THESIS

GRADUATE STUDENTS' REPRESENTATIONAL FLUENCY IN ELLIPTIC CURVES

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ABSTRACT

GRADUATE STUDENTS' REPRESENTATIONAL FLUENCY IN ELLIPTIC CURVES

Elliptic curves are an important concept in several areas of mathematics including number theory and algebraic geometry. Within these fields, three mathematical objects have each been referred to as an elliptic curve: a complex torus, a smooth projective curve of degree 3 in $\mathbb{P}^2$ with a chosen point, and a Riemann surface of genus 1 with a chosen point. In number theory and algebraic geometry, it can be beneficial to use different representations of an elliptic curve in different situations. This skill of being able to connect and translate between mathematical objects is called representational fluency. My work explores graduate students' representational fluency in elliptic curves and investigates the importance of representational fluency as a skill for graduate students. Through interviews with graduate students and experts in the field, I conclude 3 things. First, some of the connections between the above representations are made more easily by graduate students than other connections. Second, students studying number theory have higher representational fluency in elliptic curves. Third, there are numerous benefits of representational fluency for graduate students.

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TABLE OF CONTENTS

Chapter 1

Introduction

Connecting mathematical ideas by translating between different representations of these ideas is common in all levels of math. For example, in grade school mathematics classes, students are taught to connect the ideas of fractions and decimals. In undergraduate mathematics, students often connect high school concepts with what they are learning in their courses, such as rate of change and derivatives in calculus or functions and linear transformations in linear algebra [1]. It is common in graduate math courses for students to connect, and translate between, ideas from different courses and/or different areas of mathematics. This skill of creating, interpreting, connecting and translating between different representations of the same concept is called *representational fluency* [2]. Representational fluency is important for developing understanding and is used by mathematicians of all levels. In this thesis we will consider a specific example of representational fluency in graduate level mathematics.

Historically, three mathematical objects have all been referred to as "elliptic curves": complex tori, smooth projective curves of degree 3 in $\mathbb{P}^2$ with a chosen point, and Riemann surfaces of genus 1 with a chosen point. These three objects will be introduced in Chapter 2, along with proofs of their equivalence. While proving this equivalence, I will be demonstrating representational fluency, and more specifically, translating between the three representations of elliptic curves. Chapter 3 will explain in more detail the theory of representational fluency and how it will be used throughout the thesis. In Chapter 4, I will explain my methods for participant selection, data collection and analysis. This will lead into my summary of results in Chapter 5 and discussion of results in Chapter 6.

Therefore, the overall purpose of this thesis is to investigate graduate students' representational fluency in elliptic curves. More specifically, I will be investigating the following questions: How do graduate students connect and translate between different representations of elliptic curves? And what benefits does representational fluency at a graduate level have for graduate students?

Chapter 2

Math Background

Three mathematical objects have each been referred to as an elliptic curve: complex torus, smooth projective curve of degree 3 in $\mathbb{P}^2$, with chosen point, and Riemann surface of genus 1, with chosen point. This chapter will start by introducing these three objects in Section 2.1, then will go over proofs of the equivalences in Section 2.2.

2.1 Overview of Elliptic Curves

2.1.1 Complex Torus $\mathbb{C}/L$

We take $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, where $\omega_1 \neq \lambda\omega_2$ for all $\lambda \in \mathbb{R}$. We then take $L = \{\mathbb{Z}\omega_1 \bigoplus \mathbb{Z}\omega_2 = m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$. We call L a lattice. We can form the quotient group, $\mathbb{C}/L$, this is called the complex torus. We say $z_1 + L \equiv z_2 + L \iff z_2 - z_1 \in L$. We can impose the quotient topology on $\mathbb{C}/L$: a set $U \subset \mathbb{C}/L$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. We can define the fundamental parallelogram by taking the vertices to be $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$. An element z_0 being in the fundamental parallelogram tells us that for any coset $z + L$, there is another $z_0 + L \equiv z + L$. As a final note, $\mathbb{C}/L$ is an abelian group.

2.1.2 Smooth projective curves of degree 3 in projective space $\mathbb{P}^2$

We consider smooth projective curves of degree 3 in projective space $\mathbb{P}^2$. These are solutions to homogeneous equations in three variables. We dehomogenenize by setting $z = 1$ then replace x by x/z, and y by y/z. We then replace the projective space by affine space $\mathbb{A}^2_{\mathbb{C}} = \mathbb{C}^2$. This family of curves are isomorphic to curves in Weierstrass form:

$$E : y^2 = x^3 + ax + b$$

with $a, b, c \in \mathbb{C}$.

We claim that E is in fact a group. This sketch will follow [3, Section 3.2]. To define our binary operation on a pair of given points, we draw a secant line between the two points, find the third intersection point, and then reflect across $x = 0$. This works by Bezout's Theorem, which says every line (deg 1) and cubic (deg 3) intersect in 3 points in $\mathbb{P}^2$, when counted with multiplicities. This is because we can create secant lines that are tangent to the curve, giving a point with multiplicity greater than 1. The identity of the group is the point at infinity. In projective space, this is called the point $[0 : 1 : 0]$. Negatives of points are reflections across the x-axis, where the third point becomes ∞. The reflection makes the group nicer, since the sum of 3 points on the curve intersect with a line equals zero and is shown in Figure 2.1.

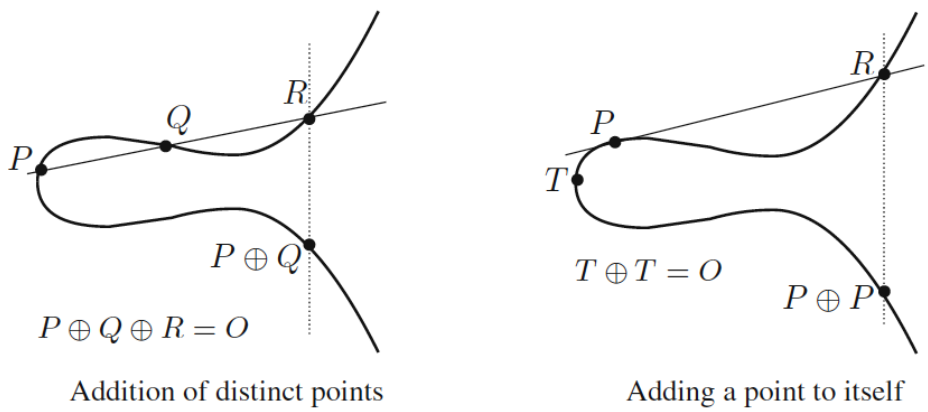

Figure 2.1: Elliptic curve addition (Image from [3, Figure 3.3])

2.1.3 Riemann Surface of genus 1 with chosen point

In this section we build the definition of a Riemann surface following [4]. The intuition here is that we want a Riemann surface to be a space which locally looks like an open set in the complex plane.

Definition 2.1.1. *A complex* ***chart*** *on X, a topological space, is a homeomorphism $\phi : U \to V$ where $U \subset X$ is an open set in X and $V \subset \mathbb{C}$ is an open set in the complex plane*

We can think of a chart on X as giving a local complex coordinate $z = \phi(x)$ for $x \in U$. We must have charts around every point of X in order for X to locally look like the complex plane everywhere. We need these charts to be compatible, which means the transition function between 2 charts is holomorphic, which leads us to the idea of complex atlas.

Definition 2.1.2. *A* ***complex atlas*** $\mathcal{A}$ *on* X*, a topological space, is a collection* $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ *of pairwise compatible charts whose domain cover* X*.*

To build our definition of a Riemann surface we need to define a few conditions for our topological space X.

Definition 2.1.3. X *is said to be* ***Hausdorff*** *if for every two distinct points x and y in* X*, there are disjoint neighborhoods U and V of x and y, respectively.*

Definition 2.1.4. *X is* ***second countable*** *if there is a countable basis for its topology.*

Definition 2.1.5. *A* ***Riemann surface*** *is a second countable connected Hausdorff topological space* X *together with a complex structure.*

We care about Riemann surfaces of genus 1, which means that our surface has one hole and is a simple torus. We also want our Riemann surface to have a chosen point.

2.2 Main Theorem

All of the objects presented above are often referred to as Elliptic curves. The reason for this is presented in the following theorem:

Theorem 2.2.1 ([4], Proposition 7.1.7, [3], [5]). *The 3 following types of objects are equivalent. Each is an abelian group.*

1. *Complex torus* $\mathbb{C}/L$*, for some lattice L*
2. *Smooth projective curve of degree 3 in* $\mathbb{P}^2$*, with chosen point*
3. *Riemann surface of genus 1, with chosen point.*

We will prove this theorem in the following subsections.

