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THE SPECTRUM OF ENERGY IN TURBULENT SHEAR FLOW

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ABSTRACT

THE SPECTRUM OF ENERGY IN TURBULENT SHEAR FLOW

The spectrum of energy in isothermal turbulent shear flow at large wave numbers is studied following the phenomenological approach used by Tchen (1953), Hinze (1959), and Panchev (1968, 1969). The considered spectrum equation consists of the dissipation, the transfer, the production, and the diffusion spectrum function.

Parametric solutions for the three-dimensional energy spectrum function, $E(k)$ are obtained firstly by using Heisenberg's type of approximations for the transfer function. Much simpler solutions for $E(k)$ are obtained with the modified Obukhov approximation (Ellison 1962). Some closed form solutions for $E(k)$ are derived by using a vorticity approximation concept.

It is shown that Tchen's k^{-1} law cannot exist if diffusional effects are taken into account. Computed one-dimensional energy spectra (isotropic relations were used) show good agreement with measurements from the viscous region of a turbulent boundary layer.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES - Continued

LIST OF SELECTED SYMBOLS

Symbol	Definition or description	Dimensions
$D(k)$	Diffusion spectrum functions	L^4T^3
E_o	$E_o = (\varepsilon \nu)^{1/4}$	L^3T^2
$E(k)$	Three-dimensional energy spectrum function	L^3T^2
$\mathcal{E}(x)$	$E(k)$ in non-dimensional form	- -
$\mathcal{F}(x_1)$	$\phi(k_1)$ in non-dimensional form	- -
$\vec{k}$	Wave number vector, $(k_1, k_2, k_3,)$	L^{-1}
k	Magnitude of $\vec{k}$	L^{-1}
k_o	Kolmogorov's wave number, $k_o = (\varepsilon/\nu^3)^{1/4}$	L^{-1}
m	Non-dimensional spectral production term	- -
n	Non-dimensional spectral diffusion term	- -
P	Instantaneous pressure, $P = \overline{P} + p$	FL^{-2}
$(1/2)\overline{q^2}$	Turbulent energy per unit mass	L^2T^{-2}
$\vec{r}$	Difference between two position vectors	L
$F(k)$	Spectrum transfer function	L^3T^{-3}
t	Time	T
$Qij(\vec{r})$	$\overline{u_i(\vec{x})uj(\vec{x} + \vec{r})}$	L^2T^{-2}
u_i	Fluctuating velocity	LT^{-1}
U_i	Instantaneous velocity, $U_i = \overline{U}_i + u_i$	LT^{-1}
$\overline{U}_i$	Mean velocity	LT^{-1}
$\vec{x}$	Position vector, (x_1, x_2, x_3)	

Symbol	Definition or description	Dimensions
x	Non-dimensional wave number, $x = k/k_o$	- -
Z^2	$\int_0^x 2\mathcal{E}(p)\, p^2\, dp$	- -
α	Kolmogorov's universal constant for the three-dimensional energy spectrum	- -
α_1	Kolmogorov's constant for the one-dimensional energy spectrum	- -
γ_H	Heisenberg's constant in relation to his approximation for $\nu_T(k)$	- -
γ_{HW}	Howells' constant in relation to his approximation for S(k)	- -
γ_{MOB}	Constant in relation to the Modified Obukhov approximation for S(k)	- -
γ_V	Constant in relation to the Vorticity approximation for S(k)	- -
ε	Total dissipation by turbulence per unit mass	L^2T^{-3}
ν	Kinematic viscosity	L^2T^{-1}
$\nu_T(k)$	Eddy viscosity in k - space	L^2T^{-1}
ρ	Instantaneous density (assumed const.)	FT^2L^{-4}
$\tau(k)$	Shear stress spectrum function	L^3T^{-2}
$\phi_{ij}(\vec{k})$	Fourier transform of $R_{ij}(\vec{r})$	L^3T^{-2}
$\phi(k_1)$	One-dimensional longitudinal energy spectrum function	L^3T^{-2}
$\phi(x_1), \mathcal{F}(x_1)$	Non-dimensional form of $\phi(k_1)$	- -

F: force, L: length, T: time

THE SPECTRUM OF ENERGY IN TURBULENT SHEAR FLOW

1. Introduction

Most of the work in the statistical theory of turbulence has been concerned with isotropic and homogeneous turbulent fields. From these studies, however, it has been possible to gain an insight into processes such as turbulent dissipation and transfer of energy between eddies of various sizes. The nonlinear process of the balance of turbulent energy in the wave number space emerges from these studies as controlled by a transfer function representing the transfer of energy from large eddies into small ones. Several forms of the transfer function have been proposed, all leading to the "-5/3" spectral law in agreement with Kolmogorov's postulates for the inertial subrange. According to the experimental research of the last decade, this law seems to be well founded and can be a starting point in any research dealing with the structure of turbulent flows.

The importance of Kolmogorov's ideas is in their applicability both to decaying turbulence, whether isotropic or not, and to maintained shear turbulence. The simultaneous application to both decaying and maintained turbulence and to isotropic and nonisotropic turbulence is achieved by concentrating attention on a domain of restricted dimensions, i.e., on a range of sufficiently large wave numbers. In this restricted domain only local velocities are considered and for a sufficiently large Reynolds number ($Re \gg 1$), local isotropy is postulated for all statistical quantities.

Except for the concept of local isotropy, little has been accomplished in the statistical theory of shear turbulence. Most attempts to obtain informations on the structure of shear turbulence have been experimental. From the statistical theory of isotropic turbulence it is known which quantities are important in describing the fluctuating field; such as turbulent intensities, correlations, spectrum, probability distribution, etc. Hence, these quantities are usually measured in experiments on shear turbulent flows.

For an analitical study of shear turbulence significant simplifications are needed due to the extreme complexity of the problem once nonisotropy is brought into the picture. Even the theories on isotropic turbulence are faced at the present time with great difficulties and, though many advances have been made in the last thirty years, a complete solution of the problem is still an unattainable goal.

While a complete mathematical solution of the problem of shear turbulence is the thing of the future, it is possible, with appropriate assumptions, to obtain a reasonable simplification of the problem enabling derivation of analytical results. At least for some of the statistical quantities a picture of their behavior in shear turbulence can be obtained.

The number of contributions to the theory of shear turbulence, due to its great complexity, is very small. Aximetrical turbulence, a type of turbulence second in simplicity to isotropic turbulence, was the first step toward nonisotropic turbulence. The kinematics of this type of turbulence has been treated as completely as has been possible with isotropic turbulence. However, although it may be nearer to reality than isotropic turbulence, aximetrical turbulence is still far away from actual turbulence. In boundary layers and other shear turbulent flows several important processes are encountered that are not studied in the isotropic or the aximetrical turbulence theory. Among these processes are those associated with mean velocity gradients and inhomogeneity of turbulence. Even in the case of supposedly isotropic turbulence generated by flow through a grid, there will exist inhomogeneity due to decay of turbulence in the direction of the mean flow.

Difficulties encountered in considering all turbulent processes simultaneously are very great. As a result, some analytical studies have been made by considering these processes separately.

Two-point equations for velocity and pressure-velocity correlations are the starting point in these studies. These generalized two-point correlation equations are constructed from the Navier-Stokes equations by methods similar to those used by von Karman and Howarth (1938) for isotropic turbulence. They are, however, too complex and simplifications are introduced to make the problem determinate.

Deissler (1961) used the weak turbulence approximation, i.e., he neglected the inertia effects represented by the triple correlations. Obviously, this assumption limits the analysis to low Reynolds numbers. Moreover, to be able to obtain solutions for spectra of such a "weak turbulence," Deissler (1961) made further simplifications. He considered first the effects of longitudinal and of transverse inhomogeneities but without a mean velocity gradient, and then he considered the effect of a uniform transverse velocity gradient on a homogeneous turbulent field.

Nevertheless, interesting conclusions were obtained from these results. The mean velocity gradient, for example, appeared (in the simplified case studied) as having the role of the usual transfer term, i.e., as affecting the transfer of energy among eddies of various sizes. This transfer was interpreted as a stretching of vortex lines by the mean velocity gradient.

In a more recent paper, Deissler (1965) again treated the case of the weak turbulence (triple correlation terms neglected) but without additional simplifications. In this case, however, the set of simultaneous correlation equations that must be considered consists of four nonlinear partial differential equations in four independent and four dependent variables. These equations are too complicated to be of practical use.

Qualitative conclusions were obtained by expanding these equations in power series in each of the space variables, valid only for small values of variables. The presence of pressure-velocity correlations and of nonlinear production terms appears as very important if a steady-state shear flow turbulence is to exist. Though the results of studies involving correlations between velocities and their derivatives are still very meager, they help in explaining some of the features of actual shear flows and can contribute to a better understanding of shear turbulence.

Deissler (1965) concludes that a system of correlation equations for turbulent shear flow, which is closed by neglecting the highest order correlations, should yield reasonable steady-state solutions. This is of course in principle, because the solution of such a set of equations is faced with almost unsurmountable difficulties.

Besides treating the correlation equations obtained from the complete Navier-Stokes equations, many efforts have been made to gain some insight into the inner workings of turbulence by using simplified equations in place of those for the real fluid. The well known Burgers' equation was proposed as a one-dimensional analog to the Navier-Stokes equation. It has been examined by many different methods. The most recent analysis is given by Kraichnan (1968) who treated it with his Lagrangian-history direct-interaction approximation.

Though there are similarities between the Burgers and Navier-Stokes equations, the sharpest difference is that Burgers' equation appears to offer no counterpart to the hierarchy of the instabilities which makes the small-scale structure of high Reynolds number turbulence chaotic and unpredictable. If the initial Reynolds number is high, Burgers' equation leads to shock fronts. The result is that Burgers' equation reduces initial chaos instead of increasing it.

In contrast to the widely accepted $-5/3$ energy spectrum of Navier-Stokes turbulence at high Reynolds numbers, the shockfront structure of evolved solutions of Burgers' equation leads to the k^{-2} inertial-range spectrum. On the other hand, the advection term in Burgers' equation tends to produce, similarly as the corresponding term in the Navier-Stokes equation, regions of steepened velocity gradients, which implies a transfer of excitation from lower to higher wave number components of the velocity field. Kraichnan (1968) concludes that Burgers' equation should be viewed as a vehicle for exploring the limits of applicability of statistical approximations designed for Navier-Stokes turbulence.

The statistical theory of turbulence at the present time is not able to produce explicit quantitative results concerning important characteristics of shear turbulent flows without making very restrictive assumptions. However, the study of some basic characteristics of shear turbulence can be made if phenomenological type approximations, similar to those made in the study of isotropic turbulence, are used.

Using a phenomenological approach Tchen (1953) studied the spectral law of small eddies for a turbulent shear flow with a given pattern of mean motion. He derived an equation for energy balance in spectral terms by means of the Fourier analysis of the Navier-Stokes equation. The deduced shear turbulence spectral equation involves five spectrum functions: the dissipation function, the transfer function, the production function, the diffusion function, and the convection function. The first two functions are common to isotropic and homogeneous turbulence without mean motion.

Depending on the relation between the vorticity of the mean motion and the vorticity of the turbulent motion, Tchen distinguishes two cases: the case of "weak turbulence' and the case of "strong turbulence". In the first case, the vorticity of the main motion is small compared with that of the turbulent motion. The interaction between the two vorticities is slight and there is no resonance. In the second case, the vorticities of the two motions are comparable. There may be a very strong interaction between them and violent resonance may occur. It should be noted that a fully developed turbulent boundary flow would represent the case of strong interaction, i.e., strong turbulence. Correspondingly, this is the case of the greatest interest.

So far, there has not been any solution of the complete shear turbulence spectrum equation for the case of strong turbulence. Tchen (1953) neglected the diffusion and the convection term and, furthermore, considered only asymptotic solutions of the remaining balance equation, which allowed him additional simplifications. Panchev (1968, 1969) was the first to consider the solution of the equation when the dissipation, the transfer, and the production term are taken into account simultaneously. He obtained parametric solutions for the three-dimensional energy spectrum function $E(k)$.

All solutions of the shear turbulence spectral equation, when the diffusion term is neglected, produce the well known k^{-1} law in the low wavenumber subrange of the equilibrium range. Tchen (1953), who was the first to obtain the k^{-1} result, assumed that the diffusional effects can be neglected. However, as will be seen, these effects must be taken into account.

In the present study, solutions of the shear turbulence spectrum equation are sought for the case when the effects of inhomogeneity are not neglected. The dissipation, the transfer, the production, and the diffusion spectrum function are treated simultaneously in finding solutions for the energy spectrum function $E(k)$.

In the first part, solutions of the spectrum equation are obtained following the approach used by Tachen (1953), Hinze (1959) and Panchev (1968, 1969); i.e., by using approximations of Heisenberg's type for the transfer spectrum function. In the second part, it is shown that the Modified Obukhov approximation (Ellison 1962) for the transfer function can be used successfully in treating the shear turbulence spectrum equation. It is employed here for the first time and leads to simple solutions for the energy spectrum function. In the third part, the vorticity approximation for the transfer spectrum function--which was used in the derivation of closed form expressions for spectra of turbulent energy and scalar fields in isotropic turbulence--is used in solutions of the shear tubulence spectrum equation. This approximation made possible the derivation of some closed form expressions for the three-dimensional energy spectrum of shear turbulence.

In the following, only an isothermal and incompressible turbulent shear flow with steady mean motion is considered. An examination of the basic spectral characteristics of turbulence in the presence of temperature gradients could be made along the lines used here for isothermal turbulence. As in the previous attempts on solving the isothermal shear spectrum equation, all works on spectra of a thermally stratified shear turbulent flow have used Heisenberg's type of approximation for the energy and the temperature transfer function (Monin 1962, Gisina 1966, Lin 1969). This type of approximation for transfer functions leads to fairly complex solutions of the spectrum equation. Solutions for the energy and the temperature spectrum function are then usually obtained by numerical integration. Asymptotic solutions for particular spectral ranges are of the greatest interest here and they are usually sought (buoyancy sub-range, for example).

The possibility of obtaining simpler solutions for spectra of thermally stratified shear turbulence by using the Modified Obukhov approximation or the vorticity approximation approach is not investigated in the present work.

2. The Energy Spectrum Equation

The derivation given here is based on general two-point equations for the velocity correlations. We start with the incompressible Navier-Stokes equation written for the points N and N', separated by the vector $\vec{r}$,

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_k}{\partial x_k} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} , \qquad (3\text{-}1)$$

$$\frac{\partial U_j'}{\partial t} + \frac{\partial U_j' U_k'}{\partial x_k'} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j'} + \nu \frac{\partial^2 U_j'}{\partial x_k' \partial x_k'} , \qquad (3\text{-}2)$$

where the subscripts can take on values of 1, 2 or 3 and a repeated subscript in a term indicates a summation. The quantities U_i and U_j' are instantaneous velocity components, x_i is a space coordinate, t is the time, ρ is the density, ν is the kinematic viscosity, and P is the instantaneous pressure:

$$U_i = \bar{U}_i + u_i , \quad U_j' = \bar{U}_j' + u_j' , \quad P = \bar{P} + p , \quad \frac{\partial \bar{U}_i}{\partial t} = \frac{\partial \bar{U}_j'}{\partial t} = 0 .$$

If the usual time averages are taken of (3-1) and (3-2) and the averaged equations are subtracted from the unaveraged ones, one obtains

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial \bar{U}_i}{\partial x_k} + \bar{U}_k \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_k}\left(u_i u_k - \overline{u_i u_k}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} , \qquad (3\text{-}3)$$

$$\frac{\partial u_j'}{\partial t} + u_k' \frac{\partial \bar{U}_j'}{\partial x_k'} + \bar{U}_k' \frac{\partial u_j'}{\partial x_k'} + \frac{\partial}{\partial x_k'}\left(u_j' u_k' - \overline{u_j' u_k'}\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j'} + \nu \frac{\partial^2 u_j'}{\partial x_k' \partial x_k'} . \qquad (3\text{-}4)$$

The next step is to multiply Eq. (3-3) by u_j and Eq. (3-4) by u_i. For Eq. (3-3), one obtains

$$u_j'\left(\frac{\partial u_i}{\partial t}\right) + u_j' u_k \frac{\partial \bar{U}_i}{\partial x_k} + u_j' \bar{U}_k \frac{\partial u_i}{\partial x_k} + u_j' \frac{\partial}{\partial x_k}\left(u_i u_k - \overline{u_i u_k}\right)$$

$$= -\frac{1}{\rho} u_j' \frac{\partial p}{\partial x_i} + \nu u_j' \frac{\partial^2 u_i}{\partial x_k \partial x_k} \qquad (3\text{-}5)$$

and, similarly, for Eq. (3-4),

$$u_i\left(\frac{\partial u_j'}{\partial t}\right) + u_i u_k' \frac{\partial \bar{U}_j'}{\partial x_k'} + u_i \bar{U}_k' \frac{\partial u_j'}{\partial x_k'} + u_i \frac{\partial}{\partial x_k'}\left(u_j' u_k' - \overline{u_j' u_k'}\right)$$

$$= -\frac{1}{\rho} u_i \frac{\partial p'}{\partial x_j'} + \nu u_i \frac{\partial^2 u_j'}{\partial x_k' \partial x_k'} \, . \qquad (3\text{-}6)$$

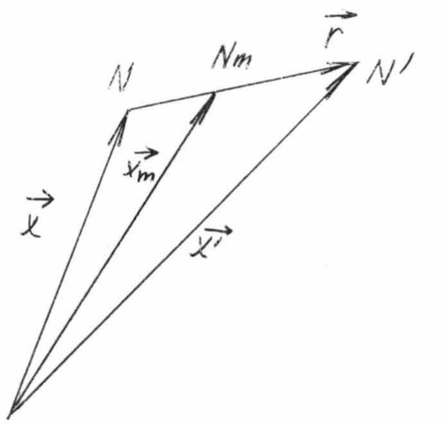
The addition of Eq. (3-5) and Eq. (3-6) and averaging with respect to time (keeping in mind that quantities at one point are independent of the position of the other point) yields

$$\frac{\partial}{\partial t}\overline{u_i u_j'} + \overline{u_k u_j'}\frac{\partial \bar{U}_i}{\partial x_k} + \overline{u_i u_k'}\frac{\partial \bar{U}_j'}{\partial x_k'} + \bar{U}_k \frac{\partial}{\partial x_k}\overline{u_i u_j'} + \bar{U}_k' \frac{\partial}{\partial x_k'}\overline{u_i u_j'}$$

$$= -\frac{\partial}{\partial x_k}\overline{u_j' u_i u_k} - \frac{\partial}{\partial x_k'}\overline{u_i u_j' u_k'}$$

$$-\frac{1}{\rho}\left[\frac{\partial}{\partial x_i}\overline{p u_j'} + \frac{\partial}{\partial x_j'}\overline{p' u_i}\right] + \nu\left[\frac{\partial^2}{\partial x_k \partial x_k} + \frac{\partial^2}{\partial x_k' \partial x_k'}\right]\overline{u_i u_j'} \, . \qquad (3\text{-}7)$$

In order to differentiate between the effects of the distance between the points N and N', and of their location (due to inhomogeneity we have to take both effects into consideration) new variables are introduced:



$$r_k = x_k' - x_k \ ,$$

$$(x_k)_m = \frac{1}{2}\left(x_k + x_k'\right) \ .$$

Thus, for a quantity R that is a function of $(x_k)_m$ and r_k, one obtains

$$\frac{\partial R}{\partial x_k} = \frac{1}{2}\frac{\partial R}{\partial (x_k)_m} - \frac{\partial R}{\partial r_k} \ ,$$

$$\frac{\partial R}{\partial x_k'} = \frac{1}{2}\frac{\partial R}{\partial (x_k)_m} + \frac{\partial R}{\partial r_k} \ ,$$

$$\frac{\partial^2 R}{\partial x_k \partial x_k} + \frac{\partial^2 R}{\partial x_k' \partial x_k'} = \frac{1}{2}\frac{\partial^2 R}{\partial (x_k)_m \partial (x_k)_m} + 2\frac{\partial^2 R}{\partial r_k \partial r_k} \ . \qquad (3\text{-}8)$$

Employing these transformations, if follows

$$\frac{\partial}{\partial x_k}\overline{u_i u_j'} = \frac{1}{2}\frac{\partial \overline{u_i u_j'}}{\partial (x_k)_m} - \frac{\partial}{\partial r_k}\overline{u_i u_j'} \ ,$$

$$\frac{\partial}{\partial x_k'}\overline{u_i u_j'} = \frac{1}{2}\frac{\partial}{\partial (x_k)_m}\overline{u_i u_j'} + \frac{\partial}{\partial r_k}\overline{u_i u_j'} ,$$

$$\frac{\partial}{\partial x_k}\overline{u_j' u_i u_k} = \frac{1}{2}\frac{\partial}{\partial x_k}\overline{u_j' u_i u_k} - \frac{\partial}{\partial r_k}\overline{u_j' u_i u_k} ,$$

$$\frac{\partial}{\partial x_k'}\overline{u_i u_j' u_k'} = \frac{1}{2}\frac{\partial}{\partial (x_k)_m}\overline{u_i u_j' u_k'} + \frac{\partial}{\partial r_k}\overline{u_i u_j' u_k} ,$$

$$\frac{\partial}{\partial x_i}\overline{p u_j'} = \frac{1}{2}\frac{\partial}{\partial (x_i)_m}\overline{p u_j'} - \frac{\partial}{\partial r_i}\overline{p u_j'} ,$$

$$\frac{\partial}{\partial x_j}\overline{p' u_i} = \frac{1}{2}\frac{\partial}{\partial (x_j)_m}\overline{p' u_i} + \frac{\partial}{\partial r_j}\overline{p' u_i} ,$$

$$\left[\frac{\partial^2}{\partial x_k \partial x_k} + \frac{\partial^2}{\partial x_k' \partial x_k'}\right]\overline{u_i u_j'} = \frac{1}{2}\frac{\partial^2(\overline{u_i u_j'})}{\partial (x_k)\partial (x_k)} + 2\frac{\partial^2(\overline{u_i u_j'})}{\partial r_k \partial r_k} .$$

Hence, Eq. (3-7) will read

$$\frac{\partial \overline{u_i u_j'}}{\partial t} + \overline{u_k u_j'}\frac{\partial \bar{U}_i}{\partial x_k} + \overline{u_i u_k'}\frac{\partial \bar{U}_j'}{\partial x_k'} + \frac{1}{2}\left(\bar{U}_k + \bar{U}_k'\right)\frac{\partial \overline{u_i u_j'}}{\partial (x_k)_m}$$
$$+ \left(\bar{U}_k' - \bar{U}_k\right)\frac{\partial \overline{u_i u_j'}}{\partial r_k} = -\frac{1}{2}\frac{\partial}{\partial (x_k)_m}\left[\overline{u_i u_j' u_k'} + \overline{u_j' u_i u_k}\right]$$
$$+ \frac{\partial}{\partial r_k}\left[\overline{u_i u_j' u_k'} - \overline{u_j' u_i u_k}\right] - \frac{1}{2\rho}\left[\frac{\partial \overline{p u_j'}}{\partial (x_i)_m} + \frac{\partial \overline{p' u_i}}{\partial (x_j)_m}\right.$$
$$\left. + \frac{1}{\rho}\left[\frac{\partial \overline{p u_j'}}{\partial r_i} - \frac{\partial \overline{p' u_i}}{\partial r_j}\right] + \frac{1}{2}\nu\frac{\partial^2 \overline{u_i u_j'}}{\partial (x_k)_m \partial (x_k)_m} + 2\nu\frac{\partial^2 \overline{u_i u_j'}}{\partial r_k \partial r_k} . \quad (3\text{-}9)$$

This is the two-point correlation equation for the general case of inhomogeneous turbulence with mean velocity gradients.

