DISSERTATION

ESTIMATES OF THE BIVARIATE DISTRIBUTION FUNCTION AT QUANTILE POINTS

Submitted by

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ABSTRACT OF DISSERTATION

ESTIMATES OF THE BIVARIATE DISTRIBUTION FUNCTION AT QUANTILE POINTS

Let $F(x,y)$ by the joint distribution function of (X,Y) possessing a continuous probability density function $f(x,y)$. Let the marginal distribution function of X be F_1 with probability density function f_1, the marginal distribution function of Y be F_2 with probability density function f_2, $\zeta_1(s) = F_1^{-1}(s)$, $\zeta_2(t) = F_2^{-1}(t)$, and $G(s,t) = F(F_1^{-1}(s),F_2^{-1}(t))$. For a random sample (X_k,Y_k), $k=1,2,\ldots,$ $X_{n1}<X_{n2}<\ldots<X_{nn}$, $Y_{n1}<Y_{n2}<\ldots<Y_{nn}$ denote the ordered values of the X's and Y's respectively in the first n vectors (X_k,Y_k). Define $Q_n(s,t)$ as the fraction of the first n vectors whose coordinates (X,Y) are such that $X<X_{ni}$, $Y<Y_{nj}$ where i and j are the smallest integers for which $\frac{i}{n}\geq s$, $\frac{j}{n}\geq t$. The following random functions are constructed

$$S_n(0,0) = S_n(0,1) = S_n(1,0) = S_n(1,1) = 0,$$

$$S_n(s,t) = \sqrt{n}[Q_n(s,t)-G(s,t)],$$

$$X_n(0) = X_n(1) = Y_n(1) = 0,$$

$$X_n(\tfrac{i}{n+1}) = \sqrt{n}f_1(\zeta_1(\tfrac{i}{n+1}))[X_{ni}-\zeta_1(\tfrac{i}{n+1})] \qquad i=1,2,\ldots,n,$$

$$Y_n(\tfrac{i}{n+1}) = \sqrt{n}f_1(\zeta_2(\tfrac{i}{n+1}))[Y_{ni}-\zeta_2(\tfrac{i}{n+1})] \qquad i=1,2,\ldots,n,$$

and $X_n(t)$ and $Y_n(t)$ are determined by linear interpolation for other values of $t\varepsilon(0,1)$ from the sequences $X_n(\frac{i}{n+1})$ and $Y_n(\frac{i}{n+1})$ respectively.

The triplet of random functions $(S_n(s,t), X_n(s), Y_n(t))$ is shown to be asymptotically a vector Gaussian process and the covariance structure is obtained.

Let $F_n(x,y), F_{1n}(x), F_{2n}(y)$ denote the usual estimates of $F(x,y)$, $F_1(x), F_2(y)$ respectively. The standardized triplet of these random functions is shown to be asymptotically a vector Gaussian process and the covariance structure is obtained.

The joint asymptotic distribution of two-sample Wilcoxon statistics is obtained for the bivariate case.

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CHAPTER I

INTRODUCTION

Let $F(x,y)$ be the distribution function of (X,Y) possessing a continuous probability density function $f(x,y)$. Let the marginal distribution function of X be denoted by $F_1(x)$ and that of Y by $F_2(y)$. Let $G(s,t) = F(F_1^{-1}(s),F_2^{-1}(t))$ be the distribution function of the variables $(F_1(X),F_2(Y))$.

The primary purpose of this paper is to investigate an estimate of $G(s,t)$ which was proposed by Siddiqui (1960). A random sample (X_k,Y_k), $k=1,2,\ldots,n$, is drawn from $F(x,y)$. The sample values of X are ordered so that $X_{n1}<X_{n2}<\ldots<X_{nn}$ and similarly $Y_{n1}<Y_{n2}<\ldots<Y_{nn}$. Let $\zeta_1(s)$ be a quantile of F_1 and $\zeta_2(t)$ be a quantile of F_2. An estimate of the combined population quantile $(\zeta_1(s),\zeta_2(t))$ is the combined sample quantile (X_{ni},Y_{nj}) where i and j are integers such that $\frac{i-1}{n} <s \leq\frac{i}{n}$, $\frac{j-1}{n} <t \leq\frac{j}{n}$. Let $M_n(s,t)$ denote the number of elements (X,Y) in the sample such that $X<X_{ni}$, $Y<Y_{nj}$. Then $Q_n(s,t) = \frac{1}{n}M_n(s,t)$ is an estimate of the distribution function at the combined population quantile point $(\zeta_1(s),\zeta_2(t))$ and hence an estimate of $G(s,t)$. Siddiqui first obtains the exact distribution of $(M_n(s,t),X_{ni},Y_{nj})$ for a fixed n. He then finds that the distribution of the standardized variates is asymptotically normal when $n,i,j\to\infty$ so that $\frac{i}{n}\to s$, $\frac{j}{n}\to t$. In Chapter II this line of reasoning is extended to generate a vector Guassian process.

If F_1 and F_2 were known, $G(s,t)$ could also be estimated by first estimating $F(x,y)$ and then using F_1 and F_2 to reduce the problem to the unit square. Similarly $F(x,y)$ could be estimated by first using Siddiqui's technique and then using F_1 and F_2. Such considerations are the subject of Chapter III.

In Chapter IV an example is given of how the second central moment of G arises when one starts extending existing theory to the bivariate case.

Throughout this paper df means distribution function and pdf means probability density function. Unless indicated otherwise all summation and integration will be over the entire space under consideration. The notation EX and $VarX$ denotes the expected value of X and the variance of X respectively.

CHAPTER II

ASYMPTOTIC DISTRIBUTION OF SIDDIQUI'S ESTIMATE OF THE BIVARIATE DISTRIBUTION FUNCTION OF QUANTILE POINTS

Let (X_i, Y_i) $i=1,2,\ldots$ be independent vectors with common df $F(x,y)$ and continuous pdf $f(x,y)$. Denote by $X_{n1}<X_{n2}<\ldots<X_{nn}$, $Y_{n1}<Y_{n2}<\ldots<Y_{nn}$ the ordered values of X's and Y's in the first n vectors $(X_i;Y_i)$ (equalities in the ordering being excluded from consideration as these form a set of probability zero). Let the marginal df of X be F_1 with pdf f_1, the marginal df of Y be F_2 with pdf f_2 and $\zeta_1(t) = F_1^{-1}(t)$, $\zeta_2(t) = F_2^{-1}(t)$. Thus $(\zeta_1(s), \zeta_2(t))$ is the combined population quantile of order (s,t), $0<s$, $t<1$.

Let i_1, j_1, i_2, j_2 be integers such that

$$\frac{i_1-1}{n} \leq s_1 < \frac{i_1}{n}; \quad \frac{j_1-1}{n} \leq t_1 < \frac{j_1}{n}; \quad \frac{i_2-1}{n} \leq s_2 < \frac{i_2}{n}; \tag{2.1}$$

$$\frac{j_2-1}{n} \leq t_2 < \frac{j_2}{n}$$

where $0<s_1, t_1, s_2, t_2 < 1$ are fixed numbers while i's and j's vary with n.

Define $Q_n(s_1,t_1)$ as the fraction of elements (X,Y) in the sample such that $X<X_{ni_1}$, $Y<Y_{nj_1}$ and $Q_n(s_2,t_2)$ as the fraction of elements (X,Y) in the sample for which $X<X_{ni_2}$, $Y<Y_{nj_2}$ where i_1, j_1, i_2, j_2, satisfy (2.1). Then $Q_n(s_1,t_1)$ is a discrete-valued

random variable with possible values $0,\frac{1}{n},\ldots,\frac{1}{n}\min(i_1-1,j_1-1)$ and $Q_n(s_2,t_2)$ is a discrete-valued random variable with possible values $0,\frac{1}{n},\ldots,\frac{1}{n}\min(i_2-1,j_2-1)$.

Siddiqui (1960) obtains the distribution of $(Q_n(s_1,t_1),X_{ni},Y_{nj})$ and shows that asymptotically $(n\to\infty)$ the joint distribution is normal. In the same paper he obtains the asymptotic distribution of $(X_{ni_1},Y_{ni_1},X_{ni_2},Y_{ni_2})$. In this chapter the asymptotic joint distribution of $(Q_n(s_1,t_1),Q_n(s_2,t_2),X_{ni_1},Y_{ni_1},X_{ni_2},Y_{ni_2})$ will be derived.

Since $Q_n(s,t)$ is an estimate of the transformed distribution function at (s,t), the joint distribution of $(Q_n(s_1,t_1),Q_n(s_2,t_2))$ is of obvious interest. We shall show asymptotically for any finite set of numbers $(s_1,t_1),(s_2,t_2),\ldots,(s_m,t_m)$ where $0<s_i,t_i<1$ $i=1,2,\ldots m$ that $(S_n(s_1,t_1),\ldots,S_n(s_m,t_m))$ has an m-variate normal distribution where $S_n(s,t) = \sqrt{n}[Q_n(s,t)-F(\zeta_1(s),\zeta_2(t))]$. Hence $S_n(s,t)$ asymptotically becomes a Gaussian process. To obtain the covariance function and hence uniquely determine this process it is sufficient to obtain the asymptotic distribution of $(S_n(s_1,t_1),S_n(s_2,t_2))$ for $s_1\le s_2$, $t_1\le t_2$.

