9645	0.5745
10	5	0.1	0.025	0.975	0.9948	0.9023
10	5	0.1	0.005	0.995	0.9865	0.8323

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Table B.1: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

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a	n	σ_A	p_L	p_U	coverage	dist
10	5	0.25	0.000	0.950	0.9395	0.3356
10	5	0.25	0.000	0.990	0.9423	0.2507
10	5	0.25	0.025	0.975	0.9665	0.4752
10	5	0.25	0.005	0.995	0.9573	0.3727
10	5	0.5	0.000	0.950	0.9463	0.2155
10	5	0.5	0.000	0.990	0.9430	0.1467
10	5	0.5	0.025	0.975	0.9675	0.2999
10	5	0.5	0.005	0.995	0.9620	0.2161
10	5	1	0.000	0.950	0.9495	0.1724
10	5	1	0.000	0.990	0.9500	0.1089
10	5	1	0.025	0.975	0.9758	0.2360
10	5	1	0.005	0.995	0.9635	0.1542
10	5	2	0.000	0.950	0.9555	0.1594
10	5	2	0.000	0.990	0.9537	0.0971
10	5	2	0.025	0.975	0.9725	0.2134
10	5	2	0.005	0.995	0.9698	0.1364
10	5	4	0.000	0.950	0.9495	0.1548
10	5	4	0.000	0.990	0.9465	0.0942
10	5	4	0.025	0.975	0.9728	0.2090
10	5	4	0.005	0.995	0.9730	0.1370
10	5	10	0.000	0.950	0.9553	0.1522
10	5	10	0.000	0.990	0.9493	0.0946
10	5	10	0.025	0.975	0.9802	0.2084
10	5	10	0.005	0.995	0.9683	0.1327
10	10	0.1	0.000	0.950	0.9635	0.5421
10	10	0.1	0.000	0.990	0.9517	0.4308
10	10	0.1	0.025	0.975	0.9798	0.7466
10	10	0.1	0.005	0.995	0.9692	0.6277
10	10	0.25	0.000	0.950	0.9460	0.2664
10	10	0.25	0.000	0.990	0.9485	0.1887
10	10	0.25	0.025	0.975	0.9705	0.3710
10	10	0.25	0.005	0.995	0.9680	0.2808
10	10	0.5	0.000	0.950	0.9460	0.1887
10	10	0.5	0.000	0.990	0.9523	0.1220
10	10	0.5	0.025	0.975	0.9718	0.2620
10	10	0.5	0.005	0.995	0.9655	0.1772
10	10	1	0.000	0.950	0.9450	0.1639
10	10	1	0.000	0.990	0.9567	0.1012
10	10	1	0.025	0.975	0.9760	0.2208
10	10	1	0.005	0.995	0.9712	0.1448
10	10	2	0.000	0.950	0.9510	0.1583
10	10	2	0.000	0.990	0.9457	0.0960
10	10	2	0.025	0.975	0.9738	0.2094
10	10	2	0.005	0.995	0.9702	0.1353
10	10	4	0.000	0.950	0.9497	0.1550
10	10	4	0.000	0.990	0.9503	0.0961
10	10	4	0.025	0.975	0.9722	0.2056
10	10	4	0.005	0.995	0.9700	0.1304
10	10	10	0.000	0.950	0.9503	0.1555
10	10	10	0.000	0.990	0.9475	0.0952
10	10	10	0.025	0.975	0.9738	0.2099
10	10	10	0.005	0.995	0.9740	0.1330

Table B.2: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for asymptotics and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

a	n	σ_A	p_L	p_U	coverage	dist	a	n	σ_A	p_L	p_U	coverage	dist
20	10	0.25	0.000	0.950	0.9547	0.1771	100	10	0.25	0.000	0.950	0.9437	0.0708
20	10	0.25	0.000	0.990	0.9417	0.1146	100	10	0.25	0.000	0.990	0.9475	0.0353
20	10	0.25	0.025	0.975	0.9677	0.2468	100	10	0.25	0.025	0.975	0.9630	0.0931
20	10	0.25	0.005	0.995	0.9667	0.1634	100	10	0.25	0.005	0.995	0.9613	0.0482
20	10	1	0.000	0.950	0.9495	0.1014	100	10	1	0.000	0.950	0.9523	0.0359
20	10	1	0.000	0.990	0.9513	0.0534	100	10	1	0.000	0.990	0.9545	0.0151
20	10	1	0.025	0.975	0.9712	0.1277	100	10	1	0.025	0.975	0.9630	0.0416
20	10	1	0.005	0.995	0.9653	0.0712	100	10	1	0.005	0.995	0.9615	0.0174
20	10	4	0.000	0.950	0.9530	0.0960	100	10	4	0.000	0.950	0.9557	0.0337
20	10	4	0.000	0.990	0.9487	0.0483	100	10	4	0.000	0.990	0.9523	0.0136
20	10	4	0.025	0.975	0.9685	0.1176	100	10	4	0.025	0.975	0.9597	0.0373
20	10	4	0.005	0.995	0.9702	0.0639	100	10	4	0.005	0.995	0.9610	0.0155
20	10	10	0.000	0.950	0.9445	0.0958	100	10	10	0.000	0.950	0.9497	0.0330
20	10	10	0.000	0.990	0.9520	0.0495	100	10	10	0.000	0.990	0.9485	0.0136
20	10	10	0.025	0.975	0.9708	0.1172	100	10	10	0.025	0.975	0.9600	0.0377
20	10	10	0.005	0.995	0.9722	0.0644	100	10	10	0.005	0.995	0.9625	0.0154
50	10	0.25	0.000	0.950	0.9523	0.1059	500	10	0.25	0.000	0.950	0.9433	0.0294
50	10	0.25	0.000	0.990	0.9480	0.0581	500	10	0.25	0.000	0.990	0.9460	0.0124
50	10	0.25	0.025	0.975	0.9675	0.1440	500	10	0.25	0.025	0.975	0.9563	0.0377
50	10	0.25	0.005	0.995	0.9585	0.0802	500	10	0.25	0.005	0.995	0.9545	0.0153
50	10	1	0.000	0.950	0.9497	0.0552	500	10	1	0.000	0.950	0.9530	0.0142
50	10	1	0.000	0.990	0.9480	0.0247	500	10	1	0.000	0.990	0.9455	0.0052
50	10	1	0.025	0.975	0.9690	0.0663	500	10	1	0.025	0.975	0.9547	0.0156
50	10	1	0.005	0.995	0.9630	0.0300	500	10	1	0.005	0.995	0.9517	0.0057
50	10	4	0.000	0.950	0.9475	0.0511	500	10	4	0.000	0.950	0.9507	0.0134
50	10	4	0.000	0.990	0.9467	0.0225	500	10	4	0.000	0.990	0.9495	0.0047
50	10	4	0.025	0.975	0.9688	0.0602	500	10	4	0.025	0.975	0.9537	0.0142
50	10	4	0.005	0.995	0.9673	0.0272	500	10	4	0.005	0.995	0.9573	0.0052
50	10	10	0.000	0.950	0.9507	0.0509	500	10	10	0.000	0.950	0.9425	0.0131
50	10	10	0.000	0.990	0.9497	0.0232	500	10	10	0.000	0.990	0.9537	0.0047
50	10	10	0.025	0.975	0.9663	0.0597	500	10	10	0.025	0.975	0.9565	0.0140
50	10	10	0.005	0.995	0.9702	0.0269	500	10	10	0.005	0.995	0.9585	0.0051

Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

a	n	σ_A	p_L	p_U	coverage	dist
2	2	0.1	0.00	0.10	0.8458	0.1234
2	2	0.1	0.00	0.30	0.9215	0.2752
2	2	0.1	0.00	0.50	0.9477	0.4444
2	2	0.1	0.00	0.70	0.9675	0.6246
2	2	0.1	0.00	0.90	0.9755	0.7976
2	2	0.1	0.00	0.95	0.9738	0.8349
2	2	0.1	0.05	0.30	1.0000	0.2499
2	2	0.1	0.05	0.50	0.9995	0.4500
2	2	0.1	0.05	0.70	0.9988	0.6494
2	2	0.1	0.05	0.90	0.9982	0.8485
2	2	0.1	0.05	0.95	0.9980	0.8981
2	2	0.1	0.10	0.30	0.9995	0.2003
2	2	0.1	0.10	0.50	0.9985	0.4002
2	2	0.1	0.10	0.70	0.9995	0.5998
2	2	0.1	0.10	0.90	0.9975	0.7983
2	2	0.1	0.30	0.50	0.9998	0.2001
2	2	0.1	0.30	0.70	0.9992	0.4001
2	2	0.25	0.00	0.10	0.9127	0.1110
2	2	0.25	0.00	0.30	0.9337	0.2790
2	2	0.25	0.00	0.50	0.9473	0.4442
2	2	0.25	0.00	0.70	0.9570	0.6016
2	2	0.25	0.00	0.90	0.9633	0.7508
2	2	0.25	0.00	0.95	0.9630	0.7682
2	2	0.25	0.05	0.30	0.9988	0.2505
2	2	0.25	0.05	0.50	0.9960	0.4496
2	2	0.25	0.05	0.70	0.9915	0.6457
2	2	0.25	0.05	0.90	0.9872	0.8362
2	2	0.25	0.05	0.95	0.9880	0.8848
2	2	0.25	0.10	0.30	0.9978	0.2010
2	2	0.25	0.10	0.50	0.9962	0.4002
2	2	0.25	0.10	0.70	0.9960	0.5983
2	2	0.25	0.10	0.90	0.9905	0.7918
2	2	0.25	0.30	0.50	0.9990	0.2005
2	2	0.25	0.30	0.70	0.9990	0.3999
2	2	0.5	0.00	0.10	0.9555	0.1030
2	2	0.5	0.00	0.30	0.9450	0.2819
2	2	0.5	0.00	0.50	0.9527	0.4413
2	2	0.5	0.00	0.70	0.9487	0.5795
2	2	0.5	0.00	0.90	0.9463	0.6715
2	2	0.5	0.00	0.95	0.9493	0.6753
2	2	0.5	0.05	0.30	0.9962	0.2506
2	2	0.5	0.05	0.50	0.9880	0.4465
2	2	0.5	0.05	0.70	0.9875	0.6360
2	2	0.5	0.05	0.90	0.9762	0.8105
2	2	0.5	0.05	0.95	0.9705	0.8435
2	2	0.5	0.10	0.30	0.9965	0.2014
2	2	0.5	0.10	0.50	0.9908	0.4000
2	2	0.5	0.10	0.70	0.9855	0.5914
2	2	0.5	0.10	0.90	0.9775	0.7701
2	2	0.5	0.30	0.50	0.9978	0.2011
2	2	0.5	0.30	0.70	0.9930	0.3996
2	2	1	0.00	0.10	0.9735	0.1001
2	2	1	0.00	0.30	0.9530	0.2764
2	2	1	0.00	0.50	0.9483	0.4208
2	2	1	0.00	0.70	0.9415	0.5411
2	2	1	0.00	0.90	0.9410	0.5958
2	2	1	0.00	0.95	0.9400	0.5804
2	2	1	0.05	0.30	0.9928	0.2503
2	2	1	0.05	0.50	0.9855	0.4403
2	2	1	0.05	0.70	0.9730	0.6135
2	2	1	0.05	0.90	0.9667	0.7663
2	2	1	0.05	0.95	0.9625	0.7887
2	2	1	0.10	0.30	0.9935	0.2016
2	2	1	0.10	0.50	0.9900	0.3945
2	2	1	0.10	0.70	0.9772	0.5739
2	2	1	0.10	0.90	0.9673	0.7327
2	2	1	0.30	0.50	0.9942	0.2015
2	2	1	0.30	0.70	0.9910	0.3967
2	2	2	0.00	0.10	0.9742	0.0978
2	2	2	0.00	0.30	0.9595	0.2662
2	2	2	0.00	0.50	0.9470	0.4031
2	2	2	0.00	0.70	0.9435	0.4974
2	2	2	0.00	0.90	0.9435	0.5463
2	2	2	0.00	0.95	0.9455	0.5407
2	2	2	0.05	0.30	0.9932	0.2458
2	2	2	0.05	0.50	0.9815	0.4303
2	2	2	0.05	0.70	0.9712	0.5914
2	2	2	0.05	0.90	0.9625	0.7253
2	2	2	0.05	0.95	0.9590	0.7542
2	2	2	0.10	0.30	0.9962	0.1988
2	2	2	0.10	0.50	0.9912	0.3869
2	2	2	0.10	0.70	0.9748	0.5579
2	2	2	0.10	0.90	0.9680	0.7007
2	2	2	0.30	0.50	0.9965	0.1993
2	2	2	0.30	0.70	0.9860	0.3892
2	2	4	0.00	0.10	0.9653	0.0939
2	2	4	0.00	0.30	0.9480	0.2530
2	2	4	0.00	0.50	0.9527	0.3824
2	2	4	0.00	0.70	0.9487	0.4758
2	2	4	0.00	0.90	0.9517	0.5245
2	2	4	0.00	0.95	0.9420	0.5209
2	2	4	0.05	0.30	0.9915	0.2411
2	2	4	0.05	0.50	0.9792	0.4197
2	2	4	0.05	0.70	0.9730	0.5809
2	2	4	0.05	0.90	0.9647	0.7174
2	2	4	0.05	0.95	0.9617	0.7376
2	2	4	0.10	0.30	0.9930	0.1957
2	2	4	0.10	0.50	0.9825	0.3790
2	2	4	0.10	0.70	0.9782	0.5440
2	2	4	0.10	0.90	0.9708	0.6873
2	2	4	0.30	0.50	0.9950	0.1970
2	2	4	0.30	0.70	0.9860	0.3808

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	2	10	0.00	0.10	0.9607	0.0906
2	2	10	0.00	0.30	0.9515	0.2472
2	2	10	0.00	0.50	0.9557	0.3673
2	2	10	0.00	0.70	0.9513	0.4606
2	2	10	0.00	0.90	0.9447	0.5176
2	2	10	0.00	0.95	0.9507	0.5178
2	2	10	0.05	0.30	0.9880	0.2385
2	2	10	0.05	0.50	0.9808	0.4132
2	2	10	0.05	0.70	0.9718	0.5705
2	2	10	0.05	0.90	0.9633	0.7061
2	2	10	0.05	0.95	0.9635	0.7365
2	2	10	0.10	0.30	0.9920	0.1935
2	2	10	0.10	0.50	0.9800	0.3733
2	2	10	0.10	0.70	0.9760	0.5338
2	2	10	0.10	0.90	0.9695	0.6790
2	2	10	0.30	0.50	0.9940	0.1926
2	2	10	0.30	0.70	0.9850	0.3726
2	5	0.1	0.00	0.10	0.8538	0.1170
2	5	0.1	0.00	0.30	0.9235	0.2759
2	5	0.1	0.00	0.50	0.9477	0.4403
2	5	0.1	0.00	0.70	0.9637	0.6149
2	5	0.1	0.00	0.90	0.9650	0.7659
2	5	0.1	0.00	0.95	0.9647	0.7982
2	5	0.1	0.05	0.30	0.9995	0.2502
2	5	0.1	0.05	0.50	0.9985	0.4500
2	5	0.1	0.05	0.70	0.9950	0.6482
2	5	0.1	0.05	0.90	0.9920	0.8437
2	5	0.1	0.05	0.95	0.9922	0.8926
2	5	0.1	0.10	0.30	1.0000	0.2000
2	5	0.1	0.10	0.50	0.9990	0.4001
2	5	0.1	0.10	0.70	0.9990	0.5997
2	5	0.1	0.10	0.90	0.9960	0.7973
2	5	0.1	0.30	0.50	1.0000	0.2000
2	5	0.1	0.30	0.70	0.9988	0.4002
2	5	0.25	0.00	0.10	0.9325	0.1070
2	5	0.25	0.00	0.30	0.9420	0.2791
2	5	0.25	0.00	0.50	0.9503	0.4341
2	5	0.25	0.00	0.70	0.9580	0.5848
2	5	0.25	0.00	0.90	0.9513	0.6867
2	5	0.25	0.00	0.95	0.9515	0.6994
2	5	0.25	0.05	0.30	0.9962	0.2515
2	5	0.25	0.05	0.50	0.9915	0.4490
2	5	0.25	0.05	0.70	0.9862	0.6409
2	5	0.25	0.05	0.90	0.9785	0.8199
2	5	0.25	0.05	0.95	0.9750	0.8605
2	5	0.25	0.10	0.30	0.9985	0.2006
2	5	0.25	0.10	0.50	0.9955	0.3999
2	5	0.25	0.10	0.70	0.9895	0.5954
2	5	0.25	0.10	0.90	0.9842	0.7814
2	5	0.25	0.30	0.50	0.9995	0.2002
2	5	0.25	0.30	0.70	0.9955	0.4006

a	n	σ_A	p_L	p_U	coverage	dist
2	5	0.5	0.00	0.10	0.9677	0.0988
2	5	0.5	0.00	0.30	0.9500	0.2757
2	5	0.5	0.00	0.50	0.9475	0.4203
2	5	0.5	0.00	0.70	0.9447	0.5382
2	5	0.5	0.00	0.90	0.9435	0.6024
2	5	0.5	0.00	0.95	0.9513	0.5982
2	5	0.5	0.05	0.30	0.9930	0.2504
2	5	0.5	0.05	0.50	0.9868	0.4425
2	5	0.5	0.05	0.70	0.9808	0.6196
2	5	0.5	0.05	0.90	0.9722	0.7713
2	5	0.5	0.05	0.95	0.9673	0.8001
2	5	0.5	0.10	0.30	0.9955	0.2016
2	5	0.5	0.10	0.50	0.9872	0.3968
2	5	0.5	0.10	0.70	0.9800	0.5811
2	5	0.5	0.10	0.90	0.9715	0.7394
2	5	0.5	0.30	0.50	0.9975	0.2005
2	5	0.5	0.30	0.70	0.9918	0.3977
2	5	1	0.00	0.10	0.9683	0.0942
2	5	1	0.00	0.30	0.9547	0.2624
2	5	1	0.00	0.50	0.9430	0.3951
2	5	1	0.00	0.70	0.9443	0.4941
2	5	1	0.00	0.90	0.9457	0.5512
2	5	1	0.00	0.95	0.9457	0.5369
2	5	1	0.05	0.30	0.9908	0.2463
2	5	1	0.05	0.50	0.9820	0.4290
2	5	1	0.05	0.70	0.9718	0.5926
2	5	1	0.05	0.90	0.9647	0.7294
2	5	1	0.05	0.95	0.9613	0.7505
2	5	1	0.10	0.30	0.9972	0.1985
2	5	1	0.10	0.50	0.9862	0.3869
2	5	1	0.10	0.70	0.9792	0.5569
2	5	1	0.10	0.90	0.9655	0.6997
2	5	1	0.30	0.50	0.9962	0.1996
2	5	1	0.30	0.70	0.9848	0.3898
2	5	2	0.00	0.10	0.9570	0.0915
2	5	2	0.00	0.30	0.9607	0.2518
2	5	2	0.00	0.50	0.9510	0.3747
2	5	2	0.00	0.70	0.9487	0.4712
2	5	2	0.00	0.90	0.9490	0.5273
2	5	2	0.00	0.95	0.9477	0.5243
2	5	2	0.05	0.30	0.9922	0.2410
2	5	2	0.05	0.50	0.9852	0.4183
2	5	2	0.05	0.70	0.9762	0.5790
2	5	2	0.05	0.90	0.9637	0.7069
2	5	2	0.05	0.95	0.9657	0.7413
2	5	2	0.10	0.30	0.9948	0.1945
2	5	2	0.10	0.50	0.9832	0.3774
2	5	2	0.10	0.70	0.9695	0.5400
2	5	2	0.10	0.90	0.9712	0.6878
2	5	2	0.30	0.50	0.9948	0.1955
2	5	2	0.30	0.70	0.9882	0.3785

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	5	4	0.00	0.10	0.9615	0.0908
2	5	4	0.00	0.30	0.9517	0.2439
2	5	4	0.00	0.50	0.9500	0.3657
2	5	4	0.00	0.70	0.9480	0.4639
2	5	4	0.00	0.90	0.9515	0.5227
2	5	4	0.00	0.95	0.9520	0.5221
2	5	4	0.05	0.30	0.9900	0.2375
2	5	4	0.05	0.50	0.9802	0.4120
2	5	4	0.05	0.70	0.9728	0.5707
2	5	4	0.05	0.90	0.9637	0.7064
2	5	4	0.05	0.95	0.9650	0.7412
2	5	4	0.10	0.30	0.9905	0.1937
2	5	4	0.10	0.50	0.9822	0.3696
2	5	4	0.10	0.70	0.9785	0.5340
2	5	4	0.10	0.90	0.9710	0.6840
2	5	4	0.30	0.50	0.9920	0.1921
2	5	4	0.30	0.70	0.9845	0.3720
2	5	10	0.00	0.10	0.9500	0.0903
2	5	10	0.00	0.30	0.9500	0.2408
2	5	10	0.00	0.50	0.9497	0.3603
2	5	10	0.00	0.70	0.9417	0.4539
2	5	10	0.00	0.90	0.9483	0.5169
2	5	10	0.00	0.95	0.9517	0.5193
2	5	10	0.05	0.30	0.9868	0.2362
2	5	10	0.05	0.50	0.9822	0.4090
2	5	10	0.05	0.70	0.9673	0.5655
2	5	10	0.05	0.90	0.9725	0.7136
2	5	10	0.05	0.95	0.9730	0.7435
2	5	10	0.10	0.30	0.9932	0.1908
2	5	10	0.10	0.50	0.9820	0.3679
2	5	10	0.10	0.70	0.9798	0.5326
2	5	10	0.10	0.90	0.9760	0.6792
2	5	10	0.30	0.50	0.9900	0.1911
2	5	10	0.30	0.70	0.9878	0.3692
2	10	0.1	0.00	0.10	0.8835	0.1164
2	10	0.1	0.00	0.30	0.9197	0.2760
2	10	0.1	0.00	0.50	0.9565	0.4397
2	10	0.1	0.00	0.70	0.9600	0.5980
2	10	0.1	0.00	0.90	0.9647	0.7459
2	10	0.1	0.00	0.95	0.9647	0.7695
2	10	0.1	0.05	0.30	0.9985	0.2507
2	10	0.1	0.05	0.50	0.9960	0.4501
2	10	0.1	0.05	0.70	0.9940	0.6477
2	10	0.1	0.05	0.90	0.9895	0.8407
2	10	0.1	0.05	0.95	0.9878	0.8873
2	10	0.1	0.10	0.30	0.9995	0.2003
2	10	0.1	0.10	0.50	0.9975	0.4003
2	10	0.1	0.10	0.70	0.9952	0.5985
2	10	0.1	0.10	0.90	0.9922	0.7942
2	10	0.1	0.30	0.50	0.9990	0.2006
2	10	0.1	0.30	0.70	0.9980	0.4003
2	10	0.25	0.00	0.10	0.9490	0.1016
2	10	0.25	0.00	0.30	0.9460	0.2758
2	10	0.25	0.00	0.50	0.9545	0.4310
2	10	0.25	0.00	0.70	0.9547	0.5578
2	10	0.25	0.00	0.90	0.9495	0.6437
2	10	0.25	0.00	0.95	0.9475	0.6327
2	10	0.25	0.05	0.30	0.9962	0.2506
2	10	0.25	0.05	0.50	0.9875	0.4477
2	10	0.25	0.05	0.70	0.9790	0.6318
2	10	0.25	0.05	0.90	0.9725	0.7955
2	10	0.25	0.05	0.95	0.9708	0.8310
2	10	0.25	0.10	0.30	0.9962	0.2012
2	10	0.25	0.10	0.50	0.9930	0.3993
2	10	0.25	0.10	0.70	0.9838	0.5893
2	10	0.25	0.10	0.90	0.9792	0.7645
2	10	0.25	0.30	0.50	0.9972	0.2015
2	10	0.25	0.30	0.70	0.9948	0.3998
2	10	0.5	0.00	0.10	0.9663	0.0956
2	10	0.5	0.00	0.30	0.9517	0.2663
2	10	0.5	0.00	0.50	0.9480	0.4080
2	10	0.5	0.00	0.70	0.9537	0.5150
2	10	0.5	0.00	0.90	0.9430	0.5650
2	10	0.5	0.00	0.95	0.9495	0.5643
2	10	0.5	0.05	0.30	0.9950	0.2471
2	10	0.5	0.05	0.50	0.9820	0.4347
2	10	0.5	0.05	0.70	0.9778	0.6055
2	10	0.5	0.05	0.90	0.9688	0.7461
2	10	0.5	0.05	0.95	0.9610	0.7677
2	10	0.5	0.10	0.30	0.9950	0.1999
2	10	0.5	0.10	0.50	0.9888	0.3919
2	10	0.5	0.10	0.70	0.9795	0.5690
2	10	0.5	0.10	0.90	0.9680	0.7127
2	10	0.5	0.30	0.50	0.9972	0.2003
2	10	0.5	0.30	0.70	0.9888	0.3949
2	10	1	0.00	0.10	0.9645	0.0932
2	10	1	0.00	0.30	0.9517	0.2531
2	10	1	0.00	0.50	0.9507	0.3839
2	10	1	0.00	0.70	0.9490	0.4811
2	10	1	0.00	0.90	0.9517	0.5342
2	10	1	0.00	0.95	0.9473	0.5284
2	10	1	0.05	0.30	0.9885	0.2428
2	10	1	0.05	0.50	0.9812	0.4217
2	10	1	0.05	0.70	0.9765	0.5834
2	10	1	0.05	0.90	0.9685	0.7194
2	10	1	0.05	0.95	0.9643	0.7440
2	10	1	0.10	0.30	0.9932	0.1972
2	10	1	0.10	0.50	0.9835	0.3806
2	10	1	0.10	0.70	0.9765	0.5476
2	10	1	0.10	0.90	0.9657	0.6916
2	10	1	0.30	0.50	0.9940	0.1979
2	10	1	0.30	0.70	0.9878	0.3817

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	10	2	0.00	0.10	0.9580	0.0908
2	10	2	0.00	0.30	0.9475	0.2449
2	10	2	0.00	0.50	0.9535	0.3692
2	10	2	0.00	0.70	0.9475	0.4631
2	10	2	0.00	0.90	0.9550	0.5245
2	10	2	0.00	0.95	0.9430	0.5136
2	10	2	0.05	0.30	0.9905	0.2382
2	10	2	0.05	0.50	0.9822	0.4156
2	10	2	0.05	0.70	0.9720	0.5730
2	10	2	0.05	0.90	0.9670	0.7086
2	10	2	0.05	0.95	0.9677	0.7391
2	10	2	0.10	0.30	0.9915	0.1921
2	10	2	0.10	0.50	0.9805	0.3725
2	10	2	0.10	0.70	0.9810	0.5397
2	10	2	0.10	0.90	0.9673	0.6813
2	10	2	0.30	0.50	0.9922	0.1942
2	10	2	0.30	0.70	0.9820	0.3744
2	10	4	0.00	0.10	0.9580	0.0906
2	10	4	0.00	0.30	0.9543	0.2408
2	10	4	0.00	0.50	0.9513	0.3645
2	10	4	0.00	0.70	0.9530	0.4587
2	10	4	0.00	0.90	0.9467	0.5170
2	10	4	0.00	0.95	0.9500	0.5170
2	10	4	0.05	0.30	0.9898	0.2362
2	10	4	0.05	0.50	0.9798	0.4102
2	10	4	0.05	0.70	0.9768	0.5696
2	10	4	0.05	0.90	0.9698	0.7104
2	10	4	0.05	0.95	0.9657	0.7339
2	10	4	0.10	0.30	0.9932	0.1903
2	10	4	0.10	0.50	0.9815	0.3696
2	10	4	0.10	0.70	0.9825	0.5329
2	10	4	0.10	0.90	0.9730	0.6795
2	10	4	0.30	0.50	0.9892	0.1910
2	10	4	0.30	0.70	0.9860	0.3690
2	10	10	0.00	0.10	0.9405	0.0894
2	10	10	0.00	0.30	0.9480	0.2398
2	10	10	0.00	0.50	0.9527	0.3623
2	10	10	0.00	0.70	0.9555	0.4603
2	10	10	0.00	0.90	0.9467	0.5170
2	10	10	0.00	0.95	0.9550	0.5204
2	10	10	0.05	0.30	0.9875	0.2364
2	10	10	0.05	0.50	0.9790	0.4077
2	10	10	0.05	0.70	0.9752	0.5668
2	10	10	0.05	0.90	0.9710	0.7110
2	10	10	0.05	0.95	0.9735	0.7408
2	10	10	0.10	0.30	0.9915	0.1901
2	10	10	0.10	0.50	0.9820	0.3678
2	10	10	0.10	0.70	0.9758	0.5314
2	10	10	0.10	0.90	0.9650	0.6784
2	10	10	0.30	0.50	0.9925	0.1899
2	10	10	0.30	0.70	0.9862	0.3689

a	n	σ_A	p_L	p_U	coverage	dist
5	2	0.1	0.00	0.10	0.8808	0.1132
5	2	0.1	0.00	0.30	0.9107	0.2742
5	2	0.1	0.00	0.50	0.9463	0.4402
5	2	0.1	0.00	0.70	0.9702	0.6105
5	2	0.1	0.00	0.90	0.9770	0.7642
5	2	0.1	0.00	0.95	0.9785	0.7959
5	2	0.1	0.05	0.30	1.0000	0.2494
5	2	0.1	0.05	0.50	1.0000	0.4489
5	2	0.1	0.05	0.70	1.0000	0.6486
5	2	0.1	0.05	0.90	1.0000	0.8478
5	2	0.1	0.05	0.95	0.9998	0.8973
5	2	0.1	0.10	0.30	1.0000	0.1996
5	2	0.1	0.10	0.50	1.0000	0.3992
5	2	0.1	0.10	0.70	1.0000	0.5987
5	2	0.1	0.10	0.90	1.0000	0.7980
5	2	0.1	0.30	0.50	1.0000	0.1997
5	2	0.1	0.30	0.70	1.0000	0.3994
5	2	0.25	0.00	0.10	0.9407	0.1023
5	2	0.25	0.00	0.30	0.9377	0.2787
5	2	0.25	0.00	0.50	0.9423	0.4348
5	2	0.25	0.00	0.70	0.9573	0.5743
5	2	0.25	0.00	0.90	0.9617	0.6530
5	2	0.25	0.00	0.95	0.9590	0.6431
5	2	0.25	0.05	0.30	1.0000	0.2479
5	2	0.25	0.05	0.50	0.9995	0.4463
5	2	0.25	0.05	0.70	0.9988	0.6425
5	2	0.25	0.05	0.90	0.9952	0.8233
5	2	0.25	0.05	0.95	0.9872	0.8540
5	2	0.25	0.10	0.30	1.0000	0.1987
5	2	0.25	0.10	0.50	0.9998	0.3974
5	2	0.25	0.10	0.70	0.9995	0.5939
5	2	0.25	0.10	0.90	0.9968	0.7846
5	2	0.25	0.30	0.50	1.0000	0.1991
5	2	0.25	0.30	0.70	1.0000	0.3979
5	2	0.5	0.00	0.10	0.9698	0.0972
5	2	0.5	0.00	0.30	0.9555	0.2763
5	2	0.5	0.00	0.50	0.9523	0.4173
5	2	0.5	0.00	0.70	0.9517	0.5062
5	2	0.5	0.00	0.90	0.9405	0.4870
5	2	0.5	0.00	0.95	0.9440	0.4386
5	2	0.5	0.05	0.30	0.9998	0.2440
5	2	0.5	0.05	0.50	0.9940	0.4303
5	2	0.5	0.05	0.70	0.9885	0.5834
5	2	0.5	0.05	0.90	0.9770	0.6697
5	2	0.5	0.05	0.95	0.9695	0.6526
5	2	0.5	0.10	0.30	1.0000	0.1963
5	2	0.5	0.10	0.50	0.9982	0.3887
5	2	0.5	0.10	0.70	0.9940	0.5612
5	2	0.5	0.10	0.90	0.9825	0.6708
5	2	0.5	0.30	0.50	1.0000	0.1972
5	2	0.5	0.30	0.70	0.9988	0.3919

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	2	1	0.00	0.10	0.9778	0.0943
5	2	1	0.00	0.30	0.9665	0.2610
5	2	1	0.00	0.50	0.9465	0.3773
5	2	1	0.00	0.70	0.9400	0.4027
5	2	1	0.00	0.90	0.9355	0.3512
5	2	1	0.00	0.95	0.9367	0.3092
5	2	1	0.05	0.30	0.9972	0.2280
5	2	1	0.05	0.50	0.9868	0.3738
5	2	1	0.05	0.70	0.9778	0.4690
5	2	1	0.05	0.90	0.9685	0.4821
5	2	1	0.05	0.95	0.9728	0.4774
5	2	1	0.10	0.30	0.9990	0.1866
5	2	1	0.10	0.50	0.9928	0.3464
5	2	1	0.10	0.70	0.9810	0.4495
5	2	1	0.10	0.90	0.9750	0.4874
5	2	1	0.30	0.50	0.9998	0.1870
5	2	1	0.30	0.70	0.9950	0.3497
5	2	2	0.00	0.10	0.9663	0.0870
5	2	2	0.00	0.30	0.9557	0.2294
5	2	2	0.00	0.50	0.9435	0.3108
5	2	2	0.00	0.70	0.9433	0.3339
5	2	2	0.00	0.90	0.9480	0.2991
5	2	2	0.00	0.95	0.9383	0.2717
5	2	2	0.05	0.30	0.9898	0.2023
5	2	2	0.05	0.50	0.9800	0.3205
5	2	2	0.05	0.70	0.9758	0.3901
5	2	2	0.05	0.90	0.9752	0.4214
5	2	2	0.05	0.95	0.9680	0.4149
5	2	2	0.10	0.30	0.9975	0.1627
5	2	2	0.10	0.50	0.9898	0.2895
5	2	2	0.10	0.70	0.9792	0.3699
5	2	2	0.10	0.90	0.9755	0.4212
5	2	2	0.30	0.50	0.9985	0.1571
5	2	2	0.30	0.70	0.9895	0.2782
5	2	4	0.00	0.10	0.9615	0.0823
5	2	4	0.00	0.30	0.9547	0.2040
5	2	4	0.00	0.50	0.9477	0.2764
5	2	4	0.00	0.70	0.9500	0.3067
5	2	4	0.00	0.90	0.9513	0.2941
5	2	4	0.00	0.95	0.9527	0.2626
5	2	4	0.05	0.30	0.9895	0.1807
5	2	4	0.05	0.50	0.9830	0.2856
5	2	4	0.05	0.70	0.9790	0.3627
5	2	4	0.05	0.90	0.9730	0.4040
5	2	4	0.05	0.95	0.9748	0.4023
5	2	4	0.10	0.30	0.9915	0.1444
5	2	4	0.10	0.50	0.9848	0.2582
5	2	4	0.10	0.70	0.9785	0.3447
5	2	4	0.10	0.90	0.9778	0.4044
5	2	4	0.30	0.50	0.9962	0.1344
5	2	4	0.30	0.70	0.9855	0.2477

a	n	σ_A	p_L	p_U	coverage	dist
5	2	10	0.00	0.10	0.9510	0.0801
5	2	10	0.00	0.30	0.9490	0.1957
5	2	10	0.00	0.50	0.9545	0.2683
5	2	10	0.00	0.70	0.9500	0.3042
5	2	10	0.00	0.90	0.9460	0.2890
5	2	10	0.00	0.95	0.9535	0.2583
5	2	10	0.05	0.30	0.9898	0.1732
5	2	10	0.05	0.50	0.9818	0.2749
5	2	10	0.05	0.70	0.9752	0.3520
5	2	10	0.05	0.90	0.9782	0.4042
5	2	10	0.05	0.95	0.9742	0.3985
5	2	10	0.10	0.30	0.9905	0.1395
5	2	10	0.10	0.50	0.9848	0.2480
5	2	10	0.10	0.70	0.9812	0.3363
5	2	10	0.10	0.90	0.9755	0.4012
5	2	10	0.30	0.50	0.9882	0.1263
5	2	10	0.30	0.70	0.9852	0.2409
5	5	0.1	0.00	0.10	0.9005	0.1113
5	5	0.1	0.00	0.30	0.9267	0.2782
5	5	0.1	0.00	0.50	0.9537	0.4383
5	5	0.1	0.00	0.70	0.9630	0.5929
5	5	0.1	0.00	0.90	0.9673	0.7165
5	5	0.1	0.00	0.95	0.9715	0.7304
5	5	0.1	0.05	0.30	1.0000	0.2491
5	5	0.1	0.05	0.50	0.9998	0.4485
5	5	0.1	0.05	0.70	0.9995	0.6475
5	5	0.1	0.05	0.90	0.9995	0.8459
5	5	0.1	0.05	0.95	0.9980	0.8930
5	5	0.1	0.10	0.30	1.0000	0.1994
5	5	0.1	0.10	0.50	1.0000	0.3989
5	5	0.1	0.10	0.70	0.9998	0.5982
5	5	0.1	0.10	0.90	0.9995	0.7963
5	5	0.1	0.30	0.50	1.0000	0.1996
5	5	0.1	0.30	0.70	1.0000	0.3991
5	5	0.25	0.00	0.10	0.9580	0.0984
5	5	0.25	0.00	0.30	0.9507	0.2753
5	5	0.25	0.00	0.50	0.9473	0.4205
5	5	0.25	0.00	0.70	0.9503	0.5239
5	5	0.25	0.00	0.90	0.9503	0.5307
5	5	0.25	0.00	0.95	0.9457	0.5006
5	5	0.25	0.05	0.30	1.0000	0.2455
5	5	0.25	0.05	0.50	0.9975	0.4392
5	5	0.25	0.05	0.70	0.9900	0.6141
5	5	0.25	0.05	0.90	0.9830	0.7256
5	5	0.25	0.05	0.95	0.9795	0.7216
5	5	0.25	0.10	0.30	0.9998	0.1975
5	5	0.25	0.10	0.50	0.9985	0.3935
5	5	0.25	0.10	0.70	0.9960	0.5817
5	5	0.25	0.10	0.90	0.9868	0.7186
5	5	0.25	0.30	0.50	1.0000	0.1981
5	5	0.25	0.30	0.70	0.9990	0.3949