Because Eq. (3-9) is too complicated and intractable, the usual further step in the shear turbulence analysis is to assume homogeneity, i.e., that all derivatives with respect to $(x_k)_m$ vanish (Hinze 1959, Deissler 1961). This assumption contributes appreciably to the simplification of the problem and is not contradictory to the existence of shear turbulence. Full homogeneity with a shear is possible if the curvature of the mean velocity profile is neglected. In other words, to create a homogeneous turbulence with shear the main motion should have a constant velocity in a given direction and a constant velocity gradient in the considered region (Hinze 1959, Lumley & Panofsky 1964). Though this type of shear turbulence may be considered a hypothetical case too, it is a reasonable approximation and is much less remote from actual turbulence than other theoretical models usually considered.

In the following analysis, we also assume that the mean velocity gradient is constant, at least in a limited region of the flow, but do not assume homogeneity in all directions. The assumption is (often used in a turbulent boundary layer analysis) that the flow is homogeneous in x_1 and x_3 directions but not in the x_2 direction. Obviously, inclusion of inhomogeneity will bring the analysis closer to the actual shear turbulent flow than in the case of full homogeneity.

Hence, derivatives of correlations do not vanish in the x_2 direction, i.e., derivatives with respect to x_2 are not neglected:

$$\bar{U}_1 = f(x_2), \; \frac{dU_1}{dx_2} \cong \text{const.}, \; \bar{U}_2 = \bar{U}_3 = 0, \; \frac{\partial}{\partial x_2} \neq 0. \qquad (3\text{-}10)$$

Writing for shortness,

$$Q_{i,j} = \overline{u_i u_j'}, \qquad K_{p,j} = \overline{p u_j'},$$

$$S_{i,kj} = \overline{u_i u_k' u_j'}, \qquad K_{i,p} = \overline{p' u_i},$$

$$S_{ik,j} = \overline{u_j' u_i u_k},$$

we obtain, from the condition (3-10),

$$Q_{k,j} \frac{\partial \bar{U}_i}{\partial x_k} = \delta_{i1} Q_{2,j} \frac{dU_1}{dx_2},$$

$$Q_{i,k} \frac{\partial \bar{U}_j'}{\partial x_k'} = \delta_{j1} Q_{i,2} \frac{d\bar{U}_1}{dx_2},$$

$$\frac{1}{2}\left(\bar{U}_k + \bar{U}_k'\right) \frac{\partial}{\partial (x_k)_m} Q_{i,j} = 0,$$

$$\left(\bar{U}_k' - \bar{U}_k\right) \frac{\partial Q_{i,j}}{\partial r_k} = \left(\bar{U}_1' - \bar{U}_1\right) \frac{\partial Q_{i,j}}{\partial r_1} = r_2 \frac{d\bar{U}_1}{dx_2} \frac{\partial Q_{i,j}}{\partial r_1},$$

where, as usual, $\delta_{ij} = 1$ for $i = j$, zero otherwise.

Eq. (3-9) now reads

$$\frac{\partial Q_{i,j}}{\partial t} + \delta_{i1} Q_{2,j} \frac{d\bar{U}_1}{dx_2} + \delta_{j1} Q_{i,2} \frac{d\bar{U}_1}{dx_2} + r_2 \frac{d\bar{U}_1}{dx_2} \frac{\partial Q_{i,j}}{\partial r_1}$$

$$= -\frac{1}{2} \frac{\partial}{\partial (x_k)_m} \left[S_{i,kj} + S_{ik,j} \right] - \frac{1}{2\rho} \left[\frac{\partial}{\partial (x_i)_m} K_{p,j} + \frac{\partial}{\partial (x_j)_m} K_{i,p} \right]$$

$$+ \frac{1}{2} \nu \frac{\partial^2 Q_{i,j}}{\partial (x_k)_m \partial (x_k)_m} + 2\nu \frac{\partial^2 Q_{i,j}}{\partial r_k \partial r_k} \quad . \qquad (3\text{-}11)$$

If new tensors are defined as

$$\frac{\partial}{\partial r_k} \left[S_{ik,j} - S_{i,kj} \right] = T_{i,j} \ ,$$

$$\frac{1}{\rho} \left[\frac{\partial}{\partial r_i} K_{p,j} - \frac{\partial}{\partial r_j} K_{i,p} \right] = P_{i,j} \ , \qquad (3\text{-}12)$$

and, having in mind that

$$\frac{\partial}{\partial (x_k)_m} \delta_{ik} K_{p,j} = \frac{\partial}{\partial (x_i)_m} K_{p,j} \ , \qquad (3\text{-}13)$$

Eq. (3-11) will read

$$\frac{\partial Q_{i,j}}{\partial t} + \left(\delta_{i1} Q_{2,j} + \delta_{j1} Q_{i,2} + r_2 \frac{\partial Q_{i,j}}{\partial r_1}\right) \frac{d\bar{U}_1}{dx_2}$$

$$= T_{i,j} + P_{i,j} + 2\nu \frac{\partial^2 Q_{i,j}}{\partial r_k \partial r_k}$$

$$- \frac{1}{2} \frac{\partial}{\partial (x_2)_m} \left[S_{i,kj} + S_{ik,j} + \frac{\delta_{ik}}{\rho} K_{p,j} + \frac{\delta_{jk}}{\rho} K_{i,p}\right]$$

$$+ \frac{1}{2} \nu \frac{\partial^2 Q_{i,j}}{\partial (x_2)_m \partial (x_2)_m} \quad . \qquad (3\text{-}14)$$

The last two terms on the right side are due to inhomogeneity and they would not have appeared had the assumption of full homogeneity been made. If, for example, an additional assumption of no shear is made, the second term on the left side drops out and Eq. (3-14) reduces to the corresponding dynamical correlation equation for isotropic-homogeneous turbulence.

Equation (3-14) is as far as we can go without further assumptions. The presence of triple correlations is the basic reason, similarly as in the case of isotropic turbulence, for insolubility of the dynamical two-point correlation equation. To proceed, suitable assumptions concerning Fourier transforms of correlation functions can be made following the approach used in studies on isotropic turbulence.

Defining a part of inhomogeneous terms in Eq. (3-14) as another tensor function,

$$\frac{1}{2}\left[S_{i,kj} + S_{ik,j} + \frac{1}{\rho}\delta_{ik} K_{p,j} + \frac{1}{\rho}\delta_{jk} K_{i,p}\right] = I_{i,j}\ , \qquad (3\text{-}14a)$$

Eq. (3-14) will read

$$\frac{\partial Q_{i,j}}{\partial t} + \left(\delta_{i1} Q_{2,j} + \delta_{j1} Q_{i,2} + r_2 \frac{\partial Q_{i,j}}{\partial r_1}\right)\frac{d\bar{U}_1}{dx_2}$$

$$= T_{i,j} + P_{i,j} + 2\nu \frac{\partial^2 Q_{i,j}}{\partial r_k \partial r_k} - \frac{\partial}{\partial (x_2)_m} I_{i,j} + \frac{1}{2}\nu \frac{\partial^2 Q_{i,j}}{\partial (x_2)_m \partial (x_2)_m} . \qquad (3\text{-}15)$$

In order to write the correlation equation in spectral form, the three-dimensional Fourier transforms are defined as

$$Q_{i,j}(\vec{r}, \vec{x}_m) = \int_{-\infty}^{\infty} E_{i,j}(\vec{k}, \vec{x}_m) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}\ ,$$

$$T_{i,j}(\vec{r}, \vec{x}_m) = \int_{-\infty}^{\infty} F_{i,j}(\vec{k}, \vec{x}_m) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}\ ,$$

$$P_{i,j}(\vec{r}, \vec{x}_m) = \int_{-\infty}^{\infty} \Pi_{i,j}(\vec{k}, \vec{x}_m) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}\ ,$$

$$I_{i,j}(\vec{r}, \vec{x}_m) = \int_{-\infty}^{\infty} D_{i,j}(\vec{k}, \vec{x}_m) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}\ . \qquad (3\text{-}16)$$

From Eq. (3-16) it follows

$$r_2 \frac{\partial}{\partial r_1} Q_{i,j} = - \int_{-\infty}^{\infty} k_1 \frac{\partial E_{i,j}}{\partial k_2} \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} \, ,$$

$$2\nu \frac{\partial^2 Q_{i,j}}{\partial r_k \partial r_k} = -2\nu \int_{-\infty}^{\infty} k^2 E_{i,j} \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} \, ,$$

$$\frac{1}{2}\nu \frac{\partial^2 Q_{i,j}}{\partial (X_2)_m \partial (X_2)_m} = \frac{1}{2} \frac{\partial^2}{\partial (X_2)_m \partial (X_2)_m} \int E_{i,j} \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} \, , \qquad (3\text{-}17)$$

and, the following dynamic equation for the energy spectrum function, $E_{i,j}$, is obtained

$$\frac{\partial E_{i,j}}{\partial t} + \left(\delta_{i1} E_{2,j} + \delta_{j1} E_{i,2} - k_1 \frac{\partial E_{i,j}}{\partial k_2} \right) \frac{d\bar{U}_1}{dX_2}$$

$$= F_{i,j} + \Pi_{i,j} - 2\nu k^2 E_{i,j} - \frac{\partial}{\partial (X_2)_m} D_{i,j} + \frac{1}{2} \frac{\partial^2 E_{i,j}}{\partial (X_2)_m \partial (X_2)_m} . \qquad (3\text{-}18)$$

Contracting the indices, $i = j = k$, one obtains

$$\frac{\partial E_{i,i}}{\partial t} + \left(2E_{1,2} - k_1 \frac{\partial E_{i,i}}{\partial k_2} \right) \frac{d\bar{U}_1}{dX_2}$$

$$= F_{i,i} - 2\nu k^2 E_{i,i} - \frac{\partial D_{i,i}}{\partial (X_2)_m} + \frac{1}{2}\nu \frac{\partial^2 E_{i,i}}{\partial (X_2)_m \partial (X_2)_m} \, , \qquad (3\text{-}19)$$

where from the condition of incompressibility $P_{i,i} = 0$ and, consequently, $\Pi_{i,i} = 0$. The spectrum functions are then averaged over all directions of the wave number k by summing over all values of i. Thus, mean values of these functions over spherical surfaces k = const. are obtained. New spectrum functions are defined as

$$E(k) = \frac{1}{2}\int E_{ii}\, dA(k),$$

$$F(k) = \frac{1}{2}\int F_{ii}\, dA(k),$$

$$\tau(k) = \frac{1}{2}\int \left(2E_{1,2} - k_1 \frac{\partial E_{ii}}{\partial k_2}\right) dA(k),$$

$$D(k) = \frac{1}{2}\int D_{ii}\, dA(k). \qquad (3\text{-}19a)$$

Application of the above averaging procedure to Eq. (3-19) yields

$$\frac{\partial E(k)}{\partial t} + \tau(k)\frac{d\overline{U}_1}{dx_2}$$

$$= F(k) - 2\nu k^2 E(k) - \frac{\partial D(k)}{\partial (x_2)_m} + \frac{1}{2}\nu \frac{\partial^2 E(k)}{\partial (x_2)_m \partial (x_2)_m} \qquad (3\text{-}20)$$

Without the second term on the left side (production term) and the last two terms on the right side (inhomogeneity), Eq. (3-20) reduces to the spectrum equation obtained for the case of isotropic-homogeneous turbulence (Batchelor 1953). Similarly as in that case, assumptions about the spectrum transfer function, $F(k)$, are needed here also. In the case of shear turbulence, however, additional assumptions must be made concerning the spectrum production function, $\tau(k)$, and the spectrum diffusion function, $D(k)$.

The physical meaning of the spectrum function $\tau(k)$ can be made clearer if the defining expression for $\tau(k)$ is integrated from 0 to ∞ :

$$\int_0^\infty \tau(k)\,dk = \frac{1}{2}\int_0^\infty \left[\int\left(2E_{1,2} - k_1\frac{\partial E_{ii}}{\partial k_2}\right)dA(k)\right]dk \; .$$

The first part of the integral on the right side can be evaluated from the relationship between $Q_{1,2}$ and $E_{1,2}$ and from the defining expression for spherical averages (3-19a) as

$$Q_{1,2}(r) = \frac{1}{2}\int Q_{1,2}\,dA = \frac{1}{2}\iint_{-\infty}^{\infty} E_{1,2}\exp(i\,\vec{k}\cdot\vec{r})\,d\vec{k}\,dA \; .$$

For $r = 0$

$$Q_{1,2}(0) = \overline{u_1 u_2} = \frac{1}{2}\iint_0^\infty 2E_{1,2}\,dA\,dk \; .$$

Thus,

$$\iint_{-\infty}^{\infty} E_{1,2}\,dA\,dk = Q_{1,2}(0) = \overline{u_1 u_2} \; . \qquad (3\text{-}21)$$

Using the first of the relations (3-17), following the same procedure, and noticing that $\left[r_2\left(\partial Q_{ii}/\partial r_1\right)\right]_{r=0} = 0$, one obtains

$$\frac{1}{2}\iint_0^\infty k_1\frac{\partial E_{ii}}{\partial k_2} = 0 \; . \qquad (3\text{-}22)$$

From (3-21) and (3-22), it follows

$$\int_0^\infty \tau(k)\,dk = \overline{u_1 u_2} \; ,$$

i.e., $\tau(k)$ is the shear stress spectrum function and $(d\bar{U}_1/dx_2)_0 \int^k \tau(p)\, dp$ is the production of turbulence in the range from k = 0 to k.

In further analysis, we assume that the turbulence Reynolds number is large enough and that there exist an equilibrium range for large wave numbers. We shall study the spectral law in the equilibrium range in a way analogous to the case of isotropic turbulence.

For a steady flow, Eq. (3-20) reads

$$\tau(k)\frac{d\bar{U}_1}{dx_2} = F(k) - 2\nu k^2 E(k) - \frac{\partial}{\partial x_2} D(k) + \frac{1}{2}\nu \frac{\partial^2 E(k)}{\partial x_2 \partial x_2} .$$

Integrating from k to ∞ :

$$\frac{d\bar{U}_1}{dx_2}\int_k^\infty \tau(p)\, dp = \int_k^\infty F(p)\, dp - 2\nu \int_k^\infty p^2 E(p)\, dp - \frac{\partial}{\partial x_2}\int_k^\infty D(p)\, dp + \frac{1}{2}\nu \frac{\partial^2 \int_k^\infty E(p)\, dp}{\partial x_2 \partial x_2} , \quad (3\text{-}24)$$

and, because

$$2\nu \int_k^\infty p^2 E(p)\, dp = 2\nu \int_0^\infty p^2 E(p)\, dp - 2\nu \int_0^k p^2 E(p)\, dp$$

$$= \varepsilon - 2\nu \int_0^k p^2 E(p)\, dp ,$$

one obtains

$$\mathcal{E} = 2\nu\int_0^k p^2 E(p)\,dp + \int_k^\infty F(p)\,dp - \frac{d\bar{U}_1}{dx_2}\int_k^\infty \tau(p)\,dp$$

$$-\frac{\partial}{\partial x_2}\int_k^\infty D(p)\,dp + \frac{1}{2}\nu\,\frac{\partial^2 \int_k^\infty E(p)\,dp}{\partial x_2\,\partial x_2}\;. \qquad (3\text{-}25)$$

The first term on the right side represents the viscous dissipation in the wave number range from 0 to k. The second term is again the transfer of turbulence energy from smaller to larger wave numbers. The transfer term arises from the nonlinear terms in the equation of motion. It does not make any contribution to the total energy, i.e., $\int_0^\infty F(k)dk = 0$. The third term is the production term. It represents the production of turbulence in the range from k to ∞. The last two terms are due to inhomogeneity in the x_2 direction. The term $-\partial/\partial x_2 \int_k^\infty D(p)\,dp$ represents turbulent diffusion in wave number space.

The last term represents diffusion of turbulent energy due to molecular viscosity. It is obvious from Eq. (3-24) that this term is very small due to small values of the molecular viscosity, ν, and small values of the integral from k to ∞ of E(k) in the equilibrium range where the inequality $\int_0^k E(p)dp \gg \int_k^\infty E(p)\,dp$ is usually assumed. Thus, the molecular diffusion term can be completely neglected.

In a more accurate analysis, however, the molecular diffusion term should be combined with the dissipation term $\int_0^\infty 2\nu k^2 E(p)\,dp$. The corresponding terms in real space, i.e., in the equation of motion for an inhomogeneous, incompressible turbulent flow are

$$\frac{1}{2}\nu\frac{\partial^2 \overline{q^2}}{\partial x_i \partial x_i} \; ; \qquad -\nu\frac{\partial \overline{u_j u_j}}{\partial x_i \partial x_i} \quad .$$

These two terms result from the term representing the work done per unit of mass and of time by the viscous shear stresses plus the term representing the dissipation per unit of mass. However, they do not have the same meaning and the last term, $-\nu\,\partial \overline{u_j u_j}/\partial x_i \partial x_i$, does not represent the total dissipation in an inhomogeneous turbulent flow (Hinze 1959). If, on the other hand, the departure from homogeneity is not too strong (what may be assumed to be true in a boundary layer except very close to the wall) the both, combined, viscous terms are nearly equal to the dissipation for a homogeneous turbulent flow, i.e., to $-\nu\,\overline{\partial u_j \partial u_j}/\partial x_i \partial x_i$. The corresponding term in wave number space is $2\nu\int_0^\infty k^2 E(k)\,dk$. Thus by equating this term with the total dissipation, ε, in Eq. (3-24), it is tacitly assumed that the molecular diffusion term is negligible.

Besides being negligible, the molecular diffusion term is not of great interest on yet another basis. In the present work, the primary interest is in studying the shape of the energy spectrum function of shear turbulence in the equilibrium region and the influence of particular spectrum functions on that shape. The molecular diffusion term, however, does not affect the basic functional shape of the energy spectrum function E(k) but affects only the total dissipation. Equation (3-25) thus reads

$$\mathcal{E} = 2\nu \int_0^{k} E(p)\, p^2\, dp + \int_k^{\infty} F(p)\, dp - \frac{d\bar{U}_1}{dx_2} \int_k^{\infty} \mathcal{T}(p)\, dp - \frac{\partial}{\partial x_2} \int_k^{\infty} \mathcal{D}(p)\, dp \,. \tag{3-26}$$

In order to determine E(k) from Eq. (3-26), the functions F(k), $\mathcal{T}$(k), and $\mathcal{D}$(k) must be expressed in terms of E(k). As the wave number k is assumed to be sufficiently large and the equilibrium region is assumed to exist, the assumptions concerning the spectrum function F(k) are similar to those used for isotropic turbulence.

For the spectrum function $\mathcal{T}$(k), Tchen (1953) suggested a mechanism of spectral transfer that enables it to be linked with the energy spectrum function E(k). Depending on the relationship between the scale of the main motion and the size of turbulent eddies, two cases are distinguished: a weak turbulence, when the transfer is determined by the gradient of the main motion, and a strong turbulence, when the transfer is determined by the vorticity of the turbulent motion. For the case of weak turbulence, Tchen (1953) assumed that $\tau(k)$ is given as

$$\int_k^\infty \tau(p)\,dp = -\nu_T(k)\frac{d\bar{U}_1}{dx_2}, \qquad (3\text{-}27)$$

which is an anolog of Boussinesq's concept concerning turbulent transfer of momentum, i.e.,

$$\int_0^\infty \tau(k)\,dk = \overline{u_1 u_2} = -\varepsilon_m \frac{d\bar{U}_1}{dx_2} .$$

Thus, in this case,

$$-\frac{d\bar{U}_1}{dx_2}\int_K^\infty \tau(k)\,dk = \nu_T(k)\left(\frac{d\bar{U}_1}{dx_2}\right)^2,$$

which can be interpreted as the dissipation of energy of the main motion by eddy viscosity.

For the case of strong turbulence with strong interaction between the main and turbulent motion, Tchen suggests that the flow of a quantity created by the diffusional mechanism of transfer is proportional to the vorticity of turbulence, instead of the main motion vorticity as in the first case. Therefore,

$$\int_K^\infty \tau(p)\,dp = -c\cdot\nu_T(k)\left\{\text{Turbulent Vorticity}\right\}$$

$$= -c\cdot\nu_T(k)\left[\int_0^k 2p^2E(p)\,dp\right]^{1/2}, \qquad (3\text{-}28)$$

where c is a constant. Hence,

$$-\frac{d\bar{U}_1}{dx_2}\int_k^\infty \tau(p)\,dp = c\cdot\nu_T(k)\frac{d\bar{U}_1}{dx_2}\left[\int_0^k 2p^2E(p)\,dp\right]^{1/2}.$$

Of main interest is the case of strong turbulence, which is the case, for example, of a turbulent boundary layer with large Reynolds number. With the relation (3-28), Eq. (3-25) will read:

$$\varepsilon = 2\nu\int_0^k p^2E(p)\,dp + \int_k^\infty F(p)\,dp$$

$$+ c\,\nu_T(k)\frac{d\bar{U}_1}{dx_2}\left[\int_0^k 2p^2E(p)\,dp\right]^{1/2} - \frac{\partial}{\partial x_2}\int_k^\infty D(p)\,dp \quad . \qquad (3\text{-}29)$$

3. Previous Solutions

Tchen (1953) was the first to study the spectrum of energy in turbulent shear flow. He used Heisenberg's approximation for the spectrum transfer function, F(k), and consequently for $\nu_T(k)$,

$$\int_k^\infty F(p)\,dp = \gamma_H^\nu \int_k^\infty \sqrt{\frac{E(p)}{p^3}}\,dp \int_0^k 2p^2E(p)\,dp \quad , \qquad (3\text{-}30)$$

where γ_H^ν is a constant, and the first integral on the right side represents turbulent viscosity, $\nu_T(k)$. Tchen considered only asymptotic solutions, i.e., he considered two particular subranges: (a) nonviscous subrange (the lowest wave number range of the equilibrium range), and (b) viscous subrange (the upper part of the equilibrium range). In this way it was possible to simplify Eq. (3-29) because some of the spectrum functions can be assumed negligible for a particular subrange.