2.1 Derivation of the Asymptotic Distribution of the Six Variables

Take a Euclidean plane (x,y) to represent the sample values. Select four numbers $(s_1,t_1),(s_2,t_2)$ such that $0<s_1<s_2<1$ and $0<t_1<t_2<1$. The selection of these numbers determines the combined population quantiles $(\zeta_1(s_1),\zeta_2(t_1))$ and $(\zeta_1(s_2),\zeta_2(t_2))$. Using the relations in (2.1), order statistics (X_{ni_1},Y_{nj_1}) and (X_{ni_2},Y_{nj_2}) are uniquely specified for any random sample of size n from $F(x,y)$ with probability one. Now take four numbers x_1,x_2,y_1,y_2 such that $x_k\le X_{ni_k}<x_k+dx_k$, $y_k\le Y_{ni_k}<Y_k+dy_k$, $k=1,2$. These numbers define four

lines which in turn divide the plane into nine disjoint regions, namely

$$R_{11} = \{(x,y) : x<x_1, y<y_1\} ,$$
$$R_{12} = \{(x,y) : x<x_1, y_1<y<y_2\},$$
$$R_{13} = \{(x,y) : x<x_1, y_2<y\},$$
$$R_{21} = \{(x,y) : x_1<x<x_2, y<y_1\} ,$$
$$R_{22} = \{(x,y) : x_1<x<x_2, y_1<y<y_2\},$$
$$R_{23} = \{(x,y) : x_1<x<x_2, y_2<y\},$$
$$R_{31} = \{(x,y) : x_2<x, y<y_1\} ,$$
$$R_{32} = \{(x,y) : x_2<x, y_1<y<y_2\} ,$$
$$R_{33} = \{(x,y) : x_2<x, y_2<y\}.$$

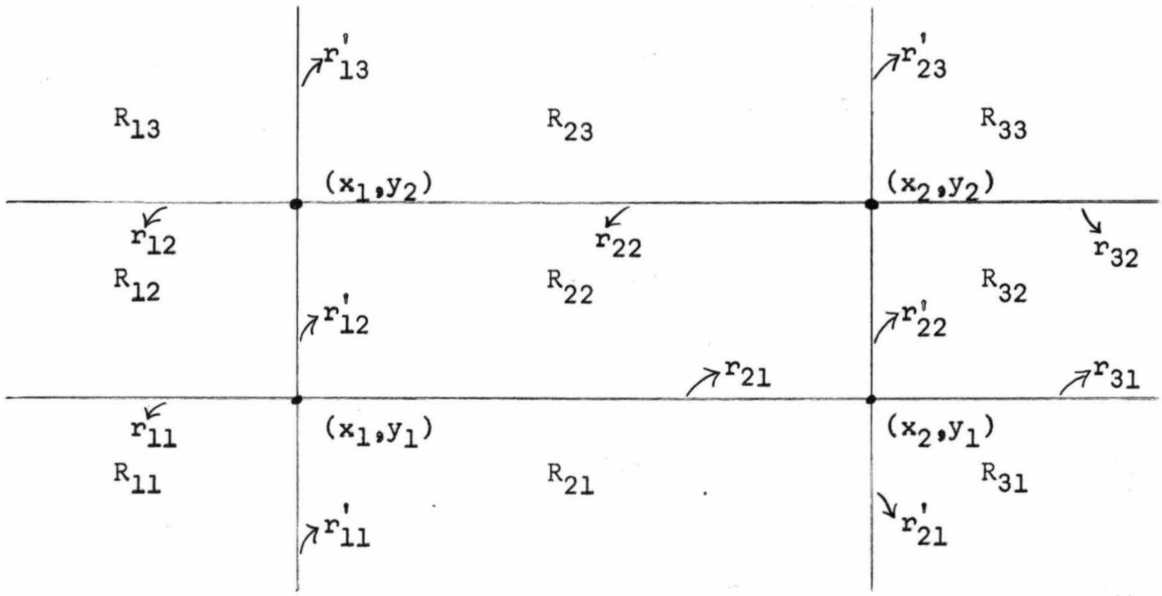
We further have the following disjoint regions

$$r_{11} = \{(x,y) : x<x_1, y=y_1\},$$
$$r_{12} = \{(x,y) : x<x_1, y=y_2\},$$
$$r_{21} = \{(x,y) : x_1<x<x_2, y=y_1\},$$
$$r_{22} = \{(x,y) : x_1<x<x_2, y=y_2\},$$
$$r_{31} = \{(x,y) : x_2<x, y=y_1\},$$
$$r_{32} = \{(x,y) : x_2<x, y=y_2\},$$
$$r'_{11} = \{(x,y) : x=x_1, y<y_1\},$$
$$r'_{12} = \{(x,y) : x=x_1, y_1<y<y_2\},$$
$$r'_{13} = \{(x,y) : x=x_1, y_2<y\},$$
$$r'_{21} = \{(x,y) : x=x_2, y<y_1\},$$
$$r'_{22} = \{(x,y) : x=x_2, y_1<y<y_2\},$$
$$r'_{23} = \{(x,y) : x=x_2, y_2<y\}.$$

The entire Euclidean plane consists of

$$\left(\bigcup_{i_j j=1}^{3} R_{ij}\right) \cup \left(\bigcup_{i=1}^{3} \bigcup_{j=1}^{2} r_{ji} \cup r'_{ij}\right) \cup \left(\bigcup_{i=1}^{2} \bigcup_{j=1}^{2} \{(x_i, y_j)\}\right).$$

Summarizing the above in a geometric form we have the following.



Let $P_{kl} = \iint_{R_{kl}} f(x,y)dxdy$, $p_{ij} = \int_{r_{ij}} f(x,y_j)dx$

$p'_{ji} = \iint_{r'_{ji}} f(x_j,y)dy \qquad k,l = 1,2,3 \quad i=1,2,3; \quad j=1,2.$

N_{kl} = the number of sample points falling in the region R_{kl}.

In finding the desired density function we must distinguish among three distinct possibilities as to the number of sample points determining the order statistics (X_{ni_1}, Y_{ni_1}), (X_{ni_2}, X_{nj_2}). These points are determined by either (a) two sample points, (b) three sample points or (c) four sample points. Thus $N_{\circ\circ} = \sum_{1}^{3} \sum_{1}^{3} N_{ij}$ is either (a) n-2, (b) n-3 or (c) n-4. Depending on how many sample points determine the order statistics under consideration the six quantities

$$N_{1\circ} = N_{11}+N_{12}+N_{13}\ , \quad N_{1\circ} = N_{11}+N_{21}+N_{31}$$

$$N_{2\circ} = N_{21}+N_{22}+N_{23}\ , \quad N_{\circ 2} = N_{12}+N_{22}+N_{32}$$

$$N_{3\circ} = N_{31}+N_{32}+N_{33}\ , \quad N_{\circ 3} = N_{13}+N_{23}+N_{33}$$

take on constant values. Thus there are in each case ((a),(b) or (c)) only four functionally independent random variables. We shall take N_{11}, N_{12}, N_{21} and N_{22} to be the four random variables to consider in deriving the desired density function. Obviously $\frac{N_{11}}{n} = Q_n(s_1,t_1)$ and $\frac{1}{n}(N_{11}+N_{12}+N_{21}+N_{22}) = Q_n(s_2,t_2)$.

Let $P(n_{11},n_{12},n_{21},n_{22},x_1,y_1,x_2,y_2)dx_1dy_1dx_2dy_2$

$$= P_r(N_{11}=n_{11},N_{12}=n_{12},N_{21}=n_{21},N_{22}=n_{22},x_1 \le X_{ni_1} \le x_1+dx_1,$$

$$y_1 \le Y_{nj_1} \le y_1+dy_1,x_2 \le X_{ni_2} \le x_2+dx_2,y_2 \le Y_{ni_2} \le y_2+dy_2).$$

Now in computing p we must add probabilities over the three distinct cases (a), (b) and (c). However the terms adding to P are each of the form

$$c \frac{n!}{\prod_{i,j} n_{ij}!} \prod_{i,j} P_{ij}^{n_{ij}}$$

where c does not depend on n. We noted previously that $N_{\circ\circ}$=n-2 in case (a),, n-3 in case (b) and n-4 in case (c). Thus in the asymptotic case we can ignore cases (a) and (b) explicitly since they contribute terms of order n^{-1} as compared to terms in case (c). So considering the eighty-one disjoint events making up case (c),

$$(2.2)\quad P(n_{ij},x_i,y_j\ i,j=1,2) = \sum_{a,b,c,d=1}^{3} p_{a1}p_{b2}p'_{1c}p'_{2d} \frac{n!}{\prod_{i,j} n_{ij}!} \prod_{i,j} p_{ij}^{n_{ij}} \left(1+O\left(\frac{1}{n}\right)\right)$$

where

$$(2.3)\quad \begin{aligned} N_{1.} &= i_1-1-\delta_{1a}-\delta_{1b}, & N_{.1} &= j_1-1-\delta_{1c}-\delta_{1d} \\ N_{2.} &= i_2-i_1-1-\delta_{2a}-\delta_{2b}, & N_{.2} &= j_2-j_1-1-\delta_{2c}-\delta_{2d} \\ N_{3.} &= n-i_2-\delta_{3a}-\delta_{3b}, & N_{.3} &= n-j_2-\delta_{3c}-\delta_{3d} \end{aligned}$$

and δ_{ij} is the Kronecker delta.