2.2.1 Proving Smooth projective curves of degree 3 in projective space $\mathbb{P}^2$ are Riemann Surfaces of genus 1 with chosen point

In this section of the thesis, we will be proving that smooth projective curves of degree 3 in $\mathbb{P}^2$ are Riemann surfaces of genus 1. The backwards direction is an application of the Riemann Roch Theorem following [6, Section 2.3] and the forwards direction uses charts to show that we have a Riemann surface.

Proof. ($\Longleftarrow$) Let X be a Riemann surface of genus 1 with a chosen point. We need to go through some definitions following [4, Chapter 5] before proceeding with this proof.

Definition 2.2.1. *A **divisor** on X is a function $D : X \to \mathbb{Z}$ such that the set of points $p \in X$ where $D(p) \neq 0$ is a discrete subset of X. Equivalently, a divisor is an element of the free abelian group generated by the points of X.*

Definition 2.2.2. *The **degree** of a divisor D on a compact Riemann surface is the sum of values of D:*

$$deg(D) = \sum_{p \in X} D(p).$$

Next, let f be a meromorphic function on X which is not zero.

Definition 2.2.3. *The **order of** f at p is*

$$ord_p(f) = \begin{cases} k & f \text{ has a zero of multiplicity } k \text{ at } p \\ -k & f \text{ has a pole of multiplicity } k \text{ at } p \end{cases}$$

Definition 2.2.4. *The **divisor of** f is the divisor defined by the order function:*

$$div(f) = \sum_p ord_p(f)p.$$

Similarly, let ω be a meromorphic 1-form on X which is not identically zero.

Definition 2.2.5. *The **canonical divisor** on X, is the divisor defined by the order function:*

$$div(\omega) = \sum_p ord_p(\omega)p.$$

Let D be a divisor on a Riemann surface X. The **space of meromorphic functions with poles bounded by** D, denoted $L(D)$, is the set

$$L(D) = \{f \text{ meromorphic } |div(f) \geq -D\}.$$

To understand this terminology, suppose that $D(p) = d > 0$. Then if $f \in L(D)$, we must have that $ord_p(f) \geq -d$, which means that f may have a pole of order at most d at p. Similarly, if $D(p) = d < 0$, then f must have a zero of order d at p. Some immediate observations are that dim$L(0) = 1$ because holomorphic functions are constant on a Riemann surface and dim$L(D) = 0$ if deg$(D) < 0$. We can now introduce a theorem that will help prove that a Riemann surface of genus 1 is a smooth projective curve of degree 3 in $\mathbb{P}^2$.

Theorem 2.2.2 (Riemann-Roch)**.** *[4, Theorem 6.3.11] Let X be a Riemann surface of genus g. Then for any divisor D and any canonical divisor K,*

$$\mathit{dim}L(D) - \mathit{dim}L(K-D) = \mathit{deg}(D) + 1 - g$$

We have a Riemann surface of genus 1, so $g = 1$. We also know that deg$(K) = 2g - 2 = 0$, therefore deg$(K - D) < 0$, if D is effective and dim$L(K - D) = 0$. Further, dim$L(D) = $ deg(D). Pick a point p in X, $L(np) = \{f$ that are allowed at most a pole of order n at $p\}$. Consider $L(np)$ for $n = 0, 1, 2, ..., 6$.

Table 2.1: Meromorphic functions for different n values

n	dim$L(np)$	functions
0	1	constants
1	1	constants
2	2	x, pole of order 2
3	3	y, pole of order 3
4	4	x^2
5	5	xy
6	6	y^2, x^3

There are 7 functions in a 6 dimensional space, therefore, there must be a linear relationship between these functions. Let us assume $ay^2+by+cxy = dx^3+ex^2+gx+h$ where $a, b, c, d, e, g, h \in \mathbb{C}$. Using a change of coordinates in the plane, we can manipulate this equation to look like $y^2 = 4x^3 + g_2x + g_3$ with $g_2, g_3 \in \mathbb{C}$. This curve is equivalent to the Weierstrass form in Section 2.1.2

($\Longrightarrow$) Let E be a smooth projective curve of degree 3 in $\mathbb{P}^2$, first, we want to put E into Weierstrass form as shown in Section 2.1.2. In order to show E is a Riemann surface we must show that for all points in E there exists a chart and that our transition functions are holomorphic. These charts are holomorphic maps from an open set in E around p to an open set in $\mathbb{C}$ and more specifically they can be chosen to be projections to the x or y axis. For all points $p \in E$, there exists a projection π_p to either the x or y axis from an open set around p, U_p along the curve such that we have the chart (U_p, π_p). Around most points both projections work as a chart, but there are a few points where only one projection works. If one of the partial derivatives of E at a point p is zero, then we can not project to the opposite axis, we must project to the axis of the variable whose partial derivative is zero. Suppose (U_α, π_α) and (U_β, π_β) are two charts such that $U_\alpha \cap U_\beta$ is nonempty. The transition function $\phi_{\alpha,\beta} : \pi_\alpha(U_\alpha \cap U_\beta) \to \pi_\beta(U_\alpha \cap U_\beta)$ is a map defined by

$$\phi_{\alpha,\beta} = \pi_\beta \circ \pi_\alpha^{-1}.$$

We know π_α^{-1} exists due to the Implicit Function Theorem. Since π_α and π_β are both holomorphic, the transition function $\phi_{\alpha,\beta}$ is also holomorphic. Therefore, we have a Riemann surface.
To show that this Riemann surface has genus one, we will use the following theorem,

Theorem 2.2.3 (Riemann-Hurwitz Formula)**.** *[7, Theorem 4.4.1] Let $f : X \to Y$ be a non-constant, degree d, holomorphic map of compact Riemann Surfaces. Denote by g_X (respectively g_Y) the genus of X (respectively Y). Then*

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} v_x,$$

where v_x is the differential length of f at x.

We know that projection on to the x axis gives a map of degree 2 to $\mathbb{P}^1$, so $d = 2$. Since Y is $\mathbb{P}^1$, $g_Y = 0$. We also know that we have 4 ramification points with differential length at each equal to 1. So we have

$$2g_X - 2 = 2(-2) + 4 \implies g_X = 1.$$

Therefore, we have shown that E is a Riemann surface of genus one.

□

2.2.2 Proving Complex Tori are Smooth projective curves of degree 3 in projective space $\mathbb{P}^2$

This section of the thesis will be sketching the proof of complex tori are equivalent to smooth projective curves of degree 3 in $\mathbb{P}^2$. The forward direction of the proof will be following [3, Section 6.3]. We will be sketching out the proof for the backwards direction following [5, Chapter 6] and [3, Section 6.1]. For the sake of clarity we have omitted some of the proofs that are technical and not as demonstrative of important ideas but refer the reader to the appropriate citations for details.

Proof. ($\Longrightarrow$) We aim to prove that complex tori are smooth projective curves of degree 3 in $\mathbb{P}^2$. Let $L = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$ be a lattice. Then the **Weierstrasss $\wp$-function** is defined on this lattice by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{2.1}$$

It has poles at the points of L and no other poles in the complex plane. Since the right hand side of (2.1) is not changed if z is replaced by $-z$ we see that $\wp(z) = \wp(-z)$ because of the squared terms in the function and the fact that we are summing over all of the points of the lattice and so the function is even. We also can observe that $\wp(z)$ is uniformly convergent, so we can compute its derivative,

$$\wp'(z) = -2 \sum_{\omega \in L\setminus\{0\}} \frac{1}{(z-\omega)^3}. \tag{2.2}$$

It is clear that $\wp'(z)$ is doubly periodic, meaning $\wp'(z+\omega) = \wp'(z)$ for all $\omega \in L$. Integrating this with respect to z gives us that $\wp(z+\omega) = \wp(z) + c(\omega)$ for all $z \in \mathbb{C}$. Let $z = -\frac{1}{2}\omega$, then $\wp(\frac{1}{2}\omega) = \wp(-\frac{1}{2}\omega) + c(\omega)$. Since $\wp(z)$ is even we have that $c(\omega) = 0$ and therefore $\wp$ is also doubly periodic.

Next, we need to define the Eisenstein series of weight 2k in order to state a theorem needed for this proof. The **Eisenstein series of weight 2k** is the series

$$G_{2k}(L) = \sum_{\omega \in L\setminus\{0\}} \omega^{-2k}.$$

It is standard to let $g_2 = 60G_4(L)$ and $g_3 = 140G_4(L)$.

Theorem 2.2.4 ([3],Theorem 5.3.5). *(a) The Laurent series for $\wp(z)$ around $z = 0$ is given by*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

(b) *The Weierstrass $\wp$ function satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

for all $z \in \mathbb{C} \setminus L$.