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	5	0.5	0.00	0.10	0.9715	0.0935
5	5	0.5	0.00	0.30	0.9630	0.2584
5	5	0.5	0.00	0.50	0.9480	0.3758
5	5	0.5	0.00	0.70	0.9420	0.4247
5	5	0.5	0.00	0.90	0.9447	0.3785
5	5	0.5	0.00	0.95	0.9437	0.3430
5	5	0.5	0.05	0.30	0.9985	0.2303
5	5	0.5	0.05	0.50	0.9915	0.3827
5	5	0.5	0.05	0.70	0.9808	0.4802
5	5	0.5	0.05	0.90	0.9722	0.5167
5	5	0.5	0.05	0.95	0.9690	0.5068
5	5	0.5	0.10	0.30	0.9995	0.1878
5	5	0.5	0.10	0.50	0.9922	0.3553
5	5	0.5	0.10	0.70	0.9872	0.4712
5	5	0.5	0.10	0.90	0.9798	0.5232
5	5	0.5	0.30	0.50	1.0000	0.1890
5	5	0.5	0.30	0.70	0.9940	0.3623
5	5	1	0.00	0.10	0.9610	0.0869
5	5	1	0.00	0.30	0.9583	0.2261
5	5	1	0.00	0.50	0.9477	0.3111
5	5	1	0.00	0.70	0.9467	0.3384
5	5	1	0.00	0.90	0.9443	0.3129
5	5	1	0.00	0.95	0.9443	0.2823
5	5	1	0.05	0.30	0.9932	0.1998
5	5	1	0.05	0.50	0.9835	0.3163
5	5	1	0.05	0.70	0.9755	0.3925
5	5	1	0.05	0.90	0.9725	0.4340
5	5	1	0.05	0.95	0.9720	0.4289
5	5	1	0.10	0.30	0.9970	0.1614
5	5	1	0.10	0.50	0.9888	0.2869
5	5	1	0.10	0.70	0.9802	0.3755
5	5	1	0.10	0.90	0.9798	0.4327
5	5	1	0.30	0.50	0.9995	0.1566
5	5	1	0.30	0.70	0.9910	0.2804
5	5	2	0.00	0.10	0.9525	0.0819
5	5	2	0.00	0.30	0.9535	0.2025
5	5	2	0.00	0.50	0.9540	0.2757
5	5	2	0.00	0.70	0.9485	0.3088
5	5	2	0.00	0.90	0.9497	0.2957
5	5	2	0.00	0.95	0.9495	0.2660
5	5	2	0.05	0.30	0.9878	0.1796
5	5	2	0.05	0.50	0.9802	0.2827
5	5	2	0.05	0.70	0.9765	0.3599
5	5	2	0.05	0.90	0.9752	0.4053
5	5	2	0.05	0.95	0.9755	0.4042
5	5	2	0.10	0.30	0.9910	0.1440
5	5	2	0.10	0.50	0.9870	0.2586
5	5	2	0.10	0.70	0.9842	0.3480
5	5	2	0.10	0.90	0.9802	0.4070
5	5	2	0.30	0.50	0.9955	0.1329
5	5	2	0.30	0.70	0.9885	0.2488
5	5	4	0.00	0.10	0.9487	0.0799
5	5	4	0.00	0.30	0.9495	0.1967
5	5	4	0.00	0.50	0.9577	0.2714
5	5	4	0.00	0.70	0.9447	0.3022
5	5	4	0.00	0.90	0.9457	0.2867
5	5	4	0.00	0.95	0.9535	0.2638
5	5	4	0.05	0.30	0.9885	0.1746
5	5	4	0.05	0.50	0.9830	0.2756
5	5	4	0.05	0.70	0.9722	0.3534
5	5	4	0.05	0.90	0.9732	0.4051
5	5	4	0.05	0.95	0.9755	0.4047
5	5	4	0.10	0.30	0.9905	0.1385
5	5	4	0.10	0.50	0.9818	0.2501
5	5	4	0.10	0.70	0.9802	0.3383
5	5	4	0.10	0.90	0.9730	0.4024
5	5	4	0.30	0.50	0.9920	0.1279
5	5	4	0.30	0.70	0.9858	0.2422
5	5	10	0.00	0.10	0.9513	0.0799
5	5	10	0.00	0.30	0.9500	0.1955
5	5	10	0.00	0.50	0.9525	0.2674
5	5	10	0.00	0.70	0.9443	0.3035
5	5	10	0.00	0.90	0.9535	0.2923
5	5	10	0.00	0.95	0.9513	0.2642
5	5	10	0.05	0.30	0.9872	0.1730
5	5	10	0.05	0.50	0.9828	0.2723
5	5	10	0.05	0.70	0.9805	0.3549
5	5	10	0.05	0.90	0.9765	0.4030
5	5	10	0.05	0.95	0.9722	0.3986
5	5	10	0.10	0.30	0.9912	0.1380
5	5	10	0.10	0.50	0.9875	0.2492
5	5	10	0.10	0.70	0.9828	0.3394
5	5	10	0.10	0.90	0.9755	0.4011
5	5	10	0.30	0.50	0.9882	0.1263
5	5	10	0.30	0.70	0.9835	0.2395
5	10	0.1	0.00	0.10	0.9253	0.1040
5	10	0.1	0.00	0.30	0.9300	0.2753
5	10	0.1	0.00	0.50	0.9540	0.4326
5	10	0.1	0.00	0.70	0.9625	0.5681
5	10	0.1	0.00	0.90	0.9613	0.6652
5	10	0.1	0.00	0.95	0.9615	0.6588
5	10	0.1	0.05	0.30	0.9998	0.2486
5	10	0.1	0.05	0.50	0.9995	0.4473
5	10	0.1	0.05	0.70	0.9992	0.6448
5	10	0.1	0.05	0.90	0.9952	0.8358
5	10	0.1	0.05	0.95	0.9912	0.8745
5	10	0.1	0.10	0.30	1.0000	0.1992
5	10	0.1	0.10	0.50	1.0000	0.3979
5	10	0.1	0.10	0.70	0.9995	0.5965
5	10	0.1	0.10	0.90	0.9980	0.7919
5	10	0.1	0.30	0.50	1.0000	0.1993
5	10	0.1	0.30	0.70	1.0000	0.3985

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	10	0.25	0.00	0.10	0.9647	0.0955
5	10	0.25	0.00	0.30	0.9560	0.2681
5	10	0.25	0.00	0.50	0.9443	0.4000
5	10	0.25	0.00	0.70	0.9475	0.4758
5	10	0.25	0.00	0.90	0.9445	0.4457
5	10	0.25	0.00	0.95	0.9437	0.4011
5	10	0.25	0.05	0.30	0.9995	0.2402
5	10	0.25	0.05	0.50	0.9952	0.4150
5	10	0.25	0.05	0.70	0.9862	0.5467
5	10	0.25	0.05	0.90	0.9718	0.6078
5	10	0.25	0.05	0.95	0.9725	0.5992
5	10	0.25	0.10	0.30	1.0000	0.1939
5	10	0.25	0.10	0.50	0.9962	0.3821
5	10	0.25	0.10	0.70	0.9882	0.5339
5	10	0.25	0.10	0.90	0.9782	0.6063
5	10	0.25	0.30	0.50	0.9998	0.1948
5	10	0.25	0.30	0.70	0.9990	0.3859
5	10	0.5	0.00	0.10	0.9630	0.0895
5	10	0.5	0.00	0.30	0.9625	0.2416
5	10	0.5	0.00	0.50	0.9507	0.3396
5	10	0.5	0.00	0.70	0.9443	0.3671
5	10	0.5	0.00	0.90	0.9447	0.3315
5	10	0.5	0.00	0.95	0.9467	0.3014
5	10	0.5	0.05	0.30	0.9965	0.2119
5	10	0.5	0.05	0.50	0.9842	0.3475
5	10	0.5	0.05	0.70	0.9808	0.4238
5	10	0.5	0.05	0.90	0.9680	0.4592
5	10	0.5	0.05	0.95	0.9738	0.4601
5	10	0.5	0.10	0.30	0.9990	0.1748
5	10	0.5	0.10	0.50	0.9935	0.3155
5	10	0.5	0.10	0.70	0.9852	0.4107
5	10	0.5	0.10	0.90	0.9770	0.4621
5	10	0.5	0.30	0.50	0.9998	0.1726
5	10	0.5	0.30	0.70	0.9928	0.3120
5	10	1	0.00	0.10	0.9463	0.0833
5	10	1	0.00	0.30	0.9503	0.2080
5	10	1	0.00	0.50	0.9523	0.2861
5	10	1	0.00	0.70	0.9460	0.3149
5	10	1	0.00	0.90	0.9487	0.2991
5	10	1	0.00	0.95	0.9553	0.2722
5	10	1	0.05	0.30	0.9925	0.1853
5	10	1	0.05	0.50	0.9820	0.2931
5	10	1	0.05	0.70	0.9758	0.3717
5	10	1	0.05	0.90	0.9752	0.4172
5	10	1	0.05	0.95	0.9710	0.4143
5	10	1	0.10	0.30	0.9948	0.1494
5	10	1	0.10	0.50	0.9888	0.2665
5	10	1	0.10	0.70	0.9842	0.3551
5	10	1	0.10	0.90	0.9798	0.4166
5	10	1	0.30	0.50	0.9982	0.1407
5	10	1	0.30	0.70	0.9892	0.2572

a	n	σ_A	p_L	p_U	coverage	dist
5	10	2	0.00	0.10	0.9497	0.0806
5	10	2	0.00	0.30	0.9443	0.1966
5	10	2	0.00	0.50	0.9535	0.2735
5	10	2	0.00	0.70	0.9525	0.3079
5	10	2	0.00	0.90	0.9457	0.2944
5	10	2	0.00	0.95	0.9515	0.2661
5	10	2	0.05	0.30	0.9845	0.1751
5	10	2	0.05	0.50	0.9810	0.2788
5	10	2	0.05	0.70	0.9778	0.3574
5	10	2	0.05	0.90	0.9760	0.4075
5	10	2	0.05	0.95	0.9712	0.3981
5	10	2	0.10	0.30	0.9908	0.1415
5	10	2	0.10	0.50	0.9862	0.2525
5	10	2	0.10	0.70	0.9820	0.3395
5	10	2	0.10	0.90	0.9765	0.4064
5	10	2	0.30	0.50	0.9912	0.1291
5	10	2	0.30	0.70	0.9835	0.2431
5	10	4	0.00	0.10	0.9470	0.0797
5	10	4	0.00	0.30	0.9477	0.1935
5	10	4	0.00	0.50	0.9485	0.2642
5	10	4	0.00	0.70	0.9473	0.3017
5	10	4	0.00	0.90	0.9485	0.2909
5	10	4	0.00	0.95	0.9463	0.2636
5	10	4	0.05	0.30	0.9878	0.1719
5	10	4	0.05	0.50	0.9838	0.2743
5	10	4	0.05	0.70	0.9812	0.3552
5	10	4	0.05	0.90	0.9742	0.4060
5	10	4	0.05	0.95	0.9692	0.3951
5	10	4	0.10	0.30	0.9918	0.1379
5	10	4	0.10	0.50	0.9858	0.2493
5	10	4	0.10	0.70	0.9795	0.3372
5	10	4	0.10	0.90	0.9795	0.4024
5	10	4	0.30	0.50	0.9900	0.1275
5	10	4	0.30	0.70	0.9878	0.2419
5	10	10	0.00	0.10	0.9485	0.0793
5	10	10	0.00	0.30	0.9493	0.1941
5	10	10	0.00	0.50	0.9550	0.2685
5	10	10	0.00	0.70	0.9475	0.3024
5	10	10	0.00	0.90	0.9455	0.2935
5	10	10	0.00	0.95	0.9445	0.2627
5	10	10	0.05	0.30	0.9865	0.1712
5	10	10	0.05	0.50	0.9820	0.2728
5	10	10	0.05	0.70	0.9760	0.3544
5	10	10	0.05	0.90	0.9715	0.3985
5	10	10	0.05	0.95	0.9745	0.3979
5	10	10	0.10	0.30	0.9882	0.1382
5	10	10	0.10	0.50	0.9900	0.2478
5	10	10	0.10	0.70	0.9798	0.3353
5	10	10	0.10	0.90	0.9798	0.4033
5	10	10	0.30	0.50	0.9888	0.1275
5	10	10	0.30	0.70	0.9885	0.2415

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	2	0.1	0.00	0.10	0.8895	0.1145
10	2	0.1	0.00	0.30	0.9105	0.2718
10	2	0.1	0.00	0.50	0.9500	0.4352
10	2	0.1	0.00	0.70	0.9702	0.5910
10	2	0.1	0.00	0.90	0.9810	0.7294
10	2	0.1	0.00	0.95	0.9800	0.7454
10	2	0.1	0.05	0.30	1.0000	0.2487
10	2	0.1	0.05	0.50	1.0000	0.4479
10	2	0.1	0.05	0.70	1.0000	0.6465
10	2	0.1	0.05	0.90	1.0000	0.8448
10	2	0.1	0.05	0.95	1.0000	0.8933
10	2	0.1	0.10	0.30	1.0000	0.1992
10	2	0.1	0.10	0.50	1.0000	0.3985
10	2	0.1	0.10	0.70	1.0000	0.5975
10	2	0.1	0.10	0.90	1.0000	0.7958
10	2	0.1	0.30	0.50	1.0000	0.1994
10	2	0.1	0.30	0.70	1.0000	0.3988
10	2	0.25	0.00	0.10	0.9437	0.1004
10	2	0.25	0.00	0.30	0.9440	0.2775
10	2	0.25	0.00	0.50	0.9495	0.4278
10	2	0.25	0.00	0.70	0.9573	0.5385
10	2	0.25	0.00	0.90	0.9573	0.5428
10	2	0.25	0.00	0.95	0.9585	0.5090
10	2	0.25	0.05	0.30	1.0000	0.2456
10	2	0.25	0.05	0.50	0.9995	0.4402
10	2	0.25	0.05	0.70	0.9982	0.6259
10	2	0.25	0.05	0.90	0.9940	0.7626
10	2	0.25	0.05	0.95	0.9872	0.7611
10	2	0.25	0.10	0.30	1.0000	0.1970
10	2	0.25	0.10	0.50	1.0000	0.3929
10	2	0.25	0.10	0.70	0.9998	0.5858
10	2	0.25	0.10	0.90	0.9965	0.7494
10	2	0.25	0.30	0.50	1.0000	0.1979
10	2	0.25	0.30	0.70	1.0000	0.3947
10	2	0.5	0.00	0.10	0.9700	0.0954
10	2	0.5	0.00	0.30	0.9657	0.2719
10	2	0.5	0.00	0.50	0.9543	0.3978
10	2	0.5	0.00	0.70	0.9433	0.4275
10	2	0.5	0.00	0.90	0.9385	0.3312
10	2	0.5	0.00	0.95	0.9475	0.2892
10	2	0.5	0.05	0.30	0.9995	0.2345
10	2	0.5	0.05	0.50	0.9945	0.3936
10	2	0.5	0.05	0.70	0.9850	0.4817
10	2	0.5	0.05	0.90	0.9748	0.4635
10	2	0.5	0.05	0.95	0.9663	0.4346
10	2	0.5	0.10	0.30	1.0000	0.1897
10	2	0.5	0.10	0.50	0.9982	0.3656
10	2	0.5	0.10	0.70	0.9895	0.4741
10	2	0.5	0.10	0.90	0.9782	0.4817
10	2	0.5	0.30	0.50	1.0000	0.1913
10	2	0.5	0.30	0.70	0.9992	0.3708

a	n	σ_A	p_L	p_U	coverage	dist
10	2	1	0.00	0.10	0.9685	0.0890
10	2	1	0.00	0.30	0.9623	0.2373
10	2	1	0.00	0.50	0.9550	0.3130
10	2	1	0.00	0.70	0.9385	0.2920
10	2	1	0.00	0.90	0.9423	0.2257
10	2	1	0.00	0.95	0.9440	0.1933
10	2	1	0.05	0.30	0.9932	0.2019
10	2	1	0.05	0.50	0.9815	0.3017
10	2	1	0.05	0.70	0.9770	0.3196
10	2	1	0.05	0.90	0.9705	0.3079
10	2	1	0.05	0.95	0.9698	0.3026
10	2	1	0.10	0.30	0.9992	0.1637
10	2	1	0.10	0.50	0.9908	0.2753
10	2	1	0.10	0.70	0.9798	0.3094
10	2	1	0.10	0.90	0.9698	0.3153
10	2	1	0.30	0.50	1.0000	0.1536
10	2	1	0.30	0.70	0.9902	0.2483
10	2	2	0.00	0.10	0.9540	0.0764
10	2	2	0.00	0.30	0.9510	0.1830
10	2	2	0.00	0.50	0.9513	0.2275
10	2	2	0.00	0.70	0.9545	0.2307
10	2	2	0.00	0.90	0.9497	0.1971
10	2	2	0.00	0.95	0.9470	0.1642
10	2	2	0.05	0.30	0.9840	0.1512
10	2	2	0.05	0.50	0.9800	0.2184
10	2	2	0.05	0.70	0.9725	0.2532
10	2	2	0.05	0.90	0.9725	0.2667
10	2	2	0.05	0.95	0.9770	0.2560
10	2	2	0.10	0.30	0.9908	0.1200
10	2	2	0.10	0.50	0.9865	0.1989
10	2	2	0.10	0.70	0.9785	0.2446
10	2	2	0.10	0.90	0.9722	0.2726
10	2	2	0.30	0.50	0.9922	0.0994
10	2	2	0.30	0.70	0.9818	0.1752
10	2	4	0.00	0.10	0.9527	0.0731
10	2	4	0.00	0.30	0.9510	0.1628
10	2	4	0.00	0.50	0.9510	0.2051
10	2	4	0.00	0.70	0.9500	0.2191
10	2	4	0.00	0.90	0.9445	0.1866
10	2	4	0.00	0.95	0.9477	0.1596
10	2	4	0.05	0.30	0.9790	0.1344
10	2	4	0.05	0.50	0.9765	0.1965
10	2	4	0.05	0.70	0.9748	0.2374
10	2	4	0.05	0.90	0.9755	0.2539
10	2	4	0.05	0.95	0.9785	0.2444
10	2	4	0.10	0.30	0.9838	0.1031
10	2	4	0.10	0.50	0.9808	0.1777
10	2	4	0.10	0.70	0.9788	0.2315
10	2	4	0.10	0.90	0.9798	0.2591
10	2	4	0.30	0.50	0.9840	0.0874
10	2	4	0.30	0.70	0.9818	0.1654

continued on next page

Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	2	10	0.00	0.10	0.9543	0.0710
10	2	10	0.00	0.30	0.9507	0.1587
10	2	10	0.00	0.50	0.9505	0.1995
10	2	10	0.00	0.70	0.9517	0.2133
10	2	10	0.00	0.90	0.9543	0.1866
10	2	10	0.00	0.95	0.9490	0.1551
10	2	10	0.05	0.30	0.9798	0.1304
10	2	10	0.05	0.50	0.9765	0.1927
10	2	10	0.05	0.70	0.9725	0.2359
10	2	10	0.05	0.90	0.9762	0.2511
10	2	10	0.05	0.95	0.9805	0.2423
10	2	10	0.10	0.30	0.9842	0.1023
10	2	10	0.10	0.50	0.9778	0.1720
10	2	10	0.10	0.70	0.9805	0.2260
10	2	10	0.10	0.90	0.9805	0.2547
10	2	10	0.30	0.50	0.9832	0.0861
10	2	10	0.30	0.70	0.9835	0.1620
10	5	0.1	0.00	0.10	0.9197	0.1054
10	5	0.1	0.00	0.30	0.9245	0.2749
10	5	0.1	0.00	0.50	0.9515	0.4276
10	5	0.1	0.00	0.70	0.9690	0.5717
10	5	0.1	0.00	0.90	0.9712	0.6613
10	5	0.1	0.00	0.95	0.9688	0.6546
10	5	0.1	0.05	0.30	1.0000	0.2479
10	5	0.1	0.05	0.50	1.0000	0.4460
10	5	0.1	0.05	0.70	0.9998	0.6430
10	5	0.1	0.05	0.90	0.9985	0.8351
10	5	0.1	0.05	0.95	0.9990	0.8801
10	5	0.1	0.10	0.30	1.0000	0.1985
10	5	0.1	0.10	0.50	1.0000	0.3972
10	5	0.1	0.10	0.70	1.0000	0.5950
10	5	0.1	0.10	0.90	0.9998	0.7894
10	5	0.1	0.30	0.50	1.0000	0.1991
10	5	0.1	0.30	0.70	1.0000	0.3978
10	5	0.25	0.00	0.10	0.9660	0.0961
10	5	0.25	0.00	0.30	0.9580	0.2695
10	5	0.25	0.00	0.50	0.9470	0.4051
10	5	0.25	0.00	0.70	0.9457	0.4573
10	5	0.25	0.00	0.90	0.9483	0.3996
10	5	0.25	0.00	0.95	0.9333	0.3385
10	5	0.25	0.05	0.30	1.0000	0.2384
10	5	0.25	0.05	0.50	0.9975	0.4132
10	5	0.25	0.05	0.70	0.9888	0.5251
10	5	0.25	0.05	0.90	0.9775	0.5441
10	5	0.25	0.05	0.95	0.9710	0.5197
10	5	0.25	0.10	0.30	1.0000	0.1924
10	5	0.25	0.10	0.50	0.9978	0.3772
10	5	0.25	0.10	0.70	0.9922	0.5204
10	5	0.25	0.10	0.90	0.9810	0.5590
10	5	0.25	0.30	0.50	1.0000	0.1939
10	5	0.25	0.30	0.70	0.9998	0.3817

a	n	σ_A	p_L	p_U	coverage	dist
10	5	0.5	0.00	0.10	0.9563	0.0883
10	5	0.5	0.00	0.30	0.9635	0.2419
10	5	0.5	0.00	0.50	0.9485	0.3222
10	5	0.5	0.00	0.70	0.9525	0.3174
10	5	0.5	0.00	0.90	0.9423	0.2484
10	5	0.5	0.00	0.95	0.9445	0.2148
10	5	0.5	0.05	0.30	0.9968	0.2039
10	5	0.5	0.05	0.50	0.9882	0.3146
10	5	0.5	0.05	0.70	0.9758	0.3495
10	5	0.5	0.05	0.90	0.9752	0.3451
10	5	0.5	0.05	0.95	0.9720	0.3372
10	5	0.5	0.10	0.30	1.0000	0.1665
10	5	0.5	0.10	0.50	0.9922	0.2873
10	5	0.5	0.10	0.70	0.9848	0.3374
10	5	0.5	0.10	0.90	0.9770	0.3465
10	5	0.5	0.30	0.50	1.0000	0.1621
10	5	0.5	0.30	0.70	0.9935	0.2722
10	5	1	0.00	0.10	0.9540	0.0788
10	5	1	0.00	0.30	0.9550	0.1821
10	5	1	0.00	0.50	0.9475	0.2283
10	5	1	0.00	0.70	0.9495	0.2374
10	5	1	0.00	0.90	0.9473	0.1978
10	5	1	0.00	0.95	0.9427	0.1712
10	5	1	0.05	0.30	0.9868	0.1552
10	5	1	0.05	0.50	0.9820	0.2228
10	5	1	0.05	0.70	0.9725	0.2599
10	5	1	0.05	0.90	0.9785	0.2797
10	5	1	0.05	0.95	0.9725	0.2651
10	5	1	0.10	0.30	0.9892	0.1214
10	5	1	0.10	0.50	0.9845	0.1989
10	5	1	0.10	0.70	0.9780	0.2497
10	5	1	0.10	0.90	0.9785	0.2815
10	5	1	0.30	0.50	0.9960	0.1015
10	5	1	0.30	0.70	0.9868	0.1818
10	5	2	0.00	0.10	0.9517	0.0729
10	5	2	0.00	0.30	0.9490	0.1619
10	5	2	0.00	0.50	0.9505	0.2067
10	5	2	0.00	0.70	0.9475	0.2204
10	5	2	0.00	0.90	0.9460	0.1855
10	5	2	0.00	0.95	0.9580	0.1622
10	5	2	0.05	0.30	0.9832	0.1358
10	5	2	0.05	0.50	0.9798	0.1983
10	5	2	0.05	0.70	0.9770	0.2433
10	5	2	0.05	0.90	0.9772	0.2608
10	5	2	0.05	0.95	0.9775	0.2463
10	5	2	0.10	0.30	0.9860	0.1068
10	5	2	0.10	0.50	0.9822	0.1776
10	5	2	0.10	0.70	0.9820	0.2332
10	5	2	0.10	0.90	0.9808	0.2646
10	5	2	0.30	0.50	0.9850	0.0882
10	5	2	0.30	0.70	0.9835	0.1675

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	5	4	0.00	0.10	0.9463	0.0704
10	5	4	0.00	0.30	0.9497	0.1573
10	5	4	0.00	0.50	0.9495	0.2007
10	5	4	0.00	0.70	0.9473	0.2179
10	5	4	0.00	0.90	0.9495	0.1869
10	5	4	0.00	0.95	0.9497	0.1575
10	5	4	0.05	0.30	0.9795	0.1319
10	5	4	0.05	0.50	0.9805	0.1925
10	5	4	0.05	0.70	0.9738	0.2362
10	5	4	0.05	0.90	0.9725	0.2481
10	5	4	0.05	0.95	0.9735	0.2413
10	5	4	0.10	0.30	0.9825	0.1020
10	5	4	0.10	0.50	0.9792	0.1737
10	5	4	0.10	0.70	0.9768	0.2273
10	5	4	0.10	0.90	0.9788	0.2574
10	5	4	0.30	0.50	0.9818	0.0865
10	5	4	0.30	0.70	0.9802	0.1649
10	5	10	0.00	0.10	0.9535	0.0716
10	5	10	0.00	0.30	0.9513	0.1573
10	5	10	0.00	0.50	0.9505	0.2017
10	5	10	0.00	0.70	0.9480	0.2158
10	5	10	0.00	0.90	0.9577	0.1849
10	5	10	0.00	0.95	0.9510	0.1554
10	5	10	0.05	0.30	0.9770	0.1288
10	5	10	0.05	0.50	0.9768	0.1951
10	5	10	0.05	0.70	0.9762	0.2364
10	5	10	0.05	0.90	0.9775	0.2524
10	5	10	0.05	0.95	0.9722	0.2389
10	5	10	0.10	0.30	0.9790	0.1004
10	5	10	0.10	0.50	0.9855	0.1723
10	5	10	0.10	0.70	0.9768	0.2252
10	5	10	0.10	0.90	0.9772	0.2568
10	5	10	0.30	0.50	0.9830	0.0854
10	5	10	0.30	0.70	0.9798	0.1608
10	10	0.1	0.00	0.10	0.9483	0.1006
10	10	0.1	0.00	0.30	0.9470	0.2768
10	10	0.1	0.00	0.50	0.9525	0.4270
10	10	0.1	0.00	0.70	0.9565	0.5352
10	10	0.1	0.00	0.90	0.9573	0.5762
10	10	0.1	0.00	0.95	0.9597	0.5391
10	10	0.1	0.05	0.30	1.0000	0.2461
10	10	0.1	0.05	0.50	1.0000	0.4417
10	10	0.1	0.05	0.70	0.9992	0.6332
10	10	0.1	0.05	0.90	0.9928	0.7879
10	10	0.1	0.05	0.95	0.9912	0.7944
10	10	0.1	0.10	0.30	1.0000	0.1976
10	10	0.1	0.10	0.50	1.0000	0.3947
10	10	0.1	0.10	0.70	0.9998	0.5888
10	10	0.1	0.10	0.90	0.9960	0.7671
10	10	0.1	0.30	0.50	1.0000	0.1983
10	10	0.1	0.30	0.70	1.0000	0.3961

a	n	σ_A	p_L	p_U	coverage	dist
10	10	0.25	0.00	0.10	0.9610	0.0923
10	10	0.25	0.00	0.30	0.9583	0.2578
10	10	0.25	0.00	0.50	0.9480	0.3649
10	10	0.25	0.00	0.70	0.9447	0.3859
10	10	0.25	0.00	0.90	0.9380	0.3020
10	10	0.25	0.00	0.95	0.9365	0.2610
10	10	0.25	0.05	0.30	0.9995	0.2232
10	10	0.25	0.05	0.50	0.9938	0.3630
10	10	0.25	0.05	0.70	0.9865	0.4334
10	10	0.25	0.05	0.90	0.9692	0.4185
10	10	0.25	0.05	0.95	0.9720	0.4027
10	10	0.25	0.10	0.30	1.0000	0.1828
10	10	0.25	0.10	0.50	0.9970	0.3381
10	10	0.25	0.10	0.70	0.9890	0.4184
10	10	0.25	0.10	0.90	0.9780	0.4309
10	10	0.25	0.30	0.50	1.0000	0.1826
10	10	0.25	0.30	0.70	0.9992	0.3434
10	10	0.5	0.00	0.10	0.9610	0.0828
10	10	0.5	0.00	0.30	0.9580	0.2042
10	10	0.5	0.00	0.50	0.9470	0.2646
10	10	0.5	0.00	0.70	0.9483	0.2586
10	10	0.5	0.00	0.90	0.9410	0.2193
10	10	0.5	0.00	0.95	0.9480	0.1895
10	10	0.5	0.05	0.30	0.9930	0.1735
10	10	0.5	0.05	0.50	0.9855	0.2546
10	10	0.5	0.05	0.70	0.9800	0.2898
10	10	0.5	0.05	0.90	0.9735	0.3031
10	10	0.5	0.05	0.95	0.9745	0.2915
10	10	0.5	0.10	0.30	0.9952	0.1386
10	10	0.5	0.10	0.50	0.9875	0.2335
10	10	0.5	0.10	0.70	0.9848	0.2822
10	10	0.5	0.10	0.90	0.9802	0.3067
10	10	0.5	0.30	0.50	0.9995	0.1237
10	10	0.5	0.30	0.70	0.9875	0.2088
10	10	1	0.00	0.10	0.9553	0.0745
10	10	1	0.00	0.30	0.9517	0.1671
10	10	1	0.00	0.50	0.9483	0.2117
10	10	1	0.00	0.70	0.9510	0.2241
10	10	1	0.00	0.90	0.9463	0.1950
10	10	1	0.00	0.95	0.9535	0.1655
10	10	1	0.05	0.30	0.9835	0.1407
10	10	1	0.05	0.50	0.9818	0.2029
10	10	1	0.05	0.70	0.9768	0.2479
10	10	1	0.05	0.90	0.9775	0.2629
10	10	1	0.05	0.95	0.9730	0.2533
10	10	1	0.10	0.30	0.9855	0.1096
10	10	1	0.10	0.50	0.9845	0.1855
10	10	1	0.10	0.70	0.9815	0.2379
10	10	1	0.10	0.90	0.9768	0.2692
10	10	1	0.30	0.50	0.9865	0.0917
10	10	1	0.30	0.70	0.9868	0.1721

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Table B.3: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for small values of a and n and Other Percentile Pairs.

continued from previous page						
a	n	σ_A	p_L	p_U	coverage	dist
10	10	2	0.00	0.10	0.9493	0.0718
10	10	2	0.00	0.30	0.9543	0.1612
10	10	2	0.00	0.50	0.9457	0.2012
10	10	2	0.00	0.70	0.9537	0.2177
10	10	2	0.00	0.90	0.9480	0.1857
10	10	2	0.00	0.95	0.9500	0.1557
10	10	2	0.05	0.30	0.9808	0.1314
10	10	2	0.05	0.50	0.9740	0.1946
10	10	2	0.05	0.70	0.9748	0.2386
10	10	2	0.05	0.90	0.9792	0.2523
10	10	2	0.05	0.95	0.9752	0.2438
10	10	2	0.10	0.30	0.9860	0.1030
10	10	2	0.10	0.50	0.9830	0.1756
10	10	2	0.10	0.70	0.9760	0.2289
10	10	2	0.10	0.90	0.9750	0.2573
10	10	2	0.30	0.50	0.9875	0.0864
10	10	2	0.30	0.70	0.9842	0.1644
10	10	4	0.00	0.10	0.9545	0.0716
10	10	4	0.00	0.30	0.9483	0.1567
10	10	4	0.00	0.50	0.9485	0.1994
10	10	4	0.00	0.70	0.9517	0.2162
10	10	4	0.00	0.90	0.9473	0.1843
10	10	4	0.00	0.95	0.9565	0.1550
10	10	4	0.05	0.30	0.9795	0.1300
10	10	4	0.05	0.50	0.9760	0.1923
10	10	4	0.05	0.70	0.9782	0.2390
10	10	4	0.05	0.90	0.9735	0.2512
10	10	4	0.05	0.95	0.9745	0.2407
10	10	4	0.10	0.30	0.9830	0.1020
10	10	4	0.10	0.50	0.9788	0.1739
10	10	4	0.10	0.70	0.9788	0.2250
10	10	4	0.10	0.90	0.9770	0.2593
10	10	4	0.30	0.50	0.9868	0.0863
10	10	4	0.30	0.70	0.9820	0.1627
10	10	10	0.00	0.10	0.9500	0.0703
10	10	10	0.00	0.30	0.9467	0.1568
10	10	10	0.00	0.50	0.9463	0.2003
10	10	10	0.00	0.70	0.9555	0.2128
10	10	10	0.00	0.90	0.9467	0.1869
10	10	10	0.00	0.95	0.9487	0.1543
10	10	10	0.05	0.30	0.9818	0.1297
10	10	10	0.05	0.50	0.9785	0.1916
10	10	10	0.05	0.70	0.9762	0.2352
10	10	10	0.05	0.90	0.9770	0.2490
10	10	10	0.05	0.95	0.9798	0.2388
10	10	10	0.10	0.30	0.9865	0.1035
10	10	10	0.10	0.50	0.9822	0.1717
10	10	10	0.10	0.70	0.9790	0.2269
10	10	10	0.10	0.90	0.9782	0.2574
10	10	10	0.30	0.50	0.9845	0.0855
10	10	10	0.30	0.70	0.9818	0.1616

Table B.4: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for Asymptotics and Other Percentile Pairs.

a	n	σ_A	p_L	p_U	coverage	dist
20	10	0.25	0.00	0.10	0.9603	0.0888
20	10	0.25	0.00	0.30	0.9600	0.2390
20	10	0.25	0.00	0.50	0.9513	0.3103
20	10	0.25	0.00	0.70	0.9403	0.2725
20	10	0.25	0.00	0.90	0.9483	0.2047
20	10	0.25	0.05	0.30	0.9968	0.2041
20	10	0.25	0.05	0.50	0.9878	0.3010
20	10	0.25	0.05	0.70	0.9758	0.3049
20	10	0.25	0.05	0.90	0.9702	0.2899
20	10	0.25	0.05	0.95	0.9730	0.2786
20	10	0.25	0.10	0.30	1.0000	0.1642
20	10	0.25	0.10	0.50	0.9945	0.2750
20	10	0.25	0.10	0.70	0.9812	0.2979
20	10	0.25	0.10	0.90	0.9730	0.2974
20	10	0.25	0.30	0.50	1.0000	0.1560
20	10	0.25	0.30	0.70	0.9962	0.2465
20	10	1	0.00	0.10	0.9540	0.0635
20	10	1	0.00	0.30	0.9473	0.1296
20	10	1	0.00	0.50	0.9515	0.1523
20	10	1	0.00	0.70	0.9523	0.1564
20	10	1	0.00	0.90	0.9515	0.1270
20	10	1	0.05	0.30	0.9750	0.1019
20	10	1	0.05	0.50	0.9755	0.1409
20	10	1	0.05	0.70	0.9708	0.1648
20	10	1	0.05	0.90	0.9710	0.1677
20	10	1	0.05	0.95	0.9722	0.1562
20	10	1	0.10	0.30	0.9778	0.0766
20	10	1	0.10	0.50	0.9710	0.1246
20	10	1	0.10	0.70	0.9738	0.1576
20	10	1	0.10	0.90	0.9748	0.1753
20	10	1	0.30	0.50	0.9808	0.0619
20	10	1	0.30	0.70	0.9782	0.1152
20	10	4	0.00	0.10	0.9510	0.0603
20	10	4	0.00	0.30	0.9515	0.1220
20	10	4	0.00	0.50	0.9533	0.1472
20	10	4	0.00	0.70	0.9445	0.1516
20	10	4	0.00	0.90	0.9447	0.1202
20	10	4	0.05	0.30	0.9705	0.0947
20	10	4	0.05	0.50	0.9725	0.1340
20	10	4	0.05	0.70	0.9725	0.1566
20	10	4	0.05	0.90	0.9762	0.1586
20	10	4	0.05	0.95	0.9705	0.1444
20	10	4	0.10	0.30	0.9730	0.0723
20	10	4	0.10	0.50	0.9732	0.1167
20	10	4	0.10	0.70	0.9700	0.1508
20	10	4	0.10	0.90	0.9742	0.1623
20	10	4	0.30	0.50	0.9748	0.0575
20	10	4	0.30	0.70	0.9792	0.1092
20	10	10	0.00	0.10	0.9477	0.0596
20	10	10	0.00	0.30	0.9503	0.1216
20	10	10	0.00	0.50	0.9575	0.1477
20	10	10	0.00	0.70	0.9430	0.1484
20	10	10	0.00	0.90	0.9513	0.1185
20	10	10	0.05	0.30	0.9710	0.0951
20	10	10	0.05	0.50	0.9718	0.1340
20	10	10	0.05	0.70	0.9700	0.1572
20	10	10	0.05	0.90	0.9705	0.1551
20	10	10	0.05	0.95	0.9690	0.1463
20	10	10	0.10	0.30	0.9738	0.0719
20	10	10	0.10	0.50	0.9740	0.1179
20	10	10	0.10	0.70	0.9685	0.1479
20	10	10	0.10	0.90	0.9785	0.1650
20	10	10	0.30	0.50	0.9718	0.0581
20	10	10	0.30	0.70	0.9760	0.1089
50	10	0.25	0.00	0.10	0.9565	0.0799
50	10	0.25	0.00	0.30	0.9615	0.1901
50	10	0.25	0.00	0.50	0.9493	0.2109
50	10	0.25	0.00	0.70	0.9433	0.1546
50	10	0.25	0.00	0.90	0.9485	0.1265
50	10	0.25	0.05	0.30	0.9948	0.1597
50	10	0.25	0.05	0.50	0.9798	0.1944
50	10	0.25	0.05	0.70	0.9728	0.1767
50	10	0.25	0.05	0.90	0.9675	0.1823
50	10	0.25	0.05	0.95	0.9620	0.1706
50	10	0.25	0.10	0.30	0.9975	0.1213
50	10	0.25	0.10	0.50	0.9898	0.1769
50	10	0.25	0.10	0.70	0.9677	0.1740
50	10	0.25	0.10	0.90	0.9655	0.1882
50	10	0.25	0.30	0.50	1.0000	0.0949
50	10	0.25	0.30	0.70	0.9815	0.1314
50	10	1	0.00	0.10	0.9485	0.0471
50	10	1	0.00	0.30	0.9525	0.0885
50	10	1	0.00	0.50	0.9475	0.0992
50	10	1	0.00	0.70	0.9495	0.0975
50	10	1	0.00	0.90	0.9513	0.0732
50	10	1	0.05	0.30	0.9645	0.0651
50	10	1	0.05	0.50	0.9692	0.0857
50	10	1	0.05	0.70	0.9683	0.0995
50	10	1	0.05	0.90	0.9660	0.0958
50	10	1	0.05	0.95	0.9645	0.0855
50	10	1	0.10	0.30	0.9673	0.0473
50	10	1	0.10	0.50	0.9728	0.0759
50	10	1	0.10	0.70	0.9665	0.0940
50	10	1	0.10	0.90	0.9635	0.0999
50	10	1	0.30	0.50	0.9708	0.0374
50	10	1	0.30	0.70	0.9640	0.0709