Using Stirling's approximation, $m! = \sqrt{2\pi}\, m^{m+1/2}e^{-m}[1+O(m^{-1})]$ for $m\geq 2$,

$$\frac{n!}{\prod n_{ij}!} = \frac{e^{-4}n^{n+1/2}}{(2\pi)^4} \prod n_{km}^{-n_{km}-1/2} + O\left(\frac{1}{n}\right)$$

since $\sum_{k,m=1}^{3} n_{km} = n-4$. Thus

$$(2.4)\quad \frac{n!}{\prod n_{ij}!} = \left[e^{-4}\ (2\pi)^4 \prod\left(\frac{n_{km}}{n}\right)^{n_{km}+1/2}\right]^{-1} + O\left(\frac{1}{n}\right).$$

Write $\frac{n_{ij}}{n} = q_{ij},\ \frac{1}{n} = dq_{ij} \qquad i,j = 1,2$

This operation is equivalent to replacing the discrete random variable $\frac{N_{ij}}{n}$ by a continuous random variable Q_{ij}. However $|Q_{ij}-N_{ij}/n| < \frac{1}{n}$.

In obtaining the asymptotic distribution we shall let $n,i_1,i_2\to\infty$ such that $i_1/n\to s_1$, $i_2/n\to s_2$, $j_1/n\to t_1$, $j_2/n\to t_2$ (since the relations in (2.1) were fundamental in setting up our method of deriving this distribution). From (2.3) we see that

$$n_{13} = i_1-n_{11}-n_{12}-(1+\delta_{1a}+\delta_{1b})$$

$$n_{31} = j_1-n_{11}-n_{21}-(1+\delta_{1c}+\delta_{1d})$$

$$n_{23} = i_2-i_1-n_{21}-n_{22}-(1+\delta_{2a}+\delta_{2b})$$

$$n_{32} = j_2-j_1-n_{12}-n_{22}-(1+\delta_{2c}+\delta_{2d})$$

$$n_{33} = n-i_2-j_2+n_{11}+n_{21}+n_{12}+n_{22}+(2-\delta_{3c}-\delta_{3d}+\delta_{2a}+\delta_{2b}+\delta_{1c}+\delta_{1d})$$

Substituting in the q's in (2.4) and denoting the transformed density function by $p(q_{11},q_{12},q_{21},q_{22},x_1,y_1,x_2,y_2)$, we can then write

$$(2.5)\quad p(q_{11},q_{12},q_{21},q_{22},x_1,y_1,x_2,y_2) = [G_n(q_{11},q_{12},q_{21},q_{22},x_1,y_1,x_2,y_2)]^n$$
$$H_n(q_{11},q_{12},q_{21},x_1,y_1,x_2,y_2)[1+O(\tfrac{1}{n})]$$

where

$$(2.6)\quad G_n = \prod_{k,l=1}^{2} [P_{kl}/q_{kl}]^{q_{kl}} \left[\frac{P_{13}}{s_1-q_{11}-q_{12}}\right]^{s_1-q_{11}-q_{12}}$$
$$\left[\frac{P_{31}}{t_1-q_{11}-q_{21}}\right]^{t_1-q_{11}-q_{21}} \left[\frac{P_{23}}{s_2-s_1-q_{21}-q_{22}}\right]^{s_2-s_1-q_{21}-q_{22}}$$
$$\left[\frac{P_{32}}{t_2-t_1-q_{12}-q_{22}}\right]^{t_2-t_1-q_{11}-q_{22}}$$
$$\left[\frac{P_{33}}{1-s_2-t_2+q_{11}+q_{12}+q_{21}+q_{22}}\right]^{1-s_2-t_2+q_{11}+q_{12}+q_{21}+q_{22}},$$

(2.7)

$$H_n = \sum_{a,b,c,d=1}^{3} \frac{P_{a1}P_{b2}P'_{1c}P'_{2d}n^4(s_1-q_{11}-q_{12})^{1/2+\delta_{1a}+\delta_{1b}}(t_1-q_{11}-q_{21})^{1/2+\delta_{1c}+\delta_{1d}}}{(q_{11}q_{12}q_{21}q_{22})^{1/2}(2\ )^4\ (s_2-s_1-q_{21}-q_{22})^{-1/2-\delta_{2a}-\delta_{2b}}}$$

$$\cdot \frac{(t_2-t_1-q_{12}-q_{22})^{1/2+\delta_{2c}+\delta_{2d}}}{(1-s_2-t_2+q_{11}+q_{12}+q_{21}+q_{22})^{5/2+\delta_{2a}+\delta_{2b}+\delta_{1c}+\delta_{1d}-\delta_{3c}-\delta_{3d}}} .$$

Let us write for $i,j=1.2$

(2.8) $F(\zeta_1(s_i),\zeta_2(t_j)) = F_{ij}, \quad f_1(\zeta_1(s_i)) = f_{1i}, \quad f_2(\zeta_2(t_i)) = f_{2i}$

$$\frac{\partial F}{\partial s_i}(\zeta_1(s_i),\zeta_2(t_j)) = b_{ij}, \quad \frac{\partial F}{\partial t_j}(\zeta_1(s_i),\zeta_2(t_j)) = c_{ij} .$$

$$g_{1i} = [s_i(1-s_i)]^{1/2}, \quad g_{2i} = [t_i(1-t_i)]^{1/2} .$$

Make the transformations

(2.9) $$u_i = \frac{n^{1/2}(x_i-\zeta_1(s_i))f_{ij}}{[s_i(1-s_i)]^{1/2}}, \quad v_i = \frac{n^{1/2}(y_i-\zeta_2(t_i))f_{2i}}{[t_i(1-t_i)]^{1/2}}, \quad i=1,2.$$

and let $p(\underline{q},\underline{x},\underline{y})$ go into $p(\underline{q},\underline{u},\underline{v})$ where p is used generically to denote a pdf and is not the same function from equation to equation.

Writing out the P_{ij}'s we have

(2.10)
$$P_{11} = F(x_1,y_1) \qquad P_{21} = F(x_2,y_1)-F(x_1,y_1)$$
$$P_{12} = F(x_1,y_2)-F(x_1,y_1) \qquad P_{22} = F(x_2,y_2)-F(x_2,y_1)-F(x_1,y_2)+F(x_1,y_1)$$
$$P_{13} = F_1(x_1)-F(x_1,y_2) \qquad P_{23} = F_1(x_2)-F(x_2,y_2)-F_1(x_1)+F(x_1,y_2)$$

$$P_{31} = F_2(y_1)-F(x_2,y_1) \quad P_{33} = 1-F_1(x_2)-F_2(y_2)+F(x_2,y_2)$$

$$P_{32} = F_2(y_2)-F(x_2,y_2)-F_2(y_1)+F(x_2,y_1).$$

Expanding the P's in powers of u's and v's, we have

(2.11)
$$P_{11} = F_{11}+g_{11}b_{11}u_1/\sqrt{n}+g_{21}c_{11}v_1/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{12} = F_{12}-F_{11}+g_{11}(b_{12}-b_{11})u_1/\sqrt{n}+g_{22}c_{12}\, v_2/\sqrt{n}-g_{21}c_{11}\, v_1/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{13} = s_1-F_{12}+g_{11}(1-b_{12})u_1/\sqrt{n}-g_{22}c_{12}\, v_2/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{21} = F_{21}-F_{11}+g_{12}b_{21}u_2/\sqrt{n}-g_{11}b_{11}u_1/\sqrt{n}+g_{21}(c_{21}-c_{11})v_1/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{22} = F_{22}-F_{21}-F_{12}+F_{11}+g_{11}(b_{11}-b_{12})u_1/\sqrt{n}+g_{12}(b_{22}-b_{21})u_2/\sqrt{n}$$
$$+g_{21}(c_{11}-c_{21})v_1/\sqrt{n}+g_{22}(c_{22}-c_{12})v_2/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{23} = s_2-s_1+F_{12}-F_{22}+g_{11}(b_{12}-1)u_1/\sqrt{n}+g_{12}(1-b_{22})u_2/\sqrt{n}$$
$$+g_{22}(c_{12}-c_{22})v_2/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{31} = t_1-F_{21}-g_{12}b_{21}u_2/\sqrt{n}+g_{21}(1-c_{21})v_1/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{32} = t_2-t_1+F_{21}-F_{22}+g_{12}(b_{21}-b_{22})u_2/\sqrt{n}+g_{21}(c_{21}-1)v_1/\sqrt{n}$$
$$+g_{22}(1-c_{22})v_2/\sqrt{n}+0(\tfrac{1}{n})$$

$$P_{33} = 1-s_2-t_2+F_{22}+g_{12}(b_{22}-1)u_2/\sqrt{n}+g_{22}(c_{22}-1)v_2/\sqrt{n}+0(\tfrac{1}{n}).$$

If we introduce the notation

$$q_{13} = s_1-q_{11}-q_{12} \qquad q_{32} = t_2-t_1-q_{12}-q_{22}$$

$$q_{31} = t_1-q_{11}-q_{21} \qquad q_{33} = 1-s_2-t_2+q_{11}+q_{12}+q_{21}+q_{22}$$

$$q_{23} = s_2-s_1-q_{21}-q_{22}$$

then $\ln G_n = \sum_{k,l=1}^{3} q_{kl}\, qn\, \frac{P_{kl}}{q_{kl}}$ and hence may be written as

$$\ln G_n = W(\underline{q})+\sum_{k,l=1}^{3}(q_{kl}-Eq_{kl})\ln\left[1+\frac{P_{kl}-Eq_{kl}}{Eq_{kl}}\right]+\sum_{k,l=1}^{3}Eq_{kl}\ln(1+\frac{P_{kl}-Eq_{kl}}{Eq_{kl}})$$

where $W(\underline{q}) = \sum_{k,l=1}^{3} q_{kl} \ln \frac{Eq_{kl}}{q_{kl}}$ with Eq_{kl} denoting the expected value of q_{kl}.

