DISSERTATION

SEARCH FOR COMMUTATIVE FUSION SCHEMES IN NONCOMMUTATIVE ASSOCIATION SCHEMES

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY JIANMIN MA ENTITLED "SEARCH FOR COMMUTATIVE FUSION SCHEMES IN NONCOMMUTATIVE ASSOCIATION SCHEMES" BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

SEARCH FOR COMMUTATIVE FUSION SCHEMES IN NONCOMMUTATIVE ASSOCIATION SCHEMES

An association scheme $\mathbb{X}$ is a finite set X together with a set of binary relations $R_0, R_1, \ldots, R_d$ that satisfy certain regularity conditions. The adjacency algebra (Bose-Mesner algebra) associated with the scheme $\mathbb{X}$ consists of integral matrices closed under matrix addition, multiplication and the Hadamard product. Call a free **Z**-module supporting two multiplications in this way a "double product" algebra.

An association scheme $\mathbb{Y}$ on X is a fusion scheme of $\mathbb{X}$ if each relation of $\mathbb{Y}$ is the union of relations of $\mathbb{X}$. The study of fusion schemes in commutative association schemes is well established method to build new association schemes with specified properties.

This work develops methods to construct commutative fusion schemes when $\mathbb{X}$ is not commutative. A principal method is to exploit "double product" homomorphic images. This leads to "lifting" issues that are addressed combinatorially.

These methods are applied to the centralizer algebra of $S_b^l \leqslant S_{lb}$, where S_n denotes the symmetric group of degree n and S_b^l is the direct product of l copies of S_b. The case $l = b = 3$ is studied in detail and several new commutative fusion schemes are found in this 55-dimensional algebra.

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To My Mother

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Chapter 1

Introduction

An association scheme $\mathbb{X}$ on a finite set X consists of a partition of $X \times X$ into binary relations $R_0 = \{(x,x)|x \in X\}, R_1, \ldots, R_d$ satisfying certain regularity properties. These properties imply that the adjacency matrices of the relations R_i form an algebra over the integers. This algebra is closed under the Hadamard (entry-wise) product and taking transpose, called the *adjacency algebra* of $\mathbb{X}$ (Section 2.1). The association scheme $\mathbb{X}$ is called *commutative* when its adjacency algebra is commutative under matrix multiplication.

Another association scheme $\mathbb{Y}$, on the same set X, is a fusion scheme of $\mathbb{X}$ if each relation of $\mathbb{Y}$ is a union of relations of $\mathbb{X}$. The adjacency algebra of $\mathbb{Y}$ is a subalgebra of that of $\mathbb{X}$. Meantime, $\mathbb{X}$ is called a fission scheme of $\mathbb{Y}$.

There is, perhaps, an analogy between this fusion/fission process and the process in chemical engineering. In chemical engineering, a compound is decomposed into components and then these components are synthesized to produce new compounds with new characteristics.

Fusion in commutative association schemes has been a powerful method to build new association schemes with specified properties. For example, in the theory of association schemes, distance regular property (d.r.p) and distance transitive property (d.t.p) are distinct but closely related [4, 13]. Distance transitivity is a symmetry property that indirectly implies the distance regular property.

Here are some explicit examples of this process. The schemes of quadratic forms are obtained by fusions [11] and are early examples of association schemes with d.r.p but not d.t.p. Also, fusion of the distance 1-or-2 relations in the distance transitive graph of a dual polar space leads to a new family of distance regular graph [18].

Fusion has also been used for other constructions. For example, the first family of distance regular digraphs of girth 4 was constructed by fusion in commutative schemes [20].

A recent example of both fission and fusion produced association schemes of very special property in [5]. This example is presented in Section 2.2.

Several authors studied fusion schemes in various commutative schemes [1, 3, 16, 14, 27, 30]. For commutative schemes, the eigenmatrices and the Bannai-Muzychuk criterion (Theorem 2.7) are main tools for studying fusion schemes.

The problem of finding/classifying all fusion schemes in an association scheme with many relations is a hard problem. This problem has been completely solved in only two families of commutative schemes: the Hamming and Johnson schemes (these schemes are defined in Section 2.1.) Their arguments require determining the basic relations in any fusion and exploitation their distance regular property [25, 26].

In the realm of noncommutative association schemes, the problem of finding/classifying fusion scheme has not been addressed in the literature. This thesis develops methods for enumerating commutative fusion schemes in noncommutative context. Analogous methods appeared in the construction of certain difference sets in nonabelian groups [21, 29]. This is also part of motivation of the current work.

The methods developed involves homomorphic images and their lifting. These images are required to support the full algebraic structure of an association scheme. At an algebraic level, the adjacency algebra is a free **Z**-module equipped with the usual matrix and Hadamard product. Call a free **Z**-module supporting two products in this way a "double product" algebra (Definition 3.1). Images that support the double product are obtained by applying modulo p reduction to the left regular representation of the adjacency algebra, where commutative "mod p" images are enumerated. One then lifts back these images. Several combinatorial principles could be used in the process of lifting (Section 3.3).

These methods are applied to the noncommutative association scheme of ordered partitions of l parts of equal size b. Its adjacency algebra has structure constants that can be put in nice form. For $l = b = 3$, its adjacency algebra has dimension 55 and several new commutative fusion schemes are found in it (see Chapter 5).

Computer algebra systems including GAP [12] and Maple play an important role in this research.

Chapter 2

Preliminaries

This chapter presents some background material about association schemes, introduces notation, and discusses some motivating examples. The reader is referred to [2, 4, 13] for the theory of commutative association schemes, and [31] for noncommutative association schemes.

To help the reader keep track of the variety of mathematical objects under discussion, a font based notation is employed to distinguish different types of objects. An upper case $\mathbb{BLACK}$ $\mathbb{BOARD}$ letter represents an association scheme, a $\mathsf{SANS\ SERIF}$ letter represents a set of relations, a $\mathfrak{fraktur}$ letter represents a set of relations/matrices indexing relations, and a **BOLDFACE** letter stands for a commutative ring with identity. An *ITALICS* letter represents a single scheme element including a relation and an adjacency matrix. Important terms are typeset in *italics* when they first appear.

2.1 Association Scheme

The *Hamming* and *Johnson* scheme are the two most important association schemes. The Hamming scheme plays a key role in the study of (nonlinear) block codes and the Johnson scheme plays a key role in the study of combinatorial designs (see [6], [10], or [13]).

Example 2.1. (Hamming scheme) Let Q be an alphabet of size q and let $X = Q^d$ be the set of ordered d-tuples of Q. Two tuples are in the i^{th} relation if their *Hamming distance* is i, that is, they differ in precisely i coordinates. Then X together with these relations is the (q-ary) Hamming scheme, denoted by $H_q(d)$. ☐

Example 2.2. (Johnson scheme) Let X be the set of all d-subsets of a fixed v-set V ($d \leq v/2$). Two d-sets are in the i^{th} relation if their intersection has size $d - i$. Then X together with these relations is the Johnson scheme, denoted by $J_v(d)$. ☐

More generally, let X be a finite set and $\mathsf{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of binary relations on X. The pair $\mathbb{X} = (X, \mathsf{R})$ is an *association scheme* of class d (d-class scheme, or d-scheme) if the following conditions hold:

(i) $R_0 = \{(x,x) | x \in X\}$.

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset (i \neq j)$.

(iii) $R_i^T = R_{i^\star}$, for some $0 \leq i^\star \leq d$, where $R_i^T = \{(y,x) | (x,y) \in R_i\}$.

(iv) Given $(x,y) \in R_k$, the integers $p_{ij}^k = |\{z \in X | (x,z) \in R_i, (z,y) \in R_j\}|$ depend only on i, j and k, not on x and y. These are referred as the *intersection numbers*.

Further, $\mathbb{X}$ is *commutative* if it satisfies

(v) For all i, j, k, $p_{ij}^k = p_{ji}^k$.

$\mathbb{X}$ is *symmetric* if it satisfies

(vi) For all i, $R_i^T = R_i$.

Symmetric schemes are necessarily commutative.

The numbers d, ν_i and p_{ij}^k are the parameter of $\mathbb{X}$. The following lemma is obtained by simple counting [2, Section 2].

Lemma 2.3. The following equations hold:

$$\sum_{\ell=0}^{d} p_{\ell i}^{j} = \nu_i \text{ and } \nu_k p_{ij}^{k} = \nu_i p_{jk^\star}^{i} = \nu_j p_{i^\star k}^{j}.$$

□

It is clear that the Hamming scheme $H_q(n)$ satisfies (i)-(ii), and (vi) which implies (iii). That $H_q(n)$ satisfies (iii) follows from Example 2.4 below.

Let $\mathbb{X} = (X, \{R_i\}_{0 \le i \le d})$ be an association scheme. For each relation R_i, let A_i be the *adjacency matrix* of the directed graph $\Gamma = (X, R_i)$. The number of in/out edges of any vertex of Γ is called the valency of R_i, denoted by ν_i. $\nu_i = p_{ii^\star}^0$. The axioms (i) - (iv) of association schemes can be expressed as

(i′) $A_0 = I$,

(ii′) $J = A_0 + A_1 + \cdots + R_d$,

(iii′) $A_i^T = R_{i^\star}$, for some $0 \le i^\star \le d$,

(iv′) $A_i A_j = \sum_{k=0}^{d} p_{ij}^k$,

where I is the identity matrix and J is the all-one matrix.

Let $\mathbf{Z}$ be the ring of integers. Then

$$\mathrm{Adj}(\mathbb{X}) = \sum_{i=0}^{d} \oplus \mathbf{Z} A_i$$

forms a $\mathbf{Z}$-algebra of dimension $d+1$, called the *adjacency algebra* (Bose-Mesner algebra) of $\mathbb{X}$ over $\mathbf{Z}$. For matrices A and B of the same size, their Hadamard product $A \circ B$ is the matrix with (i, j)-entry $a_{ij}b_{ij}$. From (ii'), Adj($\mathbb{X}$) is also closed under Hadamard product.

Each adjacency matrix A_i (resp. relation R_i) will be referred as a *basic* matrix (resp. relation) of $\mathbb{X}$.

The following example shows that association schemes are closely related to transitive permutation groups. For detailed treatment, see [2, Section 2.1] or [7, chapter 3].

Example 2.4. Let Ω be a finite set and let G be a transitive permutation group on Ω. Suppose R be the set of all orbits of G on $\Omega \times \Omega$. From the definition of orbit, $\Omega \times \Omega$ is the non-overlapping union of R. Since G is transitive, $\{(\omega, \omega) \mid \omega \in \Omega\}$ is an orbit. If R is an orbit, then $R^T = \{(\alpha, \beta) \mid (\beta, \alpha) \in R\}$ is also an orbit and hence $R^T \in \mathsf{R}$. So R satisfies conditions (i)-(iii). It remain to show it satisfies (iv).

Let R_i, R_j and R_k be three orbits in R. For $\alpha, \beta \in \Omega$, define

$$\Lambda_{i,j}(\alpha, \beta) = \{\gamma \in \Omega \mid (\alpha, \gamma) \in R_i, (\gamma, \beta) \in R_j\}.$$

For any two pairs $(\alpha', \beta'), (\alpha, \beta) \in R_k$, there exists some $g \in G$ such $g(\alpha', \beta') = (\alpha, \beta)$. For any γ' with $(\alpha', \gamma') \in R_i$ and $(\gamma', \beta') \in R_j$, we have $(g(\alpha'), g(\gamma')) \in R_i$ and $(g(\gamma'), g(\beta')) \in R_j$. Hence $g(\gamma') \in \Lambda_{i,j}(\alpha, \beta)$, i.e.,

$$\{g(\gamma') \mid \gamma' \in \Lambda_{i,j}(\alpha', \beta')\} \subset \Lambda_{i,j}(\alpha, \beta).$$

On the other hand, for any $\gamma \in \Lambda_{i,j}(\alpha, \beta)$, $g^{-1}(\gamma) \in \Lambda_{i,j}(\alpha', \beta')$. Therefore

$$\{g(\gamma') \mid \gamma' \in \Lambda_{i,j}(\alpha', \beta')\} \subset \Lambda_{i,j}(\alpha, \beta).$$

The size of $\Lambda_{i,j}(\alpha, \beta)$ depends only on the k such $(\alpha, \beta) \in R_k$, which is p_{ij}^k. So (iv) holds. Thus the principle that the symmetry property induces regularity property. ☐

Continue with Examples 2.1 and 2.2. Let S_d be the symmetric group of degree d. The wreath product $\mathrm{S}_q \operatorname{wr} \mathrm{S}_d$ acts transitively on each relation of $H_q(d)$ and it defines the Hamming scheme $H_q(d)$. The symmetric group S_v acts transitively on the set of d-subsets of V and its orbits on $X \times X$ are the relations of the Johnson scheme $J_v(d)$.

The following example is a special cases of Example 2.4 that will be needed later.

Example 2.5. Let G be a finite group with the identity e. Consider the left regular action of G on itself: that is, for any fixed g in G, $x \longrightarrow gx$ for every $x \in G$. This action is transitive and the G-orbits on $G \times G$ are indexed by elements in G. For each $g \in G$, $R_g = \{(x, y) \in G \times G \mid xy^{-1} = g\}$. The orbit R_e is the diagonal relation. Then $(G, \{R_g\}_{g \in G})$ is an association scheme of class $|G| - 1$, and denoted by $\mathbb{X}(G)$. Let A_g be adjacency matrix of R_g. Then $A_g A_h = A_{gh}$ for $g, h \in G$.

All valencies of $\mathbb{X}(G)$ are one. Conversely, any association scheme whose valencies are all one defines a group in this way. Later, any such scheme is called a *permuation scheme.* □

Fix $A_0, A_1, \ldots, A_d$ as a basis for $\mathrm{Adj}(\mathbb{X})$. Then each $A \in \mathrm{Adj}(\mathbb{X})$ define a linear map $L_A : \mathrm{Adj}(\mathbb{X}) \longrightarrow \mathrm{Adj}(\mathbb{X})$, where $L_A(M) = AM$. The map $A \longrightarrow L_A$ is called the *left regular representation* of $\mathrm{Adj}(\mathbb{X})$, and represents each matrix in $\mathrm{Adj}(\mathbb{X})$ by a $(d+1) \times (d+1)$ matrix. By (iv), this map takes A_i to B_i, where $B_i = (p_{ij}^k)$ with (j, k)-entry p_{ij}^k. The matrices $B_0, \ldots, B_d$ generate an algebra over $\mathbf{Z}$, called the *intersection algebra* of $\mathbb{X}$ and denoted by $\mathrm{Int}(\mathbb{X})$. The adjacency algebra and the intersection algebra are isomorphic under

$$\begin{array}{rccc} L: & \mathrm{Adj}(\mathbb{X}) & \longrightarrow & \mathrm{Int}(\mathbb{X}) \\ & A_i & \longrightarrow & B_i \end{array} \qquad (2.2.1)$$

Notice that the matrices in $\mathrm{Adj}(\mathbb{X})$ and $\mathrm{Int}(\mathbb{X})$ can have vastly different sizes.

Remark. The adjacency algebra and intersection algebra can be defined over other commutative rings $\mathbf{D}$ with unity, than $\mathbf{Z}$. Write $\mathrm{Adj}_{\mathbf{D}}(\mathbb{X})$ for $\mathrm{Adj}(\mathbb{X})$ to indicate the coefficient ring.

Now assume that $\mathbb{X}$ is a commutative association scheme with adjacency matrices A_i. Let $\mathbf{C}$ be the complex field. Since $\mathrm{Adj}_{\mathbf{C}}(\mathbb{X})$ consists of commuting matrices, it has a unique set of *primitive idempotents*: $E_0 = \frac{1}{|X|}J, E_1, \ldots, E_d$, where $|X|$ is the cardinality of X. Then A_i and E_i are related by

$$A_i = \sum_{j=0}^{d} p_{ji}E_j \text{ and } E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_{ji}A_j.$$

Also, the complex conjugate $\overline{E_i}$ of E_i is equal to E_{i^*} for some $0 \leq i^* \leq d$. The matrices $P = (p_{ji})$ and $Q = (q_{ji})$ with (j,i)-entry p_{ji} and q_{ji} are called the *first* and *second eigenmatrices* of $\mathbb{X}$ respectively.