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Table B.4: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for Asymptotics and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
50	10	4	0.00	0.10	0.9503	0.0442
50	10	4	0.00	0.30	0.9503	0.0813
50	10	4	0.00	0.50	0.9490	0.0948
50	10	4	0.00	0.70	0.9463	0.0934
50	10	4	0.00	0.90	0.9540	0.0695
50	10	4	0.05	0.30	0.9657	0.0605
50	10	4	0.05	0.50	0.9620	0.0819
50	10	4	0.05	0.70	0.9665	0.0942
50	10	4	0.05	0.90	0.9645	0.0884
50	10	4	0.05	0.95	0.9685	0.0790
50	10	4	0.10	0.30	0.9712	0.0453
50	10	4	0.10	0.50	0.9700	0.0725
50	10	4	0.10	0.70	0.9640	0.0898
50	10	4	0.10	0.90	0.9673	0.0937
50	10	4	0.30	0.50	0.9712	0.0355
50	10	4	0.30	0.70	0.9673	0.0659
50	10	10	0.00	0.10	0.9555	0.0447
50	10	10	0.00	0.30	0.9450	0.0814
50	10	10	0.00	0.50	0.9540	0.0936
50	10	10	0.00	0.70	0.9467	0.0929
50	10	10	0.00	0.90	0.9515	0.0694
50	10	10	0.05	0.30	0.9615	0.0611
50	10	10	0.05	0.50	0.9605	0.0818
50	10	10	0.05	0.70	0.9633	0.0944
50	10	10	0.05	0.90	0.9657	0.0888
50	10	10	0.05	0.95	0.9610	0.0787
50	10	10	0.10	0.30	0.9670	0.0450
50	10	10	0.10	0.50	0.9655	0.0709
50	10	10	0.10	0.70	0.9692	0.0909
50	10	10	0.10	0.90	0.9677	0.0946
50	10	10	0.30	0.50	0.9698	0.0355
50	10	10	0.30	0.70	0.9667	0.0648
100	10	0.25	0.00	0.10	0.9535	0.0694
100	10	0.25	0.00	0.30	0.9557	0.1347
100	10	0.25	0.00	0.50	0.9513	0.1281
100	10	0.25	0.00	0.70	0.9423	0.1065
100	10	0.25	0.00	0.90	0.9445	0.0881
100	10	0.25	0.05	0.30	0.9850	0.1050
100	10	0.25	0.05	0.50	0.9822	0.1213
100	10	0.25	0.05	0.70	0.9683	0.1228
100	10	0.25	0.05	0.90	0.9623	0.1285
100	10	0.25	0.05	0.95	0.9653	0.1199
100	10	0.25	0.10	0.30	0.9902	0.0803
100	10	0.25	0.10	0.50	0.9855	0.1078
100	10	0.25	0.10	0.70	0.9563	0.1205
100	10	0.25	0.10	0.90	0.9653	0.1364
100	10	0.25	0.30	0.50	0.9865	0.0538
100	10	0.25	0.30	0.70	0.9667	0.0922

a	n	σ_A	p_L	p_U	coverage	dist
100	10	1	0.00	0.10	0.9523	0.0361
100	10	1	0.00	0.30	0.9490	0.0631
100	10	1	0.00	0.50	0.9573	0.0698
100	10	1	0.00	0.70	0.9507	0.0682
100	10	1	0.00	0.90	0.9460	0.0490
100	10	1	0.05	0.30	0.9593	0.0453
100	10	1	0.05	0.50	0.9565	0.0605
100	10	1	0.05	0.70	0.9597	0.0682
100	10	1	0.05	0.90	0.9637	0.0625
100	10	1	0.05	0.95	0.9660	0.0560
100	10	1	0.10	0.30	0.9645	0.0332
100	10	1	0.10	0.50	0.9615	0.0519
100	10	1	0.10	0.70	0.9657	0.0662
100	10	1	0.10	0.90	0.9620	0.0675
100	10	1	0.30	0.50	0.9653	0.0259
100	10	1	0.30	0.70	0.9590	0.0497
100	10	4	0.00	0.10	0.9470	0.0341
100	10	4	0.00	0.30	0.9520	0.0598
100	10	4	0.00	0.50	0.9510	0.0672
100	10	4	0.00	0.70	0.9480	0.0659
100	10	4	0.00	0.90	0.9525	0.0462
100	10	4	0.05	0.30	0.9623	0.0424
100	10	4	0.05	0.50	0.9563	0.0567
100	10	4	0.05	0.70	0.9587	0.0641
100	10	4	0.05	0.90	0.9653	0.0594
100	10	4	0.05	0.95	0.9593	0.0510
100	10	4	0.10	0.30	0.9627	0.0309
100	10	4	0.10	0.50	0.9660	0.0495
100	10	4	0.10	0.70	0.9573	0.0612
100	10	4	0.10	0.90	0.9617	0.0630
100	10	4	0.30	0.50	0.9623	0.0246
100	10	4	0.30	0.70	0.9607	0.0456
100	10	10	0.00	0.10	0.9467	0.0336
100	10	10	0.00	0.30	0.9500	0.0594
100	10	10	0.00	0.50	0.9455	0.0653
100	10	10	0.00	0.70	0.9560	0.0659
100	10	10	0.00	0.90	0.9550	0.0461
100	10	10	0.05	0.30	0.9595	0.0427
100	10	10	0.05	0.50	0.9607	0.0571
100	10	10	0.05	0.70	0.9635	0.0650
100	10	10	0.05	0.90	0.9613	0.0582
100	10	10	0.05	0.95	0.9643	0.0506
100	10	10	0.10	0.30	0.9607	0.0312
100	10	10	0.10	0.50	0.9603	0.0488
100	10	10	0.10	0.70	0.9655	0.0611
100	10	10	0.10	0.90	0.9627	0.0622
100	10	10	0.30	0.50	0.9647	0.0245
100	10	10	0.30	0.70	0.9657	0.0462

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Table B.4: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of $\mu + A$ for Asymptotics and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
500	10	0.25	0.00	0.10	0.9510	0.0369
500	10	0.25	0.00	0.30	0.9565	0.0524
500	10	0.25	0.00	0.50	0.9493	0.0491
500	10	0.25	0.00	0.70	0.9427	0.0484
500	10	0.25	0.00	0.90	0.9455	0.0390
500	10	0.25	0.05	0.30	0.9643	0.0348
500	10	0.25	0.05	0.50	0.9640	0.0438
500	10	0.25	0.05	0.70	0.9545	0.0559
500	10	0.25	0.05	0.90	0.9607	0.0573
500	10	0.25	0.05	0.95	0.9543	0.0508
500	10	0.25	0.10	0.30	0.9725	0.0244
500	10	0.25	0.10	0.50	0.9610	0.0406
500	10	0.25	0.10	0.70	0.9575	0.0571
500	10	0.25	0.10	0.90	0.9530	0.0623
500	10	0.25	0.30	0.50	0.9570	0.0235
500	10	0.25	0.30	0.70	0.9535	0.0460
500	10	1	0.00	0.10	0.9480	0.0179
500	10	1	0.00	0.30	0.9517	0.0291
500	10	1	0.00	0.50	0.9495	0.0315
500	10	1	0.00	0.70	0.9573	0.0297
500	10	1	0.00	0.90	0.9437	0.0203
500	10	1	0.05	0.30	0.9583	0.0201
500	10	1	0.05	0.50	0.9527	0.0258
500	10	1	0.05	0.70	0.9527	0.0294
500	10	1	0.05	0.90	0.9523	0.0257
500	10	1	0.05	0.95	0.9570	0.0221
500	10	1	0.10	0.30	0.9495	0.0141
500	10	1	0.10	0.50	0.9535	0.0223
500	10	1	0.10	0.70	0.9543	0.0285
500	10	1	0.10	0.90	0.9575	0.0282
500	10	1	0.30	0.50	0.9600	0.0117
500	10	1	0.30	0.70	0.9590	0.0216

a	n	σ_A	p_L	p_U	coverage	dist
500	10	4	0.00	0.10	0.9473	0.0164
500	10	4	0.00	0.30	0.9520	0.0273
500	10	4	0.00	0.50	0.9490	0.0300
500	10	4	0.00	0.70	0.9520	0.0288
500	10	4	0.00	0.90	0.9470	0.0196
500	10	4	0.05	0.30	0.9520	0.0188
500	10	4	0.05	0.50	0.9545	0.0245
500	10	4	0.05	0.70	0.9590	0.0282
500	10	4	0.05	0.90	0.9585	0.0237
500	10	4	0.05	0.95	0.9600	0.0200
500	10	4	0.10	0.30	0.9565	0.0136
500	10	4	0.10	0.50	0.9553	0.0213
500	10	4	0.10	0.70	0.9553	0.0266
500	10	4	0.10	0.90	0.9537	0.0262
500	10	4	0.30	0.50	0.9553	0.0106
500	10	4	0.30	0.70	0.9557	0.0202
500	10	10	0.00	0.10	0.9483	0.0166
500	10	10	0.00	0.30	0.9517	0.0276
500	10	10	0.00	0.50	0.9537	0.0303
500	10	10	0.00	0.70	0.9547	0.0287
500	10	10	0.00	0.90	0.9537	0.0190
500	10	10	0.05	0.30	0.9575	0.0188
500	10	10	0.05	0.50	0.9550	0.0245
500	10	10	0.05	0.70	0.9557	0.0275
500	10	10	0.05	0.90	0.9557	0.0235
500	10	10	0.05	0.95	0.9560	0.0204
500	10	10	0.10	0.30	0.9567	0.0134
500	10	10	0.10	0.50	0.9527	0.0214
500	10	10	0.10	0.70	0.9653	0.0264
500	10	10	0.10	0.90	0.9535	0.0256
500	10	10	0.30	0.50	0.9530	0.0107
500	10	10	0.30	0.70	0.9530	0.0197

Table B.5: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

a	n	σ_A	p_L	p_U	coverage	dist
2	2	0	0.000	0.950	0.9948	0.4833
2	2	0	0.000	0.990	0.9965	0.4610
2	2	0	0.025	0.975	0.9978	0.7627
2	2	0	0.005	0.995	0.9980	0.7482
2	2	0.1	0.000	0.950	0.9960	0.4831
2	2	0.1	0.000	0.990	0.9980	0.4612
2	2	0.1	0.025	0.975	0.9988	0.7622
2	2	0.1	0.005	0.995	0.9985	0.7487
2	2	0.25	0.000	0.950	0.9950	0.4828
2	2	0.25	0.000	0.990	0.9948	0.4597
2	2	0.25	0.025	0.975	0.9980	0.7663
2	2	0.25	0.005	0.995	0.9982	0.7531
2	2	0.5	0.000	0.950	0.9950	0.4961
2	2	0.5	0.000	0.990	0.9978	0.4724
2	2	0.5	0.025	0.975	0.9978	0.7686
2	2	0.5	0.005	0.995	0.9972	0.7562
2	2	1	0.000	0.950	0.9940	0.5092
2	2	1	0.000	0.990	0.9950	0.4825
2	2	1	0.025	0.975	0.9972	0.7713
2	2	1	0.005	0.995	0.9955	0.7588
2	2	2	0.000	0.950	0.9858	0.5119
2	2	2	0.000	0.990	0.9892	0.4910
2	2	2	0.025	0.975	0.9928	0.7598
2	2	2	0.005	0.995	0.9938	0.7574
2	2	4	0.000	0.950	0.9770	0.5219
2	2	4	0.000	0.990	0.9712	0.4848
2	2	4	0.025	0.975	0.9835	0.7597
2	2	4	0.005	0.995	0.9845	0.7489
2	2	10	0.000	0.950	0.9547	0.5159
2	2	10	0.000	0.990	0.9555	0.4882
2	2	10	0.025	0.975	0.9742	0.7594
2	2	10	0.005	0.995	0.9660	0.7540
2	5	0	0.000	0.950	0.9932	0.3850
2	5	0	0.000	0.990	0.9960	0.3444
2	5	0	0.025	0.975	0.9962	0.6303
2	5	0	0.005	0.995	0.9975	0.5905
2	5	0.1	0.000	0.950	0.9945	0.3864
2	5	0.1	0.000	0.990	0.9945	0.3446
2	5	0.1	0.025	0.975	0.9958	0.6342
2	5	0.1	0.005	0.995	0.9952	0.6044
2	5	0.25	0.000	0.950	0.9910	0.3997
2	5	0.25	0.000	0.990	0.9932	0.3581
2	5	0.25	0.025	0.975	0.9968	0.6463
2	5	0.25	0.005	0.995	0.9952	0.6134
2	5	0.5	0.000	0.950	0.9892	0.4293
2	5	0.5	0.000	0.990	0.9908	0.3978
2	5	0.5	0.025	0.975	0.9960	0.6843
2	5	0.5	0.005	0.995	0.9950	0.6518
2	5	1	0.000	0.950	0.9745	0.4680
2	5	1	0.000	0.990	0.9818	0.4415
2	5	1	0.025	0.975	0.9932	0.7147
2	5	1	0.005	0.995	0.9902	0.6892
2	5	2	0.000	0.950	0.9593	0.5007
2	5	2	0.000	0.990	0.9607	0.4667
2	5	2	0.025	0.975	0.9735	0.7349
2	5	2	0.005	0.995	0.9712	0.7289
2	5	4	0.000	0.950	0.9627	0.5142
2	5	4	0.000	0.990	0.9537	0.4803
2	5	4	0.025	0.975	0.9625	0.7512
2	5	4	0.005	0.995	0.9677	0.7411
2	5	10	0.000	0.950	0.9510	0.5192
2	5	10	0.000	0.990	0.9437	0.4830
2	5	10	0.025	0.975	0.9653	0.7564
2	5	10	0.005	0.995	0.9615	0.7483
2	10	0	0.000	0.950	0.9928	0.3125
2	10	0	0.000	0.990	0.9945	0.2744
2	10	0	0.025	0.975	0.9955	0.5393
2	10	0	0.005	0.995	0.9960	0.4910
2	10	0.1	0.000	0.950	0.9895	0.3252
2	10	0.1	0.000	0.990	0.9935	0.2815
2	10	0.1	0.025	0.975	0.9965	0.5501
2	10	0.1	0.005	0.995	0.9950	0.4969
2	10	0.25	0.000	0.950	0.9872	0.3504
2	10	0.25	0.000	0.990	0.9895	0.3113
2	10	0.25	0.025	0.975	0.9968	0.5881
2	10	0.25	0.005	0.995	0.9970	0.5494
2	10	0.5	0.000	0.950	0.9805	0.3972
2	10	0.5	0.000	0.990	0.9825	0.3666
2	10	0.5	0.025	0.975	0.9968	0.6434
2	10	0.5	0.005	0.995	0.9958	0.6207
2	10	1	0.000	0.950	0.9702	0.4665
2	10	1	0.000	0.990	0.9705	0.4338
2	10	1	0.025	0.975	0.9852	0.7108
2	10	1	0.005	0.995	0.9852	0.6823
2	10	2	0.000	0.950	0.9565	0.4981
2	10	2	0.000	0.990	0.9490	0.4605
2	10	2	0.025	0.975	0.9710	0.7390
2	10	2	0.005	0.995	0.9623	0.7193
2	10	4	0.000	0.950	0.9513	0.5098
2	10	4	0.000	0.990	0.9517	0.4849
2	10	4	0.025	0.975	0.9665	0.7514
2	10	4	0.005	0.995	0.9610	0.7406
2	10	10	0.000	0.950	0.9533	0.5253
2	10	10	0.000	0.990	0.9467	0.4865
2	10	10	0.025	0.975	0.9633	0.7509
2	10	10	0.005	0.995	0.9635	0.7442
5	2	0	0.000	0.950	0.9868	0.2119
5	2	0	0.000	0.990	0.9910	0.1481
5	2	0	0.025	0.975	0.9938	0.3137
5	2	0	0.005	0.995	0.9940	0.2325
5	2	0.1	0.000	0.950	0.9882	0.2081
5	2	0.1	0.000	0.990	0.9892	0.1486
5	2	0.1	0.025	0.975	0.9942	0.3134
5	2	0.1	0.005	0.995	0.9950	0.2297

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Table B.5: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	2	0.25	0.000	0.950	0.9872	0.2145
5	2	0.25	0.000	0.990	0.9890	0.1510
5	2	0.25	0.025	0.975	0.9955	0.3162
5	2	0.25	0.005	0.995	0.9940	0.2318
5	2	0.5	0.000	0.950	0.9852	0.2244
5	2	0.5	0.000	0.990	0.9870	0.1577
5	2	0.5	0.025	0.975	0.9948	0.3268
5	2	0.5	0.005	0.995	0.9932	0.2413
5	2	1	0.000	0.950	0.9772	0.2333
5	2	1	0.000	0.990	0.9778	0.1666
5	2	1	0.025	0.975	0.9912	0.3427
5	2	1	0.005	0.995	0.9890	0.2554
5	2	2	0.000	0.950	0.9620	0.2495
5	2	2	0.000	0.990	0.9650	0.1805
5	2	2	0.025	0.975	0.9848	0.3542
5	2	2	0.005	0.995	0.9845	0.2726
5	2	4	0.000	0.950	0.9490	0.2547
5	2	4	0.000	0.990	0.9490	0.1859
5	2	4	0.025	0.975	0.9728	0.3681
5	2	4	0.005	0.995	0.9725	0.2849
5	2	10	0.000	0.950	0.9493	0.2619
5	2	10	0.000	0.990	0.9525	0.1917
5	2	10	0.025	0.975	0.9735	0.3660
5	2	10	0.005	0.995	0.9688	0.2849
5	5	0	0.000	0.950	0.9800	0.1180
5	5	0	0.000	0.990	0.9845	0.0672
5	5	0	0.025	0.975	0.9918	0.1591
5	5	0	0.005	0.995	0.9880	0.0942
5	5	0.1	0.000	0.950	0.9795	0.1185
5	5	0.1	0.000	0.990	0.9802	0.0685
5	5	0.1	0.025	0.975	0.9930	0.1630
5	5	0.1	0.005	0.995	0.9910	0.0980
5	5	0.25	0.000	0.950	0.9808	0.1300
5	5	0.25	0.000	0.990	0.9815	0.0750
5	5	0.25	0.025	0.975	0.9908	0.1770
5	5	0.25	0.005	0.995	0.9930	0.1098
5	5	0.5	0.000	0.950	0.9735	0.1587
5	5	0.5	0.000	0.990	0.9790	0.0993
5	5	0.5	0.025	0.975	0.9902	0.2199
5	5	0.5	0.005	0.995	0.9902	0.1414
5	5	1	0.000	0.950	0.9625	0.2045
5	5	1	0.000	0.990	0.9673	0.1384
5	5	1	0.025	0.975	0.9885	0.2877
5	5	1	0.005	0.995	0.9842	0.2039
5	5	2	0.000	0.950	0.9555	0.2401
5	5	2	0.000	0.990	0.9517	0.1689
5	5	2	0.025	0.975	0.9752	0.3371
5	5	2	0.005	0.995	0.9708	0.2521

a	n	σ_A	p_L	p_U	coverage	dist
5	5	4	0.000	0.950	0.9527	0.2548
5	5	4	0.000	0.990	0.9493	0.1867
5	5	4	0.025	0.975	0.9758	0.3632
5	5	4	0.005	0.995	0.9700	0.2786
5	5	10	0.000	0.950	0.9547	0.2623
5	5	10	0.000	0.990	0.9463	0.1878
5	5	10	0.025	0.975	0.9718	0.3735
5	5	10	0.005	0.995	0.9712	0.2882
5	10	0	0.000	0.950	0.9788	0.0738
5	10	0	0.000	0.990	0,9770	0.0356
5	10	0	0.025	0.975	0.9892	0.0938
5	10	0	0.005	0.995	0.9860	0.0468
5	10	0.1	0.000	0.950	0.9800	0.0782
5	10	0.1	0.000	0.990	0.9780	0.0384
5	10	0.1	0.025	0.975	0.9902	0.0985
5	10	0.1	0.005	0.995	0.9898	0.0503
5	10	0.25	0.000	0.950	0.9780	0.0919
5	10	0.25	0.000	0.990	0.9750	0.0486
5	10	0.25	0.025	0.975	0.9885	0.1205
5	10	0.25	0.005	0.995	0.9892	0.0649
5	10	0.5	0.000	0.950	0.9755	0.1330
5	10	0.5	0.000	0.990	0.9802	0.0772
5	10	0.5	0.025	0.975	0.9878	0.1763
5	10	0.5	0.005	0.995	0.9935	0.1103
5	10	1	0.000	0.950	0.9595	0.1934
5	10	1	0.000	0.990	0.9557	0.1254
5	10	1	0.025	0.975	0.9822	0.2700
5	10	1	0.005	0.995	0.9790	0.1907
5	10	2	0.000	0.950	0.9597	0.2402
5	10	2	0.000	0.990	0.9535	0.1662
5	10	2	0.025	0.975	0.9738	0.3364
5	10	2	0.005	0.995	0.9740	0.2541
5	10	4	0.000	0.950	0.9533	0.2563
5	10	4	0.000	0.990	0.9525	0.1843
5	10	4	0.025	0.975	0.9725	0.3645
5	10	4	0.005	0.995	0.9750	0.2767
5	10	10	0.000	0.950	0.9605	0.2628
5	10	10	0.000	0.990	0.9537	0.1908
5	10	10	0.025	0.975	0.9758	0.3711
5	10	10	0.005	0.995	0.9740	0.2892
10	2	0	0.000	0.950	0.9788	0.1156
10	2	0	0.000	0.990	0.9812	0.0665
10	2	0	0.025	0.975	0.9908	0.1572
10	2	0	0.005	0.995	0.9902	0.0921
10	2	0.1	0.000	0.950	0.9752	0.1159
10	2	0.1	0.000	0.990	0.9812	0.0667
10	2	0.1	0.025	0.975	0.9930	0.1560
10	2	0.1	0.005	0.995	0.9895	0.0918

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Table B.5: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and $(p_1, p_2) =$ $(0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

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a	n	σ_A	p_L	p_U	coverage	dist
10	2	0.25	0.000	0.950	0.9738	0.1196
10	2	0.25	0.000	0.990	0.9812	0.0673
10	2	0.25	0.025	0.975	0.9895	0.1590
10	2	0.25	0.005	0.995	0.9900	0.0926
10	2	0.5	0.000	0.950	0.9790	0.1245
10	2	0.5	0.000	0.990	0.9762	0.0693
10	2	0.5	0.025	0.975	0.9902	0.1640
10	2	0.5	0.005	0.995	0.9880	0.0970
10	2	1	0.000	0.950	0.9667	0.1341
10	2	1	0.000	0.990	0.9702	0.0775
10	2	1	0.025	0.975	0.9890	0.1802
10	2	1	0.005	0.995	0.9848	0.1089
10	2	2	0.000	0.950	0.9523	0.1472
10	2	2	0.000	0.990	0.9555	0.0876
10	2	2	0.025	0.975	0.9770	0.1960
10	2	2	0.005	0.995	0.9800	0.1206
10	2	4	0.000	0.950	0.9507	0.1552
10	2	4	0.000	0.990	0.9483	0.0896
10	2	4	0.025	0.975	0.9755	0.2027
10	2	4	0.005	0.995	0.9708	0.1272
10	2	10	0.000	0.950	0.9495	0.1575
10	2	10	0.000	0.990	0.9460	0.0949
10	2	10	0.025	0.975	0.9718	0.2062
10	2	10	0.005	0.995	0.9702	0.1302
10	5	0	0.000	0.950	0.9692	0.0618
10	5	0	0.000	0.990	0.9742	0.0285
10	5	0	0.025	0.975	0.9805	0.0765
10	5	0	0.005	0.995	0.9858	0.0354
10	5	0.1	0.000	0.950	0.9718	0.0625
10	5	0.1	0.000	0.990	0.9698	0.0288
10	5	0.1	0.025	0.975	0.9868	0.0776
10	5	0.1	0.005	0.995	0.9898	0.0367
10	5	0.25	0.000	0.950	0.9677	0.0675
10	5	0.25	0.000	0.990	0.9712	0.0327
10	5	0.25	0.025	0.975	0.9862	0.0825
10	5	0.25	0.005	0.995	0.9845	0.0396
10	5	0.5	0.000	0.950	0.9655	0.0822
10	5	0.5	0.000	0.990	0.9725	0.0410
10	5	0.5	0.025	0.975	0.9858	0.1033
10	5	0.5	0.005	0.995	0.9878	0.0514
10	5	1	0.000	0.950	0.9603	0.1128
10	5	1	0.000	0.990	0.9625	0.0607
10	5	1	0.025	0.975	0.9768	0.1414
10	5	1	0.005	0.995	0.9798	0.0818
10	5	2	0.000	0.950	0.9485	0.1374
10	5	2	0.000	0.990	0.9530	0.0813
10	5	2	0.025	0.975	0.9750	0.1829
10	5	2	0.005	0.995	0.9710	0.1130
10	5	4	0.000	0.950	0.9430	0.1493
10	5	4	0.000	0.990	0.9535	0.0908
10	5	4	0.025	0.975	0.9750	0.1990
10	5	4	0.005	0.995	0.9680	0.1240
10	5	10	0.000	0.950	0.9553	0.1571
10	5	10	0.000	0.990	0.9443	0.0922
10	5	10	0.025	0.975	0.9735	0.2079
10	5	10	0.005	0.995	0.9702	0.1293
10	10	0	0.000	0.950	0.9653	0.0393
10	10	0	0.000	0.990	0.9722	0.0163
10	10	0	0.025	0.975	0.9822	0.0445
10	10	0	0.005	0.995	0.9768	0.0186
10	10	0.1	0.000	0.950	0.9645	0.0408
10	10	0.1	0.000	0.990	0.9688	0.0170
10	10	0.1	0.025	0.975	0.9830	0.0467
10	10	0.1	0.005	0.995	0.9780	0.0193
10	10	0.25	0.000	0.950	0.9670	0.0473
10	10	0.25	0.000	0.990	0.9670	0.0201
10	10	0.25	0.025	0.975	0.9838	0.0533
10	10	0.25	0.005	0.995	0.9845	0.0230
10	10	0.5	0.000	0.950	0.9613	0.0676
10	10	0.5	0.000	0.990	0.9665	0.0307
10	10	0.5	0.025	0.975	0.9828	0.0764
10	10	0.5	0.005	0.995	0.9870	0.0368
10	10	1	0.000	0.950	0.9595	0.1065
10	10	1	0.000	0.990	0.9587	0.0567
10	10	1	0.025	0.975	0.9838	0.1343
10	10	1	0.005	0.995	0.9762	0.0734
10	10	2	0.000	0.950	0.9525	0.1369
10	10	2	0.000	0.990	0.9475	0.0769
10	10	2	0.025	0.975	0.9752	0.1811
10	10	2	0.005	0.995	0.9738	0.1103
10	10	4	0.000	0.950	0.9585	0.1500
10	10	4	0.000	0.990	0.9495	0.0891
10	10	4	0.025	0.975	0.9710	0.1987
10	10	4	0.005	0.995	0.9765	0.1266
10	10	10	0.000	0.950	0.9420	0.1530
10	10	10	0.000	0.990	0.9503	0.0936
10	10	10	0.025	0.975	0.9742	0.2070
10	10	10	0.005	0.995	0.9690	0.1315

Table B.6: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and $(p_1, p_2) = (0, 0.95)$, $(0, 0.99)$, $(0.025, 0.975)$, $(0.005, 0.995)$.

a	n	σ_A	p_L	p_U	coverage	dist	a	n	σ_A	p_L	p_U	coverage	dist
20	10	0	0.000	0.950	0.9585	0.0244	100	10	0	0.000	0.950	0.9535	0.0092
20	10	0	0.000	0.990	0.9637	0.0093	100	10	0	0.000	0.990	0.9547	0.0033
20	10	0	0.025	0.975	0.9690	0.0265	100	10	0	0.025	0.975	0.9610	0.0098
20	10	0	0.005	0.995	0.9673	0.0102	100	10	0	0.005	0.995	0.9477	0.0034
20	10	0.25	0.000	0.950	0.9540	0.0287	100	10	0.25	0.000	0.950	0.9507	0.0105
20	10	0.25	0.000	0.990	0.9637	0.0108	100	10	0.25	0.000	0.990	0.9535	0.0036
20	10	0.25	0.025	0.975	0.9728	0.0292	100	10	0.25	0.025	0.975	0.9620	0.0104
20	10	0.25	0.005	0.995	0.9725	0.0117	100	10	0.25	0.005	0.995	0.9625	0.0036
20	10	1	0.000	0.950	0.9593	0.0626	100	10	1	0.000	0.950	0.9505	0.0214
20	10	1	0.000	0.990	0.9537	0.0271	100	10	1	0.000	0.990	0.9530	0.0077
20	10	1	0.025	0.975	0.9705	0.0692	100	10	1	0.025	0.975	0.9665	0.0208
20	10	1	0.005	0.995	0.9735	0.0326	100	10	1	0.005	0.995	0.9630	0.0078
20	10	4	0.000	0.950	0.9443	0.0921	100	10	4	0.000	0.950	0.9557	0.0326
20	10	4	0.000	0.990	0.9517	0.0470	100	10	4	0.000	0.990	0.9540	0.0127
20	10	4	0.025	0.975	0.9708	0.1118	100	10	4	0.025	0.975	0.9560	0.0349
20	10	4	0.005	0.995	0.9720	0.0606	100	10	4	0.005	0.995	0.9627	0.0145
20	10	10	0.000	0.950	0.9573	0.0945	100	10	10	0.000	0.950	0.9450	0.0328
20	10	10	0.000	0.990	0.9480	0.0482	100	10	10	0.000	0.990	0.9470	0.0133
20	10	10	0.025	0.975	0.9722	0.1168	100	10	10	0.025	0.975	0.9593	0.0372
20	10	10	0.005	0.995	0.9725	0.0628	100	10	10	0.005	0.995	0.9660	0.0150
50	10	0	0.000	0.950	0.9565	0.0139	500	10	0	0.000	0.950	0.9483	0.0039
50	10	0	0.000	0.990	0.9560	0.0049	500	10	0	0.000	0.990	0.9447	0.0013
50	10	0	0.025	0.975	0.9643	0.0145	500	10	0	0.025	0.975	0.9567	0.0041
50	10	0	0.005	0.995	0.9633	0.0053	500	10	0	0.005	0.995	0.9560	0.0014
50	10	0.25	0.000	0.950	0.9587	0.0156	500	10	0.25	0.000	0.950	0.9577	0.0045
50	10	0.25	0.000	0.990	0.9547	0.0056	500	10	0.25	0.000	0.990	0.9487	0.0014
50	10	0.25	0.025	0.975	0.9677	0.0157	500	10	0.25	0.025	0.975	0.9547	0.0042
50	10	0.25	0.005	0.995	0.9647	0.0057	500	10	0.25	0.005	0.995	0.9560	0.0014
50	10	1	0.000	0.950	0.9563	0.0336	500	10	1	0.000	0.950	0.9555	0.0085
50	10	1	0.000	0.990	0.9530	0.0128	500	10	1	0.000	0.990	0.9510	0.0028
50	10	1	0.025	0.975	0.9692	0.0338	500	10	1	0.025	0.975	0.9605	0.0081
50	10	1	0.005	0.995	0.9660	0.0135	500	10	1	0.005	0.995	0.9565	0.0027
50	10	4	0.000	0.950	0.9515	0.0494	500	10	4	0.000	0.950	0.9463	0.0126
50	10	4	0.000	0.990	0.9503	0.0210	500	10	4	0.000	0.990	0.9587	0.0046
50	10	4	0.025	0.975	0.9667	0.0561	500	10	4	0.025	0.975	0.9537	0.0135
50	10	4	0.005	0.995	0.9635	0.0250	500	10	4	0.005	0.995	0.9583	0.0048
50	10	10	0.000	0.950	0.9493	0.0512	500	10	10	0.000	0.950	0.9465	0.0132
50	10	10	0.000	0.990	0.9535	0.0224	500	10	10	0.000	0.990	0.9447	0.0047
50	10	10	0.025	0.975	0.9685	0.0582	500	10	10	0.025	0.975	0.9577	0.0141
50	10	10	0.005	0.995	0.9655	0.0267	500	10	10	0.005	0.995	0.9583	0.0049

Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

a	n	σ_A	p_L	p_U	coverage	dist
2	2	0	0.00	0.10	0.9263	0.0792
2	2	0	0.00	0.30	0.9420	0.1980
2	2	0	0.00	0.50	0.9477	0.2882
2	2	0	0.00	0.70	0.9938	0.3835
2	2	0	0.00	0.90	0.9960	0.4728
2	2	0	0.00	0.95	0.9980	0.4832
2	2	0	0.05	0.30	0.9980	0.2195
2	2	0	0.05	0.50	0.9952	0.3864
2	2	0	0.05	0.70	0.9972	0.5504
2	2	0	0.05	0.90	0.9968	0.7085
2	2	0	0.05	0.95	0.9985	0.7435
2	2	0	0.10	0.30	0.9988	0.1772
2	2	0	0.10	0.50	0.9982	0.3496
2	2	0	0.10	0.70	0.9985	0.5180
2	2	0	0.10	0.90	0.9980	0.6773
2	2	0	0.30	0.50	0.9990	0.1782
2	2	0	0.30	0.70	0.9992	0.3516
2	2	0.1	0.00	0.10	0.9243	0.0785
2	2	0.1	0.00	0.30	0.9405	0.1992
2	2	0.1	0.00	0.50	0.9473	0.2889
2	2	0.1	0.00	0.70	0.9898	0.3811
2	2	0.1	0.00	0.90	0.9970	0.4745
2	2	0.1	0.00	0.95	0.9952	0.4772
2	2	0.1	0.05	0.30	0.9988	0.2193
2	2	0.1	0.05	0.50	0.9978	0.3882
2	2	0.1	0.05	0.70	0.9980	0.5502
2	2	0.1	0.05	0.90	0.9968	0.7082
2	2	0.1	0.05	0.95	0.9980	0.7417
2	2	0.1	0.10	0.30	0.9988	0.1772
2	2	0.1	0.10	0.50	0.9985	0.3495
2	2	0.1	0.10	0.70	0.9982	0.5186
2	2	0.1	0.10	0.90	0.9982	0.6793
2	2	0.1	0.30	0.50	0.9992	0.1783
2	2	0.1	0.30	0.70	0.9995	0.3509
2	2	0.25	0.00	0.10	0.9250	0.0788
2	2	0.25	0.00	0.30	0.9345	0.2002
2	2	0.25	0.00	0.50	0.9527	0.2940
2	2	0.25	0.00	0.70	0.9920	0.3868
2	2	0.25	0.00	0.90	0.9960	0.4761
2	2	0.25	0.00	0.95	0.9952	0.4845
2	2	0.25	0.05	0.30	0.9988	0.2202
2	2	0.25	0.05	0.50	0.9975	0.3871
2	2	0.25	0.05	0.70	0.9982	0.5525
2	2	0.25	0.05	0.90	0.9985	0.7101
2	2	0.25	0.05	0.95	0.9962	0.7434
2	2	0.25	0.10	0.30	0.9992	0.1778
2	2	0.25	0.10	0.50	0.9988	0.3510
2	2	0.25	0.10	0.70	0.9988	0.5160
2	2	0.25	0.10	0.90	0.9982	0.6781
2	2	0.25	0.30	0.50	0.9992	0.1784
2	2	0.25	0.30	0.70	0.9982	0.3511

a	n	σ_A	p_L	p_U	coverage	dist
2	2	0.5	0.00	0.10	0.9315	0.0801
2	2	0.5	0.00	0.30	0.9350	0.2025
2	2	0.5	0.00	0.50	0.9483	0.2981
2	2	0.5	0.00	0.70	0.9925	0.3981
2	2	0.5	0.00	0.90	0.9935	0.4820
2	2	0.5	0.00	0.95	0.9935	0.4922
2	2	0.5	0.05	0.30	0.9980	0.2220
2	2	0.5	0.05	0.50	0.9982	0.3910
2	2	0.5	0.05	0.70	0.9968	0.5549
2	2	0.5	0.05	0.90	0.9972	0.7150
2	2	0.5	0.05	0.95	0.9978	0.7459
2	2	0.5	0.10	0.30	0.9980	0.1790
2	2	0.5	0.10	0.50	0.9982	0.3512
2	2	0.5	0.10	0.70	0.9978	0.5186
2	2	0.5	0.10	0.90	0.9975	0.6799
2	2	0.5	0.30	0.50	0.9990	0.1790
2	2	0.5	0.30	0.70	0.9995	0.3540
2	2	1	0.00	0.10	0.9255	0.0826
2	2	1	0.00	0.30	0.9447	0.2122
2	2	1	0.00	0.50	0.9470	0.3177
2	2	1	0.00	0.70	0.9800	0.4126
2	2	1	0.00	0.90	0.9918	0.4992
2	2	1	0.00	0.95	0.9932	0.5079
2	2	1	0.05	0.30	0.9975	0.2236
2	2	1	0.05	0.50	0.9985	0.3952
2	2	1	0.05	0.70	0.9978	0.5579
2	2	1	0.05	0.90	0.9968	0.7116
2	2	1	0.05	0.95	0.9972	0.7489
2	2	1	0.10	0.30	0.9978	0.1812
2	2	1	0.10	0.50	0.9988	0.3553
2	2	1	0.10	0.70	0.9982	0.5235
2	2	1	0.10	0.90	0.9972	0.6813
2	2	1	0.30	0.50	0.9988	0.1815
2	2	1	0.30	0.70	0.9988	0.3568
2	2	2	0.00	0.10	0.9365	0.0855
2	2	2	0.00	0.30	0.9370	0.2208
2	2	2	0.00	0.50	0.9527	0.3348
2	2	2	0.00	0.70	0.9738	0.4338
2	2	2	0.00	0.90	0.9858	0.5085
2	2	2	0.00	0.95	0.9848	0.5130
2	2	2	0.05	0.30	0.9948	0.2279
2	2	2	0.05	0.50	0.9930	0.3975
2	2	2	0.05	0.70	0.9930	0.5639
2	2	2	0.05	0.90	0.9945	0.7159
2	2	2	0.05	0.95	0.9958	0.7418
2	2	2	0.10	0.30	0.9952	0.1832
2	2	2	0.10	0.50	0.9948	0.3585
2	2	2	0.10	0.70	0.9962	0.5257
2	2	2	0.10	0.90	0.9950	0.6807
2	2	2	0.30	0.50	0.9968	0.1837
2	2	2	0.30	0.70	0.9972	0.3596