Proof of Theorem 2.2.4. (a) for z with $|z| < |\omega|$ we have

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1-\frac{z}{\omega})^2} - 1\right) = \sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^{n+2}} = \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}$$

(b) To see this we write out the first few terms of the Laurent expansions around 0:

$$\wp'(z)^2 = 4z^{-6} - 24G_4z^{-2} - 80G_6 + \cdots$$

$$\wp(z)^3 = z^{-6} + 9G_4z^{-2} + 15G_6 + \cdots$$

$$\wp(z) = z^{-2} + 3G_4z^2 + \cdots$$

Comparing these expansions, we see that the function $f(z) = \wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6$ is holomorphic at $z = 0$ and satisfies $f(0) = 0$. We then have the following facts: (1) (2.1) converges absolutely and uniformly on every compact subset of $\mathbb{C}/L$. The series defines a meromorphic function on $\mathbb{C}$ having a double pole at each lattice point and no other poles [3, Theorem 3.1]; (2) a holomorphic doubly periodic function is constant [3, Proposition 2.1]. From these two facts and that $f(z)$ is holomorphic, we know that $f(z)$ is constant and therefore $f(z) = 0$. Therefore, $\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$.

□

Returning to proving that complex tori are smooth projective curves of degree 3 in $\mathbb{P}^2$, we now want to show that the polynomial $f(x) = 4x^3 - g_2x - g_3$ has distinct roots, which is equivalent to the Weierstrass form introduced in Section 2.1.2. Let $\{\omega_1, \omega_2\}$ be a basis for L and let $\omega_3 = \omega_1 + \omega_2$. Since $\wp'(z)$ is an odd and doubly periodic function, we know that $\wp'(\frac{\omega_i}{2}) = -\wp'(\frac{-\omega_i}{2}) = -\wp'(\frac{\omega_i}{2})$, therefore $\wp'(\frac{\omega_i}{2}) = 0$ for $i \in \{1, 2, 3\}$. Then, $f(x) = 0$ at each of the values $x = \wp(\frac{\omega_i}{2})$. We want to show that these three values are distinct. Consider the function $\wp(z) - \wp(\frac{\omega_i}{2})$. This function is even, so it has a double zero at $z = \frac{\omega_i}{2}$. Since it is a double periodic function with only a pole of order 2 at the lattice points, it only has these zeros in the fundamental parallelogram. So, $\wp(\frac{\omega_i}{2}) \neq \wp(\frac{\omega_j}{2})$ for $j \neq i$. Finally, we can introduce the last proposition needed for this direction of the proof.

Proposition 2.2.1. *[3, Proposition 3.6] Let E be the curve*

$$E : y^2 = 4x^2 - g_2x - g_3,$$

which is an elliptic curve. Then the map

$$\begin{aligned} \phi : \mathbb{C}/L &\to E(\mathbb{C}) \subset \mathbb{P}^2 \\ z &\mapsto [\wp(z) : \wp'(z) : 1] \end{aligned}$$

is an isomorphism of Riemann surfaces.

Proof of 2.2.1. Let $(x, y) \in E(\mathbb{C})$. Then $\wp(z) - x$ is a nonconstant meromorphic doubly periodic function, so it has a zero, say $z = a$. Then, $\wp'(a)^2 = y^2$, so replacing a by $-a$ if necessary, we obtain $\wp'(a) = y$. Then $\phi(a) = (z, y)$ and we see that the map ϕ is surjective.

Suppose $\phi(z_1) = \phi(z_2)$. Assume that $2z_1 \notin L$. Then the function $\wp(z) - \wp(z_1)$ is an elliptic function of order 2 that vanishes at $z_1, -z_1$ and z_2. Two of these values must be congruent modulo L. Therefore $z_2 \equiv \pm z_1 \mod L$ for some choice of sign. Then $\wp'(z_1) = \wp'(z_2) = \wp'(\pm z_1) = \pm\wp'(z_1)$ implies that $z_2 \equiv z_1 \mod L$. If $2z_1 \in L$, then $\wp(z) - \wp(z_1)$ would have a double zero at z_1 and a zero at z_2, concluding again that $z_2 \equiv z_1 \mod L$. Therefore, ϕ is injective. □

We have shown that $\phi : \mathbb{C}/L \to E(\mathbb{C}) \subset \mathbb{P}^2$ is an isomorphism of Riemann surfaces, finishing our proof that complex tori are smooth projective curves of degree 3 in $\mathbb{P}^2$.

($\Longleftarrow$) In this proof we aim to define a map from E to $\mathbb{C}/L$. In order to do we start with supposing that we have constructed a lattice L that corresponds to E. Then we define a function that we want to show is essentially $\wp^{-1}$, which ends up locally being an integral containing a square root. In order to make this integral well-defined, we take two branch cuts and glue two copies of $\mathbb{P}^1$ together to form a Riemann surface which is essentially E. After compactifying this Riemann surface, we use facts from analytic continuation to justify that our integral is well defined. Integrating along different paths on this Riemann surface end up being equal modulo some lattice L, which gives us our map. The following shows a sketch of the details to accomplish this.

Any elliptic curve over $\mathbb{C}$ can be expressed in the form $E : y^2 = 4x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{C}$. There is a map

$$E \to \mathbb{P}^1$$
$$(x, y) \mapsto x$$

which is a double cover ramified at $0, 1, \lambda$ and ∞.

We are essentially trying to find an inverse to the Weierstrass $\wp$ function. Once we construct the inverse, we will also have our parameters ω_1 and ω_2 for our lattice. Let $w = \wp(z)$, then by Theorem 2.2.4, we know w satisfies the following:

$$\begin{aligned} \left(\frac{dw}{dz}\right)^2 &= 4w(w-1)(w-\lambda) \\ \frac{dw}{2\sqrt{w(w-1)(w-\lambda)}} &= dz \\ z(w) = \int_0^w \frac{dw}{2\sqrt{w(w-1)(w-\lambda)}}&. \end{aligned} \tag{2.3}$$

Since the square root is not single valued, this line integral is not path independent. To solve this problem, it is necessary to make branch cuts. We take two copies of $\mathbb{P}^1$ and make branch cuts. When we glue the two copies together along the branch cuts, we form a Riemann surface. To show that this is a Riemann surface, we will define the charts.

Let our double cover of $\mathbb{C} \setminus \{0, 1, \lambda\}$ be U, and compactify it by adding the preimages of $0, 1, \lambda$ and ∞ which we denote by α, β, γ and ∞, respectively. We denote this compactified space

$\mathcal{R}$. We now want to show that U and $\mathcal{R}$ are Riemann surfaces. Let $e : U \to \mathbb{C} \setminus \{0, 1, \lambda\}$ be the covering map and fix $\zeta_0 \in U$ with $e(\zeta_0) = z_0$. Consider an open disk $D \subset \mathbb{C} \setminus \{0, 1, \lambda\}$, because U is a double cover, $e^{-1}(D)$ is a disjoint union of two open sets $\Delta_1 \cup \Delta_2$. We have (Δ_j, e) for $j \in \{1, 2\}$ as charts. Therefore U is a Riemann surface. For $\mathcal{R}$ we extend our map e so that $e : \mathcal{R} \to \mathbb{C} \cup \{\infty\}$. We now just need to define charts about α, β, γ and ∞. Let $D_0^\times$ be a punctured disc centered at 0. An open disc of α is then given by $\{\alpha\} \cup e^{-1}(D_0^\times)$ and we define a chart $(\{\alpha\} \cup e^{-1}(D_0^\times), \sqrt{\zeta - \alpha})$. Similarly, we can define the following compatible charts: $(\{\beta\} \cup e^{-1}(D_1^\times), \sqrt{\zeta - \beta})$, $(\{\gamma\} \cup e^{-1}(D_\lambda^\times), \sqrt{\zeta - \gamma})$ and $(\{\infty\} \cup e^{-1}(D_\infty^\times), \frac{1}{\sqrt{\zeta}})$ where $D_\infty^\times$ is of the form $z > R$ for some $R \geq \max\{|0|, |1|, |\lambda|\}$. Therefore $\mathcal{R}$ is a compact Riemann surface.

We now want to show that $\sqrt{z(z-1)(z-\lambda)}$ can be analytically continued to be a well defined function on U.