The matrices A_i and E_i act as *dual bases* in the adjacency algebras, see [2], [4] or [6] for details.

2.2 Fusion in association schemes

Let $\mathbb{X} = (X, \{R_i\}_{0\leq i\leq d})$ be an association scheme with adjacency matrices A_i. Another association scheme $\mathbb{Y} = (X, \{R'_i\}_{0\leq i\leq e})$ is a *fusion* scheme of $\mathbb{X}$ if each relation of $\mathbb{Y}$ is a union of the relations of $\mathbb{X}$ while $\mathbb{X}$ is called a *fission* scheme of $\mathbb{Y}$. The following example is take from [5].

Example 2.6. In the context of Example 2.1, the alphabet is a finite field $\mathbf{F}_4$ with 4 elements and the vertex set $X = \mathbf{F}_4^3$, all the vectors of length 3. Recall for $x, y \in X$, $(x, y) \in R_i$ if x and y differ in precisely i coordinates, for $i = 0, 1, 2, 3$. This is the Hamming scheme $H_4(3)$. It has the following fission: splitting R_3 into

3 relations according to the value of $(x_1-y_1)(x_2-y_2)(x_3-y_3)$ for all $(x,y) \in R_3$. Denote these three new relations $C_i (i = 1,2,3)$. Since $-1 = 1$ in $\mathbf{F}_4$, $C_i = C_i^T$. These new relations together R_0, R_1 and R_2 satisfy axioms (i)-(iv) of Section 2.1 and (iv) can be verified with direct calculation. So this fission scheme is symmetric with six relations. This example is continued in Example 2.8 below. $\square$

Now assume that $\mathbb{X}$ is commutative with adjacency matrices A_i and primitive idempotents E_i. Let $\mathbb{Y}$ be a fusion e-scheme with adjacency matrices C_i and primitive idempotents F_i. Then each C_i (dually F_i) is the 01-linear combination of the A_j's (respectively E_j's). This induces two partitions of $\{0,1,\ldots,d\}$: $\tau = \{T_0 = \{0\}, \ldots, T_e\}$ and $\pi = \{\Pi_0 = \{0\}, \ldots, \Pi_e\}$ such that

$$C_i = \sum_{j \in T_i} A_j, \text{ and } F_i = \sum_{j \in \Pi_i} E_j, \ i = 0,1,\ldots,e.$$

The equation $R_i^T = R_{i^\star}$ defines a map $i \mapsto i^\star$ on $\{0,1,\ldots,d\}$. For $S \subset \{0,1,\ldots,d\}$, let

$$S^\star = \{i^\star \mid i \in S\}.$$

The following theorem is known as *Bannai-Muzychuk criterion*, proved independently in [1] and [24].

Theorem 2.7. (Bannai-Muzychuk criterion) Let $\mathbb{X} = (X, \mathsf{R})$ be a commutative scheme, $\mathbb{Y}$ be a fusion scheme of $\mathbb{X}$, and τ and π be two partitions defined above. Then the following conditions hold

(i) $T_\alpha^\star = T_{\alpha^\star}$ for some $0 \le \alpha^\star \le e$ (dually $\Pi_\alpha^\star = \Pi_{\alpha^\star}$);

(ii) for each $0 \le \alpha, \beta \le e$ and for any pair $i, j \in T_\alpha$,

$$\sum_{h \in \Pi_\beta} q_{ih} = \sum_{h \in \Pi_\beta} q_{jh},$$

(dually $\sum_{h \in T_\alpha} p_{ih} = \sum_{h \in T_\alpha} p_{jh}$ for any $i, j \in \Pi_\beta$).

Conversely, if two partitions τ and π of $\{0,1,\ldots,d\}$ satisfy the above conditions, they determine a fusion scheme of $\mathbb{X}$. $\square$

Example 2.8. (Example 2.6 continued [5]) The fission scheme has the first eigenmatrix with ordering R_0, R_1, R_2, C_1, C_2 and C_3

$$P = \begin{bmatrix} 1 & 9 & 27 & 9 & 9 & 9 \\ 1 & 5 & 3 & -3 & -3 & -3 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & -3 & 3 & 5 & -3 & -3 \\ 1 & -3 & 3 & -3 & -3 & 5 \\ 1 & -3 & 3 & -3 & 5 & -3 \end{bmatrix}.$$

This scheme has an interesting fusion scheme: merge R_1 with any of C_i, and merge the remaining two C_i. By Theorem 2.7, this gives a 3-scheme, denoted by $\mathbb{A}$. Moreover, $\mathbb{A}$ has further fusion: combining any two non-diagonal relations gives arise a two class scheme. $\square$

Remark. The reader familiar with finite geometries may recognize that the three non-diagonal relations of $\mathbb{A}$ come from a partition of the points of projective plane PG(2, 4) into two hyperovals and a unital. This leads to one of many constructions of the sporadic simple group M_{24}.

Theorem 2.9. Let $\mathbb{X} = (X, \{R_i\}_{0\le i\le d})$ be an association scheme, not necessarily commutative. And let $\tau = \{T_0 = \{0\}, \ldots, T_e\}$ be a partition of $\{0,1,\ldots,d\}$. Then $(X, \{S_i\}_{0\le i\le e})$ is a scheme with $S_i = \cup_{j\in T_i} R_j$ if and only if

(i) $S_i^T = S_{i^\star}$ for some $0 \le i^\star \le e$;

(ii) for any $0 \le \alpha, \beta, \gamma \le e$,

$$\sum_{i\in T_\alpha}\sum_{j\in T_\beta} p_{ij}^k$$

is a constant for any $k \in T_\gamma$.

Proof. That the new relations R_i' satisfy the scheme axioms (i)-(ii) is automatic. (i) and (ii) above is a restatement of the scheme axioms (iii) and (iv), respectively. $\square$

2.3 Permutation relations

The material in this section has not appeared in the literature, and it is developed further in Chapter 6.

A relation of valency one is called a *permutation relation.* If each relation of an association scheme has valency one, this scheme is called a *permutation scheme.* The reason for these terms will become clear soon. Some authors call a relation of valency one "thin", and hence a thin scheme [15].

Let $\mathbb{X} = (X, \mathsf{R} = \{R_0, \dots, R_d\})$ be an association scheme and R_i be a permutation relation. Then for every $x \in X$, there is a unique $y \in X$ such that $(x, y) \in R_i$. Write $y = xR_i$. The relation R_i induces a permutation $x \rightarrow xR_i$ on X.

Further, the map $x \rightarrow xR_i$ induces a permutation on R. Let R_j be any relation of $\mathbb{X}$. Suppose $(x, y) \in R_j$. By Lemma 2.3, $\sum_{l=0}^{d} p_{li}^{j} = \nu_i = 1$. The relation containing (yR_i, x) is uniquely determined by i and j. Say $(yR_i, x) \in R_l$. Similarly, the relation containing (xR_i, yR_j) is uniquely determined by l and i, hence by i and j. Therefore, the mapping $x \rightarrow xR_i$ on X induces a permutation on the relations R.

Let $\mathfrak{p}$ be the set of permutation relations in R. Then $\mathfrak{p}$ induces a permutation group on R.

Theorem 2.10. With the above notation, the union of orbits of $\mathfrak{p}$ on R forms a fusion scheme on X.

Proof. Any permutation relation maps R_0 to itself. The orbits induces a partition τ of $\{0, 1, \dots, d\}$. Let $\tau = \{T_0 = \{0\}, \dots, T_e\}$. Set $S_i = \cup_{j \in T_i} R_i$ for $i = 0, 1, \dots, e$.

For each S_i, choose any relation R_j in it. Then S_i^T is the the orbit containing R_j^T. For any $k, l \in T_\gamma$, there exists an element of $\mathfrak{p}$ that maps R_k to R_l. It follows that, for any $0 \leq \alpha, \beta \leq e$,

$$\sum_{i \in T_\alpha} \sum_{j \in T_\beta} p_{ij}^k = \sum_{i \in T_\alpha} \sum_{j \in T_\beta} p_{ij}^l.$$

The conditions (i) and (ii) of Theorem 2.9 hold, and the proof is completed. ☐

Further fusion can lead to commutative fusion schemes of $\mathbb{X}$. Of course, every association scheme has one permutation relation, namely, the diagonal relation R_0. If there are no other permutation relations, the fusion scheme from Theorem 2.10 is $\mathbb{X}$ itself.

Chapter 3

General Methods

This chapter develops methods for finding commutative fusion schemes in noncommutative association schemes. We introduce the double product algebra that includes the adjacency algebra, the intersection algebra and the group algebra as examples, define the homomorphism of the double product algebras, and finally outline how to use the homomorphic images of the intersection algebra to find commutative fusion schemes.

3.1 Double product algebra

Let $\mathbb{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. The adjacency algebra $\mathrm{Adj}(\mathbb{X})$ has basic matrices $A_0, A_1, \ldots, A_d$ and the intersection algebra $\mathrm{Int}(\mathbb{X})$ has basic matrices $B_0, B_1, \ldots, B_d$.

From Section 2.1, $\mathrm{Adj}(\mathbb{X})$ supports the usual matrix and Hadamard multiplications and is closed under taking the matrix transpose. The following definition captures these properties.

Definition 3.1. Let $\mathbf{D}$ be a commutative ring with identity. Let $\mathcal{A}$ be a $(d+1)$-dimensional $\mathbf{D}$-module equipped with two multiplications ($\bullet$ and $\circ$) and a unary operation $\flat$. If $\mathcal{A}$ satisfies the following conditions: for all $x, y \in \mathcal{A}$,

(i) $(\mathcal{A}, \bullet)$ is an associative algebra with identity;

(ii) $(\mathcal{A},\ \circ)$ is a commutative associative algebra with identity,

(iii) $\flat(\flat(x)) = x$;

(iv) $\flat(x \bullet y) = \flat(y) \bullet \flat(x)$;

(v) $\flat(x \circ y) = \flat(x) \circ \flat(y)$;

(vi) $(\mathcal{A},\ \bullet)$ has a basis $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d$ such that

  (a) $\mathbf{a}_0 = 1$ is the identity for multiplication $\bullet$;

  (b) for all i and j $\mathbf{a}_i \circ \mathbf{a}_j = \mathbf{a}_i \delta_{ij}$, where δ_{ij} is the Kronecker delta;

  (c) $\jmath = \mathbf{a}_0 + \mathbf{a}_1 + \cdots + \mathbf{a}_d$ is the identity for multiplication $\circ$.

Then $\mathcal{A}$ is called a *double product* algebra, abbreviated as $\mathfrak{D}$-algebra.

A basis that satisfies condition (vi) is called a $\mathfrak{D}$-basis of $\mathcal{A}$. Call $\mathcal{A}$ *commutative* if $(\mathcal{A}, \bullet)$ is a commutative algebra. Notice that the Fraktur character $\mathfrak{D}$ represents a type of algebra (and later a related type of homomorphism), other than a relation set.

Example 3.2. The adjacency algebra $\mathrm{Adj}(\mathbb{X})$ is a $\mathfrak{D}$-algebra with the usual matrix and Hadamard multiplication and the unary operation $\flat : A \to A^T$. The identity for the Hadamard product is the all-one matrix J. The conditions (iii)-(v) hold since $(A^T)^T = A$, $(AB)^T = B^T A^T$, and $(A \circ B)^T = A^T \circ B^T$ for all $A, B \in \mathrm{Adj}(\mathbb{X})$. □

Example 3.3. Let $\mathbf{Z}$ be the ring of integers and G be a finite group with identity e. Let $\mathbf{Z}G$ be the group of G over $\mathbf{Z}$. Define the second product $\circ$ in $\mathbf{Z}G$: for $g, h \in G$, $g \circ h = g\delta_{gh}$. The product $\circ$ is extended linearly to $\mathbf{Z}G$ and this product is commutative with identity $\sum_{g\in G} g$. The unary operation is define by $\flat(x) = x^{-1}$, which is extended linearly to $\mathbf{Z}G$. Then $\mathbf{Z}G$ is a $\mathfrak{D}$-algebra.

Let $\mathbb{X}(G)$ be the association scheme defined in Example 2.5 with adjacency matrices $A_g(g \in G)$. For $g, h \in G$, the Hadamard $A_g \circ A_h$ agrees $g \circ h$ defined above. The map $g \to A_g$ from $\mathbf{Z}G$ to the adjacency algebra preserves the $\mathfrak{D}$-algebra operations. In other word, this map is $\mathfrak{D}$-isomorphism (see Definition 3.4 below). $\square$

Definition 3.4. Let $\mathbf{D}$ and $\mathbf{E}$ be two commutative rings with identity. Suppose that $\hat{}: a \longrightarrow \hat{a}$ is a ring homomorphism from $\mathbf{D}$ to $\mathbf{E}$. Let $\mathcal{A}$ be a $\mathfrak{D}$-algebra over $\mathbf{D}$ and $\mathcal{B}$ be a $\mathfrak{D}$-algebra over $\mathbf{E}$. A map ϕ from $\mathcal{A}$ to $\mathcal{B}$ is $\mathfrak{D}$-homomorphism associated with $\hat{}$ if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ and $a \in \mathbf{D}$,

(i) $\phi(\mathbf{x}+\mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$;

(ii) $\phi(\mathbf{x} \bullet \mathbf{y}) = \phi(\mathbf{x}) \bullet \phi(\mathbf{y})$;

(iii) $\phi(\mathbf{x} \circ \mathbf{y}) = \phi(\mathbf{x}) \circ \phi(\mathbf{y})$;

(iv) $\phi(\flat(\mathbf{x})) = \flat(\phi(\mathbf{x})$;

(v) $\phi(a\mathbf{x}) = \hat{a}\phi(\mathbf{x})$ for a in $\mathbf{D}$ and $\mathbf{x}$ in $\mathcal{A}$.

Further, suppose that $\hat{}$ is an isomorphism between $\mathbf{D}$ and $\mathbf{E}$. Then ϕ is called a $\mathfrak{D}$-isomorphism between $\mathcal{A}$ and $\mathcal{B}$ if ϕ is bijective. $\square$

Example 3.5. Let $\mathbb{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an imprimitive association scheme with adjacency matrices $A_0, A_1, \ldots, A_d$. Suppose that $\cup_{i=0}^{r} R_i (r < d)$ is an equivalence relation on X and S is an equivalence class. Then $\mathbb{S} = (S, \{R_i\}_{0 \leq i \leq r})$ is a subscheme with adjacency matrices $A'_0, A'_1, \ldots, A'_r$ (see Section 6.1 for the definition of subschemes). The map $\phi(A'_i) = A_i (0 \leq i \leq r)$ induces a $\mathfrak{D}$-homomorphism from the adjacency algebra of $\mathbb{S}$ to that of $\mathbb{X}$. $\square$

A fusion scheme of $\mathbb{X}$ determines a partition of $\{1, \ldots, d\}$. When d is very small, one can check all partitions of $\{1, \ldots, d\}$ to determine all fusion schemes of $\mathbb{X}$. When d is large, this approach is not feasible.

Another approach attempts to determine the basic relations in any fusion scheme. This approach was used to classify of fusion schemes in the Hamming and Johnson schemes [25, 26]. This work also relied heavily on distance regular property of these schemes.

For noncommutative schemes, there is no analogous property to the distance regular property. A different approach is required. The double product homomorphism is useful for this purpose. In the following example, the Hadamard product is exported to the intersection algebra.

Example 3.6. Continue with Example 3.2. In the intersection algebra $\mathrm{Int}(\mathbb{X})$, define the second product $\circ$ by $B_i \circ B_j = B_i \delta_{ij}$ for $0 \leq i, j \leq d$. The equations $A_i^T = A_{i^*}$ induce a map on the intersection matrices defined $\flat(B_i) = B_{i^*}$. Extend the product $\circ$ and the unary operation $\flat$ linearly. The map $L : A_i \mapsto B_i$ in (2.2.1) gives a $\mathfrak{D}$-isomorphism (associated with the identity isomorphism) between $\mathrm{Adj}(\mathbb{X})$ and $\mathrm{Int}(\mathbb{X})$. ☐

The $\mathfrak{D}$-homomorphic images of the adjacency algebra obtained by the intersection algebra modulo a positive integer p are a focus point of this thesis. This involves the reductions of the matrix entries modulo p and the coefficient ring modulo p. The homomorphic images thus produced are an algebras over $\mathbf{Z}_p$, the ring of integers modulo p. This reduction preserves the operations in the intersection algebra.