The simplest solutions are obtained for the case of weak turbulence. In this case the production term is simply

$$- \frac{d\bar{U}_1}{dx_2} \int_k^\infty \tau(p)\,dp = c \cdot \nu_T(k) \left(\frac{d\bar{U}_1}{dx_2}\right)^2 .$$

Neglecting the diffusion term, a solution of Eq. (3-29) for this case can be easily obtained. However, the nature of this solution (Hinze 1959) is entirely the same as of the solution obtained for the equilibrium range of isotropic turbulence. The weak turbulence case does not affect the shape of the energy spectrum that is obtained for isotropic turbulence. There is only a larger dissipation. This means that the solution of Eq. (3-29) for the weak turbulence case can be obtained if ε in the solution for isotropic turbulence is replaced with $\varepsilon + \nu(dU_1/dx_2)^2$. Considering the underlying assumptions made for the weak turbulence case, this final result is not surprising.

For the strong turbulence case (the case of strong interaction between the main and turbulent motion) in which we are primarily interested, an asymptotic solution for nonviscous subrange (lower wave number region of the equilibrium range) can be obtained if the dissipation and the transfer spectrum function are assumed negligible. If the diffusion term is neglected too, Eq. (3-29) reduces to the simple relation

$$\mathcal{E} = c\,\frac{d\bar{U}_1}{dx_2}\,\nu_T(k)\left[\int_0^k 2p^2 E(p)\,dp\right]^{1/2}. \qquad (3\text{-}31)$$

Using Heisenberg's expression for $\nu_T(k)$, a solution for E(k) is readily obtained as

$$E(k) = \left[\frac{\mathcal{E}}{c\; d\bar{U}_1/dx_2}\right]\cdot k^{-1}. \qquad (3\text{-}32)$$

This is the original asymptotic solution obtained by Tchen (1953) for this subrange.

This asymptotic solution, however, does not show how much a strong production term affects the inertial subrange. For this information, a solution for E(k) that would connect the low wave number region with the "-5/3" inertial subrange is needed.

Panchev (1968, 1969) considered solutions for E(k) in the case of strong turbulence when the dissipation, the transfer, and the production function are taken into account simultaneously. The effects of inhomogeneities were, however, neglected. He also used Heisenberg's approximation for the transfer spectrum function, but for $\nu_T(k)$ he used a generalized form that reduces to Heisenberg's and Howells' form for $s = 1$ and $s = 2$, respectively:

$$\int_k^\infty F(p)\,dp = \gamma_s \left[\int_k^\infty E^{s/2}(p)\, p^{-\frac{s}{2}-1}\, dp \right]^{1/s} \int_0^k 2E(p)\,p^2\,dp \,. \quad (3\text{-}33)$$

With Heisenberg's and Howell's expression for $\nu_T(k)$, solutions for the three-dimensional energy spectrum were obtained in parametric forms. For corresponding ranges of wave numbers, the k^{-1} range, Eq. (3-32), the inertial $k^{-5/3}$ subrange, and the viscous k^{-7} range are obtained. The k^{-7} result is obviously a consequence of the Heisenberg concept for the spectrum transfer function.

The sparce data on energy spectra measured in shear turbulent flows show sometimes an approximate k^{-1} variation of $E(k)$ in the lower subrange of the equilibrium range as suggested by Eq. (3-32). The conditions for the existence of this law, namely, a strong velocity gradient and, consequently, a large production term at the point in consideration usually mean that a strong inhomogeneity is created in the considered region.

The existing data on energy spectra in shear turbulent flows show many inconsistencies and irregularities in the wave number region where the effect of shear is expected to be the strongest. This indicates that the behavior of the lower part of the equilibrium range of shear turbulence cannot be explained as a result of the effects of the velocity gradient only. The effects of inhomogeneities must be accounted for if a more realistic description of the spectrum behavior is to be obtained.

4. Solutions for the Complete Spectrum Equation

Before an attempt is made to solve the complete spectrum equation, Eq. (3-29), additional assumptions must be made concerning the diffusion spectrum function, D(k).

To see more clearly the physical meaning of the inhomogeneous term in Eq. (3-29), it is integrated from 0 to ∞ and from the defining expression for sperical averages, Eq. (3-19a), one obtains

$$\int_0^\infty D(k)\,dk = \frac{1}{2}\int_0^\infty \left[\int D_{ii}\,dA\right] dk \quad . \tag{3-34}$$

On the other hand, using again the definition for spherical averages,

$$I_{i,j}(r) = \frac{1}{2}\int I_{i,j}\,dA \quad ,$$

and, using the Fourier transform relations, (3-16),

$$I_{i,j}(r) = \frac{1}{2}\int\left[\int_{-\infty}^{\infty} D_{i,j}\exp(i\vec{k}\cdot\vec{r})\,d\vec{k}\right] dA \quad .$$

For $r = 0$, and contracting indices, $i = j = k$,

$$I_{i,i}(0) = \frac{1}{2}\int\int_{-\infty}^{\infty} D_{ii}\,dk\,dA$$

$$= \int\int_0^\infty D_{ii}\,dA\,dk \quad . \tag{3-35}$$

Thus, from Eq. (3-34) and Eq. (3-35)

$$\int_0^\infty D(k)\,dk = \frac{1}{2}\iint_0^\infty D_{ii}\,dA\,dk = \frac{1}{2} I_{ii}(0) .$$

Going back to Eq. (3-14a),

$$I_{i,j} = \frac{1}{2}\left[S_{i,kj} + S_{ik,j} + \frac{\delta_{ik}}{\rho} K_{p,j} + \frac{\delta_{jk}}{\rho} K_{i,p}\right] ;$$

for $i = j = k$

$$I_{i,i} = \frac{1}{2}\left(\overline{u_i u_i' u_i'} + \overline{u_i u_i u_i'} + \frac{1}{\rho}\overline{p u_i'} + \frac{1}{\rho}\overline{u_i p'}\right) ,$$

and, for $r = 0$,

$$I_{ii}(0) = \frac{1}{2}\left[2\overline{u_i u_i^2} + \frac{2}{\rho}\overline{u_i p}\right] = \left[\overline{u_i\left(u_i^2 + \frac{p}{\rho}\right)}\right] . \quad (3\text{-}36)$$

Thus,

$$-\frac{\partial}{\partial x_i}\int_0^\infty D(k)\,dk = -\frac{\partial}{\partial x_i}\frac{1}{2} I_{i,i}(0)$$

$$= -\frac{\partial}{\partial x_i}\frac{1}{2}\left[\overline{u_i\left(u_i^2 + \frac{p}{\rho}\right)}\right] , \quad (3\text{-}37)$$

which corresponds to the term representing inhomogeneity in the turbulence energy equation (obtained by multiplying the one-point equation, Eq. (3-1), by u_i and taking an average). This term represents the convective diffusion by turbulence of the total turbulence energy. An analogy with diffusional processes is used to write an expression for the corresponding inhomogeneous term in wave number space.

From Eq. (3-37) it is possible to see that $\int_0^\infty D(k)\,dk$

represents a flux of turbulent energy. Accordingly, it is assumed that $\int_k^\infty D(p)\,dp$ can be written in wave number space as the product of a turbulent viscosity and the energy gradient:

$$\int_k^\infty D(p)\,dp = C_1 \nu_T(k) \frac{\partial}{\partial x_2}\left[\int_0^k E(p)\,dp\right]$$

Thus, the diffusion term reads

$$-\frac{\partial}{\partial x_2}\int_k^\infty D(p)\,dp = -C_1 \frac{\partial}{\partial x_2}\left\{\nu_T(k)\frac{\partial}{\partial x_2}\int_0^k E(p)\,dp\right\}. \qquad (3\text{-}37a)$$

As the primary interest is in the equilibrium range, the usual assumption is that $\int_0^k E(p)\,dp \gg \int_k^\infty E(p)\,dp$, i.e.,

$$\int_0^k E(p)\,dp \sim \int_0^\infty E(k)\,dk = \frac{1}{2}\overline{u_i^2} = \frac{1}{2}\overline{q^2}.$$

Hence, Eq. (3-29) reads now

$$\mathcal{E} = 2\nu\int_0^k E(p)p^2\,dp + \int_k^\infty F(p)\,dp$$
$$+ C\nu_T(k)\frac{d\bar{U}_1}{dx_2}\left[\int_0^k 2E(p)p^2\,dp\right]^{1/2} - C_1\frac{\partial}{\partial x_2}\left\{\nu_T(k)\frac{\partial}{\partial x_2}\left(\frac{1}{2}\overline{q^2}\right)\right\}, \qquad (3\text{-}38)$$

where writing the total turbulence energy as $\frac{1}{2}\overline{q^2}$ in the expression for the diffusion term means that the energy diffusion due to existence of velocity-pressure correlations is neglected.

The diffusion term complicates in this form the solution of Eq. (3-38). A further simplification is needed. Expanding the term, one obtains:

$$C_1 \frac{\partial}{\partial x_2}\left\{ \nu_T(k) \frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x_2} \right\} = C_1 \nu_T(k) \frac{\partial^2(\frac{1}{2}\overline{q^2})}{\partial x_2^2} + C_1 \frac{\partial \nu_T(k)}{\partial x_2} \frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x_2} . \qquad (3\text{-}38a)$$

We assume now that the gradient $\partial\nu_T(k)/\partial x_2$ is negligible and that consequently the second term on the right side can be neglected altogether.

Thus, the diffusion term reads

$$\frac{\partial}{\partial x_2} \int_k^\infty D(p)\, dp = C_2 \nu_T(k) \frac{\partial^2(\frac{1}{2}\overline{q^2})}{\partial x_2^2} . \qquad (3\text{-}39)$$

Obviously, this form can be viewed as describing the diffusion of turbulent energy in a locally homogeneous turbulent field, i.e., $\nu_T(k)$ is assumed approximately constant in the considered region and the expression (3-39) follows from (3-38a).

In considering a shear turbulent flow where the gradient of the total turbulent energy decreases along the axis x_2 (as in a turbulent boundary layer) the second derivative $\partial^2(\frac{1}{2}\overline{q^2})/\partial x_2^2$ will be negative and the diffusion term will be therefore positive.

Finally, for the case of strong turbulence, Eq. (3-38) reads

$$\mathcal{E} = 2\nu \int_0^k E(p) p^2 + \int_k^\infty F(p) dp$$
$$+ c \frac{d\bar{U}_1}{dx_2} \nu_T(k) \left[\int_0^k 2E(p) p^2 dp \right]^{1/2} + C_2 \nu_T(k) \frac{\partial^2 (\frac{1}{2}\bar{q}^2)}{\partial x_2^2} \quad . \qquad (3\text{-}40)$$

Thus, the equilibrium in the exchange of energy is expressed as

$$\mathcal{E} = \text{Dissipation}\,(0,k) + \text{Transfer}\,(k,\infty)$$
$$+ \text{Production}\,(k,\infty) + \text{Diffusion}\,(k,\infty) \quad .$$

In the following paragraphs, solutions of the balance equation, Eq. (3-40), are obtained when the complete equation is considered, i.e., when all terms are taken into account. Following previous approaches, solutions are first derived using Heisenberg's type of approximation for the transfer function $F(k)$. Several simple solutions for the energy spectrum are then derived using the Modified Obukhov approximation (Ellison 1962). In the last part, the vorticity approximation approach is used for the derivation of closed form expressions for $E(k)$.

The solutions are then computed for different production and diffusion parameters. A discussion of these results and comparision with data from actual shear turbulent flows are presented in the last section.

4.1 Solutions with Heisenberg's Approximation

In this case (Heisenberg 1948),

$$\int_k^\infty F(k)\,dk = \gamma_H \int_k^\infty \sqrt{\frac{E(p)}{p^3}}\,dp \int_0^k 2p^2 E(p)\,dp ,$$

where γ_H is a constant and the first integral is the turbulent viscosity ν_T (k), i.e.,

$$\nu_T(k) = \gamma_H \int_k^\infty \sqrt{\frac{E(p)}{p^3}} .$$

The spectrum equation, Eq. (3-40), becomes now

$$\mathcal{E} = 2\nu \int_0^k E(p)\,p^2\,dp + \gamma_H \nu_T(k) \int_0^k 2p^2 E(p)\,dp$$

$$+ c_1 \nu_T(k) \frac{d\bar{U}_1}{dx_2} \left[\int 2p^2 E(p)\,dp\right]^{1/2} + c_2 \nu_T(k) \frac{\partial^2 \frac{1}{2} q^2}{\partial x_2^2} ,$$

or,

$$\mathcal{E} = 2\nu \int_0^k p^2 E(p)\,dp + \gamma_H \nu_T(k) \left\{ \int_0^k 2p^2 E(p)\,dp \right.$$

$$\left. + \frac{c_1}{\gamma_H} \frac{d\bar{U}_1}{dx_2} \left[\int_0^k 2p^2 E(p)\,dp\right]^{1/2} + \frac{c_2}{\gamma} \frac{\partial^2 (\frac{1}{2} \overline{q^2})}{\partial x_2^2} \right\} .$$

With the expression for $\nu_T(k)$,

$$\mathcal{E} = 2\nu \int p^2 E(p)\,dp + \gamma_H \int_k^\infty \sqrt{\frac{E(p)}{p^3}}\,dp \left\{ \int_0^k 2p^2 E(p)\,dp \right.$$

$$\left. + \frac{c_1}{\gamma_H} \frac{d\bar{U}_1}{dx_2} \left[\int_0^k 2p^2 E(p)\,dp \right]^{1/2} + \frac{c_2}{\gamma_H} \frac{\partial^2 \left(\frac{1}{2}\overline{q^2} \right)}{\partial x_2^2} \right\}. \quad (3\text{-}41)$$

The constant γ_H should be determined from the condition that in the inertial subrange the following relationship holds:

$$E(k) = \alpha \mathcal{E}^{2/3} k^{-5/3}.$$

Consequently, one obtains

$$\gamma_H = \frac{8}{9} \alpha^{-3/2},$$

where α is the Kolmogorov constant for the inertial subrange.

Thus,

$$\mathcal{E} = 2\nu \int_0^k p^2 E(p)\,dp + \frac{8}{9} \alpha^{-3/2} \int_k^\infty \sqrt{\frac{E(p)}{p^3}}\,dp \left\{ \int_0^k 2p^2 E(p)\,dp \right.$$

$$\left. + \frac{c_1}{\gamma_H} \frac{d\bar{U}_1}{dx_2} \left[\int_0^k 2p^2 E(p)\,dp \right]^{1/2} + \frac{c_2}{\gamma_H} \frac{\partial^2 \left(\frac{1}{2}\overline{q^2} \right)}{\partial x_2^2} \right\}. \quad (3\text{-}42)$$

A nondimensional form is more convenient for handling and computation of the results; hence, the following non-dimensional variables are introduced:

$$x = k/k_0 \;, \quad \mathcal{E}(x) = E(k)/E_0 \;,$$

where

$$k_0 = \alpha^{-3/4}(\varepsilon/\nu^3)^{1/4} \;, \quad E_0 = \alpha^{9/4}(\varepsilon\nu^5)^{1/4} . \qquad (3\text{-}43)$$

Substituting into Eq. (3-42), one obtains

$$2\nu\int_0^k p^2 E(p)\,dp = \varepsilon\int_0^x \mathcal{E}(p)p^2\,dp \;,$$

$$\int_k^\infty \sqrt{\frac{E(p)}{p^3}}\,dp = \alpha^{3/2}\int_x^\infty \sqrt{\frac{\mathcal{E}(p)}{p^3}}\,dp \;,$$

$$\int_0^k 2p^2 E(p)\,dp = \left(\frac{\varepsilon}{\nu}\right)\int_0^x \mathcal{E}(p)p^2\,dp \,.$$

Hence, Eq. (3-42) in nondimensional form reads

$$1 = \int_0^x 2\mathcal{E}(p)p^2\,dp + \left(\frac{8}{9}\right)\int_x^\infty \sqrt{\frac{\mathcal{E}(p)}{p^3}}\,dp \left\{\int_0^x 2\mathcal{E}(p)p^2\,dp\right.$$

$$\left. + \frac{c_1}{\delta_H}\left(\frac{\nu}{\varepsilon}\right)^{1/2}\frac{d\bar{U}_1}{dx_2}\left[\int_0^x 2p^2\mathcal{E}(p)\,dp\right]^{1/2} + \frac{c_2}{\delta_H}\left(\frac{\nu}{\varepsilon}\right)\frac{\partial^2(\frac{1}{2}q^2)}{\partial x_2^2}\right. , \qquad (3\text{-}44)$$

or, writing

$$\left(\frac{C_1}{\gamma_H}\right)\left(\frac{\nu}{\varepsilon}\right)^{1/2}\frac{d\bar{U}_1}{dX_2} = m,$$

$$\left(\frac{C_2}{\gamma_H}\right)\left(\frac{\nu}{\varepsilon}\right)\frac{\partial^2(\frac{1}{2}\bar{q^2})}{\partial X_2^2} = n,$$

where m and n represent the production and the diffusion parameter, respectively, at the point in the flow where the energy spectrum is measured.

Eq. (3-42) now reads

$$1 = \int_0^x 2\mathcal{E}(p)p^2dp + \left(\frac{8}{9}\right)\int_x^\infty \sqrt{\frac{\mathcal{E}(p)}{p^3}}\left\{\int_0^x 2\mathcal{E}(p)p^2dp + m\left[\int_0^x 2\mathcal{E}(p)p^2dp\right]^{1/2} + n\right\}. \quad (3\text{-}45)$$

Let
$$Z^2 = \int_0^x 2\mathcal{E}(p)p^2dp,$$

so that
$$\mathcal{E}(x) = \frac{Z}{x^2}\frac{dZ}{dx}. \quad (3\text{-}46)$$

Eq. (3-45) now becomes

$$1 = Z^2 + \left(\frac{8}{9}\right) \int_x^{\infty} \sqrt{\frac{\mathcal{E}(p)}{p^3}}\, dp \left\{ Z^2 + mZ + n \right\}, \qquad (3\text{-}47)$$

or

$$\frac{1 - Z^2}{Z^2 + mZ + n} = \left(\frac{8}{9}\right) \int_x^{\infty} \sqrt{\frac{\mathcal{E}(p)}{p^3}}\, dp \ .$$

Differentiating with respect to x, and after some rearrangement, one obtains

$$\frac{dZ}{dx} = \left(\frac{2}{3}\right)^4 \frac{Z\left[mZ + Z^2 + n\right]^4}{\left[\frac{m}{2} + (1+n)Z + \frac{m}{2} Z^2\right]^2} \cdot x^{-5} . \qquad (3\text{-}48)$$

Hence, this ordinary differential equation with separated variables and the relationship (3-46) represent a parametric solution for the three-dimensional energy spectrum, $\mathcal{E}(x)$, when the complete spectrum equation, Eq. (3-45), is considered.

If we concentrate now our attention on the nonviscous region and, consequently, neglect the dissipation term, Eq. (3-47) becomes

$$1 = \left(\frac{8}{9}\right) \int_x^{\infty} \sqrt{\frac{\mathcal{E}(p)}{p^3}}\, dp \left\{ Z^2 + mZ + n \right\} . \qquad (3\text{-}49)$$

After differentiation with respect to x, one obtains

$$\frac{dZ}{dx} = \left(\frac{8}{9}\right)^2 \frac{Z\,[Z^2 + mZ + n]^4}{(2Z+m)^2}\, x^{-5} \qquad (3\text{-}50)$$

Though in this case the differential ratio dz/dx is somewhat simplified, we still have a fairly complex parametric solution for $\mathcal{E}(x)$, and an assessment of the effects of the production and the diffusion term on the energy spectrum in the nonviscous region is impossible without a numerical solution for $\mathcal{E}(x)$.

4.2 Solutions with Howells' Approximation

Another Heisenberg's type of approximation for the turbulent viscosity $\nu_T(k)$, and consequently for the transfer function F(k), is the approximation due to Howells (1960). He originally proposed it in an article on spectra of scalar fields. It reads

$$\nu_T(k) = \delta_{HW} \left[\int_k^{\infty} E(p)\, p^{-2}\, dp \right]^{1/2}. \qquad (3\text{-}51)$$

Both Howell's and Heisenberg's approximations can be viewed as particular cases of a general expression for $\nu_T(k)$ that must satisfy the following requirements: (a) its dimensions must be square of length divided by time, and (b) the turbulent viscosity $\nu_T(k)$ is expressed as an integral over all wave numbers greater than k. There are several of these general expressions for $\nu_T(k)$. Stewart and Townsned (1951) showed that this particular form for $\nu_T(k)$ in combination with the second integral in the expression for the transfer function leads to a power law energy spectrum in the dissipation range.

With Howell's expression for $\nu_T(k)$, Eq. (3-40) becomes

$$\varepsilon = 2\nu \int_0^k E(p)p^2 dp + \gamma_{HW}\left[\int_k^\infty E(p)p^{-2}dp\right]^{1/2}\left\{\int_0^k 2p^2 E(p)dp + \frac{c_1}{\gamma_{HW}}\frac{d\bar{U}_1}{dx_2}\left[\int_0^k 2p^2 E(p)dp\right]^{1/2} + \frac{c_2}{\gamma_{HW}}\frac{\partial^2(k\bar{q}^2)}{\partial x_2^2}\right\}.$$

For Howell's approximation one finds, in the same way as in the case of Heisenberg's approximation,

$$\gamma_{HW} = \frac{2}{3}\left(\frac{8}{9}\right)^{1/2}\alpha^{-3/2}.$$

Using nondimensional variables (3-43), relations (3-44a), and making again the substitution

$$Z^2 = \int_0^x 2\,\mathcal{E}(p)p^2 dp\,, \quad \mathcal{E}(x) = \frac{Z}{x^2}\frac{dZ}{dx}\,,$$

Eq. (3-40) becomes

$$1 = Z^2 + \frac{2}{3}\left[\frac{8}{3}\int_x^\infty \left(Z\frac{dZ}{dx}\right)p^{-4}dp\right]^{1/2}\left\{Z^2 + mZ + n\right\}, \quad (3\text{-}53)$$

or

$$\frac{1-Z^2}{Z^2+mZ+n} = \frac{2}{3}\left[\frac{8}{3}\int_x^\infty \left(Z\frac{dZ}{dx}\right)p^{-4}dp\right]^{1/2}.$$

Squaring both sides and differentiating with respect to x, one obtains

$$\frac{Z(mZ+Z^2+n)^3}{(1-Z^2)\left[m+2Z+mZ^2+2nZ\right]} = \left(\frac{27}{16}\right)x^4 . \qquad (3\text{-}54)$$

Thus, because the differential ratio dz/dx cancels on both sides, we obtain directly x = f(z) which, together with the relation (3-46), represents a parametric solution for the energy spectrum function $\mathcal{E}(x)$, i.e.,

$$x = f(Z),$$
$$\mathcal{E}(x) = \frac{Z}{x^2}\frac{dZ}{dx} .$$

If the dissipation spectrum function is neglected, Eq. (3-53) becomes

$$1 = \frac{2}{3}\left[\frac{8}{3}\int_x^\infty \mathcal{E}(p)p^{-2}dp\right]^{1/2}\left\{Z^2+mZ+n\right\}, \qquad (3\text{-}55)$$

and, following the above procedure, a parametric solution valid in the nonviscous region is obtained as

$$\frac{Z(Z^2 + mZ + n)^3}{(m + 2Z)} = \frac{27}{16} x^4 ,$$

$$\mathcal{E}(x) = \frac{Z}{x^2} \frac{dZ}{dx} . \qquad (3\text{-}56)$$

Again, as in the previous case of Heisenberg's approximation, nothing can be concluded about the effects of the production and the diffusion term on the energy spectrum. For this, the complete numerical computation of the parametric solution is needed.