Siddiqui (1960) shows that $q_{kl} - Eq_{kl}$ is of order $n^{-1/2}$ and also $P_{kl} - Eq_{kl}$ is of order $n^{-1/2}$ since from (2.11), Eq_{kl} is the first term in the expansion of P_{kl}. Furthermore $\sum_{k,l=1}^{3} q_{kl} = \sum_{k,l=1}^{3} P_{kl} = 1$ and since $\ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + \ldots$,

(2.12) $$\ln G_n = W(\underline{q}) + \sum_{k,l=1}^{3} (q_{kl} - Eq_{kl}) \frac{P_{kl} - Eq_{kl}}{Eq_{kl}} - \frac{1}{2} \sum_{k,l=1}^{3} \frac{(P_{kl} - Eq_{kl})^2}{Eq_{kl}} + O(n^{-3/2}) .$$

Let us define $w_{ij} = \sqrt{n}\,(q_{ij} - Eq_{ij})$ with a similar transformation on Q_{ij}, $i,j = 1,2$. From the expansion of P_{kl}, we have defining $(Eq_{ij})^{-1} = \nu_{ij}$ $i,j = 1,2,3$,

(2.13) $$\begin{aligned} n \sum_{k,l=1}^{3} (q_{kl} - Eq_{kl}) \frac{P_{kl} - Eq_{kl}}{Eq_{kl}} &= \nu_{11} w_{11} [g_{11} b_{11} u_1 + g_{21} c_{11} v_1] \\ &+ \nu_{12} w_{12} [g_{11}(b_{12} - b_{11}) u_1 + g_{22} c_{12} v_2 - g_{21} c_{11} v_1] \\ &- \nu_{13} (w_{11} + w_{12}) [g_{11}(1 - b_{12}) u_1 - g_{22} c_{12} v_2] + \nu_{21} w_{21} [g_{12} b_{21} u_2 \\ &- g_{11} b_{11} u_1 + g_{21}(c_{21} - c_{11}) v_1] + \nu_{22} w_{22} [g_{11}(b_{11} - b_{12}) u_1 \\ &+ g_{12}(b_{22} - b_{21}) u_2 + g_{21}(c_{11} - c_{21}) v_1 + g_{22}(c_{22} - c_{12}) v_2] \\ &- \nu_{31} (w_{11} + w_{21}) [-g_{12} b_{21} u_2 + g_{21}(1 - c_{21}) v_1] \\ &- \nu_{23} (w_{21} + w_{22}) [g_{11}(b_{12} - 1) u_1 + g_{12}(1 - b_{22}) u_2 + g_{22}(c_{12} - c_{22}) v_2] \end{aligned}$$

$$- \nu_{32}(w_{12}+w_{22})[g_{12}(b_{21}-b_{22})u_2 + g_{21}(c_{21}-1)v_1 + g_{22}(1-c_{22})v_2]$$

$$+ \nu_{33}[w_{11}+w_{12}+w_{21}+w_{22})[g_{12}(b_{22}-1)u_2+g_{22}(c_{22}-1)v_2]+0(n^{-1/2}).$$

Also

$$(2.14)\quad n \sum_{k,l=1}^{3} \frac{(P_{kl}-Eq_{kl})^2}{Eq_{kl}} = \nu_{11}[g_{11}b_{11}u_1+g_{21}c_{11}v_1]^2 + \nu_{12}[g_{11}(b_{12}-b_{11})u_1$$

$$+ g_{22}c_{12}v_2 - g_{21}c_{11}v_1]^2 + \nu_{13}[g_{11}(1-b_{12})u_1 - g_{22}c_{12}v_2]^2$$

$$+ \nu_{21}[g_{12}b_{21}u_2 - g_{11}b_{11}u_1 + g_{21}(c_{21}-c_{11})v_1]^2 + \nu_{22}[g_{11}(b_{11}-b_{12})u_1$$

$$+ g_{12}(b_{22}-b_{21})u_2 + g_{21}(c_{11}-c_{21})v_1 + g_{22}(c_{22}-c_{12})v_2]^2$$

$$+ \nu_{31}[-g_{12}b_{21}u_2 + g_{21}(1-c_{21})v_1]^2 + \nu_{23}[g_{11}(b_{12}-1)u_1$$

$$+ g_{12}(1-b_{22})u_2 + g_{22}(c_{12}-c_{22})v_2]^2 + \nu_{32}[g_{12}(b_{21}-b_{22})u_2$$

$$+ g_{21}(c_{21}-1)v_1 + g_{22}(1-c_{22})v_2]^2 + \nu_{33}[g_{12}(b_{22}-1)u_2$$

$$+ g_{22}(c_{22}-1)v_2]^2 + 0(n^{-1/2}).$$

Now $W(\underline{q})$ may be written as

$$-W(\underline{q}) = \sum_{k,l=1}^{3} (q_{kl}-Eq_{kl})\ln[1 + \frac{q_{kl}-Eq_{kl}}{Eq_{kl}}]$$

$$+ \sum_{k,l=1}^{3} Eq_{kl} \ln[1 + \frac{q_{kl}-Eq_{kl}}{Eq_{kl}}]$$

$$= \frac{1}{2} \sum_{k,l=1}^{3} \frac{(q_{kl}-Eq_{kl})^2}{Eq_{kl}} + 0(n^{-3/2})$$

$$\therefore -nW(\underline{q}) = \frac{1}{2} \sum_{k,l=1}^{3} {}_{kl}[\sqrt{n}(q_{kl}-Eq_{kl})]^2 + O(n^{-1/2}).$$

Substituting in the w's for the q's and denoting the resulting function by $W'(\underline{w})$, we have

$$-2nW'(\underline{w}) = \nu_{11}w_{11}^2+\nu_{12}w_{12}^2+\nu_{21}w_{21}^2+\nu_{22}w_{22}^2+\nu_{13}(w_{11}+w_{12})^2$$

$$+\nu_{31}(w_{11}+w_{21})^2+\nu_{23}(w_{21}+w_{22})^2+\nu_{32}(w_{12}+w_{22})^2+\nu_{33}(w_{11}+w_{12}+w_{21}+w_{22})^2$$

$$+ O(n^{-1/2}).$$

(2.15) $$-2nW'(\underline{w}) = (\nu_{10}+\nu_{13}+\nu_{33})w_{11}^2+(\nu_{12}+\nu_{13}+\nu_{32}+\nu_{33})w_{12}^2$$

$$+(\nu_{21}+\nu_{31}+\nu_{23}+\nu_{33})w_{21}^2+(\nu_{22}+\nu_{23}+\nu_{32}+\nu_{33})w_{22}^2+2(\nu_{13}+\nu_{33})w_{11}w_{12}$$

$$+2(\nu_{31}+\nu_{33})w_{11}w_{21}+2\nu_{33}w_{11}w_{22}+2\nu_{33}w_{12}w_{21}+2(\nu_{32}+\nu_{33})w_{12}w_{22}$$

$$+2(\nu_{23}+\nu_{33})w_{21}w_{22} + O(n^{-1/2}).$$

Combining all of the above we have

$$p(\underline{w},\underline{u},\underline{v}) = (2\pi)^{-4}|V^*|\exp[-\tfrac{1}{2}\,\underline{Z}^{*\prime}V^{*-1}\underline{Z}^*][1+O(n^{-1/2})]$$

where

$$\underline{Z}^* = \begin{matrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$

and the elements of V^{*-1} are obtained from (2.13), (2.14), and (2.15).