Example 3.7. Let $\mathcal{A}$ be the intersection algebra of an association scheme and let p be a positive integer. The mapping $\phi : A \rightarrow (A \mod p)$ for each $A \in \mathcal{A}$ is a $\mathfrak{D}$-homomorphism of $\mathcal{A}$ with the operations inherited from $\mathcal{A}$. ☐

3.2 Upper bound on the dimension of commutative fusion

The adjacency algebra is free **Z**-module with two ring structures associated with the usual matrix and Hadamard multiplications. This section focuses on the first ring structure and ignoring the second.

Let $\mathbb{X} = (X, \mathsf{R})$ be an association scheme with adjacency algebra $\mathcal{A}$. Let $\mathbf{C}X$ be a vector space of dimension $|X|$ over the complex field. Each matrix in $\mathcal{A}$ can be regarded as a linear transformation of $\mathbf{C}X$ to itself and thus $\mathbf{C}X$ becomes a left $\mathcal{A}$-module. Then $\mathbf{C}X$ is isomorphic to a direct sum of irreducible $\mathcal{A}$-modules:

$$\mathbf{C}X = \sum_{i=0}^{r} f_i N_i$$

where N_i are distinct irreducible $\mathcal{A}$-modules that occur in $\mathbf{C}X$ and f_i is the multiplicity of N_i in $\mathcal{A}$. Since the all-one vector generates a submodule of $\mathbf{C}X$, we may assume that $f_0 = 1$. Thus $\mathcal{A}$ is isomorphic to a direct sum of matrix algebras:

$$\mathcal{A} \approx \sum_{i=0}^{r} Mat_{f_i, f_i}(\mathbf{C}). \tag{3.3.1}$$

The decomposition in (3.3.1) can be used to bound the dimensions of commutative subalgebras in $\mathcal{A}$. When $\mathbb{X}$ is commutative, all $f_i = 1$. In this case, the bound is the dimension of $\mathcal{A}$. For each matrix algebra in (3.3.1), there is an upper bound on the dimensions of its commutative subalgebras. The sum of these upper bound is an upper bound on the numbers of classes in any commutative fusion schemes in $\mathbb{X}$.

The decomposition in (3.3.1) produces a set of *block idempotents.* These block idempotents are used in next section to organize the search for commutative $\mathfrak{D}$-algebras.

The direct sum in (3.3.1) provides homomorphisms from $\mathcal{A}$ to matrix algebras. These homomorphisms might be used with other tools to locate commutative fusion in $\mathbb{X}$.

3.3 Searching principles

Suppose that $\mathcal{A}$ is a $\mathfrak{D}$-algebra with $\mathfrak{D}$-basis $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d$. A $\mathfrak{D}$-subalgebra $\mathcal{F}$ in $\mathcal{A}$ is called a *fusion* algebra of $\mathcal{A}$ if there exists a partition $\pi = \{\Pi_0 = \{0\}, \Pi_1, \ldots, \Pi_e\}$ of $\{0, 1, \ldots, d\}$ such that $f_0, f_1, \ldots, f_e$ is a basis for $\mathcal{F}$ with $f_i = \sum_{j \in \Pi_i} \mathbf{a}_j$.

Suppose that S is a set. Let τ be a set of subsets of S. We call τ a *partial partition* of S if, for any distinct subsets $A, B \in \tau$, $A \cap B = \emptyset$. The set of partial partitions of S forms a partially ordered set, where, for two partitions τ and σ, $\tau \leq \sigma$ if τ is a refinement of σ.

All of the methods in this work are based on elements than subalgebras, This implies that we start with a partial partition and perform a test, if the test fails, then no refinement of this partial partition supports a fusion algebras (or fusion scheme).

This rest of this section discusses several general techniques that are used in searching for commutative fusion schemes. These techniques can be generalized to commutative $\mathfrak{D}$-algebras with minor modification.

Let $\mathbb{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be association scheme with adjacency matrices A_i and intersection numbers p_{ij}^k.

We use $[i..j]$ to denote the set of integers from i to j inclusive. For any subset $S \subseteq [0..d]$, we say S supports $\sum_{i \in S} A_i$ and $\cup_{j \in S} R_i$. For convenience, we don't distinguish the subset S, $\sum_{i \in S} A_i$ and $\cup_{i \in S} R_i$.

For x and y in a $\mathfrak{D}$-algebra of $\mathbb{X}$, the element $\mathbf{x} \circ \mathbf{y}$ is called the *Hadamard shadow* of $\mathbf{x}$ and $\mathbf{y}$. The following is referred the *Hadamard shadow principle*, which is used in sections 5.4 and 5.5.

Principle 3.8. (Hadamard shadow) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two elements in a $\mathfrak{D}$-algebra. If $\mathbf{x}$ and $\mathbf{y}$ are in a commutative subalagebra, then $\mathbf{x} \circ \mathbf{y}$ must commute with $\mathbf{x}$ and $\mathbf{y}$, i.e.,

$$\mathbf{x} \bullet (\mathbf{x} \circ \mathbf{y}) = (\mathbf{x} \circ \mathbf{y}) \bullet \mathbf{x} \text{ and } \mathbf{y} \bullet (\mathbf{x} \circ \mathbf{y}) = (\mathbf{x} \circ \mathbf{y}) \bullet \mathbf{y}.$$

$\square$

For a subset $S \subseteq [1..d]$, define Σ_S as the sum $\Sigma_S = \sum_{i \in S} A_i$ in $\mathrm{Adj}(\mathbb{X})$. For $S, T \subseteq \{1, \dots, d\}$, set

$$c_k(S,T) = \sum_{i \in S} \sum_{j \in T} p_{ij}^k.$$

In other words, $c_k(S,T)$ is the coefficient of A_k in the product

$$\Sigma_S \Sigma_T = \sum_{k=0}^{d} c_k(S,T) A_k.$$

Suppose that $c_k(S,T)$ takes r different values $v_1, v_2, \dots, v_r$. We define a partition of $[1..d]$: for $1 \le i \le r$,

$$\Pi_i = \{k \mid 1 \le k \le d, c_k(S,T) = v_i\}.$$

We set

$$\mathrm{spt}(S,T) = \{\Pi_0 = \{0\}, \Pi_1, \dots, \Pi_r\}.$$

Let $\pi = \{\Pi_0 = \{0\}, \Pi_1, \dots, \Pi_s\}$ be a partition of $[0..d]$ that supports a fusion scheme. Then (ii) of Theorem 2.9 can restated as, for any $\alpha, \beta \in [0..s]$, $c_k(\Pi_\alpha, \Pi_\beta)$ depends on Π_γ with $k \in \Pi_\gamma$.

The following question is of interest: given $S, T \subseteq [1..d]$ with $S = T$ or $S \cap T = \emptyset$, is there any commutative fusion scheme containing S and T?

This question can be answered by the following procedure. Compute $\pi = \operatorname{spt}(S, T)$. Exactly one of three cases can happen to π:

(i) π does not contain S and T.

(ii) π does contain S and T, but elements of π do not commute with each other.

(iii) None of above.

If case (i) or (ii) occurs, then the answer to the above question is NO. If case (iii) occurs, set $\sigma = \pi$. Suppose $\pi = \{\Pi_0 = \{0\}, \Pi_1, \ldots, \Pi_r\}$. For each pair (i, j) with $1 \leq i < j \leq r$, compute $\operatorname{spt}(\Pi_i, \Pi_j)$. If either case (i) or (ii) happens to $\operatorname{spt}(\Pi_i, \Pi_j)$, then the answer to the above question is NO. Otherwise, say $\operatorname{spt}(\Pi_i, \Pi_j) = \{\Lambda_0 = \{0\}, \Lambda_1, \ldots, \Lambda_s\}$. Form a new partition which consists of nonempty intersections of $\Pi_i \cap \Lambda_j$ for all $0 \leq i \leq r, 0 \leq j \leq s$. This partition refines τ and σ, which is again called σ. Once all pairs (i, j) are done, compare σ with τ. If $\sigma \neq \tau$, we let $\tau = \sigma$ and repeat the above procedure. (One can be smart the next round by avoiding to the pairs originally from π). Else exit with the partition τ that supports a commutative fusion scheme. This fusion contains the basis relation S.

The above procedure can be extended to a partial partition of $[1..d]$ with minor modification. This is used in Step (Lift) of Section 3.4. We refer this procedure as the *scheme element principle.*

Principle 3.9. (Scheme element) Given a partial partition of $[1..d]$, one can decide if it support basic relations in a commutative fusion scheme. $\square$

Recall that $i \rightarrow i^\star$ is a bijective function from $[0..d]$ to itself (Example 3.6). For any subset $S \subset [0..d]$, define

$$S^\star = \{i^\star \mid i \in S\}.$$

If $S = S^\star$, S is *symmetric* and *nonsymmetric* otherwise. The pair $\{S, S^\star\}$ is called *symmetric* if $S \cap S^\star = \emptyset$. In particular, if $S = \{i\}$ is symmetric, then i is symmetric. A partial partition τ of $[0..d]$ is called *admissible* if $T^\star \in \tau$ whenever $T \in \tau$. In the rest of this section, τ denotes an admissible partial partition.

The last step of our search for commutative fusions involves the lifting of a partial partition τ of $[0..d]$. Let τ^c be the set of elements of $[0..d]$ which do not appear in τ. This lifting process requires appending some subsets in τ with the elements in τ^c. The following principle is used in the lifting algorithm in Section 3.4, called the *pairing principle.*

Principle 3.10. (Pairing) Let τ be an admissible partial partition. Suppose that $i \in \tau^c$ is be added to some subset of τ. If i is symmetric, it can only be added a symmetric subset of τ; if i is nonsymmetric, then either $\{i, i'\}$ is add to some symmetric subset of τ, or i is added to some T in τ and $i^\star$ is added to $T^\star$, where $T \cap T^\star = \emptyset$. ☐

The following principle explains how to generate a scheme element from a partial partition. It is quite effective in the lift process.

Principle 3.11. Suppose that $\tau \backslash \{0\}$ has s symmetric subsets and t symmetric pairs, and τ^c has s' symmetric subsets and t' symmetric pairs.

By the Pigeon hole principle, at most $\lfloor \frac{s'+t'}{s} \rfloor$ $(= m)$ symmetric subsets/pairs of τ^c is added to some symmetric subset of τ, where $\lfloor x \rfloor$ is the largest integer smaller than x. Thus for any symmetric subset of τ, at most $\sum_{i=0}^{m} \binom{s'+t'}{i}$ cases have to be tested. Again, by the same principle, for any symmetric pair in τ, we test at most $\sum_{i=0}^{m} \binom{t'}{i} 2^i$ cases, where $m = \lfloor t'/t \rfloor$. ☐

3.4 Approaches for finding fusion schemes

Our search for commutative fusion schemes generally involve the following three steps. Sections 5.3, 5.4 and 5.5 implement these three steps. It may be helpful to understand these methods to read this section in parallel with those sections.

Let $\mathbb{X} = (X, \{R_i\}_{0 \le i \le d})$ be an association scheme. Its intersection algebra $\mathcal{A}$ has a basis the intersection matrices $B_0, B_1, \ldots, B_d$.

Step (Mod) Seek $\mathfrak{D}$-homomorphisms of the intersection algebra.

These $\mathfrak{D}$-homomorphisms are obtained by reducing $\mathcal{A}$ modulo some positive integer p.

There are actually two reductions involved. Since $\mathcal{A}$ is a matrix ring over $\mathbf{Z}$, one can apply modulo p reduction to the entries of any matrix in $\mathcal{A}$. This a ring homomorphism. Since $\mathcal{A}$ is a left $\mathbf{Z}$-module and $\mathbf{Z}_p$ is a right $\mathbf{Z}$-module; one can form tensor product $\mathbf{Z}_p \otimes \mathcal{A}$.

These two modulo p reductions chained together result in a matrix algebra over $\mathbf{Z}_p$, denoted by $\bar{\mathcal{A}}$. Let $\bar{B}_i = B_i \mod p$. Since the first rows of B_i add to the all-one vector, the matrices $\bar{B}_0, \bar{B}_1, \ldots, \bar{B}_d$ form a basis of $\bar{\mathcal{A}}$. The operations inherited from $\mathcal{A}$ makes $\bar{\mathcal{A}}$ a $\mathfrak{D}$-algebra.

Examine the matrices $\bar{B}_i$ to see if they have the block triangular form. Look for blocks that inherit the $\mathfrak{D}$-algebra structure, i.e., the second product and unary operation. (See Section 5.3 for such an example.) The projection of $\bar{\mathcal{A}}$ into such a block is a $\mathfrak{D}$-homomorphism. Its homomorphic image will be denoted by $\widetilde{\mathcal{A}}$. The homomorphism from $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ obtained from the above process is a $\mathfrak{D}$-homomorphism.

It turns out that the most useful homomorphisms are a map that takes B to a principal submatrix of B modulo p for every $B \in \mathcal{A}$. This map is required to preserve the operations of the adjacency algebra.

A natural question to ask is: which positive integer p will produce a desired $\mathfrak{D}$-homomorphism? This question has been answered for difference sets in [22]: any prime p divides the determinant of the "inversion matrix". If p is greater than all valencies, then it will not help the question. One should look for primes that are smaller than the largest valency, especially any prime that divides some but not all valencies because $\nu_k p_{ij}^k = \nu_i p_{jk^*}^i$.

Suppose we have a desired $\mathfrak{D}$-homomorphism from $\mathcal{A}$ to $\widetilde{\mathcal{A}}$. Let $\mathbb{F}$ be the commutative fusion scheme of $\mathbb{X}$ from a partition π of $[0..d]$. The image of $\mathbb{F}$ in $\widetilde{\mathcal{A}}$ is a commutative $\mathfrak{D}$-subalgebra supported by a partial partition $\tilde{\pi}$ of $[0..d]$. Clearly, $\tilde{\pi} \leq \pi$. This leads to search for all commutative $\mathfrak{D}$-subalgebras in $\widetilde{\mathcal{A}}$.

Step (Split) Seek all commutative $\mathfrak{D}$-subalgebras of $\widetilde{\mathcal{A}}$ from partial partitions.

Since $\widetilde{\mathcal{A}}$ is a finite dimensional algebra over $\mathbf{Z}_p$, it can be written as a direct sum of two-side ideals from representation theory, e.g. [9, Theorem 55.2]:

$$\widetilde{\mathcal{A}} = \mathscr{I}_1 \oplus \mathscr{I}_2 \oplus \ldots \oplus \mathscr{I}_s, \tag{3.3.2}$$

where $\mathscr{I}_j$ are nonzero two-sided ideals. This decomposition determines a set of *block idempotents* $e_1, \ldots, e_s$ satisfying

$$1 = e_1 + \cdots + e_s, \quad e_i e_j = e_i \delta_{ij}, \text{ and } \widetilde{\mathcal{A}} e_i = e_i \widetilde{\mathcal{A}} = \mathscr{I}_i,$$

where δ_{ij} is the Kronecker delta.

Each two sided ideal in (3.3.2) is isomorphic to a (not necessarily full) matrix algebra. So $\mathcal{A}$ is isomorphic to a direct sum of matrix algebras:

$$\widetilde{\mathcal{A}} \approx \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_s. \tag{3.3.3}$$

The problem of finding all commutative $\mathfrak{D}$-subalgebras in $\widetilde{\mathcal{A}}$ is reduced to the problem of finding all corresponding subalgebras in a direct sum of matrix algebras. The latter can be addressed with two complementing approaches. Let us illustrate these approaches work with $\widetilde{\mathcal{A}} \approx \mathcal{M}_1 \oplus \mathcal{M}_2$. Each element m of $\widetilde{\mathcal{A}}$ determines a matrix m_1 in $\mathcal{M}_1$ and m_2 in $\mathcal{M}_1$. We write $m = m_1 \oplus m_2$. See Sections 5.4 and 5.5 for more details.

According to the block idempotents they contain, the commutative $\mathfrak{D}$-subalgebras in $\widetilde{\mathcal{A}}$ is divided into two non-overlapping families: the first family consists these subalgebras that contain e_1 and e_2, and the second family consists the rest.