Even if the lowest part of the equilibruim range is considered and assuming that the transfer function can be neglected, one still obtains only a parametric solution, i.e.,

$$1 = \frac{2}{3} \left[\frac{8}{3} \int_x^\infty \left(Z \frac{dZ}{dx} \right) p^{-4} dp \right]^{1/2} \left\{ mZ + n \right\} . \qquad (3\text{-}57)$$

After squaring and differentiating w.r.t.x, it follows

$$Z(mZ + n)^3 = \frac{27}{16} m x^4 ,$$

$$\mathcal{E}(x) = \frac{Z}{x^2} \frac{dZ}{dx} . \qquad (3\text{-}58)$$

Though Howell's approximation for $\nu_T(k)$, similar to Heisenberg's, does not lead to explicit expressions for the energy spectrum function, $\mathcal{E}(x)$, it does produce parametric solutions that are much less complex. Computation of these solutions and discussion of the result is given in sections 5. and 6.

4.2 Solutions with the Modified Obukhov Approximation

In the study of isotropic turbulence, the transfer theories of Obukhov, Heisenberg, and Kovasznay have played an important role. The particular analytical form of each theory is obtained by considering a plausible physical mechanism of energy transfer and by satisfying certain dimensional requirements. On physical grounds, however, there has been a tendency to favor Heisenberg's theory and its consequences have been worked out in more detail than of any other theory. Some justification for this lies in the fact that Obukhov's approximation, for example, leads to the physically impossible solution for the energy spectrum (Batchelor 1953).

Ellison's modification of the original Obukhov approximation (Ellison 1962) overcomes this difficulty and leads to very interesting solutions for the energy spectrum. It was the first approximation which produced the energy spectrum that trails off in the viscous region as $\sim e^{-k^2}$.

The Modified Obukhov approximation is based on the physical argument that the flux $\int_k^\infty F(p)\,dp$ is a consequence of a Reynolds stress taken as proportional to $k \cdot E(k)$ and acting on the root-mean square rate of strain of all smaller wave numbers, i.e.,

$$\dot{S}(k) = \int_k^\infty F(k)\,dk = \gamma_M' k \cdot E(k) \left[\int_0^k 2E(p) p^2 \, dp \right]^{1/2}, \quad (3\text{-}59)$$

where (from $E(k) = \alpha \varepsilon^{2/3} k^{-5/3}$ for the inertial subrange)

$$\gamma_M' = \sqrt{\frac{2}{3}} \alpha^{-3/2} .$$

The modified Obukhov approximation for the transfer function does not have an explicit expression for $\nu_T(k)$ needed in the spectrum balance equation, Eq. (3-40). This can be easily overcome if the modified Obuhkov approximation is written in a slightly different form, i.e.,

$$S'(k) = \gamma_M' \left\{ \frac{k E(k)}{\left[\int_0^k 2E(p) p^2 \, dp \right]^{1/2}} \right\} \int_0^k 2E(p) p^2 \, dp , \quad (3\text{-}60)$$

where the ratio between the energy $k \cdot E(k)$ and the vorticity $\left[\int_0^k 2E(p) p^2 \, dp \right]^{1/2}$ represents the turbulent viscosity

$$\nu_T(k) = \gamma_M' \frac{k E(k)}{\left[\int_0^k 2E(p) p^2 \, dp \right]^{1/2}} . \quad (3\text{-}61)$$

It is interesting to note that in the derivation of the turbulent diffusion equation by using a probabilistic, random walk approach (Monin & Yaglom 1965) the diffusion coefficient arises as a limit of the ratio W^2/A, where W^2 is the energy, $[L^2/T^2]$, of the diffusing particle and A is a characteristic frequency, $[T^{-1}]$.

Hence, Eq. (3-40) now reads

$$\mathcal{E} = 2\nu \int_0^k E(p)\,p^2\,dp + \gamma_M \frac{k E(k)}{\left[2\int_0^k E(p)\,p^2\,dp\right]^{1/2}} \left\{ \int_0^k 2p^2 E(p)\,dp + \frac{C_1}{\gamma_M}\frac{d\overline{U}_1}{dx_2}\left[\int_0^k 2E(p)\,p^2\,dp\right]^{1/2} + \frac{C_2}{\gamma_M}\frac{\partial^2(k\overline{q^2})}{\partial x_2^2} \right\}. \tag{3-62}$$

Using nondimensional variables (3-43) and the expression for γ_M, Eq. (3-62) becomes

$$1 = \int_0^X 2\mathcal{E}(p)\,p^2\,dp + \sqrt{\tfrac{2}{3}}\, X\, \mathcal{E}(X) \left\{ \left[\int_0^X 2\mathcal{E}(p)\,p^2\,dp\right]^{1/2} + \frac{C_1}{\gamma_M}\left(\frac{\nu}{\varepsilon}\right)^{1/2}\frac{d\overline{U}_1}{dx_2} + \frac{C_2}{\gamma_M}\left(\frac{\nu}{\varepsilon}\right)\left[\int_0^X 2\mathcal{E}(p)\,p^2\,dp\right]^{1/2}\frac{\partial^2(k\overline{q^2})}{\partial x_2^2} \right\}, \tag{3-63}$$

or,

$$1 = Z^2 + \sqrt{\tfrac{2}{3}}\, X\, \mathcal{E}(X) \left\{ Z + m + \frac{n}{Z} \right\}, \tag{3-64}$$

where, as before,

$$\mathcal{Z}^2 = \int_0^x 2\mathcal{E}(p)\,p^2\,dp\,,$$

$$\mathcal{E}(x) = \frac{\mathcal{Z}}{x^2}\frac{d\mathcal{Z}}{dx}\,. \qquad (3\text{-}64a)$$

The complete solution in this case is very simple. For, from (3-64)

$$1-\mathcal{Z}^2 = \sqrt{\tfrac{2}{3}}\,x\cdot\frac{\mathcal{Z}}{x^2}\frac{d\mathcal{Z}}{dx}\left\{\frac{\mathcal{Z}^2+m\mathcal{Z}+n}{\mathcal{Z}}\right\},$$

$$x\,dx = \sqrt{\tfrac{2}{3}}\left\{\frac{\mathcal{Z}^2+m\mathcal{Z}+n}{1-\mathcal{Z}^2}\right\}d\mathcal{Z}\,.$$

Hence,

$$\frac{x^2}{2} = \sqrt{\tfrac{2}{3}}\left\{(1+n)\,\mathrm{Arctan}\,\mathcal{Z} - \frac{m}{2}\ln(1-\mathcal{Z}^2) - \mathcal{Z}\right\}, \qquad (3\text{-}65)$$

and the integration constant is zero from the condition: $x = 0$, $\mathcal{Z} = 0$.

This parametric solution, represented by Eq. (3-65) and Eq. (3-64a) is much simpler than the other two obtained in the previous paragraphs.

In this case, moreover, explicit asymptotic solutions for the energy spectrum function, $\mathcal{E}(x)$, are possible.

Neglecting only the dissipation term, one obtains

$$1 = \sqrt{\tfrac{2}{3}}\; x\,\mathcal{E}(x)\left\{\frac{Z^2+mZ+n}{Z}\right\} ,$$

and, with (3-64a),

$$x^2 = 2\sqrt{\tfrac{2}{3}}\left\{\frac{Z^3}{3}+\frac{mZ^2}{2}+nZ\right\}.$$

The only possible explicit solution for Z would be for $m=2\sqrt{n}$:

$$\frac{(\sqrt{n}+Z)^3}{3} = \frac{Z^3}{3}+\sqrt{n}\,Z^2+nZ+\frac{n^{3/2}}{2} .$$

Thus,

$$\frac{x^2}{2}\sqrt{\tfrac{3}{2}} + \frac{n^{3/2}}{3} = \frac{(\sqrt{n}+Z)^3}{3} ,$$

and

$$Z\left[\left(\tfrac{3}{2}\right)^{3/2}x^2+n^{3/2}\right]^{1/3}-\sqrt{n} .$$

Hence,

$$\mathcal{E}(x) = \frac{Z}{x^2}\frac{dZ}{dx} = \sqrt{\tfrac{3}{2}}\,\frac{\left[(3/2)^{3/2}+n^{3/2}\right]^{1/3}-\sqrt{n}}{\left[(3/2)^{3/2}x^2+n^{3/2}\right]^{2/3}x} , \qquad (3\text{-}66)$$

$$\sqrt{n} = m/2 .$$

For $n = m = 0$, the spectral law of the inertial subrange is recovered:

$$\underset{m=n=0}{\mathcal{E}(x)} = (3/2)^{1/2}(3/2)^{1/2}(3/2)^{-1}x^{2/3}x^{-7/3} = x^{-5/3}.$$

In considering the nonviscous region where $x \ll 1$, it follows from Eq. (3-66) that for smaller and smaller wave numbers as the difference $[(3/2)^{3/2} + m^{3/2}]^{1/3} - \sqrt{n}$ becomes smaller, the exponent representing the behavior of the energy spectrum function, $\mathcal{E}(x) \sim x^{-a}$, becomes less than one. Thus, the x^{-1} law of shear turbulence for the lower wave number region--found for the case when the effects of inhomogeneities are neglected--is affected if diffusion of energy is taken into account. Only for $[(3/2)^{3/2} + m^{3/2}]^{1/3} \gg \sqrt{n}$, will $\mathcal{E}(x) \sim x^{-1}$.

If only the production term due to shear and the diffusion term are considered--the transfer term is assumed negligible--Eq. (3-64) then becomes

$$1 = x\,\mathcal{E}(x)\left\{\frac{mZ + n}{Z}\right\}, \qquad (3\text{-}67)$$

and, with $\mathcal{E}(x) = (Z/x^2)\,dZ/dx$, the solution is

$$x^2 = mZ^2 + 2nZ \quad .$$

Hence,

$$Z_{1,2} = -\frac{n}{m} \pm \sqrt{\left(\frac{n}{m}\right)^2 + \frac{X^2}{m}} \quad .$$

As the value of Z cannot be negative, only the first root is considered. Thus,

$$\mathcal{E}(x) = \frac{Z}{X^2}\frac{dZ}{dX} = \left\{1 - \frac{1}{\left(1 + \frac{m}{n^2}X^2\right)^{1/2}}\right\}\left(\frac{1}{m}\right)X^{-1} . \qquad (3\text{-}68)$$

Obviously, only for $n = 0$ would $\mathcal{E}(x) \sim x^{-1}$, as was obtained originally by Tchen (1953). It is again apparent that the k^{-1} law cannot be maintained if the diffusion due to inhomogeneity is taken into account, i.e., when $m \neq 0$.

Solutions with the Vorticity Approximation

An explicit expression for the energy spectrum function, $\mathcal{E}(x,m,n)$, as a solution of the complete spectral equation is possible if a vorticity approximation approach is employed (Kesic 1969).

In the universal equilibrium range, for the range of wave numbers far from the dissipation range, $k \ll k_o$, we can make the usual assumption that the viscosity ν does not play any important role and the vorticity of turbulence can be expressed, on dimensional grounds (ε and k being the only pertinent variables), as

$$\left[\int_0^k 2E(p)p^2 dp\right]^{1/2} \sim (\varepsilon k^2)^{1/3} .$$

The same result follows as the solution of the vorticity integral if $E(k) \sim k^{-5/3}$.

With this approximation for the vorticity, the transfer function becomes

$$S'(k) = \gamma_v' \, kE(k) \left[\varepsilon k^2\right]^{1/3} ,$$

$$\gamma_v' = 1/\alpha . \qquad (3\text{-}69)$$

The case when the diffusion term is neglected is considered first. Equation (3-40) then becomes

$$\varepsilon = 2\nu \int_0^k E(p)p^2 dp + \gamma_v' kE(k)(\varepsilon k^2)^{1/3} + c_1 \frac{d\bar{U}_1}{dx_2} kE(k). \qquad (3\text{-}70)$$

Because the inertial subrange approximation is employed for the vorticity, i.e., $\gamma_v'(\varepsilon k^2)^{1/3}$, the final expression for the energy spectrum function is not expected to be strictly valid in the dissipation range. This inaccuracy, however, can be accepted because the main impact of shear is expected to be in the lower part of the inertial subrange and it is our main interest here.

Using the nondimensional variables (3-43), Eq. (3-70) reads

$$1 = \int_0^x 2\mathcal{E}(p)p^2 dp + x\mathcal{E}(x)\{x^{2/3} + m\}. \quad (3\text{-}71)$$

With

$$Z^2 = \int_0^x 2\mathcal{E}(p)p^2 dp \ ,$$
$$\mathcal{E}(x) = \frac{Z}{x^2}(dZ/dx) \ ,$$

it follows

$$\frac{x\,dx}{x^{2/3} + m} = \frac{Z\,dZ}{(1 - Z^2)} . \qquad (3\text{-}72)$$

Using the following substitution:

$$x^{1/3} = u \ ,$$
$$dx = 3u^2 du \ ,$$

Eq. (3-72) reduces to the case of table integrals and integration yields

$$\ln\left[\frac{1 - Z^2}{C}\right] = -\left\{\frac{3u^4}{2} - 3m\left[u^2 - m\ln(u^2 + m)\right]\right\} , \quad (3\text{-}73)$$

where C is the integration constant to be determined from the condition: $x = 0$, $Z = 0$.

Thus,

$$C = e^{3m^2 \ln(m)} = (m)^{3m^2}.$$

From Eq. (3-73), and with $u = x^{1/3}$,

$$Z^2 = 1 - C e^{-3\left\{\frac{x^{4/3}}{2} - m x^{2/3} + m^2 \ln\left(x^{2/3} + m\right)\right\}}.$$

Hence,

$$\mathcal{E}(x) = \frac{1}{2x^2}\frac{dZ^2}{dx},$$

$$\mathcal{E}(x) = \frac{1}{2x^2}\frac{d}{dx}\left\{-C e^{-3\left\{\frac{x^{4/3}}{2} - m x^{2/3} + m^2 \ln\left(x^{2/3} + m\right)\right\}}\right\}.$$

Finally,

$$\mathcal{E}(x) = \frac{x^{-1}}{x^{2/3} + m}\left(\frac{m}{x^{2/3} + m}\right)^{3m^2} e^{-3\left\{\frac{x^{4/3}}{2} - m x^{2/3}\right\}}. \qquad (3\text{-}74)$$

In the low wave-number region, Eq. (3-74) yields $\mathcal{E}(x) \sim x^{-1}$. For negligible shear, m = 0, it is reduced to the expression for the energy spectrum in isotropic turbulence obtained by Pao (1965).

A more accurate solution for $\mathcal{E}(x,m)$ should use an approximation for the vorticity valid in the dissipation range also.

By using such a vorticity approximation (Kesic 1969) Eq. (3-70) then becomes

$$1 = \int_0^x 2\mathcal{E}(p)p^2dp + x\mathcal{E}(x)\left\{\frac{x^{2/3}}{1+x^{2/3}} + m\right\}. \qquad (3\text{-}75)$$

Following the same procedure as in the derivation of Eq. (3-74), one finally obtains

$$\mathcal{E}(x) = \frac{x^{-1}+x^{-1/3}}{m(1+x^{2/3})+x^{2/3}}\,\frac{m}{m+bx^{2/3}}\,e^{-\frac{3}{b^2}\left\{\frac{b}{3}x^2+\frac{1}{2}x^{4/3}-\frac{m}{b}x^{2/3}\right\}}, \qquad (3\text{-}76)$$

where b = m + 1. In the low wave number region, Eq. (3-76) gives the same result as Eq. (3-74). The difference shows only in the dissipation range.

The complete spectral equation, with both the production and the diffusion terms, reads

$$\begin{aligned}\varepsilon &= 2\nu\int_0^k E(p)p^2dp + \delta_\nu\, kE(k)(\varepsilon k^2)^{1/3} \\ &+ C_1\frac{d\bar{u}_1}{dx_2}kE(k) + C_2\frac{\partial^2(k\bar{q}^2)}{\partial x_2^2}\frac{kE(k)}{(\varepsilon k^2)^{1/3}},\end{aligned} \qquad (3\text{-}77)$$

or, in nondimensional form, using (3-43) and (3-44a),

$$1 = \int_0^x 2\mathcal{E}(p)p^2dp + x\mathcal{E}(x)\left\{x^{2/3}+m+nx^{-2/3}\right\}. \qquad (3\text{-}78)$$

Before solving the complete Eq. (3-77), an analysis of the behavior of the energy spectrum in the nonviscous range is possible. Concentrating attention to this region, the dissipation term can be neglected and directly from Eq. (3-77) it follows that

$$\mathcal{E}(x) = \frac{1}{x^{5/3} + mx + nx^{1/3}} \,. \tag{3-79}$$

Obviously, for $m = n = 0$, the $-5/3$ inertial subrange law results. For $n = 0$,

$$\mathcal{E}(x) = \frac{1}{x^{5/3} + mx} = \frac{x^{-1}}{x^{2/3} + m} \,,$$

and for small x, such that $m \gg x^{2/3}$, the x^{-1} nonviscous region follows. However, if $n \neq 0$ the x^{-1} result will be definitely affected.

In the case when $n \gg mx^{2/3}$ (For very small x, for example), Eq. (3-79) gives the asymptotic solution as $\sim x^{-1/3}$. This, however, cannot be valid. In the low wave number region of the nonviscous subrange, where apparently the $-5/3$ spectral law is affected, the approximation for the vorticity cannot be $\sim (\mathcal{E} k^2)^{1/3}$ which is strictly valid for the inertial subrange only. For small departure from the $-5/3$ inertial subrange, however, this can be accepted as a first approximation.

By using the substitution (3-46), Eq. (3-78) becomes

$$1 = Z^2 + \frac{Z}{x}\frac{dZ}{dx}\left\{x^{2/3} + m + n x^{-2/3}\right\}. \qquad (3\text{-}81)$$

Hence,

$$\frac{x^{5/3}\,dx}{x^{4/3} + m x^{2/3} + n} = \frac{Z\,dZ}{1 - Z^2}\,. \qquad (3\text{-}82)$$

Making the following substitution:

$$x^{2/3} = u, \quad dx = 3/2\, x^{1/3}\, du,$$

Eq. (3-82) is reduced to

$$\frac{3/2\, w^3\, dw}{w^2 + m w + n} = \frac{Z\,dZ}{1 - Z^2},$$

and, a solution for Z^2 is obtained as

$$Z^2 = 1 - C \exp\left\{-3\left[\frac{x^{4/3}}{2} - m x^{2/3} + \frac{m^2 - n}{2} \ln\left(x^{4/3} + m x + n\right)\right.\right.$$

$$\left.\left. + \frac{3mn - m^3}{2\sqrt{m^2 - 4n}} \ln \frac{2x^{2/3} + m - \sqrt{m^2 - 4n}}{2x^{2/3} + m + \sqrt{m^2 - 4n}}\right]\right\}, \qquad (3\text{-}83)$$

valid for $m^2 > 4n$. In the case of $m^2 < 4n$ or $m^2 = 4n$, only the last term in the exponent would be changed to

$$\frac{3mn - m^3}{\sqrt{4n - m^2}} \operatorname{Arctan} \frac{2x^{2/3} + m}{\sqrt{4n - m^2}} ,$$

or
$$- \frac{3mn - m^3}{2x^{2/3} + m} ,$$

respectively.

The integration constant, C, in (3-83) is determined from the condition: x = 0, Z = 0. Thus, for $m^2 > 4n$,

$$C = \exp\left\{3\left[\frac{m^2 - n}{2} \ln(m) + \frac{3mn - m^3}{2\sqrt{m^2 - 4n}} \ln \frac{m - \sqrt{m^2 - 4n}}{m + \sqrt{m^2 - 4n}}\right]\right\} .$$

The energy spectrum function then follows as

$$\mathcal{E}(x) = \frac{Z}{x^2} \frac{dZ}{dx} = \frac{1}{2x^2} \frac{dZ^2}{dx} ,$$

and, with (3-83),

$$\mathcal{E}(x) = \frac{C}{x^{5/3} + mx + nx^{1/3}} \exp\left\{-\frac{3}{2}\left[x^{4/3} - 2mx^{2/3} + (m^2 - n)\ln\left(x^{4/3} + mx^{2/3} + n\right) + \frac{3mn}{\sqrt{m^2 - 4n}} \ln \frac{2x^{2/3} + m - \sqrt{m^2 - 4n}}{2x^{2/3} + m + \sqrt{m^2 - 4n}}\right]\right\} . \qquad (3\text{-}84)$$

5. <u>Computation of Solutions for the Energy Spectrum</u>

Among the solutions obtained with Heisenberg's type approximation for $\nu_T(k)$, Howell's approximation, Eq.

(3-51), leads to a simpler solution of the spectral equation (3-40) than the original Heisenberg approximation. In the latter case the parametric solution (3-48) was obtained. Though the differential equation (3-48) can be readily integrated by use of table integrals, the final result is very complex. Even a solution valid in the non-viscous range only (the dissipation term neglected) is impractically long.

Howell's approximation, on the other hand, yields a much simpler solution. It is given, Eq. (3-54), as

$$x^4 = \left(\frac{16}{27}\right)\frac{Z(Z^2 + mZ + n)^3}{(1-Z^2)(mZ^2 + 2(n+1)Z + m)} ,$$

and the energy spectrum function follows as

$$\mathcal{E}(x) = \frac{(64/27)(Z^3/x^7)(Z^2 + mZ + n)^4}{(1-Z^2)\cdot A\cdot B + (Z^2+mZ+n)\cdot C\cdot Z} ,$$

$$A = mZ^2 + 2(1+n) + m ,$$
$$B = 7Z^2 + 4m + 2n ,$$
$$C = 4mZ^3 + 6(1+n)Z - 2n . \qquad (3\text{-}85)$$

This parametric solution is computed for different values of the parameters m and n and the results are plotted in Figs. 8 and 9.

The Modified Obukhov approximation for the transfer spectrum function leads to especially simple solutions of Eq. (3-40). In this case the parametric solution is, Eq.

(3-65),

$$x^2 = 2\sqrt{\tfrac{2}{3}}\left\{(1+n)\,\mathrm{Arctan}\,Z - \frac{n}{2}\ln(1-Z^2) - Z\right\},$$

$$\mathcal{E}(x) = \frac{Z}{x^2}\frac{dZ}{dx}\ .$$

Hence,

$$\frac{dx^2}{dZ} = 2\sqrt{\tfrac{2}{3}}\left\{\frac{1+n-m}{2(1+Z)} + \frac{1+n+m}{2(1-Z)} - 1\right\},$$

and

$$\mathcal{E}(x) = \frac{\sqrt{\tfrac{3}{2}}\ Z(1-Z^2)}{x(Z^2+mZ+n)}\ . \tag{3-86}$$

The results, computed for different values of m and n, are plotted in Figs. 1-7.

