Solutions of the linearized Richards equation with arbitrary boundary and initial conditions: flux and soil moisture respectively

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Abstract. Analytical solutions of differential equations describe physical problems and provide general insight of the studied natural mechanisms. Although they may be not suitable to solve complex hydrological problems, they are fast and useful to test numerical procedures. The solutions proposed in this work are obtained for arbitrary flux boundary conditions and arbitrary soil moisture initial conditions. This permits to use standard meteorological data: precipitation data (incoming flux) and Bowen ratio data (outgoing flux), which are very common, while soil volumetric water content measurements are usually not available exactly at the soil-atmosphere interface. A first class of solutions is obtained with a uniform initial condition for the soil moisture and a time dependent surface flux, which well represents experimental precipitation/ evaporation cases. A solution with a more general boundary condition is derived using a sum of simple solutions obtained for constant boundary conditions. Finally the same technique is applied to the soil moisture initial condition too. The vertical profiles of the soil water content computed by this simple sum of solutions are compared with the results of the aforementioned analytical solutions.

1. Introduction

It is well recognized that analytical solutions of differential equations describing physical problems provide general insights and concisely identify the relationships among the variables of the studied problems allowing rational approximations and simplifications. Therefore, although numerical methods are powerful in solving complex non linear problems, analytical solutions conserve their utility and can also provide a useful check to numerical procedures. Exact and approximated analytical solutions of the non linear differential equation governing the water flow in unsaturated soils (Richards equation) have been derived by Sanders et al. (1988), Hogarth et al. (1989, 1992),

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Parlange et al. (1992), Ross and Parlange (1994), Parlange et al. (1997), Hogarth and Parlange (2000) among others. Moreover, analytical solutions of the linearized Richards equation have been derived in integral form by Warrick (1975) and Basha (1999), but their results give closed form solutions only for constant flux. Chen et al. (2001) derived analytical solutions of the linearized Richards equation for a variety of time dependent fluxes, before surface saturation.

A different approach to obtain analytical solutions of the linearized Richards equation was utilized by Menziani et al. (in press) assuming arbitrary initial and boundary conditions for the water content. The evolution of the last quoted study is here presented. Firstly, a class of analytical solutions is obtained assuming a uniform initial condition and a known time dependent flux at the surface, which well represents experimental precipitation/evaporation cases. Secondly, a new solution is obtained for any surface flux boundary condition and any soil water content initial condition. This is the result of the sum of simple solutions obtained for constant complementary conditions. The vertical profiles of the soil water content computed by this simple sum of solutions are compared with the results of the aforementioned class of analytical solutions. The time behaviour of the incoming flux at the surface describes rainfall infiltration or sprinkle irrigation whose intensity are lower then the infiltration capacity of the soil. Moreover, precipitation measurements are much more common then soil moisture measurements so, solutions of the flow equation, obtained assuming a time dependent surface flux boundary condition, can be very useful in situations of interest in hydrology.

2. Theory

Consider the linearized Richards equation satisfied by the soil water content θ ranging from 0 to 1:

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial z^2} - K\frac{\partial \theta}{\partial z} \tag{1}$$

with the following arbitrary conditions:

$$\begin{aligned} &\theta = \theta_i(z) && (z\rangle 0; t = 0) \\ &\theta = \theta_0(t) && (z = 0; t\rangle 0) \end{aligned} \tag{2}$$

where D is the hydraulic diffusivity and $K = \partial k / \partial \theta$; they are assumed constant. k is the soil hydraulic conductivity.

On the basis of the work described in Menziani et al. (in press), the space and time evolution of the soil water content is given by the sum of two solutions: $\theta(z,t) = \theta_1(z,t) + \theta_2(z,t)$ where:

$$\theta_1(z,t) = \frac{e^{\frac{K\cdot z}{2\cdot D} - \frac{K^2\cdot t}{4\cdot D}}}{\sqrt{\pi}\sqrt{4\cdot D\cdot t}} \cdot \int_0^{\infty} \theta_i(z') \cdot e^{-\frac{K\cdot z'}{2\cdot D}} \left[e^{-\left(\frac{z-z'}{\sqrt{4\cdot D\cdot t}}\right)^2} - e^{-\left(\frac{z+z'}{\sqrt{4\cdot D\cdot t}}\right)^2} \right] \cdot dz'$$

$$\theta_2(z,t) = \frac{2}{\sqrt{\pi}} \cdot \int_0^{t} \theta_0(t') \cdot e^{-\left[\frac{K(t-t')-z}{\sqrt{4\cdot D\cdot (t-t')}}\right]^2} \cdot \frac{4\cdot D\cdot z\cdot dt'}{2\left[4\cdot D\cdot (t-t')\right]^{3/2}} \tag{3}$$

The solution $\theta_1(z,t)$ derives from the initial condition of the problem and a null boundary condition, while the solution $\theta_2(z,t)$ derives from the boundary condition of the problem and an initial condition equal to zero. In particular, if $\theta_i(z) = 0$ the solution $\theta(z,t)$ coincides with $\theta_2(z,t)$.

