A Study of Potential Flow From the Point of View of Differential Geometry

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In this note three theorems concerning a study of potential flow of fluids from the point of view of differential geometry are presented. One has first the following definitions:

Def. 1. The surface $H = \phi(x,y)$, where $\nabla^2\phi = 0$, is called a harmonic surface

Def. 2. The surfaces $H = \phi(x,y)$ and $H = \psi(x,y)$, where $\phi(x,y)$ and $\psi(x,y)$ are conjugate harmonic functions, are called conjugate harmonic surfaces.

Now consider the surface $H = \phi(x,y)$ in the Euclidean 3-space with coordinates x, y, H. Planes $x =$ const. and $y =$ const. will cut the surface in two families of lines, which will serve as the coordinate lines of the surface considered as a Riemannian 2-space, with the new coordinates of a point on the surface identical as its $x - y$ coordinates. For convenience let these families be called x-lines and y-lines, respectively. It is then obvious that the fundamental quadratic is

$$\cancel{ds^2 = (1+\phi_x^2)\,dx^2 + 2\phi_x\phi_y\,dx\,dy + (1+\phi_y^2)\,dy^2}$$

$$ds^2 = g_{11}\,dx^2 + g_{12}\,dx\,dy + g_{21}\,dy\,dx + g_{22}\,dy^2$$

with

$$g_{11} = 1+\phi_x^2, \quad g_{12} = g_{21} = \phi_x\phi_y, \quad g_{22} = 1+\phi_y^2 \qquad (1)$$

and

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1+\phi_x^2+\phi_y^2 = 1+q^2 \qquad (2)$$

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where subscripts denote partial differentiation, and where

$$q^2 = \phi_x^2 + \phi_y^2$$

is in fact the square of the steepest slope at any point.

The Gaussian curvature K at any point is equal to R_{1212}/g, where R_{1212} is the only distinct non-vanishing term of the Riemannian tensor in 2-space and is defined as

$$R_{1212} = \frac{1}{2}\left(2\frac{\partial^2 g_{12}}{\partial x \partial y} - \frac{\partial^2 g_{11}}{\partial y^2} - \frac{\partial^2 g_{22}}{\partial x^2}\right) + g_{\alpha\beta}\left[\left\{{\alpha \atop 12}\right\}\left\{{\beta \atop 12}\right\} - \left\{{\alpha \atop 11}\right\}\left\{{\beta \atop 22}\right\}\right]$$

where $\left\{{\alpha \atop 12}\right\}$ etc. stand for the Christoffel symbols of the second kind, the scripts 1 and 2 being associated with x and y, respectively, and where repeated dummy scripts (α and β) call for summations over 1 and 2.

Substituting (1) and (2) in R_{1212}/g, one obtains,

$$K = \frac{1}{(1+q^2)^2}\left(\phi_{xx}\phi_{yy} - \phi_{xy}^2\right) = -\frac{1}{(1+q^2)^2}\left(\phi_{xx}^2 + \phi_{xy}^2\right) \qquad (3)$$

since $\phi_{yy} = -\phi_{xx}$. From (3), one has

Theorem 1. The Gaussian curvature of a harmonic surface is never positive.

Since a soap film with no pressure difference on the two sides is a harmonic surface, and since a positive Gaussian curvature would mean a pressure difference, this theorem is indeed to be expected.

If $K = 0$, then it follows that $\phi_{xx} = \phi_{yy} = 0$ and $\phi_{xy} = 0$. That is

$$\phi = ax + by + c$$

Hence one obtains the

Corallary. A harmonic surface cannot be isometric with a plane unless it is actually a plane.

Take now the conjugate harmonic surface

$$H = \psi(x,y)$$

where ψ is conjugate to ϕ and satisfies the Cauchy-Riemann equations

$$\phi_x = \psi_y \quad , \qquad \phi_y = -\psi_x \tag{4}$$

The Gaussian curvature for the ψ-surface is, by a derivation similar to that of (3),

$$K' = \frac{1}{(1+\psi_x^2+\psi_y^2)}(\psi_{xx}\psi_{yy} - \psi_{xy}^2) = -\frac{\psi_{xy}^2 + \psi_{yy}^2}{(1+\psi_x^2+\psi_y^2)} \tag{5}$$

In view of (4), one has, from (3) and (5)

$$K = K'$$

Hence, one has

Theorem 2. The Gaussian curvatures of conjugate harmonic surfaces are equal at the same (x,y) coordinates.

It must however be noted that the conjugate harmonic surfaces are in general not isometric.

In the plane flow of a viscous fluid, the rate of dissipation of energy per unit volume is given by (Lamb, H.: Hydrodynamics, Dover, p. 580)

$$\Phi = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\} \tag{6}$$

where μ is the dynamic viscosity of the fluid and, with ψ denoting the stream-function,

$$u = \frac{\partial \psi}{\partial y} \tag{7}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{8}$$

are the velocity-components in the x- and y-directions, respectively.

Substituting (7) and (8) into (6), one obtains

$$\Phi = \mu [4 \psi_{xy}^2 + (\psi_{yy} - \psi_{xx})^2] \tag{9}$$

If the flow is irrational (for example, radial flow of a viscous fluid from a line source or to a line sink, or potential flow of a viscous fluid outside of the boundary layer), $\psi_{xx} + \psi_{yy} = 0$, and $\psi_{yy} - \psi_{xx} = -2\psi_{xx}$, and (9) becomes

$$\Phi = 4\mu (\psi_{xx}^2 + \psi_{xy}^2) = -4\mu (1+q^2) K$$

where K is the Gaussian curvature of the ψ-surface. Hence one has

Theorem 3. The rate of dissipation of energy in the potential flow of a viscous fluid is, in a domain D of the x-y plane, equal to

$$-4\mu \iint_D (1+q^2)^2 K \, dA$$

where K is the Gaussian curvature and q the magnitude of the steepest slope of the ψ-surface, ψ being the stream-function.

The well-known Gauss-Bonnett theorem states that the surface integral

$$\iint_S K \, d\sigma$$

is precisely the angle $\Delta\theta$ which a vector at any point on the boundary of the surface domain S makes with itself after it has been transported in the positive direction (S always lying on the left) around S parallelly in the sense of Levi-Civita.

When a harmonic ψ-surface is very flat, $q << 1$, and

$$-4\mu \iint_D (1+q^2)^2 K \, dA \doteq -4\mu \iint_S K \, d\sigma = -4\mu \Delta\theta$$

Hence, one has, for very flat harmonic surfaces, the

Corallary 1. The rate of dissipation of energy in a region D of potential flow of viscous fluids is proportional to the angle of deviation of a vector from its original position, after it has been transported around the corresponding curved region S on the

ψ-surface (or ϕ-surface, by virtue of Thm. 2) parallelly in the sense of Levi-Civita. The constant of proportionality is -4μ when the transportation is in the positive direction (S always lying on the left).

By virtue of Theorem 1 and its corollary, and remembering that a plane 4-surface corresponds to parallel flow, one has further Corollary 2. In the potential flow of viscous fluids there is always dissipation of energy except when the flow is parallel.

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