In the case of the vorticity approximation, explicit expressions for the energy spectrum $\mathcal{E}(x)$ are obtained. Thus, the values of $\mathcal{E}(x)$ are calculated for different values of m and n directly from Eq. (3-84).

All these solutions, obtained by using different approximations for the transfer spectrum function, are for the three-dimensional energy spectrum. So far, there has not been any experimental data on the three-dimensional energy spectrum. All measurements refer to the one-dimensional spectrum functions. In literature, the one-dimensional spectra are calculated by exact relations derived for the case of isotropic turbulence. In the case of shear turbulence, however, the present knowledge of the spectral densities is insufficient for an exact calculation of the one-dimensional spectral functions. Monin (1962) suggested that within the equilibrium range the one-dimensional spectrum functions could be calculated--at least as a crude estimate--by means of the formulas valid for locally isotropic turbulence. He used this procedure to calculate the one-dimensional spectra from relations derived for the three-dimensional spectra in a thermally stratified atmosphere.

The same proceudre is used here to calculate the one-dimensional energy spectra from the expressions for the three-dimensional energy spectrum $\mathcal{E}(x)$. Calculation is carried out for two cases: (a) $\mathcal{E}(x)$ obtained with the use of the Modified Obukhov approximation, and (b) $\mathcal{E}(x)$ obtained with the vorticity approximation.

The relationship between the one-dimensional energy spectrum $F(x_1)$ and the three-dimensional energy spectrum $\mathcal{E}(x)$ is given (Hinze 1959) for the $\overline{u_1^2}$ spectrum component as

$$\mathcal{F}(x_1) = \int_{x_1}^{\infty} \left(1 - \frac{x_1^2}{x^2}\right) \frac{\mathcal{E}(x)}{x} dx . \qquad (3\text{-}87)$$

In the case of an explicit expression for $\mathcal{E}(x)$, as was obtained by using vorticity approximation, Eq. (3-87) is transformed into a form that is more convenient for numerical integration by substituting a new variable

$$y = x_1/x \ .$$

Thus,

$$\mathcal{F}(x_1) = -\int_{y=1}^{y=0} \left(1 - \frac{x_1^{\,2}}{\left(\frac{x_1}{y}\right)^2}\right) \frac{\mathcal{E}\left(\frac{x_1}{y}\right)}{\left(\frac{x_1}{y}\right)} \left(\frac{x_1}{y}\right) dy \ ,$$

and

$$\mathcal{F}(x_1) = \int_0^1 (1-y^2) \frac{\mathcal{E}\left(\frac{x_1}{y}\right)}{y} dy \ . \qquad (3\text{-}88)$$

Writing the derived expression for $\mathcal{E}(x)$, Eq. (3-84), as a function of (x_1/y), the above relation is used to calculate the one-dimensional energy spectrum $F(x_1)$. The results are plotted in Figs. 17 and 18.

In the case of the Modified Obukhov approximation, the solution for $\mathcal{E}(x)$ was obtained in a parametric form, Eq. (3-65). Substituting (3-46) into Eq. (3-87), one obtains

$$\mathcal{F}(x_1) = \int_{x=x_1}^{x=\infty} \left(1 - \frac{x_1^{\,2}}{x^2}\right) \frac{1}{x} \frac{Z}{x^2} \frac{dZ}{dx} dx \ ,$$

or,

$$\mathcal{F}(x_1) = \int_{Z(x=x_1)}^{Z(x=\infty)} \left(1 - \frac{x_1^{\,2}}{x^2}\right) \frac{Z}{x^3} dZ = \int_{Z(x=x_1)}^{1} \left(1 - \frac{x_1^{\,2}}{x^2}\right) \frac{Z}{x^3} dZ \ ,$$

where

$$x^2 = 2\sqrt{\frac{2}{3}}\left\{(1+n)\,\mathrm{Arctan}\,Z - \frac{m}{2}\ln\left(1-Z^2\right) - Z\right\}.$$

These two relations are used to calculate $F(x_1)$. The results are plotted in Figs. 19-28.

6. <u>Results and Discussion</u>

The results obtained from the solutions derived in this chapter are plotted in Figs. 1 - 28. In Figs. 1 - 11 the three-dimensional energy spectrum $\mathcal{E}(x)$ is plotted for different values of the production parameter, m, and for different approximations for the transfer spectrum function. All these approximations lead to the x^{-1} solution for the low-wave number range if the diffusion spectrum function is neglected in Eq. (3-40). The effect of the shear is clearly displayed. As the production parameter increases, the inertial $x^{-5/3}$ subrange is affected more and more--up to a larger and larger wave number. For a very strong production, i.e., for large values of the parameter m, the shear effects penetrate the energy spectrum even through the dissipation range.

The impact of the diffusion term on the three-dimensional energy spectrum in the low wave-number range is shown in Figs. 4 - 9. In Figs. 1 - 7 the solutions for the three-dimensional energy spectrum obtained with the Modified Obukhov approximation and for different values of the production parameter, m, and the diffusion parameter, n, are shown. It is clear that by taking the diffusion spectrum function into account the x^{-1} range is strongly affected. Only for very small values of the diffusion parameter, n, (relative to the production parameter m) is it possible to obtain a solution for the energy spectrum function, $\mathcal{E}(x)$, that gives the variation of a part of the spectrum in the low wave number range as x^{-1}. However, even with as large a value of the production parameter as m = 2, Fig. 5, the diffusion parameter as small as n = 0.005 will destroy the x^{-1} range completely.

The relation between the terms of the spectral equation (3-40) and their relative magnitudes for different values of the production and the diffusion parameter is shown in Figs. 12 - 16. It is clear from these diagrams that with the penetration of the shear and the diffusional effects toward larger wave numbers, the dissipation is moved toward larger wave numbers too--a fact that has already been noticed in experiments on turbulent boundary layers (Hinze 1959, Tielman 1967). At the same time, the transfer term becomes smaller and smaller indicating a shrinkage of the inertial $k^{-5/3}$ subrange.

In Fig. 7 a composite plot of the three-dimensional energy spectrum is given. Together with an increase of the value of the production parameter, m, an increase of the diffusion parameter, n, is assumed also. This hypothetical case represents the actual situation in a turbulent boundary layer. The increase of m represents an increase of the velocity gradient, $d\bar{u}/dx_2$, and the increase of n represents an increase of the diffusional effects as the distance from the wall is decreased. The behavior of the three-dimensional energy spectrum in the presence of shear and inhomogeneities, as given by Fig. 7 , is in a good qualitative agreement with the experimental data on energy spectra in turbulent boundary layers. These data are, however, still very meager.

The first systematic measurements of the energy spectrum were reported by Klebanoff (1951) and Laufer (1954). Klebanoff's data were taken in an isothermal turbulent boundary layer with zero pressure gradient. The measured energy spectra (Klebanoff 1951, Fig. 6) show that the contribution to the turbulence energy in the low-wave number range decreases as the wall is approached, but that the contribution in the high-wave number range is increased. This is the same picture as given by the solutions of the spectral equation, Eq. (3-40), and shown in Figs. 1 – 28.

The most recent measurements of the energy spectrum in an isothermal boundary layer were done by Tielman (1967). He plotted Heisenberg's isotropic relation for comparison but found, in agreement with the solutions of Eq. (3-40) presented in Fig. 7 , that each spectrum branches away from Heisenberg's theoretical solution at a different point and, consequently, the $k^{-5/3}$ inertial subrange shrinks more and more as the wall is approached. Tielman's data, plotted in the same nondimensional variables as the spectra in Fig. 7, show striking similarity to the calculated spectra. However, a comparison of the solutions of Eq. (3-40) and Tielman's data (Fig. 56, cit. ref.) cannot be made directly because the data represent the one-dimensional energy spectra and the solutions, Fig. 7, represent the three-dimensional energy spectra.

It is not known, however, how the one-dimensional energy spectrum is related to the three-dimensional energy spectrum in shear turbulence. One is then forced, as already pointed out, to use the relationship valid for locally isotropic turbulence. As this relationship is used for the high-wave number energy spectrum where an approximate local isotropy may exist and local homogeneity is assumed (Monin 1962) it is hoped that a reasonable representation for the one-dimensional energy spectrum can be obtained. The basic condition is, of course, that the solution for the three-dimensional energy spectrum is in itself a good mathematical model for the real phenomenon.

The computed one-dimensional spectra in Fig. 12 -- corresponding to the three-dimensional spectra in Fig. 7 -- immediately show that the main characteristics of the three-dimensional spectra (already observed as in a good qualitative agreement with the actual data) are preserved.

Obviously, the computed one-dimensional spectra from Fig. 22 could be compared with experimental measurements and an assessment of the values of the production parameter, m, and the diffusion parameter, n, can be made. Intuitively, it is expected that there should be some dependence between these parameters and the turbulence Reynolds number.

It is known that in the regions of both strong shear and inhomogeneities, as in the region close to the wall, the turbulence Reynolds number is very small and that for larger Re_λ (greater than 500, say), the effects of shear and inhomogeneities on the energy spectrum can be neglected. From (3-44a),

$$m = C_1\left(\frac{\nu}{\varepsilon}\right)^{1/2}\frac{d\bar{U}_1}{dx_2}\,, \qquad m = C_2\left(\frac{\nu}{\varepsilon}\right)\frac{\partial^2(\frac{1}{2}q^2)}{\partial x_2^2}\,.$$

Using the definition for the turbulence Reynolds number, i.e.,

$$Re_\lambda = \frac{(\overline{u^2})^{1/2}\lambda}{\nu}\,,$$

where λ is the microscale or the dissipation scale, and the isotropic relation:

$$\varepsilon = 15\nu\frac{\overline{u^2}}{\lambda^2}\,,$$

one obtains

$$\left(\frac{\nu}{\varepsilon}\right)^{1/2} = \frac{1}{\sqrt{15}}\frac{\lambda^2}{\nu}\left(\frac{1}{Re_\lambda}\right),$$

and

$$m = B\left(\frac{1}{Re_\lambda}\right)\lambda^2 \frac{d\overline{U}_1}{dX_2}\ ,$$

where B is a constant.

In a turbulent boundary layer with the increase of the mean velocity gradient (approaching the wall) the microscale and the turbulence Reynolds number decrease and vice versa. The form of the variation of $\lambda^2 d\overline{U}_1/dX_2$ is the same for any fully developed boundary layer (going from zero to some finite value) and it can be assumed that the value of $\lambda^2 d\overline{U}_1/dX_2$ is the same if Re_λ at the points in question is the same (for two different mean velocities, for example). Thus,

$$\left(\frac{B}{Re_{\lambda}}\right)_1\left(\lambda^2 \frac{d\overline{U}_1}{dX_2}\right)_1 = \left(\frac{B}{Re_\lambda}\right)_2\left(\lambda^2 \frac{d\overline{U}_1}{dX_2}\right)_2\ ,\quad Re_{\lambda_1} = Re_{\lambda_2}\ .$$

Moreover, in a part of the boundary layer one can expect that $(\lambda^2 d\overline{U}_1/dX_2) \sim$ const. In that case, the production parameter would vary as $\sim$ (const /Re_λ). Similarly, for the diffusion parameter, it is assumed that

$$n = C_2\left(\frac{\nu}{\varepsilon}\right)\frac{\partial^2(\frac{1}{2}\overline{q^2})}{\partial X_2^2} = f(Re_\lambda)\ ,\quad n_1 = n_2 \text{ if } Re_{\lambda_1} = Re_{\lambda_2}\ .$$

Hence, according to the above assumptions, the parameters m and n for two different turbulent boundary

layer flows have the same values if Re_λ is the same. This is an interesting result as it indicates a similarity of the energy spectra in a turbulent boundary layer. In the case of isotropic turbulence, similarity exists for any energy spectrum if plotted in nondimensional variables :
$\chi = k/(\varepsilon/\nu^3)^{1/4}$, $\mathcal{E}(\chi) = E(k)/(\varepsilon\nu^5)^{1/4}$. In the case of shear turbulence, similarity would then exist if in addition to the same nondimensional variables the turbulence Reynolds number is the same.

The only measurements of the $\overline{u_1^2}$ spectra in a boundary layer that are calculated and plotted in a nondimensional form using the above mentioned variables are those by Tielman (1967). He obtained the energy spectrum data for two different velocities: U_∞ = 20 ft/sec , and 40 ft/sec. A check of his results (Fig. 56, cit. ref.) strongly indicates that the energy spectra fall on the same curve for the same Reynolds number irrespective of the distance from the wall or the mean velocity, U_∞ . This supports the above assumption of similarity of energy spectra if scaled with the turbulence Reynolds number.

Comparison of the computed one-dimensional spectra from Fig. 22 with Tielman's measurements shows that agreement exists. Though originally the data were intended to be used only as an illustration of the qualitative correctness of the analytical solutions, it was not difficult to obtain a good fit to Tielman's data with the Kolmogorov constant $\alpha = 1.7$ and varying the values of m and n. The calculated one-dimensional spectra are presented in Figs. 19 - 28.

Comparing the values of m and n, used in plotting the fitting curves to Tielman's data, with the turbulence Reynolds number for corresponding points, it is clear that m and n vary in a regular manner as Re_λ decreases, i.e., as the wall is approached. It was found that a good approximation for the production parameter, m, can be written as:

$$m \sim \frac{10 \div 12}{Re_\lambda}$$

for the part of the boundary layer where Re_λ varies from ~ 100 to ~ 3, and for the diffusion parameter, n, as

$$n \sim \frac{0.5}{Re_\lambda} .$$

These results support the above assumption that in a turbulent boundary layer the production term (in nondimensional form) may vary as $\sim C/Re$, and the diffusion parameter, n, as a function of Re_λ also. This enables a similarity of the energy spectra, for the same Re_λ, to exist.

For $Re_\lambda \longrightarrow \infty$, the isotropic solution ($m = 0$, $n = 0$) is obtained from Eq. (3-40). In Figs. 23 - 28 Tielman's data are plotted together with the isotropic solution for comparison. It can be seen that for $m > 0.1$ (i.e., $Re < 100$), the $k^{-5/3}$ inertial subrange is nonexistent. Only for $Re \geq 200$ does the inertial range becomes a sizeable part of the spectrum.

From the obtained results it can be concluded that the derived solution for the three-dimensional energy spectrum, Eq. (3-65), gives an accurate picture of the behavior of the energy spectrum function in a turbulent boundary layer and that the one-dimensional energy spectrum calculated by means of Eq. (3-87) from the three-dimensional spectrum, Eq. (3-65), gives a good quantitative agreement with the actual measurements from turbulent boundary layers.

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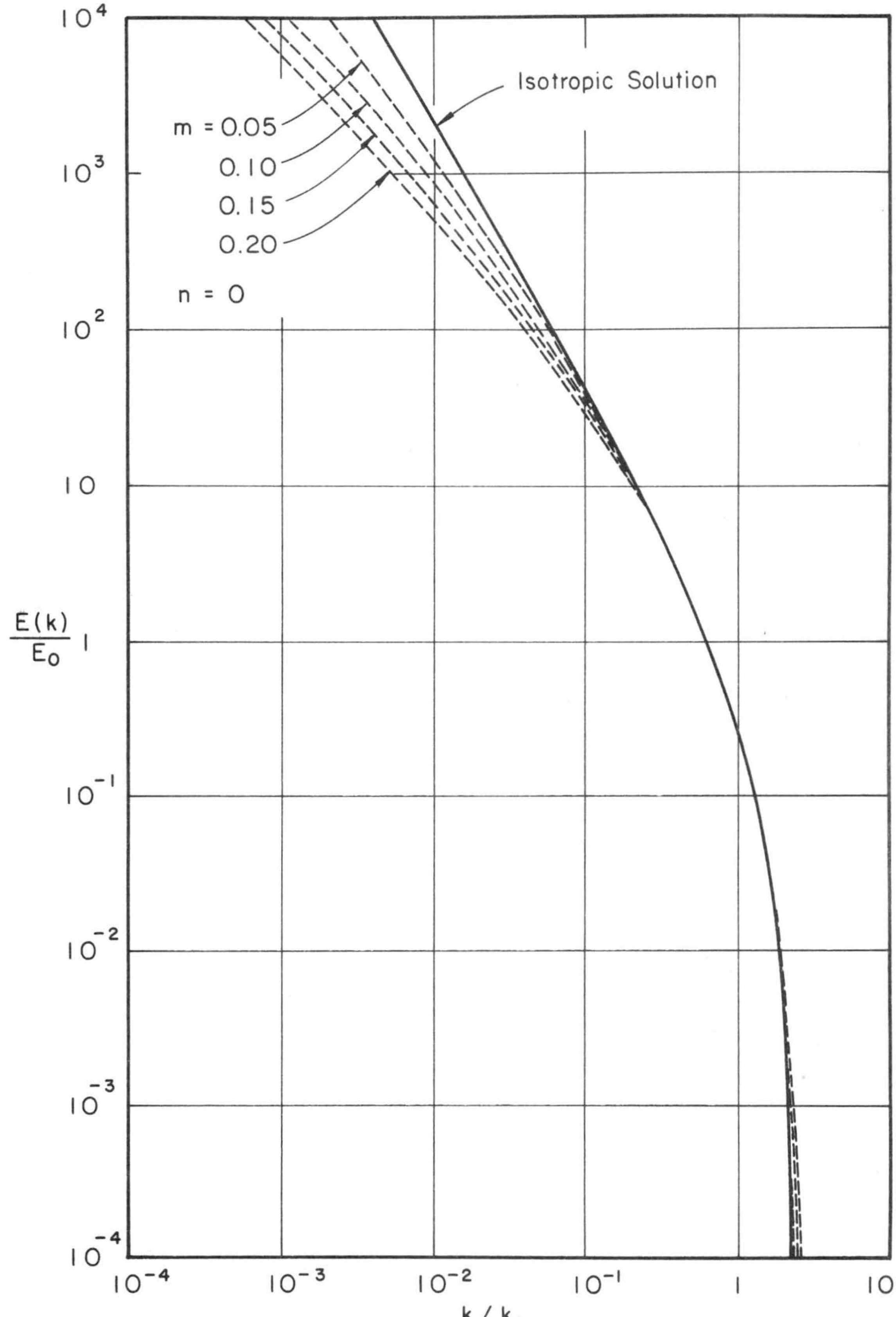

Figure 1. Three-dimensional energy spectrum.
MOB approximation; m=0.05,0.1,0.15,0.2; n=0

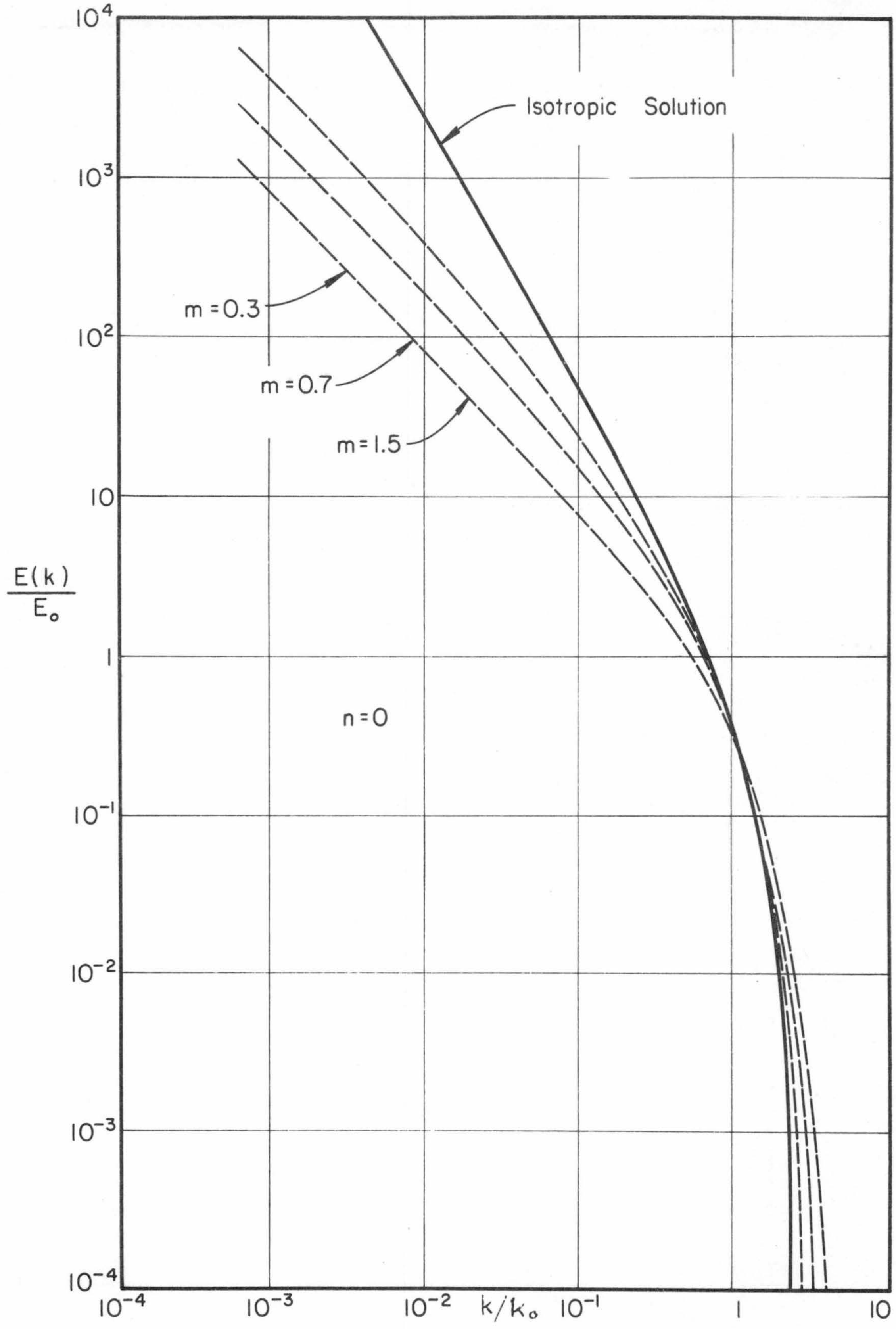

Figure 2. Three-dimensional energy spectrum. MOB approximation; m=0.3,0.7,1.5; n=0

