Civil Engineering Department
Colorado Agricultural and Mechanical College
Fort Collins, Colorado

ATMOSPHERIC DIFFUSION FROM A LINE AND POINT SOURCE OF MASS ABOVE THE GROUND

by
C. S. Yih
Associate Professor

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FOREWORD

This report is No. 6 of a series written for the Diffusion Project presently being conducted by the Colorado Agricultural and Mechanical College for the Office of Naval Research. The experimental phase of this project is being carried out in a wind-tunnel at the Fluid Mechanics Laboratory of the College. The project is under the general supervision of Dr. M. L. Albertson, Head of Fluid Mechanics Research of the Civil Engineering Department.

To Dr. M. L. Albertson, and to Dr. D. F. Peterson, Head of the Civil Engineering Department and Chief of the Civil Engineering Section of the Experiment Station, as well as to Professor T. H. Evans, Dean of the Engineering School and Chairman of the Engineering Division of the Experiment Station, the writer wants to express his appreciation for their kind interest in the present work.

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Atmospheric Diffusion from a Line or Point Source

of Mass Above the Ground

by

Chia-Shun Yih

Abstract

Under the assumption that the wind velocity and the diffusivities vary as certain power functions of height, the mass distribution in the atmosphere resulting from a line or point source above the ground is calculated. It is obvious that the result obtained has a direct bearing on the problem of smog control.

1. Atmospheric Diffusion from a Line Source of Mass Above the Ground

Supposing that a horizontal line source of mass with strength G (mass per length per unit time) is situated at a height h above the ground, and that a wind is blowing horizontally in a direction normal to the length of the line source, it is proposed to calculate the mass distribution in the atmosphere under the assumption that the wind velocity and the vertical diffusivity vary as power functions of height.

One chooses for convenience the wind velocity $u(h)$ at the height h as the reference velocity. The wind velocity at any height y can be written as

$$u = u(h)\left(\frac{y}{h}\right)^m$$

Similarly, using D to denote the vertical diffusivity, one has

$$D = D(h)\left(\frac{y}{h}\right)^n$$

where $D(h)$ is the value of D at the height h.

*Associate Professor, Department of Civil Engineering, Colorado Agricultural and Mechanical College. At present on leave at the University of Nancy, Nancy, France.

With the origin chosen on the ground directly under the source, the diffusion equation is

$$u \frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(D \frac{\partial c}{\partial y}\right) \tag{1}$$

where c is the concentration and x is measured in the downwind direction. Denoting the ambient concentration by c_0, one can form the dimensionless parameter

$$\Theta = \frac{c - c_0}{c_0}$$

and use Θ instead of c in (1).

The quantity

$$\frac{h\, u(h)}{D(h)}$$

can be considered as the Reynolds number at h and will be denoted by R(h). Choosing the new variables

$$\xi = \frac{1}{R(h)} \frac{x}{h}, \qquad \eta = \frac{y}{h}$$

(1) can be written as

$$\eta^m \frac{\partial \Theta}{\partial \xi} = \frac{\partial}{\partial \eta}\left(\eta^n \frac{\partial \Theta}{\partial \eta}\right) \tag{2}$$

One now proceeds to solve this equation with the boundary conditions

(a) $\Theta \to 0$ as $\eta \to \infty$

(b) $\dfrac{\partial \Theta}{\partial \eta} = 0$ for $\eta = 0$ (impermeable ground)

(c) $\Theta \to 0$ as $\xi \to \infty$

(d) $\Theta = \dfrac{G}{c_0 h\, u(h)} \delta_1(\eta)$ for $\xi = 0$

where $\delta_1(\eta)$ is the Dirac measure defined as follows

$$\delta_1(\eta) = 0 \text{ for } \eta \neq 1$$

$$\int_0^\infty \delta_1(\eta)\, d\eta = 1$$

The condition (d) is obtained by considering the following equation of continuity in integral form:

$$\int_0^\infty u(c - c_0)\, dy = G$$

which in terms of the new variables assumes the form

$$\int_0^\infty \eta^m \Theta\, d\eta = \frac{G}{c_0 h\, u(h)}$$

Since, at $\xi = 0$, $\Theta = 0$ everywhere except at $\eta = 1$, the last equation shows that Θ is indeed a multiple of $\delta_1(\eta)$ at $\xi = 0$, as required by (d).