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	2	4	0.00	0.10	0.9473	0.0877
2	2	4	0.00	0.30	0.9475	0.2329
2	2	4	0.00	0.50	0.9557	0.3474
2	2	4	0.00	0.70	0.9665	0.4445
2	2	4	0.00	0.90	0.9667	0.5130
2	2	4	0.00	0.95	0.9718	0.5172
2	2	4	0.05	0.30	0.9898	0.2313
2	2	4	0.05	0.50	0.9872	0.4022
2	2	4	0.05	0.70	0.9848	0.5606
2	2	4	0.05	0.90	0.9840	0.7057
2	2	4	0.05	0.95	0.9838	0.7390
2	2	4	0.10	0.30	0.9918	0.1864
2	2	4	0.10	0.50	0.9868	0.3614
2	2	4	0.10	0.70	0.9872	0.5247
2	2	4	0.10	0.90	0.9890	0.6778
2	2	4	0.30	0.50	0.9958	0.1854
2	2	4	0.30	0.70	0.9920	0.3617
2	2	10	0.00	0.10	0.9383	0.0886
2	2	10	0.00	0.30	0.9457	0.2358
2	2	10	0.00	0.50	0.9523	0.3575
2	2	10	0.00	0.70	0.9553	0.4530
2	2	10	0.00	0.90	0.9550	0.5177
2	2	10	0.00	0.95	0.9505	0.5135
2	2	10	0.05	0.30	0.9900	0.2339
2	2	10	0.05	0.50	0.9828	0.4054
2	2	10	0.05	0.70	0.9730	0.5618
2	2	10	0.05	0.90	0.9740	0.7081
2	2	10	0.05	0.95	0.9718	0.7396
2	2	10	0.10	0.30	0.9908	0.1884
2	2	10	0.10	0.50	0.9845	0.3638
2	2	10	0.10	0.70	0.9802	0.5283
2	2	10	0.10	0.90	0.9738	0.6759
2	2	10	0.30	0.50	0.9925	0.1871
2	2	10	0.30	0.70	0.9840	0.3641
2	5	0	0.00	0.10	0.9353	0.0675
2	5	0	0.00	0.30	0.9477	0.1609
2	5	0	0.00	0.50	0.9533	0.2385
2	5	0	0.00	0.70	0.9805	0.3154
2	5	0	0.00	0.90	0.9902	0.3820
2	5	0	0.00	0.95	0.9895	0.3778
2	5	0	0.05	0.30	0.9912	0.1832
2	5	0	0.05	0.50	0.9942	0.3280
2	5	0	0.05	0.70	0.9968	0.4681
2	5	0	0.05	0.90	0.9978	0.6021
2	5	0	0.05	0.95	0.9968	0.6277
2	5	0	0.10	0.30	0.9942	0.1488
2	5	0	0.10	0.50	0.9982	0.3001
2	5	0	0.10	0.70	0.9960	0.4438
2	5	0	0.10	0.90	0.9962	0.5740
2	5	0	0.30	0.50	0.9975	0.1541
2	5	0	0.30	0.70	0.9980	0.3063
2	5	0.1	0.00	0.10	0.9327	0.0676
2	5	0.1	0.00	0.30	0.9460	0.1619
2	5	0.1	0.00	0.50	0.9497	0.2399
2	5	0.1	0.00	0.70	0.9848	0.3214
2	5	0.1	0.00	0.90	0.9898	0.3848
2	5	0.1	0.00	0.95	0.9935	0.3865
2	5	0.1	0.05	0.30	0.9920	0.1843
2	5	0.1	0.05	0.50	0.9952	0.3281
2	5	0.1	0.05	0.70	0.9968	0.4775
2	5	0.1	0.05	0.90	0.9978	0.6043
2	5	0.1	0.05	0.95	0.9975	0.6296
2	5	0.1	0.10	0.30	0.9952	0.1514
2	5	0.1	0.10	0.50	0.9975	0.3004
2	5	0.1	0.10	0.70	0.9978	0.4489
2	5	0.1	0.10	0.90	0.9975	0.5818
2	5	0.1	0.30	0.50	0.9975	0.1546
2	5	0.1	0.30	0.70	0.9970	0.3098
2	5	0.25	0.00	0.10	0.9363	0.0688
2	5	0.25	0.00	0.30	0.9453	0.1685
2	5	0.25	0.00	0.50	0.9525	0.2469
2	5	0.25	0.00	0.70	0.9772	0.3314
2	5	0.25	0.00	0.90	0.9905	0.4038
2	5	0.25	0.00	0.95	0.9932	0.3982
2	5	0.25	0.05	0.30	0.9882	0.1879
2	5	0.25	0.05	0.50	0.9928	0.3345
2	5	0.25	0.05	0.70	0.9955	0.4810
2	5	0.25	0.05	0.90	0.9975	0.6177
2	5	0.25	0.05	0.95	0.9980	0.6433
2	5	0.25	0.10	0.30	0.9925	0.1538
2	5	0.25	0.10	0.50	0.9978	0.3057
2	5	0.25	0.10	0.70	0.9965	0.4576
2	5	0.25	0.10	0.90	0.9968	0.5952
2	5	0.25	0.30	0.50	0.9965	0.1582
2	5	0.25	0.30	0.70	0.9975	0.3125
2	5	0.5	0.00	0.10	0.9293	0.0714
2	5	0.5	0.00	0.30	0.9447	0.1778
2	5	0.5	0.00	0.50	0.9523	0.2680
2	5	0.5	0.00	0.70	0.9755	0.3585
2	5	0.5	0.00	0.90	0.9912	0.4331
2	5	0.5	0.00	0.95	0.9880	0.4350
2	5	0.5	0.05	0.30	0.9882	0.1975
2	5	0.5	0.05	0.50	0.9905	0.3494
2	5	0.5	0.05	0.70	0.9958	0.5032
2	5	0.5	0.05	0.90	0.9982	0.6445
2	5	0.5	0.05	0.95	0.9965	0.6743
2	5	0.5	0.10	0.30	0.9930	0.1602
2	5	0.5	0.10	0.50	0.9940	0.3166
2	5	0.5	0.10	0.70	0.9952	0.4746
2	5	0.5	0.10	0.90	0.9972	0.6184
2	5	0.5	0.30	0.50	0.9975	0.1614
2	5	0.5	0.30	0.70	0.9970	0.3221

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	5	1	0.00	0.10	0.9213	0.0769
2	5	1	0.00	0.30	0.9377	0.1984
2	5	1	0.00	0.50	0.9483	0.3023
2	5	1	0.00	0.70	0.9655	0.3946
2	5	1	0.00	0.90	0.9772	0.4730
2	5	1	0.00	0.95	0.9768	0.4673
2	5	1	0.05	0.30	0.9810	0.2108
2	5	1	0.05	0.50	0.9845	0.3651
2	5	1	0.05	0.70	0.9915	0.5280
2	5	1	0.05	0.90	0.9925	0.6717
2	5	1	0.05	0.95	0.9930	0.7012
2	5	1	0.10	0.30	0.9862	0.1711
2	5	1	0.10	0.50	0.9885	0.3354
2	5	1	0.10	0.70	0.9938	0.4951
2	5	1	0.10	0.90	0.9912	0.6388
2	5	1	0.30	0.50	0.9932	0.1707
2	5	1	0.30	0.70	0.9938	0.3378
2	5	2	0.00	0.10	0.9450	0.0852
2	5	2	0.00	0.30	0.9477	0.2207
2	5	2	0.00	0.50	0.9500	0.3331
2	5	2	0.00	0.70	0.9570	0.4253
2	5	2	0.00	0.90	0.9553	0.4965
2	5	2	0.00	0.95	0.9635	0.5029
2	5	2	0.05	0.30	0.9838	0.2231
2	5	2	0.05	0.50	0.9800	0.3895
2	5	2	0.05	0.70	0.9772	0.5465
2	5	2	0.05	0.90	0.9815	0.6971
2	5	2	0.05	0.95	0.9770	0.7213
2	5	2	0.10	0.30	0.9860	0.1800
2	5	2	0.10	0.50	0.9880	0.3504
2	5	2	0.10	0.70	0.9822	0.5134
2	5	2	0.10	0.90	0.9812	0.6674
2	5	2	0.30	0.50	0.9878	0.1776
2	5	2	0.30	0.70	0.9848	0.3488
2	5	4	0.00	0.10	0.9407	0.0888
2	5	4	0.00	0.30	0.9495	0.2313
2	5	4	0.00	0.50	0.9547	0.3510
2	5	4	0.00	0.70	0.9560	0.4491
2	5	4	0.00	0.90	0.9560	0.5161
2	5	4	0.00	0.95	0.9463	0.5099
2	5	4	0.05	0.30	0.9872	0.2304
2	5	4	0.05	0.50	0.9780	0.4015
2	5	4	0.05	0.70	0.9752	0.5566
2	5	4	0.05	0.90	0.9728	0.7017
2	5	4	0.05	0.95	0.9667	0.7313
2	5	4	0.10	0.30	0.9888	0.1848
2	5	4	0.10	0.50	0.9810	0.3594
2	5	4	0.10	0.70	0.9780	0.5256
2	5	4	0.10	0.90	0.9718	0.6729
2	5	4	0.30	0.50	0.9860	0.1842
2	5	4	0.30	0.70	0.9838	0.3581
2	5	10	0.00	0.10	0.9480	0.0899
2	5	10	0.00	0.30	0.9467	0.2378
2	5	10	0.00	0.50	0.9523	0.3606
2	5	10	0.00	0.70	0.9505	0.4562
2	5	10	0.00	0.90	0.9515	0.5201
2	5	10	0.00	0.95	0.9570	0.5194
2	5	10	0.05	0.30	0.9882	0.2335
2	5	10	0.05	0.50	0.9842	0.4057
2	5	10	0.05	0.70	0.9780	0.5675
2	5	10	0.05	0.90	0.9685	0.7065
2	5	10	0.05	0.95	0.9623	0.7321
2	5	10	0.10	0.30	0.9915	0.1892
2	5	10	0.10	0.50	0.9792	0.3651
2	5	10	0.10	0.70	0.9765	0.5254
2	5	10	0.10	0.90	0.9688	0.6742
2	5	10	0.30	0.50	0.9888	0.1872
2	5	10	0.30	0.70	0.9812	0.3655
2	10	0	0.00	0.10	0.9425	0.0575
2	10	0	0.00	0.30	0.9510	0.1367
2	10	0	0.00	0.50	0.9485	0.2028
2	10	0	0.00	0.70	0.9735	0.2698
2	10	0	0.00	0.90	0.9898	0.3229
2	10	0	0.00	0.95	0.9900	0.3122
2	10	0	0.05	0.30	0.9878	0.1601
2	10	0	0.05	0.50	0.9875	0.2887
2	10	0	0.05	0.70	0.9970	0.4194
2	10	0	0.05	0.90	0.9972	0.5287
2	10	0	0.05	0.95	0.9968	0.5439
2	10	0	0.10	0.30	0.9885	0.1305
2	10	0	0.10	0.50	0.9940	0.2671
2	10	0	0.10	0.70	0.9948	0.3952
2	10	0	0.10	0.90	0.9978	0.5101
2	10	0	0.30	0.50	0.9970	0.1390
2	10	0	0.30	0.70	0.9978	0.2766
2	10	0.1	0.00	0.10	0.9440	0.0576
2	10	0.1	0.00	0.30	0.9435	0.1368
2	10	0.1	0.00	0.50	0.9580	0.2072
2	10	0.1	0.00	0.70	0.9772	0.2805
2	10	0.1	0.00	0.90	0.9892	0.3278
2	10	0.1	0.00	0.95	0.9918	0.3202
2	10	0.1	0.05	0.30	0.9828	0.1619
2	10	0.1	0.05	0.50	0.9928	0.2959
2	10	0.1	0.05	0.70	0.9935	0.4244
2	10	0.1	0.05	0.90	0.9972	0.5409
2	10	0.1	0.05	0.95	0.9978	0.5589
2	10	0.1	0.10	0.30	0.9890	0.1328
2	10	0.1	0.10	0.50	0.9938	0.2706
2	10	0.1	0.10	0.70	0.9965	0.4022
2	10	0.1	0.10	0.90	0.9978	0.5227
2	10	0.1	0.30	0.50	0.9965	0.1406
2	10	0.1	0.30	0.70	0.9965	0.2785

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
2	10	0.25	0.00	0.10	0.9315	0.0600
2	10	0.25	0.00	0.30	0.9453	0.1455
2	10	0.25	0.00	0.50	0.9503	0.2221
2	10	0.25	0.00	0.70	0.9683	0.2982
2	10	0.25	0.00	0.90	0.9862	0.3523
2	10	0.25	0.00	0.95	0.9892	0.3492
2	10	0.25	0.05	0.30	0.9810	0.1717
2	10	0.25	0.05	0.50	0.9878	0.3092
2	10	0.25	0.05	0.70	0.9960	0.4469
2	10	0.25	0.05	0.90	0.9962	0.5627
2	10	0.25	0.05	0.95	0.9968	0.5872
2	10	0.25	0.10	0.30	0.9882	0.1400
2	10	0.25	0.10	0.50	0.9952	0.2826
2	10	0.25	0.10	0.70	0.9972	0.4225
2	10	0.25	0.10	0.90	0.9962	0.5465
2	10	0.25	0.30	0.50	0.9958	0.1469
2	10	0.25	0.30	0.70	0.9962	0.2902
2	10	0.5	0.00	0.10	0.9300	0.0647
2	10	0.5	0.00	0.30	0.9407	0.1627
2	10	0.5	0.00	0.50	0.9565	0.2552
2	10	0.5	0.00	0.70	0.9667	0.3392
2	10	0.5	0.00	0.90	0.9802	0.4029
2	10	0.5	0.00	0.95	0.9828	0.4055
2	10	0.5	0.05	0.30	0.9782	0.1884
2	10	0.5	0.05	0.50	0.9832	0.3335
2	10	0.5	0.05	0.70	0.9922	0.4851
2	10	0.5	0.05	0.90	0.9950	0.6187
2	10	0.5	0.05	0.95	0.9955	0.6377
2	10	0.5	0.10	0.30	0.9852	0.1534
2	10	0.5	0.10	0.50	0.9898	0.3037
2	10	0.5	0.10	0.70	0.9955	0.4532
2	10	0.5	0.10	0.90	0.9952	0.5948
2	10	0.5	0.30	0.50	0.9948	0.1565
2	10	0.5	0.30	0.70	0.9948	0.3078
2	10	1	0.00	0.10	0.9340	0.0751
2	10	1	0.00	0.30	0.9447	0.1908
2	10	1	0.00	0.50	0.9517	0.2959
2	10	1	0.00	0.70	0.9560	0.3898
2	10	1	0.00	0.90	0.9645	0.4609
2	10	1	0.00	0.95	0.9718	0.4631
2	10	1	0.05	0.30	0.9788	0.2080
2	10	1	0.05	0.50	0.9778	0.3672
2	10	1	0.05	0.70	0.9820	0.5209
2	10	1	0.05	0.90	0.9860	0.6662
2	10	1	0.05	0.95	0.9852	0.6947
2	10	1	0.10	0.30	0.9805	0.1678
2	10	1	0.10	0.50	0.9815	0.3282
2	10	1	0.10	0.70	0.9885	0.4852
2	10	1	0.10	0.90	0.9865	0.6341
2	10	1	0.30	0.50	0.9898	0.1672
2	10	1	0.30	0.70	0.9882	0.3292
2	10	2	0.00	0.10	0.9323	0.0837
2	10	2	0.00	0.30	0.9407	0.2155
2	10	2	0.00	0.50	0.9553	0.3309
2	10	2	0.00	0.70	0.9520	0.4251
2	10	2	0.00	0.90	0.9555	0.4946
2	10	2	0.00	0.95	0.9583	0.4942
2	10	2	0.05	0.30	0.9815	0.2212
2	10	2	0.05	0.50	0.9805	0.3875
2	10	2	0.05	0.70	0.9698	0.5454
2	10	2	0.05	0.90	0.9702	0.6879
2	10	2	0.05	0.95	0.9745	0.7207
2	10	2	0.10	0.30	0.9818	0.1784
2	10	2	0.10	0.50	0.9795	0.3481
2	10	2	0.10	0.70	0.9788	0.5111
2	10	2	0.10	0.90	0.9748	0.6602
2	10	2	0.30	0.50	0.9828	0.1773
2	10	2	0.30	0.70	0.9812	0.3489
2	10	4	0.00	0.10	0.9360	0.0867
2	10	4	0.00	0.30	0.9487	0.2328
2	10	4	0.00	0.50	0.9483	0.3488
2	10	4	0.00	0.70	0.9460	0.4471
2	10	4	0.00	0.90	0.9507	0.5149
2	10	4	0.00	0.95	0.9510	0.5147
2	10	4	0.05	0.30	0.9858	0.2310
2	10	4	0.05	0.50	0.9810	0.4011
2	10	4	0.05	0.70	0.9750	0.5595
2	10	4	0.05	0.90	0.9667	0.7025
2	10	4	0.05	0.95	0.9718	0.7348
2	10	4	0.10	0.30	0.9880	0.1848
2	10	4	0.10	0.50	0.9838	0.3593
2	10	4	0.10	0.70	0.9740	0.5203
2	10	4	0.10	0.90	0.9718	0.6700
2	10	4	0.30	0.50	0.9880	0.1835
2	10	4	0.30	0.70	0.9812	0.3584
2	10	10	0.00	0.10	0.9523	0.0893
2	10	10	0.00	0.30	0.9535	0.2399
2	10	10	0.00	0.50	0.9510	0.3567
2	10	10	0.00	0.70	0.9505	0.4538
2	10	10	0.00	0.90	0.9437	0.5163
2	10	10	0.00	0.95	0.9543	0.5195
2	10	10	0.05	0.30	0.9868	0.2343
2	10	10	0.05	0.50	0.9798	0.4064
2	10	10	0.05	0.70	0.9683	0.5628
2	10	10	0.05	0.90	0.9690	0.7081
2	10	10	0.05	0.95	0.9670	0.7354
2	10	10	0.10	0.30	0.9908	0.1893
2	10	10	0.10	0.50	0.9792	0.3630
2	10	10	0.10	0.70	0.9775	0.5288
2	10	10	0.10	0.90	0.9677	0.6741
2	10	10	0.30	0.50	0.9880	0.1879
2	10	10	0.30	0.70	0.9828	0.3634

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	2	0	0.00	0.10	0.9307	0.0672
5	2	0	0.00	0.30	0.9387	0.1486
5	2	0	0.00	0.50	0.9513	0.1968
5	2	0	0.00	0.70	0.9752	0.2296
5	2	0	0.00	0.90	0.9828	0.2327
5	2	0	0.00	0.95	0.9892	0.2109
5	2	0	0.05	0.30	0.9908	0.1337
5	2	0	0.05	0.50	0.9928	0.2185
5	2	0	0.05	0.70	0.9942	0.2970
5	2	0	0.05	0.90	0.9955	0.3491
5	2	0	0.05	0.95	0.9945	0.3443
5	2	0	0.10	0.30	0.9940	0.1082
5	2	0	0.10	0.50	0.9935	0.2029
5	2	0	0.10	0.70	0.9965	0.2885
5	2	0	0.10	0.90	0.9955	0.3504
5	2	0	0.30	0.50	0.9965	0.1064
5	2	0	0.30	0.70	0.9930	0.2069
5	2	0.1	0.00	0.10	0.9247	0.0666
5	2	0.1	0.00	0.30	0.9405	0.1476
5	2	0.1	0.00	0.50	0.9537	0.1992
5	2	0.1	0.00	0.70	0.9770	0.2310
5	2	0.1	0.00	0.90	0.9878	0.2351
5	2	0.1	0.00	0.95	0.9868	0.2118
5	2	0.1	0.05	0.30	0.9900	0.1332
5	2	0.1	0.05	0.50	0.9932	0.2187
5	2	0.1	0.05	0.70	0.9940	0.2973
5	2	0.1	0.05	0.90	0.9960	0.3498
5	2	0.1	0.05	0.95	0.9968	0.3478
5	2	0.1	0.10	0.30	0.9950	0.1085
5	2	0.1	0.10	0.50	0.9970	0.2016
5	2	0.1	0.10	0.70	0.9972	0.2883
5	2	0.1	0.10	0.90	0.9960	0.3514
5	2	0.1	0.30	0.50	0.9978	0.1065
5	2	0.1	0.30	0.70	0.9975	0.2091
5	2	0.25	0.00	0.10	0.9285	0.0670
5	2	0.25	0.00	0.30	0.9447	0.1528
5	2	0.25	0.00	0.50	0.9457	0.2048
5	2	0.25	0.00	0.70	0.9772	0.2333
5	2	0.25	0.00	0.90	0.9838	0.2321
5	2	0.25	0.00	0.95	0.9858	0.2135
5	2	0.25	0.05	0.30	0.9880	0.1357
5	2	0.25	0.05	0.50	0.9935	0.2224
5	2	0.25	0.05	0.70	0.9915	0.2989
5	2	0.25	0.05	0.90	0.9968	0.3521
5	2	0.25	0.05	0.95	0.9965	0.3471
5	2	0.25	0.10	0.30	0.9978	0.1100
5	2	0.25	0.10	0.50	0.9940	0.2021
5	2	0.25	0.10	0.70	0.9942	0.2914
5	2	0.25	0.10	0.90	0.9958	0.3527
5	2	0.25	0.30	0.50	0.9965	0.1075
5	2	0.25	0.30	0.70	0.9960	0.2077
5	2	0.5	0.00	0.10	0.9327	0.0689
5	2	0.5	0.00	0.30	0.9327	0.1548
5	2	0.5	0.00	0.50	0.9533	0.2085
5	2	0.5	0.00	0.70	0.9715	0.2460
5	2	0.5	0.00	0.90	0.9840	0.2463
5	2	0.5	0.00	0.95	0.9835	0.2227
5	2	0.5	0.05	0.30	0.9900	0.1396
5	2	0.5	0.05	0.50	0.9898	0.2270
5	2	0.5	0.05	0.70	0.9948	0.3084
5	2	0.5	0.05	0.90	0.9955	0.3608
5	2	0.5	0.05	0.95	0.9945	0.3548
5	2	0.5	0.10	0.30	0.9908	0.1114
5	2	0.5	0.10	0.50	0.9930	0.2102
5	2	0.5	0.10	0.70	0.9968	0.2984
5	2	0.5	0.10	0.90	0.9955	0.3612
5	2	0.5	0.30	0.50	0.9960	0.1099
5	2	0.5	0.30	0.70	0.9965	0.2133
5	2	1	0.00	0.10	0.9323	0.0706
5	2	1	0.00	0.30	0.9440	0.1674
5	2	1	0.00	0.50	0.9493	0.2287
5	2	1	0.00	0.70	0.9675	0.2663
5	2	1	0.00	0.90	0.9732	0.2621
5	2	1	0.00	0.95	0.9828	0.2375
5	2	1	0.05	0.30	0.9855	0.1497
5	2	1	0.05	0.50	0.9900	0.2409
5	2	1	0.05	0.70	0.9878	0.3212
5	2	1	0.05	0.90	0.9932	0.3749
5	2	1	0.05	0.95	0.9930	0.3698
5	2	1	0.10	0.30	0.9885	0.1197
5	2	1	0.10	0.50	0.9942	0.2208
5	2	1	0.10	0.70	0.9912	0.3094
5	2	1	0.10	0.90	0.9930	0.3742
5	2	1	0.30	0.50	0.9952	0.1142
5	2	1	0.30	0.70	0.9945	0.2203
5	2	2	0.00	0.10	0.9415	0.0756
5	2	2	0.00	0.30	0.9420	0.1807
5	2	2	0.00	0.50	0.9453	0.2470
5	2	2	0.00	0.70	0.9545	0.2863
5	2	2	0.00	0.90	0.9607	0.2787
5	2	2	0.00	0.95	0.9600	0.2514
5	2	2	0.05	0.30	0.9858	0.1608
5	2	2	0.05	0.50	0.9860	0.2582
5	2	2	0.05	0.70	0.9850	0.3373
5	2	2	0.05	0.90	0.9860	0.3917
5	2	2	0.05	0.95	0.9842	0.3843
5	2	2	0.10	0.30	0.9860	0.1279
5	2	2	0.10	0.50	0.9858	0.2360
5	2	2	0.10	0.70	0.9902	0.3242
5	2	2	0.10	0.90	0.9878	0.3880
5	2	2	0.30	0.50	0.9938	0.1203
5	2	2	0.30	0.70	0.9898	0.2294

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	2	4	0.00	0.10	0.9485	0.0780
5	2	4	0.00	0.30	0.9475	0.1907
5	2	4	0.00	0.50	0.9547	0.2627
5	2	4	0.00	0.70	0.9477	0.2967
5	2	4	0.00	0.90	0.9435	0.2866
5	2	4	0.00	0.95	0.9545	0.2622
5	2	4	0.05	0.30	0.9832	0.1686
5	2	4	0.05	0.50	0.9810	0.2699
5	2	4	0.05	0.70	0.9832	0.3473
5	2	4	0.05	0.90	0.9810	0.4006
5	2	4	0.05	0.95	0.9760	0.3945
5	2	4	0.10	0.30	0.9882	0.1347
5	2	4	0.10	0.50	0.9818	0.2444
5	2	4	0.10	0.70	0.9812	0.3316
5	2	4	0.10	0.90	0.9835	0.4000
5	2	4	0.30	0.50	0.9885	0.1240
5	2	4	0.30	0.70	0.9880	0.2377
5	2	10	0.00	0.10	0.9533	0.0799
5	2	10	0.00	0.30	0.9487	0.1937
5	2	10	0.00	0.50	0.9473	0.2652
5	2	10	0.00	0.70	0.9513	0.3009
5	2	10	0.00	0.90	0.9513	0.2903
5	2	10	0.00	0.95	0.9540	0.2580
5	2	10	0.05	0.30	0.9865	0.1706
5	2	10	0.05	0.50	0.9805	0.2743
5	2	10	0.05	0.70	0.9772	0.3511
5	2	10	0.05	0.90	0.9710	0.3999
5	2	10	0.05	0.95	0.9780	0.3937
5	2	10	0.10	0.30	0.9852	0.1369
5	2	10	0.10	0.50	0.9848	0.2489
5	2	10	0.10	0.70	0.9830	0.3340
5	2	10	0.10	0.90	0.9808	0.3982
5	2	10	0.30	0.50	0.9902	0.1262
5	2	10	0.30	0.70	0.9868	0.2379
5	5	0	0.00	0.10	0.9350	0.0537
5	5	0	0.00	0.30	0.9375	0.1088
5	5	0	0.00	0.50	0.9503	0.1410
5	5	0	0.00	0.70	0.9650	0.1579
5	5	0	0.00	0.90	0.9788	0.1422
5	5	0	0.00	0.95	0.9772	0.1166
5	5	0	0.05	0.30	0.9770	0.0905
5	5	0	0.05	0.50	0.9808	0.1424
5	5	0	0.05	0.70	0.9898	0.1868
5	5	0	0.05	0.90	0.9940	0.2019
5	5	0	0.05	0.95	0.9902	0.1918
5	5	0	0.10	0.30	0.9838	0.0724
5	5	0	0.10	0.50	0.9892	0.1302
5	5	0	0.10	0.70	0.9922	0.1825
5	5	0	0.10	0.90	0.9902	0.2082
5	5	0	0.30	0.50	0.9912	0.0691
5	5	0	0.30	0.70	0.9935	0.1346

a	n	σ_A	p_L	p_U	coverage	dist
5	5	0.1	0.00	0.10	0.9413	0.0540
5	5	0.1	0.00	0.30	0.9497	0.1105
5	5	0.1	0.00	0.50	0.9515	0.1426
5	5	0.1	0.00	0.70	0.9657	0.1569
5	5	0.1	0.00	0.90	0.9802	0.1443
5	5	0.1	0.00	0.95	0.9755	0.1209
5	5	0.1	0.05	0.30	0.9700	0.0910
5	5	0.1	0.05	0.50	0.9832	0.1445
5	5	0.1	0.05	0.70	0.9875	0.1881
5	5	0.1	0.05	0.90	0.9922	0.2026
5	5	0.1	0.05	0.95	0.9945	0.1962
5	5	0.1	0.10	0.30	0.9840	0.0721
5	5	0.1	0.10	0.50	0.9888	0.1301
5	5	0.1	0.10	0.70	0.9928	0.1856
5	5	0.1	0.10	0.90	0.9912	0.2127
5	5	0.1	0.30	0.50	0.9932	0.0699
5	5	0.1	0.30	0.70	0.9942	0.1357
5	5	0.25	0.00	0.10	0.9423	0.0551
5	5	0.25	0.00	0.30	0.9450	0.1161
5	5	0.25	0.00	0.50	0.9520	0.1542
5	5	0.25	0.00	0.70	0.9620	0.1733
5	5	0.25	0.00	0.90	0.9752	0.1577
5	5	0.25	0.00	0.95	0.9810	0.1298
5	5	0.25	0.05	0.30	0.9768	0.0970
5	5	0.25	0.05	0.50	0.9865	0.1543
5	5	0.25	0.05	0.70	0.9872	0.1998
5	5	0.25	0.05	0.90	0.9915	0.2214
5	5	0.25	0.05	0.95	0.9902	0.2081
5	5	0.25	0.10	0.30	0.9812	0.0762
5	5	0.25	0.10	0.50	0.9890	0.1412
5	5	0.25	0.10	0.70	0.9932	0.1956
5	5	0.25	0.10	0.90	0.9910	0.2274
5	5	0.25	0.30	0.50	0.9895	0.0739
5	5	0.25	0.30	0.70	0.9928	0.1441
5	5	0.5	0.00	0.10	0.9317	0.0584
5	5	0.5	0.00	0.30	0.9435	0.1302
5	5	0.5	0.00	0.50	0.9513	0.1733
5	5	0.5	0.00	0.70	0.9520	0.2007
5	5	0.5	0.00	0.90	0.9732	0.1834
5	5	0.5	0.00	0.95	0.9762	0.1595
5	5	0.5	0.05	0.30	0.9698	0.1110
5	5	0.5	0.05	0.50	0.9770	0.1774
5	5	0.5	0.05	0.70	0.9860	0.2335
5	5	0.5	0.05	0.90	0.9898	0.2585
5	5	0.5	0.05	0.95	0.9912	0.2505
5	5	0.5	0.10	0.30	0.9745	0.0882
5	5	0.5	0.10	0.50	0.9822	0.1615
5	5	0.5	0.10	0.70	0.9910	0.2287
5	5	0.5	0.10	0.90	0.9938	0.2623
5	5	0.5	0.30	0.50	0.9965	0.0848
5	5	0.5	0.30	0.70	0.9912	0.1627

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	5	1	0.00	0.10	0.9385	0.0665
5	5	1	0.00	0.30	0.9423	0.1545
5	5	1	0.00	0.50	0.9507	0.2132
5	5	1	0.00	0.70	0.9560	0.2435
5	5	1	0.00	0.90	0.9633	0.2326
5	5	1	0.00	0.95	0.9663	0.2097
5	5	1	0.05	0.30	0.9715	0.1331
5	5	1	0.05	0.50	0.9790	0.2167
5	5	1	0.05	0.70	0.9848	0.2853
5	5	1	0.05	0.90	0.9875	0.3266
5	5	1	0.05	0.95	0.9868	0.3157
5	5	1	0.10	0.30	0.9798	0.1064
5	5	1	0.10	0.50	0.9845	0.1956
5	5	1	0.10	0.70	0.9858	0.2753
5	5	1	0.10	0.90	0.9858	0.3232
5	5	1	0.30	0.50	0.9915	0.1017
5	5	1	0.30	0.70	0.9890	0.1955
5	5	2	0.00	0.10	0.9453	0.0732
5	5	2	0.00	0.30	0.9447	0.1762
5	5	2	0.00	0.50	0.9545	0.2405
5	5	2	0.00	0.70	0.9530	0.2805
5	5	2	0.00	0.90	0.9495	0.2665
5	5	2	0.00	0.95	0.9593	0.2439
5	5	2	0.05	0.30	0.9862	0.1567
5	5	2	0.05	0.50	0.9755	0.2488
5	5	2	0.05	0.70	0.9740	0.3249
5	5	2	0.05	0.90	0.9745	0.3734
5	5	2	0.05	0.95	0.9760	0.3728
5	5	2	0.10	0.30	0.9835	0.1232
5	5	2	0.10	0.50	0.9872	0.2275
5	5	2	0.10	0.70	0.9795	0.3124
5	5	2	0.10	0.90	0.9792	0.3712
5	5	2	0.30	0.50	0.9848	0.1156
5	5	2	0.30	0.70	0.9848	0.2227
5	5	4	0.00	0.10	0.9475	0.0785
5	5	4	0.00	0.30	0.9463	0.1908
5	5	4	0.00	0.50	0.9437	0.2585
5	5	4	0.00	0.70	0.9470	0.2930
5	5	4	0.00	0.90	0.9525	0.2800
5	5	4	0.00	0.95	0.9443	0.2570
5	5	4	0.05	0.30	0.9842	0.1654
5	5	4	0.05	0.50	0.9828	0.2675
5	5	4	0.05	0.70	0.9790	0.3441
5	5	4	0.05	0.90	0.9782	0.3989
5	5	4	0.05	0.95	0.9752	0.3861
5	5	4	0.10	0.30	0.9862	0.1321
5	5	4	0.10	0.50	0.9860	0.2402
5	5	4	0.10	0.70	0.9785	0.3298
5	5	4	0.10	0.90	0.9772	0.3911
5	5	4	0.30	0.50	0.9880	0.1224
5	5	4	0.30	0.70	0.9850	0.2334

a	n	σ_A	p_L	p_U	coverage	dist
5	5	10	0.00	0.10	0.9507	0.0805
5	5	10	0.00	0.30	0.9533	0.1932
5	5	10	0.00	0.50	0.9477	0.2633
5	5	10	0.00	0.70	0.9515	0.3023
5	5	10	0.00	0.90	0.9485	0.2905
5	5	10	0.00	0.95	0.9515	0.2601
5	5	10	0.05	0.30	0.9842	0.1712
5	5	10	0.05	0.50	0.9868	0.2714
5	5	10	0.05	0.70	0.9772	0.3486
5	5	10	0.05	0.90	0.9752	0.4035
5	5	10	0.05	0.95	0.9745	0.3976
5	5	10	0.10	0.30	0.9905	0.1378
5	5	10	0.10	0.50	0.9862	0.2474
5	5	10	0.10	0.70	0.9842	0.3374
5	5	10	0.10	0.90	0.9798	0.3984
5	5	10	0.30	0.50	0.9900	0.1258
5	5	10	0.30	0.70	0.9868	0.2395
5	10	0	0.00	0.10	0.9467	0.0424
5	10	0	0.00	0.30	0.9420	0.0842
5	10	0	0.00	0.50	0.9580	0.1051
5	10	0	0.00	0.70	0.9647	0.1139
5	10	0	0.00	0.90	0.9740	0.0959
5	10	0	0.00	0.95	0.9758	0.0739
5	10	0	0.05	0.30	0.9640	0.0659
5	10	0	0.05	0.50	0.9788	0.1014
5	10	0	0.05	0.70	0.9850	0.1249
5	10	0	0.05	0.90	0.9912	0.1303
5	10	0	0.05	0.95	0.9892	0.1173
5	10	0	0.10	0.30	0.9732	0.0513
5	10	0	0.10	0.50	0.9868	0.0910
5	10	0	0.10	0.70	0.9858	0.1248
5	10	0	0.10	0.90	0.9878	0.1341
5	10	0	0.30	0.50	0.9895	0.0479
5	10	0	0.30	0.70	0.9900	0.0934
5	10	0.1	0.00	0.10	0.9443	0.0433
5	10	0.1	0.00	0.30	0.9407	0.0881
5	10	0.1	0.00	0.50	0.9555	0.1093
5	10	0.1	0.00	0.70	0.9645	0.1208
5	10	0.1	0.00	0.90	0.9758	0.0988
5	10	0.1	0.00	0.95	0.9790	0.0775
5	10	0.1	0.05	0.30	0.9643	0.0683
5	10	0.1	0.05	0.50	0.9752	0.1042
5	10	0.1	0.05	0.70	0.9828	0.1323
5	10	0.1	0.05	0.90	0.9902	0.1348
5	10	0.1	0.05	0.95	0.9910	0.1242
5	10	0.1	0.10	0.30	0.9748	0.0538
5	10	0.1	0.10	0.50	0.9858	0.0953
5	10	0.1	0.10	0.70	0.9872	0.1285
5	10	0.1	0.10	0.90	0.9890	0.1416
5	10	0.1	0.30	0.50	0.9878	0.0502
5	10	0.1	0.30	0.70	0.9908	0.0958