Definition 2.2.6 ([5], 6.6). *Let f be a function element at a, and let $C = \{C(t) | t \in [\alpha, \beta]\} \in \mathbb{C}$ be a path from a to b. A function element g at b is an* ***analytic continuation*** *of f along C if there exists a partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

and function elements f_j on open disks D_j centered at $C(t_j)$ for $0 \leq n \leq n$, such that

1. $f_0 = f$
2. $f_n = g$
3. $C([t_{j-1}, t_j]) \subset D_{j-1}$
4. *f_j is a direct analytic continuation of f_{j-1}*

We see that $\sqrt{z(z-1)(z-\lambda)}$ is an analytic function on D_0 with $\sqrt{z_0(z_0-1)(z_0-\lambda)} = q_0$. We have the following fact: there exists a unique analytic function $F : U \to \mathbb{C}$ such that $F(\zeta_0) = q_0$ and the following holds: for any $\zeta_1 \in U$, any path $C \subset U$ from ζ_0 to ζ_1, and any open disk $D \subset \mathbb{C} \setminus \{0, 1, \lambda\}$ containing $z_1 = e(\zeta_1)$, if e^{-1} is a map from D to the component Δ of $e^{-1}(D)$

that contains ζ_1, then $(F \circ e^{-1}, D)$ is an analytic continuation of $(\sqrt{z(z-1)(z-\lambda)}, D_0)$ along $e(C)$ [5], Proposition 6.24. We write

$$F(\zeta) = \sqrt{(\zeta-\alpha)(\zeta-\beta)(\zeta-\gamma)}$$

for the function produced by this proposition where α, β and γ are the same as what was introduced earlier. The reciprocal of this is given by

$$F(\zeta)^{-1} = \frac{1}{\sqrt{(\zeta-\alpha)(\zeta-\beta)(\zeta-\gamma)}}$$

where $F(\zeta_0)^{-1} = \frac{1}{q_0}$, $F(\zeta)^{-1}$ is analytic in U. By Riemann's removable singularity theorem, $F(\zeta)^{-1}$ is a meromorphic function on $\mathcal{R}$. We now look at its behavior near each of α, β, γ and ∞. The local coordinate near α is $s = \sqrt{\zeta - \alpha}$. We have $s^2 = z - \alpha$, which gives us $(z-1)(z-\lambda) = (s^2 + -1)(s^2 - \lambda)$. Therefore

$$F(\zeta)^{-1} = s^{-1} \frac{1}{\sqrt{(s^2-1)(s^2-\lambda)}}.$$

$F(\zeta)^{-1}$ has simple poles at α, β and γ.

The local coordinate near ∞ is $s = \frac{1}{\sqrt{\zeta}}$. We have $z = s^{-2}$, which means we have

$$F(\zeta)^{-1} = s^3 \frac{1}{\sqrt{(1-s^2)(1-\lambda s^2)}}.$$

We can write the integral of (2.3) as

$$\widehat{w}(C) = \int_C \frac{1}{2} F(\zeta)^{-1} d\zeta. \tag{2.4}$$

For $z \in \mathbb{C} \setminus \{0, 1, \lambda\}$ consider a loop $C \subset \mathbb{C} \setminus \{0, 1, \lambda\}$ based at z_0. All piecewise smooth loops close to C will have the same total winding number. We are interested in only looking at loops with even total winding number because every loop brings us across the branch cut so loops with even winding number will end on the same part of the double cover. We will call the subgroup of even total winding number H and it is generated by the equivalences classes of $\Gamma_0^2, \Gamma_1^2, \Gamma_\lambda^2, \Gamma_0\Gamma_1,$ and $\Gamma_1\Gamma_\lambda$ where Γ_i is a piecewise smooth loop based at z_0 with winding number equal to 1 about i. We can lift these loops to the respective loops $\Gamma_\alpha, \Gamma_\beta, \Gamma_\gamma, \overline{\Gamma_1}$ and $\overline{\Gamma_2}$ in $\mathcal{R}$ with base point ζ_0. Due to $\Gamma_\alpha, \Gamma_\beta$ and Γ_γ being loops in small disks around α, β and γ respectively, integrating 2.4 along these paths, the integral depends only on the endpoints and is therefore equal to zero [5]. The loops $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ are two closed paths for which the integral 2.4 is nonzero. Define

$$\omega_1 = \int_{\overline{\Gamma_1}} \frac{1}{2} F(\zeta)^{-1} d\zeta \text{ and } \omega_2 = \int_{\overline{\Gamma_2}} \frac{1}{2} F(\zeta)^{-1} d\zeta. \tag{2.5}$$

We let $L \subset \mathbb{C}$ be given by

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

Now, let $w : \mathcal{R} \to \mathbb{C}/L$ be defined by $w(\zeta) = \widehat{w}(C) \mod L$ for any piecewise smooth path $C \subset \mathcal{R}$ from ζ_0 to ζ_1, where $\widehat{w}(C)$ is from (2.4). To see that this map is well defined, let C and C' be two piecewise smooth paths from ζ_0 to ζ_1. We want to show that $w(C') - w(C) \in L$. Without loss of generality, we assume the following: if $\zeta \notin \{\alpha, \beta, \gamma, \infty\}$, then neither C nor C' meets any point in $\{\alpha, \beta, \gamma, \infty\}$, whereas if $\zeta \in \{\alpha, \beta, \gamma, \infty\}$, then C and C' meet $\{\alpha, \beta, \gamma, \infty\}$ only at their final points. We have $w(C') - w(C) = w(C'C^{-1})$, where $C'C^{-1}$ is a loop with base point ζ_0. We know the following fact: if C and C' are homotopic in $\mathcal{R}$, then $w(C) = w(C')$ [5, Lemma 6.26]. Therefore, we can homotopically perturb $C'C^{-1}$ near ζ, so that $C'C^{-1}$ does not meet any point in $\{\alpha, \beta, \gamma, \infty\}$. We know by our fact above that $w(C'C^{-1})$ depends only on equivalence classes of $C'C^{-1}$ in $\mathcal{R}$. The equivalence classes of loops in $\mathcal{R}$ are generated by $\Gamma_\alpha, \Gamma_\beta, \Gamma_\gamma, \overline{\Gamma_1}$ and $\overline{\Gamma_2}$. But, we have shown that $w(\Gamma_\alpha) = w(\Gamma_\beta) = w(\Gamma_\gamma) = 0$. Therefore, $w(C'C^{-1})$ is an integer combination of $w(\overline{\Gamma_1})$ and $w(\overline{\Gamma_2})$. This shows that $w(C') - w(C) \in L$, proving w is well defined. This map is also continuous and open, for details of the proof of this fact, see [5, Proposition 6.28]. Therefore, ω_1 and ω_2 defined in (2.5) generate a lattice that gives rise to an elliptic curve over $\mathbb{C}$.

$\square$

2.2.3 Proving Complex Tori are Riemann Surfaces of genus 1 with chosen point

In this section we prove that complex tori are Riemann surfaces of genus one with a chosen point. The forward direction of the this proof is following [4, Section 1.2] and the backwards direction is a sketch following [4, Section 7.1].

sketch of proof. ($\Longrightarrow$) Let E be a complex torus, we want to show that E is a Riemann Surface of genus one. We know that every open set in $E = \mathbb{C}/L$ is the image of an open set in $\mathbb{C}$ because our projection map $\pi : \mathbb{C} \to \mathbb{C}/L$ is a covering space. Since L is a subset of $\mathbb{C}$ there exists an $\epsilon > 0$ such that $2\epsilon < |\omega|$ for $\omega \in L \setminus \{0\}$. For any $z \in \mathbb{C}$, we can define the open disk $D_z = D(z, \epsilon)$ around z with radius ϵ. Our choice of ϵ tells us that no two points in D_z can differ by an element of the lattice. The restriction of π to D_z maps D_z homeomorphically onto the open set $\pi(D_z)$. For each $z \in \mathbb{C}$, define $\phi_z : \pi(D_z) \to D_z$ to be the inverse of the map $\pi|_{D_z}$. These are complex charts on E. We now want to check that the transition functions are holomorphic. Choose two points α and β with charts ϕ_α and ϕ_β. Let $U = \pi(D_\alpha) \cap \pi(D_\beta)$. If U is not empty let $T(z_0) = \phi_\beta(\phi_\alpha^{-1}(z_0)) = \phi_\beta(\pi(z_0))$ for $z_0 \in \phi_\alpha(U)$. $T(z_0) = z_0 + \omega$ for some $\omega \in L$, which is holomorphic. This is a translation by some element in the lattice and is therefore a biholomorphic function. Thus E is a Riemann surface.

Next, we can look at the fundamental parallelogram P, as defined in Section 2.1.1. The opposite edges are identified together, giving us the Riemann surface with genus one.