For the first family, we find commutative subalgebras in $\mathcal{M}_1$ and $\mathcal{M}_2$. Techniques similar to Principles 3.8 and 3.9 can be used in this searching process. Then matching all pairs of subalgebras in $\mathcal{M}_1$ and $\mathcal{M}_2$ finds the first family.

Suppose that $\mathcal{S}$ is a subalgebra in the second family. Let $m = m_1 \oplus m_2$ in $\mathcal{S}$ be a matrix in $\mathcal{S}$. Then the minimal polynomials of m_1 and m_2 can not be relatively prime to each other by Theorem 5.5. From this more elaborate combinatorial and algebraic techniques are employed to find all members in the second family (see Section 5.5).

Suppose that $\widetilde{\mathcal{S}}$ is a commutative $\mathfrak{D}$-subalgebra in $\widetilde{\mathcal{A}}$. Then $\widetilde{\mathcal{S}}$ determines a partial partition $\tilde{\tau}$ of $[0..d]$. Let $\tilde{}$ be the homomorphism form $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ above. For any $T \in \tau$, we have

$$\sum_{i^\star \in T^\star} \tilde{B}_{i^\star} = \sum_{i \in T} B_i.$$

Thus, $T^\star \in \tau$, i.e., τ is admissible.

Step (Lift) Lift an admissible partial partition to produce commutative fusion scheme.

The last step involves the lifting of the candidates from Step (Split) back to $\mathcal{A}$ to produce commutative fusion schemes of $\mathbb{X}$. This issue will be addressed combinatorially. Each candidate from Step (Split) produces an admissible partial partition $\tilde{\sigma}$ of $[0..d]$ with $0 \in \tilde{\sigma}$. First, apply principle 3.11 to $\tilde{\sigma}$ and produce a partial partition $\tilde{\tau}$ of scheme element candidates. Second, apply principle 3.9 to $\tilde{\tau}$. If this produces a partition σ of $[0..d]$, then it supports a commutative fusion scheme. Compare σ with $\tilde{\sigma}$. If any subset of $\tilde{\sigma}$ splits into more than one subsets in σ, then $\tilde{\sigma}$ has no lift.

Suppose that σ lifts $\tilde{\sigma}$. The fusion scheme it supports may still have commutative fission schemes that also lift $\tilde{\sigma}$. The following procedure is applied further to find these fissions. Let $\tilde{\sigma} = \{\tilde{\sigma}_0 = \{0\}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_r\}$. Arrange the subsets in σ such that $\tilde{\sigma}_i \subset \sigma_i (0 \leq i \leq r)$. Find the first i such that $\tilde{\sigma}_i \neq \sigma_i$. Apply Principle 3.9 to each subset that contains $\tilde{\sigma}_i$ and is contained in σ_i, together with the rest of subsets in σ. The result is either no lift, or a new partition that support a commutative fission of the original scheme. This process is repeated till no further commutative fissions are produced. The procedure is referred as the *fission process.*

Here is some notations we use to describe the lift algorithms. Suppose τ is a partial partition of $[0..d]$ from Step (Lift). Let $\tau^{\times} = \tau \backslash \{0\}$. and τ^{c} be the set of elements in $[0..d]$ that are not in any subset of τ. If Ω is a set, we denote by $\binom{\Omega}{i..j}$ the collection of subsets of Ω of cardinality between i and j.

The input of the lift algorithm is a partial partition τ and the output is a set of partitions of $[0..d]$ that lift τ.

Algorithm 1 Lifting the partial partition τ

```
if $\tau$ is a full partition of $[0..d]$ then
  test if $\tau$ produces a commutative fusion and exit
end if
result =[ ]
compute $\tau^c$
$S$ = symmetric parts in $\tau^\times$
$s$ = cardinality of $S$
$T$ = nonsymmetric pairs of $\tau^\times$
$t$ = cardinality of $T$
$T'$ = set representatives from each pair in $T$
$t'$ = cardinality of $T'$
$S^c$ = symmetric part in $\tau^c$
$T^c$ = nonsymmetric pairs in $\tau^c$
$m = \lfloor (s^c + t^c)/s \rfloor$
$m' = \lfloor t^c/t \rfloor$
for each set $e$ in $S$ do
  for each set $e^c$ in $\binom{S \cup T}{0..m}$ do
    compute the partition ptn based on $e \cup e^c$; if ptn is NOT compatible with $e \cup e^c$, continue the loop; else if ptn is compatible with $\tau$, add ptn to result
  end for
end for
for for each $e$ in $T'$ do
  for for each nonsymmetric subset of $T^c$ do
    compute the partition ptn based on $e \cup e^c$ and its transpose; if ptn is NOT compatible with them, continue the loop; else if ptn is compatible with $\tau$, add ptn to result
  end for
end for
For each partition in result, find all commutative fission that lifts $\tau$.
```

Chapter 4

Association schemes of equipotent partitions

This chapter defines association schemes $\mathrm{E}_n(b)$ (resp. $\mathrm{U}_n(b)$) on the set of ordered (resp. unordered) equipotent partitions with parameters n and b. These schemes are shown to be noncommutative in general.

Notation 4.1. The following notations are fixed in this and next chapters.

$\mathbf{N}$	nonnegative integers
λ	a fixed equipotent partition
Λ	the set of integers from 1 to n
Ω	the set of ordered λ-partitions
Ξ	the set of unordered λ-partitions
$\mathfrak{m}$	the set of $\ell \times \ell$ matrices over $\mathbf{N}$ whose row/column sum is λ
R	$\{R_M \mid M \in \mathfrak{m}\}$
$\mathrm{E}_n(b)$	the association scheme from the action S_n on Ω
$\mathrm{U}_n(b)$	the association scheme from the action of S_n on Ξ

□

A sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a *partition* of n if $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $\sum_i \lambda_i = n$, and λ is *equipotent* if $\lambda_1 = \cdots = \lambda_\ell$.

Let Λ be the set of integers from 1 to n. An *ordered partition* α *of* Λ *of type* λ (or simply, λ-partition) is a list $(\alpha_1, \dots, \alpha_\ell)$ of subsets of Λ such that $|\alpha_i| = \lambda_i$ and $\cup_i \alpha_i = \Lambda$. Call α equipotent if λ is equipotent. In the rest of this chapter, λ is a fixed equipotent partition of n with all $\lambda_i = b$.

Let Ω be the set of ordered λ-partitions of Λ. The symmetric group S_n acts transitively on Ω. By Example 2.4, Ω together with all orbits of S_n on $\Omega \times \Omega$ forms an association scheme, denoted by $\mathrm{E}_n(b)$.

For two partitions α and β in Ω, define $\alpha.\beta$ to be the $\ell \times \ell$ matrix whose (i,j)-entry is $|\alpha_i \cap \beta_j|$, where $\alpha = (\alpha_1, \cdots, \alpha_\ell)$ and $\beta = (\beta_1, \cdots, \beta_\ell)$. For any permutation f in S_n, $f(\alpha).f(\beta) = \alpha.\beta$. Conversely, for any two pairs (α, β) and (α', β') in $\Omega \times \Omega$ with $\alpha.\beta = \alpha'.\beta'$, form two arrays A and A' with ℓ rows and ℓ columns such that (i,j)-entry of A is the set $\alpha_i \cap \beta_j$ and (i,j)-entry of A' is the set $\alpha'_i \cap \beta'_j$. The sets $\alpha_i \cap \beta_j$ are disjoint and their union is Λ. For each pair (i,j), choose a permutation of S_n such that $f_{ij}(\alpha_i \cap \beta_j) = \alpha'_i \cap \beta'_j$ and $f_{ij}(x) = x$ for each $x \in \Omega \backslash (\alpha_i \cap \beta_j)$. In particular, if $\alpha_i \cap \beta_j$ is the empty set, f_{ij} is the identity of S_n. The permutation $f = \prod_{i,j} f_{ij}$ maps (α, β) to (α', β'). Therefore, (α, β) and (α', β') are in the same orbit if and only if $\alpha.\beta = \alpha'.\beta'$.

Denote by $\mathfrak{m}$ be the set of $\ell \times \ell$ matrices of nonnegative integer entries whose rows and columns sum to λ. Then $\mathfrak{m}$ is in one-to-one correspondence with the orbits of S_n on $\Omega \times \Omega$. For $M \in \mathfrak{m}$, set

$$R_M = \{(\alpha, \beta) \in \Omega \times \Omega \mid \alpha.\beta = M\}.$$

In this case, we say M parametrizes/indexes the relation R_M. The diagonal matrix bI indexes the diagonal relation R_0. A permutation relations is parametrized by a monomial matrix in $\mathfrak{m}$. (Recall a matrix is *monomial* if it has exactly one nonzero entry on each row and column.) Conversely, for each monomial matrix $M \in \mathfrak{m}$ and a partition $\alpha \in \Omega$, there is exactly one partition $\beta \in \Omega$ such that $(\alpha, \beta) \in R_M$. The relation R_M has valency one and hence a permutation relation.

Example 4.2. When $b = 1$, Ω consists of permutations of Λ. The relations of $\mathrm{E}_n(1)$ are parametrized by $\ell \times \ell$ permutation matrices. Moreover, $\mathrm{E}_n(1)$ isomorphic the permutation scheme $\mathbb{X}(S_n)$. Hence this scheme is not commutative when $n \geq 3$.

When $n = 2b$, $\mathrm{E}_{2b}(b)$ is the Johnson scheme $J_{2b}(b)$ and thus is commutative.

The following theorem states that taking transposes in R and in $\mathfrak{m}$ reconcile.

Theorem 4.3. $R_M^T = R_{M^T}$ for $M \in \mathfrak{m}$.

Proof. If $\alpha.\beta = M$, then $\beta.\alpha = M^T$. Since $R_M^T = \{(\beta, \alpha) \mid \alpha.\beta = M\} = \{\{(\beta, \alpha) \mid \beta.\alpha = M^T\}$, the proof is completed. ☐

Theorem 4.4. The scheme $\mathrm{E}_n(b)$ is not commutative if $n > 2b$.

Proof. Consider the permutation relations. The corresponding adjacency matrices form a group with matrix multiplication. This group is isomorphic to the symmetric group S_ℓ. Thus if $n > 2b$, then $\ell > 2$. Since S_ℓ is not commutative for $\ell > 2$, the adjacency algebra of $\mathrm{E}_n(b)$ is not commutative. ☐

Remark. The λ-partitions of Λ are called *tabloids* of type λ, or λ-tabloids in the theory of the symmetric group [19].

Forgetting the ordering within an ordered λ-partition results in an unordered partition. In other words, a λ-*unordered partition* of Ω is a set of disjoint b-sets of Ω whose union is Ω.

Let Ξ be the set of unordered λ-partitions of Λ. The symmetric group S_n acts transitively on Ξ and the S_n-orbits on $\Xi \times \Xi$ forms an association scheme. This scheme is denoted by $\mathrm{U}_n(b)$.

Example 4.5. The scheme $\mathrm{U}_9(3)$ is a symmetric 4-scheme. It was defined in [23] and was shown to have a fusion 2-scheme (a strongly regular graph). This fusion also has a fission 3-scheme.

The scheme $\mathrm{U}_{10}(2)$ is a symmetric 6-scheme. It was defined in [27] and was shown to have only trivial fusion scheme. ☐

Theorem 4.6. If $b > 3$ and $n > 15$, then $\mathrm{U}_n(b)$ is not commutative.

Proof. [1] Fix a partition γ in Ξ. Let H be the stabilizer of γ in S_n and π be the permutation character of S_n on Ξ. By [2, Theorem 1.4, p49], it is sufficient to show that π is multiplicity-free.

Let π_k be permutation character of S_n on the set of k-subsets of Λ. Then the inner product of π_i and π_j is $(\pi_i, \pi_j)_{S_n} = i + 1\,(0 \leq i \leq j \leq n/2)$ by [2, Lemma 2.4, p.212]. Moreover, $\pi_i - \pi_{i-1}$ is an irreducible character of S_n by [2, Theorem 2.5, p.212]. Now $(\pi, \pi_3)_{S_n} = (1, \pi_3)_H$ by Frobenius Reciprocity. $(1, \pi_3)_H$ is the number of orbits of H on $\binom{\Lambda}{3}$, the 3-sets of Λ. Now we show how to compute the orbit number. For any 3-set , form the multiset $\gamma = \{|\cap \gamma_i| \mid 1 \leq i \leq \ell\}$. There are three possible multisets: $\{3, 0, \ldots, 0\}, \{2, 1, 0, \ldots, 0\}$ and $\{1, 1, 1, 0, \ldots, 0\}$. For any two 3-sets and of Λ, if $\gamma = \gamma$, then there is a permutation $f \in H$ such $f(s) = t$. Therefore, H has three orbits on $\binom{\Lambda}{3}$. Similarly, we can show there are 5 H-orbits on $\binom{\Lambda}{4}$. Therefore, $(\pi_4 - \pi_3, \pi) = 2$ and hence π is not multiplicity-free. $\square$

The scheme $\mathrm{U}_n(1)$ has only one vertex and hence it is trivial; $\mathrm{U}_n(2)$ is commutative [27, 28], and $\mathrm{U}_n(3)$ is commutative for $n = 3, 6, 9, 12$, and 15. It is shown that $\mathrm{U}_n(b)$ is a quotient scheme of $\mathrm{E}_n(b)$ in Section 6.1.

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[1] Part of proof is due to private communication with Dr. Jan Saxl

Chapter 5

Commutative fusion schemes in $\mathrm{E}_9(3)$

In this chapter, commutative fusion schemes in $\mathrm{E}_9(3)$ are enumerated with the methods outlined in Chapter 3.

5.1 The association scheme $\mathrm{E}_9(3)$

The scheme $\mathrm{E}_9(3)$ has vertices the 1680 ordered partitions of type $(3, 3, 3)$. Its relations are parametrized by the fifty-five 3×3 matrices of nonnegative integral entries whose row and column sum is 3.

The matrices in $\mathfrak{m}$ break into five groups according to entries counting multiplicity. The relations are grouped accordingly. The number of matrices in these groups are 6, 18, 12, 18 and 1. Representatives from each group are listed below:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad (5.5.1)$$

The relations from the same group have the same valency. The valencies of the relations indexed by matrices in (5.5.1) are 1, 9, 27, 54, and 216. There are 32 nonsymmetric matrices in $\mathfrak{m}$ and hence $\mathrm{E}_9(3)$ has 32 nonsymmetric relations by Theorem 4.3. By Theorem 4.4, $\mathrm{E}_9(3)$ is a noncommutative scheme of class 54.

A full list of matrices in $\mathfrak{m}$ is attached in the Appendix A and it determines an ordering that is used throughout this chapter. All fusion schemes are described by positions of these matrices.

5.2 Commutative fusion from permutation relations

As in Section 2.3, $\mathfrak{p}$ is the set of permutation relations in $\mathrm{E}_9(3)$. By Theorem 2.10, the orbits of $\mathfrak{p}$ on R yields a fusion scheme. This fusion scheme happens to be symmetric with 16 classes, denoted by $\mathrm{E}_9(3)^{\mathfrak{p}}$. Its basic relations are associated with the following partition:

$$[1], [2,6], [3,4,5], [7,9,10,19,20,23], [8,14,15,17,22,24], [11,13,16],$$
$$[12,18,21], [25,26], [27,29,31,33,35,36], [28,30], [32,34], [37,38,46],$$
$$[41,43,47], [39,44,48,50,53,52], [40,42,45,49,51,54], [55].$$

All fusions in $\mathrm{E}_9(3)^{\mathfrak{p}}$ are found using principles 3.9 and 3.11: the set of integers from i and j is abbreviated as $[i..j]$.

(i) $[1], [2,6], [3..5] \cup [7..55]$

(ii) $[1], [2..6], [7..55]$

(iii) $[1], [2,6], [3..5], [7..55]$

(iv) $[1], [2..6], [7..36] \cup [55], [37..54]$

(v) $[1], [2..6] \cup [55], [7..24], [25..36], [37..54]$

(vi) $[1],[2..6],[7..24],[25..36],[37..54],[55]$

(vii) $[1],[2,6], [3,4,5],[7,9,10,11,13,16,19,20,23], [8,12,14,15,17,18,21,22,24],[25,26,28,30,32,34], [27,29,31,33,35,36],[37,38,40,42,45,46,49,51,54], [39,41,43,44,47,48,50,52,53], [55]$

Notice that (vi) and (iii) are fusions of (vii), (v) and (iv) are fusions of (vi), and (i) and (ii) are fusions of (iii).