Assuming

$$\Theta = X(\xi)\, Y(\eta)$$

and using primes to denote differentiation, one has, upon substitution into (2),

$$\frac{X'}{X} = \frac{(\eta^n Y')'}{\eta^m Y} = -\lambda^2$$

which gives

$$X = e^{-\lambda^2 \xi} \tag{3}$$

$$(\eta^n Y')' + \lambda^2 \eta^m Y = 0 \tag{4}$$

where λ is real because of (c). It is (4) that will be investigated in detail.

Substituting

$$f = \eta^p Y \qquad \phi = \eta^q \tag{5}$$

in (4), one finds that if

$$p = (n-1)/2 \qquad q = (m-n+2)/2$$

(4) assumes the following form

$$f'' + \frac{f'}{\phi} + (\sigma^2 - \frac{\nu^2}{\phi})f = 0 \tag{6}$$

where

$$\sigma = |\lambda/q| \qquad |\nu| = |(n-1)/(m-n+2)|$$

and where the primes denote differentiation with respect to ϕ.

The solutions of (6) are Bessel functions of the ν-th order:

$$J_{|\nu|}\left(\frac{\phi}{\sigma}\right), \qquad J_{-|\nu|}\left(\frac{\phi}{\sigma}\right)$$

For easy reference one will name

$$Y_1 = \eta^{-p} J_{|\nu|}\left(\frac{\phi}{\sigma}\right), \quad Y_2 = \eta^{-p} J_{-|\nu|}\left(\frac{\phi}{\sigma}\right)$$

respectively the first and the second solution.

The boundary conditions will now be investigated. Since asymptotically the Bessel functions are of the order of

$$\phi^{-\frac{1}{2}} = \eta^{-\frac{q}{2}}$$

both solutions will be of the order of

$$\eta^{-(p+\frac{q}{2}}$$

for large η . Since

$$p + \frac{q}{2} = \frac{1}{4}(m+n)$$

is always positive (m and n being always positive), the condition at $\eta = \infty$ is always satisfied. For the boundary condition at $\eta = 0$ one notes that for small η and a non-entire ν ,

$$J_{|\nu|}\left(\frac{\phi}{\sigma}\right) = J_{|\nu|}\left(\frac{\eta^q}{\sigma}\right) \sim \eta^{q|\nu|}, \quad J_{-|\nu|}\left(\frac{\phi}{\sigma}\right) = J_{-|\nu|}\left(\frac{\eta^q}{\sigma}\right) = \eta^{-q|\nu|}$$

With only positive values of q considered (In practice q is always positive),

$$q|\nu| = \frac{1}{2}|1-n|$$

and one sees that in order that Θ and its derivative with respect to η are bounded at $\eta = 0$, the first or the second solution should be used depending on whether p is positive or negative. Take for instance the case when p is positive, the first term in the first solution is a constant, and the second term is of the order of η^{2q+p} . Since $2q+p-1 = (2m-n+1)/2$ is always positive in practice, one will consider only positive values of this quantity. For such values (b) is then satisfied. The second solution is excluded because it is not bounded at $\eta = 0$. Similarly if p is negative it can be easily shown that the second solution should be used. The first solution is excluded because its derivative becomes infinite at $\eta = 0$. For convenience of exposition, one will simply denote the solution to be used by $Y = \eta^{-p} J_{\nu}\left(\frac{\phi}{\sigma}\right)$, where ν is positive or negative depending on whether p is positive or negative.

It remains to satisfy (d). The expansion formula to be used is due to MacRobert (1931):

$$f(\phi) = \int_0^\infty \sigma \, d\sigma \int_0^\infty f(\rho) \, J_\alpha(\sigma\phi) \, J_\alpha(\sigma\phi)\rho \, d\rho \tag{8}$$

Where the value of α (real) must be greater than -1. This condition is satisfied by ν or $-\nu$, since in all practical cases ν is less than 1.