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
5	10	0.25	0.00	0.10	0.9415	0.0463
5	10	0.25	0.00	0.30	0.9397	0.0953
5	10	0.25	0.00	0.50	0.9483	0.1250
5	10	0.25	0.00	0.70	0.9587	0.1376
5	10	0.25	0.00	0.90	0.9730	0.1169
5	10	0.25	0.00	0.95	0.9745	0.0948
5	10	0.25	0.05	0.30	0.9650	0.0787
5	10	0.25	0.05	0.50	0.9762	0.1210
5	10	0.25	0.05	0.70	0.9828	0.1530
5	10	0.25	0.05	0.90	0.9900	0.1582
5	10	0.25	0.05	0.95	0.9902	0.1457
5	10	0.25	0.10	0.30	0.9677	0.0609
5	10	0.25	0.10	0.50	0.9838	0.1098
5	10	0.25	0.10	0.70	0.9875	0.1481
5	10	0.25	0.10	0.90	0.9892	0.1655
5	10	0.25	0.30	0.50	0.9905	0.0568
5	10	0.25	0.30	0.70	0.9942	0.1091
5	10	0.5	0.00	0.10	0.9330	0.0522
5	10	0.5	0.00	0.30	0.9430	0.1179
5	10	0.5	0.00	0.50	0.9467	0.1560
5	10	0.5	0.00	0.70	0.9545	0.1804
5	10	0.5	0.00	0.90	0.9700	0.1612
5	10	0.5	0.00	0.95	0.9748	0.1307
5	10	0.5	0.05	0.30	0.9688	0.0980
5	10	0.5	0.05	0.50	0.9718	0.1564
5	10	0.5	0.05	0.70	0.9855	0.2053
5	10	0.5	0.05	0.90	0.9870	0.2183
5	10	0.5	0.05	0.95	0.9888	0.2079
5	10	0.5	0.10	0.30	0.9718	0.0782
5	10	0.5	0.10	0.50	0.9832	0.1425
5	10	0.5	0.10	0.70	0.9855	0.1948
5	10	0.5	0.10	0.90	0.9900	0.2259
5	10	0.5	0.30	0.50	0.9925	0.0739
5	10	0.5	0.30	0.70	0.9942	0.1430
5	10	1	0.00	0.10	0.9300	0.0625
5	10	1	0.00	0.30	0.9383	0.1459
5	10	1	0.00	0.50	0.9560	0.2040
5	10	1	0.00	0.70	0.9540	0.2379
5	10	1	0.00	0.90	0.9623	0.2240
5	10	1	0.00	0.95	0.9570	0.1912
5	10	1	0.05	0.30	0.9730	0.1280
5	10	1	0.05	0.50	0.9722	0.2090
5	10	1	0.05	0.70	0.9730	0.2717
5	10	1	0.05	0.90	0.9800	0.3095
5	10	1	0.05	0.95	0.9822	0.3029
5	10	1	0.10	0.30	0.9730	0.1023
5	10	1	0.10	0.50	0.9815	0.1887
5	10	1	0.10	0.70	0.9800	0.2623
5	10	1	0.10	0.90	0.9810	0.3107
5	10	1	0.30	0.50	0.9838	0.0965
5	10	1	0.30	0.70	0.9822	0.1862

a	n	σ_A	p_L	p_U	coverage	dist
5	10	2	0.00	0.10	0.9403	0.0738
5	10	2	0.00	0.30	0.9433	0.1744
5	10	2	0.00	0.50	0.9505	0.2422
5	10	2	0.00	0.70	0.9575	0.2778
5	10	2	0.00	0.90	0.9505	0.2687
5	10	2	0.00	0.95	0.9490	0.2374
5	10	2	0.05	0.30	0.9800	0.1521
5	10	2	0.05	0.50	0.9785	0.2485
5	10	2	0.05	0.70	0.9822	0.3231
5	10	2	0.05	0.90	0.9825	0.3685
5	10	2	0.05	0.95	0.9750	0.3662
5	10	2	0.10	0.30	0.9828	0.1215
5	10	2	0.10	0.50	0.9818	0.2235
5	10	2	0.10	0.70	0.9782	0.3097
5	10	2	0.10	0.90	0.9778	0.3679
5	10	2	0.30	0.50	0.9832	0.1149
5	10	2	0.30	0.70	0.9830	0.2197
5	10	4	0.00	0.10	0.9453	0.0784
5	10	4	0.00	0.30	0.9500	0.1873
5	10	4	0.00	0.50	0.9455	0.2556
5	10	4	0.00	0.70	0.9497	0.2930
5	10	4	0.00	0.90	0.9453	0.2816
5	10	4	0.00	0.95	0.9417	0.2511
5	10	4	0.05	0.30	0.9835	0.1653
5	10	4	0.05	0.50	0.9792	0.2677
5	10	4	0.05	0.70	0.9760	0.3415
5	10	4	0.05	0.90	0.9770	0.3944
5	10	4	0.05	0.95	0.9752	0.3895
5	10	4	0.10	0.30	0.9858	0.1333
5	10	4	0.10	0.50	0.9848	0.2396
5	10	4	0.10	0.70	0.9805	0.3298
5	10	4	0.10	0.90	0.9810	0.3971
5	10	4	0.30	0.50	0.9902	0.1223
5	10	4	0.30	0.70	0.9848	0.2346
5	10	10	0.00	0.10	0.9443	0.0801
5	10	10	0.00	0.30	0.9547	0.1932
5	10	10	0.00	0.50	0.9493	0.2625
5	10	10	0.00	0.70	0.9557	0.3057
5	10	10	0.00	0.90	0.9590	0.2909
5	10	10	0.00	0.95	0.9525	0.2631
5	10	10	0.05	0.30	0.9842	0.1703
5	10	10	0.05	0.50	0.9815	0.2716
5	10	10	0.05	0.70	0.9772	0.3527
5	10	10	0.05	0.90	0.9778	0.4026
5	10	10	0.05	0.95	0.9808	0.4001
5	10	10	0.10	0.30	0.9875	0.1369
5	10	10	0.10	0.50	0.9842	0.2446
5	10	10	0.10	0.70	0.9800	0.3380
5	10	10	0.10	0.90	0.9790	0.3989
5	10	10	0.30	0.50	0.9885	0.1254
5	10	10	0.30	0.70	0.9842	0.2387

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	2	0	0.00	0.10	0.9403	0.0564
10	2	0	0.00	0.30	0.9345	0.1150
10	2	0	0.00	0.50	0.9507	0.1434
10	2	0	0.00	0.70	0.9600	0.1581
10	2	0	0.00	0.90	0.9768	0.1405
10	2	0	0.00	0.95	0.9765	0.1177
10	2	0	0.05	0.30	0.9755	0.0929
10	2	0	0.05	0.50	0.9848	0.1427
10	2	0	0.05	0.70	0.9888	0.1839
10	2	0	0.05	0.90	0.9888	0.1997
10	2	0	0.05	0.95	0.9878	0.1862
10	2	0	0.10	0.30	0.9830	0.0725
10	2	0	0.10	0.50	0.9888	0.1299
10	2	0	0.10	0.70	0.9918	0.1810
10	2	0	0.10	0.90	0.9905	0.2051
10	2	0	0.30	0.50	0.9910	0.0682
10	2	0	0.30	0.70	0.9948	0.1325
10	2	0.1	0.00	0.10	0.9365	0.0569
10	2	0.1	0.00	0.30	0.9410	0.1154
10	2	0.1	0.00	0.50	0.9485	0.1439
10	2	0.1	0.00	0.70	0.9643	0.1564
10	2	0.1	0.00	0.90	0.9715	0.1383
10	2	0.1	0.00	0.95	0.9782	0.1164
10	2	0.1	0.05	0.30	0.9790	0.0937
10	2	0.1	0.05	0.50	0.9868	0.1433
10	2	0.1	0.05	0.70	0.9880	0.1866
10	2	0.1	0.05	0.90	0.9912	0.2020
10	2	0.1	0.05	0.95	0.9910	0.1889
10	2	0.1	0.10	0.30	0.9848	0.0735
10	2	0.1	0.10	0.50	0.9905	0.1318
10	2	0.1	0.10	0.70	0.9918	0.1800
10	2	0.1	0.10	0.90	0.9920	0.2049
10	2	0.1	0.30	0.50	0.9915	0.0686
10	2	0.1	0.30	0.70	0.9922	0.1333
10	2	0.25	0.00	0.10	0.9313	0.0572
10	2	0.25	0.00	0.30	0.9443	0.1180
10	2	0.25	0.00	0.50	0.9480	0.1480
10	2	0.25	0.00	0.70	0.9605	0.1622
10	2	0.25	0.00	0.90	0.9755	0.1437
10	2	0.25	0.00	0.95	0.9795	0.1200
10	2	0.25	0.05	0.30	0.9735	0.0954
10	2	0.25	0.05	0.50	0.9798	0.1464
10	2	0.25	0.05	0.70	0.9868	0.1867
10	2	0.25	0.05	0.90	0.9905	0.2022
10	2	0.25	0.05	0.95	0.9932	0.1888
10	2	0.25	0.10	0.30	0.9850	0.0747
10	2	0.25	0.10	0.50	0.9890	0.1332
10	2	0.25	0.10	0.70	0.9905	0.1832
10	2	0.25	0.10	0.90	0.9900	0.2064
10	2	0.25	0.30	0.50	0.9950	0.0689
10	2	0.25	0.30	0.70	0.9948	0.1351

a	n	σ_A	p_L	p_U	coverage	dist
10	2	0.5	0.00	0.10	0.9323	0.0577
10	2	0.5	0.00	0.30	0.9425	0.1227
10	2	0.5	0.00	0.50	0.9515	0.1529
10	2	0.5	0.00	0.70	0.9625	0.1690
10	2	0.5	0.00	0.90	0.9712	0.1492
10	2	0.5	0.00	0.95	0.9738	0.1220
10	2	0.5	0.05	0.30	0.9710	0.0990
10	2	0.5	0.05	0.50	0.9812	0.1504
10	2	0.5	0.05	0.70	0.9878	0.1907
10	2	0.5	0.05	0.90	0.9898	0.2053
10	2	0.5	0.05	0.95	0.9892	0.1947
10	2	0.5	0.10	0.30	0.9795	0.0775
10	2	0.5	0.10	0.50	0.9888	0.1367
10	2	0.5	0.10	0.70	0.9895	0.1885
10	2	0.5	0.10	0.90	0.9918	0.2127
10	2	0.5	0.30	0.50	0.9912	0.0711
10	2	0.5	0.30	0.70	0.9920	0.1365
10	2	1	0.00	0.10	0.9407	0.0614
10	2	1	0.00	0.30	0.9435	0.1332
10	2	1	0.00	0.50	0.9547	0.1733
10	2	1	0.00	0.70	0.9590	0.1907
10	2	1	0.00	0.90	0.9667	0.1625
10	2	1	0.00	0.95	0.9690	0.1347
10	2	1	0.05	0.30	0.9775	0.1097
10	2	1	0.05	0.50	0.9762	0.1660
10	2	1	0.05	0.70	0.9822	0.2073
10	2	1	0.05	0.90	0.9875	0.2222
10	2	1	0.05	0.95	0.9862	0.2095
10	2	1	0.10	0.30	0.9800	0.0843
10	2	1	0.10	0.50	0.9822	0.1479
10	2	1	0.10	0.70	0.9835	0.2007
10	2	1	0.10	0.90	0.9892	0.2254
10	2	1	0.30	0.50	0.9862	0.0752
10	2	1	0.30	0.70	0.9892	0.1452
10	2	2	0.00	0.10	0.9445	0.0664
10	2	2	0.00	0.30	0.9495	0.1461
10	2	2	0.00	0.50	0.9507	0.1877
10	2	2	0.00	0.70	0.9513	0.2068
10	2	2	0.00	0.90	0.9503	0.1771
10	2	2	0.00	0.95	0.9555	0.1480
10	2	2	0.05	0.30	0.9708	0.1188
10	2	2	0.05	0.50	0.9802	0.1803
10	2	2	0.05	0.70	0.9772	0.2215
10	2	2	0.05	0.90	0.9800	0.2381
10	2	2	0.05	0.95	0.9792	0.2241
10	2	2	0.10	0.30	0.9808	0.0938
10	2	2	0.10	0.50	0.9810	0.1616
10	2	2	0.10	0.70	0.9835	0.2149
10	2	2	0.10	0.90	0.9838	0.2413
10	2	2	0.30	0.50	0.9848	0.0815
10	2	2	0.30	0.70	0.9830	0.1541

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	2	4	0.00	0.10	0.9500	0.0697
10	2	4	0.00	0.30	0.9533	0.1548
10	2	4	0.00	0.50	0.9400	0.1936
10	2	4	0.00	0.70	0.9515	0.2052
10	2	4	0.00	0.90	0.9420	0.1781
10	2	4	0.00	0.95	0.9530	0.1532
10	2	4	0.05	0.30	0.9798	0.1273
10	2	4	0.05	0.50	0.9762	0.1901
10	2	4	0.05	0.70	0.9780	0.2328
10	2	4	0.05	0.90	0.9770	0.2464
10	2	4	0.05	0.95	0.9762	0.2330
10	2	4	0.10	0.30	0.9845	0.0979
10	2	4	0.10	0.50	0.9825	0.1692
10	2	4	0.10	0.70	0.9850	0.2238
10	2	4	0.10	0.90	0.9782	0.2529
10	2	4	0.30	0.50	0.9838	0.0837
10	2	4	0.30	0.70	0.9842	0.1593
10	2	10	0.00	0.10	0.9517	0.0713
10	2	10	0.00	0.30	0.9525	0.1563
10	2	10	0.00	0.50	0.9435	0.1980
10	2	10	0.00	0.70	0.9470	0.2135
10	2	10	0.00	0.90	0.9510	0.1834
10	2	10	0.00	0.95	0.9490	0.1591
10	2	10	0.05	0.30	0.9768	0.1306
10	2	10	0.05	0.50	0.9750	0.1905
10	2	10	0.05	0.70	0.9785	0.2335
10	2	10	0.05	0.90	0.9760	0.2515
10	2	10	0.05	0.95	0.9720	0.2362
10	2	10	0.10	0.30	0.9840	0.0998
10	2	10	0.10	0.50	0.9810	0.1702
10	2	10	0.10	0.70	0.9825	0.2286
10	2	10	0.10	0.90	0.9798	0.2559
10	2	10	0.30	0.50	0.9838	0.0856
10	2	10	0.30	0.70	0.9828	0.1606
10	5	0	0.00	0.10	0.9397	0.0426
10	5	0	0.00	0.30	0.9443	0.0798
10	5	0	0.00	0.50	0.9540	0.0977
10	5	0	0.00	0.70	0.9513	0.1000
10	5	0	0.00	0.90	0.9718	0.0808
10	5	0	0.00	0.95	0.9688	0.0610
10	5	0	0.05	0.30	0.9650	0.0606
10	5	0	0.05	0.50	0.9738	0.0877
10	5	0	0.05	0.70	0.9862	0.1095
10	5	0	0.05	0.90	0.9885	0.1072
10	5	0	0.05	0.95	0.9850	0.0968
10	5	0	0.10	0.30	0.9667	0.0470
10	5	0	0.10	0.50	0.9820	0.0800
10	5	0	0.10	0.70	0.9830	0.1060
10	5	0	0.10	0.90	0.9865	0.1146
10	5	0	0.30	0.50	0.9888	0.0415
10	5	0	0.30	0.70	0.9862	0.0793

a	n	σ_A	p_L	p_U	coverage	dist
10	5	0.1	0.00	0.10	0.9403	0.0426
10	5	0.1	0.00	0.30	0.9407	0.0805
10	5	0.1	0.00	0.50	0.9505	0.1004
10	5	0.1	0.00	0.70	0.9613	0.1050
10	5	0.1	0.00	0.90	0.9665	0.0825
10	5	0.1	0.00	0.95	0.9718	0.0637
10	5	0.1	0.05	0.30	0.9645	0.0624
10	5	0.1	0.05	0.50	0.9722	0.0906
10	5	0.1	0.05	0.70	0.9755	0.1102
10	5	0.1	0.05	0.90	0.9868	0.1088
10	5	0.1	0.05	0.95	0.9855	0.0987
10	5	0.1	0.10	0.30	0.9712	0.0471
10	5	0.1	0.10	0.50	0.9792	0.0809
10	5	0.1	0.10	0.70	0.9855	0.1080
10	5	0.1	0.10	0.90	0.9845	0.1168
10	5	0.1	0.30	0.50	0.9828	0.0423
10	5	0.1	0.30	0.70	0.9875	0.0810
10	5	0.25	0.00	0.10	0.9393	0.0436
10	5	0.25	0.00	0.30	0.9435	0.0864
10	5	0.25	0.00	0.50	0.9527	0.1064
10	5	0.25	0.00	0.70	0.9597	0.1127
10	5	0.25	0.00	0.90	0.9745	0.0900
10	5	0.25	0.00	0.95	0.9663	0.0685
10	5	0.25	0.05	0.30	0.9655	0.0666
10	5	0.25	0.05	0.50	0.9738	0.0988
10	5	0.25	0.05	0.70	0.9805	0.1186
10	5	0.25	0.05	0.90	0.9825	0.1168
10	5	0.25	0.05	0.95	0.9868	0.1055
10	5	0.25	0.10	0.30	0.9740	0.0517
10	5	0.25	0.10	0.50	0.9800	0.0865
10	5	0.25	0.10	0.70	0.9875	0.1148
10	5	0.25	0.10	0.90	0.9875	0.1224
10	5	0.25	0.30	0.50	0.9852	0.0448
10	5	0.25	0.30	0.70	0.9892	0.0848
10	5	0.5	0.00	0.10	0.9425	0.0478
10	5	0.5	0.00	0.30	0.9413	0.0983
10	5	0.5	0.00	0.50	0.9493	0.1249
10	5	0.5	0.00	0.70	0.9573	0.1370
10	5	0.5	0.00	0.90	0.9585	0.1057
10	5	0.5	0.00	0.95	0.9702	0.0837
10	5	0.5	0.05	0.30	0.9677	0.0778
10	5	0.5	0.05	0.50	0.9740	0.1137
10	5	0.5	0.05	0.70	0.9792	0.1400
10	5	0.5	0.05	0.90	0.9855	0.1397
10	5	0.5	0.05	0.95	0.9862	0.1271
10	5	0.5	0.10	0.30	0.9645	0.0593
10	5	0.5	0.10	0.50	0.9748	0.1017
10	5	0.5	0.10	0.70	0.9865	0.1337
10	5	0.5	0.10	0.90	0.9850	0.1432
10	5	0.5	0.30	0.50	0.9878	0.0517
10	5	0.5	0.30	0.70	0.9878	0.0981

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	5	1	0.00	0.10	0.9420	0.0553
10	5	1	0.00	0.30	0.9357	0.1176
10	5	1	0.00	0.50	0.9485	0.1574
10	5	1	0.00	0.70	0.9520	0.1691
10	5	1	0.00	0.90	0.9590	0.1430
10	5	1	0.00	0.95	0.9590	0.1126
10	5	1	0.05	0.30	0.9688	0.0973
10	5	1	0.05	0.50	0.9698	0.1461
10	5	1	0.05	0.70	0.9760	0.1800
10	5	1	0.05	0.90	0.9822	0.1840
10	5	1	0.05	0.95	0.9800	0.1738
10	5	1	0.10	0.30	0.9657	0.0745
10	5	1	0.10	0.50	0.9750	0.1316
10	5	1	0.10	0.70	0.9802	0.1718
10	5	1	0.10	0.90	0.9835	0.1918
10	5	1	0.30	0.50	0.9855	0.0658
10	5	1	0.30	0.70	0.9842	0.1252
10	5	2	0.00	0.10	0.9460	0.0648
10	5	2	0.00	0.30	0.9427	0.1402
10	5	2	0.00	0.50	0.9483	0.1818
10	5	2	0.00	0.70	0.9593	0.2016
10	5	2	0.00	0.90	0.9470	0.1692
10	5	2	0.00	0.95	0.9550	0.1399
10	5	2	0.05	0.30	0.9725	0.1149
10	5	2	0.05	0.50	0.9752	0.1731
10	5	2	0.05	0.70	0.9758	0.2127
10	5	2	0.05	0.90	0.9745	0.2270
10	5	2	0.05	0.95	0.9785	0.2142
10	5	2	0.10	0.30	0.9715	0.0890
10	5	2	0.10	0.50	0.9808	0.1558
10	5	2	0.10	0.70	0.9768	0.2059
10	5	2	0.10	0.90	0.9785	0.2304
10	5	2	0.30	0.50	0.9805	0.0777
10	5	2	0.30	0.70	0.9825	0.1488
10	5	4	0.00	0.10	0.9473	0.0685
10	5	4	0.00	0.30	0.9433	0.1512
10	5	4	0.00	0.50	0.9500	0.1942
10	5	4	0.00	0.70	0.9480	0.2098
10	5	4	0.00	0.90	0.9573	0.1804
10	5	4	0.00	0.95	0.9467	0.1513
10	5	4	0.05	0.30	0.9805	0.1255
10	5	4	0.05	0.50	0.9745	0.1869
10	5	4	0.05	0.70	0.9772	0.2285
10	5	4	0.05	0.90	0.9775	0.2450
10	5	4	0.05	0.95	0.9742	0.2325
10	5	4	0.10	0.30	0.9780	0.0972
10	5	4	0.10	0.50	0.9778	0.1672
10	5	4	0.10	0.70	0.9758	0.2183
10	5	4	0.10	0.90	0.9830	0.2508
10	5	4	0.30	0.50	0.9858	0.0840
10	5	4	0.30	0.70	0.9800	0.1581

a	n	σ_A	p_L	p_U	coverage	dist
10	5	10	0.00	0.10	0.9500	0.0697
10	5	10	0.00	0.30	0.9553	0.1571
10	5	10	0.00	0.50	0.9545	0.1969
10	5	10	0.00	0.70	0.9467	0.2124
10	5	10	0.00	0.90	0.9517	0.1868
10	5	10	0.00	0.95	0.9495	0.1548
10	5	10	0.05	0.30	0.9778	0.1304
10	5	10	0.05	0.50	0.9745	0.1916
10	5	10	0.05	0.70	0.9778	0.2359
10	5	10	0.05	0.90	0.9742	0.2485
10	5	10	0.05	0.95	0.9772	0.2405
10	5	10	0.10	0.30	0.9792	0.0995
10	5	10	0.10	0.50	0.9760	0.1703
10	5	10	0.10	0.70	0.9780	0.2229
10	5	10	0.10	0.90	0.9760	0.2533
10	5	10	0.30	0.50	0.9838	0.0849
10	5	10	0.30	0.70	0.9802	0.1612
10	10	0	0.00	0.10	0.9440	0.0332
10	10	0	0.00	0.30	0.9437	0.0594
10	10	0	0.00	0.50	0.9525	0.0706
10	10	0	0.00	0.70	0.9547	0.0724
10	10	0	0.00	0.90	0.9643	0.0539
10	10	0	0.00	0.95	0.9683	0.0390
10	10	0	0.05	0.30	0.9635	0.0443
10	10	0	0.05	0.50	0.9720	0.0626
10	10	0	0.05	0.70	0.9760	0.0724
10	10	0	0.05	0.90	0.9812	0.0690
10	10	0	0.05	0.95	0.9785	0.0602
10	10	0	0.10	0.30	0.9620	0.0331
10	10	0	0.10	0.50	0.9742	0.0548
10	10	0	0.10	0.70	0.9768	0.0714
10	10	0	0.10	0.90	0.9762	0.0739
10	10	0	0.30	0.50	0.9825	0.0281
10	10	0	0.30	0.70	0.9838	0.0535
10	10	0.1	0.00	0.10	0.9485	0.0339
10	10	0.1	0.00	0.30	0.9445	0.0622
10	10	0.1	0.00	0.50	0.9470	0.0731
10	10	0.1	0.00	0.70	0.9540	0.0745
10	10	0.1	0.00	0.90	0.9677	0.0552
10	10	0.1	0.00	0.95	0.9695	0.0405
10	10	0.1	0.05	0.30	0.9547	0.0454
10	10	0.1	0.05	0.50	0.9660	0.0648
10	10	0.1	0.05	0.70	0.9782	0.0749
10	10	0.1	0.05	0.90	0.9798	0.0699
10	10	0.1	0.05	0.95	0.9832	0.0619
10	10	0.1	0.10	0.30	0.9603	0.0337
10	10	0.1	0.10	0.50	0.9740	0.0566
10	10	0.1	0.10	0.70	0.9780	0.0738
10	10	0.1	0.10	0.90	0.9780	0.0743
10	10	0.1	0.30	0.50	0.9765	0.0289
10	10	0.1	0.30	0.70	0.9820	0.0545

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Table B.7: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for small values of a and n and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
10	10	0.25	0.00	0.10	0.9453	0.0359
10	10	0.25	0.00	0.30	0.9515	0.0698
10	10	0.25	0.00	0.50	0.9560	0.0852
10	10	0.25	0.00	0.70	0.9565	0.0885
10	10	0.25	0.00	0.90	0.9663	0.0655
10	10	0.25	0.00	0.95	0.9660	0.0481
10	10	0.25	0.05	0.30	0.9563	0.0533
10	10	0.25	0.05	0.50	0.9700	0.0761
10	10	0.25	0.05	0.70	0.9788	0.0883
10	10	0.25	0.05	0.90	0.9868	0.0800
10	10	0.25	0.05	0.95	0.9852	0.0711
10	10	0.25	0.10	0.30	0.9643	0.0403
10	10	0.25	0.10	0.50	0.9730	0.0667
10	10	0.25	0.10	0.70	0.9790	0.0826
10	10	0.25	0.10	0.90	0.9862	0.0858
10	10	0.25	0.30	0.50	0.9830	0.0325
10	10	0.25	0.30	0.70	0.9835	0.0613
10	10	0.5	0.00	0.10	0.9397	0.0420
10	10	0.5	0.00	0.30	0.9540	0.0886
10	10	0.5	0.00	0.50	0.9515	0.1119
10	10	0.5	0.00	0.70	0.9560	0.1192
10	10	0.5	0.00	0.90	0.9595	0.0920
10	10	0.5	0.00	0.95	0.9655	0.0676
10	10	0.5	0.05	0.30	0.9607	0.0684
10	10	0.5	0.05	0.50	0.9655	0.1002
10	10	0.5	0.05	0.70	0.9732	0.1167
10	10	0.5	0.05	0.90	0.9848	0.1106
10	10	0.5	0.05	0.95	0.9865	0.1004
10	10	0.5	0.10	0.30	0.9645	0.0523
10	10	0.5	0.10	0.50	0.9708	0.0871
10	10	0.5	0.10	0.70	0.9782	0.1126
10	10	0.5	0.10	0.90	0.9895	0.1159
10	10	0.5	0.30	0.50	0.9840	0.0432
10	10	0.5	0.30	0.70	0.9858	0.0800
10	10	1	0.00	0.10	0.9470	0.0532
10	10	1	0.00	0.30	0.9425	0.1147
10	10	1	0.00	0.50	0.9505	0.1497
10	10	1	0.00	0.70	0.9540	0.1620
10	10	1	0.00	0.90	0.9547	0.1320
10	10	1	0.00	0.95	0.9525	0.1056
10	10	1	0.05	0.30	0.9643	0.0914
10	10	1	0.05	0.50	0.9705	0.1403
10	10	1	0.05	0.70	0.9745	0.1703
10	10	1	0.05	0.90	0.9782	0.1733
10	10	1	0.05	0.95	0.9788	0.1599
10	10	1	0.10	0.30	0.9728	0.0718
10	10	1	0.10	0.50	0.9768	0.1248
10	10	1	0.10	0.70	0.9795	0.1635
10	10	1	0.10	0.90	0.9810	0.1809
10	10	1	0.30	0.50	0.9768	0.0613
10	10	1	0.30	0.70	0.9808	0.1179

a	n	σ_A	p_L	p_U	coverage	dist
10	10	2	0.00	0.10	0.9487	0.0639
10	10	2	0.00	0.30	0.9473	0.1404
10	10	2	0.00	0.50	0.9505	0.1777
10	10	2	0.00	0.70	0.9540	0.1967
10	10	2	0.00	0.90	0.9480	0.1648
10	10	2	0.00	0.95	0.9493	0.1363
10	10	2	0.05	0.30	0.9730	0.1139
10	10	2	0.05	0.50	0.9738	0.1707
10	10	2	0.05	0.70	0.9720	0.2127
10	10	2	0.05	0.90	0.9768	0.2225
10	10	2	0.05	0.95	0.9742	0.2119
10	10	2	0.10	0.30	0.9772	0.0878
10	10	2	0.10	0.50	0.9800	0.1546
10	10	2	0.10	0.70	0.9772	0.2023
10	10	2	0.10	0.90	0.9768	0.2268
10	10	2	0.30	0.50	0.9765	0.0762
10	10	2	0.30	0.70	0.9810	0.1470
10	10	4	0.00	0.10	0.9497	0.0683
10	10	4	0.00	0.30	0.9497	0.1510
10	10	4	0.00	0.50	0.9537	0.1928
10	10	4	0.00	0.70	0.9535	0.2122
10	10	4	0.00	0.90	0.9563	0.1803
10	10	4	0.00	0.95	0.9533	0.1502
10	10	4	0.05	0.30	0.9772	0.1254
10	10	4	0.05	0.50	0.9755	0.1844
10	10	4	0.05	0.70	0.9790	0.2286
10	10	4	0.05	0.90	0.9760	0.2408
10	10	4	0.05	0.95	0.9728	0.2318
10	10	4	0.10	0.30	0.9812	0.0971
10	10	4	0.10	0.50	0.9785	0.1668
10	10	4	0.10	0.70	0.9775	0.2191
10	10	4	0.10	0.90	0.9765	0.2505
10	10	4	0.30	0.50	0.9818	0.0831
10	10	4	0.30	0.70	0.9812	0.1567
10	10	10	0.00	0.10	0.9467	0.0707
10	10	10	0.00	0.30	0.9453	0.1545
10	10	10	0.00	0.50	0.9513	0.1986
10	10	10	0.00	0.70	0.9535	0.2162
10	10	10	0.00	0.90	0.9477	0.1825
10	10	10	0.00	0.95	0.9547	0.1554
10	10	10	0.05	0.30	0.9788	0.1314
10	10	10	0.05	0.50	0.9785	0.1923
10	10	10	0.05	0.70	0.9735	0.2345
10	10	10	0.05	0.90	0.9745	0.2493
10	10	10	0.05	0.95	0.9765	0.2369
10	10	10	0.10	0.30	0.9835	0.1005
10	10	10	0.10	0.50	0.9802	0.1711
10	10	10	0.10	0.70	0.9820	0.2258
10	10	10	0.10	0.90	0.9780	0.2549
10	10	10	0.30	0.50	0.9805	0.0844
10	10	10	0.30	0.70	0.9820	0.1617

Table B.8: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and Other Percentile Pairs.

a	n	σ_A	p_L	p_U	coverage	dist
20	10	0	0.00	0.10	0.9447	0.0247
20	10	0	0.00	0.30	0.9477	0.0428
20	10	0	0.00	0.50	0.9513	0.0484
20	10	0	0.00	0.70	0.9525	0.0479
20	10	0	0.00	0.90	0.9515	0.0340
20	10	0	0.05	0.30	0.9590	0.0306
20	10	0	0.05	0.50	0.9645	0.0412
20	10	0	0.05	0.70	0.9708	0.0476
20	10	0	0.05	0.90	0.9732	0.0428
20	10	0	0.05	0.95	0.9748	0.0369
20	10	0	0.10	0.30	0.9593	0.0222
20	10	0	0.10	0.50	0.9680	0.0363
20	10	0	0.10	0.70	0.9673	0.0455
20	10	0	0.10	0.90	0.9735	0.0458
20	10	0	0.30	0.50	0.9698	0.0184
20	10	0	0.30	0.70	0.9698	0.0346
20	10	0.25	0.00	0.10	0.9447	0.0273
20	10	0.25	0.00	0.30	0.9480	0.0507
20	10	0.25	0.00	0.50	0.9515	0.0591
20	10	0.25	0.00	0.70	0.9545	0.0586
20	10	0.25	0.00	0.90	0.9623	0.0405
20	10	0.25	0.05	0.30	0.9585	0.0372
20	10	0.25	0.05	0.50	0.9623	0.0503
20	10	0.25	0.05	0.70	0.9705	0.0551
20	10	0.25	0.05	0.90	0.9732	0.0470
20	10	0.25	0.05	0.95	0.9818	0.0407
20	10	0.25	0.10	0.30	0.9553	0.0273
20	10	0.25	0.10	0.50	0.9617	0.0434
20	10	0.25	0.10	0.70	0.9637	0.0519
20	10	0.25	0.10	0.90	0.9765	0.0510
20	10	0.25	0.30	0.50	0.9745	0.0207
20	10	0.25	0.30	0.70	0.9758	0.0377
20	10	1	0.00	0.10	0.9463	0.0433
20	10	1	0.00	0.30	0.9507	0.0878
20	10	1	0.00	0.50	0.9533	0.1095
20	10	1	0.00	0.70	0.9480	0.1131
20	10	1	0.00	0.90	0.9480	0.0831
20	10	1	0.05	0.30	0.9583	0.0671
20	10	1	0.05	0.50	0.9680	0.0960
20	10	1	0.05	0.70	0.9698	0.1091
20	10	1	0.05	0.90	0.9755	0.1015
20	10	1	0.05	0.95	0.9725	0.0892
20	10	1	0.10	0.30	0.9615	0.0509
20	10	1	0.10	0.50	0.9645	0.0824
20	10	1	0.10	0.70	0.9660	0.1034
20	10	1	0.10	0.90	0.9740	0.1047
20	10	1	0.30	0.50	0.9725	0.0396
20	10	1	0.30	0.70	0.9802	0.0742

a	n	σ_A	p_L	p_U	coverage	dist
20	10	4	0.00	0.10	0.9473	0.0577
20	10	4	0.00	0.30	0.9490	0.1174
20	10	4	0.00	0.50	0.9575	0.1434
20	10	4	0.00	0.70	0.9433	0.1444
20	10	4	0.00	0.90	0.9513	0.1146
20	10	4	0.05	0.30	0.9698	0.0915
20	10	4	0.05	0.50	0.9708	0.1297
20	10	4	0.05	0.70	0.9685	0.1521
20	10	4	0.05	0.90	0.9702	0.1490
20	10	4	0.05	0.95	0.9692	0.1402
20	10	4	0.10	0.30	0.9732	0.0691
20	10	4	0.10	0.50	0.9735	0.1140
20	10	4	0.10	0.70	0.9685	0.1430
20	10	4	0.10	0.90	0.9788	0.1588
20	10	4	0.30	0.50	0.9712	0.0562
20	10	4	0.30	0.70	0.9755	0.1053
20	10	10	0.00	0.10	0.9555	0.0592
20	10	10	0.00	0.30	0.9460	0.1202
20	10	10	0.00	0.50	0.9517	0.1462
20	10	10	0.00	0.70	0.9463	0.1491
20	10	10	0.00	0.90	0.9505	0.1179
20	10	10	0.05	0.30	0.9675	0.0945
20	10	10	0.05	0.50	0.9695	0.1333
20	10	10	0.05	0.70	0.9715	0.1565
20	10	10	0.05	0.90	0.9670	0.1576
20	10	10	0.05	0.95	0.9738	0.1457
20	10	10	0.10	0.30	0.9742	0.0712
20	10	10	0.10	0.50	0.9725	0.1174
20	10	10	0.10	0.70	0.9770	0.1502
20	10	10	0.10	0.90	0.9700	0.1624
20	10	10	0.30	0.50	0.9730	0.0570
20	10	10	0.30	0.70	0.9748	0.1081
50	10	0	0.00	0.10	0.9460	0.0163
50	10	0	0.00	0.30	0.9530	0.0276
50	10	0	0.00	0.50	0.9473	0.0304
50	10	0	0.00	0.70	0.9530	0.0292
50	10	0	0.00	0.90	0.9505	0.0198
50	10	0	0.05	0.30	0.9617	0.0187
50	10	0	0.05	0.50	0.9530	0.0251
50	10	0	0.05	0.70	0.9650	0.0286
50	10	0	0.05	0.90	0.9647	0.0246
50	10	0	0.05	0.95	0.9560	0.0209
50	10	0	0.10	0.30	0.9593	0.0138
50	10	0	0.10	0.50	0.9633	0.0219
50	10	0	0.10	0.70	0.9617	0.0274
50	10	0	0.10	0.90	0.9637	0.0266
50	10	0	0.30	0.50	0.9655	0.0111
50	10	0	0.30	0.70	0.9655	0.0206

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Table B.8: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
50	10	0.25	0.00	0.10	0.9470	0.0182
50	10	0.25	0.00	0.30	0.9557	0.0326
50	10	0.25	0.00	0.50	0.9430	0.0374
50	10	0.25	0.00	0.70	0.9583	0.0359
50	10	0.25	0.00	0.90	0.9615	0.0234
50	10	0.25	0.05	0.30	0.9533	0.0233
50	10	0.25	0.05	0.50	0.9485	0.0306
50	10	0.25	0.05	0.70	0.9660	0.0328
50	10	0.25	0.05	0.90	0.9692	0.0262
50	10	0.25	0.05	0.95	0.9665	0.0222
50	10	0.25	0.10	0.30	0.9613	0.0168
50	10	0.25	0.10	0.50	0.9607	0.0263
50	10	0.25	0.10	0.70	0.9627	0.0307
50	10	0.25	0.10	0.90	0.9698	0.0286
50	10	0.25	0.30	0.50	0.9692	0.0121
50	10	0.25	0.30	0.70	0.9700	0.0218
50	10	1	0.00	0.10	0.9447	0.0303
50	10	1	0.00	0.30	0.9487	0.0580
50	10	1	0.00	0.50	0.9553	0.0711
50	10	1	0.00	0.70	0.9510	0.0682
50	10	1	0.00	0.90	0.9510	0.0468
50	10	1	0.05	0.30	0.9595	0.0432
50	10	1	0.05	0.50	0.9613	0.0597
50	10	1	0.05	0.70	0.9643	0.0642
50	10	1	0.05	0.90	0.9722	0.0534
50	10	1	0.05	0.95	0.9722	0.0462
50	10	1	0.10	0.30	0.9573	0.0317
50	10	1	0.10	0.50	0.9670	0.0506
50	10	1	0.10	0.70	0.9653	0.0596
50	10	1	0.10	0.90	0.9700	0.0575
50	10	1	0.30	0.50	0.9647	0.0230
50	10	1	0.30	0.70	0.9680	0.0422
50	10	4	0.00	0.10	0.9515	0.0425
50	10	4	0.00	0.30	0.9537	0.0803
50	10	4	0.00	0.50	0.9515	0.0913
50	10	4	0.00	0.70	0.9493	0.0906
50	10	4	0.00	0.90	0.9497	0.0659
50	10	4	0.05	0.30	0.9595	0.0577
50	10	4	0.05	0.50	0.9610	0.0790
50	10	4	0.05	0.70	0.9620	0.0898
50	10	4	0.05	0.90	0.9650	0.0842
50	10	4	0.05	0.95	0.9692	0.0751
50	10	4	0.10	0.30	0.9680	0.0436
50	10	4	0.10	0.50	0.9670	0.0680
50	10	4	0.10	0.70	0.9670	0.0859
50	10	4	0.10	0.90	0.9675	0.0892
50	10	4	0.30	0.50	0.9718	0.0340
50	10	4	0.30	0.70	0.9698	0.0636
50	10	10	0.00	0.10	0.9495	0.0437
50	10	10	0.00	0.30	0.9475	0.0811
50	10	10	0.00	0.50	0.9515	0.0950
50	10	10	0.00	0.70	0.9503	0.0934
50	10	10	0.00	0.90	0.9453	0.0686
50	10	10	0.05	0.30	0.9593	0.0599
50	10	10	0.05	0.50	0.9657	0.0807
50	10	10	0.05	0.70	0.9660	0.0929
50	10	10	0.05	0.90	0.9653	0.0875
50	10	10	0.05	0.95	0.9708	0.0783
50	10	10	0.10	0.30	0.9657	0.0441
50	10	10	0.10	0.50	0.9673	0.0703
50	10	10	0.10	0.70	0.9700	0.0895
50	10	10	0.10	0.90	0.9677	0.0929
50	10	10	0.30	0.50	0.9657	0.0349
50	10	10	0.30	0.70	0.9692	0.0654
100	10	0	0.00	0.10	0.9450	0.0119
100	10	0	0.00	0.30	0.9505	0.0193
100	10	0	0.00	0.50	0.9493	0.0215
100	10	0	0.00	0.70	0.9493	0.0204
100	10	0	0.00	0.90	0.9550	0.0136
100	10	0	0.05	0.30	0.9497	0.0134
100	10	0	0.05	0.50	0.9505	0.0175
100	10	0	0.05	0.70	0.9563	0.0196
100	10	0	0.05	0.90	0.9593	0.0167
100	10	0	0.05	0.95	0.9570	0.0140
100	10	0	0.10	0.30	0.9587	0.0096
100	10	0	0.10	0.50	0.9610	0.0154
100	10	0	0.10	0.70	0.9603	0.0188
100	10	0	0.10	0.90	0.9570	0.0181
100	10	0	0.30	0.50	0.9575	0.0075
100	10	0	0.30	0.70	0.9595	0.0142
100	10	0.25	0.00	0.10	0.9483	0.0132
100	10	0.25	0.00	0.30	0.9460	0.0227
100	10	0.25	0.00	0.50	0.9493	0.0260
100	10	0.25	0.00	0.70	0.9487	0.0246
100	10	0.25	0.00	0.90	0.9530	0.0157
100	10	0.25	0.05	0.30	0.9535	0.0164
100	10	0.25	0.05	0.50	0.9510	0.0216
100	10	0.25	0.05	0.70	0.9577	0.0224
100	10	0.25	0.05	0.90	0.9593	0.0176
100	10	0.25	0.05	0.95	0.9607	0.0148
100	10	0.25	0.10	0.30	0.9520	0.0118
100	10	0.25	0.10	0.50	0.9540	0.0180
100	10	0.25	0.10	0.70	0.9560	0.0208
100	10	0.25	0.10	0.90	0.9570	0.0189
100	10	0.25	0.30	0.50	0.9613	0.0082
100	10	0.25	0.30	0.70	0.9635	0.0148