($\Longleftarrow$) This full proof requires the Abel-Jacobi map and is given in [4, Chapter 8], we will provide a sketch. Let X be a Riemann surface of genus 1. Our goal is to show that the universal cover of X called Y is isomorphic to $\mathbb{C}$. From Riemann-Roch, we know that for the canonical divisor K, deg$(K) = 2g - 2$. For our genus 1 curve, we let ω_0 be a meromorphic 1-form and $K_0 = div(\omega_0)$ be a canonical divisor on X. We know that deg$(K_0) = 0$ and from Riemann-Roch $h^0(K_0) - h^1(K_0) = 0$, then by Serre Duality, $h^0(K_0) - h^0(K_0 - K_0) = 0$. We know that $h^0(\emptyset) = 1$

because only the constant functions are holomorphic, therefore $h^0(K_0) = 1$. Let

$$L(D) = \{f \text{ meromorphic } | div(f) + D \geq 0\}$$

and let ω_0 be a 1-form with divisor K_0. If $f \in L(D)$ then $div(f\omega_0) = div(f) + K_0 \geq 0$. The divisor $div(f\omega_0)$ has degree 0, therefore all coefficients of $div(f\omega_0)$ equal zero so $f\omega_0$ has no zeros or poles. We use ω to define $\phi : Y \to \mathbb{C}$ and let π be the covering map. Consider the pullback $\pi^*\omega$, this is a holomorphic 1-form on Y with no zeros. Fix $p_0 \in Y$ and for $p \in Y$, choose a path γ from p_0 to p, then

$$\phi(p) = \int_\gamma \pi^*\omega.$$

This integral does not depend on the path because Y is simply connected, meaning every path is homotopic to each other and $\pi^*\omega$ is holomorphic, so the integral is well defined. Therefore, ϕ is holomorphic and is the isomorphism between Y and $\mathbb{C}$.

$\square$

Chapter 3

Theoretical Framing

The goal of this research is to understand how experts and graduate students understand elliptic curves. We saw in the math background section that there are three mathematical objects equivalent to each other that are all referred to as elliptic curves. In this study I am particularly interested in the six connections between these three mathematical objects, and to what degree students and faculty understand these connections. In other words, I am interested in exploring faculty and students' *representational fluency* surrounding elliptic curves. In this section I define representational fluency and how it relates to this study.

Fonger defines representational fluency as "the ability to create, interpret, translate between, and connect multiple representations" of mathematical objects and defines a representation as "a symbolized from that stands for somethings from a person's point of view" [2]. Since there are three mathematical objects that are all equivalent to elliptic curves, each object can be viewed as a representation of elliptic curves from the point of view of different areas of mathematics. A complex torus $\mathbb{C}/L$, for some lattice L, is a representation from the point of view of the complex analysis. From the algebraic/arithmetic geometry point of view, an elliptic curve is represented by a smooth projective curve E of degree 3 in $\mathbb{P}^2$, with chosen point. Finally, a Riemann surface of genus 1, with a chosen point is a representation from a complex algebraic geometry point of view. I will be looking at how graduate students can create, interpret, translate between and connect these three representations of elliptic curves. The theory of representational fluency provides the language needed to articulate and investigate my research questions:

1. How do graduate students connect and translate between different representations of elliptic curves?

2. What benefits does representational fluency at a graduate level have for graduate students?

A *translation* is describes as a student's movement from one representation to another in which the meaning of the mathematical object is interpreted with respect to the target representation [8]. The process of translating includes creating the target representation and interpreting properties of the original representation in the target representation [2]. In terms of elliptic curves, I am interpreting translating as a student proving that their original representation is equivalent to their target representation. This means that I will be looking for 6 translations between the 3 representations of elliptic curves. In order for a student to make a *connection* between representations, "a student must articulate an invariant feature of the mathematical object being represented across representational forms" [2]. For example, a student can represent a line with the formula $y = mx + b$ or a graph, and in order to connect the representations they might identify the slope of both to be equal to m, which would be the invariant feature. For elliptic curves, examples of invariant features are that all representations are complex manifolds and are abelian groups.

Representational fluency is prevalent in literature in a variety of ways. It most often is used around the Rule of Four, which outlines students' translations between four representations: verbal, graphic, numeric and symbolic [9]. Nathan et al. used representational fluency and the Rule of Four to investigate middle school students' ability to solve problems using tabular, graphical, verbal and symbolic representations and to translate between these [10]. They found that students succeeded more with graphical representations and that instructions about intuitions from instructors can enhance students' abilities to translate. Adu-Gyamfi, Stiff and Bossé investigated translation errors made by undergraduate students in college algebra translating between table, graph and equation representations [11]. This shows that even at the college level students make errors when translating. Finally, [1] demonstrates representational fluency being used not with the Rule of Four. They examined linear algebra students' representational fluency of function as a high school function and function as a linear transformation. They found that higher representational fluency relied on metaphorical thinking.

The National Council of Teachers of Mathematics (NCTM) outlines that it is important for students at all levels to connect mathematical ideas and be able to translate among different mathematical representations [12]. The mathematics community agrees on the importance of a student's ability to translate for mathematical understanding and problem solving skills [13]. There is research linked to representational fluency at levels up through undergraduate math [11], but there is limited research on representational fluency and its importance at the graduate level of mathematics. I aim to fill this gap by investigating graduate students' representational fluency around elliptic curves. Also, I will explore its importance by asking professors and graduate students about their opinions about why it matters.

Chapter 4

Methodology

4.1 Participants

This study occurred at an university's department of mathematics[1]. Four professors and ten graduate students agreed to participate. The professors were chosen based on their area of research. All four professors study algebraic geometry and/or number theory. There was one professor who studies more number theory, 3 who study more algebraic geometry and one who studies an area between the two, called arithmetic geometry. Graduate students were chosen based on if they took the elliptic curves course offered at the university or if they studied number theory or algebraic geometry. There was one master's student and nine PhD students. Graduate students ranged from second year to sixth year in the program. Participants' general area of research and whether or not

Table 4.1: Overview of graduate student participants

Pseudonym	Area of Research	Completed Elliptic Curves course
Alex	Number Theory	Yes
Taylor	Other	Yes
Cameron	Other	Yes
Logan	Number Theory	Yes
Ryan	Number Theory	Yes
Sam	Algebraic Geometry	No
Quinn	Algebraic Geometry	No
Drew	Number Theory	Yes
Jordan	Other	Yes
Dakota	Algebraic Geometry	Yes

[1]This study was determined to be not human reseearch by the IRB committee at Colorado State University. I still had informed consent, took steps to protect identities, and took steps for quality research. I followed the 8 markers of quality resesearch: (a) worthy topic, (b) rich rigor, (c) sincerity, (d) credibility, (e) resonance, (f) significant contribution, (g) ethics, and (h) meaningful coherence, which are outlined in [14]. For example, I used methods and procedures that fit my research goals and questions, and I meaningfully connected literature, research questions, results and discussion.

they took the elliptic curves course is displayed in (4.1). An individual's specific area of research, year in school, and what program they were in were left out to preserve anonymity. All participants will be referred to using they/them/theirs pronouns.

4.2 Data collection

This study took place in two rounds. During the first round, all professors participated in a semistructured interview either in person or over zoom. The goal of these interviews was to understand how experts in the field understand the connections and translations between these mathematical objects representing elliptic curves. The goal here was both to understand the experts' representational fluency and to identify what translating between the representations looked like in practice, since it was not reasonable to expect full proofs as I wrote out in Chapter 2 for the translations. Therefore, I did these interviews first so that I could go into interviews with graduate students knowing what was reasonable. These interviews with professors lasted between 14 and 33 minutes. The next round of the study included semistructured interviews with all graduate student participants [15]. These interviews occurred in person or over zoom a year after the elliptic curves course had occurred. The goal of these interviews was to investigate graduate students' representational fluency around elliptic curves by seeing what connections and translations the graduate students understood. These interviews lasted between 4 and 31 minutes. All interviewees were asked the following questions:

1. What is your familiarity/experience with Elliptic curves?
2. How do you think about a Riemann surface of genus 1 with a chosen point?
3. How do you think about a complex torus?
4. How do you think about a smooth projective curve E of degree 3 in $\mathbb{P}^2$, with chosen point?
5. How do you think about these objects as being connected, if at all?

6. What do you think graduate students gain by understanding/learning these connections? OR What do you think graduate students gain by being able to connect different areas of mathematics?

Questions 1 through 4 aimed to see how participants create these representations. Question 5 aimed to see how participants connected the representations then with some prompting got participants to show their understanding of translations. The last questions get at participant's' perspectives on the value of representational fluency, both in mathematics in general, and specific to elliptic curves. Examples of follow up questions included:

- What do you mean by that?
- Can you explain that in more detail?

I started the process with a trial pilot interview with a professor and a pilot interview with a graduate student, but the interviews were left in because no questions were changed after them.

4.3 Data Analysis

There were three stages of data analysis in this study: analysis of professor interviews, analysis of graduate student interviews and a cross-case analysis. All interviews were video recorded and transcribed for this process. I used thematic analysis techniques, which is the process to identify and organize data into recurring patterns and themes [16], for all stages. Starting with the interviews with professors, I read through the transcripts and summarized their responses to find common themes among their responses using open coding. I used these common themes to build my coding rubric for the graduate student interviews. My rubric was built on what experts said for each translation. In analyzing the graduate student interviews, I read through each transcript to again summarize the interviews. From there, I was able to apply my rubric to see which translations and connections each graduate student was able to complete. After this, I identified themes within my data specifically looking at which translations were made and by whom. This process resulted in a few themes I highlight in Chapter 6.