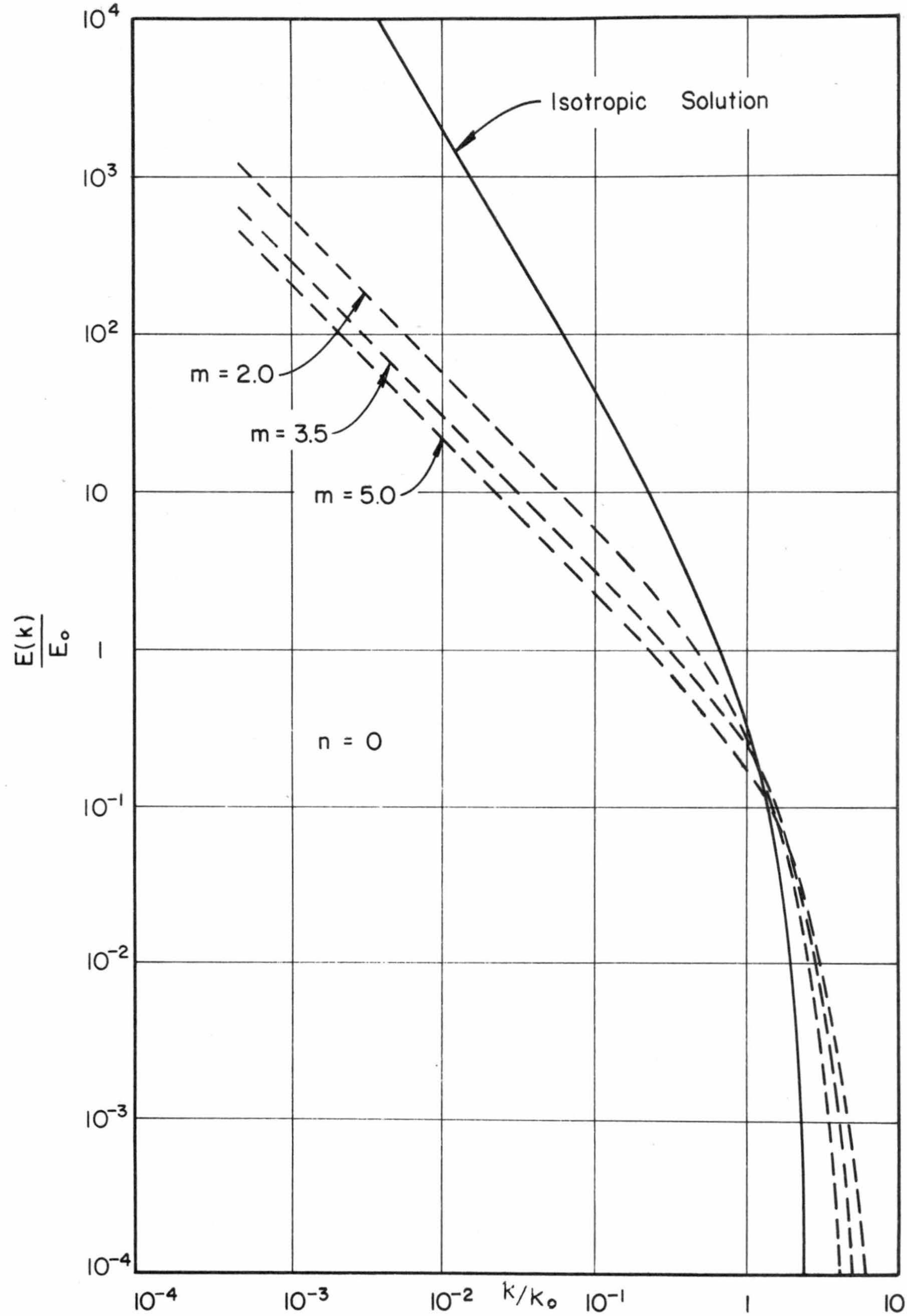

Figure 3. Three-dimensional energy spectrum. MOB approximation; m=2.0,3.5,5.0 ; n=0

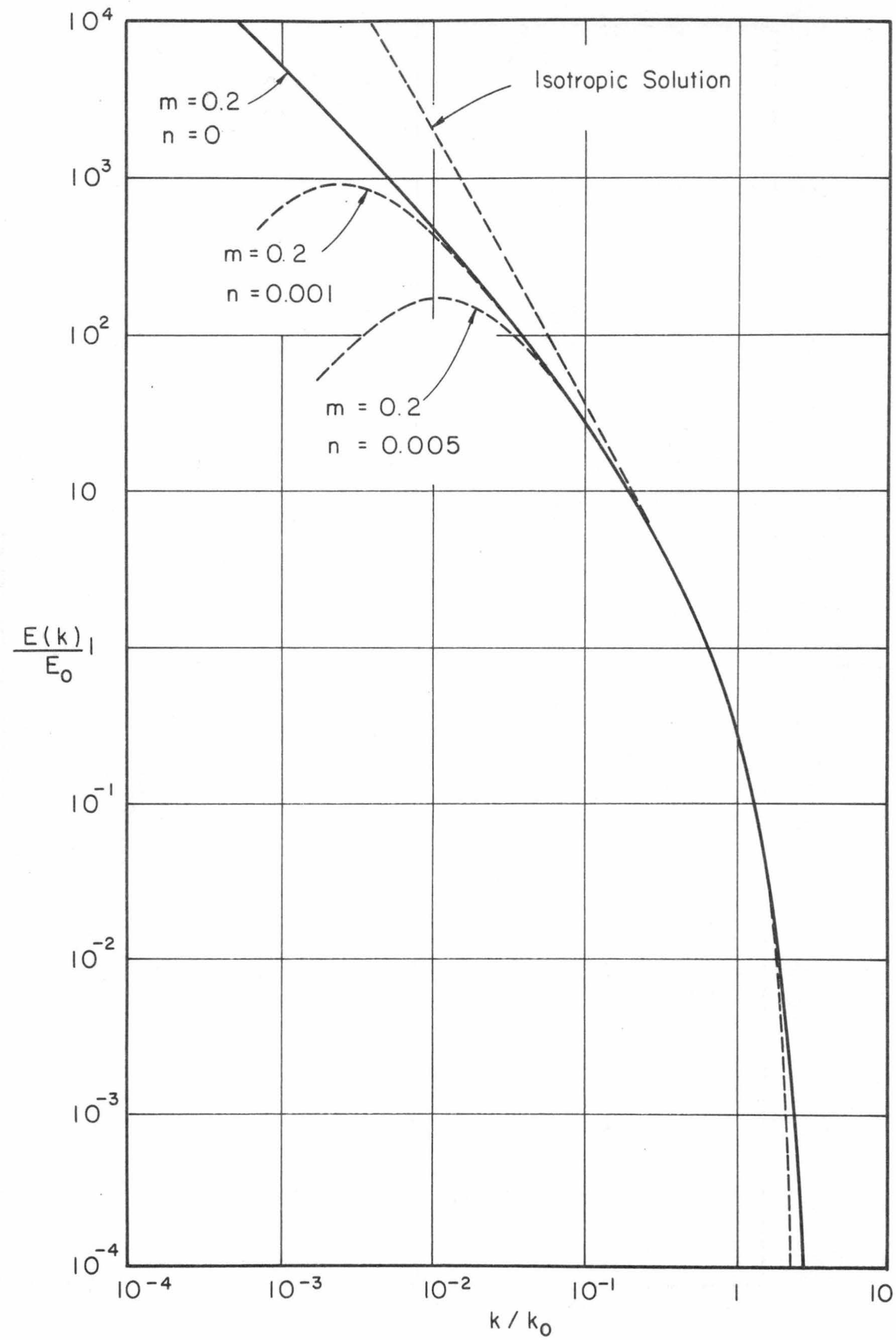

Figure 4. Three-dimensional energy spectrum.
MOB approximation; m=0.2; n=0,0.001, 0.005

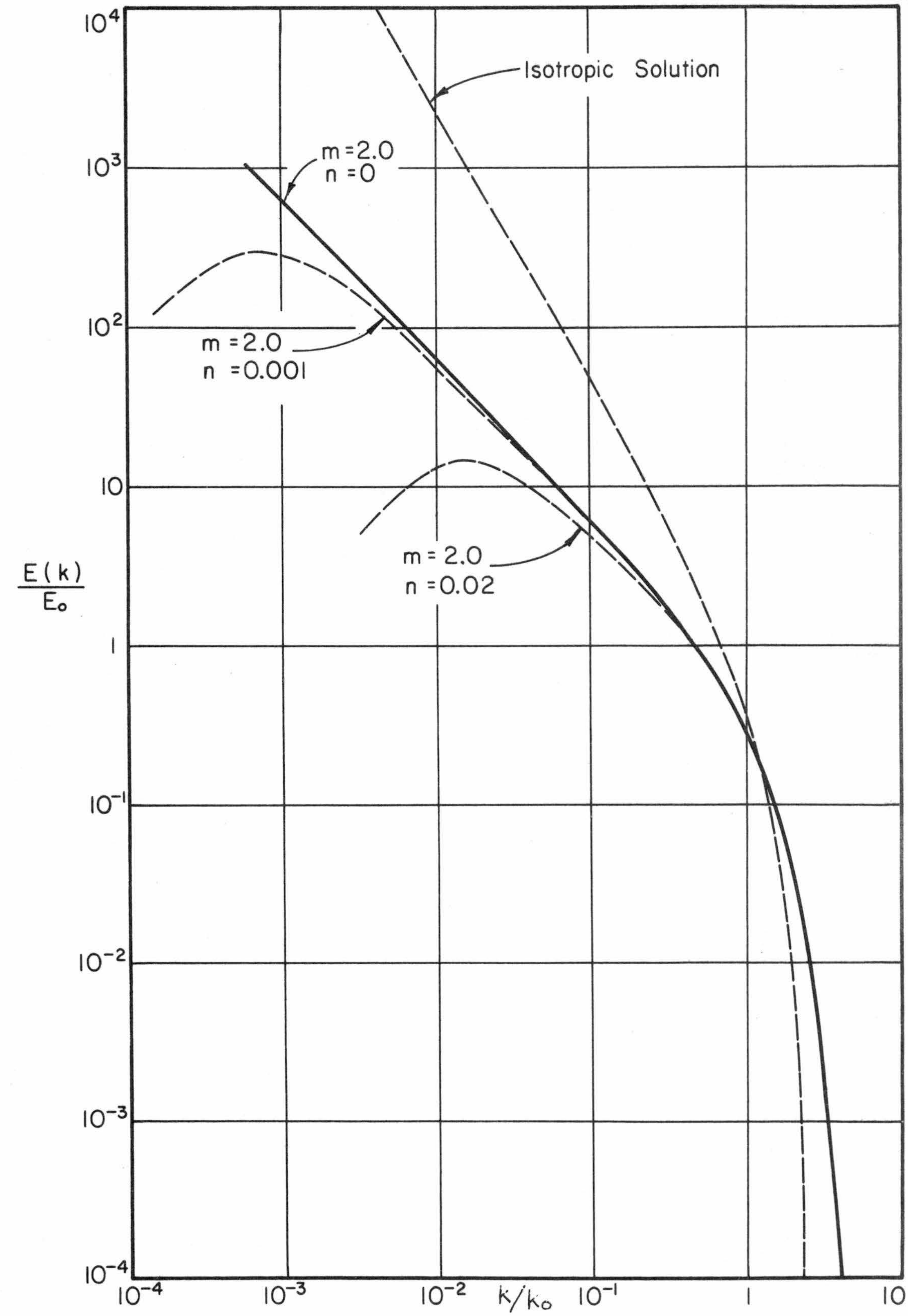

Figure 5. Three-dimensional energy spectrum. MOB approximation; m=2.0; n=0,0.001,0.02

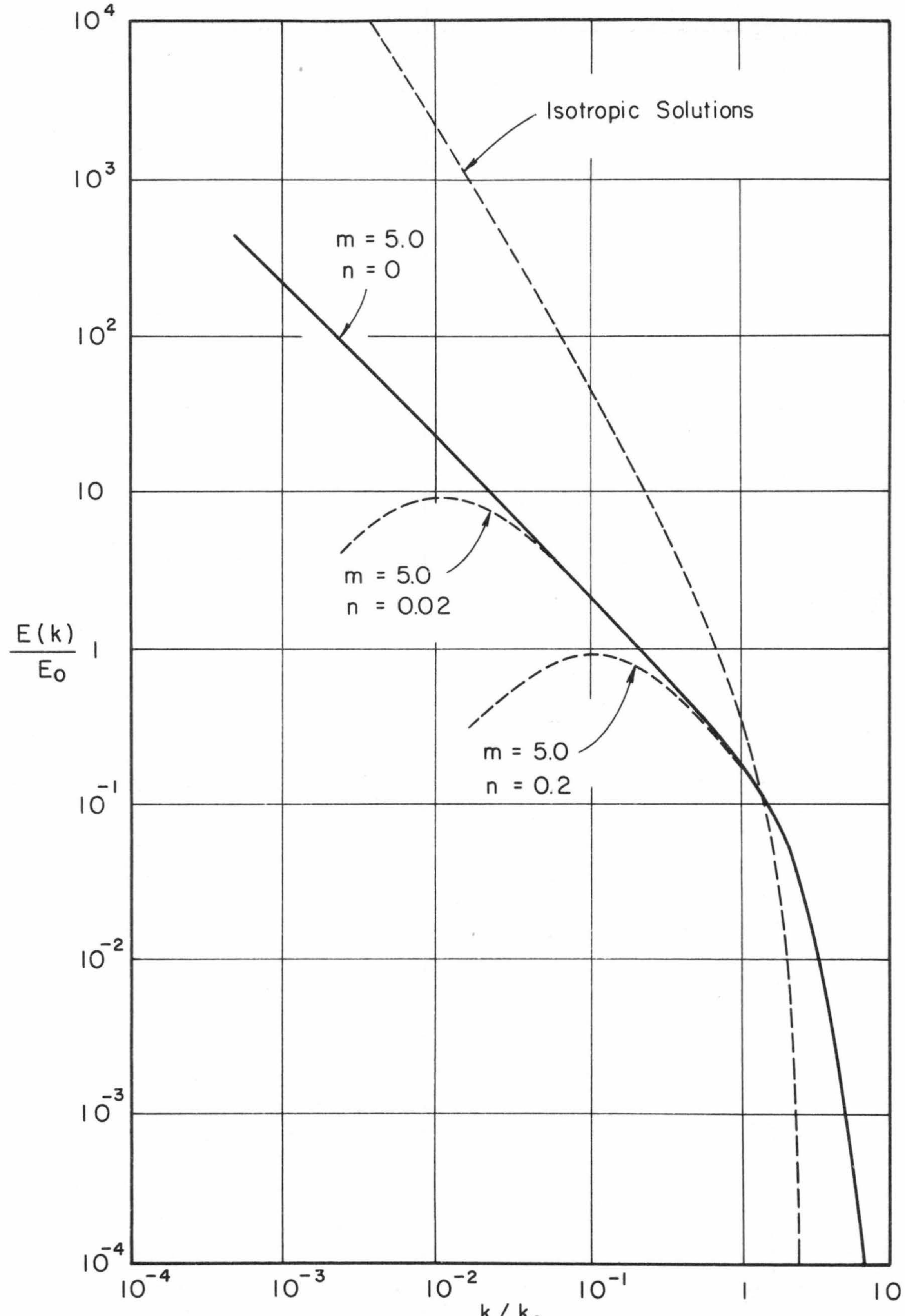

Figure 6. Three-dimensional energy spectrum. MOB approximation; m=5; n=0,0.02,0.2

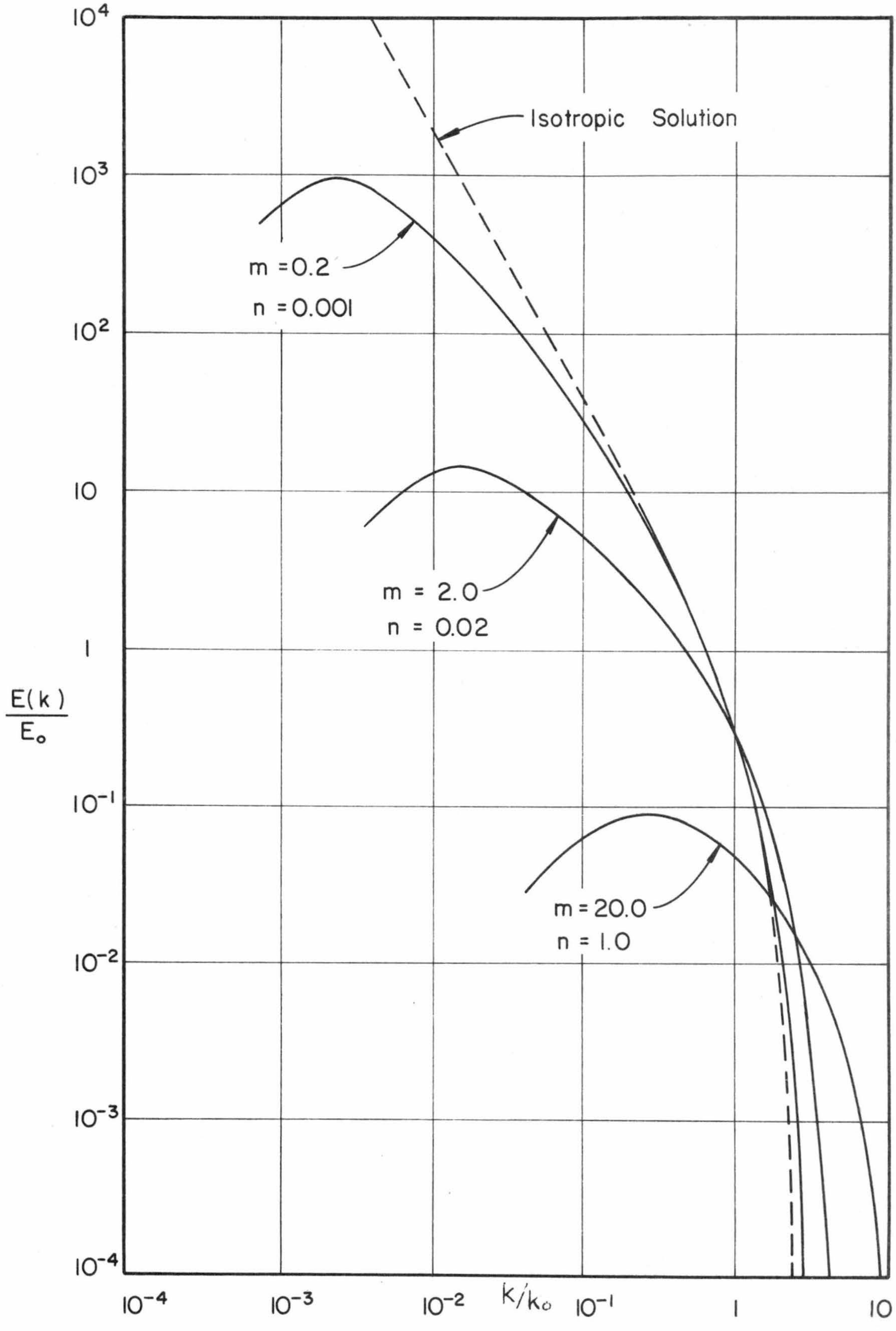

Figure 7. Three-dimensional energy spectrum. MOB approx.; m=0.2,2.0,20; n=0.001,0.02,1

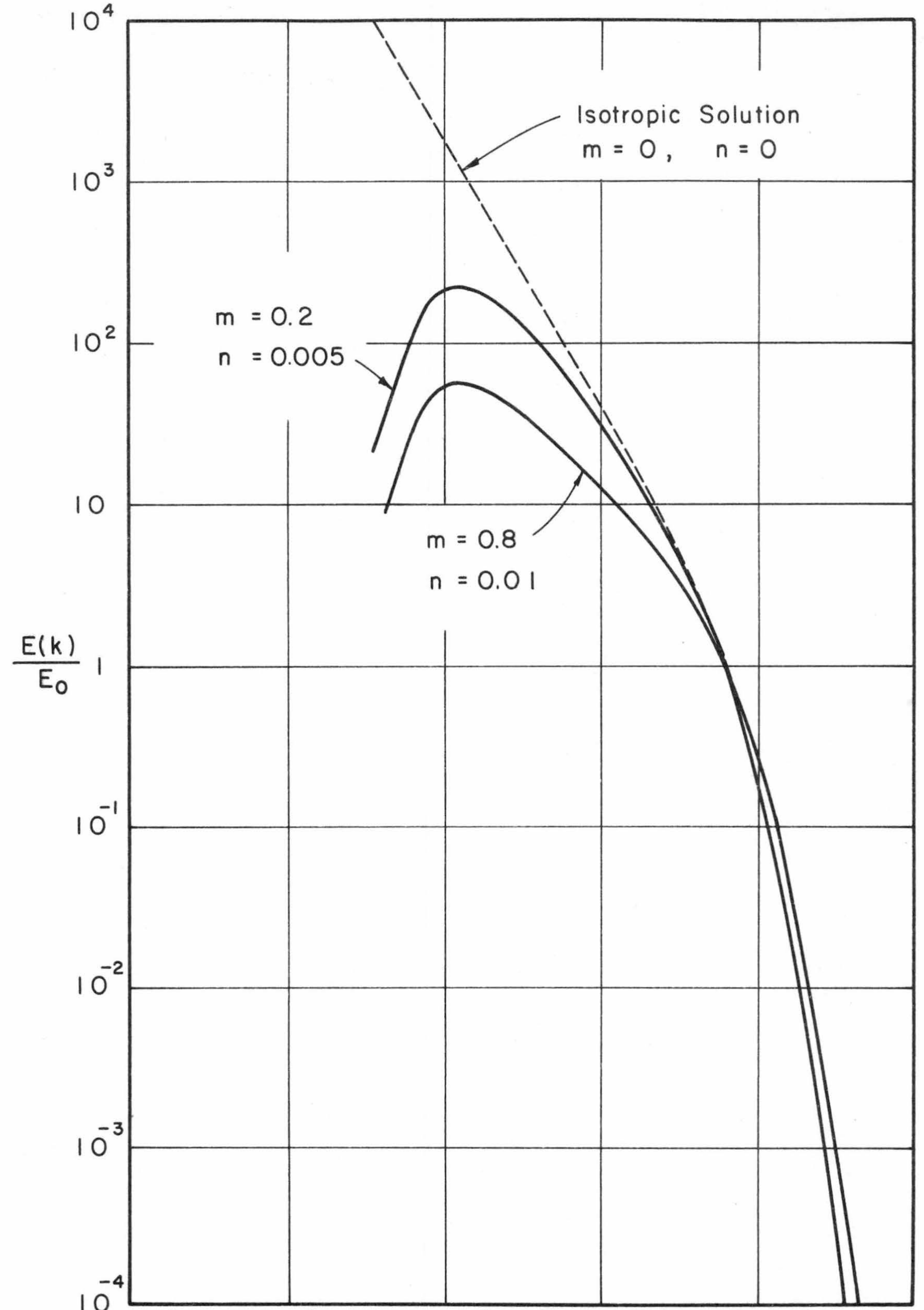

Figure 8. Three-dimensional energy spectrum. Howells' approximation; m=0.2,0.8; n=0.01,0.05