In the present investigation one seeks a density function g ( ) such that

$$\delta_1(\eta) = \int_0^\infty \sigma g(\sigma) \eta^{-p} J_\nu(\sigma\eta^q) \, d\sigma \tag{9}$$

or

$$\eta^p \delta_1(\eta) = \int_0^\infty \sigma g(\sigma) \, J_\nu(\sigma\eta^q) \, d\sigma \tag{9}$$

Since the argument of the Bessel function is not $\sigma\eta$ but $\sigma\phi$, it is necessary to transform $\delta_1(\eta)$ to $\delta_1(\phi)$. For this purpose one notes that

$$\int_0^\infty \delta_1(\eta) \, d\eta = \int_0^\infty \frac{1}{q} \phi^{\frac{1-q}{q}} \delta_1(\eta) \, d\phi = \int_0^\infty \frac{1}{q} \delta_1(\eta) \, d\phi = 1$$

so that

$$\delta_1(\eta) = q\,\delta_1(\phi)$$

Then from (9) and (8)

$$q\delta_1(\phi) = \int_0^\infty \sigma g(\sigma) J_\nu(\sigma\phi) \, d\sigma = \int_0^\infty \sigma \, d\sigma \int_0^\infty q\delta_1(\rho) J_\nu(\sigma\phi) J_\nu(\sigma\rho) \rho \, d\rho$$

and by comparison

$$= \int_0^\infty q\sigma \, J_\nu(\sigma) \, J_\nu(\sigma\phi) \, d\sigma$$

$$g(\sigma) = q\,J_\nu(\sigma)$$

The solution is therefore

$$\theta = \int_0^\infty q \, e^{-\sigma^2 q^2 \xi} J_\nu(\sigma) \, J_\nu(\sigma\eta^q) \, \sigma \, d\sigma \tag{10}$$

It should be remarked that although the integral in (9) is not convergent, the limit of the integral in (10) as $\xi \to 0$ to converges everywhere to zero for $\eta \neq 1$, and it is as the limit of the integral in (10) that the one in (9) should be considered. The proof is rather lengthy. Suffice it here to cite the much simpler and analogous case of the Fourier integral. The integral

$$\int_0^\infty \cos \sigma\rho \, d\sigma \tag{11}$$

is obviously not convergent for any value of ρ, but the integral

$$\int_0^\infty e^{-\sigma^2 t} \cos\sigma\rho \, d\sigma = \sqrt{\frac{\pi}{4t}} \; e^{-\frac{\rho^2}{4t}} \tag{12}$$

converges everywhere to zero as $t \to 0$ except at $\rho = 0$. It can be easily shown that the limit is actually the Dirac measure $\delta_0(\rho)$. When the integral in (11) is used, it should always be understood to mean the limit of that in (12).

2. Atmospheric Diffusion from a Point Source of Mass Above the Ground

Supposing that a point source of mass with strength G (mass per unit time) is situated at a height h above the ground, and that a wind is blowing horizontally, it is proposed to calculate the mass distribution in the atmosphere on the assumption that the wind velocity and the diffusivities vary as power functions of height, the power for the wind velocity and the lateral diffusivity being the same. Concerning the equality of the power of u and that of the lateral diffusivity E, reference is made to the works of D. R. Davies. See, for instance, Davies (1950).

One chooses the projection of the point source on the ground as the origin, and measures x, y, and z respectively in the downwind, vertical, and the cross-wind direction. The expressions for u and for D being the same as before, one has in addition

$$E = E(h)\left(\frac{y}{h}\right)^m$$

where E(h) is the value of E at the height h. Retaining the meanings of θ, ξ, η, and R(h), and writing

$$\frac{D(h)}{E(h)} = \beta^2, \quad \varsigma = \beta\left(\frac{z}{h}\right)$$

the diffusion equation is

$$\eta^m \frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \eta}\left(\eta^n \frac{\partial \theta}{\partial \eta}\right) + \eta^m \frac{\partial^2 \theta}{\partial \varsigma^2} \tag{13}$$

The boundary conditions are

(a) $\theta \to 0$ as $\eta \to \infty$

(b) $\frac{\partial \theta}{\partial \eta} = 0$ for $\eta = 0$

(c) $\theta \to 0$ as $\varsigma \to \infty$

(d) $\frac{\partial \theta}{\partial \varsigma}$ 0 for $\varsigma = 0$

(e) $\theta = 0$ as $\xi \to \infty$

(f) $\theta = \frac{\beta G}{c_0 h^2 u(h)} \delta_{1.0}(\eta\,\varsigma)$ at $\xi = 0$

where $\delta_{1.0}(\eta\,\varsigma)$ is the Dirac measure defined as follows

$$\delta_{1.0}(\eta\,\varsigma) = 0 \text{ for } (\eta\,\varsigma) \neq (1\,0)$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \delta_{1.0}(\eta\,\varsigma)\, d\eta\, d\varsigma = 1$$

It should be obvious from the definition that $\eta^m \delta_{1.0}(\eta\,\varsigma) = \delta_{1.0}(\eta\,\varsigma)$. The explanation for (f) is the same as for (d) in the previous section.