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Table B.8: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
100	10	1	0.00	0.10	0.9457	0.0231
100	10	1	0.00	0.30	0.9487	0.0428
100	10	1	0.00	0.50	0.9523	0.0497
100	10	1	0.00	0.70	0.9515	0.0480
100	10	1	0.00	0.90	0.9527	0.0315
100	10	1	0.05	0.30	0.9497	0.0301
100	10	1	0.05	0.50	0.9537	0.0409
100	10	1	0.05	0.70	0.9577	0.0432
100	10	1	0.05	0.90	0.9647	0.0348
100	10	1	0.05	0.95	0.9680	0.0298
100	10	1	0.10	0.30	0.9467	0.0223
100	10	1	0.10	0.50	0.9637	0.0346
100	10	1	0.10	0.70	0.9635	0.0399
100	10	1	0.10	0.90	0.9637	0.0369
100	10	1	0.30	0.50	0.9623	0.0157
100	10	1	0.30	0.70	0.9627	0.0284
100	10	4	0.00	0.10	0.9493	0.0330
100	10	4	0.00	0.30	0.9543	0.0571
100	10	4	0.00	0.50	0.9510	0.0654
100	10	4	0.00	0.70	0.9435	0.0625
100	10	4	0.00	0.90	0.9487	0.0445
100	10	4	0.05	0.30	0.9625	0.0416
100	10	4	0.05	0.50	0.9698	0.0551
100	10	4	0.05	0.70	0.9583	0.0618
100	10	4	0.05	0.90	0.9600	0.0551
100	10	4	0.05	0.95	0.9685	0.0487
100	10	4	0.10	0.30	0.9577	0.0305
100	10	4	0.10	0.50	0.9653	0.0476
100	10	4	0.10	0.70	0.9653	0.0592
100	10	4	0.10	0.90	0.9643	0.0587
100	10	4	0.30	0.50	0.9680	0.0236
100	10	4	0.30	0.70	0.9585	0.0430
100	10	10	0.00	0.10	0.9490	0.0332
100	10	10	0.00	0.30	0.9500	0.0589
100	10	10	0.00	0.50	0.9497	0.0660
100	10	10	0.00	0.70	0.9533	0.0654
100	10	10	0.00	0.90	0.9423	0.0460
100	10	10	0.05	0.30	0.9573	0.0428
100	10	10	0.05	0.50	0.9603	0.0562
100	10	10	0.05	0.70	0.9623	0.0640
100	10	10	0.05	0.90	0.9607	0.0566
100	10	10	0.05	0.95	0.9617	0.0506
100	10	10	0.10	0.30	0.9590	0.0310
100	10	10	0.10	0.50	0.9657	0.0486
100	10	10	0.10	0.70	0.9637	0.0610
100	10	10	0.10	0.90	0.9600	0.0613
100	10	10	0.30	0.50	0.9595	0.0239
100	10	10	0.30	0.70	0.9627	0.0451

a	n	σ_A	p_L	p_U	coverage	dist
500	10	0	0.00	0.10	0.9573	0.0055
500	10	0	0.00	0.30	0.9490	0.0088
500	10	0	0.00	0.50	0.9547	0.0095
500	10	0	0.00	0.70	0.9530	0.0088
500	10	0	0.00	0.90	0.9490	0.0059
500	10	0	0.05	0.30	0.9433	0.0058
500	10	0	0.05	0.50	0.9447	0.0077
500	10	0	0.05	0.70	0.9545	0.0086
500	10	0	0.05	0.90	0.9565	0.0072
500	10	0	0.05	0.95	0.9543	0.0060
500	10	0	0.10	0.30	0.9535	0.0042
500	10	0	0.10	0.50	0.9540	0.0067
500	10	0	0.10	0.70	0.9560	0.0082
500	10	0	0.10	0.90	0.9497	0.0078
500	10	0	0.30	0.50	0.9543	0.0033
500	10	0	0.30	0.70	0.9473	0.0062
500	10	0.25	0.00	0.10	0.9465	0.0063
500	10	0.25	0.00	0.30	0.9455	0.0105
500	10	0.25	0.00	0.50	0.9560	0.0117
500	10	0.25	0.00	0.70	0.9460	0.0109
500	10	0.25	0.00	0.90	0.9515	0.0067
500	10	0.25	0.05	0.30	0.9445	0.0074
500	10	0.25	0.05	0.50	0.9567	0.0095
500	10	0.25	0.05	0.70	0.9533	0.0097
500	10	0.25	0.05	0.90	0.9630	0.0075
500	10	0.25	0.05	0.95	0.9580	0.0062
500	10	0.25	0.10	0.30	0.9565	0.0052
500	10	0.25	0.10	0.50	0.9560	0.0079
500	10	0.25	0.10	0.70	0.9487	0.0089
500	10	0.25	0.10	0.90	0.9570	0.0082
500	10	0.25	0.30	0.50	0.9555	0.0035
500	10	0.25	0.30	0.70	0.9575	0.0064
500	10	1	0.00	0.10	0.9540	0.0114
500	10	1	0.00	0.30	0.9460	0.0199
500	10	1	0.00	0.50	0.9513	0.0225
500	10	1	0.00	0.70	0.9477	0.0207
500	10	1	0.00	0.90	0.9540	0.0128
500	10	1	0.05	0.30	0.9470	0.0136
500	10	1	0.05	0.50	0.9537	0.0177
500	10	1	0.05	0.70	0.9505	0.0182
500	10	1	0.05	0.90	0.9550	0.0138
500	10	1	0.05	0.95	0.9527	0.0115
500	10	1	0.10	0.30	0.9547	0.0098
500	10	1	0.10	0.50	0.9537	0.0150
500	10	1	0.10	0.70	0.9575	0.0165
500	10	1	0.10	0.90	0.9515	0.0148
500	10	1	0.30	0.50	0.9595	0.0066
500	10	1	0.30	0.70	0.9610	0.0119

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Table B.8: Empirical Coverage Probabilities and "Expected Distance" of Lower Confidence Bounds for the Proportion of Conformance of Y for Asymptotics and Other Percentile Pairs.

continued from previous page

a	n	σ_A	p_L	p_U	coverage	dist
500	10	4	0.00	0.10	0.9507	0.0160
500	10	4	0.00	0.30	0.9537	0.0268
500	10	4	0.00	0.50	0.9517	0.0292
500	10	4	0.00	0.70	0.9503	0.0278
500	10	4	0.00	0.90	0.9485	0.0184
500	10	4	0.05	0.30	0.9553	0.0182
500	10	4	0.05	0.50	0.9557	0.0243
500	10	4	0.05	0.70	0.9557	0.0264
500	10	4	0.05	0.90	0.9595	0.0224
500	10	4	0.05	0.95	0.9525	0.0190
500	10	4	0.10	0.30	0.9563	0.0132
500	10	4	0.10	0.50	0.9533	0.0206
500	10	4	0.10	0.70	0.9555	0.0253
500	10	4	0.10	0.90	0.9570	0.0245
500	10	4	0.30	0.50	0.9640	0.0103
500	10	4	0.30	0.70	0.9563	0.0190
500	10	10	0.00	0.10	0.9510	0.0166
500	10	10	0.00	0.30	0.9523	0.0274
500	10	10	0.00	0.50	0.9485	0.0297
500	10	10	0.00	0.70	0.9527	0.0284
500	10	10	0.00	0.90	0.9535	0.0190
500	10	10	0.05	0.30	0.9497	0.0189
500	10	10	0.05	0.50	0.9533	0.0245
500	10	10	0.05	0.70	0.9570	0.0271
500	10	10	0.05	0.90	0.9527	0.0233
500	10	10	0.05	0.95	0.9560	0.0196
500	10	10	0.10	0.30	0.9590	0.0134
500	10	10	0.10	0.50	0.9597	0.0212
500	10	10	0.10	0.70	0.9553	0.0261
500	10	10	0.10	0.90	0.9537	0.0254
500	10	10	0.30	0.50	0.9553	0.0105
500	10	10	0.30	0.70	0.9530	0.0196

Appendix C

COMPUTER PROGRAMS

This appendix contains the FORTRAN computer programs used in the simulation studies discussed in main body.

C.1 FORTRAN Program for Simulation Study of the Imputation Method Applied to Tolerance Intervals for Distributions from Unbalanced One-way Random Effects Model

This FORTRAN program is for the simulation study to test the performance of the method discussed in Section 3.2.

```
! Filename  C:\1Thesis\Generalized\ImputeFactorA\ProgramAM.f90
!
!  Purpose: to do simulation of ave method for unbalanced one-way
!  mixed anova model, using Generlized method
!
!Order of input values
!Number of rho to be inputed
!The rho's
!number of levels
!The replicates per level, there may be multiples lines of replicates.
!The last replicate must be equal to max size

MODULE Constants;
  CHARACTER (len=45), PARAMETER :: &
      input_fylename = "i_3levels_AM.txt", &
      !unit = 13
      up_output_fylename ="o_upper_bounds_AM.txt";
      !unit = 16

!DON'T FORGET TO CHANGE THE WRITE STATEMENT TO THE OUTPUT FILE AND THE
!OUTPUT TO THE SCREEN STATEMENT WITH THE LEVELS
  DOUBLE PRECISION, PARAMETER :: sigma2_e = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  INTEGER, PARAMETER :: num_runs=10000;
  INTEGER, PARAMETER :: num_dist_runs=10000;
```

```
  INTEGER, PARAMETER :: num_conf_levels=3;
  DOUBLE PRECISION , DIMENSION(1:3), PARAMETER :: conf=(/.9, .95, .99 /);
  DOUBLE PRECISION, PARAMETER :: content=.90

  INTEGER :: N, num_levels, size_max_level, N_extend, &
    num_missing, run,i_b;
  DOUBLE PRECISION, DIMENSION (1:2) :: phi, phi_hat;
  DOUBLE PRECISION :: up_true_quantile, z_cont, R, &
                      y_bar, s2_1, s2_2;
  INTEGER, DIMENSION (1:num_runs) :: reml_info;
  DOUBLE PRECISION, DIMENSION (1:2) :: df_ave;
  DOUBLE PRECISION, DIMENSION (1:num_conf_levels,1:num_runs) :: &
                         up_tol_bound, low_tol_bound;
  TYPE cov_stats
  DOUBLE PRECISION :: actual_coverage, expect_tol_bound, expect_abs_dist;
  END TYPE cov_stats;
  TYPE (cov_stats), DIMENSION (1:num_conf_levels) :: Upper_results;
  INTEGER, DIMENSION (1:2) :: df_ave_count;
END MODULE Constants;

PROGRAM OneWayAve;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: levels;
  DOUBLE PRECISION, DIMENSION (:), ALLOCATABLE :: rho_s, y_data
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE ::  &
       one_N,X_alpha,I_N,V_alpha,D,D_inv,y_ext_hat;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end,rho_count,rho_index;

  OPEN(UNIT=12, FILE=low_output_fylename, ACTION="READWRITE", &
      POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  OPEN(UNIT=16, FILE=up_output_fylename, ACTION="READWRITE", &
      POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");

  iseed=12345;
  CALL RNSET(iseed);
  z_cont=DNORIN(content);
  phi(2)=sigma2_e;

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) rho_count;
  ALLOCATE (rho_s(1:rho_count), stat = allocation_status);allocation=1;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) rho_s;
```

```
READ(13,*,iostat=is_end) num_levels;
ALLOCATE (levels(1:num_levels), stat = allocation_status);allocation=1;
CALL AllocationCheck(allocation_status,allocation);

WRITE(16,"(2(a,i6))") "# coverage runs:", num_runs, &
   "; # dist runs:", num_dist_runs;
WRITE(16,"(a,f5.2,a,3f5.2)") "content=",content, &
   "; confidence levels",conf;
WRITE(16,"(a)") "Output format is:";
WRITE(16,"(a)") "Replicates";
WRITE(16,"(a)") "Rho, actual coverages, theta, E[tol bound]s, &
   E[abs dist]";
ReadLevels:&
DO
  READ(13,*,iostat=is_end) levels;
  ReadFyle:&
  IF (is_end .lt. 0) THEN
    EXIT;
  ELSE
!*****  MAY NEED TO CHANGE THE NUMBER IN FRONT OF i
    Print "(3f6.3)", confs;
    PRINT "(a,10i3)", "Replicates", levels;
    WRITE(16,"(10i3)") levels;
    N=0;
    DO i=1,num_levels;
      N=N+levels(i);
    END DO;
    size_max_level=levels(num_levels);
    N_extend=size_max_level*num_levels;
    num_missing=N_extend-N;

    ALLOCATE (I_N(1:N,1:N), stat = allocation_status);allocation=2;
    CALL AllocationCheck(allocation_status,allocation);
    I_N(1:N,1:N)=0;
    CALL Identity(I_N,N);
    ALLOCATE (one_N(1:N,1), stat = allocation_status);allocation=3;
    CALL AllocationCheck(allocation_status,allocation);
    one_N(1:N,1)=1;
    ALLOCATE (X_alpha(1:N,1:num_levels), stat = allocation_status);
           allocation=4;
    CALL AllocationCheck(allocation_status,allocation);
    X_alpha(1:N,1:num_levels)=0;
    CALL DiagonalOnes(X_alpha,levels,N);

    ALLOCATE (V_alpha(1:N,1:N), stat = allocation_status);allocation=5;
    CALL AllocationCheck(allocation_status,allocation);
    V_alpha(1:N,1:N)=0;
    CALL DiagonalJs(V_alpha,levels,N);
```

```
ALLOCATE (D(1:N_extend,1:N_extend), stat = allocation_status);
     allocation=6;
CALL AllocationCheck(allocation_status,allocation);
ALLOCATE (D_inv(1:N_extend,1:N_extend), stat = allocation_status);
    allocation=7;
CALL AllocationCheck(allocation_status,allocation);
D(1:N_extend,1:N_extend)=0;
CALL MakeD(D,levels);
CALL Transpose(D,D_inv,N_extend,N_extend);

ALLOCATE (y_data(1:N), stat = allocation_status);allocation=7
CALL AllocationCheck(allocation_status,allocation);
ALLOCATE (y_ext_hat(1:N_extend,1:1), stat = allocation_status);
      allocation=8
CALL AllocationCheck(allocation_status,allocation);

RhoLoop:&
DO rho_index=1,rho_count;
  PRINT "(a,f10.6)", "rho=", rho_s(rho_index);
  R=rho_s(rho_index)/(1-rho_s(rho_index));
  phi(1)=R*phi(2);
  up_true_quantile=mu+z_cont*dsqrt(phi(1));
  df_ave_count(1:2)=0;

  RunLoop:&
  DO run=1,num_runs;
  if ((run==50) .OR. (mod(run,2000)==0)) then; print *, &
      "run", run; end if;
    CALL GenerateY(y_data,levels);
    reml_info(run)=1;
    CALL REML(one_N,I_N,V_alpha,y_data,levels);
    RemlDecision:&
    IF (reml_info(run) .ne. 1) THEN
      up_tol_bound(1:3,run)=99999;
    ELSE
      CheckBalance:&
      IF (num_missing == 0) THEN
        DO i_b=1,N;
          y_ext_hat(i_b,1)=y_data(i_b);
        END DO
        df_ave(1)=num_levels-1;
        df_ave(2)=num_levels*(size_max_level-1);
      ELSE
        CALL GetY_ext_hat(one_N,I_N,V_alpha,y_data,D,D_inv&
             &,y_ext_hat, levels);
        CALL CalcAveDF(y_ext_hat);
      END IF CheckBalance
```

```
          CALL CalcStats(y_ext_hat);
          CALL CalcToleranceLimit;
        END IF RemlDecision;
      END DO RunLoop;
      CALL CalcUpCoverage;
      WRITE(16,"(f6.2,3f7.3,f7.2,3f7.2,3f7.2)") &
        rho_s(rho_index),&
        Upper_results(1:3)%actual_coverage,&
        up_true_quantile,&
        Upper_results(1:3)%expect_tol_bound,&
        Upper_results(1:3)%expect_abs_dist;
    END DO RhoLoop;
    WRITE(16,"(a)") "";
  END IF ReadFyle;
  DEALLOCATE(I_N,one_N,X_alpha,V_alpha,y_data,D,D_inv,y_ext_hat);
 END DO ReadLevels;
 CLOSE(STATUS = "KEEP", UNIT = 16);
 PRINT *, "end of program";
END PROGRAM OneWayAve;


!***********  GENERALIZED CONFIDENCE INTERVAL  *****************
!                    SUBROUTINES

SUBROUTINE CalcToleranceLimit;
  USE Constants;

  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: N_0_1, Chi_1, Chi_2, &
      sigma_y_hat,sigma_a_hat, T_star_up;
  DOUBLE PRECISION :: DF,zero_cutoff;
  INTEGER :: index;
  INTEGER :: a,reps;

  a=num_levels;
  reps=size_max_level;
  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=df_ave(1);
  CALL DRNCHI(num_dist_runs,DF,Chi_1);
  DF=df_ave(2)
  CALL DRNCHI(num_dist_runs,DF,Chi_2);

  zero_cutoff=s2_2/s2_1;
  DO index=1,num_dist_runs;
    IF ( Chi_2(index)/Chi_1(index) < zero_cutoff ) THEN
      sigma_a_hat(index)=0;
      sigma_y_hat(index)=dsqrt( s2_1 /Chi_1(index) );
    ELSE
      sigma_a_hat(index)=dsqrt( (s2_1/Chi_1(index) - &
```

```
      s2_2/Chi_2(index))/reps );
    sigma_y_hat(index)=dsqrt( s2_1/Chi_1(index) );
  END IF;
END DO;
T_star_up = y_bar - N_0_1*sigma_y_hat/ &
      dsqrt(dble(reps*a)) + z_cont*sigma_a_hat;
CALL DSVRGN(num_dist_runs,T_star_up,T_star_up);
DO index=1,num_conf_levels;
  up_tol_bound(index,run)=T_star_up(ceiling(conf(index)*num_dist_runs));
END DO;
END SUBROUTINE CalcToleranceLimit;

SUBROUTINE CalcUpCoverage;
  USE Constants;
  INTEGER :: index, num_valid,conf_index;
  DOUBLE PRECISION :: tol_bound
  INTEGER, DIMENSION(1:num_conf_levels) :: num_ok;
  DOUBLE PRECISION, DIMENSION(1:num_conf_levels)::sum_bound,sum_abs_dist;

  num_valid=0;
  num_ok(1:num_conf_levels)=0;
  sum_bound(1:num_conf_levels)=0;
  sum_abs_dist(1:num_conf_levels)=0;
  DO index=1,num_runs;
    IF (up_tol_bound(1,index) .lt. 9999) THEN
      num_valid=num_valid+1;
      DO conf_index=1,num_conf_levels;
        tol_bound=up_tol_bound(conf_index,index);
        sum_bound(conf_index)=sum_bound(conf_index)+tol_bound;
        sum_abs_dist(conf_index)=sum_abs_dist(conf_index)+ &
           abs(up_true_quantile-tol_bound)
        IF (up_true_quantile .le. tol_bound) THEN
           num_ok(conf_index)=num_ok(conf_index)+1;
    END IF;
  END DO;
END IF;
  END DO;
  DO conf_index=1,num_conf_levels;
    Upper_results(conf_index)%actual_coverage = &
       dble(num_ok(conf_index))/dble(num_valid);
    Upper_results(conf_index)%expect_tol_bound = &
       sum_bound(conf_index)/dble(num_valid);
    Upper_results(conf_index)%expect_abs_dist = &
       sum_abs_dist(conf_index)/dble(num_valid);
  END DO;
  IF (num_valid .lt. num_runs) THEN
    WRITE(14,"(a,f6.2,a,i4)") "R = ", R, &
      " # runs with nonconvergent reml is",num_runs-num_valid;
```

```
  END IF;
END SUBROUTINE CalcUpCoverage;

!**************    MISCELLANEOUS  *****************
!                  SUBROUTINTES

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;


!*************** GENERATES DATA, EXTENDS DATA, REML ************
!                          SUBROUTINES
!GenerateY: generates the data used in the simulation
!REML: calculates the REML estimates
!GetY_ext_hat: calculates the imputed values needed to make the
! Y data vector a balanced.
!CalcAveDF: Calculates Hockings suggested Degrees of Freedom.


SUBROUTINE GenerateY(y_data,levels);
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:N) ::  y_data;
  INTEGER, DIMENSION (1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:num_levels) :: alphas;
  DOUBLE PRECISION, DIMENSION (1:N) :: errors;
  INTEGER :: i,j,place ;

  CALL DRNNOR(num_levels,alphas);
  CALL DRNNOR(N,errors);
  place=0;
  DO i=1,num_levels;
    DO j=1,levels(i);
      place=place+1
      y_data(place)=mu+sqrt(phi(1))*alphas(i)+errors(place);
    END DO;
  END DO;
END SUBROUTINE GenerateY;

SUBROUTINE REML(one_N,I_N,V_alpha,y_data,levels);
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:N) ::  y_data;
  DOUBLE PRECISION, DIMENSION (1:N,1) :: one_N, test_m, ty_data;
  DOUBLE PRECISION, DIMENSION (1:N,1:N) :: I_N;
```

```
DOUBLE PRECISION, DIMENSION (1:N,1:N) :: V_alpha;
INTEGER, DIMENSION(1:num_levels) :: levels;
DOUBLE PRECISION, DIMENSION(1:N,1:N) :: V,V_inv,M,V_invM,t_alpha,temp;
DOUBLE PRECISION, DIMENSION (1:2) :: phi_n, phi_n_1;
DOUBLE PRECISION, DIMENSION (1:2) :: rho;
DOUBLE PRECISION, DIMENSION(1:2,1:2) :: omega;
DOUBLE PRECISION :: distance, determinant,sumsq, sumlin;
INTEGER :: iteration,IPATH,index;

phi_n = phi;
iteration=1;
DO
  IF (iteration==1000) THEN
    reml_info(run)=-1;
    EXIT;
  END IF;
  V=phi_n(1)*V_alpha+phi_n(2)*I_N;
  V_inv(1:N,1:N)=0;
  CALL DiagonalMultJs(V_inv,levels,N,phi_n(1),phi_n(2));
  V_inv=(1/phi_n(2))*(I_N-V_inv);

  CALL CalM;
!Calculates V_inv*M
  CALL DMRRRR(N,N,V_inv,N,N,N,M,N,N,N,V_invM,N);
!Calculates the transpose of the y_data vector
  CALL DTRNRR(1,N,y_data,1,N,1,ty_data,N);
!Calculates V_inv*M*V_alpha
  CALL DMRRRR(N,N,V_invM,N,N,N,V_alpha,N,N,N,t_alpha,N);
!Calculates omega(1,2) and rho(1)
  CALL DMRRRR(N,N,t_alpha,N,N,N,V_invM,N,N,N,temp,N);
  omega(1,2)=Trace(temp)
  CALL DMRRRR(N,N,temp,N,N,1,ty_data,N,N,1,test_m,N);
  CALL DMRRRR(1,N,y_data,1,N,1,test_m,N,1,1,rho(1),1);
!Calculates omega(2,2) and rho(2)
  CALL DMRRRR(N,N,V_invM,N,N,N,V_invM,N,N,N,temp,N);
  omega(2,2)=Trace(temp);
  CALL DMRRRR(N,N,temp,N,N,1,ty_data,N,N,1,test_m,N);
  CALL DMRRRR(1,N,y_data,1,N,1,test_m,N,1,1,rho(2),1);
  CALL DMRRRR(N,N,t_alpha,N,N,N,t_alpha,N,N,N,temp,N);
  omega(1,1)=Trace(temp);
  CALL DMRRRR(N,N,V_invM,N,N,N,t_alpha,N,N,N,temp,N);
  omega(2,1)=Trace(temp);
  IPATH=1;

  determinant=omega(1,1)*omega(2,2)-omega(1,2)*omega(2,1);
  IF ((-.000001 .lt. determinant).and.(determinant .lt. .000001)) THEN
    reml_info(run)=0;
    EXIT;
```

```
  END IF;
  phi_n_1(1)=(rho(1)*omega(2,2)-rho(2)*omega(1,2))/determinant;
  phi_n_1(2)=(-rho(1)*omega(1,2)+rho(2)*omega(1,1))/determinant;
  distance=sqrt( (phi_n(1)-phi_n_1(1))**2 + (phi_n(2)-phi_n_1(2))**2 );
  IF (distance .le. .0000001)  THEN
    phi_hat=phi_n_1;
    IF (phi_hat(1) .le. 0) THEN
      phi_hat(1)=0;
      sumsq=0;sumlin=0;
      DO index=1,N;
        sumsq=sumsq+y_data(index)**2;
        sumlin=simlin+y_data(index);
      END DO;
      phi_hat(2)=(sumsq-((sumlin)**2)/N)/(N-1);
    END IF;
    EXIT;
  ELSE
    phi_n=phi_n_1;
    iteration=iteration+1;
  END IF;
END DO;

CONTAINS;
SUBROUTINE CalM;
  DOUBLE PRECISION, DIMENSION(1:N,1:N) :: tem;
  DOUBLE PRECISION, DIMENSION(1,1:N) :: oneTvinv;
  DOUBLE PRECISION :: stuff;

!Calculates 1_N'*V_inv
  CALL DMXTYF(N,1,one_N,N,N,N,V_inv,N,1,N,oneTvinv,1);
! Calculates 1_N'*V_inv*1_N  this is a scaler
  CALL DMRRRR(1,N,oneTvinv,1,N,1,one_N,N,1,1,stuff,1);
! Calculates 1_N*1_N*V_inv
  CALL DMRRRR(N,1,one_N,N,1,N,oneTvinv,1,N,N,tem,N);
! finally we get M
  M=I_N-tem/stuff;
END SUBROUTINE CalM;

FUNCTION Trace(matrix) result(sum);
  INTEGER :: i;
  DOUBLE PRECISION :: sum;
  DOUBLE PRECISION, DIMENSION(1:N,1:N) :: matrix;

  sum=0;
  DO i=1,N;
    sum=sum+matrix(i,i);
  END DO;
END FUNCTION Trace;
```

```
END SUBROUTINE REML;

SUBROUTINE GetY_ext_hat(one_N,I_N,V_alpha,y_data,D,D_inv,&
      y_ext_hat,levels);
  USE Constants;

  DOUBLE PRECISION, DIMENSION (1:N,1) :: one_N;
  DOUBLE PRECISION, DIMENSION (1:N,1:N) :: I_N,V_alpha;
  DOUBLE PRECISION, DIMENSION(1:N) ::  y_data;
  DOUBLE PRECISION, DIMENSION (1:N_extend,1:N_extend) :: D,D_inv;
  DOUBLE PRECISION, DIMENSION (1:N_extend,1) :: y_ext_hat;
  INTEGER, DIMENSION(1:num_levels) :: levels;
  DOUBLE PRECISION :: mu_hat;
  DOUBLE PRECISION, DIMENSION(1:num_missing,1:N) :: beta_hat;
  DOUBLE PRECISION, DIMENSION(1:num_missing,1) :: y_missing&
       &,one_num_missing,temp_missing;
  DOUBLE PRECISION, DIMENSION (1:N,1) :: ty_data,temp_N;
  DOUBLE PRECISION, DIMENSION (1:num_missing,1:N) :: V_mo
  DOUBLE PRECISION, DIMENSION(1:N,1:N) :: Voo,Voo_inv;
  DOUBLE PRECISION, DIMENSION(1:N_extend,1) :: y_star;

 !Calculates Voo and Voo_inv
  Voo=phi_hat(1)*V_alpha+phi_hat(2)*I_N;
!CALL DLINRG(N,Voo,N,Voo_inv,N);
  Voo_inv(1:N,1:N)=0;
  CALL DiagonalMultJs(Voo_inv,levels,N,phi_hat(1),phi_hat(2));
  Voo_inv=(1/phi_hat(2))*(I_N-Voo_inv);

 !Calculates the transpose of the y_data vector
  CALL DTRNRR(1,N,y_data,1,N,1,ty_data,N);
  CALL GetMuHat;
  CALL GetV_mo;
  one_num_missing=1;
  temp_N=ty_data-mu_hat*one_N;
  CALL DMRRRR(num_missing,N,V_mo,num_missing,N,N,Voo_inv,N, &
          num_missing,N,beta_hat,num_missing);
  CALL DMRRRR(num_missing,N,beta_hat,num_missing,N,1,temp_N,N, &
          num_missing,1,temp_missing,num_missing);
  y_missing=mu_hat*one_num_missing+temp_missing;
  y_star=0;
  y_star(1:N,1)=ty_data(1:N,1);
  y_star(N+1:N_extend,1)=y_missing(1:num_missing,1);
  CALL DMRRRR(N_extend,N_extend,D_inv,N_extend,N_extend,1,y_star, &
     N_extend, N_extend,1,y_ext_hat,N_extend);
  CONTAINS;
  SUBROUTINE GetV_mo;
    DOUBLE PRECISION, DIMENSION (1:N_extend,1:N_extend) :: V_a,temp;
    INTEGER, DIMENSION(1:num_levels) :: levels;
```

```
    INTEGER :: i,j;

    DO i=1,num_levels;
      levels(i)=size_max_level;
    END DO;
    V_mo(1:num_missing,1:N)=0;
    V_a(1:N_extend,1:N_extend)=0;
    CALL DiagonalJs(V_a,levels,N_extend);
    CALL DMRRRR(N_extend,N_extend,D,N_extend,N_extend,N_extend,V_a,&
      N_extend,N_extend,N_extend,temp,N_extend);
    CALL DMRRRR(N_extend,N_extend,temp,N_extend,N_extend,N_extend,D_inv,&
      N_extend,N_extend,N_extend,V_a,N_extend);
    V_mo(1:num_missing,1:N)=phi_hat(1)*V_a(N+1:N_extend,1:N);
  END SUBROUTINE GetV_mo;

  SUBROUTINE GetMuHat;
    DOUBLE PRECISION, DIMENSION(1,1:N) :: oneTvinv;
    DOUBLE PRECISION :: stuff;

    mu_hat=0;
  !Calculates 1_N'*V_inv
    CALL DMXTYF(N,1,one_N,N,N,N,Voo_inv,N,1,N,oneTvinv,1);
  !Calculates 1_N'*V_inv*1_N  this is a scaler
    CALL DMRRRR(1,N,oneTvinv,1,N,1,one_N,N,1,1,stuff,1);
  !Calculate mu_hat
    CALL DMRRRR(1,N,oneTvinv,1,N,1,ty_data,N,1,1,mu_hat,1);
    mu_hat=mu_hat/stuff;
  END SUBROUTINE GetMuHat;
END SUBROUTINE GetY_ext_hat;


SUBROUTINE CalcAveDF(y_ext_hat);
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:N_extend,1) ::  y_ext_hat;
  INTEGER, DIMENSION (1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:N_extend,1:N_extend) :: &
         P_calc,Ident,J_alpha,J_extend;
  DOUBLE PRECISION :: temp,EMS;
  INTEGER :: projection,i,j;

  Ident(1:N_extend,1:N_extend)=0;
  CALL Identity(Ident,N_extend);
  levels(1:num_levels)=size_max_level;
  J_alpha(1:N_extend,1:N_extend)=0;
  CALL DiagonalJs(J_alpha,levels,N_extend);
  J_alpha=J_alpha*(1/real(size_max_level));
  J_extend(1:N_extend,1:N_extend)=1/real(N_extend);
  DO projection=1,2;
```

```
    IF (projection == 1) THEN
      P_calc=J_alpha-J_extend;
      EMS=real(size_max_level)*phi_hat(1)+phi_hat(2);
    ELSE
      P_calc=Ident-J_alpha;
      EMS=phi_hat(2);
    END IF;
    temp=0;
    DO i=1,N_extend;
      DO j=1,N_extend;
        temp=temp+P_calc(i,j)*y_ext_hat(i,1)*y_ext_hat(j,1)
      END DO;
    END DO;
    df_ave(projection)=temp/EMS;
  END DO;
END SUBROUTINE CalcAveDF;

SUBROUTINE CalcStats(y_ext_hat);
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:N_extend,1) ::  y_ext_hat;
  INTEGER :: level, replicate, index;
  DOUBLE PRECISION :: sum_ij, sum_ij_sq, sum_i_sq;
  DOUBLE PRECISION :: sum_level;

  sum_ij=0; sum_ij_sq=0;
  sum_i_sq=0
  index=0;
  DO level=1,num_levels;
    sum_level=0;
    DO replicate=1,size_max_level;
      index=index+1;
      sum_ij=sum_ij+y_ext_hat(index,1);
      sum_ij_sq=sum_ij_sq+y_ext_hat(index,1)**2
      sum_level=sum_level+y_ext_hat(index,1);
    END DO;
    sum_i_sq=sum_i_sq+sum_level**2;
  END DO;

  s2_1=sum_i_sq/size_max_level-(sum_ij**2)/N_extend;
  s2_2=sum_ij_sq-(sum_i_sq/size_max_level);
  y_bar=sum_ij/N_extend;
END SUBROUTINE CalcStats;


!**************    MATRIX OPERATIONS  *****************
!                    SUBROUTINES

SUBROUTINE IntializeMatrix(matrix,num_rows,num_cols);
```

```
  INTEGER :: size;
  DOUBLE PRECISION, DIMENSION (1:num_rows,1:num_cols) :: matrix;

  DO i=1,num_rows;
    DO j=1,num_cols;
      matrix(i,j)=0;
    END DO;
  END DO;
END SUBROUTINE IntializeMatrix;

SUBROUTINE Transpose(matrix,matrix_T,num_rows,num_columns);
  DOUBLE PRECISION, DIMENSION (1:num_rows,1:num_columns) :: matrix;
  DOUBLE PRECISION, DIMENSION (1:num_columns,1:num_rows) :: matrix_T;
  INTEGER :: now_rows, num_columns;
  INTEGER :: i,j;

  DO i=1,num_columns;
    DO j=1,num_rows;
     matrix_T(i,j)=matrix(j,i);
    END DO;
  END DO;
END SUBROUTINE Transpose;

SUBROUTINE Identity(matrix,size);
  INTEGER :: size;
  DOUBLE PRECISION, DIMENSION (1:size,1:size) :: matrix;
  INTEGER :: i,j;

  DO i=1,size;
    DO j=1,size;
      IF (i == j)  THEN
          matrix(i,j)=1;
      END IF;
    END DO;
  END DO;
END SUBROUTINE Identity;

SUBROUTINE OneN(matrix,size);
  INTEGER :: size;
  DOUBLE PRECISION, DIMENSION (1:size,1) :: matrix;
  INTEGER :: i;

  DO i=1,size;
    matrix(i,1)=1;
  END DO;
END SUBROUTINE OneN;

SUBROUTINE DiagonalOnes(matrix,levels,size);
```

```
  USE Constants;

  INTEGER :: size;
  INTEGER, DIMENSION(1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:size,1:num_levels) :: matrix;
  INTEGER :: i,k;
  INTEGER :: lower_limit,upper_limit;

  lower_limit=0;
  DO k=1,num_levels;
    IF (k == 1) THEN
      lower_limit=1;
      upper_limit=levels(1);
    ELSE
      lower_limit=lower_limit+levels(k-1);
      upper_limit=upper_limit+levels(k);
    END IF;
    DO i=lower_limit,upper_limit;
      matrix(i,k)=1;
    END DO;
  END DO;
END SUBROUTINE DiagonalOnes;

SUBROUTINE DiagonalJs(matrix,levels,size);
  USE Constants;
  INTEGER :: size;
  INTEGER, DIMENSION(1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:size,1:size) :: matrix;
  INTEGER :: i,j,k;
  INTEGER :: lower_limit,upper_limit;

  lower_limit=0;
  DO k=1,num_levels;
    IF (k == 1) THEN
      lower_limit=1;
      upper_limit=levels(1);
    ELSE
      lower_limit=lower_limit+levels(k-1);
      upper_limit=upper_limit+levels(k);
    END IF;
    DO i=lower_limit,upper_limit;
      DO j=lower_limit,upper_limit;
         matrix(i,j)=1;
      END DO;
    END DO;
  END DO;
END SUBROUTINE DiagonalJs;
```

```
SUBROUTINE DiagonalMultJs(matrix,levels,size,phi1,phi2);
  USE Constants;
  INTEGER :: size;
  INTEGER, DIMENSION(1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:size,1:size) :: matrix;
  DOUBLE PRECISION :: phi1,phi2;
  INTEGER :: i,j,k;
  INTEGER :: lower_limit,upper_limit;

  lower_limit=0;
  DO k=1,num_levels;
    IF (k == 1) THEN
      lower_limit=1;
      upper_limit=levels(1);
    ELSE
      lower_limit=lower_limit+levels(k-1);
      upper_limit=upper_limit+levels(k);
    END IF;
    DO i=lower_limit,upper_limit;
      DO j=lower_limit,upper_limit;
         matrix(i,j)=phi1/(phi2+levels(k)*phi1);
      END DO;
    END DO;
  END DO;
END SUBROUTINE DiagonalMultJs;

SUBROUTINE MakeD(matrix,levels);
  USE Constants;
  INTEGER, DIMENSION(1:num_levels) :: levels;
  DOUBLE PRECISION, DIMENSION (1:N_extend,1:N_extend) :: matrix;
  INTEGER :: i,j,k ;
  INTEGER :: v_start,v_end,h_position;

! make the part of D that puts the non missing data at the top
  DO k=1,num_levels;
    IF (k == 1) THEN
      v_start=1;
      v_end=levels(1);
    ELSE
      v_start=v_start+levels(k-1);
      v_end=v_end+levels(k);
    END IF;
    h_position=size_max_level*(k-1);
    DO i=v_start,v_end;
      h_position=h_position+1;
      matrix(i,h_position)=1;
    END DO;
  END DO;
```

```
! make the part of d that puts the missing data at the bottom
  DO k=1,num_levels;
    IF (k == 1) THEN
      v_start=N+1;
      v_end=N+(size_max_level-levels(1));
    ELSE
      v_start=v_start+(size_max_level-levels(k-1));
      v_end=v_end+(size_max_level-levels(k));
    END IF;
    h_position=(size_max_level*(k-1))+levels(k);
    DO i=v_start,v_end;
      h_position=h_position+1;
      matrix(i,h_position)=1;
    END DO;
  END DO;
END SUBROUTINE MakeD;
```

C.2 FORTRAN Programs for Simulation Study for Tolerance Intervals for Distributions Associated with Two-way Random Effects Model with Interaction

Section C.2.1 contains the complete program used to test the performance of the proposed tolerance intervals for the distribution of $Y = \mu + A + B + AB + e$ for the model with interaction, see Section 3.4.1. The program used to test the performance of tolerance intervals for the distribution of $\mu + A$ (see Section 3.4.2) is nearly identical to the one used for $Y = \mu + A + B + AB + e$. In Section C.2.2 we explain the differences and then present the code that needs to be substituted into the program for $Y = \mu + A + B + AB + e$ to produce the program used for $\mu + A$. Section C.2.2 also contains the input file of the parameter settings used in the simulation.