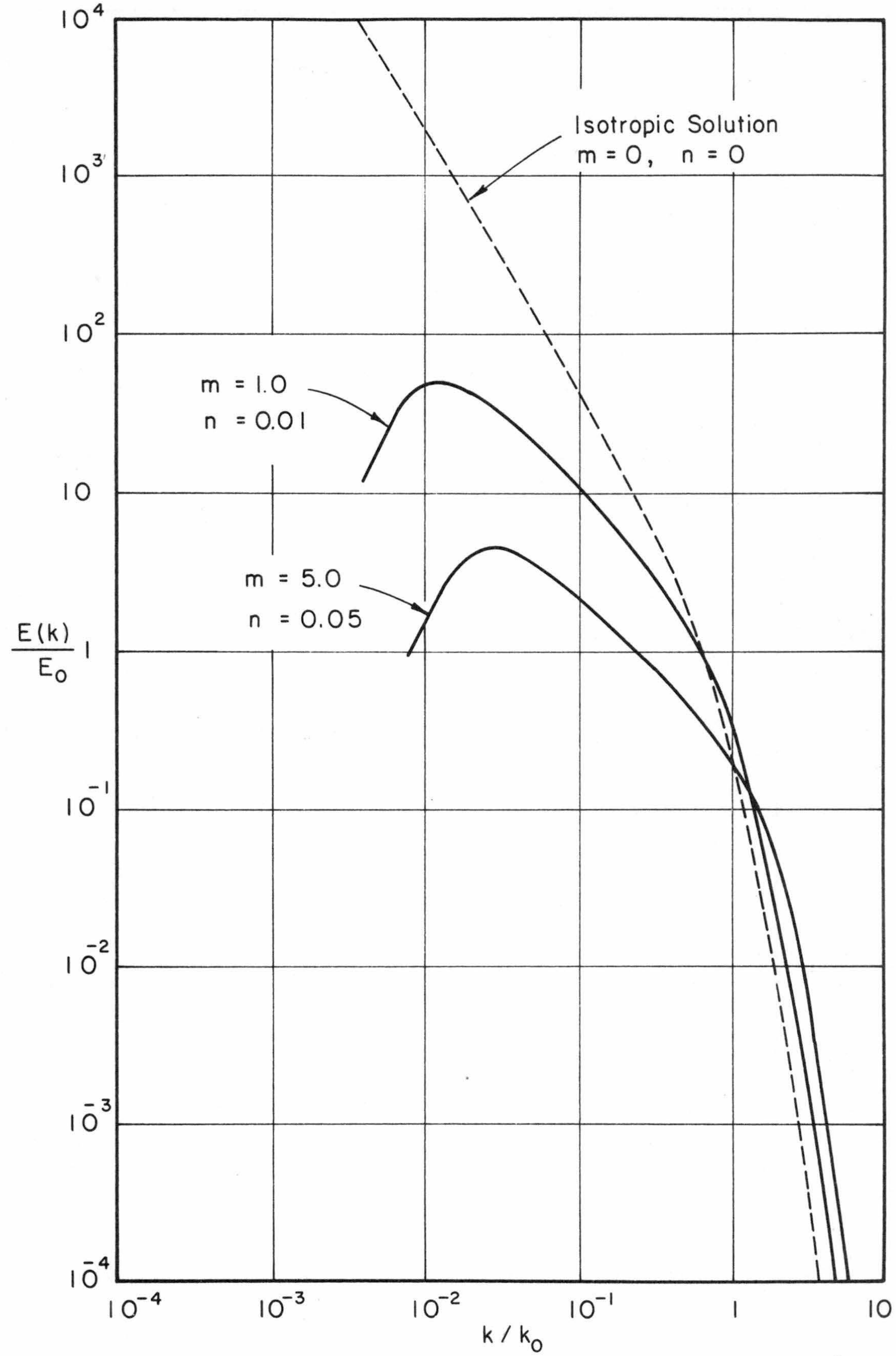

Figure 9. Three-dimensional energy spectrum. Howells' approximation; m=1,5; n=0.01,0.05

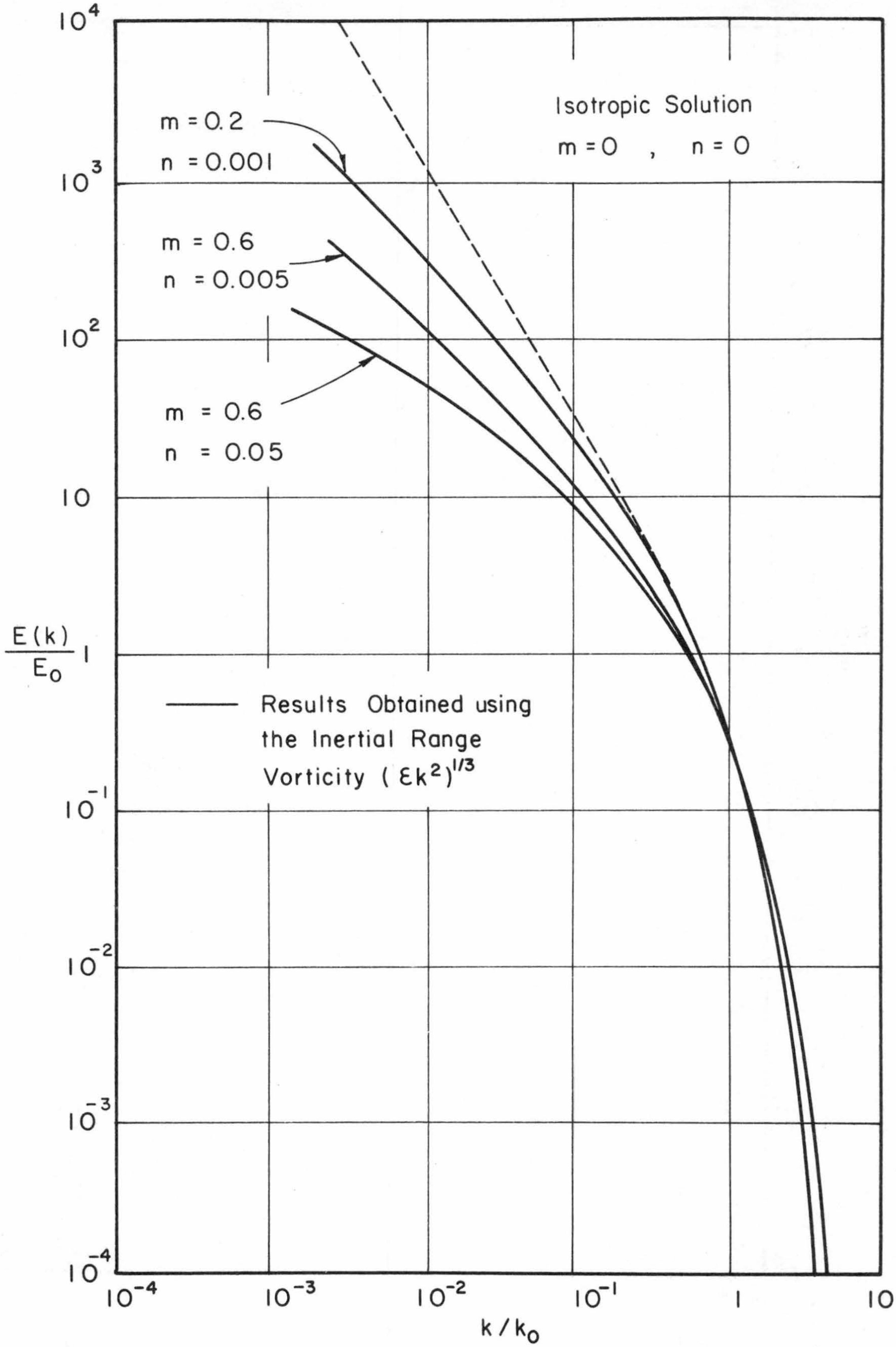

Figure 10. Three-dimensional energy spectrum. Vorticity approximation; m=0.2,0.6; n=0.001,0.005,0.05

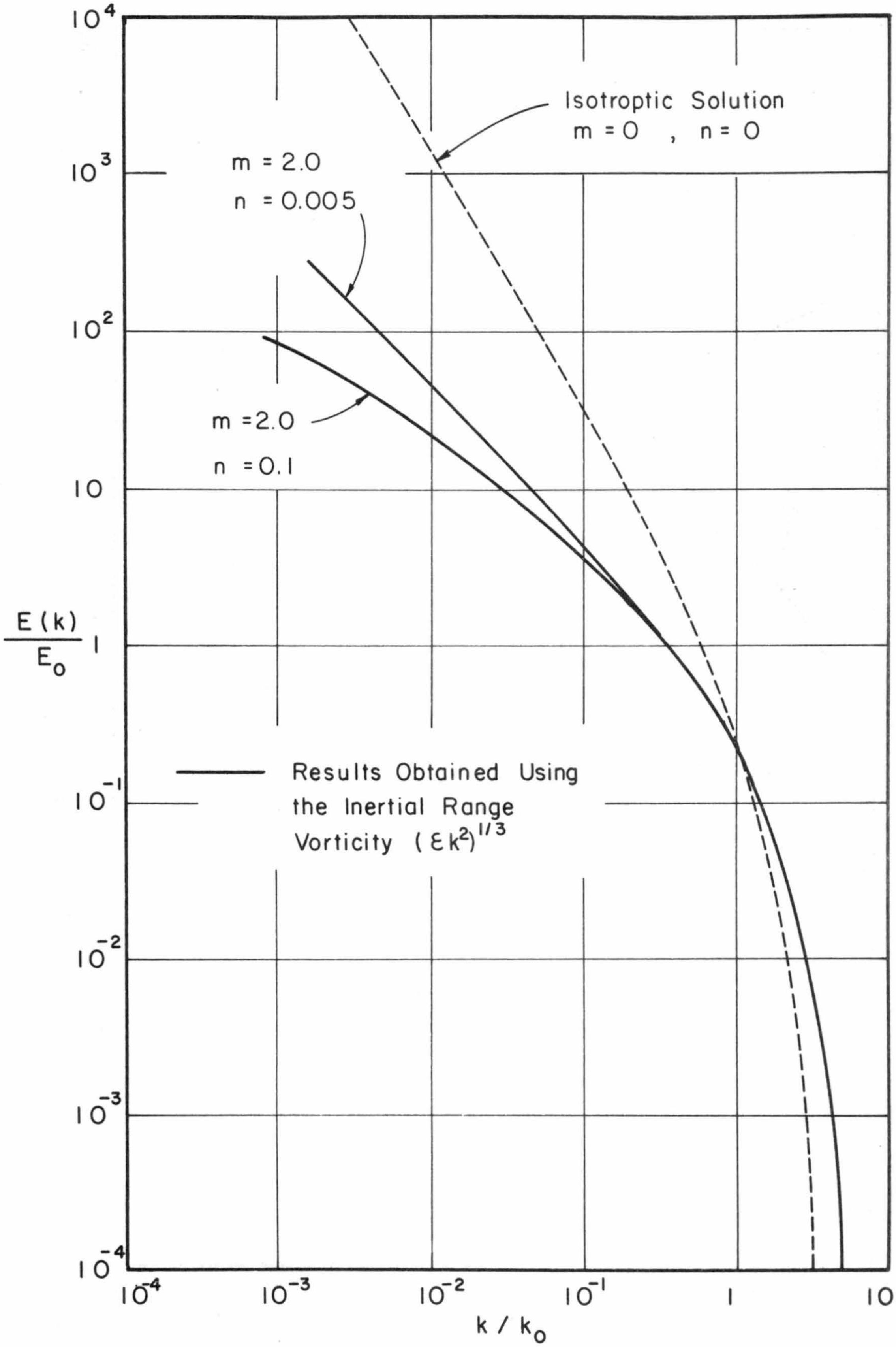

Figure 11. Three-dimensional energy spectrum. Vorticity approximation; m=2; n=0.005,0.1