4.4 Positionality Statement

I am a graduate student in the mathematics department at the university where the study occurred. I had taken the elliptic curves course that was offered and became interested in the connections that were in introduced in the course. Doing the project allowed me to increase my understanding on elliptic curves. Due to being in the department, I had relationships with all participants prior to the interviews. I had all of the faculty members as professors prior to the interviews and had been in classes or seminars with every graduate student participant. This resulted in me having a collegial relationship with each participant, so interviewees felt comfortable sharing what they knew as well as what they did not know.

Chapter 5

Results

In this chapter, I will be summarizing interviews with all participants. The first section summarizes interviews with professors who are experts in the field, then I summarize interviews with graduate students. The final section of this chapter summarizes my results with professors by creating a coding rubric for what I was looking for in my interviews with graduate students and summarizes results from the graduate student interviews into a figure.

5.1 Professor Interviews

Professor 1 first thinks of an elliptic curve as "a donut. So complex torus with one hole, genus, genus one." They started out by explaining that from $\mathbb{C}/L$ one can identify opposite edges and wrap up to get the shape of the Riemann surface of genus one. Then, going from the complex torus to the projective cubic, Professor 1 described using L invarient functions to find that we need a function of order two at every lattice point. This then gives the Weierstrass $\wp$-function. For the other direction, Professor 1 explained that you use elliptic integrals to figure out what lattice we need. From there, they described using the Weierstrass $\wp$-function and it's derivative to get the equation for the cubic. They then explained that proving that a Riemann surface of genus one is equivalent to $\mathbb{C}/L$ involves showing that the universal cover of the Riemann surface is $\mathbb{C}$, and then we have to mod out by a lattice because of the loops of the fundamental group. Professor 1 explains that you can use Riemann Roch Theorem to prove that a Riemann surface of genus one is a smooth projective cubic. For the other direction, they explained that there is a theorem that says that algebraic equations describe abstract things like the Riemann surface of genus one. When asked about why it is important for graduate students to be able to make these connections and translations, professor 1 explained that in graduate school students learn a lot of big topics and it is important to see that they are all connected so that a student doesn't get stuck thinking that what they do is the best.

Professor 2 has less experience with elliptic curves. They state that they do not work directly with elliptic curves but see them show up frequently in talks. They explained that they do not distinguish between a Riemann surface of genus one with a fixed point and a complex torus, although they were able to explain their reasoning as translations between the two. They explained that we can view the Riemann surface as $\mathbb{C}$ and then we are able to mod out by a lattice to get the complex torus. From a torus to the Riemann surface, one can identify opposite edges with each other to get the donut shape. For the translation from the Riemann surface to a cubic in projective space, Professor 2 explained that there is a formula that tells you for what functions the degree of the canonical bundle is zero. This formula ends up telling us that we need a degree 3 curve in projective space. Professor 2 finished up by explaining the importance of this representational fluency by explaining that when a mathematician understands these connections and can translate between representations, they are able to listen to talks on something and translate it to the area of mathematics they are familiar with. Therefore, this skill gives mathematicians the dictionary to be able to understand more talks. Professor 2 explained "I go to a lot of talks where someone talks about elliptic curves and they write down like y^2 equals something. And that's a language that I'm less comfortable with. And I, in my mind, I like go, okay, they're just talking about a lattice or a donut with a complex structure." Also, answering the question more generally, they said given a problem it is good to translate it into different ways of thinking because one may be easier to prove.

Professor 3 explained that when they teach a course on elliptic curves they start with the complex tori representation. They explain that from there the class discovers the Weierstrass $\wp$-function and can see that this function and its derivative satisfy a cubic equation, getting us into the projective cubic representation. For the translation from complex torus to Riemann surface, Professor 3 explained that the complex torus is a Riemann surface by definition and we can see that because the charts on the Riemann surface are inherited from the charts of the complex plane. When translating from a smooth projective cubic in projective space to a Riemann surface, they explained that this uses the Implicit Function Theorem to find charts which are the projections onto the coordinate axes, this gives us a Riemann surface. Professor 3 says that it is much more difficult to translate the other direction and continues to explain that this translation involves finding functions that are meromorphic and satisfy the Cauchy Riemann equations. When asked about the importance of representational fluency of elliptic curves for graduate students, Professor 3 enthusiastically said that "it really does illustrate one of sort of, what are the unifying principles of mathematics" and further explained that every area of math has an elliptic curve problem at the foundation of it.

When asked to describe elliptic curves, Professor 4 said that there are two different ways that they think of them, as either cubics in a plane or as complex tori. As they continued talking they also mentioned that one can represent an elliptic curve with a Riemann of genus one with a point. When describing the translation between the representation as complex torus to the representation as a projective cubic, Professor 4 described a theorem that says that any meromorphic periodic function can be written as a combination of the Weierstrass $\wp$-function and its derivative. Then, we can use these functions to a smooth projective cubic. In order to translate the other direction, Professor 4 explains that we want to produce $\mathbb{C}$ and the lattice from our cubic. To produce $\mathbb{C}$, one can think of an elliptic curve as a Riemann surface and then can show that $\mathbb{C}$ is its universal cover. From there, they explained that to find the lattice you use elliptic integrals. Translating from a complex torus to a Riemann surface of genus one, Professor 4 explained that when they cover the complex torus with analytic charts and check that the transition functions are holomorphic, which shows that the complex torus is a Riemann surface. For translating the other direction, we know that $\mathbb{C}$ is the universal cover of the Riemann surface, then when we integrate along loops from a point, as the loops differ we see that the integrals are equal modulo periods, which gives us our lattice. Switching over to the translations between projective cubics and Riemann surfaces, Professor 4 explained that from a projective cubic, we can write y as a function of x or x as a function of y due to the inverse function theorem. From this fact we can use x or y as our charts and then we see that we have a Riemann surface. Professor 4 used ideas that we had already discussed to show that a Riemann surface is a complex torus which is a projective cubic, showing the last translation. When asked about the importance for graduate students to understand these connections and translations, Professor 4 explained that it is easy to make mistakes when you don't understand the symbols from other areas of mathematics.

5.2 Graduate Student Interviews

Alex is a graduate student studying number theory, who has taken the elliptic curves course. They first think of an elliptic curve as the complex plane modulo a lattice, then as a smooth projective curve and finally as a Riemann surface. Alex thinks about a Riemann surface as being a complex torus by cutting the Riemann surface along two circles to get the lattice. They think of going the other way by gluing the lattice together to first get a tube then glue the ends together and get the donut shape of the Riemann surface. Alex thinks of translating from complex torus to the projective curve by using the Weierstrass $\wp$-function and the other direction by using the Weierstrass $\wp$-function and Uniformization Theorem. They were not sure on how to prove that the projective curve is a Riemann surface but guessed that you have to projectivize and dehomogenize the other direction. Alex believes that it is important to be able to connect different areas of math because your results might be related to other things that give you a bigger perspective. Having a broader idea can help attack your problem from a different angle. Although, they stated "along this specific strain of ideas, unless you actually have to write down the details, I don't think very many people think about the details very often, because it's kind of convoluted and doesn't necessarily help you intuitively understand the story."

Taylor is a graduate student who took the elliptic curves course but is not studying something related to number theory or algebraic geometry. When first asked what an elliptic curve is, Taylor knew that there were different ways to think of elliptic curves and started out by drawing the cubic in complex space. Taylor described the projective curves and complex tori as being equivalent by looking at tessellations of the complex plane. When asked what they meant by a tessellation they explained "the complex tori is defined so we take those two complex numbers, I'll call them omega one, omega two, and then do the gluing. Like this to you know, the tori is defined by this looping. And so these two complex numbers can be used to define the particular tori. And then this just leads to a tessellation. So there was two of these three, passing that to the board that continues on to to omega two and then this would be minus omega two from here and down like that. So, creating this tessellation." I interpreted this as Taylor describing the lattice for the complex torus. Taylor then described drawing lines on a Riemann surface of genus one to see that it is equivalent to the cubic in projective space. Those were all the connections that Taylor attempted to make. Taylor believes that being able to connect different areas of math is important because a student will often have one area that they are more confident in that becomes their "crutch", in this example the complex tori was Taylor's crutch. From the area that the student is more confident in, a student can learn the translations and move to some other area to learn the tools and theorems and algorithms in that area of math to solve problems.