Let us assume now, again the initial condition for θ, but the boundary condition for the flux defined as: $\Phi = -D\frac{\partial \theta}{\partial z} + K\theta$. Taking into account the linearity of the flux relationship and of the differential equation (1), the following equation (4), with Φ instead of θ as unknown, can be written:

$$\frac{\partial \Phi}{\partial t} = D\frac{\partial^2 \Phi}{\partial z^2} - K\frac{\partial \Phi}{\partial z} \tag{4}$$

Therefore, given the conditions:

$$\begin{aligned} &\theta = \theta_i(z); \text{ corresponding to:} \\ &\Phi = \Phi_i(z) = -D\frac{d\theta_i}{dz} + K\theta_i && (z\rangle 0; t=0) \\ &\Phi = \Phi_0(t) && (z=0; t\rangle 0) \end{aligned} \tag{5}$$

for the flux the following solution $\Phi(z,t) = \Phi_1(z,t) + \Phi_2(z,t)$ is obtained, where:

$$\Phi_1(z,t) = \frac{e^{\frac{K\cdot z}{2\cdot D} - \frac{K^2\cdot t}{4\cdot D}}}{\sqrt{\pi}\sqrt{4\cdot D\cdot t}} \cdot \int_0^{\infty} \Phi_i(z') \cdot e^{-\frac{K\cdot z'}{2\cdot D}} \left[e^{-\left(\frac{z-z'}{\sqrt{4\cdot D\cdot t}}\right)^2} - e^{-\left(\frac{z+z'}{\sqrt{4\cdot D\cdot t}}\right)^2} \right] \cdot dz'$$

$$\Phi_2(z,t) = \frac{2}{\sqrt{\pi}} \cdot \int_0^{t} \Phi_0(t') \cdot e^{-\left[\frac{K(t-t')-z}{\sqrt{4\cdot D\cdot (t-t')}}\right]^2} \cdot \frac{4\cdot D\cdot z\cdot dt'}{2\left[4\cdot D\cdot (t-t')\right]^{3/2}} \tag{6}$$

Finally, with simple considerations and remembering that $\lim_{z\to\infty} e^{-K\cdot z/D} \cdot \theta(z,t) = 0$, the unknown function θ is obtained:

$$\theta(z,t) = \frac{e^{\frac{K \cdot z}{D}}}{D} \cdot \int_{z}^{\infty} e^{-\frac{K \cdot z'}{D}} \cdot \Phi(z',t) \cdot dz' \qquad (7)$$

A priori equation (7) gives the distribution of θ(z,t) for any $\Phi_0(t)$ and $\theta_i(z)$ but the integrals in equations (6) and (7) may be difficult, or impossible, to be solved analytically.

3. Solution with exponential flux at the surface

Since rain gauge data are widely collected it follows that, during precipitation events, the incoming water flux trend is known in many places while this is not the case of the soil volumetric content. During the last years, strong flood events happened both in meteorologically well-predicted situations and also in unpredicted local summer storms (always more frequently). In any case the floods are strongly related to the state of the soil moisture (Obled and Djerboua, 2000).

Looking at recent precipitation events it can be seen that many isolated local summer cases can be described with a surface flux represented with the sum of few exponential functions. Fig. 1 shows four examples corresponding to a flux given by: 1) a simple exponential function (curve 1); 2) the sum of two exponentials with null precipitation at t=0 (curve 2); 3) the sum of two exponentials with precipitation not null at t=0 (curve 3); 4) the sum of three exponentials (curve 4) corresponding to a null precipitation at t=0 and with a null derivative at t=0. The integral (from zero to infinity) of all the four curves gives a total precipitation of 50 mm. The parameters of the four quoted functions are reported in table 1.