Before we write out the elements of V^{*-1}, let us make a transformation

$$t_{11} = w_{11},\quad t_{12} = w_{11}+w_{12},\quad t_{21} = w_{11}+w_{21},\quad t_{22} = w_{11}+w_{12}+w_{21}+w_{22}$$

with a similar transformation on the random variables. Denoting the transformed vector by $\underline{Z}$ where $\underline{Z}' = (t_{11},t_{12},t_{21},u_1,v_1,u_2,v_2)$, we have for the density function of $\underline{Z}$,

$$f(\underline{Z}) = (2\pi)^{-4}|V|\exp[-\tfrac{1}{2}\,\underline{Z}'V^{-1}\underline{Z}][1+O(n^{-1/2})].$$

Letting

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix},\qquad V^{*-1} = \left(\begin{array}{c|c} V^{*11} & V^{*12} \\ \hline V^{*21} & V^{*22} \end{array}\right),$$

since

$$\underline{Z} = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right)\underline{Z}^*,\qquad V^{-1} = \left(\begin{array}{c|c} B'V^{*11}B & B'V^{*12} \\ \hline B'V^{*12} & V^{*22} \end{array}\right) = \left(\begin{array}{c|c} V^{11} & V^{12} \\ \hline V^{21} & V^{22} \end{array}\right).$$

Now,

$$V^{*11} = \begin{bmatrix} \nu_{11}+\nu_{13}+\nu_{31}+\nu_{33} & \nu_{13}+\nu_{33} & \nu_{31}+\nu_{33} & \nu_{33} \\ \nu_{13}+\nu_{33} & \nu_{12}+\nu_{13}+\nu_{32}+\nu_{33} & \nu_{33} & \nu_{32}+\nu_{33} \\ \nu_{31}+\nu_{33} & \nu_{33} & \nu_{21}+\nu_{23}+\nu_{31}+\nu_{33} & \nu_{23}+\nu_{33} \\ \nu_{33} & \nu_{32}+\nu_{33} & \nu_{23}+\nu_{33} & \nu_{22}+\nu_{23}+\nu_{32}+\nu_{33} \end{bmatrix}$$

The first two columns of $-V^{*12}$ are

$$\begin{bmatrix} g_{11}[\nu_{11}b_{11}-\nu_{13}(1-b_{12})] & g_{21}[\nu_{11}c_{11}-\nu_{31}(1-c_{21})] \\ g_{11}[\nu_{12}(b_{12}-b_{11})-\nu_{13}(1-b_{12})] & g_{21}[-\nu_{12}c_{11}+\nu_{32}(1-c_{21})] \\ g_{11}[-\nu_{21}b_{11}+\nu_{23}(1-b_{12})] & g_{21}[\nu_{21}(c_{21}-c_{11})-\nu_{31}(1-c_{21})] \\ g_{11}[\nu_{22}(b_{11}-b_{12})+\nu_{23}(1-b_{12})] & g_{21}[\nu_{22}(c_{11}-c_{21})+\nu_{32}(1-c_{21})] \end{bmatrix},$$

and the second two columns of $-V^{*12}$ are

$$\begin{bmatrix} g_{12}[\nu_{31}b_{21}-\nu_{33}(1-b_{22})] & g_{22}[\nu_{13}c_{12}-\nu_{33}(1-c_{22})] \\ g_{12}[\nu_{32}(b_{22}-b_{21})-\nu_{33}(1-b_{22})] & g_{22}[(\nu_{13}+\nu_{12})c_{12}-(\nu_{32}+\nu_{33})(1-c_{22})] \\ g_{12}[(\nu_{21}+\nu_{31})b_{21}-(\nu_{23}+\nu_{33})(1-b_{22})] & g_{22}[\nu_{23}(c_{22}-c_{12})-\nu_{33}(1-c_{22})] \\ g_{12}[(\nu_{22}+\nu_{32})(b_{22}-b_{21})-(\nu_{23}+\nu_{33})(1-b_{22})] & g_{22}[(\nu_{22}+\nu_{23})(c_{22}-c_{12})-(\nu_{32}+\nu_{33})(1-c_{22})] \end{bmatrix}.$$

Hence

(2.16)
$$V^{11} = \begin{bmatrix} \nu_{11}+\nu_{12}+\nu_{21}+\nu_{22} & -\nu_{12}-\nu_{22} & -\nu_{21}-\nu_{22} & \nu_{22} \\ -\nu_{12}-\nu_{22} & \nu_{12}+\nu_{13}+\nu_{23}+\nu_{22} & \nu_{22} & -\nu_{22}-\nu_{23} \\ -\nu_{21}-\nu_{22} & \nu_{22} & \nu_{21}+\nu_{31}+\nu_{32}+\nu_{22} & -\nu_{22}-\nu_{32} \\ \nu_{22} & -\nu_{22}-\nu_{23} & -\nu_{22}-\nu_{32} & \nu_{22}+\nu_{23}+\nu_{32}+\nu_{33} \end{bmatrix}$$

and letting $(V^{11})_{ij} = v^{ij}$, fortunately V^{12} simplifies as follows. Denoting the columns of V^{12} by V_i^{12}, i=1,2,3,4, we have after some algebra,

(2.17)

$$-V_1^{12} = g_{11}\begin{bmatrix} b_{11}v^{11}+b_{12}v^{12} \\ b_{11}v^{21}+b_{12}v^{22}-(\nu_{13}+\nu_{23}) \\ b_{11}v^{31}+b_{12}v^{32} \\ b_{11}v^{41}+b_{12}v^{42}+\nu_{23} \end{bmatrix}$$

$$-V_2^{12} = g_{21}\begin{bmatrix} c_{11}v^{11}+c_{21}v^{13} \\ c_{11}v^{21}+c_{21}v^{23} \\ c_{11}v^{31}+c_{21}v^{33}-(\nu_{31}+\nu_{32}) \\ c_{11}v^{41}+c_{21}v^{43}+\nu_{32} \end{bmatrix}$$

$$-V_3^{12} = g_{12}\begin{bmatrix} b_{21}v^{13}+b_{22}v^{14} \\ b_{21}v^{23}+b_{22}v^{24}+\nu_{23} \\ b_{21}v^{33}+b_{22}v^{34} \\ b_{21}v^{43}+b_{22}v^{44}-(\nu_{23}+\nu_{33}) \end{bmatrix}$$

$$-V_4^{12} = g_{22}\begin{bmatrix} c_{12}v^{12}+c_{22}v^{14} \\ c_{12}v^{22}+c_{22}v^{24} \\ c_{12}v^{32}+c_{22}v^{34}+\nu_{32} \\ c_{12}v^{42}+c_{22}v^{44}-(\nu_{32}+\nu_{33}) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} g_{11}b_{11} & g_{21}c_{11} & 0 & 0 \\ g_{11}b_{12} & 0 & 0 & g_{22}c_{12} \\ 0 & g_{21}c_{21} & g_{12}b_{21} & 0 \\ 0 & 0 & g_{12}b_{22} & g_{22}c_{22} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ g_{11}(\nu_{13}+\nu_{23}) & 0 & -g_{12}\nu_{23} & 0 \\ 0 & g_{21}(\nu_{31}+\nu_{32}) & 0 & -g_{22}\nu_{32} \\ -g_{11}\nu_{23} & -g_{21}\nu_{32} & g_{12}(\nu_{23}+\nu_{33}) & g_{22}(\nu_{32}+\nu_{33}) \end{bmatrix}$$

then we may write V^{12} as $V^{12} = -(V^{11}A-L)$. Similarly with sufficient patience one will discover that V^{22} may be expressed as $V^{22} = A'(V^{11}A-L)+M$ where the first two columns of M are

$$\begin{bmatrix} g_{11}^2(1-b_{12})(\nu_{13}+\nu_{23}) & 0 \\ 0 & g_{21}^2(1-c_{21})(\nu_{31}+\nu_{32}) \\ -g_{11}g_{12}\nu_{23}(1-b_{12}) & 0 \\ 0 & -g_{21}g_{22}(1-c_{21})\nu_{32} \end{bmatrix}$$

and the last two columns of M are

$$\begin{bmatrix} -g_{11}g_{12}(1-b_{22})\nu_{23} & g_{11}g_{22}[(c_{22}-c_{12})\nu_{23}-c_{12}\nu_{13}] \\ g_{21}g_{12}[b_{22}\nu_{32}-b_{21}(\nu_{32}+\nu_{31})] & -g_{21}g_{22}(1-c_{22})\nu_{32} \\ g_{12}^2(1-b_{22})(\nu_{23}+\nu_{33}) & g_{12}g_{22}[(1-c_{22})\nu_{33}-(c_{22}-c_{12})\nu_{23}] \\ g_{12}g_{22}[(1-b_{22})\nu_{33}-(b_{22}-b_{21})\nu_{32}] & g_{22}^2(1-c_{22})(\nu_{32}+\nu_{33}) \end{bmatrix}.$$

Now since

$$\left(\begin{array}{c|c} V_{11} & V_{12} \\ \hline V'_{12} & V_{22} \end{array}\right) \left(\begin{array}{c|c} V'' & -(V''A-L) \\ \hline -(V''A-1)' & A'(V''A-L)+M \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right),$$

$$-V'_{12}(V''A-L)+V_{22}(A'V''A-A'L+M) = I$$

(2.18) $$V'_{12}V''-V_{22}(A'V''-L') = 0.$$

Hence

(2.19) $$V'_{12}L+V_{22}(L'A-A'L+M) = I.$$

Also

(2.20) $$V_{11}V''-V_{12}(A'V''-L') = I$$

$$V_{11}(V''A-L)+V_{12}(A'V''A-A'L+M) = 0$$

so

(2.21) $$V_{11}L+V_{12}(L'A-A'L+M) = A.$$

The first two columns of $L'A-A'L+M$ are

$$\begin{array}{ll} g_{11}^2(1-b_{12})(\nu_{13}+\nu_{23}) & 0 \\ 0 & g_{21}^2(1-c_{21})(\nu_{31}+\nu_{32}) \\ -g_{11}g_{12}(1-b_{22})\nu_{23} & g_{21}g_{12}[b_{22}\nu_{32}-b_{21}(\nu_{31}+\nu_{32})] \\ g_{11}g_{22}[c_{22}\nu_{23}-c_{12}(\nu_{13}+\nu_{23})] & -g_{21}g_{22}(1-c_{22})\nu_{32} \end{array}$$

and the last two columns are

$$\begin{matrix} -g_{11}g_{12}(1-b_{12})\nu_{23} & 0 \\ 0 & -g_{21}g_{22}(1-c_{21})\nu_{32} \\ g_{12}^2(1-b_{22})(\nu_{23}+\nu_{33}) & g_{12}g_{22}[(b_{21}-b_{22})\nu_{32}+\nu_{33}(1-b_{22})] \\ g_{12}g_{22}[\nu_{23}(c_{12}-c_{22})+\nu_{33}(1-c_{22})] & g_{22}^2(1-c_{22})(\nu_{32}+\nu_{33}) \end{matrix} .$$