5.3 Intersection matrices modulo p

As outlined in Section 3.4, the first step towards finding commutative fusions is to seek a $\mathfrak{D}$-homomorphism obtained by applying modulo p reduction to the intersection algebra for some positive integer p. As pointed out in Section 3.4, one seeks the primes that divides some but not all valencies of $E_9(3)$. Since $E_9(3)$ has valencies 1, 9, 29, 54, and 216, $p = 2$, 3 and 7. It turns out that $p = 2$ works better.

Let B_i be the intersection matrices of $E_9(3)$. For each matrix B_i, write $\bar{B}_i$ for the matrix of B_i modulo 2. Then $\bar{B}_i$ has a nice block structure. Each matrix can be partitioned into 3×3 blocks:

$$\left[\begin{array}{c|c|c} 36 & & \\ \hline & 18 & \\ \hline & & 1 \end{array}\right],$$

where the lines indicates the partition of the matrix and the diagonal entries indicate block sizes. Let us agree to write $[C]_{ij}$ for (i, j)-block of C. The blocks have the following pattern:

(I1) For $1 \le i \le 36$, the $(1,1)$-blocks $[\bar{B}_i]_{11}$ and $(2,2)$-blocks $[\bar{B}_i]_{22}$ are permutation matrices for $1 \le i \le 36$; for $37 \le i \le 55$, these blocks are the zero matrix. Moreover, the Hadamard product $[\bar{B}_i]_{11}$ and $[\bar{B}_j]_{11}$ is the zero matrix for $i \ne j$.

(I2) For $1 \le i \le 55$, the $(2,1)$-blocks $[\bar{B}_i]_{21}$ and $(3,1)$-blocks $[\bar{B}_i]_{31}$ are the zero matrix.

(I3) For $0 \le i \le 36$, $[\bar{B}_{i^*}]_{11}$ is the inverse of $[\bar{B}_i]_{11}$, where i^* is defined by $R_{i^*} = R_i^T$.

Write $\tilde{B}_i$ for $[B_i]_{11}$, and $\mathfrak{B}$ for the set matrices $\tilde{B}_i (i = 1, \dots, 36)$.

Theorem 5.1. Let $\mathbf{Z}_2$ be the field of two elements.

(i) The set $\mathfrak{B}$ forms a group under the usual matrix multiplication.

(ii) $\mathbf{Z}_2\mathfrak{B}$ is a $\mathfrak{D}$-algebra under the usual matrix and Hadamard multiplications, and the unary operation being taking inverse.

(iii) The map $B_i \to \tilde{B}_i$ defines a $\mathfrak{D}$-homomorphism from $\mathrm{Int}_{\mathbf{Z}}(\mathrm{E}_9(3))$ to $\mathbf{Z}_2\mathfrak{B}$.

Proof. Let ^ be the mod 2 map from $\mathbf{Z}$ to $\mathbf{Z}_2$. Since $B_1 = I$, $\tilde{B}_1$ is the identity of $\tilde{\mathfrak{B}}$. From (I1), $\tilde{B}_i$ is a permutation matrix for $i = 1, \dots, 36$ and is the zero matrix for $i \geq 37$. Therefore, for $1 \leq i, j \leq 36$, the product $\tilde{B}_i\tilde{B}_j$ is also a permutation matrix. Since $B_iB_j = \sum_k p_{ij}^k B_k$, there is precisely one k with $\widehat{p_{ij}^k} = 1$ and hence $\tilde{B}_i\tilde{B}_j = \tilde{B}_k$ for some $k \leq 36$.

Since $\tilde{\mathfrak{B}}$ is finite and closed under matrix multiplication, $\tilde{\mathfrak{B}}$ forms group. Assertions (i) and (ii) follows.

Let $\phi : B_i \to \tilde{B}_i$. Since ^ is a ring homomorphism, it implies that ϕ satisfies (i), (ii), (iii) and (v) of Definition 3.4. Since the valency ν_i takes values 1, 9 and 27 for $i \leq 36$, $\phi(\nu_i B_1) = \tilde{B}_1$. Also $\phi(B_iB_{i^*}) = \tilde{B}_i\tilde{B}_{i^*} = \tilde{B}_k$ for some $k \leq 36$ since $\tilde{B}$ is a group. This forces that $\phi(B_iB_{i^*}) = \tilde{B}_1$, since ν_i is odd and

$$\phi(B_iB_{i^*}) = \phi(\nu_i B_1) + \phi(\sum_{k=2}^{55} p_{ij}^k B_k).$$

Therefore, $\phi(B_{i^*}) = \tilde{B}_i^{-1}$. Thus ϕ satisfies (vi) of Definition 3.4 by examples 3.3 and 3.6. Assertion (iii) follows. ☐

For brevity, write i for $\tilde{B}_i$ in this paragraph. Direct calculation shows that $\mathfrak{B}$ has 9 conjugacy classes. The elements of order 3 fall into 3 classes: $\{28, 30\}, \{32, 34\}$ and $\{2, 6, 25, 26\}$. The order 2 elements fall into 3 classes:

$\{11,13,16\},\{12,18,21\}$ and $\{3,4,5,29,31,33,35,36\}$. The sets $\{1\}\cup\{28,30\}\cup\{11,13,16\}$ and $\{1\}\cup\{32,34\}\cup\{12,18,21\}$ form two normal subgroups of $\mathcal{B}$ and both are isomorphic to S_3. The group $\mathcal{B}$ is checked to be the product of these two normal subgroups. Thus $\mathcal{B}$ is isomorphic to $S_3 \times S_3$. The multiplication table of $\mathcal{B}$ is given below. The various "–"s serve dual purposes: transpose in the index matrices $\mathfrak{m}$ and inverse in $\mathcal{B}$. For example, "$2-6$" shows that the second matrix in $\mathfrak{m}$ has transpose the sixth matrix and $\tilde{B}_6$ is the inverse of $\tilde{B}_2$ in $\mathcal{B}$.

$$\begin{bmatrix} 1 & 28 & - & 30 & 13 & 16 & 11 \\ 32 & 6 & & 25 & 19 & 7 & 23 \\ | & & \times & & | & | & | \\ 34 & 26 & & 2 & 20 & 10 & 9 \\ 12 & 15 & - & 17 & 3 & 31 & 29 \\ 21 & 8 & - & 14 & 35 & 4 & 27 \\ 18 & 22 & - & 24 & 33 & 36 & 5 \end{bmatrix} \tag{5.5.2}$$

This multiplication table reveals an interesting pattern: one of factor S_3 of $\widetilde{\mathcal{B}}$ is first row. It is indexed by the following matrices in $\mathfrak{m}$.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The first matrix indexes the identity of $\mathcal{B}$. The second and third have 2's on the main diagonal, which parametrize the order 3 elements in the first row in (5.5.2). The remaining three matrices has one 3 and two 2's on the main diagonal and parametrize the order 2 elements. The other factor S_3 of $\mathcal{B}$ is first column of the table. The following matrices in $\mathfrak{m}$ parametrize the first column.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Notice that the similar pattern occurs but with 1 and 2 switching roles.

Suppose that $\tau = \{T_0 = \{1\}, \ldots, T_r\}$ is a partition of [1..55] that produces a commutative fusion scheme of $\mathrm{E}_9(3)$. Then $\left\{\sum_{j \in T_i} \tilde{B}_j \mid 1 \leq i \leq r\right\}$ is a basis for a commutative $\mathfrak{D}$-algebra of $\mathbf{Z}_2\mathfrak{B}$ by discarding the zero matrix. This basis produces a partitions of [1..36].

The search for $\mathfrak{D}$-subalgebras in $\mathbf{Z}_2\mathfrak{B}$ leads to the 2-modular representation of $\mathbf{F}\mathfrak{B}$. By Section 3.4, $\mathbf{Z}_2\mathfrak{B}$ can be written as a sum of two-sided ideals. The next theorem determines the two-sided ideals and its proof is postponed to the end of this chapter.

Theorem 5.2. $\mathbf{Z}_2\mathfrak{B}$ is isomorphic to a direct sum of four matrix algebras:

$$\mathbf{Z}_2\mathfrak{B} \approx \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3 \oplus \mathscr{A}_4$$

with $\mathscr{A}_1$ is the first matrix algebra, $\mathscr{A}_2$ and $\mathscr{A}_3$ the second, and $\mathscr{A}_4$ the third of following forms

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & x_1 & 0 & x_3 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & x_1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_5 & x_6 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix}.$$

□

Fix an isomorphism Ψ from the direct sum of matrix algebras in Theorem 5.2 onto $\mathbf{Z}_2\mathfrak{B}$. In general, Ψ is not necessarily a $\mathfrak{D}$-isomorphism.

Our search for commutative $\mathfrak{D}$-algebras in $\mathbf{Z}_2\mathfrak{B}$ consists of two steps. First, search all commutative algebras in the direct sum of matrix algebra in Theorem 5.2. Second, apply Ψ to any algebra from the first step and then check if it is a $\mathfrak{D}$-algebra in $\mathbf{Z}_2\mathfrak{B}$. These two steps actually intertwine in order to search effectively (see Sections 5.4 and 5.5).

The block idempotents come from the elements of order 3 in $\mathfrak{B}$ (the 2-regular elements).

Lemma 5.3. Let $H = Sym(\{a, b, c\}) \times Sym(\{1, 2, 3\})$ with identity e, where $Sym(\{a, b, c\})$ is the symmetric group on the set $(\{a, b, c\})$. Then the following are the four block idempotents of $\mathbf{Z}_2H$:

$$\begin{aligned} e_1 &= [e + (abc) + (acb)][e + (123) + (132)] \\ e_2 &= [e + (abc) + (acb)][(123) + (132)] \\ e_3 &= [e + (123) + (132)][(abc) + (acb)] \\ e_4 &= [(123) + (132)][(abc) + (acb)] \end{aligned}$$

$\square$

Corollary 5.4. In the context of lemma 5.3, we have

$$e_1 + e_2 + e_3 + e_4 = 1, e_1 \circ e_i = e_i,\ e_2 \circ e_3 = e_4,\ \text{and } e_4 \circ e_i = e_4\ (1 \leq i \leq 4).$$

$\square$

The commutative $\mathfrak{D}$-subalgebras of $\mathbf{Z}_2\mathfrak{B}$ can be divided into non-overlapping families according to the block idempotents they contain. By Corollary 5.4, there are four families:

(C1) All block idempotents occur.

(C2) Exactly two block idempotents occur, From Corollary 5.4, only the following pairs can occur: $\{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}$.

(C3) Exactly one block idempotent occur.

(C4) None of block idempotents occurs.

Any algebra in (C1) is called *totally split*, while any algebra in (C4) is *totally nonsplit*.

There are two complementing approaches in finding the commutative $\mathfrak{D}$-algebras in $\mathbf{F}\mathfrak{B}$. Let $\mathcal{S}$ be such an algebra. First, if the block idempotent e_i occur in $\mathcal{S}$, $\Psi^{-1}(\mathcal{S})$ contains a subalgebra from $\mathscr{A}_i$. This allows to build $\mathfrak{D}$-algebras in

$\mathbf{F}\mathfrak{B}$ from subalgebras in $\mathscr{A}_i$. Second, if the idempotents e_i and e_j do not occur, then the following lemma put a restriction on the minimal polynomials of the i^{th} and j^{th} component of any matrix in $\Psi^{-1}(\mathcal{S})$.

Lemma 5.5. Let A and B be two square matrices with minimal polynomials $\mu(x)$ and $\nu(x)$. If $\mu(x)$ and $\nu(x)$ are relative prime to each other, then the block diagonal matrices $\operatorname{diag}(I, O)$ and $\operatorname{diag}(O, I)$ are polynomials of $\operatorname{diag}(A, B)$, where I is the identity matrix and O is the zero matrix.

All commutative $\mathfrak{D}$-subalgebras in (C1) and (C4) are found in Sections 5.4 and 5.5. These algebras are lifted using the algorithm on page 19.

In searching for commutative $\mathfrak{D}$-subalgebras of $\mathbf{Z}_2\mathfrak{B}$, Principle 3.8 is used frequently: for two 4-tuple matrices m and n, if $\Psi(m)$ and $\Psi(n)$ are in a $\mathfrak{D}$-algebra, they must commute with the the Hadamard shadow $\Psi(m) \circ \Psi(n)$. This is also referred as the *Hadamard test.*

5.4 Totally split $\mathfrak{D}$-algebras

In this section, we determine all totally split commutative $\mathfrak{D}$-algebras in $\mathbf{F}\mathfrak{B}$. Since all block idempotents are present, this allows us to build these algebra from subalgebras in each matrix algebra $\mathscr{A}_i$.

In the following, we adopt the convention that a zero block is omitted in writing a matrix. The identity matrix and the zero matrix are written as I and O.

5.4.1 Algebras in $\mathscr{A}_1$

The algebra $\mathscr{A}_1$ is commutative and consists of matrices of the form:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ & x_1 & 0 & x_3 \\ & & x_1 & x_2 \\ & & & x_1 \end{bmatrix}.$$

All matrices in $\mathscr{A}_1$ have minimal polynomials of degree 1 or 2. We enumerate its subalgebras by their dimensions. Since any subalgebra contains the identity matrix, any non-identity basis matrix can be normalized to to have all 0 on its main diagonal.

The algebras of dimension 1 and 4 are trivial, namely $\{O, I\}$ and $\mathscr{A}_1$.

In any 2-dimensional algebra, each non-identity basis element has minimal polynomial of degree 2. It is easy to check each of seven nonzero matrix with diagonal 0 generates a 2-dimension subalgebra. There are 7 such subalgebras.

Any subalgebra of dimension 3 is generate by a pair of elements whose minimal polynomial have degree 2. Direct computation find 3 subalgebras of dimension 3.

In total, there are 12 subalgebras in the first block. All of them pass the Hadamard test.

5.4.2 Algebras in $\mathscr{A}_2$ and $\mathscr{A}_3$

The algebras $\mathscr{A}_2$ and $\mathscr{A}_3$ is of dimension 8 and consists of matrices of the form:

$$\left[\begin{array}{cc|cc} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ \hline & & x_1 & x_2 \\ & & x_5 & x_6 \end{array}\right].$$

This algebra is noncommutative of dimension 8. Any matrix can be partitioned into blocks as indicated above.

There is only one algebra of dimension 1. Consider all commutative algebras of dimension > 1 in $\mathscr{A}_2$.

For any algebra $\mathcal{S}$ in $\mathscr{A}_2$, Let $\mathcal{S}_D$ denote the projection of $\mathcal{S}$ onto the diagonal blocks. Then $\dim(S_D) = 1$, or 2.

If $\dim(\mathcal{S}_D) = 1$, then $\mathcal{S}$ is necessarily commutative. Any matrix in S has the form

$$\begin{bmatrix} a & 0 & b & c \\ & a & d & e \\ & & a & 0 \\ & & & a \end{bmatrix}.$$

Identify the first two rows of the matrix above with a point in PG(3, 2), the three dimensional projective space over $\mathbf{Z}_2$. The algebras of dimensions 2, 3, 4 and 5 correspond to points, lines and planes in PG(3, 2). Therefore, there are 15 algebras of dimension 2, 35 algebras of dimension 3, 15 algebras of dimension 4 and one algebra of dimension 5.

Now suppose $\dim(\mathcal{S}_D) = 2$. Then $\dim(\mathcal{S}) = 2, 3$, or 4.

(i) If $\dim(\mathcal{S}) = 2$, then $\mathcal{S}$ has a generator of the form

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix},$$

where $A \neq O, I$. Since $\mathcal{S}$ contains the identity matrix, C can be normalized so that $A_{11} = 0$. Moreover, since $\dim(\mathcal{S}) = 2$,

$$C^2 = \begin{bmatrix} A^2 & AB + BA \\ & A^2 \end{bmatrix}$$

is linear combination of I and C. Direct calculation finds 43 such algebras.

(ii) If $\dim(\mathcal{S}) = 3$, then $\mathcal{S}$ has a basis:

$$\begin{bmatrix} I & \\ & I \end{bmatrix}, \begin{bmatrix} A & B \\ & A \end{bmatrix}, \begin{bmatrix} O & C \\ & O \end{bmatrix},$$

where $A_{11} = 0, A \neq O$ and $C \neq O$. Of course, the submatrices A, B and C satisfy certain equations. Direct calculation finds 36 such algebras.