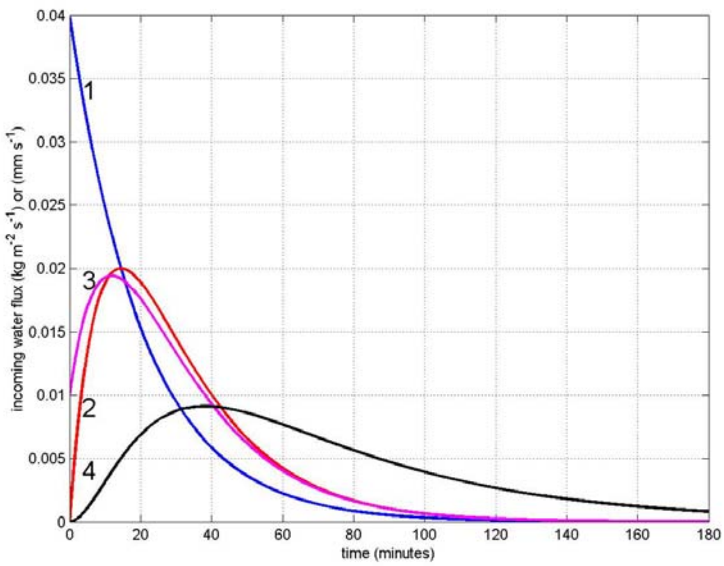

Figure 1. Water Flux at the surface (Examples of Boundary Conditions)

$$\Phi_0(t) = q_1 \cdot e^{-\beta_1 \cdot t} - q_2 \cdot e^{-\beta_2 \cdot t} + q_3 \cdot e^{-\beta_3 \cdot t}$$

N	q_1 (mm s^{-1})	β_1 (s^{-1})	q_2 (mm s^{-1})	β_2 (s^{-1})	q_3 (mm s^{-1})	β_3 (s^{-1})
1	0.04	$8\ 10^{-4}$	0	0	0	0
2	0.08	$8\ 10^{-4}$	0.08	$1.6\ 10^{-3}$	0	0
3	0.08	$8\ 10^{-4}$	0.07	$1.4\ 10^{-3}$	0	0
4	0.02	$2\ 10^{-3}$	0.05	$1.0\ 10^{-3}$	0.03	$3.3\ 10^{-4}$

Table 1. Parameters of the four functions reported in figure 1.

The solution of (4) assuming a null uniform initial condition for the soil water content ($\theta_i(z)=0$) and a flux boundary condition given by: $\Phi_0(t) = q \cdot e^{-\beta \cdot t}$ is:

$$\theta(z,t) = \frac{q \cdot e^{\frac{K \cdot z}{D} - \beta \cdot t}}{2 \cdot D} \cdot \left\{ \frac{e^{-\left(\frac{K}{2 \cdot D} - 2 \cdot \gamma\right) \cdot z}}{\frac{K}{2 \cdot D} - 2 \cdot \gamma} \cdot erfc\left(\frac{z + 4 \cdot \gamma \cdot D \cdot t}{\sqrt{4 \cdot D \cdot t}} \right) + \right.$$
$$\left. + \frac{e^{-\left(\frac{K}{2 \cdot D} + 2 \cdot \gamma\right) \cdot z}}{\frac{K}{2 \cdot D} + 2 \cdot \gamma} \cdot erfc\left(\frac{z - 4 \cdot \gamma \cdot D \cdot t}{\sqrt{4 \cdot D \cdot t}} \right) - \frac{K}{\beta} \cdot e^{\beta \cdot t} \cdot erfc\left(\frac{z + K \cdot t}{\sqrt{4 \cdot D \cdot t}} \right) \right\} \quad (8)$$

Where: $\gamma = \sqrt{\left(\frac{K}{4 \cdot D}\right)^2 - \frac{\beta}{4 \cdot D}}$. Clearly equation (8) is valid only if $\beta \le K^2/(4 \cdot D)$. If $\beta > K^2/(4 \cdot D)$ the solution of (4) is more complicate and involves the error function of complex argument (Abramowitz and Stegun, 1965).
Due to linearity of (4), using a boundary condition which is the sum of two or three exponential functions, the solution is given by the sum of two or three solutions like (8).
Fig. 3 shows (solid lines) the trend of the vertical profile of the soil volumetric water content θ obtained assuming a uniform initial condition ($\theta_i(z)=0$) and $\Phi_0(t) = q_1 \cdot e^{-\beta_1 \cdot t} - q_2 \cdot e^{-\beta_2 \cdot t}$ as boundary condition. In this theoretical example, five vertical profiles of the soil volumetric water content can be seen. The thin vertical solid line is the initial condition while the other solid curves are obtained with a constant time step. The water content profile trend shows the time increase of the soil moisture as the water enters the soil. The last two curves show an inflection point moving downwards while this is not evident in the first two. The circles describe the solution obtained using the approximating function presented in the following paragraph.