Let us define for $i,j=1,2$

(2.22)

$$\rho_{ij} = \frac{F_{ij}-s_i t_j}{g_{1i}g_{2i}}, \quad \rho_s = \frac{s_1(1-s_2)}{g_{11}g_{12}}, \quad \rho_t = \frac{t_1(1-t_2)}{g_{21}g_{22}}$$

$$d_{ij} = \frac{F_{ij}(t_j-F_{ij})(1-t_j)}{g_{1i}^2 g_{2j}^2(1-\rho_{ij}^2)}$$

$$D_{ij} = \frac{F_{ij}(s_i-F_{ij})(1-s_i)}{g_{1i}^2 g_{2j}^2(1-\rho_{ij}^2)}$$

$$h_{ij} = g_{1i}(b_{ij}-d_{ij}), \quad H_{ij}=g_{2j}(c_{ij}-D_{ij})$$

$$K_{ij} = \frac{F_{ij}(s_i-F_{ij})(t_j-F_{ij})(1-s_i-t_j+F_{ij})}{g_{1i}^2 g_{2j}^2(1-\rho_{ij}^2)} .$$

Then from Siddiqui (1960) the known elements of V are

(2.23) $v_{11} = K_{11}+h_{11}^2+H_{11}^2+2\rho_{11}h_{11}H_{11}$

$v_{22} = K_{12}+h_{12}^2+H_{12}^2+2\rho_{12}h_{12}H_{12}$

$v_{33} = K_{21}+h_{21}^2+H_{21}^2+2\rho_{21}h_{21}H_{21}$

$v_{44} = K_{22}+h_{22}^2+H_{22}^2+2\rho_{22}h_{22}H_{22}$

$v_{15} = h_{11}+\rho_{11}H_{11}$, $v_{16} = H_{11}+\rho_{11}h_{11}$, $v_{25} = h_{12}+\rho_{12}h_{12}$,

$$v_{28} = H_{12}+\rho_{12}h_{12}, \quad v_{36} = H_{21}+\rho_{21}h_{21}, \quad v_{37} = h_{21}+\rho_{21}H_{21} \quad ,$$

$$v_{47} = h_{22}+\rho_{22}H_{22}, \quad v_{48} = H_{22}+\rho_{22}h_{22}$$

$$V_{22} = \begin{bmatrix} 1 & \rho_{11} & \rho_s & \rho_{12} \\ \rho_{11} & 1 & \rho_{21} & \rho_t \\ \rho_s & \rho_{21} & 1 & \rho_{22} \\ \rho_{12} & \rho_t & \rho_{22} & 1 \end{bmatrix} .$$

Now from equation (2.19),

$$v_{36}g_{21}(\nu_{31}+\nu_{32}) - v_{46}g_{21}\nu_{32}+g_{21}^2(1-c_{21})(\nu_{31}+\nu_{32})$$

$$+\rho_{21}g_{21}g_{12}[b_{22}\nu_{32}-b_{21}(\nu_{31}+\nu_{32})] - \rho_t g_{21}g_{22}(1-c_{22})\nu_{32} = 1 .$$

Simplifying and noting that for $i,j=1,2$

$$g_{2j}^2(1-D_{ij}) - g_{2j}g_{1i}\rho_{ij}d_{ij} = (t_j-F_{ij})(1-t_j)$$

$$g_{1i}^2(1-d_{ij}) - g_{2j}g_{1i}\rho_{ij}D_{ij} = (s_i-F_{ij})(1-s_i),$$

(2.24) $v_{46} = \text{cov}(t_{22},v_1) = g_{21}^{-1}[t_1F_{22}-F_{21}+t_1(1-t_2)c_{22}+(F_{21}-s_2t_1)b_{22}]$

Similarly v_{45} may be obtained from (2.19) or by interchanging s and t in (2.24). Using (2.18) we obtain an interesting contrast to (2.24),

(2.25) $v_{18} = \text{cov}(t_{11},v_2) = g_{22}^{-1}[b_{11}(F_{12}-s_1t_2)+c_{11}t_1(1-t_2) - F_{11}(1-t_2)]$.

Also we have

(2.26) $v_{38} = \text{cov}(t_{21},v_2) = g_{22}^{-1}[b_{21}(F_{22}-s_2t_2)+c_{21}t_1(1-t_2) - F_{21}(1-t_2)]$.

The elements of V_{11} are computed similarly using (2.20) and (2.21).

After simplification

$$v_{24} = n\ \mathrm{cov}(Q_n(s_1,t_2),Q_n(s_2,t_2)) = b_{12}[(F_{12}-s_1t_2)c_{22}+s_1(1-s_2)b_{22}$$

$$+s_1F_{22}-F_{12}]+c_{12}[t_2(1-t_2)c_{22}+(F_{22}-s_2t_2)b_{22}-F_{22}(1-t_2)]$$

$$-F_{12}[b_{22}(1-s_2)+c_{22}(1-t_2)-(1-F_{22})]$$

$$v_{14} = n\ \mathrm{cov}(Q_n(s_1,t_1),Q_n(s_2,t_2)) = -[b_{22}(1-s_2)+c_{22}(1-t_2)+F_{22}-1]F_{11}$$

$$+b_{11}g_{11}v_{45}+c_{11}g_{21}v_{46}\ .$$

Before summarizing the above results let us prove the following theorem.

Theorem 2.1

If under the conditions in Section 2.1 one selects k_1k_2 points (s_i,t_j) $i=1,\ldots,k_{ij}$ $j=1,\ldots,k_2$ and defines the corresponding $(Q_n(s_i,t_j),U_n(s_i),V_n(t_j))$ $i=1,\ldots,k_1;$ $j=1,\ldots,k_2$ then asymptotically the vector $(\sqrt{n}[Q_n(s_1,t_2)-F_{11}],U_n(s_1),V_n(t_1),\ldots,\sqrt{n}[Q_n(s_{k_1},t_{k_2})-F_{k_1k_2}],$ $U_n(s_{k_1}),V_n(t_{k_2}))$ is distributed as a multivariate normal.

Proof. Proceed as was the case for $k_1,k_2=2$. One has k_1k_2 functionally independent variates and need only concern himself with the case when the order statistics are determined by k_1+k_2 sample points. The equation corresponding to (2.2) is of the same form with the only change being a change in summation and product limits. Use Stirling's approximation and obtain an equation of the same form as (2.5). The corresponding G_n is of the same form and simplifies to $n\ln G_n = -\frac{1}{2}\Sigma v_{kl}(w_{kl}-P_{kl}+Eq_{kl})^2+O(n^{-3/2})$ as was the case for $k_1k_2=2$. The corresponding H_n serves essentially to determine the constant of integration for the resulting density function. Hence the limiting distribution is a multivariate normal.

A stochastic process is said to be a Gaussian process if for any finite number of points in the index set the resulting variates have a multivariate normal distribution. Hence we have proven the following theorem.

Theorem 2.2

Let $F(x,y)$ be the df of (X,Y) possessing a pdf $f(x,y)$. We will assume that $\{(x,y):f(x,y)>0\}$ is either the entire plane or a simply connected convex region. Let (X_i,Y_i), $i=1,2,\ldots$ be independent vectors with common continuous pdf $f(x,y)$ and $X_{n1}<X_{n2}<\ldots<X_{nn}$, $Y_{n1}<Y_{n2}<\ldots<Y_{nn}$, the ordered values of X's and Y's in the first n vectors (X_i,Y_i). Let the marginal df of X be F_1 with pdf f_1, the marginal df of Y be F_2 with pdf f_2, $\zeta_1(t) = F_1^{-1}(t)$, $\zeta_2(t) = F_2^{-1}(t)$, $G(s,t) = F(\zeta_1(s), {}_2(t))$.

Following Siddiqui (1965), let us construct the following random functions

$$X_n(0) = X_n(1) = Y_n(0) = Y_n(1) = 0$$

$$X_n(\tfrac{i}{n+1}) = n^{1/2} f_1(\zeta_1(\tfrac{i}{n+1}))[X_{ni} - \zeta_1(\tfrac{i}{n+1})] \quad i=1,2,\ldots,n,$$

$$Y_n(\tfrac{i}{n+1}) = n^{1/2} f_2(\zeta_2(\tfrac{i}{n+1}))[Y_{ni} - \zeta_2(\tfrac{i}{n+1})] \quad i=1,2,\ldots,n,$$

and determine $X_n(t)$ and $Y_n(t)$ by linear interpolation for other values of $t\varepsilon(0,1)$ from the sequences $X_n(\frac{i}{n+1})$ and $Y_n(\frac{i}{n+1})$ respectively.