Assuming

$$\theta = X(\xi)\, Y(\eta)\, Z(\varsigma)$$

one finds that the fundamental solution to be used is

$$e^{-(\lambda^2+\mu^2)\xi}\, \eta^{-p} J_\nu(\sigma \eta^q) \cos \mu \varsigma$$

where all symbols already employed have their meanings as before (in particular ν is positive or negative depending on whether p is positive or negative), and where μ is real in view of (e). The conditions (a), (b), (d), and (e) being satisfied (For the satisfaction of (a) and (b), the arguments are the same as in the previous section), one proceeds to demand the satisfaction of (c) and (f).

Remembering $\phi = \eta^q$, one has, as before

$$\delta_{1,0}(\eta,\xi) = q\,\delta_{1,0}(\phi,\xi)$$

The Fourier-Bessel integral formula to be used is

$$f(\phi,\xi) = \frac{2}{\pi}\int_0^\infty \cos\mu\xi\,d\mu\int_0^\infty \sigma J_\nu(\sigma\phi)\,d\sigma\int_0^\infty\int_0^\infty \cos\mu\gamma\,d\gamma\, f(\rho,\gamma)\, J_\nu(\sigma\rho)\,\rho\,d\rho \quad (14)$$

To satisfy (f), one seeks a weight function $F(\sigma,\mu)$ such that

$$\delta_{1,0}(\eta,\xi) = \int_0^\infty \cos\mu\xi\,d\mu\int_0^\infty \sigma F(\sigma,\mu)\,\eta^{-p} J_\nu(\sigma\eta^q)\,d\sigma$$

or

$$q\eta^{-p}\delta_{1,0}(\phi,\xi) = q\,\delta_{1,0}(\phi,\xi) = \int_0^\infty \cos\mu\xi\,d\mu\int_0^\infty \sigma F(\sigma,\mu)\, J_\nu(\sigma\phi)\,d\sigma \quad (15)$$

But this is equal to

$$\frac{2}{\pi}\int_0^\infty \cos\mu\xi\,d\mu\int_0^\infty \sigma J_\nu(\sigma\phi)\,d\sigma\int_0^\infty\int_0^\infty q\,\delta_{1,0}(\rho,\gamma)\cos\mu\gamma\, J_\nu(\sigma\rho)\,\rho\,d\rho\,d\gamma$$

$$= \frac{1}{\pi}\int_0^\infty \cos\mu\xi\,d\mu\int_0^\infty q\sigma J_\nu(\sigma)\, J_\nu(\sigma\phi)\,d\sigma$$

By comparison,

$$F(\sigma,\mu) = \frac{\pi}{2}\,q\,J_\nu(\sigma) \quad (16)$$

and the solution is

$$\Theta = \frac{1}{\pi}\,q\int_0^\infty\int_0^\infty e^{-(\sigma^2q^2+\mu^2)\xi}\,\sigma J_\nu(\sigma)\, J_\nu(\sigma\phi)\cos\mu\xi\,d\sigma\,d\mu \quad (17)$$

As before, the integral in (15) should be considered as the limit of that in (17), as $\xi \to 0$.

The solution given in (17) can be written, in virtue of (12):

$$\Theta = \left(\frac{1}{\pi\xi}\right)^{\frac{1}{2}} q\int_0^\infty\int_0^\infty e^{-\sigma^2q^2\xi-\frac{\xi^2}{4\xi}}\,\sigma J_\nu(\sigma)\, J_\nu(\sigma\phi)\,d\sigma \quad (18)$$

from which it is obvious that (c) is satisfied.

3. General Remark

It is hoped that the results given in this paper will play a role in specifying the location of smog-causing factories, urban or suburban, and the height of their chimneys. They should be so specified that the smog concentration at any part of the city calculated by the formulas given in this paper is below a certain harmless amount.

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