4. Solution with step functions approximating the flux at the surface

Not always the experimental surface flux can be represented by a simple function. In such a case, may be very difficult or even impossible to solve the integral in equations (6) and (7). Vice versa, the solution of equation (4) is simple for the following complementary conditions:

$$\begin{aligned} &\theta_i(z) = 0 && (z \rangle 0; t = 0) \\ &\Phi_0(t) = q_0 && (z = 0; t \rangle 0) \end{aligned} \tag{9}$$

(where q_0 is a constant); that is a uniform (zero) initial condition for the soil volumetric water content and a constant flux at the surface. From equations (6= and (7) the solution of equation (4) results:

$$\theta(z,t) = \frac{q_0}{K} \cdot \left[1 - \frac{1}{2} \cdot erfc\left(\frac{K \cdot t - z}{\sqrt{4 \cdot D \cdot t}} \right) \right] + \\ + q_0 \cdot \frac{e^{\frac{K \cdot z}{D}}}{2} \cdot \left[\frac{\sqrt{4 \cdot D \cdot t}}{D} \cdot Ierfc\left(\frac{K \cdot t + z}{\sqrt{4 \cdot D \cdot t}} \right) - \frac{1}{K} \cdot erfc\left(\frac{K \cdot t + z}{\sqrt{4 \cdot D \cdot t}} \right) \right] \tag{10}$$

Ierfc is the iterated complementary error function.
Approximating any arbitrary boundary condition with a sum of step functions the solution of the problem is given by the sum of expressions similar to equation (10). In fact, assuming a uniform initial condition $\theta_i(z) = 0$ (i.e. $\Phi_i(z) = 0$) and a boundary condition as sketched in Fig. 2 the solution, in the time interval ($t_{M-1} - t_M$), results:

$$\theta(z,t) = \sum_{j=1}^{M} \frac{q_j - q_{j-1}}{2 \cdot K} \cdot \left\{ 1 + erf\,\frac{K \cdot (t - t_{j-1}) - z}{\sqrt{4 \cdot D \cdot (t - t_{j-1})}} + e^{\frac{K \cdot z}{D}} \cdot \right. \\ \left. \left[2 \cdot \sqrt{\frac{K^2 \cdot (t - t_{j-1})}{D}} \cdot Ierfc\,\frac{K \cdot (t - t_{j-1}) + z}{\sqrt{4 \cdot D \cdot (t - t_{j-1})}} - erfc\,\frac{K \cdot (t - t_{j-1}) + z}{\sqrt{4 \cdot D \cdot (t - t_{j-1})}} \right] \right\} \tag{11}$$

M is the number of discontinuities (at t_1, t_2, ... t_M ....) where the boundary condition assumes the values: q_1, q_2, ... q_M ..... (besides q_0).

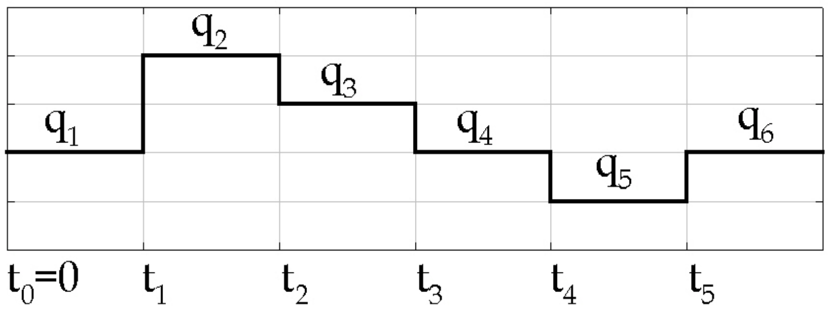

Figure 2. Step functions used to approximate the surface water flux.

In Fig. 2 a constant flux q_1 lasts from t_0 to t_1, a constant flux q_2 lasts from t_1 and t_2 and so on. In equation (11) the difference ($q_j - q_{j-1}$) represents the height of the step function starting at t_{j-1} (clearly q_0=0).