Cameron is a graduate student who took the elliptic curves course and is studying differential geometry. They remembered from the class that an elliptic curve is a projective curve of degree three. Cameron did not recall that elliptic curves are also complex tori and Riemann surfaces of genus one with a chosen point, but was able to make a few connections and translations when reminded that the three objects are equivalent. Cameron stated that given a Riemann surface of genus one, one can cut along two circles to get the lattice or torus. Given a complex torus, Cameron knew that you can use the Weierstrass $\wp-$function to get a map to the projective curve. Going the other direction, they hypothesized that there must be some computations that someone can do to find the lattice generators, but was not sure how to do this. The final connection that Cameron attempted to make was from the cubic in projective space to the Riemann surface of genus one. They guessed that if you parameterize the cubic, you would get the donut shape of the Riemann surface. When asked about if graduate students benefit from the ability to connect different areas of math, Cameron stated that "I think anyone who studies math has an innate bias to think it's beautiful when there's like a unifying result." They believe that it is nice to look and think of things in different ways because making connections can help make pictures in your head. Although, they do think that connecting to other areas of math may not be necessary for graduate students in certain specializations.

Jordan is a graduate student who took the elliptic curves course but is not studying number theory or algebraic geometry. Jordan was able to draw an elliptic curve over the complex numbers but was not sure what a complex torus or a Riemann surface was so was unable to make any connections or translations. Jordan believes that making connections can help graduate students see something that they might not expect, which can help solve your problem.

Logan is a graduate student studying number theory who took the elliptic curves course. They first think of elliptic curves as the projective curve with the group law and then as the complex numbers modulo a lattice. Logan stated that given a complex torus you can identify the edges with the opposing edge and wrap the complex torus up to see the Riemann surface. They were not sure how to go the other direction. The connections between the projective curve and the Riemann surface were not clear to Logan; they knew they were connected but were not sure how. Showing the connections between the projective curve and the complex torus, Logan stated that you can use the Weierstrass $\wp$-function for both directions. When asked about the benefits for graduate students to be able to connect different areas of math, Logan stated "I think it is important because no subject or idea in mathematics is isolated. Concepts in mathematics only make sense within a broader context, and in fact, have to make sense in a broader context to be a worthwhile concept! Practically, making connections is good for learning and remembering. In research you often need to have various ways of attacking a problem to make progress-if we only understand topics from one viewpoint, we may never see a solution, proof, etc."

Ryan is a graduate student studying number theory who took the elliptic curves course. They first think about elliptic curves as a projective curve and then are familiar with the equivalences to the other two objects. Ryan said that a smooth projective curve is a Riemann surface because the curve is projective, then it being genus one comes from looking at the differentials. Going the other direction they remember that there is some theorem about compact Riemann surfaces always being projective curves but they are not sure how to get the exact equation. Translating from complex torus to cubic in projective space, Ryan was not sure. They did identify the other direction as using the Weierstrass $\wp$-function and the L function. Ryan thinks that the connection between the complex torus and the Riemann surface is clear because the torus wraps up to a donut with one hole and then has the two loops from wrapping up. For the other direction Ryan thinks that one can look at differential forms and homology to see that the Riemann surface is a complex torus. Ryan believes that having different points of view is always productive because if you can't make progress from one point of view, moving to another point of view your question might be obvious or at least more accessible. Different areas of math also have different tools to help solve the problem. For this specific example of connecting different areas of math, Ryan said that "it always confused me when I was an undergrad that they would talk about an elliptic curve and write down an equation. And then, like, the next minute, they'd be talking about a torus".

Dakota is a graduate student studying algebraic geometry who took the elliptic curves course. They were familiar with the three mathematical objects and knew they were all equivalent but hadn't thought about elliptic curves in a while so felt a bit rusty. Dakota stated that given a curve in projective space, you can use the Riemann Roch Theorem to see that the curve is a genus one Riemann surface. From the complex torus to the Riemann surface of genus one, they said it is clear looking at the lattice that you get a genus one Riemann surface. The last connection that Dakota attempted to make was from a complex torus to the projective curve. They said that there is some formula to get A and B or the discriminant for the curve. Dakota was not sure about the rest of the connections. Dakota believes that being able to connect different areas of math is important because it allows you to" reformulate your problem into another language, and then read and then get to use tools from that language can help answer questions from the original perspective that you were working with".

Sam is a graduate student studying algebraic geometry that has not taken the elliptic curves course. They have heard elliptic curves come up frequently but were not confident on what they are. Sam thinks of a genus one Riemann surface as being equivalent to a complex torus because they are both complex and torus shaped. The other connection Sam attempted was the cubic in projective space to the complex torus. They said that if you look at the curve in projective space it should look like a torus. Sam believes that connecting different areas of math makes for better collaborations. They explained that the more you and your collaborator know about each other's areas of math the easier it is to find a middle ground to collaborate on. For themselves, they find that getting a new perspective can help give a better understanding of the original definition and its place in mathematics.

Quinn is a graduate student studying algebraic geometry that has not taken the elliptic curves course. They were fairly unfamiliar with elliptic curves, and had only heard them mentioned in talks. They view Riemann surfaces of genus one and complex tori as the same thing. Quinn was not able to make any of the connections. Quinn thinks that it is important for graduate students to be able to connect different areas of math because "it seems like in math, we, as humans, we're trying to like categorize things and draw these boundaries between different fields. And so I think it's important to continue to realize that those boundaries are illusions, and they're just human constructs".

Drew is a graduate student studying number theory who took the elliptic curves course. They felt very confident on the topic of elliptic curves. Drew was comfortable with these three mathematics objects being equivalent, they stated that they have not thought about these translations in detail much. They were unsure about all translations. Drew believes that it is very important to be able to connect different areas of math and explained that "you don't really fully understand a mathematical object until you can pin it down from all of the major angles. I personally find that that's a bit of a shortcoming with off of how we introduce material in a general sense, is you're given a new mathematical object and you're given a very narrow window on what it is, like a algebraist will introduce a mathematical object to be like, Oh, yes, here's the group structure, whereas a topologists will tell you about the topology... none of these are really the whole picture, right? There's a lot of details that are missing." They continued to say that as a graduate student, one perspective is not always enough to solve a problem. Even when it is enough, you may not get the easiest or most graceful solution.

5.3 Summary of Results

From the interviews with professors, I was able to determine a rubric outlining what I was looking for from graduate students to determine if they successfully connected and translated between the three mathematical objects. These mathematical objects are

1. Complex torus $\mathbb{C}/L$, for some lattice L
2. Smooth projective curve E of degree 3 in $\mathbb{P}^2$, with chosen point
3. Riemann surface of genus 1, with chosen point.

From (1) to (2) a complete translation included mentioning using the Weierstrass $\wp-$function. Translating the other way, from (2) to (1) I was looking for the interviewee to talk about uniformization or the use of branch cuts and elliptic integrals. When translating between (1) and (3) students needed to use charts and show that the transition functions are holomorphic or discuss how to identify opposite edges with one another and glue together to get the donut shape, thus showing that the genus is one. From (3) to (1), I was looking for graduate students to show that $\mathbb{C}$ is the universal cover of the Riemann surface and then when integrating along different loops, the integrals are equal modulo periods of a lattice. When translating from (2) to (3), I was looking for graduate students to use charts and show that the transition functions are holomorphic. For the final translation, from (3) to (2), students need to talk about Riemann Roch Theorem to make the translation. This rubric for translations is summarized in Figure 5.1.

In order to make a connection, a graduate student had to express that the two representations were connected and make an attempt at the translation. This attempt could be intuition or not completely correct. For example, from (1) to (2) the connection made by Dakota was "I remember some form, there's like some formula that you know, you you that goes between the two that you can calculate like your A and B or like your discriminate or something, I just don't remember what that formula is". From (3) to (2), an example of a connection from Ryan is "I've just seen the theorem that Riemann surfaces, compact Riemann surfaces are always curves. I don't actually know how you like take a complex Riemann surface and get like that equation, you know. But somehow you do that". Any time a participant made a connection, they were on the right track but were not able to accurately state the translation.

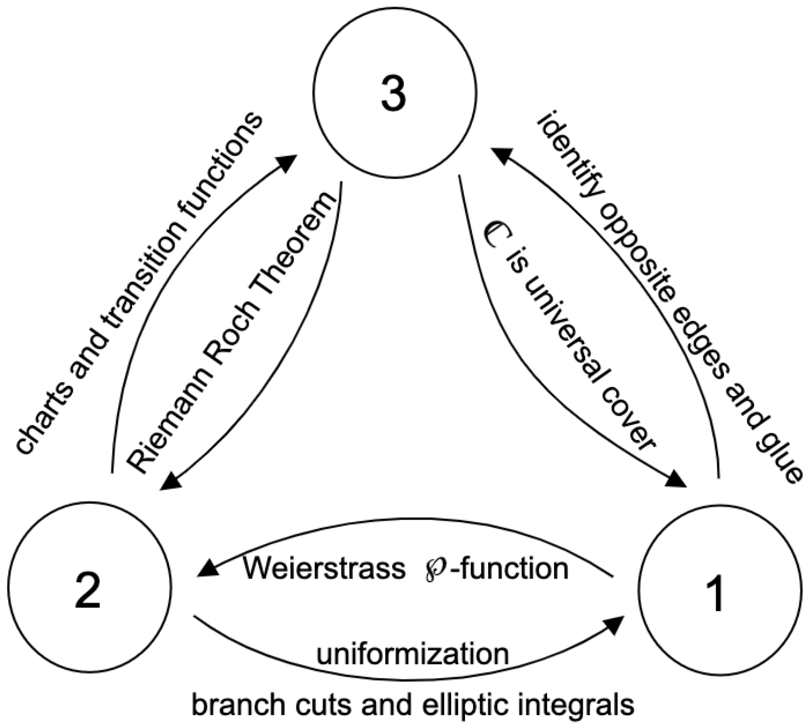

Figure 5.1: Summary of key for translations

In Figure 5.2, I summarize the representational fluency of the graduate students. They received a solid arrow if they were able to make that translation between representations and a dotted arrow if they were able to connect the two representations but were not able to state the translation correctly.