(iii) If $\dim(\mathcal{S}) = 4$, then $\mathcal{S}$ has a basis of the form:

$$\begin{bmatrix} I & \\ & I \end{bmatrix}, \begin{bmatrix} A & B \\ & A \end{bmatrix}, \begin{bmatrix} O & C \\ & O \end{bmatrix}, \begin{bmatrix} O & D \\ & O \end{bmatrix},$$

where $A_{11} = 0, A \neq O$, $C \neq O, D \neq 0$, and $C \neq D$. There are 28 commutative algebras.

In total, there are 174 commutative algebras and 32 of them pass the Hadamard test.

5.4.3 Commutative algebra in $\mathscr{A}_4$

Let $\mathcal{S}$ be a commutative algebra in $\mathscr{A}_4$. Suppose $A, B \in \mathcal{S}$. If $\Psi(0,0,0,A)$ and $\Psi(0,0,0,B)$ fail to commute with $\Psi(0,0,0,A) \circ \Psi(0,0,0,B)$, then $\mathcal{S}$ is said to fails the *Hadamard test*. In particular, if $\mathcal{S}$ is generated by a matrix A, then A is said to fail the Hadamard test.

If $\mathcal{S}$ fails the Hadamard test, then so do all algebras containing $\mathcal{S}$. This implies that one should focus matrices of minimal polynomials of lower degrees in $\mathscr{A}_4$.

Let us denote by $\mu_A(x)$ the minimal polynomial of matrix A. The following are pairs of minimal polynomials for A and $I + A$ for all $A \in \mathcal{A}_4$:

$$[x, 1+x], [x^2, 1+x^2], [x+x^2], [1+x+x^2], [x^3, 1+x+x^2+x^3],$$
$$[1+x^3, x+x^2+x^3], [x+x^3, x^2+x^3], [x^4, 1+x^4], [x+x^4],$$
$$[1+x+x^4], [x^2+x^4], [1+x^2+x^4], [x+x^2+x^4, 1+x+x^2+x^4],$$
$$[x^3+x^4, x+x^2+x^3+x^4], [1+x^3+x^4, 1+x+x^2+x^3+x^4],$$
$$[x+x^3+x^4, 1+x^2+x^3+x^4], [1+x+x^3+x^4, x^2+x^3+x^4].$$

Since every algebra contains the identity, we just need to consider one of each pair. Since x and $x+1$ are minimal polynomials of O and I, we just need to consider the polynomials of degree 2 and higher.

Let G be the general linear group $\mathrm{GL}(4, \mathbf{Z}_2)$. Matrices in the same G-orbit have the same minimal polynomials.

Consider x^2+x+1 first. Let $A \in \mathscr{A}_4$ with $\mu_A(x) = x^2+x+1$. As an element in G, A has order 3 and generates a subgroup of order 3. Direct computation shows that are 56 subgroups of order 3 in G and that none of these subgroups pass the Hadamard test. Since any subalgebra containing a matrix of minimal polynomial x^4+x+1, x^4+x^3+1 or $x^4+x^3+x^2+x+1$ has a matrix of minimal polynomial x^2+x+1, there is no need to consider these polynomials.

Consider x^2+x. Any matrix A with minimal polynomial x^2+x is *idempotent* since $A^2 = A$. These matrices fall into three G-orbits of sizes 240, 280 and 280, with representatives $\mathrm{diag}(1, 1, 0, 0)$, $\mathrm{diag}(1, 0, 0, 0)$, and $\mathrm{diag}(0, 1, 1, 1)$, where $\mathrm{diag}(a, b, c, d)$ is the diagonal matrix with entries a, b, c, and d.

The map $A \rightarrow A+I$ divides the matrices in the first orbits into 240/2=120 pairs. Computation shows that none of the pairs pass the Hadamard test.

The map $A \rightarrow A+I$ is bijective between the last two orbits. We consider only the second orbit. There are 280 matrices in this orbit, 36 of them pass the Hadamard test. Each of 36 matrices generates a two dimensional algebras. Let A be one of the 36 matrices. Compute its centralizer $\mathcal{C}$ in $\mathscr{A}_4$ and generate the set of algebras that properly contains $\langle A \rangle$, the algebra generate by A. These algebras are then further tested. As a result, we find a three-dimensional subalgebra that passes the Hadamard test. In summary, we have 37 subalgebra candidates.

Consider matrices of minimal polynomial x^2. Any such matrix A is *nilpotent* since $A^2 = O$. These matrices fall into two orbits with representatives

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

Using the same procedure in the above paragraph, we find 153 two dimensional subalgebras from these matrices that pass the Hadamard test. For each subalgebra, no algebras that properly contain it pass the Hadamard test.

No matrices with the rest minimal polynomials can pass the Hadamard test. We summarize the results above in the following theorem.

Theorem 5.6. $\mathscr{A}_4$ has the following commutative subalgebras that pass the Hadamard test: (i) 36 two-dimensional subalgebras each of which is generated by an idempotent matrix; (ii) 153 two-dimensional subalgebras each of which is generated by a nilpotent matrix. (iii) one subalgebra of dimension 3.

When lifted to the the original algebra, the subalgebras in the above produced one fusion scheme. It happens to be the fusion scheme found in Section 5.2.

5.5 Totally nonsplit $\mathfrak{D}$-algebras

Two methods are described for searching nonsplit $\mathfrak{D}$-subalgebras. The first is for the purpose of experiment. It searches those satisfying the hypothesis 5.7 below, using a backtracking algorithm. The second deals with the general case. It organizes the search with a graph model. This graph is constructed on the 4-tuples of minimal polynomials of matrices in Theorem 5.2.

5.5.1 Algebras containing a single element

In this section, we search for $\mathfrak{D}$-algebras in $\mathbf{Z}_2\mathfrak{B}$ under the following assumption:

Hypothesis 5.7. Assume each totally nonsplit $\mathfrak{D}-$algebra in discussion is symmetric and contains an element of $\mathfrak{B}$. □

The elements in $\mathcal{B}$ we choose is $s = \tilde{B}_{13}$ or $\tilde{B}_4$. For any $\mathfrak{D}$-algebra $\mathcal{S}$ containing s, each basis element of $\mathcal{S}$ is a union of conjugate classes by s.

A backtracking algorithm is employed to find all $\mathfrak{D}$-algebras containing s satisfying the hypothesis 5.7. It algorithm is implemented as a parallel algorithm, consisting of a master process and 32 slave processes.

Our computing facility consists a 32 1GHz dual-processor loosely connected machine with 1GB memory on each, running Linux operating system. A node is a dual process with local memory.

One node is designated as the master node and the master process runs on one of its processors. One slave process runs on the other processor of the master node and the rest 31 processes runs the remaining 31 node each.

Each solution is an unordered partition τ of $\mathcal{B}$ that supports a $\mathfrak{D}$-algebra. Let Λ be the partition of $\mathcal{B}$ from the symmetrization of the conjugacy classes by s. Since $s = s^{-1}$, $\{s\} \in \Lambda$ and let $\Lambda_0 = \{B_0\}$ and $\Lambda_1 = \{s\}$.

Start with the partial partition $\tau = \{\Lambda_0, \Lambda_1\}$. The next step produces a set of partial partitions that extend τ. Let τ^c be the partial partition obtained from Λ by removing the subsets in τ. Identify each subset of τ^c with its characteristic vector v. The master process distributes all vectors v with $v[1] = 1$ evenly to the 32 slave processes. The slave tests if the subset corresponding to v can be added to τ. The above process is repeated for each new partial partition.

Here is how each the slave process i tests if a given subset v can be added to τ. If v commutes with elements in τ in $\mathbf{Z}_2\mathcal{B}$, then v is added to τ.

A partition produced this way does not necessarily support a $\mathfrak{D}$-algebra. Another procedure using Theorem 2.9 is applied to find these partition that support a $\mathfrak{D}$-algebra.

The algorithm 2 generates all partitions of $\mathcal{B}$ satisfying Hypothesis 5.7. It is implemented in Parallel GAP [8].

Algorithm 2 Generating algebras with backtracking

```
$S_1 := [\{s\}]$
$k := 1$
while $k > 0$ do
  while $S_1 \neq \emptyset$ do
    $a_k$ = the next element from $S_k$ {*advance*}
    $min$= the smallest entries in the remaining entities
    if $v$ is a partition of [1..36] then
      report it
    end if
    $S_k := S_k - \{a_k\}$
    master assigns vectors to the 32 slave processes and waiting for the result.
    slave process extends the current partition by add one more subset.
    $k := k + 1$
    update $S_k$ based the results returned by slaves.
  end while
  $k := k - 1$ {*backtracking*}
end while
```

We find 26277 symmetric commutative $\mathfrak{D}$-algebras which contains the element $\tilde{B}_4$. Lifting these algebra leads two commutative fusion schemes with partitions:

(i) $[1], [2,6], [3,5], [4], [7..24], [25..36], [37..54], [55]$.

(ii) $[1], [2,3,5,6], [4], [7..24], [25..36], [37..54], [55]$.

Note (ii) is a fusion of (i). Further (i) has two more fusions:

(iii) $[1], [2,6], [3,4], [4], [7..24, 55], [25..36], [37..54]$.

(iv) $[1], [2,3,5,6], [4], [7..24, 55], [25..36], [37..54]$.

We find there are 6207 symmetric commutative partition algebras which contains the element $\tilde{B}_{13}$, but none of them leads to commutative fusion schemes.

5.5.2 Totally nonsplit algebras: general case

Let $m = (m_1, m_2, m_3, m_4)$ be a matrix tuple from the representation of $\mathcal{B}$. We denote by $\mu_i(x)$ the minimal polynomial of m_i and $\bar{\mu}_i$ the gcd of minimal polynomials of the rest three matrices. By Lemma 5.5, we have

$$\gcd(\mu_i(x), \bar{\mu}_i(x)) \neq 1 \tag{5.5.3}$$

We generate all possible 4-tuples of minimal polynomials that satisfy the restriction (5.5.3). Let us denote the set of these 4-tuples by $\mathcal{P}$. We define a directed graph on $\mathcal{P}$: for $f, g \in \mathcal{P}$, define $(f, g) \in \mathcal{E}$ if the presence of f implies that of g. We use $\mathcal{P}$ again to denote this graph. We contract this graph on its cycles. For any cycle $f_1, f_2, \ldots, f_n$, that is, a sequence that satisfies $(f_n, f_1), (f_i, f_{i+1}) \in \mathcal{E}$ for $1 \leq i \leq n-1$, we contract the graph by replacing all $(f, f_i)(2 \leq i \leq n) \in \mathcal{E}$ with (f, f_1), all $(f_i, g)(2 \leq i \leq n) \in \mathcal{E}$ with (f_1, g), then remove all $f_2, \ldots, f_n$. As a result of reduction process, we have a cycle-free graph, called it again $\mathcal{P}$.

For each $g \in \mathcal{B}$, the map $x \mapsto g^{-1}xg$ is an inner automorphism. The inner automorphisms of $\mathcal{B}$ forms a group, denoted by $\mathrm{Inn}(\mathcal{B})$. An inner automorphism maps $\mathfrak{D}$-subalgebras to $\mathfrak{D}$-subalgebras.

For every node $r = (r_1, r_2, r_3, r_4)$ in $\mathcal{P}$, we could find all 4-tuples of matrices with minimal polynomials r. Divide these matrices into orbits of $\mathrm{Inn}(\mathcal{B})$. Choose one element from each orbit. For each element, find all $\mathfrak{D}$-subalgebras containing it.

It turns out that computation above is very expensive and slow. For any $i\,(i = 1, 2, 3, 4)$, collect all polynomial in the i-th component of nodes in $\mathcal{P}$. For each polynomial p, form a pair $[p, M_i(p)]$, where $M_i(p)$ consists matrices in $\mathscr{A}_i$ of minimal polynomial p. For $i = 4$, partition $M_4(p)$ into $\mathrm{Inn}(\mathcal{B})$-orbits. Choose one element from each orbit and form a set, called it again $M_r(p)$.

For each node $r = (r_1, r_2, r_3, r_4)$ in $\mathcal{P}$, form the Cartesian product

$$M(r) = M_1(r_1) \times M_2(r_2) \times M_3(r_3) \times M_4(r_4).$$

For each element in $M(r)$, find all commutative $\mathfrak{D}$-subalgebras containing it as a basis element. We find 6715 commutative $\mathfrak{D}$-subalgebras of dimensions are 3, 4 and 5. These algebras are listed in Appendix B. Applying Inn($\mathcal{B}$), we would find all totally nonsplit commutative $\mathfrak{D}$-subalgebras.

Remark. One can further reduce the above computation with a bottom-up approach if he needs. Apply the procedure in the above paragraph to the leaf nodes and delete them. (A node is a leaf if it has no children.) If nothing comes out from a leaf, then one can remove all its parent nodes. Then repeat this for the new graph till it is null.

5.6 Proof of Theorem 5.2

The terms and notation is referred to James and Kerber [19] or Curtis and Reiner [9] if they are not defined.

Let p be a prime number and G be a finite group. An element $x \in G$ is called p-regular if p divides the order of x. A conjugacy class of G is call p-regular if it contains p-regular elements. A p-modular irreducible representation of G is call a p-block. .

The following theorem can be found in any representation book (for instance, see [9, Theorem 55.2]).

Theorem 5.8. Let K be field and G be a finite group. Then the group algebra KG isomorphic to a direct sum of blocks.

Since the representation of a direct product of groups is the tensor product of representations of the factor group, we consider the 2-blocks of S_3.

From now on p=2. Since $\mathbf{Z}_p$ is a split field of S_3, there are just as many p-modular irreducible representation of S_3 as p-regular classes of S_3. The three ordinary representations of S_3 are the trivial, sign, and the degree 2 representations. These are represented by [3], $[1^3]$ and [2, 1]. The p-modular decomposition matrix of S_3 is given in [19, Theorem 6.1.43]:

$$\begin{array}{c} [3] \\ [1^3] \\ [2,1] \end{array} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the power of p dividing $|S_3|(=6)$ is the same as that of p dividing the degree of [2, 1], [2, 1] remain to be 2-modularly irreducible representation by Theorem [19, 6.1.18, p.244], denoted by K_2. The other p-block of S_3 comes from the quotient group S_3/A_3. We write $S_3/A_3 = \{1, e\}$ with $e^2 = 1$. The left regular (ordinary) representation of C_2 is equivalent to the representation defined by

$$e \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Reduce the matrix entries modulo p is p-modular representation, denoted by K_1. Now one can check there is no 2×2 matrix P over $\mathbf{Z}_2$ such that

$$P \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

Therefore, K_1 is a p-modularly irreducible representation of C_2. Lifting to S_3, K_1 is a p-modularly irreducible representation of of S_3. One can further check that K_1 and K_2 are inequivalent p-modularly irreducible representation of of S_3.

The p-blocks of S_3 are given in terms of generators of S_3:

$$K_1 : \begin{array}{c|c} (123) & (12) \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}, \qquad K_2 : \begin{array}{c|c} (123) & (12) \\ \hline \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}.$$

$S_3 \times S_3$ has four p-regular classes and hence it is has four p-blocks. The tensor products of $\{K_1, K_2\}$ with itself gives four inequivalent p-blocks of $S_3 \times S_3$. This completes the proof of Theorem 5.2.

Chapter 6

General association schemes

This chapter discusses general association schemes, including primitivity and imprimitivity, subschemes and quotient schemes, and automorphism of various levels. The scheme $\mathrm{E}_n(b)$ is studies in this context.

6.1 Primitivity and imprimitivity

Let $\mathbb{X} = (X, \mathsf{R} = \{R_0, \ldots, R_d\})$ be an association scheme. The scheme $\mathbb{X}$ is *imprimitive* if some union of relations of $\mathbb{X}$ is an equivalence relation on X distinct form R_0 and $X \times X$, and $\mathbb{X}$ is *primitive* otherwise.