Let $Q_n(s,t)$ denote the fraction of the first n vectors whose coordinates (X,Y) are such that $X<X_{ni_1}$, $Y<Y_{nj_1}$, where the integers i_1, j_1 satisfy the relations $\frac{i_1-1}{n} \leq s < \frac{i_1}{n}$, $\frac{j_1-1}{n} \leq t < \frac{j_1}{n}$. Let

$$S_n(0,0) = S_n(0,1) = S_n(1,0) = S_n(1,1) = 0,$$

$$S_n(s,t) = \sqrt{n}[Q_n(s,t) - G(s,t)] ,$$

then as $n\to\infty$ $(S_n(s,t), X_n(s), Y_n(t)) \to (S(s,t), X(s),Y(t))$ in distribution where $(S(s,t), X(s), Y(t))$, $s,t\varepsilon(0,1)$, is a vector Gaussian process with

(2.27) $$ES(s,t) = EX(s) = EY(t) = 0$$

$$ES(s_1,t_1)S(s_2,t_2) = -\beta(\max(s_1,s_2),\max(t_1,t_2)G(\min(s_1,s_2),$$

$$\min(t_1,t_2)) + b(\min(s_1,s_2),\min(z_1,t_2))\alpha_1(\max(s_1,s_2),$$

$$\max(t_1,t_2),\min(s_1,s_2) + c(\min(s_1,s_2),\min(t_1,t_2))$$

$$\alpha_2(\max(s_1,s_2),\max(t_1,t_2),\min(t_1,t_2),$$

where

$$b(u,v) = \frac{\partial G(u,v)}{\partial u}, \quad C(u,v) = \frac{\partial G(u,v)}{\partial v},$$

$$\beta(u,v) = b(u,v)(1-u)+c(u,v)(1-v)+G(u,v)-1,$$

$$\alpha_1(u,v,w) = c(u,v)[G(w,v)-vw]+b(u,v)[\min(u,w)-uw] + wG(u,v)$$
$$-G(\min(u,w),v),$$

$$\alpha_2(u,v,2) = b(u,v)[G(u,w)-uw]+c(u,v)[\min(v,w)-vw] + wG(u,v)$$
$$-G(u,\min(v,w)),$$

$$E\ S(s,t)X(u) = \alpha_1(s,t,u)$$

$$E\ S(s,t)Y(v) = \alpha_2(s,t,v)$$

$$E\,X(s)Y(t) = G(s,t) - st$$

$$E\,X(s)X(t) = EY(s)Y(t) = \min(s,t) - st$$

The preceding theorem appears to have potential for use in many areas of statistics. For example, in a paper to be published Siddiqui and Crow use the $X(s)$ process to obtain the asymptotic distribution of special symmetrical linear combinations of order statistics such as trimmed means, Winsorized means, "linearly weighted" means, and a combination of the median and two other order statistics. The purpose of the paper was to obtain a robust estimate of location. One can similarly define a measure of spread for a distribution and obtain a robust estimate of this measure. Unlike measures of location and spread, measures of statistical dependence are not widely agreed upon. However one can define measures of dependency which utilize $S(s,t)$ as estimates and thereby obtain asymptotically the joint distribution of estimates of location, spread and statistical dependency. Theorem 2.2 also can be utilized in constructing tests of independence and possibly as a starting point for "goodness of fit" tests.

When one is interested only in statistical dependency between two random variables, the fact that a transformation which leaves the order statistics of each random variable invariant also leaves $G(s,t)$ invariant is a strong argument for using functionals of $G(s,t)$ as measures of statistical dependency. It is not the purpose of this paper to investigate such measures of dependency and, indeed, such considerations soon lead one to rather difficult problems.

From Theorem 2.2 we note that when X and Y are independent: $b(u,v) = v$, $c(u,v) = u$, $\alpha_1(u,v,w) = \alpha_2(u,v,w) = 0$, and

$\beta(u,v) = -(1-u)(1-v)$.

Hence under independence

$$E\, S(s_1,t_1)S(s_2,t_2) = [\min(s_1,s_2) - s_1s_2][\min(t_1,t_2)-t_1t_2],$$

$$E\, S(s,t)X(u) = ES(s,t)Y(u) = 0, \tag{2.28}$$

$$E\, X(s)Y(t) = 0.$$

Also from Theorem 2.2

$$\begin{aligned} E\, S^2(s,t) = {} & G(s,t)[1-G(s,t)]+b(s,t)\{c(s,t)[G(s,t)-st] \\ & +(1-s)[b(s,t)s-2G(st)]\}+c(s,t)\{b(s,t)[G(s,t)-st] \\ & +(1-t)[c(s,t)t-2G(s,t)]\}. \end{aligned} \tag{2.29}$$

It is easily shown that $E\, S^2(s,t) \leq 5/4$ hence $\mathrm{Var}Q_n(s,t) \leq 5/4n$. A closer bound of $\mathrm{Var}Q_n(s,t) \leq 1/4n$ is found when $F_1 = F_2 = 1/2$.

As is obvious from the definition of $Q_n(s,t)$, as either s or t approaches one, $E\, S^2(s,t)$ approaches zero.

CHAPTER III

ASYMPTOTIC DISTRIBUTION OF THE USUAL ESTIMATE OF THE BIVARIATE DISTRIBUTION FUNCTION

In addition to the topic suggested by the title of this chapter, we shall compare two ways of estimating the distribution function at quantile points.

3.1 The Usual Estimate

Let (X_i', Y_i') $i=1,2,\ldots$ be independent vectors with common df $F(x,y)$ and continuous pdf $f(s,y)$. Let the marginal df of X' be F_1 with pdf f_1, the marginal df of Y' be F_2 with pdf f_2. For the first n vectors (X_i', Y_i'), define the following random functions

$$(3.1) \qquad I_u(v) = 1 \text{ if } v<u$$
$$= 0 \text{ if } v \geq u,$$

$$(3.2) \qquad F_{1n}(x) = \frac{1}{n} \sum_{i=1}^{n} I_x(X_i'),$$

$$(3.3) \qquad F_{2n}(y) = \frac{1}{n} \sum_{i=1}^{n} I_y(Y_i'),$$

$$(3.4) \qquad F_n(x,y) = \frac{1}{n} \sum_{i=1}^{n} I_x(X_i') I_y(Y_i'),$$

$$(3.5) \qquad Z_{1n}'(x) = \sqrt{n}[F_{1n}(x) - F_1(x)],$$

$$(3.6) \qquad Z_{2n}'(y) = \sqrt{n}[F_{2n}(y) - F_2(y)],$$

(3.7) $\quad Z_n'(x,y) = \sqrt{n}[F_n(x,y) - F(x,y)]$,

$F_n(x,y)$ is the classical estimate of the bivariate df at any point (x,y), $-\infty<x,\ y<\infty$.

Theorem 3.1

The vector $(Z_{1n}'(x), Z_{2n}'(y), Z_n'(x,y))$ is asymptotically $(n\to\infty)$ a vector valued Gaussian process $(Z_1'(x), Z_2'(y), Z'(x,y))$ $x,y\varepsilon(-\infty,\infty)$ with

(3.8) $\quad EZ_1'(x) = EZ_2'(y) = EZ'(x,y) = 0$

(3.9) $\quad EZ_i'(x_1)Z_i'(x_2) = \min(F_i(x_1),F_i(x_2)) - F_i(x_1)F_i(x_2)$, $i=1,2$

(3.10) $\quad EZ_1'(x)Z_2'(y) = F(x,y) - F_1(x)F_2(y)$

(3.11) $\quad EZ'(x,y)Z_1'(x') = F(\min(x,x'),y)[1-F_1(x')]$

(3.12) $\quad EZ'(x,y)Z_2'(y') = F(x,\min(y,y'))[1-F_2(y')]$

(3.13) $\quad EZ'(x,y)Z'(x',y') = F(\min(x,x'),\min(y,y')) - F(x,y)F(x',y')$

Proof.

The asymptotic normality is obvious from the multivariate central limit theorem. We need only establish that the quoted covariances are correct since (3.8) is also obvious. In the following products of sums, the cross-terms have expectation zero so for $x_2 \geq x_1$, $y_2 \geq y_1$,

$$EZ_n'(x_1,y_1)Z_n'(x_2,y_2) = \frac{1}{n}\sum_{i=1}^{n} E[I_{x_1}(X_i)I_{y_1}(Y_i)I_{x_2}(X_i)I_{y_2}(Y_i)]$$

$$- F(x_1,y_1)F(x_2,y_2)$$

$$= F(x_1,y_1) - F(x_1,y_1)F(x_2,y_2)$$

Since $F_{1n}(x) = F_n(x,\infty)$, $F_{2n}(y) = F_n(\infty,y)$, the remaining covariances can be obtained by taking appropriate limits.

If in Theorem 3.1 we define the random vectors (X_i,Y_i) $i=1,2,\ldots$ by $X_i = F_1(X_i')$, $Y_i = F_2(Y_i')$, then X and Y have uniform marginal distributions on $(0,1)$ with joint df $G(x,y) = F(F_1^{-1}(x),F_2^{-1}(y))$ and pdf $g(x,y)$. Letting for $s,t\varepsilon(0,1)$

$$Z_{1n}(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [I_s(X_i)-s],$$

$$Z_{2n}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [I_t(Y_i)-t],$$

$$Z_n(s,t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [I_s(X_i)I_t(Y_i)-G(s,t)],$$

we have the following corollary.