C.2.1 Program for $Y = \mu + A + B + AB + e$

```
!Filename
!C:\Thesis\Two_Way\Y_WI_R\Program.f90
!
!Purpose:  To test generalized confindence interval technique
!for finding tolerance limits
!
!ORDER OF INPUT VALUES
```

```
!number of different factor levels for Factor A
!factor levels for A
!
!number of different factor levels for B
!factor levels for B
!
!number of different levels of replicates
!replicate levels
!
!Number of R1's to be inputed
!The R_1's
!
!Number of R2's to be inputed
!The R_2's
!
!Number of R3's to be inputed
!The R_3's

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
     input_fylename = "iY_WI_R1.txt", &
         !unit = 13
     output_fylename = "oY_WI_R2nd.txt";
         !unit = 12
!DON'T FORGET TO CHANGE THE WRITE STATEMENT TO THE OUTPUT FILE AND THE
!OUTPUT TO THE SCREEN STATEMENT WITH THE LEVELS
  DOUBLE PRECISION, PARAMETER :: sigma2_e = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  DOUBLE PRECISION, PARAMETER :: content =.90;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;

  INTEGER :: coverage_run,a,b,n;
  DOUBLE PRECISION, DIMENSION (1:4) :: phi;
  DOUBLE PRECISION, DIMENSION (1:3) :: R;
  DOUBLE PRECISION :: true_quantile;
  TYPE output_data
    DOUBLE PRECISION :: tolerance_limit;
    INTEGER :: relative_to_true_quantile;
    DOUBLE PRECISION :: distance;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: z_cont,actual_coverage;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) :: y_bar, &
    ssa, ssb, ssab, ssr;
END MODULE Constants;
```

```
PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: az,bz,nz;
  DOUBLE PRECISION, DIMENSION (:), ALLOCATABLE ::  r1z, r2z, r3z;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, &
    num_az, a_index, &
    num_bz, b_index, &
    num_nz, n_index, &
    num_r1z, r1_index, &
    num_r2z, r2_index, &
    num_r3z, r3_index;

  iseed=12345;
  CALL RNSET(iseed);
  z_cont=DNORIN(content);
  phi(4)=sigma2_e;

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_az;
  ALLOCATE (az(1:num_az), stat = allocation_status);allocation=1;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) az;

  READ(13,*,iostat=is_end) num_bz;
  ALLOCATE (bz(1:num_bz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) bz;

  READ(13,*,iostat=is_end) num_nz;
  ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) nz;

  READ(13,*,iostat=is_end) num_r1z;
  ALLOCATE (r1z(1:num_r1z), stat = allocation_status);allocation=3;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) r1z;

  READ(13,*,iostat=is_end) num_r2z;
  ALLOCATE (r2z(1:num_r2z), stat = allocation_status);allocation=4;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) r2z;

  READ(13,*,iostat=is_end) num_r3z;
```

```
ALLOCATE (r3z(1:num_r3z), stat = allocation_status);allocation=5;
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) r3z;

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
     POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,f5.2)") "content=", content;
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")"  a  b  n    R1     R2     R3    theta    coverage";
CLOSE(UNIT=12, STATUS= "KEEP");

ALoop:&
DO a_index=1,num_az;
a=az(a_index);

BLoop:&
DO b_index=1,num_bz;
b=bz(b_index);

NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
    POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
R1Loop:&
DO r1_index=1,num_r1z;
  R(1)=r1z(r1_index);
  phi(1)=R(1)*phi(4);
  R2Loop:&
  DO r2_index=1,num_r2z;
IF (r2z(r2_index) .gt. R(1)) THEN
  EXIT
END IF;
PRINT "(9(a,i2))",&
       "r2 run:", r2_index,&
       ", r1 run:", r1_index," of",num_r1z,&
       ",  n run:", n_index," of",num_nz,&
       ",  b run:", b_index," of",num_bz,&
       ",  a run:", a_index," of",num_az;
    R(2)=r2z(r2_index);
    phi(2)=R(2)*phi(4);
    R3Loop:&
    DO r3_index=1,num_r3z;
      PRINT "(a,i2)","     r3 run:",r3_index;
      R(3)=r3z(r3_index)
      phi(3)=R(3)*phi(4);
```

```
        true_quantile=mu-z_cont*dsqrt( phi(1)+phi(2)+phi(3)+phi(4) )
        results(1:num_coverage_runs)%relative_to_true_quantile=0;
        CALL GenerateData;
        CoverageLoop:&
        DO coverage_run=1,num_coverage_runs;
!          IF (mod(coverage_run,500) == 0) THEN
!           PRINT *,"    coverage run ", coverage_run;
!          END IF;
          CALL CalcToleranceLimit;
          IF(true_quantile .ge. results(coverage_run)%tolerance_limit)THEN
             results(coverage_run)%relative_to_true_quantile=1;
          END IF;
        END DO CoverageLoop;
        CALL CalcCoverage;
        WRITE(12,"(3i3,f8.2,2(f8.2),a,f7.3,a,f7.4)") &
            a,b,n, R,"  ",true_quantile,"   ",actual_coverage;
      END DO R3Loop
      WRITE(12,"(a)") "";
    END DO R2Loop;
  END DO R1Loop;
  CLOSE(UNIT=12, STATUS= "KEEP");
  END DO NLoop;
  END DO BLoop;
  END DO ALoop;
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;

SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;

  CALL DRNNOR(num_coverage_runs,y_bar);
  y_bar=mu+dsqrt( (b*n*phi(1)+a*n*phi(2)+n*phi(3)+phi(4))/(n*a*b) )*y_bar;

  DF=dble(a-1);
  CALL DRNCHI(num_coverage_runs,DF,ssa);
  ssa=(b*n*phi(1)+n*phi(3)+phi(4))*ssa;
  DF=dble(b-1);
  CALL DRNCHI(num_coverage_runs,DF,ssb);
  ssb=(a*n*phi(2)+n*phi(3)+phi(4))*ssb;
  DF=dble((a-1)*(b-1));
```

```
  CALL DRNCHI(num_coverage_runs,DF,ssab);
  ssab=(n*phi(3)+phi(4))*ssab;
  DF=dble( a*b*(n-1) );
  CALL DRNCHI(num_coverage_runs,DF,ssr);
  ssr=phi(4)*ssr;
END SUBROUTINE GenerateData;

SUBROUTINE CalcToleranceLimit;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
      N_0_1, Chi_a, Chi_b, Chi_ab, Chi_r, &
      sigma_y_hat,sigma_ybar_hat,T_star;
  DOUBLE PRECISION :: DF,zero_cutoff;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_a);
  DF=dble(b-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_b);
  DF=dble( (b-1)*(a-1));
  CALL DRNCHI(num_dist_runs,DF,Chi_ab);
  DF=dble( a*b*(n-1) );
  CALL DRNCHI(num_dist_runs,DF,Chi_r);

  DO index=1,num_dist_runs;
    sigma_ybar_hat(index)=max(&
        ssa(coverage_run)/Chi_a(index) + &
        ssb(coverage_run)/Chi_b(index) - &
        ssab(coverage_run)/Chi_ab(index)&
        ,dble(0));
     sigma_ybar_hat(index)=dsqrt( 1/dble(a*b*n) * sigma_ybar_hat(index) );
  END DO;

  sigma_y_hat=&
     dsqrt( 1/dble(a*b*n) *& !sqrt
       ( & !sum
         a*ssa(coverage_run)/Chi_a + &
         b*ssb(coverage_run)/Chi_b + &
         (a*b-a-b)*ssab(coverage_run)/Chi_ab + &
         a*b*(n-1)*ssr(coverage_run)/Chi_r &
       ) & ! end  of sum
     ); !end of sqrt

  T_star=y_bar(coverage_run) - N_0_1*sigma_ybar_hat-z_cont*sigma_y_hat;
  CALL DSVRGN(num_dist_runs,T_star,T_star);
  results(coverage_run)%tolerance_limit= &
       T_star(floor((alpha)*num_dist_runs));
```

```
END SUBROUTINE CalcToleranceLimit;

SUBROUTINE CalcCoverage;
  USE Constants;
  INTEGER :: index, count;

  count=0;
  DO index=1,num_coverage_runs;
    IF (results(index)%relative_to_true_quantile == 1) THEN
       count=count+1;
    END IF;
  END DO;
  actual_coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;
```

C.2.2 Modifications Needed for $\mu + A$ and Example of Input File

Beyond naming of input and output files there are three major differences between the programs for Y and $\mu+A$. They are– 1) the calculation of the true $(1-p)^{\text{th}}$ quantile, 2) the calculation of the generalized lower tolerance bound and 3) the parameter range for the R2 loop. The true quantile calculation occurs at the beginning of the R3 loop. While the calculation of the generalized tolerance bound is done by the Subroutine `CalcToleranceLimit`. This section contains the two pieces of code that can be substituted at the above points to modify the program for Y to produce the program for $\mu + A$. In the program for $\mu + A$ the range for the R2 loop needs to be modified by removing the `IF (r2z(r2_index) .gt. R(1)) THEN EXIT END IF;` from right after the start of the R2Loop.

At the end of the section is the input file with the parameter combinations used in the simulation study.

```
true_quantile=mu-z_cont*dsqrt(phi(1))

SUBROUTINE CalcToleranceLimit;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
     N_0_1, Chi_a, Chi_b, Chi_ab, Chi_r, &
     sigma_ybar_hat,sigma_a_hat,T_star;
  DOUBLE PRECISION :: DF,zero_cutoff;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
```

```
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_a);
  DF=dble(b-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_b);
  DF=dble( (b-1)*(a-1));
  CALL DRNCHI(num_dist_runs,DF,Chi_ab);
  DF=dble( a*b*(n-1) );
  CALL DRNCHI(num_dist_runs,DF,Chi_r);

  zero_cutoff=ssab(coverage_run)/ssa(coverage_run);
  DO index=1,num_dist_runs;
    sigma_ybar_hat(index)=max(&
       ssa(coverage_run)/Chi_a(index) + &
       ssb(coverage_run)/Chi_b(index) - &
       ssab(coverage_run)/Chi_ab(index)&
       ,dble(0));
    sigma_ybar_hat(index)=dsqrt( 1/dble(a*b*n) * sigma_ybar_hat(index) );
    IF ( Chi_ab(index)/Chi_a(index) < zero_cutoff ) THEN
       sigma_a_hat(index)=0;
    ELSE
       sigma_a_hat(index)=dsqrt( (ssa(coverage_run)/Chi_a(index) - &
          ssab(coverage_run)/Chi_ab(index))/(b*n) );
    END IF;
  END DO;
  T_star = y_bar(coverage_run) - N_0_1*sigma_ybar_hat-z_cont*sigma_a_hat;
  CALL DSVRGN(num_dist_runs,T_star,T_star);
  results(coverage_run)%tolerance_limit= &
        T_star(floor((alpha)*num_dist_runs));
END SUBROUTINE CalcToleranceLimit;
```

```
3
2 5 10
3
2 5 10
3
2 5 10
9
0 .1 .25 .5 1 2 4 10 100
9
0 .1 .25 .5 1 2 4 10 100
5
.25 .5 1 2 4
```

C.3 FORTRAN Programs for Simulation Study of Tolerance Intervals for Distribution Associated with Two-way Random Effects Models Without Interaction

C.3.1 Program for $\mu + A$

```
!Filename
!C:\Thesis\Two_Way\A_WOInt_Rand\Program.f90
!
!Purpose:  To a simulation to test generalized confindence interval
! technique for finding tolerance limits
!
!ORDER OF INPUT VALUES
!number of different factor levels for A
!factor levels for A
!
!number of different factor levels for B
!factor levels for B
!
!number of different levels of replicates
!replicate levels
!
!Number of rho_1's to be inputed
!The rho_1's
!
!Number of rho_2's to be inputed
!The rho_2's

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
input_fylename = "in_A_WOI_R1.txt", &
            !unit = 13
output_fylename = "o_A_WOI_R2nd.txt";
             !unit = 12

!DON'T FORGET TO CHANGE THE WRITE STATEMENT TO THE OUTPUT FILE AND THE
!OUTPUT TO THE SCREEN STATEMENT WITH THE LEVELS

  DOUBLE PRECISION, PARAMETER :: sigma2_e = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  DOUBLE PRECISION, PARAMETER :: content =.90;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;

  INTEGER :: coverage_run,a,b,n;
  DOUBLE PRECISION, DIMENSION (1:3) :: phi;
  DOUBLE PRECISION, DIMENSION (1:2) :: R;
  DOUBLE PRECISION :: true_quantile;
  TYPE output_data
    DOUBLE PRECISION :: tolerance_limit;
    INTEGER :: relative_to_true_quantile;
```

```
    DOUBLE PRECISION :: distance;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: z_cont,actual_coverage;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) :: y_bar, &
    ssa, ssb, ssr;
  INTEGER, DIMENSION (1:num_coverage_runs) :: positive_sigma_a_hat;
END MODULE Constants;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: az,bz,nz;
  DOUBLE PRECISION, DIMENSION (:), ALLOCATABLE ::  r1z, r2z;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, &
    num_az, a_index, &
    num_bz, b_index, &
    num_nz, n_index, &
    num_r1z, r1_index, &
    num_rz, r2_index;

  iseed=12345;
  CALL RNSET(iseed);
  z_cont=DNORIN(content);
  phi(3)=sigma2_e;

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_az;
  ALLOCATE (az(1:num_az), stat = allocation_status);allocation=1;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) az;

  READ(13,*,iostat=is_end) num_bz;
  ALLOCATE (bz(1:num_bz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) bz;

  READ(13,*,iostat=is_end) num_nz;
  ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) nz;

  READ(13,*,iostat=is_end) num_r1z;
  ALLOCATE (r1z(1:num_r1z), stat = allocation_status);allocation=3;
  CALL AllocationCheck(allocation_status,allocation);
```

```
READ(13,*,iostat=is_end) r1z;

READ(13,*,iostat=is_end) num_r2z;
ALLOCATE (r2z(1:num_r2z), stat = allocation_status);allocation=4;
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) r2z;

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
     POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,f5.2)") "content=", content;
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")"   a   b   n     R1      R2     theta     coverage";
CLOSE(UNIT=12, STATUS= "KEEP");

ALoop:&
DO a_index=1,num_az;
a=az(a_index);

BLoop:&
DO b_index=1,num_bz;
b=bz(b_index);

NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
      POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
R1Loop:&
DO r1_index=1,num_r1z;
  PRINT "(8(a,i2))",&
      "r1 run", r1_index, " of",num_r1z, &
      ", n run:", n_index," of",num_nz,&
      ", b run:", b_index," of",num_bz,&
      ", a run:", a_index," of",num_az
  R(1)=r1z(r1_index);
  phi(1)=R(1)*phi(3);
R2Loop:&
DO r2_index=1,num_r2z;
  PRINT "(a,i3)","  r2 run:", r2_index;
    R(2)=r2z(r2_index);
    phi(2)=R(2)*phi(3);
    true_quantile=mu-z_cont*dsqrt(phi(1))
    results(1:num_coverage_runs)%relative_to_true_quantile=0;
    CALL GenerateData;
    CoverageLoop:&
    DO coverage_run=1,num_coverage_runs;
```

```
!         IF (mod(coverage_run,500) == 0) THEN
!          PRINT *,"    coverage run ", coverage_run;
!         END IF;
        CALL CalcToleranceLimit;
        IF (true_quantile .ge. results(coverage_run)%tolerance_limit) THEN
          results(coverage_run)%relative_to_true_quantile=1;
        END IF;
      END DO CoverageLoop;
      CALL CalcCoverage;
     WRITE(12,"(3i4,f8.2,f8.2,a,f7.3,a,f7.4)") &
          a,b,n,&
          R,"  ",true_quantile,&
          "   ",actual_coverage;
    END DO R2Loop;
    WRITE(12,"(a)") "";
  END DO R1Loop;

  CLOSE(UNIT=12, STATUS= "KEEP");
  END DO NLoop;
  END DO BLoop;
  END DO ALoop;
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;

SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;

  CALL DRNNOR(num_coverage_runs,y_bar);
  y_bar=mu+dsqrt( (b*n*phi(1)+a*n*phi(2)+phi(3))/(n*a*b) )*y_bar;

  DF=dble(a-1);
  CALL DRNCHI(num_coverage_runs,DF,ssa);
  ssa=(b*n*phi(1)+phi(3))*ssa;

  DF=dble(b-1);
  CALL DRNCHI(num_coverage_runs,DF,ssb);
  ssb=(a*n*phi(2)+phi(3))*ssb;

  DF=dble( (a-1)*(b-1) + a*b*(n-1) );
  CALL DRNCHI(num_coverage_runs,DF,ssr);
```

```
  ssr=phi(3)*ssr;
END SUBROUTINE GenerateData;

SUBROUTINE CalcToleranceLimit;
  USE Constants;

  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
      N_0_1, Chi_a, Chi_b, Chi_r, &
      sigma_ybar_hat,sigma_a_hat,T_star;
  DOUBLE PRECISION :: DF,zero_cutoff;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_a);
  DF=dble(b-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_b);
  DF=dble( (a-1)*(b-1) + a*b*(n-1) );
  CALL DRNCHI(num_dist_runs,DF,Chi_r);

  zero_cutoff=ssr(coverage_run)/ssa(coverage_run);
  DO index=1,num_dist_runs;
    sigma_ybar_hat(index)=max(&
        ssa(coverage_run)/Chi_a(index) + &
        ssb(coverage_run)/Chi_b(index) - &
        ssr(coverage_run)/Chi_r(index)&
        ,dble(0));
    sigma_ybar_hat(index)=dsqrt( 1/dble(a*b*n) * sigma_ybar_hat(index) );
    IF ( Chi_r(index)/Chi_a(index) < zero_cutoff ) THEN
      sigma_a_hat(index)=0;
    ELSE
      sigma_a_hat(index)=dsqrt( (ssa(coverage_run)/Chi_a(index) - &
        ssr(coverage_run)/Chi_r(index))/(b*n) );
     END IF;
  END DO;
  T_star = y_bar(coverage_run) - N_0_1*sigma_ybar_hat-z_cont*sigma_a_hat;
  CALL DSVRGN(num_dist_runs,T_star,T_star);
  results(coverage_run)%tolerance_limit= &
          T_star( floor((alpha)*num_dist_runs) );
END SUBROUTINE CalcToleranceLimit;

SUBROUTINE CalcCoverage;
  USE Constants;
  INTEGER :: index, count;

  count=0;
  DO index=1,num_coverage_runs;
    IF (results(index)%relative_to_true_quantile == 1) THEN
```

```
      count=count+1;
    END IF;
  END DO;
  actual_coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;
```

C.3.2 Modifications Needed for $Y = \mu + A + B + e$ and example of Input File

Beyond naming of input and output files there are three major differences between the programs for $\mu + A$ and Y. They are– 1) the calculation of the true $(1-p)^{\text{th}}$ quantile, 2) the calculation of the generalized lower tolerance bound and 3) the parameter range for the R2 loop. The true quantile calculation occurs at the beginning of the R3 loop. While the calculation of the generalized tolerance bound is done by the Subroutine `CalcToleranceLimit`. This section contains the two pieces of code that can be substituted at the above points to modify the program for $\mu + A$ to produce the simulation program for Y. Due to the symmetry between the parameters for the distribution of Y, in the program for Y the range for the R2 loop needs to be modified by adding the `IF (r2z(r2_index) .gt. R(1)) THEN EXIT END IF;` from right before the start of the R2Loop (also see C.2.1).

At the end of the section is the input file with the parameter combinations used in the simulation study.

```
true_quantile=mu-z_cont*dsqrt(phi(1)+phi(2)+phi(3))

SUBROUTINE CalcToleranceLimit;
  USE Constants;

  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
      N_0_1, Chi_a, Chi_b, Chi_r, &
      sigma_y_hat,sigma_ybar_hat,T_star;
  DOUBLE PRECISION :: DF,zero_cutoff;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_a);
  DF=dble(b-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_b);
  DF=dble( (a-1)*(b-1) + a*b*(n-1) );
  CALL DRNCHI(num_dist_runs,DF,Chi_r);
```

```
  DO index=1,num_dist_runs;
    sigma_ybar_hat(index)=max(&
       ssa(coverage_run)/Chi_a(index) + &
       ssb(coverage_run)/Chi_b(index) - &
       ssr(coverage_run)/Chi_r(index),dble(0));
    sigma_ybar_hat(index)=dsqrt( 1/dble(a*b*n) * sigma_ybar_hat(index) );
  END DO;

  sigma_y_hat=dsqrt( 1/dble(a*b*n) *& !sqrt
  ( & !sum
       a*ssa(coverage_run)/Chi_a + &
       b*ssb(coverage_run)/Chi_b + &
       (a*b*n-a-b)*ssr(coverage_run)/Chi_r &
       ) & ! end  of sum
     ); !end of sqrt
  T_star = y_bar(coverage_run) - N_0_1*sigma_ybar_hat-z_cont*sigma_y_hat;
  CALL DSVRGN(num_dist_runs,T_star,T_star);
  results(coverage_run)%tolerance_limit=&
          T_star( floor((alpha)*num_dist_runs) );
END SUBROUTINE CalcToleranceLimit;
```

```
3
2 5 10
3
2 5 10
3
2 5 10
9
0 .1 .25 .5 1 2 4 10 100
9
0 .1 .25 .5 1 2 4 10 100
```

C.4 FORTRAN Programs for Simulation Study of Lower Confidence Intervals for Proportion of Conformance of IID Sample

C.4.1 Program L2 method of Wang and Lam [54] Using Quantiles as Inputs

```
!Filename
!srv_jack_program.f90
!
!Purpose:  A simulation to test generalized confindence interval technique
!for finding proportion of conformance
!
!ORDER OF INPUT VALUES
!
!number of different levels of replicates
```

```
!replicate levels
!
!Number of quantile pairs; pairs are input as
!  (lower_quantile, upper_qunatile)  quantile pairs, one per each line.
!

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
      input_fylename = "in_srv_jack.txt", &
       !unit = 13
      output_fylename = "o_srv_jack.txt";
        !unit = 12

  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;


  INTEGER :: coverage_run, n;
  DOUBLE PRECISION :: lq, uq; !lq is lower quantile bound,uq uppper QB
  DOUBLE PRECISION :: true_proportion;
  TYPE output_data
    DOUBLE PRECISION :: gen_est, lt_true_prop;
    DOUBLE PRECISION :: gen_dist;
    DOUBLE PRECISION :: L2_jack_est, L2_jack_dist;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: gen_LCB_coverage, L2_coverage;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) :: e_1, e_2;
END MODULE Constants;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: nz;
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE :: kappaz;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, num_nz, n_index, num_k_pairs, k_index;

  iseed=12345;
  CALL RNSET(iseed);

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_nz;
  ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
```

```
READ(13,*,iostat=is_end) nz;

READ(13,*,iostat=is_end) num_k_pairs;
ALLOCATE (kappaz(1:num_k_pairs,1:2),&
   stat = allocation_status);allocation=-1;
CALL AllocationCheck(allocation_status,allocation);
DO k_index=1,num_k_pairs;
  READ(13,*,iostat=is_end) kappaz(k_index,1:2);
END DO;
CLOSE(UNIT=13, STATUS= "KEEP");

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
   POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")" n   L_Q   U_Q   gen_cov   L2_cov  true proportion";
CLOSE(UNIT=12, STATUS= "KEEP");
 NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);
PRINT "(2(a,i2))", "N Loop:", n_index, " of", num_nz;
OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
    POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  KappaLoop:&
  DO k_index=1,num_k_pairs;
    lq=-kappaz(k_index,1);
    uq=kappaz(k_index,2);
    PRINT "(2(a,i2))",&
       "conformance limits:", k_index, " of", num_k_pairs;
    true_proportion=DNORDF(uq)-DNORDF(lq);
    results(1:num_coverage_runs)%gen_est=0;
    results(1:num_coverage_runs)%gen_dist=0;
    results(1:num_coverage_runs)%L2_jack_est=0;
    results(1:num_coverage_runs)%L2_jack_dist=0;
    CALL GenerateData;
    CoverageLoop:&
    DO coverage_run=1,num_coverage_runs;
      !IF (mod(coverage_run,500) == 0) THEN
        !PRINT *,"    coverage run ", coverage_run;
      !END IF;
      CALL GCBProporConforCoverage;
      CALL L2_jack;
    END DO CoverageLoop;
    CALL CalcCoverage( results(1:num_coverage_runs)%gen_est,&
         gen_LCB_coverage);
    CALL CalcCoverage( results(1:num_coverage_runs)%L2_jack_est,&
    L2_coverage);
   WRITE(12,"(i3,a,i4,a,i4,3(a,f7.4))") &
```

```
          n," ",-int(lq),"  ",int(uq),"   ",gen_LCB_coverage, &
             "   ", L2_coverage, "  ", true_proportion;
      CALL coverage_stats
    END DO KappaLooP;
  CLOSE(UNIT=12, STATUS= "KEEP");
  END DO NLoop;
  OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
        POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(12,"(a)") "";
  CLOSE(UNIT=12, STATUS= "KEEP");
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;

!Generates the e's, note these are not the sufficient statistics.
!It was done this way since there is no sigma involved.
!The sufficient statistics are
!calculated when the lower confidence bounds are calculated.

SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;

  CALL DRNNOR(num_coverage_runs,e_1);
  DF=dble(n-1);
  CALL DRNCHI(num_coverage_runs,DF,e_2);
END SUBROUTINE GenerateData;

SUBROUTINE GCBProporConforCoverage;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
     N_0_1, Chi_2, fid_mu, fid_denom, R_PC_star;
  DOUBLE PRECISION :: DF;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(n-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_2);
  fid_denom=dsqrt( e_2(coverage_run)/Chi_2 );
  fid_mu=e_1(coverage_run)/dsqrt(dble(n))- &
      N_0_1*fid_denom/dsqrt(dble(n));
  DO  index=1,num_dist_runs;
```

```
    R_PC_star(index) = &
      DNORDF( (uq-fid_mu(index))/fid_denom(index) ) - &
      DNORDF( (lq-fid_mu(index))/fid_denom(index) ) ;
  END DO;
  CALL DSVRGN(num_dist_runs,R_PC_star,R_PC_star);
  results(coverage_run)%gen_est=R_PC_star(floor(alpha*num_dist_runs));
END SUBROUTINE GCBProporConforCoverage;

SUBROUTINE L2_jack;
  USE Constants;
  DOUBLE PRECISION :: K1, K2, chi_table_value, DF;

  K1 = ( -lq+e_1(coverage_run)/dsqrt(dble(n)) ) / &
       dsqrt(e_2(coverage_run)/dble(n-1));
  K2 = ( uq-e_1(coverage_run)/dsqrt(dble(n)) ) / &
       dsqrt(e_2(coverage_run)/dble(n-1));
  DF=dble(n-1)
  chi_table_value = DCHIIN(alpha,DF);
  results(coverage_run)%L2_jack_est = &
     DNORDF( 1/dsqrt(dble(n)) + max(K1,K2)*&
    dsqrt(chi_table_value/dble(n-1)) ) - &
     DNORDF( 1/dsqrt(dble(n)) - min(K1,K2)*&
   dsqrt(chi_table_value/dble(n-1)) );
END SUBROUTINE L2_jack;

SUBROUTINE CalcCoverage(est_array,coverage);
  USE Constants;
  INTEGER :: index, count;
  DOUBLE PRECISION :: coverage;
  DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in)::est_array;

  count=0;
  DO index=1,num_coverage_runs;
    IF (est_array(index) .lt. true_proportion) THEN
       count=count+1;
END IF;
  END DO;
  coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;

SUBROUTINE coverage_stats;
  USE constants;
  DOUBLE PRECISION :: mean_dist, var_dist, mean_est, var_est;

  CALL calcdistance(results(1:num_coverage_runs)%gen_est, &
     results(1:num_coverage_runs)%gen_dist);
  CALL calcdistance(results(1:num_coverage_runs)%L2_jack_est, &
     results(1:num_coverage_runs)%L2_jack_dist);
```

```
  WRITE(12,"(a)"), "          m(est)   sd(est)   m(dist)  sd(dist) ";
  CALL mean_var(results(1:num_coverage_runs)%gen_est,mean_est,var_est);
  CALL mean_var(results(1:num_coverage_runs)%gen_dist,mean_dist,var_dist);
  WRITE(12,"(4(a,f7.4))"),"gen_LCB: ", mean_est, "   ", dsqrt(var_est),&
     "   ", mean_dist, "   ", dsqrt(var_dist);
  CALL mean_var(results(1:num_coverage_runs)%&
     L2_jack_est,mean_est,var_est);
  CALL mean_var(results(1:num_coverage_runs)%&
     L2_jack_dist,mean_dist,var_dist);
  WRITE(12,"(4(a,f7.4))"),"L2     : ", mean_est, "   ", dsqrt(var_est),&
     "   ", mean_dist, "   ", dsqrt(var_dist);
  WRITE(12,"(a)"),"";

  CONTAINS;
    SUBROUTINE mean_var(data_array,mean,var) ;
      DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in) :: &
         data_array;
      DOUBLE PRECISION :: sum, sum_sqs, mean, var;

      sum=0; sum_sqs=0;
      DO index=1,num_coverage_runs;
        sum = sum+data_array(index);
        sum_sqs = sum_sqs + data_array(index)**2;
      END DO;
      mean = sum/num_coverage_runs;
      var= (sum_sqs - sum**2/num_coverage_runs)/(num_coverage_runs);
    END SUBROUTINE mean_var;

    SUBROUTINE calcdistance(est_array,dist_array);
      INTEGER :: index, count;
       DOUBLE PRECISION, DIMENSION(1:num_coverage_runs),INTENT(in) :: &
         est_array;
       DOUBLE PRECISION, DIMENSION (1:num_coverage_runs):: dist_array;
      dist_array = abs(est_array-true_proportion);
    END SUBROUTINE calcdistance;
END SUBROUTINE coverage_stats;
```

C.4.2 Program Using Percentile Pairs as Inputs

```
!Filename
!Prog_s_r_v.f90
!
!Purpose:  To a simulation to test generalized confindence
!interval technique for finding proportion of conformance
!
!ORDER OF INPUT VALUES
!
```

```
!number of different levels of replicates
!replicate levels
!
!
!Number of percentiles pairs; pairs are input as
!(lower_percentile, upper_percentile) percentile pairs,
! one per each line.
!

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
     input_fylename = "in_srv.txt", &
       !unit = 13
     output_fylename = "o_srv.txt";
       !unit = 12
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  INTEGER, PARAMETER :: num_coverage_runs = 4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;

  INTEGER :: coverage_run, n;
  DOUBLE PRECISION :: lp, up; !lp is lower percentage bound, up uppper PB
  DOUBLE PRECISION :: true_proportion;
  TYPE output_data
     DOUBLE PRECISION :: estimate, dist_est;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: actual_coverage, mean_distance, var_distance;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) :: e_1, e_2;

END MODULE Constants;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: nz;
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE :: percentiles;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, num_nz, n_index, num_p_pairs, p_index;

  iseed=12345;
  CALL RNSET(iseed);

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_nz;
  ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
```

```
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) nz;

READ(13,*,iostat=is_end) num_p_pairs;
ALLOCATE (percentiles(1:num_p_pairs,1:2),&
   stat = allocation_status);allocation=-1;
CALL AllocationCheck(allocation_status,allocation);
DO p_index=1,num_p_pairs;
   READ(13,*,iostat=is_end) percentiles(p_index,1:2);
END DO;
CLOSE(UNIT=13, STATUS= "KEEP");

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
   POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")"  n      L_%     U_%     coverage    m(dist)";
CLOSE(UNIT=12, STATUS= "KEEP");

NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);
PRINT "(2(a,i2))", "N Loop:", n_index, " of", num_nz;
OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
     POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  PercentileLoop:&
  DO p_index=1,num_p_pairs;
    lp=percentiles(p_index,1);
    up=percentiles(p_index,2);
    PRINT "(2(a,i2))",&
      "conformance limits:", p_index, " of", num_p_pairs;
    true_proportion=up-lp;
    results(1:num_coverage_runs)%estimate=0;
    CALL GenerateData;
    CoverageLoop:&
    DO coverage_run=1,num_coverage_runs;
      !IF (mod(coverage_run,500) == 0) THEN
        !PRINT *,"    coverage run ", coverage_run;
      !END IF;
      CALL GCBProporConforCoverage;
    END DO CoverageLoop;
    CALL CalcCoverage( results(1:num_coverage_runs)%estimate,&
       actual_coverage);
    CALL coverage_stats;
    WRITE(12,"(i3,a,2f7.3,2(a,f7.4))") &
        n,"  ",lp,up,"   ",actual_coverage,"   ",mean_distance ;
  END DO PercentileLooP;
 CLOSE(UNIT=12, STATUS= "KEEP");
```

```
  END DO NLoop;
  OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
      POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(12,"(a)") "";
  CLOSE(UNIT=12, STATUS= "KEEP");
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;

SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;

  CALL DRNNOR(num_coverage_runs,e_1);
  DF=dble(n-1);
  CALL DRNCHI(num_coverage_runs,DF,e_2);
END SUBROUTINE GenerateData;

SUBROUTINE GCBProporConforCoverage;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
     N_0_1, Chi_2, fid_mu, fid_denom, R_theta;
  DOUBLE PRECISION :: DF, lcl, ucl;
  INTEGER :: index, num_lt;

  num_lt=0;
  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(n-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_2);
  fid_denom=dsqrt( e_2(coverage_run)/Chi_2 );
  fid_mu=e_1(coverage_run)/dsqrt(dble(n))- &
       N_0_1*fid_denom/dsqrt(dble(n));
  IF (lp == 0) THEN
    ucl= mu + DNORIN(up);
    DO  index=1,num_dist_runs;
      R_theta(index) = DNORDF((ucl-fid_mu(index))/fid_denom(index));
END DO;
  ELSE
    lcl= mu + DNORIN(lp);
    ucl= mu + DNORIN(up);
    DO  index=1,num_dist_runs;
   R_theta(index) = DNORDF((ucl-fid_mu(index))/fid_denom(index))-&
```

```
        DNORDF( (lcl-fid_mu(index))/fid_denom(index) );
      END DO;
    END IF;
    CALL DSVRGN(num_dist_runs,R_theta,R_theta);
    results(coverage_run)%estimate = R_theta(floor(alpha*num_dist_runs))
  END SUBROUTINE GCBProporConforCoverage;

  SUBROUTINE CalcCoverage(est_array,coverage);
    USE Constants;
    INTEGER :: index, count;
    DOUBLE PRECISION :: coverage;
    DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in)::est_array;

    count=0;
    DO index=1,num_coverage_runs;
      IF (est_array(index) .lt. true_proportion) THEN
        count=count+1;
  END IF;
    END DO;
    coverage=dble(count)/dble(num_coverage_runs);
  END SUBROUTINE CalcCoverage;

  SUBROUTINE coverage_stats;
    USE constants;

    CALL calcdistance(results(1:num_coverage_runs)%estimate, &
        results(1:num_coverage_runs)%dist_est);
    CALL mean_var(results(1:num_coverage_runs)%&
        dist_est,mean_distance,var_distance);

    CONTAINS;
      SUBROUTINE mean_var(data_array,mean,var) ;
        DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in)::&
       data_array;
        DOUBLE PRECISION :: sum, sum_sqs, mean, var;

        sum=0; sum_sqs=0;
        DO index=1,num_coverage_runs;
          sum = sum+data_array(index);
          sum_sqs = sum_sqs + data_array(index)**2;
        END DO;
        mean = sum/num_coverage_runs;
        var= (sum_sqs - sum**2/num_coverage_runs)/(num_coverage_runs);
  END SUBROUTINE mean_var;

      SUBROUTINE calcdistance(est_array,dist_array);
        INTEGER :: index, count;
        DOUBLE PRECISION, DIMENSION(1:num_coverage_runs),INTENT(in)::&
```

```
      est_array;
    DOUBLE PRECISION, DIMENSION(1:num_coverage_runs):: dist_array;
    dist_array = abs(est_array-true_proportion);
  END SUBROUTINE calcdistance;
END SUBROUTINE coverage_stats;
```