Example 6.1. Let $X = D_8 = \{a^i b^j \mid a^4 = b^2 = abab = e\}$, the dihedral group of order 8 with identity e. The conjugacy classes of D_8 are

$$C_0 = \{e\}, C_1 = \{a^2\}, C_2 := \{a, a^3\}, C_3 = \{b, a^2 b\}, \text{ and } C_4 = \{ab, a^3 b\}.$$

For $0 \leq i \leq 4$, define

$$R_i = \{(x, y) \in X \times X \mid x^{-1} y \in C_i\}.$$

Direct calculation shows that $(X, \{R_i\}_{0 \le i \le 4})$ is a commutative 4-scheme. Make a relation table with (x, y)-entry i with $(x, y) \in R_i$:

	e	a^2	a	a^3	b	a^2b	ab	a^3b
e	0	1	2	2	3	3	4	4
a^2	1	0	2	2	3	3	4	4
a	2	2	0	1	4	4	3	3
a^3	2	2	1	0	4	4	3	3
b	3	3	4	4	0	1	2	2
a^2b	3	3	4	4	1	0	2	2
ab	4	4	3	3	2	2	0	1
a^3b	4	4	3	3	2	2	1	0

(6.6.1)

Notice that the lines partition this table into 2×2 blocks and every block appears precisely once on each row and column. This is characteristic of an imprimitive scheme.

One may also notice that $R_0 \cup R_1$ is an equivalence relation on X and hence this scheme is imprimitive. The equivalence classes are

$$\Sigma = \{\{e, a^2\}, \{a, a^3\}, \{b, a^2b\}, \{ab, a^3b\}\}.$$

Notice that Σ labels the block matrices in (6.6.1). We define an association scheme on Σ. Label the 4 block matrices in the first row of (6.6.1) from 0 to 3. For $(x, y) \in \Sigma$, define

$$(x, y) \in \tilde{R}_i \text{ if } (x, y) \text{ submatrix is the } i^{\text{th}} \text{ block.}$$

It can be checked that $(\Sigma, \{\tilde{R}_i\}_{i=0,1,2,3})$ is an association scheme. This is the quotient scheme with respect to the equivalence relation $R_0 \cup R_1$.

Of course $\{e, a^2\}$ is a normal subgroup N of D_8 and the equivalence classes are the cosets of N. The quotient scheme can be identified with the quotient group of D_8 by N by Example 2.5. ☐

Let $\mathbb{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an imprimitive scheme with intersection number p_{ij}^k and valencies ν_i. By rearranging indices, let $\cup_{i=0}^s R_i$ be an equivalent relation on X with Σ the set of equivalence classes.

For $x, y \in R_i$ and $S, T \in \Sigma$, set

$$\partial(x, y) = i \text{ and } \partial(S, T) = \{j \mid 0 \leq j \leq d,\ R_j \cap (S \times T) \neq \emptyset\}. \tag{6.6.2}$$

Now assume that $\mathbb{X}$ is commutative schemes, and define the subschemes and quotient schemes (see, e.g., [2, Section 2.9] for details).

On each $X' \in \Sigma$ we find an association scheme $(X', \{R_i'\}_{0 \leq i \leq s})$, called a *subscheme* of $\mathbb{X}$, where $R_i' = R_i \cap (X' \times X')$.

Define a relation $\sim$ on $\{0, 1, \ldots, d\}$ by

$$i \sim j \text{ if and only if } p_{j\alpha}^i \neq 0 \text{ for some } 0 \leq \alpha \leq s. \tag{6.6.3}$$

Then $\sim$ is an equivalence relation. Let $T_0 = \{0, 1, \ldots, s\}, T_1, \ldots, T_r$ be the equivalence classes. For each T_i, define a relation on Σ:

$$\tilde{R}_i = \{(\tilde{x}, \tilde{y}) \mid \text{ there exist } x \in \tilde{x} \text{ and } y \in \tilde{y} \text{ with } \partial(x, y) \in T_i\}. \tag{6.6.4}$$

Then $\tilde{\mathbb{X}} = (\Sigma, \{\tilde{R}_i\}_{0 \leq i \leq r})$ is an association scheme, called the quotient scheme of $\mathbb{X}$ with respect to $\cup_{i=0}^s R_i$. The parameters $\tilde{\mathbb{X}}$ can be expressed in terms of these of $\mathbb{X}$.

Now assume that $\mathbb{X}$ is a general scheme, not necessarily commutative. Its subschemes are defined in an analogous way, while (6.6.3) is not sufficient to define the quotient scheme. Example 6.1 suggests one can define the quotient scheme directly on the equivalence classes.

Lemma 6.2. Let Σ be the set of equivalence classes of $\cup_{i=0}^s R_i$ on X. Then

$$\{\partial(S, T) \mid S, T \in \Sigma\}$$

forms a partition of $\{0, 1, \ldots, d\}$.

Proof. Since $\cup_{S,T\in\Sigma}(S\times T) = X\times X$, $\cup_{S,T\in\Sigma}\partial(S\times T) = \{0,1,\dots,d\}$. It suffices to show that, for any $S,T,S',T'\in\Sigma$, the sets $\partial(S,T)$ and $\partial(S',T')$ are either equal or disjoint.

Suppose $i\in\partial(S,T)\cap\partial(S,T)$. Then there exist $(x,y)\in S\times T$ and $(x',y')\in S'\times T'$ with $\partial(x,y) = i = \partial(x',y')$. Suppose also $j\in\partial(S,T)$. Then exists $(z,w)\in S\times T$ with $j=\partial(z,w)$. Let $k=\partial(x,w)$ and $\alpha=\partial(w,y)$. Then $p^i_{k\alpha}\neq 0$. Therefore, there exists some $w'\in X$ with $\partial(x',w')=k$ and $\partial(w',y')=\alpha$. Since $\alpha\le s$, $w'\in T'$. Similarly, let $\beta=\partial(x,z)$. Then $p^k_{\beta j}\neq 0$ and there exist some $z'\in S'$ with $\partial(x',z')=\beta$ and $\partial(z',w')=j$. Therefore $j\in\partial(S',T')$. Hence $\partial(S,T)\subset\partial(S',T')$. By a symmetric argument $\partial(S',T')\subset\partial(S,T)$. Thus $\partial(S,T)=\partial(S',T')$. This completes the proof. □

Now Let $T_0=\{0,1,\dots,s\},T_1,\dots,T_r$ be the partition in the lemma above. Define

$$\tilde{R}_i = \{\partial(S,T)=T_i \mid S,T\in\Sigma\}.$$

Then $\widetilde{\mathbb{X}} = (\Sigma,\{\tilde{R}_i\}_{0\le i\le r})$ is an association scheme, the *quotient scheme* of $\mathbb{X}$ with respect to $\cup_{i=0}^{s}R_i$.

Now we discuss association schemes with nondiagonal permutation relations. This is motivated by the scheme $\mathrm{E}_n(b)$. Let $\mathbb{X}=(X,\{R_i\}_{0\le i\le d})$ be an association scheme. Again, denote by $\mathfrak{p}$ the set of permutation relations in $\mathbb{X}$. With loss of generality, let $\mathfrak{p}=\{R_0,R_1,\dots,R_s\}(s>0)$.

Lemma 6.3. The union $\cup_{i=0}^{s}R_i$ is an equivalence relation on X.

Proof. Let $\mathcal{E}=\cup_{i=0}^{s}R_i$. Since $R_0\subset\mathcal{E}$, $(x,x)\in\mathcal{E}$ for every $x\in X$. Since R_α^T is a permutation relation whenever R_α is a permutation relation, $(y,x)\in\mathcal{E}$ whenever $(x,y)\in\mathcal{E}$.

Suppose $(x, y) \in \mathcal{E}$ and $(y, z) \in \mathcal{E}$. Let $\alpha = \partial(x, y)$, $\beta = \partial(y, z)$ and $k = \partial(x, z)$. Since for each $i (0 \leq i \leq d)$, $\sum_{\ell=0}^{d} p_{\alpha\ell}^{i} = 1$, α and β uniquely determines k. Moreover, $p_{\alpha\beta}^{k} = 1$. Since $\nu_k p_{\alpha\beta}^{k} = \nu_\beta p_{\alpha^* k}^{\beta}$, $\nu_k = 1$. Therefore $(x, z) \in \mathcal{E}$. This completes the proof. □

The next theorem follows from Lemma 6.3 and the definition of imprimitivity.

Theorem 6.4. Let $\mathbb{X} = (X, \mathsf{R} = \{R_0, \ldots, R_d\})$ be an association scheme. Then $\mathbb{X}$ is imprimitive if $\mathsf{R} \gneq \mathfrak{p} \gneq \{R_0\}$. □

In the rest of this section, we determined the quotient scheme with respect to $\mathfrak{p}$. Let R_α be a permutation relation of $\mathbb{X}$. Since $\sum_{\ell=0}^{d} p_{\alpha\ell}^{k} = \nu_\alpha = 1$, $p_{\alpha\ell}^{k} \leq 1$. Therefore, for each $k\,(0 \leq k \leq d)$, there exists a unique a i with $p_{\alpha i}^{k} = 1$. Write $k = \alpha i$. Similarly, there exists a unique k with $p_{i\alpha}^{k} = 1$ and write $k = i\alpha$. Moreover, for any permutation relation R_β, $(\alpha i)\beta = \alpha(i\beta)$.

For every $i\,(0 \leq i \leq d)$, let

$$\tilde{i} = \{\alpha i \beta \mid 0 \leq \alpha, \beta \leq s\}.$$

In particular, $\tilde{0} = \{0, 1, \ldots, s\}$. One can show $\{\tilde{i} \mid 0 \leq i \leq d\}$ forms a partition of $\{0, 1, \ldots, d\}$ with argument analogous to proof of Lemma 6.3. Let $T_0 = \{0, 1, \ldots, s\}, T_1, \ldots, T_r$ be the partition of $\{0, 1, \ldots, d\}$.

Theorem 6.5. Suppose $\mathbb{X} = (X, \mathsf{R} = \{R_0, \ldots, R_d\})$ be an association scheme with $\mathfrak{p} = \{R_0, R_1, \ldots, R_s\} (s > 0)$ the set of permutation relations. Let $\tilde{X}$ be the set of equivalence class of $\cup_{i=0}^{s} R_i$ and Σ be any fixed equivalence class.

(i) $(X', \{R_i'\}_{\leq i \leq s})$ is a permutation scheme with $R_i' = R_i \cap (X' \times X')$.

(ii) $(\Sigma, \{\tilde{R}_i\}_{0 \leq i \leq r})$ is an association scheme with $\tilde{R}_i$ defined by (6.6.4). □

Example 6.6. Recall from Chapter 4, $\mathrm{E}_n(b)$ has $\ell!$ permutations. These relations are indexed by permutation matrices in $\mathfrak{m}$. Again, $\mathfrak{p}$ denotes the set of permutation relations in $\mathrm{E}_n(b)$. By abuse of notation, $\mathfrak{p}$ is also used to denote the union of relation in $\mathfrak{p}$. For each ordered partition $\alpha = (\alpha_1, \dots, \alpha_\ell)$, the equivalence class $\tilde{\alpha}$ of α consists of $\ell!$ partitions each of which is obtained by part permutation of α. Thus $\tilde{\alpha}$ can be identity with the unordered partition $\{\alpha_1, \dots, \alpha_\ell\}$.

Let M be any matrix in $\mathfrak{m}$. For each matrix in $\mathfrak{m}$ that indexes a permutation relation in $\mathfrak{p}$, there exists a unique matrix N in $\mathfrak{m}$ with $p_{P,M}^N = 1$. Moreover, N is the product $M(P/b)$. Similarly, $p_{M,P}^N = 1$ determines N uniquely with $N = (P^T/b)M$. Therefore,

$$\widetilde{M} = \{P^T M Q/b^2 \mid P, Q \in \mathfrak{m} \text{ index permutation relations in } \mathfrak{p}\}.$$

Suppose α, β are two unordered partitions in $\mathrm{U}_n(b)$. First fix an ordering for α and β. As ordered partitions, α and β determines the matrix $\alpha.\beta$. Let $M = \alpha.\beta$. Now a rearrangement of $\alpha_1, \dots, \alpha_\ell$ induces a row permutation of M. And a rearrangement of $\beta_1, \dots, \beta_\ell$ induces a column permutation of M. Therefore, the orbit of S_n in $\Xi \times \Xi$ containing (α, β) corresponds $\widetilde{M}$. Hence $\mathrm{U}_n(b)$ can be identified with the quotient scheme of $\mathrm{E}_n(b)$ with respect to $\mathfrak{p}$. ☐

Remark. Zieschang [31] discussed the primitivity, subschemes and quotient schemes for general association schemes in terms of *complex product.* Our treatment of Theorem 6.5 follows his approach.

6.2 Automorphisms

Suppose that $\mathbb{X} = (X, \mathsf{R})$ and $\mathbb{Y} = (Y, \mathsf{S})$ are association schemes. A map $\theta : X \to Y$ is call an *isomorphism* between $\mathbb{X}$ and $\mathbb{Y}$ if θ is bijective and θ

induces a bijective function from R onto S. In this case, $\mathbb{X}$ and $\mathbb{Y}$ are said to be isomorphic to each other. When $\mathbb{X} = \mathbb{Y}$, θ will be an *automorphism* of $\mathbb{X}$.

Let $\mathbb{X} = (X, \mathsf{R})$ be an association scheme with adjacency matrices A_i and adjacency algebra $\mathrm{Adj}(\mathbb{X})$. The set of automorphism of $\mathbb{X}$ forms a group under function composition, denoted by $\mathrm{Aut}(\mathbb{X})$. An automorphism θ of $\mathbb{X}$ will be an an *inner automorphism* of $\mathbb{X}$ if it fixes each relation of $\mathbb{X}$. The group of inner automorphisms of $\mathbb{X}$ is denote by $\mathrm{Inn}(\mathbb{X})$. Following group theory, an automorphism that is not an inner automorphism will be called an *outer* automorphism. An pseudo-automorphism of $\mathbb{X}$ is an automorphism of the adjacency algebra $\mathrm{Adj}(\mathbb{X})$ with respect to both ordinary and Hadamard product [17]. The group of pseudo-automorphisms is denoted by $\mathrm{Pseu}(\mathbb{X})$.

An outer automorphism (pseudo-automorphism) of $\mathbb{X}$ induces a permutation on R and an inner automorphism induces the identity permutation on R. Then $\mathrm{Inn}(\mathbb{X})$ is a normal subgroup of $\mathrm{Aut}(\mathbb{X})$; $\mathrm{Aut}(\mathbb{X})/\mathrm{Inn}(\mathbb{X})$ can be viewed as a subgroup $\mathrm{Pseu}(\mathbb{X})$.

Remark. A pseudo-automorphism of $\mathbb{X}$ is nothing but a $\mathfrak{D}$-automorphism of $\mathrm{Adj}(\mathbb{X})$.

Theorem 6.7. Let $\mathbb{X} = (X, \mathsf{R})$ be a scheme. Let G be a subgroup of $\mathrm{Aut}(\mathbb{X})$ or $\mathrm{Pseu}(\mathbb{X})$. Let $\Pi = \{\Pi_0 = \{R_0\}, \Pi_1, \dots, \Pi_r\}$ be orbits of G on R. The $(X, \{\bar{R}_i\}_{0 \le i \le f})$ with $\bar{R}_i = \cup_{R \in \Pi_i} R$ is a fusion scheme.

Proof. For any $\phi \in G$, it induces a permutation on R and thus can be regarded as a permutation on the index set $\{0, 1, \dots, d\}$. Since ϕ is an algebra isomorphism of the adjacency algebra of $\mathbb{X}$. $\phi(A_0) = A_0$. Thus $\Pi_0 = \{0\}$. For any $0 \le i, j, k \le d$, we have $p_{ij}^k = p_{i^\phi j^\phi}^{k^\phi}$ and hence $\phi(i^*) = \phi(i)^*$. Therefore, for any $0 \le \alpha, \beta, \gamma \le r$

and any $k, l \in \Pi_\gamma$,

$$\sum_{i\in\Pi_\alpha}\sum_{j\in\Pi_\beta} p_{ij}^k = \sum_{i\in\Pi_\alpha}\sum_{j\in\Pi_\beta} p_{ij}^l.$$

By theorem 2.9, $(X, \{\bar{R}_i\}_{0\leq i\leq f})$ is a fusion scheme. ☐

Suppose R_i is a permutation relation in $\mathbb{X}$. Then $x \longrightarrow \bar{x}$ defined by $(x, \bar{x}) \in R_i$ is a bijection on X. It induces an outer automorphism of $\mathbb{X}$. Let A_i be adjacency matrix of R_i. Then $x \longrightarrow \bar{x}$ induces an automorphism of $\mathrm{Adj}(\mathbb{X})$ $A \longrightarrow A_i^T A A_i$ for every $A \in \mathrm{Adj}(\mathbb{X})$.