Corollary 3.1

As $n\to\infty(Z_{1n}(s),Z_{2n}(t),Z_n(s,t))$ converges in distribution to a vector Gaussian process $(Z_1(s),Z_2(t),Z(s,t))$, $s,t\varepsilon(0,1)$, with

$$Z_i(s) = Z_i'(F_i^{-1}(s)) \quad i=1,2,$$

$$Z(s,t) = Z'(F_1^{-1}(s),\ F_2^{-1}(t)) \ .$$

Consequently the variance of $F_n(F_1^{-1}(s),F_2^{-1}(t))$ is $G(s,t)[1-G(s,t)]$, a result that we shall use later. Clearly one can utilize Corollary 3.1 only when the marginal distribution functions are known.

3.2 Correspondence Between Siddiqui's Estimate and the Usual Estimate

Theorem 3.2 $Q_n(F_{1n}(x), F_{2n}(y)) = F_n(x,y)$ for $0<F_{1n}(x),\ F_{2n}(y)<1$ where Q_n is defined in Theorem 2.2 and F_n is defined in Equation (3.4).

Proof. $Q_n(F_{1n}(x), F_{2n}(y)) =$ the fraction of sample points (X,Y) such that $X<X_{ni}$, $Y<Y_{nj}$ where $\frac{i-1}{n} \leq F_{1n}(x) < \frac{i}{n}$, $\frac{j-1}{n} \leq F_{2n}(y) < \frac{j}{n}$.

However from Equations (3.2) and (3.3), $\frac{i-1}{n} \leq F_{1n}(x) < \frac{i}{n}$ if and only if $X_{n,i-1} \leq x < X_{ni}$ and similarly $Y_{n,j-1} \leq y < Y_{nj}$. Consequently $Q_n(F_{1n}(x), F_{2n}(y)) = q$ if and only if $F_n(x,y) = q$.

Suppose we modify our definition of the marginal empirical distribution functions so that they are continuous i.e. define $F^*_{1n}(x)=F_{1n}(x)$ for $x=\frac{i}{n}$, $i=1,2,\ldots,n$ and linearly interpolate for other values. Suppose we similarly define $F^*_{2n}(y)$, then Theorem 3.2 still holds for the starred functions i.e. $Q_n(F^*_{1n}(x), F^*_{2n}(y)) = F_n(x,y)$ for $0<F^*_{1n}(x)$, $F^*_{2n}(y)<1$. This result follows since $\frac{i-1}{n} \leq F^*_{1n}(x) < \frac{i}{n}$ if $X_{n,j-1} \leq x < X_{ni}$ and similarly for $F^*_{2n}(y)$. We now have $F^{*\,-1}_{in}$ $i=1,2$, well defined hence the following corollary.

Corollary 3.2

$$Q_n(s,t) = F_n(F^{*\,-1}_{1n}(s),\ F^{*\,-1}_{2n}(t))$$

Comparing Corollaries 3.1 and 3.2 the distinction between the two estimates is particularly vivid. If, as is usually the case, one does not know the marginal distribution functions, then by Corollary 3.2 we can still map the estimation problem to the unit square and use Theorem 2.2. Using the classical technique one would need to know the marginal distribution functions. If one knows the marginals then the two techniques of mapping the problem to the unit square will not in general yield the same results. In this case the estimate with the smallest variance apparently depends on properties of F but at least in the case of independence, Siddiqui's technique has uniformly smaller variance.

CHAPTER IV

EXTENSION OF THE WILCOXON TEST TO THE BIVARIATE CASE

Let (X_i, Y_i), $i=1,2,\ldots$ be independent vectors with common df $F(x,y)$. Let (X_i', Y_i'), $i=1,2,\ldots$ be independent vectors with common df $G(x',y')$. Let $F(x,\infty) = F_1(x)$, $F(\infty,y) = F_2(y)$, $G(x,\infty) = G_1(x)$, $G(\infty,y) = G_2(y)$. We shall take the first m vectors (X_i, Y_i), $i=1,2,\ldots,m$, the first n vectors (X_i', Y_i'), $i=1,2,\ldots,n$, and then define the following statistics

$$U_{ij} = \begin{cases} 1 & X_i < X_j' \\ 0 & X_i \geq X_j' \end{cases} \qquad V_{ij} = \begin{cases} 1 & Y_i < Y_j' \\ 0 & Y_i \geq Y_j' \end{cases}$$

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij} \qquad V = \sum_{i=1}^{m} \sum_{j=1}^{n} V_{ij} .$$

Since U and V are the univariate two-sample Wilcoxon statistics for the X's and Y's respectively, it is well known that

$$EU = mn\int F_1 dG_1, \qquad EV = mn\int F_2 dG_2 ,$$

$$\text{Var } U = mn[\int F_1 dG_1 + (n-1)\int(1-G_1)^2 dF_1 + (m-1)\int F_1^2 dG_1 - (m+n-1)(\int F_1 dG_1)^2],$$

and similarly for V. Now,

$$EUV = \sum_{ijk\ell} P(x_i < x_j', y_k < y_\ell')$$

$$= \sum_{\substack{i \neq k \\ j \neq \ell}} P(x_i < x_j', y_k < y_\ell') + \sum_{i \neq k} P(x_i < x_j', y_k < y_j') + \sum_{j \neq \ell} P(x_i < x_j', y_i < y_\ell')$$

$$+\Sigma P(x_i<x_j',y_i<y_j').$$

$$P(x_i<x_j',y_k<y_\ell') = \iint F_{ij}(x_j',y_\ell')dG_{j\ell}(x_j',y_\ell')$$

where $F_{ik}(u,v) = F(u,v)$ for $i=k$

$= F_1(u)F_2(v)$ for $i\neq k$,

$G_{j\ell}(u,v) = G(u,v)$ for $j=\ell$

$= G_1(u)G_2(v)$ for $j\neq\ell$.

There are m^2n^2 terms in EUV with $mn(mn-m-n+1)$ terms for which $i\neq k$ and $j\neq\ell$, $mn(n-1)$ terms for which $i=k$ and $j\neq\ell$, $mn(m-1)$ terms for which $i\neq k$ and $j=\ell$, and mn terms for which $i=k$ and $j=\ell$. Hence

$$\begin{aligned}Cov(U,V) = mn[&\iint F(u,v)dF(u,v)+(m-1)\iint F_1(u)F_2(v)dG(u,v)\\&+(n-1)\iint F(u,v)dG_1(u)dG_2(v)-(m+n-1)\int F_1(u)dG_1(u)\\&\int F_2(v)dG_2(v)].\end{aligned}$$

Both U and V are what Hoeffding (1951) calls U-statistics and it follows from theorem 7.1 of that paper that U and V are asymptotically $(n,m\to\infty)$ distributed as a bivariate normal with the means, variances, and covariances given above.

For $F=G$, the above reduces to

$$\begin{aligned}Cov(U,V) = mn[&\iint F(u,v)dF(u,v)+(m-1)\iint F_1(u)F_2(v)dF(u,v)\\&+(n-1)\iint F(u,v)dF_1(u)dF_2(v) - \frac{m+n-1}{4}]\end{aligned}$$

and $\mathrm{Var}\ U = \mathrm{Var}\ V = \dfrac{mn(m+n+1)}{12}$

Integration by parts reveals that

$$I(F) = \int\int F_1(u)F_2(v)dF(u,v) = \int\int F(u,v)dF_1(u)dF_2(v).$$

Hence as $n,m\to\infty$, the correlation coefficient $\rho(U,V)$ is

$$(4.1) \qquad \rho(U,V) = 12[I(F)-1/4].$$

If we consider the random variables $F_1(X)$ and $F_2(Y)$ we see that the correlation between $F_1(X)$ and $F_2(Y)$ is given by $\rho(U,V)$. Hence asymptotically the correlation between the univariate two-sample Wilcoxon statistics is simply the correlation between $F_1(X)$ and $F_2(Y)$ which, as in the preceding chapters, motivates us to consider the transformed df $F(F_1^{-1}(s),F_2^{-1}(t))$ when investigating measures of statistical dependency between two random variables.

If the distribution of (X,Y) is the standarized bivariate normal then we can compute $\rho(U,V)$ by expanding the density function using Tchebycheff-Hermite polynomials. We have

$$I(F) = \int\int F_1(x)F_2(y)dF(x,y)$$

$$= \int\int F_1(x)F_2(y) \sum_{k=0}^{\infty} \frac{H_k(x)H_k(y)}{k!} r^k dF_1(x)dF_2(y)$$

where r is the correlation coefficient and $H_k(x)$ is the Tchebycheff-Hermite polynomial. The first four Tchebycheff-Hermite polynomials are $H_0(x)=1$, $H_1(x)=x$, $H_2(x)=x^2-1$, $H_3(x)=x^3-3x$, and $H_4=x^4-6x^2+3$.

To evaluate $I(F)$ we note that

$$\int_{-\infty}^{\infty} F_1(x)H_1(x)dF_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xF_1(x)e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{-e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} F_1(x)\Big|_{-\infty}^{\infty} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{1}{2\sqrt{\pi}} .$$

Using integration by parts we similarly evaluate the other terms and find that

$$\rho(U,V) = \frac{3}{\pi} r + \frac{1}{8\pi} r^3 + O(r^5)$$

$$= .9549r + .0398r^3 + O(r^5).$$

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