C.5 FORTRAN Programs for Simulation Study of Lower Confidence Intervals for Proportion of Conformance of Population Characteristics of the One-way Random Effects Model

C.5.1 Program for $Y = \mu + A + e$

```
!Filename
!Prog_y_r.f90
!
!Purpose:  A simulation to test generalized confindence interval
!technique for finding proportion of conformance
!
!ORDER OF INPUT VALUES
!
!number of different factor levels for A
!factor levels for A
!
!number of different levels of replicates
!replicate levels
!
!number of sigmaA
!The sigmaAz
!
!Number of percentiles pairs; pairs are input as
!(lower_percentile, upper_percentile) percentile pairs,
!one per each line.
!

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
     input_fylename = "in_y_r.txt", &
       !unit = 13
     output_fylename = "o_y_r.txt";
       !unit = 12
  DOUBLE PRECISION, PARAMETER :: sigma2_e = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;
```

```
  INTEGER :: coverage_run, a,n;
  DOUBLE PRECISION ::R, lp, up;!lp is lower percentage bound,up uppper PB
  DOUBLE PRECISION :: sigma_y_bar, sigma_y, sigma_A, sigma2_A;
  DOUBLE PRECISION :: true_proportion;
  TYPE output_data
     DOUBLE PRECISION :: estimate, dist_est;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: actual_coverage, mean_distance, var_distance;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) :: y_bar, ssa, ssr;
END MODULE Constants;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: az, nz;
  DOUBLE PRECISION, DIMENSION (:), ALLOCATABLE ::  sigma_Az;
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE :: percentiles;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, num_az, a_index, num_nz, n_index, &
    num_sigma_Az, sigma_A_index, num_p_pairs, p_index;

  iseed=12345;
  CALL RNSET(iseed);

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_az;
  ALLOCATE (az(1:num_az), stat = allocation_status);allocation=1;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) az;

  READ(13,*,iostat=is_end) num_nz;
  ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) nz;

  READ(13,*,iostat=is_end) num_sigma_Az;
  ALLOCATE (sigma_Az(1:num_sigma_Az), &
      stat = allocation_status);allocation=3;
  CALL AllocationCheck(allocation_status,allocation);
  READ(13,*,iostat=is_end) sigma_Az;

  READ(13,*,iostat=is_end) num_p_pairs;
  ALLOCATE (percentiles(1:num_p_pairs,1:2), &
      stat = allocation_status);allocation=-1;
  CALL AllocationCheck(allocation_status,allocation);
```

```
DO p_index=1,num_p_pairs;
  READ(13,*,iostat=is_end) percentiles(p_index,1:2);
END DO;
CLOSE(UNIT=13, STATUS= "KEEP");

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
   POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")"   a    n   sigma_A    L_%     U_%     coverage   m(dist)";
CLOSE(UNIT=12, STATUS= "KEEP");

ALoop:&
DO a_index=1,num_az;
a=az(a_index);

NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
    POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
Sigma_ALoop:&
DO sigma_A_index=1,num_sigma_Az;
  PRINT "(6(a,i2))",&
    "sigma_A run:", sigma_A_index, " of",num_sigma_Az, &
    ", n run:", n_index," of",num_nz,&
    ", a run:", a_index," of",num_az;
  sigma_A = sigma_Az(sigma_A_index);
  sigma2_A = sigma_A*sigma_A;
  sigma_y_bar=dsqrt( (n*sigma2_A+sigma2_e)/(n*a) );
  sigma_y = dsqrt(sigma2_A+sigma2_e);
  PercentileLoop:&
  DO p_index=1,num_p_pairs;
    lp=percentiles(p_index,1);
    up=percentiles(p_index,2);
    PRINT "(2(a,i2))",&
       "conformance limits:", p_index, " of", num_p_pairs;
    true_proportion=up-lp;
    results(1:num_coverage_runs)%estimate=0;
    results(1:num_coverage_runs)%estimate=0;
    CALL GenerateData;
    CoverageLoop:&
    DO coverage_run=1,num_coverage_runs;
      !IF (mod(coverage_run,500) == 0) THEN
         !PRINT *,"     coverage run ", coverage_run;
      !END IF;
      CALL GCBProporConforCoverage;
```

```
      END DO CoverageLoop;
      CALL CalcCoverage( results(1:num_coverage_runs)%estimate,&
         actual_coverage);
      CALL coverage_stats;
      WRITE(12,"(2i4,f7.2,a,2f7.3,2(a,f7.4))") a,n,&
         sigma_A,"  ",lp,up," ",actual_coverage,"   ",mean_distance;
    END DO PercentileLooP;
  END DO Sigma_ALoop;
  CLOSE(UNIT=12, STATUS= "KEEP");
  END DO NLoop;
  END DO ALoop;
  OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
       POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(12,"(a)") "";
  CLOSE(UNIT=12, STATUS= "KEEP");
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;


SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;

  CALL DRNNOR(num_coverage_runs,y_bar);
  y_bar=mu+sigma_y_bar*y_bar;
  DF=dble(a-1);
  CALL DRNCHI(num_coverage_runs,DF,ssa);
  ssa=(n*sigma2_A+sigma2_e)*ssa;
  DF=dble(a*(n-1));
  CALL DRNCHI(num_coverage_runs,DF,ssr);
  ssr=sigma2_e*ssr;
END SUBROUTINE GenerateData;

SUBROUTINE GCBProporConforCoverage;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
     N_0_1, Chi_a, Chi_r, &
     sigma_x_hat, mu_hat, R_theta;
  DOUBLE PRECISION :: DF, lcl, ucl;
  INTEGER :: index, num_lt;
```

```
  num_lt=0;
  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_a);
  DF=dble(a*(n-1));
  CALL DRNCHI(num_dist_runs,DF,Chi_r);

  mu_hat=y_bar(coverage_run)-N_0_1*&
       dsqrt( ssa(coverage_run)/dble(a*n)/Chi_a );
  sigma_x_hat=dsqrt( (ssa(coverage_run)/n)/Chi_a + &
    dble(n-1)/dble(n)*ssr(coverage_run)/Chi_r );
  IF (lp == 0) THEN
    ucl= mu + DNORIN(up)*sigma_y;
DO  index=1,num_dist_runs;
      R_theta(index) = DNORDF((ucl-mu_hat(index))/sigma_x_hat(index));
END DO;
  ELSE
    lcl= mu + DNORIN(lp)*sigma_y;
    ucl= mu + DNORIN(up)*sigma_y;
    DO  index=1,num_dist_runs;
     R_theta(index) = DNORDF((ucl-mu_hat(index))/sigma_x_hat(index))- &
       DNORDF( (lcl-mu_hat(index))/sigma_x_hat(index) );
    END DO;
  END IF;
  CALL DSVRGN(num_dist_runs,R_theta,R_theta);
  results(coverage_run)%estimate=R_theta(floor(alpha*num_dist_runs));
END SUBROUTINE GCBProporConforCoverage;

SUBROUTINE CalcCoverage(est_array,coverage);
  USE Constants;
  INTEGER :: index, count;
  DOUBLE PRECISION :: coverage;
  DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in) ::est_array;

  count=0;
  DO index=1,num_coverage_runs;
     IF (est_array(index) .lt. true_proportion) THEN
       count=count+1;
    END IF;
  END DO;
  coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;

SUBROUTINE coverage_stats;
  USE constants;

  CALL calcdistance(results(1:num_coverage_runs)%estimate, &
     results(1:num_coverage_runs)%dist_est);
```

```
  CALL mean_var(results(1:num_coverage_runs)%&
     dist_est,mean_distance,var_distance);

  CONTAINS;
    SUBROUTINE mean_var(data_array,mean,var) ;
      DOUBLE PRECISION,DIMENSION (1:num_coverage_runs),&
        INTENT(in)::data_array;
      DOUBLE PRECISION :: sum, sum_sqs, mean, var;

      sum=0; sum_sqs=0;
      DO index=1,num_coverage_runs;
        sum = sum+data_array(index);
        sum_sqs = sum_sqs + data_array(index)**2;
  END DO;
      mean = sum/num_coverage_runs;
      var= (sum_sqs - sum**2/num_coverage_runs)/(num_coverage_runs);
END SUBROUTINE mean_var;

    SUBROUTINE calcdistance(est_array,dist_array);
  INTEGER :: index, count;
      DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),&
        INTENT(in)::est_array;
      DOUBLE PRECISION,DIMENSION(1:num_coverage_runs)::dist_array;

      dist_array = abs(est_array-true_proportion);
END SUBROUTINE calcdistance;
END SUBROUTINE coverage_stats;
```

C.5.2 Program for $\mu + A$

```
!Filename
!A_r_program.f90
!
!Purpose: A simulation to test generalized confindence interval
!technique for finding proportion of conformance
!
!ORDER OF INPUT VALUES
!
!number of different factor levels for A
!factor levels for A
!
!number of different levels of replicates
!replicate levels
!
!Number of sigma_A
!The sigma_Az
```

```
!
!Number of percentiles pairs; pairs are input as
!(lower_percentile, upper_percentile) percentile pairs,
!one per each line.
!

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
    input_fylename = "in_A_r.txt", &
       !unit = 13
    output_fylename = "o_A_r.txt";
       !unit = 12
  DOUBLE PRECISION, PARAMETER :: sigma2_e = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;


  INTEGER :: coverage_run, a,n;
  DOUBLE PRECISION :: lp, up;!lp is lower percentage bound,up uppper PB
  DOUBLE PRECISION :: sigma_x_bar, sigma_A, sigma2_A;
  DOUBLE PRECISION :: true_proportion;
  TYPE output_data
    DOUBLE PRECISION :: estimate, dist_est;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: results;
  DOUBLE PRECISION :: actual_coverage, mean_distance, var_distance;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs) ::x_bar, ssb, ssw;
END MODULE Constants;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:), ALLOCATABLE :: az, nz;
  DOUBLE PRECISION, DIMENSION (:), ALLOCATABLE ::  sigma_Az;
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE :: percentiles;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, num_az, a_index, num_nz, n_index, &
    num_sigma_Az, sigma_A_index, num_p_pairs, p_index;

  iseed=12345;
  CALL RNSET(iseed);

  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_az;
```

```
ALLOCATE (az(1:num_az), stat = allocation_status);allocation=1;
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) az;

READ(13,*,iostat=is_end) num_nz;
ALLOCATE (nz(1:num_nz), stat = allocation_status);allocation=2;
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) nz;

READ(13,*,iostat=is_end) num_sigma_Az;
ALLOCATE (sigma_Az(1:num_sigma_Az), stat = allocation_status);
   allocation=3;
CALL AllocationCheck(allocation_status,allocation);
READ(13,*,iostat=is_end) sigma_Az;

READ(13,*,iostat=is_end) num_p_pairs;
ALLOCATE (percentiles(1:num_p_pairs,1:2), stat = allocation_status);
   allocation=-1;
CALL AllocationCheck(allocation_status,allocation);
DO p_index=1,num_p_pairs;
  READ(13,*,iostat=is_end) percentiles(p_index,1:2);
END DO;
CLOSE(UNIT=13, STATUS= "KEEP");

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
    POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
WRITE(12,"(a)")"   a   n  sigma_A   L_%    U_%    coverage  m(dist)";
CLOSE(UNIT=12, STATUS= "KEEP");

ALoop:&
DO a_index=1,num_az;
a=az(a_index);

NLoop:&
DO n_index=1,num_nz;
n=nz(n_index);

OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
    POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
Sigma_ALoop:&
DO sigma_A_index=1,num_sigma_Az;
   PRINT "(6(a,i2))",&
     "sigma_A run:", sigma_A_index, " of",num_sigma_Az, &
     ", n run:", n_index," of",num_nz,&
     ", a run:", a_index," of",num_az;
  sigma_A=sigma_Az(sigma_A_index);
```

```
    sigma2_A = sigma_A*sigma_A;
    sigma_x_bar=dsqrt( (n*sigma2_A+sigma2_e)/(n*a) );
    PercentileLoop:&
    DO p_index=1,num_p_pairs;
      lp=percentiles(p_index,1);
      up=percentiles(p_index,2);
      PRINT "(2(a,i2))",&
         "conformance limits:", p_index, " of", num_p_pairs;
      true_proportion=up-lp;
      results(1:num_coverage_runs)%estimate=0;
      results(1:num_coverage_runs)%dist_est=0
      CALL GenerateData;
      CoverageLoop:&
      DO coverage_run=1,num_coverage_runs;
        !IF (mod(coverage_run,500) == 0) THEN
           !PRINT *,"    coverage run ", coverage_run;
        !END IF;
        CALL GCBProporConforCoverage;
      END DO CoverageLoop;
      CALL CalcCoverage( results(1:num_coverage_runs)%estimate,&
         actual_coverage);
      CALL coverage_stats;
      WRITE(12,"(2i4,f7.2,a,2f7.3,2(a,f7.4))") a,n,sigma_A,"  ",&
        lp,up,"   ",actual_coverage,"  ",mean_distance ;
    END DO PercentileLooP;
  END DO Sigma_ALoop;
  CLOSE(UNIT=12, STATUS= "KEEP");
  END DO NLoop;
  END DO ALoop;
  OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
       POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(12,"(a)") "";
  CLOSE(UNIT=12, STATUS= "KEEP");
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;


SUBROUTINE GenerateData;
  USE Constants;

  DOUBLE PRECISION :: DF;
```

```
  CALL DRNNOR(num_coverage_runs,x_bar);
  x_bar=mu+sigma_x_bar*x_bar;
  DF=dble(a-1);
  CALL DRNCHI(num_coverage_runs,DF,ssb);
  ssb=(n*sigma2_A+sigma2_e)*ssb;
  DF=dble(a*(n-1));
  CALL DRNCHI(num_coverage_runs,DF,ssw);
  ssw=sigma2_e*ssw;
END SUBROUTINE GenerateData;

SUBROUTINE GCBProporConforCoverage;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: N_0_1, Chi_b, Chi_w, &
     R_sigma_A, R_mu, R_theta;
  DOUBLE PRECISION :: DF, lcl, ucl;
  INTEGER :: index;

  CALL DRNNOR(num_dist_runs,N_0_1);
  DF=dble(a-1);
  CALL DRNCHI(num_dist_runs,DF,Chi_b);
  DF=dble(a*(n-1));
  CALL DRNCHI(num_dist_runs,DF,Chi_w);
  R_mu=x_bar(coverage_run)-N_0_1*&
       dsqrt( ssb(coverage_run)/dble(a*n)/Chi_b );
  R_sigma_A= MAX(&
      dble(0), ( ssb(coverage_run)/Chi_b - ssw(coverage_run)/Chi_w )&
      /dble(n) );
  R_sigma_A = dsqrt(R_sigma_A);
  IF (lp == 0) THEN
    ucl= mu + DNORIN(up)*sigma_A;
    DO  index=1,num_dist_runs;
      IF (R_sigma_A(index) .lt. .000000000000001) THEN
        IF (ucl-R_mu(index) .lt. 0 ) THEN; R_theta(index) = 0;
        ELSE; R_theta(index) = 1; END IF;
      ELSE
        R_theta(index) = DNORDF( (ucl-R_mu(index))/R_sigma_A(index) );
      END IF;
    END DO;
  ELSE
    lcl= mu + DNORIN(lp)*sigma_A;
    ucl= mu + DNORIN(up)*sigma_A;
    DO  index=1,num_dist_runs;
      IF (R_sigma_A(index) .lt. .000000000000001) THEN
        IF ((ucl-R_mu(index) .gt. 0) .and. (lcl-R_mu(index) .lt. 0))THEN
          R_theta(index) = 1;
        ELSE
          R_theta(index) = 0;
```

```
      END IF;
    ELSE
      R_theta(index)= DNORDF((ucl-R_mu(index))/R_sigma_A(index))- &
      DNORDF( (lcl-R_mu(index))/R_sigma_A(index) );
    END IF;
  END DO;
END IF;
CALL DSVRGN(num_dist_runs,R_theta,R_theta);
results(coverage_run)%estimate=R_theta( floor(alpha*num_dist_runs) );
END SUBROUTINE GCBProporConforCoverage;

SUBROUTINE CalcCoverage(est_array,coverage);
  USE Constants;

  INTEGER :: index, count;
  DOUBLE PRECISION :: coverage;
  DOUBLE PRECISION,DIMENSION(1:num_coverage_runs),INTENT(in)::est_array;

  count=0;
  DO index=1,num_coverage_runs;
    IF (est_array(index) .lt. true_proportion) THEN
      count=count+1;
    END IF;
  END DO;
  coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;

SUBROUTINE coverage_stats;
  USE constants;

  CALL calcdistance(results(1:num_coverage_runs)%estimate, &
        results(1:num_coverage_runs)%dist_est);
  CALL mean_var(results(1:num_coverage_runs)% dist_est,&
        mean_distance,var_distance);

  CONTAINS;
    SUBROUTINE mean_var(data_array,mean,var) ;
      DOUBLE PRECISION,DIMENSION (1:num_coverage_runs),INTENT(in)::&
         data_array;
  DOUBLE PRECISION :: sum, sum_sqs, mean, var;

      sum=0; sum_sqs=0;
      DO index=1,num_coverage_runs;
        sum = sum+data_array(index);
        sum_sqs = sum_sqs + data_array(index)**2;
      END DO;
      mean = sum/num_coverage_runs;
      var= (sum_sqs - sum**2/num_coverage_runs)/(num_coverage_runs);
```

```
    END SUBROUTINE mean_var;

    SUBROUTINE calcdistance(est_array,dist_array);
      INTEGER :: index, count;
      DOUBLE PRECISION, DIMENSION(1:num_coverage_runs),INTENT(in):: &
         est_array;
      DOUBLE PRECISION, DIMENSION(1:num_coverage_runs):: dist_array;

      dist_array = abs(est_array-true_proportion);
    END SUBROUTINE calcdistance;
END SUBROUTINE coverage_stats;
```

C.6 FORTRAN Program for the Simulation Study of Three Lower Tolerance Intervals for the Common Mean Problem

```
!Filename
!C:\1_Thesis\common_means\three_labs\program_3_labs.f90
!
!INPUT VALUES
!
!The first line is the number of five tuples which are:
!
!The five tuples are (n_1, n_2, n_3, sigma_2, sigma_3). Since sigma_1 is
!assumed to be 1, it is not entered. The values can be entered on one line
!or on two lines sigma's on the first line and n's on the second line. The
!sigma's are read in as double precision data types and n's are read in as
!integers data type.
!
!OUTPUT FILES
!
!There are two output files:
!1) o_3_labs has the coverages for each of the input triplicates, also
! the SUBROUTINE coverage_stats writes the mean and sd of the
! absolute distance from the estimates to the true value of the parameter.
!2) o_3_labs_details has the estimates generated for each of the coverage
! runs.  If this is not desired then in the main program the subroutine
!   calls
!   CALL HeaderCoverageDistFile; and
!   CALL WriteCoverageDist(sigmaz(inputs_index,2:num_labs));
!   need to be commented out, plus the the two OPEN and one CLOSE
!   statements for unit=15 that are at the end of the main program.
!
!To switch program to n-labs, with n .gt. 2, the needed changes are:
!1) change the value of the PARAMETER num_labs to n.
!2) change the last WRITE statement in the two Header SUBROUTINES by
!   adding or subtracting the requisite number of sigma_* and n_*.
!3) in the two Write**Ouput SUBROUTINES change the replicates in the
```

```
!   formats for the sigma's and n's to  n-1 and n respectively
!4) and if wished change the input and output file names in the MODULE
!   Constants.
!5) Input files are (2k-1)-tuples (n_1,...,n_k,sigma_2,....,sigma_k).

MODULE Constants;
  CHARACTER (len=55), PARAMETER :: &
        input_fylename = "in_3_labs.txt", &
            !unit = 13
        output_fylename = "o_3_labs.txt", &
            !unit = 12
         output_details_fylename = "o_3_labs_details.txt";
            !unit = 15
!DON'T FORGET TO CHANGE THE WRITE STATEMENT TO THE OUTPUT FILE FOR THE
!NUMBER OF LABS

  INTEGER ,PARAMETER :: num_labs = 3;
  DOUBLE PRECISION, PARAMETER :: sigma_1 = 1.0;
  DOUBLE PRECISION, PARAMETER ::  mu=0;
  DOUBLE PRECISION , PARAMETER :: alpha=.05;
  INTEGER, PARAMETER :: num_coverage_runs=4000;
  INTEGER, PARAMETER :: num_dist_runs=10000;
  DOUBLE PRECISION, PARAMETER :: int_error = .00001;
  DOUBLE PRECISION, PARAMETER :: h_int = .0005;

  INTEGER :: coverage_run;
  INTEGER, DIMENSION (1:num_labs) :: n;
  DOUBLE PRECISION, DIMENSION(1:num_labs) :: sigma2;
! For simplicity of programing used sums of squares instead of S^2.
  DOUBLE PRECISION, DIMENSION(1:num_labs,1:num_coverage_runs)::x_bar, ssx;
  TYPE output_data
    DOUBLE PRECISION :: two_stage_est, two_stage_dist;
    DOUBLE PRECISION :: gen_construct_est,gen_construct_dist;
    DOUBLE PRECISION :: fisher_est, fisher_dist;
  END TYPE output_data;
  TYPE (output_data), DIMENSION (1:num_coverage_runs) :: &
             results;
  DOUBLE PRECISION :: two_stage_coverage, gen_construct_coverage,&
     fisher_coverage;
! Used in integration to find General Construction estimate
  DOUBLE PRECISION, DIMENSION (1:num_labs) :: x_b, sig2_hat;
  DOUBLE PRECISION :: dist_constant;
END MODULE Constants;

MODULE functions_mod;
  USE constants;

  CONTAINS;
```

```
  FUNCTION unconstrain_dist(t) RESULT(f_result);
    DOUBLE PRECISION, INTENT (in) :: t;
    DOUBLE PRECISION :: f_result;
    INTEGER :: i;

    f_result = 1;
DO i=1,num_labs;
  f_result = f_result * &
     ( 1 + (x_b(i)-t)**2/sig2_hat(i) )**(dble(n(i))/2);
END DO;
f_result = 1/f_result;
  END FUNCTION unconstrain_dist;

  FUNCTION distrib(t) RESULT(f_result);
    DOUBLE PRECISION, INTENT (in) :: t;
    DOUBLE PRECISION :: f_result;
    INTEGER :: i;

    f_result = 1;
    DO i=1,num_labs;
      f_result = f_result * &
          ( 1 + (x_b(i)-t)**2/sig2_hat(i) )**(dble(n(i))/2);
    END DO;
    f_result = dist_constant/f_result;
  END FUNCTION distrib;
END MODULE functions_mod;

PROGRAM CheckGeneralized;
  USE NUMERICAL_LIBRARIES;
  USE IMSLF90;
  USE Constants;

  INTEGER, DIMENSION (:,:), ALLOCATABLE :: nz;
  DOUBLE PRECISION, DIMENSION (:,:), ALLOCATABLE ::  sigmaz;
  INTEGER :: allocation_status,allocation,itring;
  INTEGER :: i, iseed,is_end, &
     num_inputs, inputs_index, temp;

  iseed=12345;
  CALL RNSET(iseed);
!  OPEN(UNIT=15, FILE=data_fylename, ACTION="READWRITE", &
!       POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  OPEN(UNIT=13, FILE=input_fylename, STATUS='UNKNOWN');
  READ(13,*,iostat=is_end) num_inputs;
  ALLOCATE (nz(1:num_inputs,1:num_labs), stat = allocation_status);
     allocation=2;
  CALL AllocationCheck(allocation_status,allocation);
  ALLOCATE (sigmaz(1:num_inputs,2:num_labs), stat = allocation_status);
```

```
    allocation=4;
  CALL AllocationCheck(allocation_status,allocation);

  DO inputs_index=1,num_inputs;
    READ(13,*,iostat=is_end) nz(inputs_index,1:num_labs), &
                    sigmaz(inputs_index,2:num_labs);
  END DO;
  CLOSE(UNIT=13, STATUS="KEEP");
  CALL HeaderCoverageOutput;
!  CALL HeaderCoverageDistFile;
  PRINT "(a,f12.8)", "integral step is H=", h_int;

  InputsLoop:&
  DO inputs_index=1,num_inputs;
    PRINT "(2(a,i3))", "inputs run:", inputs_index, " of", num_inputs;
    sigma2(1)=sigma_1**2;
    DO i=2,num_labs;
       sigma2(i)=sigmaz(inputs_index,i)**2;
    END DO;
    DO i=1,num_labs;
      n(i)=nz(inputs_index,i);
    END DO;
    results(1:num_coverage_runs)%two_stage_est=0;
    results(1:num_coverage_runs)%two_stage_dist=0;
    results(1:num_coverage_runs)%gen_construct_est=0;
    results(1:num_coverage_runs)%gen_construct_dist=0;
    results(1:num_coverage_runs)%fisher_est=0;
    results(1:num_coverage_runs)%fisher_dist=0;
    CALL GenerateData;
    CoverageLoop:&
    DO coverage_run=1,num_coverage_runs;
        IF (mod(coverage_run,500) == 0) THEN
          PRINT *,"    coverage run ", coverage_run;
        END IF;
          CALL CalcTwoStage;
          CALL CalcGenConstruct;
          CALL fisher;
    END DO CoverageLoop;
    CALL CalcCoverage( results(1:num_coverage_runs)%two_stage_est,&
              two_stage_coverage);
    CALL CalcCoverage( results(1:num_coverage_runs)%gen_construct_est,&
              gen_construct_coverage);
    CALL CalcCoverage( results(1:num_coverage_runs)%fisher_est,&
              fisher_coverage);
    CALL WriteCoverageOutput(sigmaz(inputs_index,2:num_labs));
    CALL coverage_stats;
!    CALL WriteCoverageDist(sigmaz(inputs_index,2:num_labs));
    IF (MOD(inputs_index,5) == 0) THEN
```

```
      CLOSE(UNIT=12, STATUS= "KEEP");
      OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
          POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
!      CLOSE(UNIT=15, STATUS= "KEEP");
!      OPEN(UNIT=15, FILE=output_fylename, ACTION="READWRITE", &
!          POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
    END IF;
  END DO InputsLoop;
  WRITE(12,"(a)") ""
  CLOSE(UNIT=12, STATUS= "KEEP");
!  CLOSE(UNIT=15, STATUS= "KEEP");
END PROGRAM CheckGeneralized;

SUBROUTINE AllocationCheck(check, allocation);
  INTEGER :: check, allocation;
  IF (check < 0) THEN
    PRINT *, "Allocation error at allocation", allocation;
    STOP;
  END IF;
END SUBROUTINE AllocationCheck;

SUBROUTINE GenerateData;
  USE Constants;
  DOUBLE PRECISION :: DF;
  INTEGER :: i;

  DO i=1,num_labs;
    CALL DRNNOR(num_coverage_runs,x_bar(i,1:num_coverage_runs));
    x_bar(i,1:num_coverage_runs)=mu+&
       dsqrt( sigma2(i)/dble(n(i)) )*x_bar(i,1:num_coverage_runs);
  END DO;
  DO i=1,num_labs;
    DF=dble(n(i)-1);
    CALL DRNCHI(num_coverage_runs,DF,ssx(i,1:num_coverage_runs));
    ssx(i,1:num_coverage_runs)=(sigma2(i))*ssx(i,1:num_coverage_runs);
  END DO;
END SUBROUTINE GenerateData;

SUBROUTINE CalcCoverage(est_array,coverage);
  USE Constants;
  INTEGER :: index, count;
  DOUBLE PRECISION :: coverage;
  DOUBLE PRECISION, DIMENSION(1:num_coverage_runs),INTENT(in)::est_array;

  count=0;
  DO index=1,num_coverage_runs;
    IF (est_array(index) .gt. mu) THEN
  count=count+1;
```

```
END IF;
  END DO;
  coverage=dble(count)/dble(num_coverage_runs);
END SUBROUTINE CalcCoverage;

SUBROUTINE coverage_stats;
  USE constants;
  DOUBLE PRECISION :: two_stage_mean, two_stage_var, &
           gen_construct_mean, gen_construct_var, mean, var;

  CALL calcdistance(results(1:num_coverage_runs)%two_stage_est, &
        results(1:num_coverage_runs)%two_stage_dist);
  CALL calcdistance(results(1:num_coverage_runs)%gen_construct_est, &
        results(1:num_coverage_runs)%gen_construct_dist);
  CALL calcdistance(results(1:num_coverage_runs)%fisher_est, &
        results(1:num_coverage_runs)%fisher_dist);
  WRITE(12,"(a)"), "          mean      sd ";
  CALL mean_var(results(1:num_coverage_runs)%two_stage_dist,mean,var);
  WRITE(12,"(2(a,f7.4))"), "2 stage: ", mean, "  ", dsqrt(var);
  CALL mean_var(results(1:num_coverage_runs)%gen_construct_dist,mean,var);
  WRITE(12,"(2(a,f7.4))"), "gen_con: ", mean, "  ", dsqrt(var);
  CALL mean_var(results(1:num_coverage_runs)%fisher_dist,mean,var);
  WRITE(12,"(2(a,f7.4))"), "fisher:  ", mean, "  ", dsqrt(var);

  CONTAINS;
    SUBROUTINE mean_var(data_array,mean,var) ;
      DOUBLE PRECISION, DIMENSION (1:num_coverage_runs), &
         INTENT(in)::data_array;
      DOUBLE PRECISION :: sum, sum_sqs, mean, var;

      sum=0; sum_sqs=0;
      DO index=1,num_coverage_runs;
         sum = sum+data_array(index);
sum_sqs = sum_sqs + data_array(index)**2;
      END DO;
      mean = sum/num_coverage_runs;
      var= (sum_sqs - sum**2/num_coverage_runs)/(num_coverage_runs);
END SUBROUTINE mean_var;

    SUBROUTINE calcdistance(est_array,dist_array);
  INTEGER :: index, count;
      DOUBLE PRECISION,D IMENSION (1:num_coverage_runs),&
          INTENT(in)::est_array;
  DOUBLE PRECISION, DIMENSION (1:num_coverage_runs):: dist_array;

        dist_array = abs(est_array-mu);
    END SUBROUTINE calcdistance;
END SUBROUTINE coverage_stats;
```

```
SUBROUTINE HeaderCoverageOutput;
  USE constants;

  OPEN(UNIT=12, FILE=output_fylename, ACTION="READWRITE", &
       POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(12,"(a,f12.8)") "the integration step is h=:", h_int;
  WRITE(12,"(a,i6)") "number of coverage runs is:", num_coverage_runs;
  WRITE(12,"(a,i7)") "number of fiducial runs is:", num_dist_runs;
  WRITE(12,"(a)") &
      " n_1  n_2  n_3  sigma1 sigma2 sigma3 2_stage  gen_cons  Fisher";
END SUBROUTINE HeaderCoverageOutput;

SUBROUTINE WriteCoverageOutput(sigma_array);
  USE constants;
  DOUBLE PRECISION, DIMENSION(2:num_labs) :: sigma_array;

  WRITE(12,"(i3,2(i5),3(f7.2),3(a,f7.4))") &
     n(1:num_labs),sigma_1, sigma_array(2:num_labs),&
      "  ",two_stage_coverage,&
      "  ",gen_construct_coverage,&
      "   ",fisher_coverage;
END SUBROUTINE WriteCoverageOutput;

SUBROUTINE HeaderCoverageDistFile;
  USE constants;

  OPEN(UNIT=15, FILE=output_details_fylename, ACTION="READWRITE", &
        POSITION="APPEND", STATUS='OLD', ACCESS="SEQUENTIAL");
  WRITE(15,"(a)")" n_1 n_2 n_3 sig_1 sig_2 sig_3 &
       2_stage  gen_cons  Fisher";
END SUBROUTINE HeaderCoverageDistFile;

SUBROUTINE WriteCoverageDist(sigma_array);
  USE constants;
  DOUBLE PRECISION, DIMENSION(2:num_labs) :: sigma_array;
  INTEGER :: i;

  DO i=1,num_coverage_runs;
    WRITE(15, "(3(i4),3(f6.2),3(a,f7.4))") &
n(1:num_labs),sigma_1,sigma_array(2:num_labs),&
            "   ",results(i)%two_stage_est,&
            "  ",results(i)%gen_construct_est,&
            "  ",results(i)%fisher_est;
  END DO;
END SUBROUTINE WriteCoverageDist;
!
! CALCULATES TWO STAGE FIDUCIAL GENERALIZED CONFINDENCE BOUND
```

```
!
SUBROUTINE CalcTwoStage;
  USE Constants;
  DOUBLE PRECISION, DIMENSION(1:num_dist_runs) :: &
        E_1, sigma2_pooled, S_1, R_theta_star;
  DOUBLE PRECISION, DIMENSION(1:num_labs,1:num_dist_runs) :: E_x;
  DOUBLE PRECISION :: DF;
  INTEGER :: i;

  CALL DRNNOR(num_dist_runs,E_1);
  DO i=1,num_labs
    DF=dble(n(i)-1);
    CALL DRNCHI( num_dist_runs,DF,E_x(i,1:num_dist_runs) );
  END DO;
  S_1=0;sigma2_pooled=0;
  DO i=1,num_labs;
    S_1 = S_1 + &
  dble(n(i))*x_bar(i,coverage_run)* &
         E_x(i,1:num_dist_runs)/ssx(i,coverage_run);
    sigma2_pooled = sigma2_pooled + &
      dble(n(i))*E_x(i,1:num_dist_runs)/ssx(i,coverage_run)
  END DO;
  sigma2_pooled=1/sigma2_pooled;
  R_theta_star = sigma2_pooled*S_1-E_1*dsqrt(sigma2_pooled);
  CALL DSVRGN(num_dist_runs,R_theta_star,R_theta_star);
  results(coverage_run)%two_stage_est= &
      R_theta_star( floor((1-alpha)*num_dist_runs) );
END SUBROUTINE CalcTwoStage;
!
! SUBROUTINE CalGenConstruct CALCULATES THE GCI USING THE GENERALIZED
! CONSTRUCTION TECHNIQUE.  IT CALLS THE FUNCTIONS f_simpson_integral AND
! f_gen_construt.
!
SUBROUTINE CalcGenConstruct;
  USE Constants;
  USE functions_mod
  DOUBLE PRECISION :: t_min, t_max, max_sig, min_xbar, max_xbar;
  INTEGER :: i;

  max_sig=0;min_xbar=0;max_xbar=0;
  DO i=1,num_labs;
   sig2_hat(i)=ssx(i,coverage_run)/n(i);
   x_b(i)=x_bar(i,coverage_run);
   max_sig= MAX(max_sig,sig2_hat(i));
   min_xbar = MIN(min_xbar,x_b(i));
   max_xbar = MAX(max_xbar,x_b(i));
  END DO;
  max_sig = sqrt( max_sig );
```

```
  t_min = min_xbar - max_sig/(int_error)**( dble(1)/(2*num_labs-1) ) ;
  t_max = max_xbar + max_sig/(int_error)**( dble(1)/(2*num_labs-1) ) ;
  dist_constant = 1/f_simpson_integral(unconstrain_dist,t_min,t_max);
  t_min = min_xbar - &
          max_sig/(int_error/dist_constant)**( dble(1)/(2*num_labs-1) ) ;
  results(coverage_run)%gen_construct_est=f_gen_construct(distrib,t_min);
END SUBROUTINE CalcGenConstruct;

FUNCTION f_simpson_integral(f,x_min,x_max) RESULT(f_result);
  USE constants;
  DOUBLE PRECISION, INTENT(in) :: x_min, x_max;
  DOUBLE PRECISION :: t, est,f,f_result;

  est=0;
  t=x_min;
  integral_loop: DO
    est=est+ (h_int/3)*( f(t)+4*f(t+h_int)+f(t+2*h_int) );
    IF (t+3*h_int .gt. x_max) THEN
      EXIT integral_loop;
    ELSE
      t=t+2*h_int;
    END IF;
  END DO integral_loop;
  f_result = est;
END FUNCTION F_simpson_integral;

FUNCTION f_gen_construct(f,x_min) RESULT(f_result);
  USE constants;
  DOUBLE PRECISION, INTENT(in) :: x_min;
  DOUBLE PRECISION :: t, est_l, est_u, f, f_result;

  est_l=0;est_u=0;
  t=x_min;
  integral_loop: DO
    est_l=est_u;
    est_u=est_u + (h_int/3)*( f(t)+4*f(t+h_int)+f(t+2*h_int) );
    IF ( (est_l .lt. (1-alpha)) .AND. (est_u .gt. (1-alpha)) ) THEN
      EXIT integral_loop;
    ELSE
      t=t+2*h_int;
    END IF;
  END DO integral_loop;
  f_result = t;
END FUNCTION f_gen_construct;
!
! SUBROUTINE CALCULATES FISHER'S EXACT TEST.
! SUBROUTINE CALLS FUNCTION, fishers_p.
!
```

```
SUBROUTINE fisher;
  USE constants;

  DOUBLE PRECISION :: start,finish, cutoff, p, df, half_way;
  INTEGER :: index;

  p=1-alpha;  df=DBLE(2*num_labs);
  cutoff=DCHIIN(p,df);
  start= x_bar(1,coverage_run);
  DO index=2,num_labs;
    start= MIN( start, x_bar(index,coverage_run) );
  END DO;
  finish=start+1;
  DO
    IF (fishers_p(finish) .gt. cutoff) THEN
  EXIT;
ELSE
  start=finish; finish=finish+1;
END IF;
  END DO;
  DO index = 1,20;
    half_way = (start+finish)/2;
    IF (fishers_p(half_way) .gt. cutoff) THEN
  finish=half_way;
ELSE
  start=half_way;
    END IF;
  END DO;
  results(coverage_run)%fisher_est=(start+finish)/2;
END SUBROUTINE fisher;

FUNCTION fishers_p(mu_0) RESULT(f_result);
  USE constants;
  DOUBLE PRECISION, INTENT(IN) :: mu_0;
  DOUBLE PRECISION :: f_result,t,df;
  INTEGER :: i ;

  f_result=0;
  DO i=1,num_labs;
    t = &
     (x_bar(i,coverage_run)-mu_0)/ &
          dsqrt( ssx(i,coverage_run)/DBLE((n(i)-1)*n(i)) );
    df=DBLE(n(i)-1);
    f_result = f_result + LOG(DTDF(t,df));
  END DO;
  f_result= -2*(f_result);
END FUNCTION fishers_p;
```