Appendix A

List of indexing matrices of $E_9(3)$

$$\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\quad\begin{bmatrix}0&0&3\\3&0&0\\0&3&0\end{bmatrix}\quad\begin{bmatrix}3&0&0\\0&0&3\\0&3&0\end{bmatrix}\quad\begin{bmatrix}0&0&3\\0&3&0\\3&0&0\end{bmatrix}\quad\begin{bmatrix}0&3&0\\3&0&0\\0&0&3\end{bmatrix}\quad\begin{bmatrix}0&3&0\\0&0&3\\3&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&2&1\\0&1&2\\3&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&2\\0&2&1\\3&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&2\\3&0&0\\0&2&1\end{bmatrix}\quad\begin{bmatrix}0&0&3\\2&1&0\\1&2&0\end{bmatrix}\quad\begin{bmatrix}2&1&0\\1&2&0\\0&0&3\end{bmatrix}\quad\begin{bmatrix}3&0&0\\0&1&2\\0&2&1\end{bmatrix}$$
$$\begin{bmatrix}3&0&0\\0&2&1\\0&1&2\end{bmatrix}\quad\begin{bmatrix}0&0&3\\1&2&0\\2&1&0\end{bmatrix}\quad\begin{bmatrix}2&1&0\\0&0&3\\1&2&0\end{bmatrix}\quad\begin{bmatrix}2&0&1\\0&3&0\\1&0&2\end{bmatrix}\quad\begin{bmatrix}2&0&1\\1&0&2\\0&3&0\end{bmatrix}\quad\begin{bmatrix}1&2&0\\2&1&0\\0&0&3\end{bmatrix}$$
$$\begin{bmatrix}1&2&0\\0&0&3\\2&1&0\end{bmatrix}\quad\begin{bmatrix}1&0&2\\2&0&1\\0&3&0\end{bmatrix}\quad\begin{bmatrix}1&0&2\\0&3&0\\2&0&1\end{bmatrix}\quad\begin{bmatrix}0&3&0\\2&0&1\\1&0&2\end{bmatrix}\quad\begin{bmatrix}0&3&0\\1&0&2\\2&0&1\end{bmatrix}\quad\begin{bmatrix}0&2&1\\3&0&0\\0&1&2\end{bmatrix}$$

$$\begin{bmatrix}0&2&1\\1&0&2\\2&1&0\end{bmatrix}\quad\begin{bmatrix}0&1&2\\2&0&1\\1&2&0\end{bmatrix}\quad\begin{bmatrix}0&1&2\\1&2&0\\2&0&1\end{bmatrix}\quad\begin{bmatrix}2&1&0\\0&2&1\\1&0&2\end{bmatrix}\quad\begin{bmatrix}2&1&0\\1&0&2\\0&2&1\end{bmatrix}\quad\begin{bmatrix}2&0&1\\1&2&0\\0&1&2\end{bmatrix}$$
$$\begin{bmatrix}2&0&1\\0&1&2\\1&2&0\end{bmatrix}\quad\begin{bmatrix}1&2&0\\0&1&2\\2&0&1\end{bmatrix}\quad\begin{bmatrix}1&2&0\\2&0&1\\0&1&2\end{bmatrix}\quad\begin{bmatrix}1&0&2\\2&1&0\\0&2&1\end{bmatrix}\quad\begin{bmatrix}1&0&2\\0&2&1\\2&1&0\end{bmatrix}\quad\begin{bmatrix}0&2&1\\2&1&0\\1&0&2\end{bmatrix}$$

$$\begin{bmatrix}0&2&1\\2&0&1\\1&1&1\end{bmatrix}\quad\begin{bmatrix}0&1&2\\1&1&1\\2&1&0\end{bmatrix}\quad\begin{bmatrix}0&2&1\\1&1&1\\2&0&1\end{bmatrix}\quad\begin{bmatrix}2&1&0\\0&1&2\\1&1&1\end{bmatrix}\quad\begin{bmatrix}2&1&0\\1&1&1\\0&1&2\end{bmatrix}\quad\begin{bmatrix}2&0&1\\1&1&1\\0&2&1\end{bmatrix}$$
$$\begin{bmatrix}2&0&1\\0&2&1\\1&1&1\end{bmatrix}\quad\begin{bmatrix}1&2&0\\1&0&2\\1&1&1\end{bmatrix}\quad\begin{bmatrix}1&2&0\\1&1&1\\1&0&2\end{bmatrix}\quad\begin{bmatrix}1&1&1\\1&0&2\\1&2&0\end{bmatrix}\quad\begin{bmatrix}1&1&1\\1&2&0\\1&0&2\end{bmatrix}\quad\begin{bmatrix}1&1&1\\2&0&1\\0&2&1\end{bmatrix}$$
$$\begin{bmatrix}1&1&1\\2&1&0\\0&1&2\end{bmatrix}\quad\begin{bmatrix}1&1&1\\0&1&2\\2&1&0\end{bmatrix}\quad\begin{bmatrix}1&1&1\\0&2&1\\2&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&2\\1&1&1\\1&2&0\end{bmatrix}\quad\begin{bmatrix}0&1&2\\2&1&0\\1&1&1\end{bmatrix}\quad\begin{bmatrix}1&0&2\\1&2&0\\1&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

Appendix B

List of totally nonsplit $\mathfrak{D}$-subalgebras

We found 6175 commutative totally nonsplit $\mathfrak{D}$-subalgebras of $S_3 \times S_3$ modulo 2 in Section 5.5. Each algebra can be represented as a 6 by 6 array which corresponds to $S_3 \times S_3$ in matrix form (5.5.2) on page 26. For example, the matrix

011222
113222
131222
222241
222412
222121

represents the algebra coming the partition: $[1],[2,4,5,6,28,29,30,32,33,34]$, $[3,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,27,36],[25,26],[35]$. These arrays are further divided into three categories: the first one consists of symmetric matrices; the second one consists of matrices that are not symmetric but whose transpose is in the the list; and the third the rest of the list.

Further we divide each category into G-orbits by conjugation, where G is the group of the diagonal elements of (5.5.2). (Remember these elements correspond to six permutation relations in $E_9(3)$.) Representatives from orbits in each category are listed.

I. symmetric matrices

011222	011111	011111	011111	011111	011111	011111	011111	011111	011111
113222	111121	111121	111212	111213	111222	112111	112111	112212	112222
131222	111121	111121	111212	111312	111222	121111	121111	121212	121222
222241	111112	111121	122111	123111	122112	111111	111111	122112	122122
222412	122112	122212	111111	111111	122112	111113	111112	111121	122212
222121	111221	111121	122111	132111	122221	111131	111121	122211	122221

011111	011111	011111	011111	011111	011111	011111	011111	011111	011111
112223	112333	121111	121111	121121	121212	121212	122111	122111	122121
121223	121333	112111	112111	112121	112212	112212	122111	122111	122121
122133	133133	111311	111211	111221	122211	122212	111211	111212	111211
122313	133313	111131	111121	122212	111111	111112	111122	111111	122112
133331	133331	111113	111112	111122	122112	122222	111122	111212	111122

011111 011111 011111 011111 011111 011112 011112 011112 011112 011112
122121 122212 122222 122222 122222 111112 111112 111122 111211 111221
122121 122212 122222 122222 122222 111112 111112 111122 111211 111221
111222 122212 122211 122222 122222 111112 111121 111122 122121 122111
122212 111111 122112 122212 122222 111112 111222 122221 111222 122111
111222 122212 122122 122222 122222 222221 222121 222211 211121 211111

011112 011112 011112 011112 011112 011112 011112 011112 011112 011112
111221 112122 112211 121112 121221 123112 122112 122122 122211 123221
111221 121122 121211 112112 112221 132112 122112 122122 122211 132221
122122 111112 122111 111212 122211 111212 111222 111221 122222 122211
122221 122121 111122 111122 122121 111122 111222 122221 111222 122121
211211 222211 211121 222222 211112 222222 222222 222112 211222 211112

011112 011112 011121 011121 011121 011121 011121 011121 011121 011121
122221 123332 111111 111121 111212 111222 112111 112222 122111 122222
122221 132332 111111 111121 111212 111222 121111 121222 122111 122222
122221 133212 111111 111121 122111 122121 111112 122122 111212 122222
122221 133122 211121 222222 211121 222222 211121 222222 211121 222222
211112 222222 111111 111121 122111 122121 111211 122221 111212 122222

011122 011122 011122 011122 011122 011122 011122 011122 011122 011122
111112 111122 111211 111221 112122 113122 112211 113211 121112 121221
111112 111122 111211 111221 121122 131122 121211 131211 112112 112221
111111 111122 122111 122122 111122 111122 122111 122111 111212 122221
211111 222211 211111 222211 222212 222212 211112 211112 211112 222212
222111 222211 211111 211211 222221 222221 211121 211121 222222 211122

011122 011122 011122 011122 011122 011122 011212 011212 011212 011212
122112 122122 122122 122211 122211 122221 111111 111222 121111 121222
122112 122122 122122 122211 122211 122221 111111 111222 112111 112222
111212 111222 111221 122211 122212 122221 211121 222111 211221 222211
211111 222222 222211 211122 211111 222211 111212 122111 111212 122111
222212 222222 222112 211122 211212 211112 211121 222111 211122 222112

011212 011212 011212 011212 011221 011222 011222 011222 011222 011222
122111 122121 122212 122222 113331 111111 111111 111121 111121 111121
122111 122121 122212 122222 131331 111111 111111 111121 111121 111121
211222 211212 222222 222212 233122 211112 211121 211111 211111 211112
111212 122111 111212 122111 233212 211112 211222 222111 222121 222122
211222 211212 222222 222212 111221 211221 211121 211111 211111 211221

011222 011222 011222 011222 011222 011222 011222 011222 011222 011232
111212 111222 111222 111222 112121 112212 112212 121111 121112 121112
111212 111222 111222 111222 121121 121212 121212 112111 112112 112112
222121 222111 222111 222112 211112 222111 222122 211212 211222 211222
211222 222111 222121 222122 222121 211122 211222 211122 211222 311232
222121 222111 222111 222221 211211 222121 222221 211222 222222 222222

011233 011222 011222 011222 011222 011222 011223 011222 011222 011222
131112 121222 121222 121222 131222 131332 141424 123111 122111 122121
113112 112222 112222 112222 113222 113332 114424 132111 122111 122121
211323 222222 222233 222211 222211 233211 244211 211212 211222 211211
311222 222222 222323 222121 222121 233121 222121 211122 211222 222122
322323 222222 222332 222112 222112 222112 344113 211222 211222 211122

011222 011222 011222 011222 011222 011222 011222
122121 122121 122212 123331 123331 122222 122222
122121 122121 122212 132331 132331 122222 122222
211212 211212 222222 233212 233221 222211 222212
222111 222121 211222 233122 233222 222122 222111
211212 211212 222222 211222 211122 222122 222212

II. Nonsymmetric matrices I

011221 011111 011111 011111 011111 011111 011111 011111 011111 011111
133414 111111 111112 111112 111112 111112 111112 111121 111121 111121
133414 111111 111112 111112 111112 111112 111112 111121 111121 111121
233424 122212 122121 122122 122211 122212 211122 111121 111122 111211
111221 122212 122211 211212 122121 211122 122212 122121 211122 122211
233424 111111 122212 211122 122122 211212 211122 111212 222122 111122

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011111 011111 011111 011111 011111 011111 011111 011111 011111 011111
111121 111121 111121 111222 111222 111222 111222 111222 111222 111222
111121 111121 111121 111222 111222 111222 111222 111222 111222 111222
111212 222122 222212 111111 111111 111112 111221 111221 111222 122121
211212 211122 122212 111222 222111 111112 111112 222221 111222 122121
222212 111122 222212 122121 211212 122122 122211 211122 122212 122212

011111 011111 011111 011111 011111 011111 011111 011111 011111 011111
111222 111222 111222 112111 112111 112111 112111 112111 112111 112212
111222 111222 111222 121111 121111 121111 121111 121111 121111 121212
122122 222111 222221 111112 111131 111221 133133 122122 122122 211221
211122 222111 222221 111222 111133 222111 133131 122121 211212 222212
211122 122212 122122 111122 111333 222122 133113 122112 211221 211122

011111 011111 011111 011111 011111 011111 011111 011111 011111 011111
112333 112222 121111 121121 121212 121212 121212 121212 121212 121212
121333 121222 112111 112121 112212 112212 112212 112212 112212 112212
111313 211211 222122 222112 111112 111112 111221 122121 122122 122122
111133 211121 222212 211121 122121 122212 211212 111112 111222 222111
111111 122221 222221 111122 111122 111211 222122 122111 122112 211221

011111 011111 011111 011111 011111 011111 011111 011111 011111 011111
121212 121212 122111 122111 122111 122111 122111 122111 122111 122111
112212 112212 122111 122111 122111 122111 122111 122111 122111 122111
211212 222221 111121 111121 111122 122111 122111 122222 211111 222112
222221 122212 111212 222121 111122 122111 122111 211222 211222 222112
122111 222122 111212 222121 222122 122222 211111 211222 211222 222221

011111 011111 011111 011111 011111 011111 011111 011111 011111 011111
122121 122121 122121 122121 122121 122121 122121 122121 122121 122121
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111111 111111 111112 111221 122112 122121 122122 122212 222111 222221
122121 211212 122211 211122 122112 122121 211122 211212 122212 211122
111222 222111 111221 222221 122221 122212 211122 211212 222111 111221

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122212 122222 122222 122222 122222 122222 122222 122222 122222 122222
111111 111112 111121 111122 122111 122111 122112 122221 122222 211111
111111 111112 122121 211122 122121 211212 122211 211122 122222 122212
111111 222221 111212 222122 122222 211111 122221 211221 211111 211111

011111 011111 011111 011111 011111 011112 011112 011112 011112 011112
122222 122222 122222 122222 122222 111112 111112 111112 111112 111112
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211111 211111 211221 222122 222212 111112 111112 111112 111112 122211
211111 211111 211122 222122 122212 122112 211112 222221 222221 122112
122222 211111 122221 222122 222212 222112 111112 111112 222221 211211

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122211 122222 211111 211122 211122 211211 211211 222112 222212 222221
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111122 111211 111211 111211 111211 111211 111211 111211 111211 111211
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222211 222211 111112 111112 111221 122111 122222 211222 222112 122222
122112 211221 111221 222112 111221 122111 211222 211222 222112 111211
111211 222122 111221 222112 222221 122222 211222 211111 222221 211121

011112 011112 011112 011112 011112 011112 011112 011112 011112 011112
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121122 121122 121122 121122 121122 121122 121211 121211 112112 112221
122222 211111 211111 211222 211222 222221 122111 211222 111212 111121
222211 222122 222211 111211 222122 211212 222211 111122 222211 111211
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111121 211122 222121 222212 111211 111211 111212 122112 122121 122122
222211 211212 222122 222122 111211 222211 222211 122112 122121 122212
111222 211112 222111 222222 211112 122112 111111 211112 222222 211112

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122122 211112 211212 222111 222122 222122 222211 222211 111112 111112
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211111 222112 222112 222221 222221 111121 111122 122211 211122 211122
222111 122112 211221 211112 211221 222211 111122 211121 211121 211212
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211221 222122 122221 211112 211112 222121 222121 222222 211121 122121

011121 011121 011121 011121 011121 011121 011121 011121 011121 011121
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222222 111222 222111 222222 211121 211212 211211 122122 211211 222112

011121 011121 011121 011121 011121 011121 011121 011121 011121 011121
122211 122211 122211 122211 122211 122211 122211 122211 122211 122211
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122121 211121 211121 211212 122121 211212 111222 222111 222111 222222
211211 122211 211122 211211 122211 211122 111112 222221 222112 222221

011121 011121 011121 011121 011121 011121 011121 011121 011121 011121
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111111 111111 111112 122212 211121 222221 222222 111121 111121 122222
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011121 011121 011121 011121 011122 011122 011122 011122 011122 011122
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II. Nonsymmetric matrices II

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