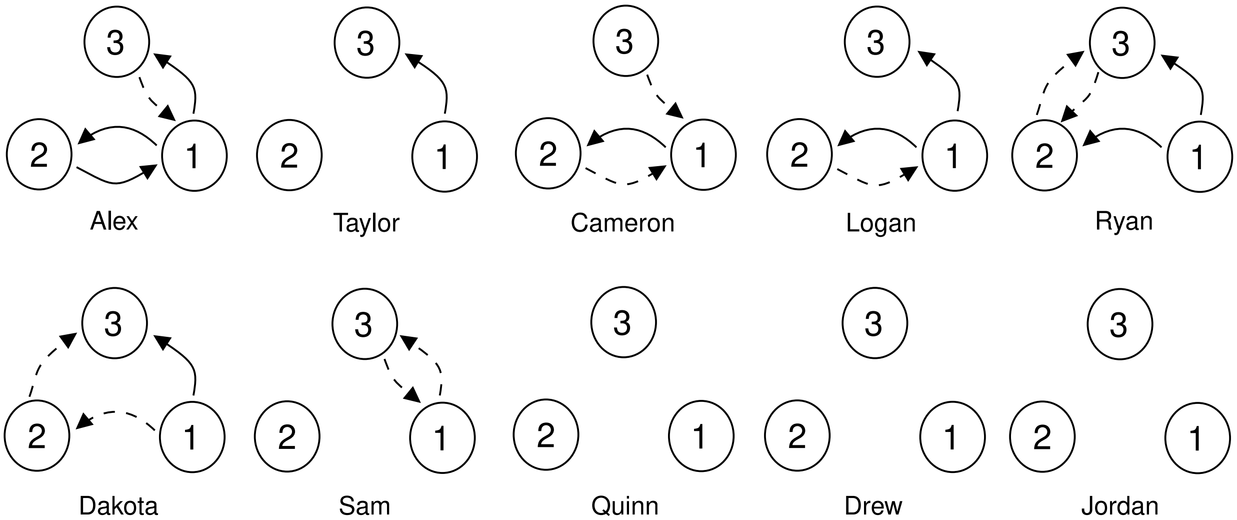

Figure 5.2: Summary of graduate students' translations and connections

Chapter 6

Discussion and Conclusion

From Chapter 2, I described three mathematical objects and then tested the understanding graduate students have regarding the connections and translations between the objects. Restated, these mathematical objects are

1. Complex torus $\mathbb{C}/L$, for some lattice L

2. Smooth projective curve E of degree 3 in $\mathbb{P}^2$, with chosen point

3. Riemann surface of genus 1, with chosen point.

Now, in this section, we will discuss our findings of how graduate students associate these objects with each other. In particular, there is a high prevalence of the translation of object (1) to (3), and object (1) to (2); conversely, the translations of object (2) to (3), (3) to (2), and (3) to (1) were less commonly made.

For object (1) to (3), I looked for graduate students to use charts and show that the transition functions are holomorphic. It then follows that the complex torus is a Riemann surface, after which I look for the student to talk about how to identify opposite edges with one another and glue the edges together to get the donut shape, thus showing that the genus is one. This connection was made by five of the ten graduate students. That being said, if the graduate student only talked about the genus one deduction, then I counted this as a full understanding of the translation of object (1) to (3) because some of the professors also stated this as the translation. Hence, this explanation also warranted a solid arrow.

Similarly, the translation of object (1) to (2) was also expressed by five graduate students. In this case, I looked for graduate students to mention the Weierstrass $\wp$-function. Four students saw this relationship and hence were interpreted as for fully understanding, earning a solid arrow. One student, instead, stated they believed that there was a formula to get A and B or the discriminant from the lattice; they received a dotted arrow since I considered knowledge of the existence of a formula sufficient for connecting the two objects but not complete for translating between representations.

On the other hand, the translations of object (2) to (3), (3) to (2), (3) to (1) were not expressed by most of the graduate students. First, the proof from object (2) to (3) was partially constructed by two students. They both were unsure of their work, as evidenced by one participant using "probably" in their explanation and the other saying "but I don't know the details so well" demonstrating an incomplete understanding of the translation, but it is noteworthy that they were both on the right track and were therefore able to make the connection. The opposite direction, object (3) to (2), was only attempted by three students, of which only one student demonstrated a partially correct understanding.

The most translations made by a graduate student was three. Only Alex was able to complete three translations, along with one additional connection. Ryan completed 2 translations as well as 2 more connections and Logan completed 2 translations with one additional connection. All three of these interviewees with the highest levels of representational fluency in elliptic curves are studying number theory. Overall, number theorists knew this mathematics more than the graduate students who studied other areas of mathematics. This may be due to the fact that the course covering elliptic curves was advertised as a number theory course and taught by a number theorist. There was student, Drew, who studies number theory but made zero connections or translations. They stated that they knew these three mathematical objects are equivalent but had not thought through the translations and did not attempt to during the interview.

The importance of having representational fluency between different areas of mathematics was consistent across all interviews except 1. Almost all interviewees agreed that the ability to make connections and translations in mathematics is crucial for success as a graduate student. A few different trends in reasoning showed up throughout the rest of the interviews. The first trend was the notion that being able to connect different areas of math helps a student develop a broader perspective and understanding of the math concept that they are studying. Drew stated "you don't really fully understand a mathematical object until you can pin it down from all of the major angles." This trend also showed up among the professors interviewed. Professor 1 stated that in graduate school, students learn big topics and it is important to see that they all connect together. In doing so, students do not get stuck thinking that what they do is the best thing in the world. Adaptability was the next major trend in participants' interviews. Coming from a broader perspective and having the ability to connect and be able to translate between different areas of math, students are able to approach their problems in different ways. Dakota stated that "you can reformulate your problem into a different language to use tools from that language to help solve your problem." Many of the interviewees mentioned the fact that different areas of math have different languages and tools that allow mathematicians to attack their problem in different ways. Professor 4 said that if you do not understand the symbols from another area of math then it can be easy to make mistakes. Ryan stated that "it just happens that like, you can't make progress from one point of view. But from the other point of view, it might even just be obvious, if not just like, easier to get your hands on." Going along with having different tools when in a different area of mathematics, Professor 2 and Sam mentioned that understanding the other area's tools and language can help a mathematician collaborate with others or understand more talks. While every person interviewed felt that connecting different areas of math as an important skill for graduate students to be able to do, only one person interviewed talked about this specific example. Professor 3 stated that this "really illustrated one of sort of, what are the unifying principles of mathematics." Professor 3 continued by explaining that almost every area of mathematics has an elliptic curve problem at the foundation of it.

Overall, despite a majority of interviewees believing that there are benefits for graduate students to have representational fluency between different areas of mathematics, no graduate students had complete representational fluency in elliptic curves. The importance and difficulties are consistent with studies done at lower levels of mathematics [9, 10]. In the case of this thesis our findings may be due to the fact that the course given on elliptic curves at the university where this study occurred did not teach the translations explicitly. The students that completed translations had thought through them on their own. This work leads to some implications for learning and teaching graduate level mathematics. Learners should be aware of the connections that the mathematical concepts they are learning about have to other areas of mathematics as the NCTM outlines that representational fluency is important for students at all levels [12]. Graduate students who want to improve their understanding should work through the connections and translations that appear in their own work and courses. On the teaching side, Shulman states that instructors need to have "at hand a veritable armamentarium of alternative forms of representation" and apply the most powerful ones to make the concepts more understandable to students [17]. Instructors of courses on elliptic curves could do more to encourage their students to think through the connections introduced in Chapter 2 and help introduce the mathematical ideas that the interviewees were lacking which lead to their inability to connect and translate. Another implication for instructors of graduate courses in general is to take time to emphasize the importance of representational fluency for graduate students in mathematics. This thesis discussed one example but there are many other examples of connections in mathematics.

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