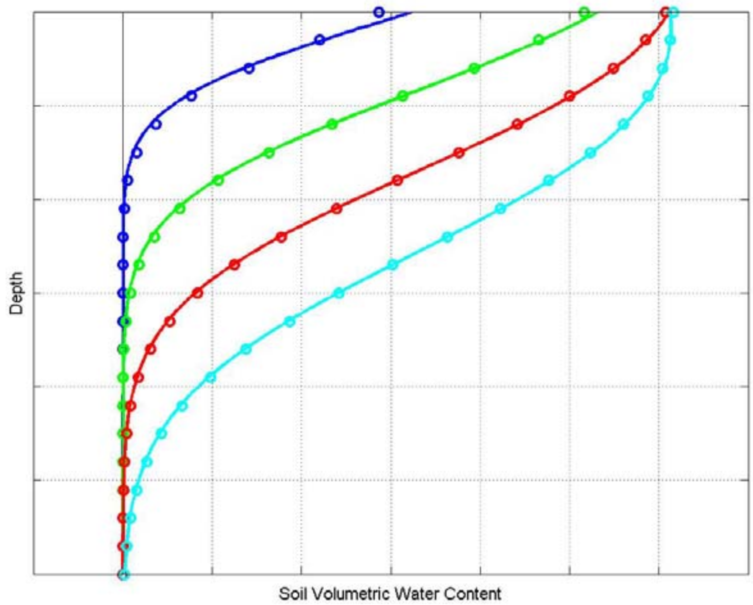

Figure 3. Soil volumetric water content profiles obtained using equation (8) (solid lines) and the approximated profiles from equation (11).

In Fig. 3 the solid lines are the results obtained from equation (8), according to the description previously done; they are compared with the approximated solution (11) (circles). The soil moisture results agree satisfactory even though a raw time resolution was used to approximate the incoming flux. In real experimental cases this technique can be used to choose a proper rain gauge acquisition time.

5. Solution with step functions approximating the initial condition

In the previous paragraph the boundary condition (i.e. the flux at the air-soil interface) was approximated by a sum of step functions. In a similar way, here, an arbitrary initial condition is approximated by:

$$\theta_i(z) = \theta_0 + \sum_{n=1}^{N} (\theta_n - \theta_{n-1}) \cdot H(z - z_n) \tag{12}$$

N is the total number of discontinuities (at z_1, z_2, ... z_N) where the initial condition assumes the values: θ_1, θ_2, ... θ_N (besides θ_0). H(x) is the Heaviside function with argument x (Jones, 1966).

Now, assuming a null flux as boundary condition, the solution of equation (4) says that the soil volumetric water content is given by the sum of (N+1) solutions:

$$\theta(z,t) = \theta^{(0)}(z,t) + \sum_{n=1}^{N} \theta^{(n)}(z,t) \tag{13}$$

where, from equation (6) and (7), one obtains:

$$\theta^{(0)}(z,t) = \frac{\theta_0}{2} \cdot \left\{ erfc\left(\frac{K \cdot t - z}{\sqrt{4 \cdot D \cdot t}} \right) + e^{\frac{K \cdot z}{D}} \cdot \left[erfc\left(\frac{K \cdot t + z}{\sqrt{4 \cdot D \cdot t}} \right) - \frac{K \cdot \sqrt{4 \cdot D \cdot t}}{D} \cdot Ierfc\left(\frac{K \cdot t + z}{\sqrt{4 \cdot D \cdot t}} \right) \right] \right\} \tag{14}$$

and, in a similar way, but with some more difficulties, one obtaines:

$$\theta^{(n)}(z,t) = \frac{\theta_n - \theta_{n-1}}{2} \cdot \left\{ erfc\left(\frac{K \cdot t + z_n - z}{\sqrt{4 \cdot D \cdot t}} \right) + e^{\frac{K \cdot z}{D}} \cdot \left[erfc\left(\frac{K \cdot t + z_n + z}{\sqrt{4 \cdot D \cdot t}} \right) - \frac{K \cdot \sqrt{4 \cdot D \cdot t}}{D} \cdot Ierfc\left(\frac{K \cdot t + z_n + z}{\sqrt{4 \cdot D \cdot t}} \right) \right] \right\} \tag{15}$$

Generally, approximating any arbitrary boundary and initial condition with a series of step functions the solution of the Richards equation is given by the sum of equations (11) and (13).

6. Conclusion

In this work the linearized Richards equation with a boundary condition on the flux and a soil water content initial condition has been solved in integral form.

A class of closed form analytical solutions has been derived for a flux boundary condition, which is the sum of exponential functions. On the other hand, many real precipitation events may be represented as the sum of few exponential functions.

A more general solution is obtained approximating the flux boundary condition by the sum of step functions and with a null uniform soil water content initial condition. For this solution the mathematical constrain discussed in paragraph (3) doesn't exist.

Finally, a solution is obtained for a null flux at the surface and an initial condition approximated by the sum of step functions. The expression obtained adding the last two solutions permits to solve the linearized Richard equation for any arbitrary boundary and initial condition.

The expression obtained from the described procedure is not exactly an analytical solution but it can be very useful to solve hydrological problems. In particular the procedure allows using experimental rain gauge data, which are very common. In fact, these data may be assumed as the incoming water flux at the atmosphere-soil interface if the precipitation rate doesn't exceed the soil infiltrability.

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