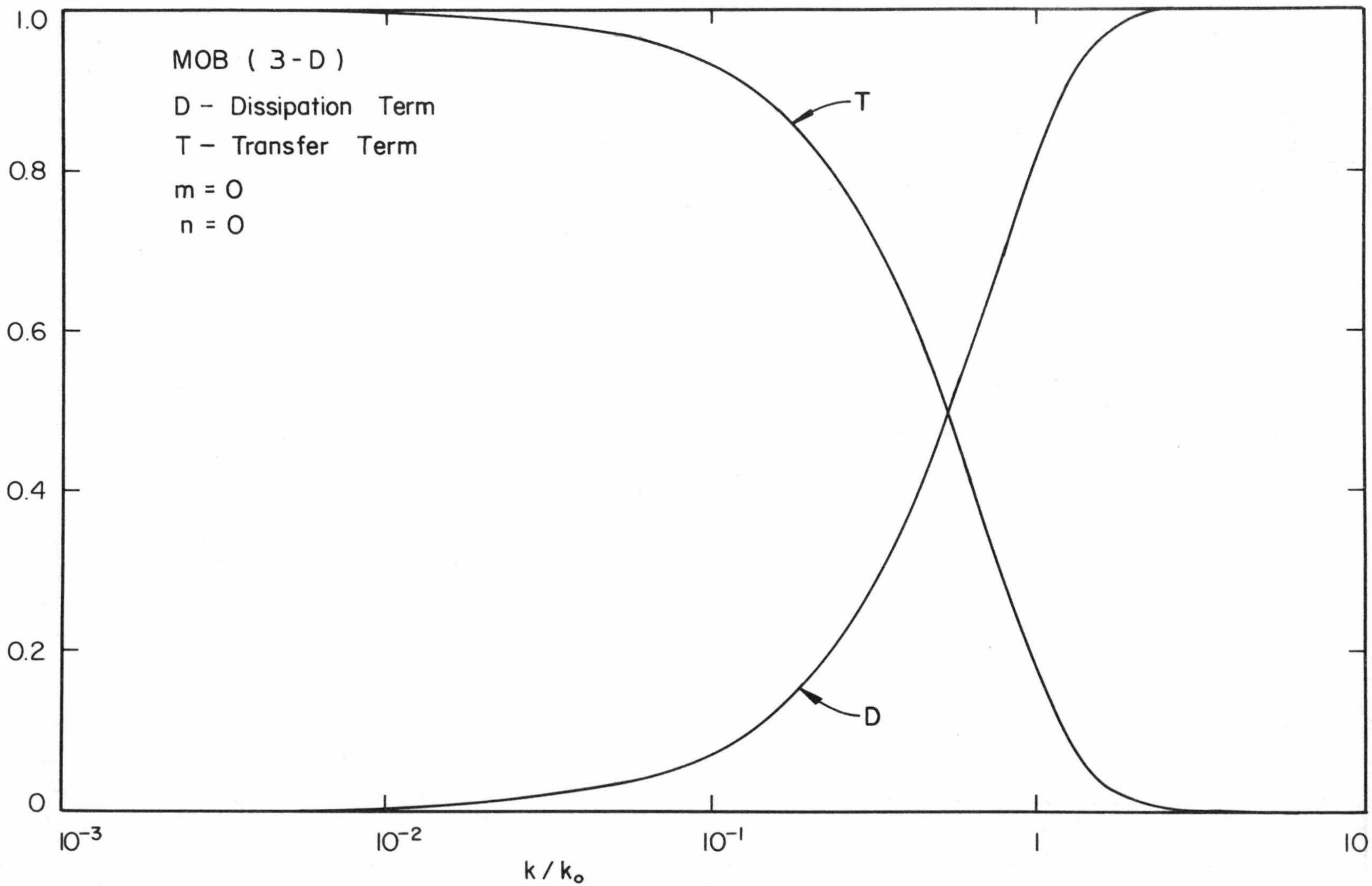

Figure 12. Relative variation of the spectrum equation terms; m=0; n-0

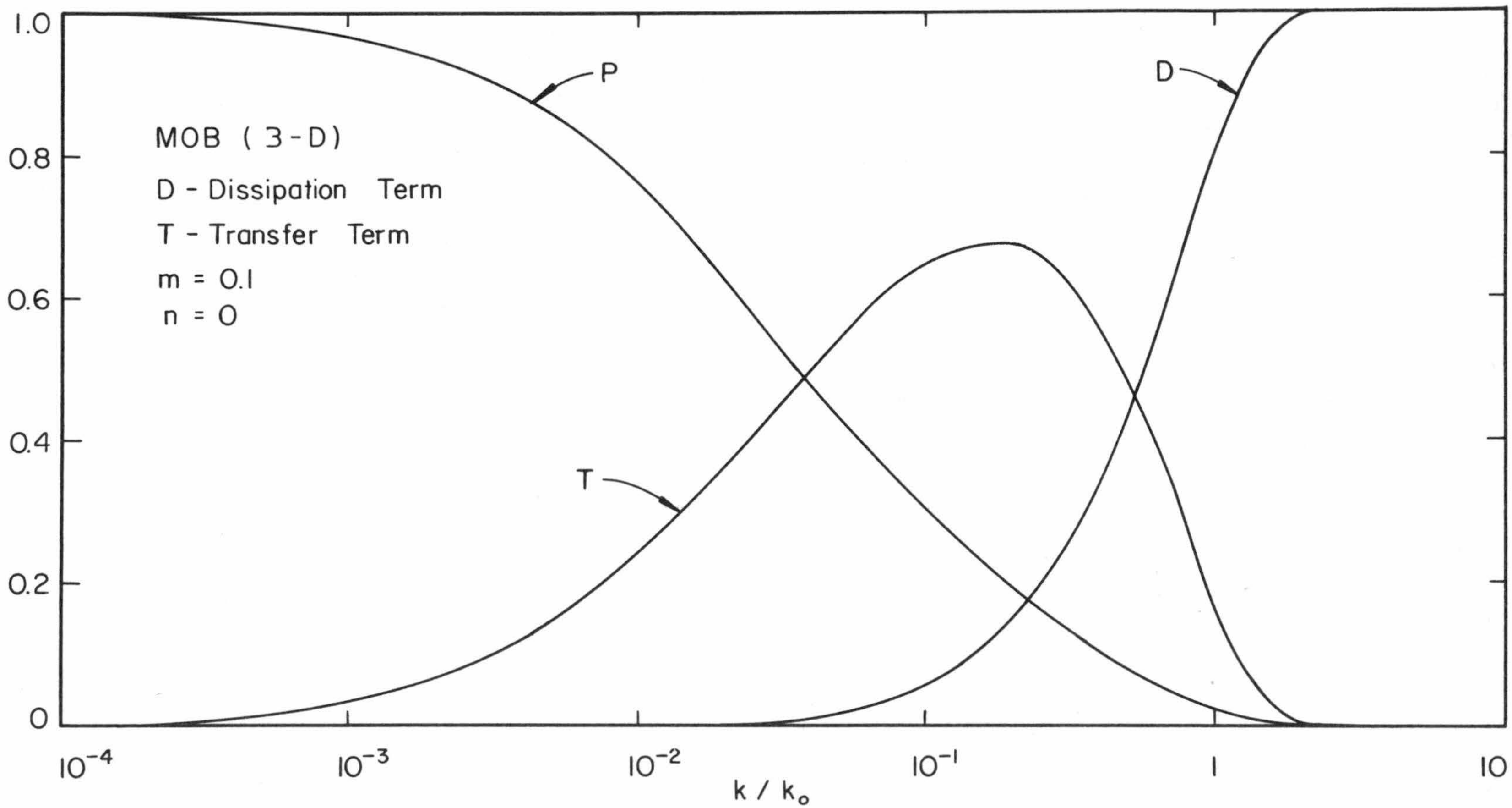

Figure 13. Relative variation of the spectrum equation terms; m=0.1; n=0

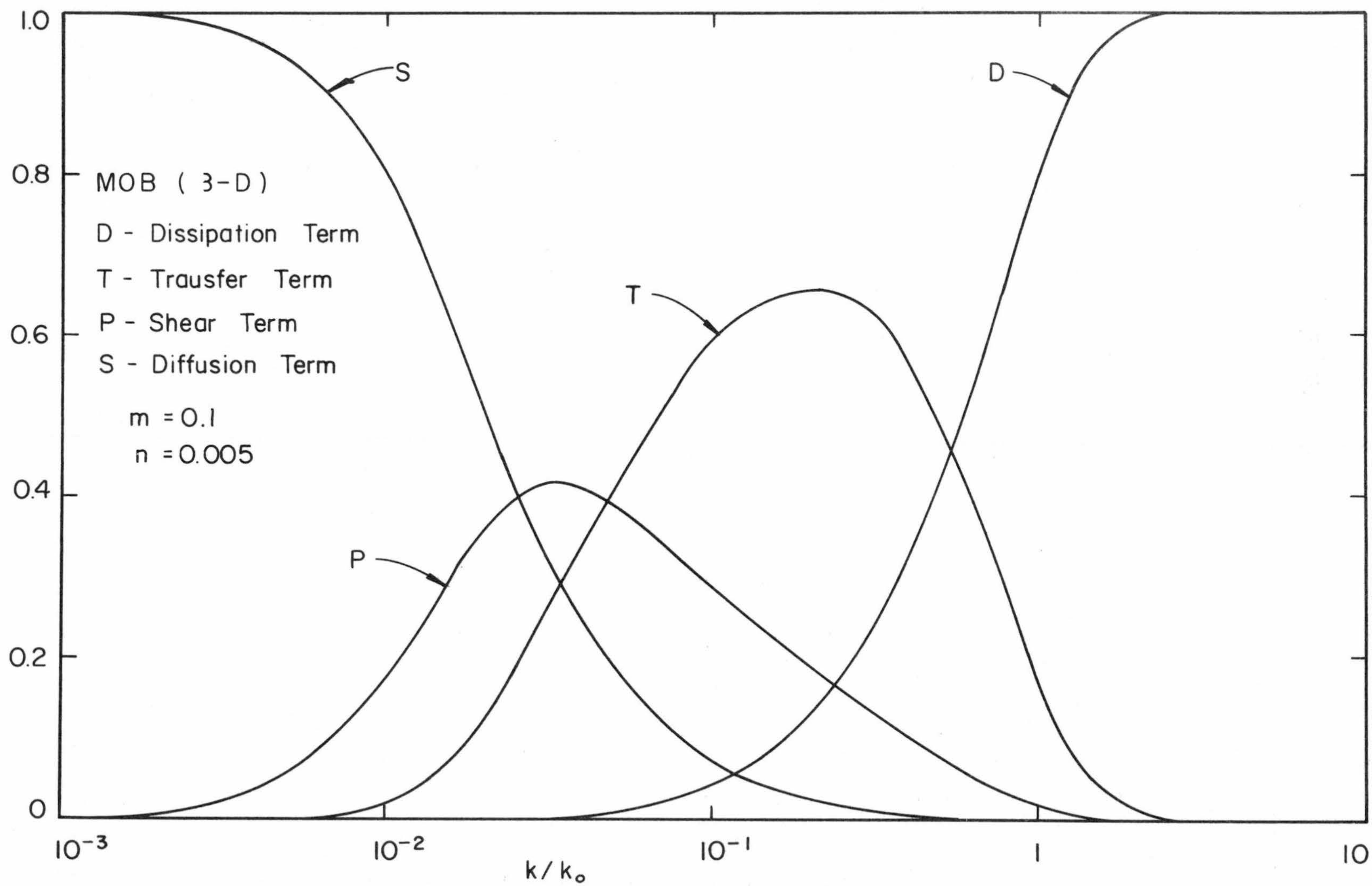

Figure 14. Relative variation of the spectrum equation terms; m=0.1; n=0.005

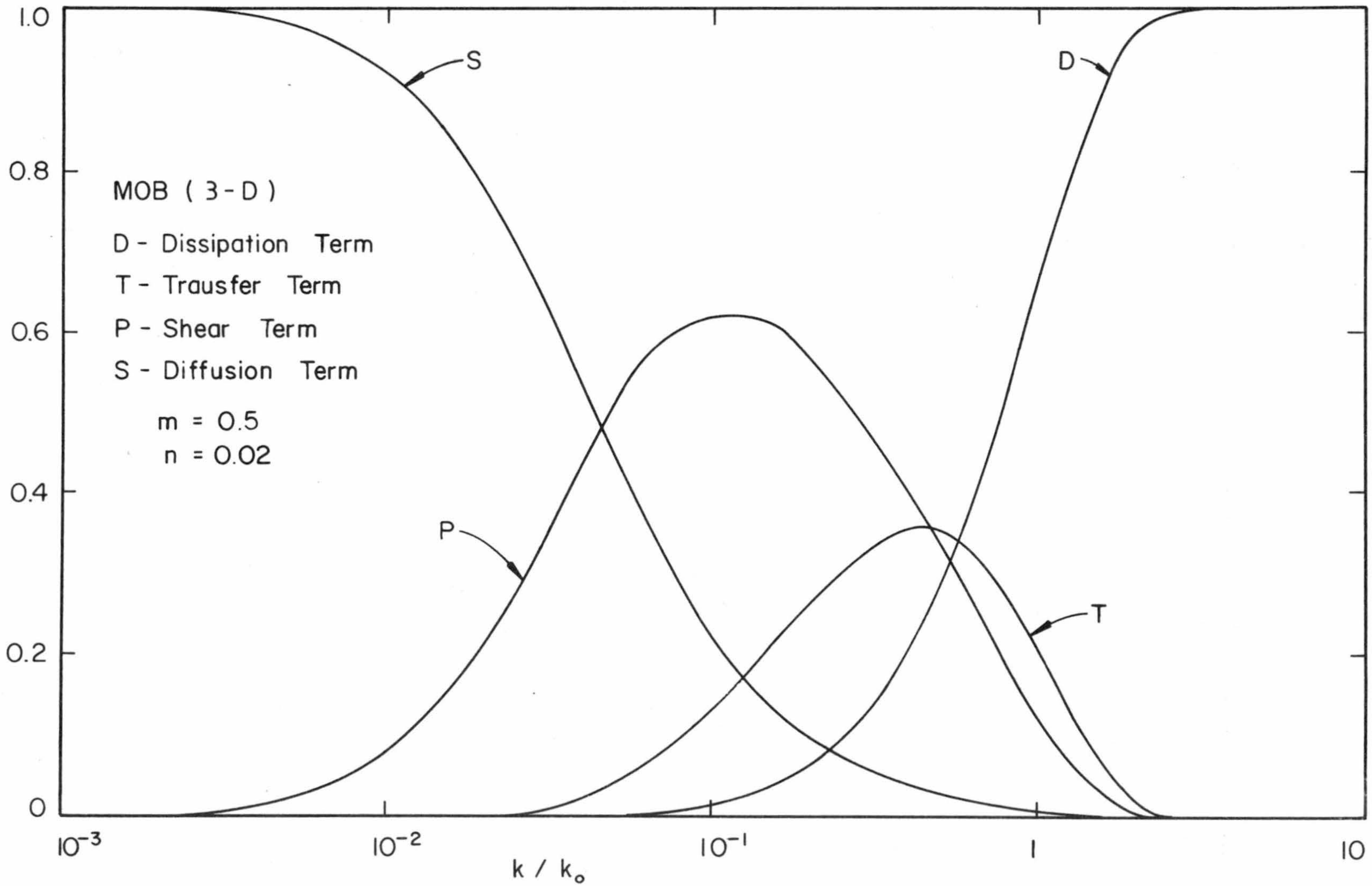

Figure 15. Relative variation of the spectrum equation terms; m=0.5; n=0.02

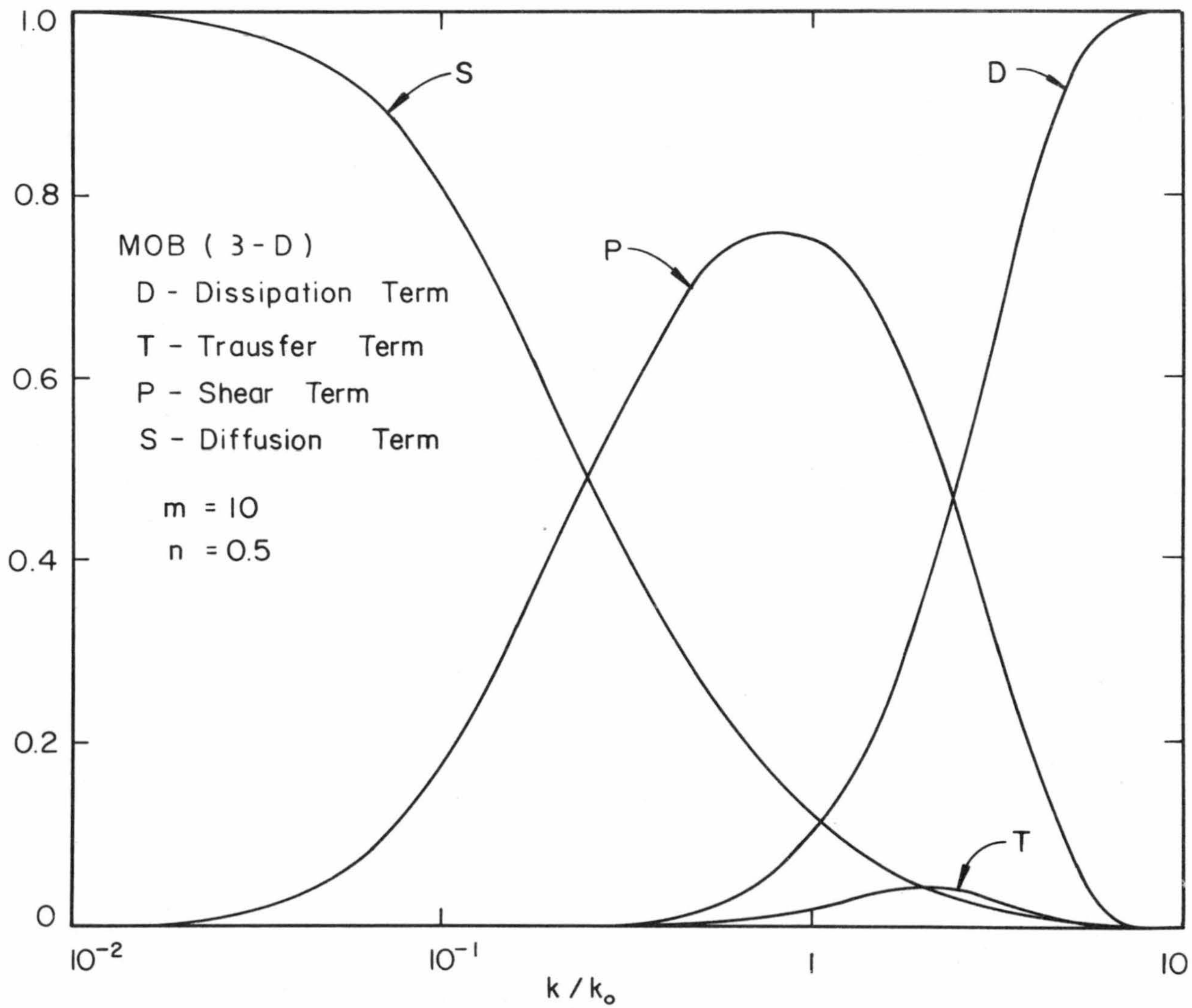

Figure 16. Relative variation of the spectrum equation terms; m=10.0; n=0.5

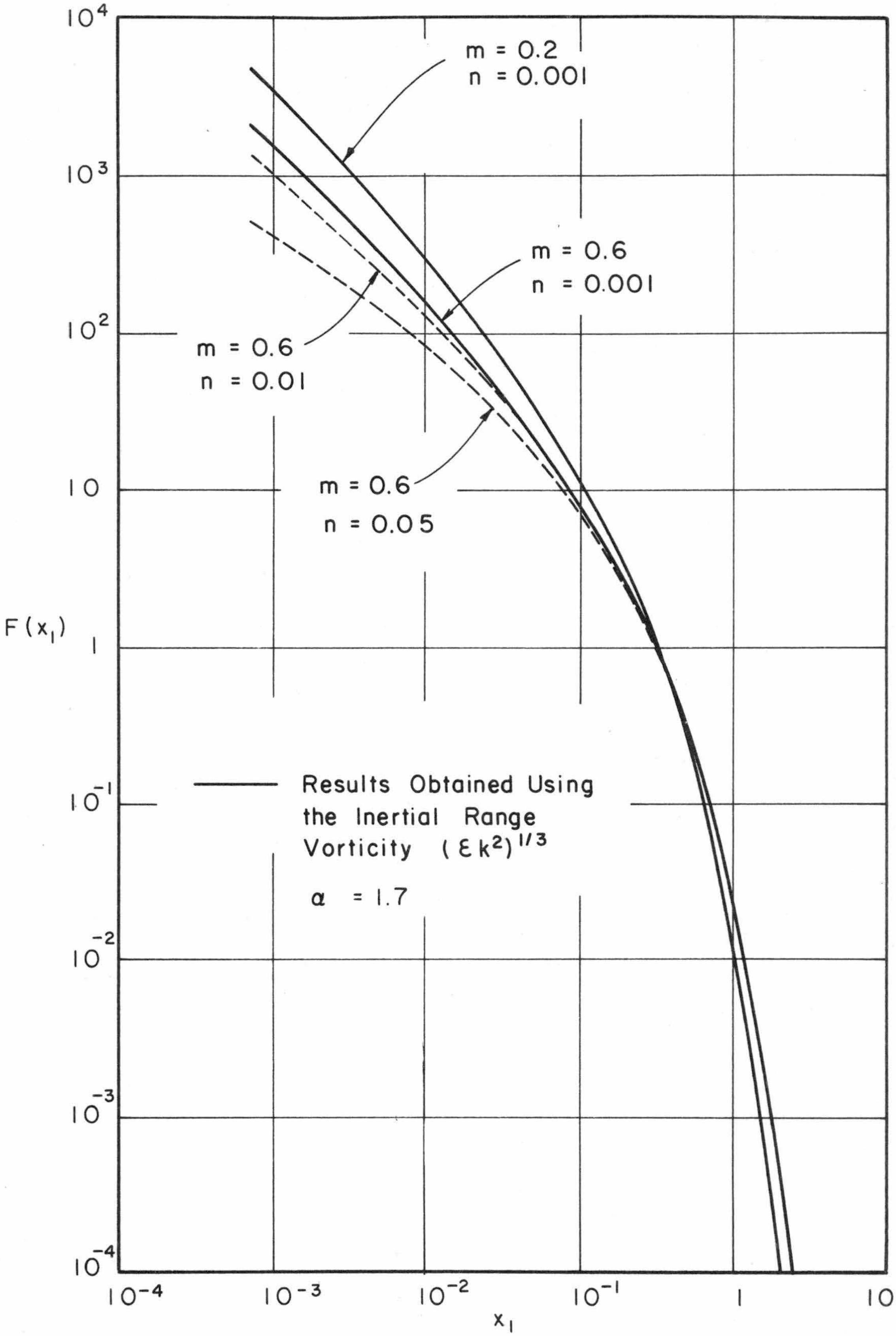

Figure 17. One-dimensional energy spectrum. Vorticity approximation; m=0.2,0.6; n=0.001,0.01,0.05

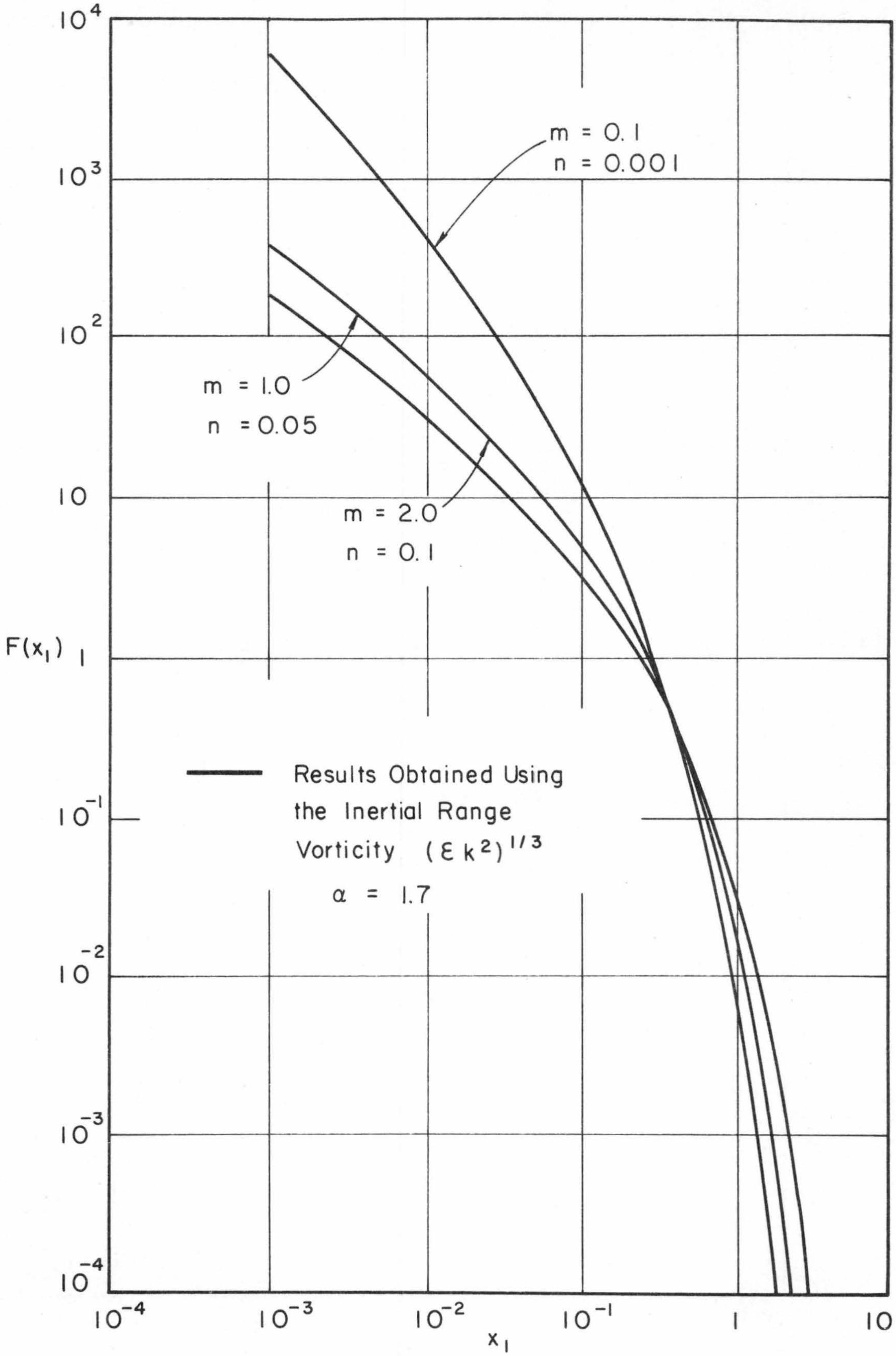

Figure 18. One-dimensional energy spectrum. Vorticity approximation; m=0.1,1,2; n=0.001,0.05,0.1

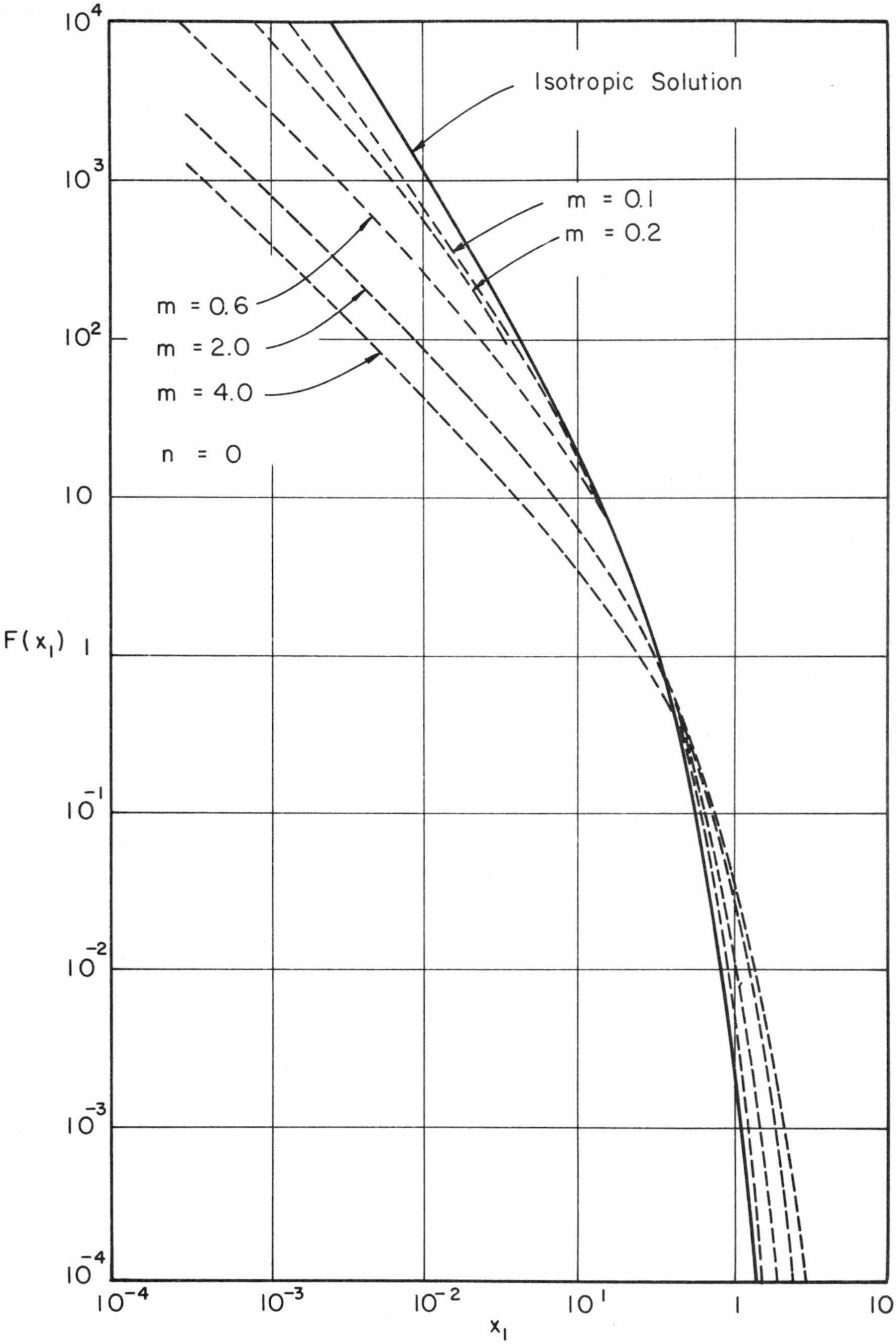

Figure 19. One-dimensional energy spectrum.
MOB approximation; m=0.1,0.2,0.6,2,4; n=0

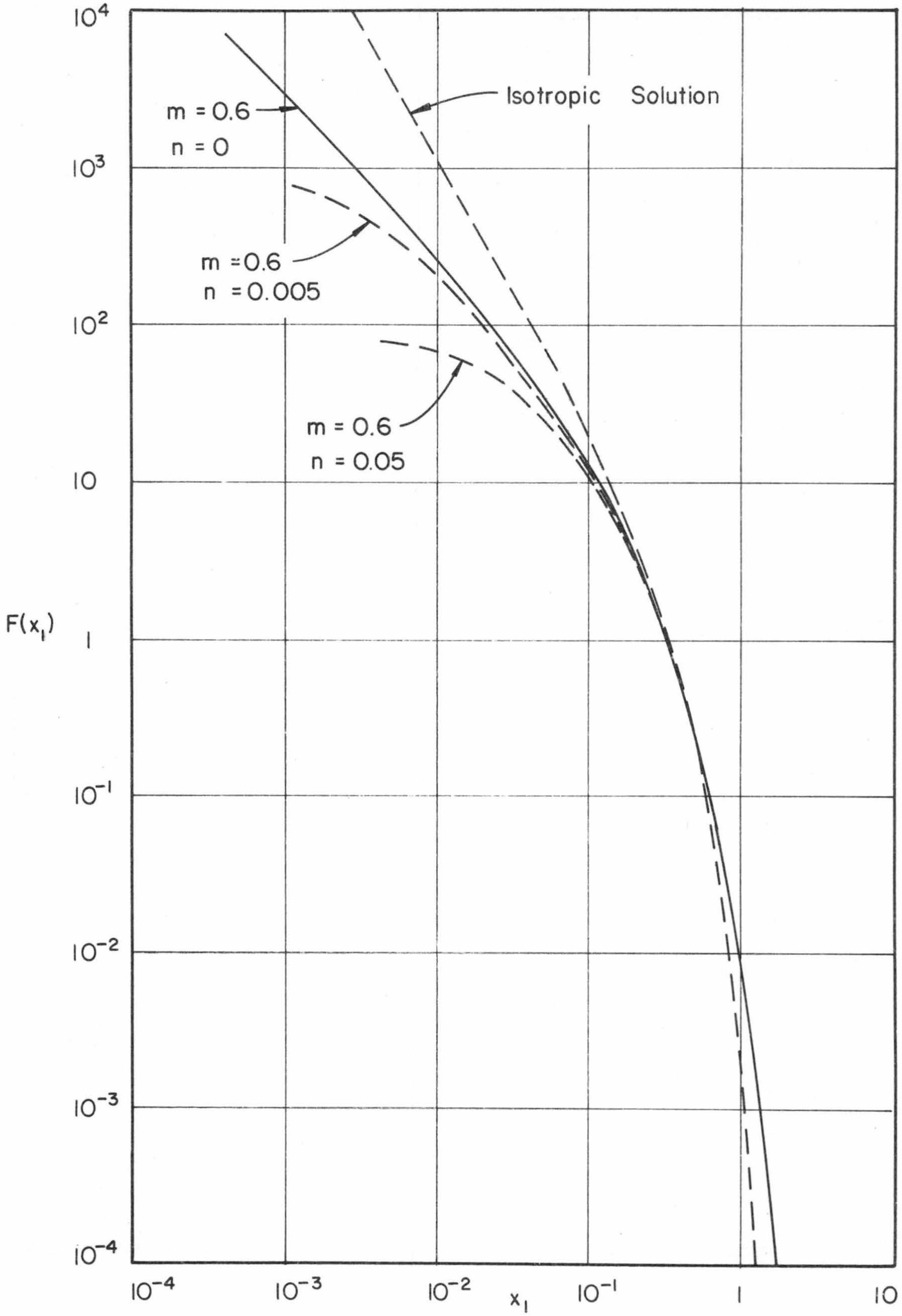

Figure 20. One-dimensional energy spectrum.
MOB approximation; m=0.6; n=0,0.005,0.05

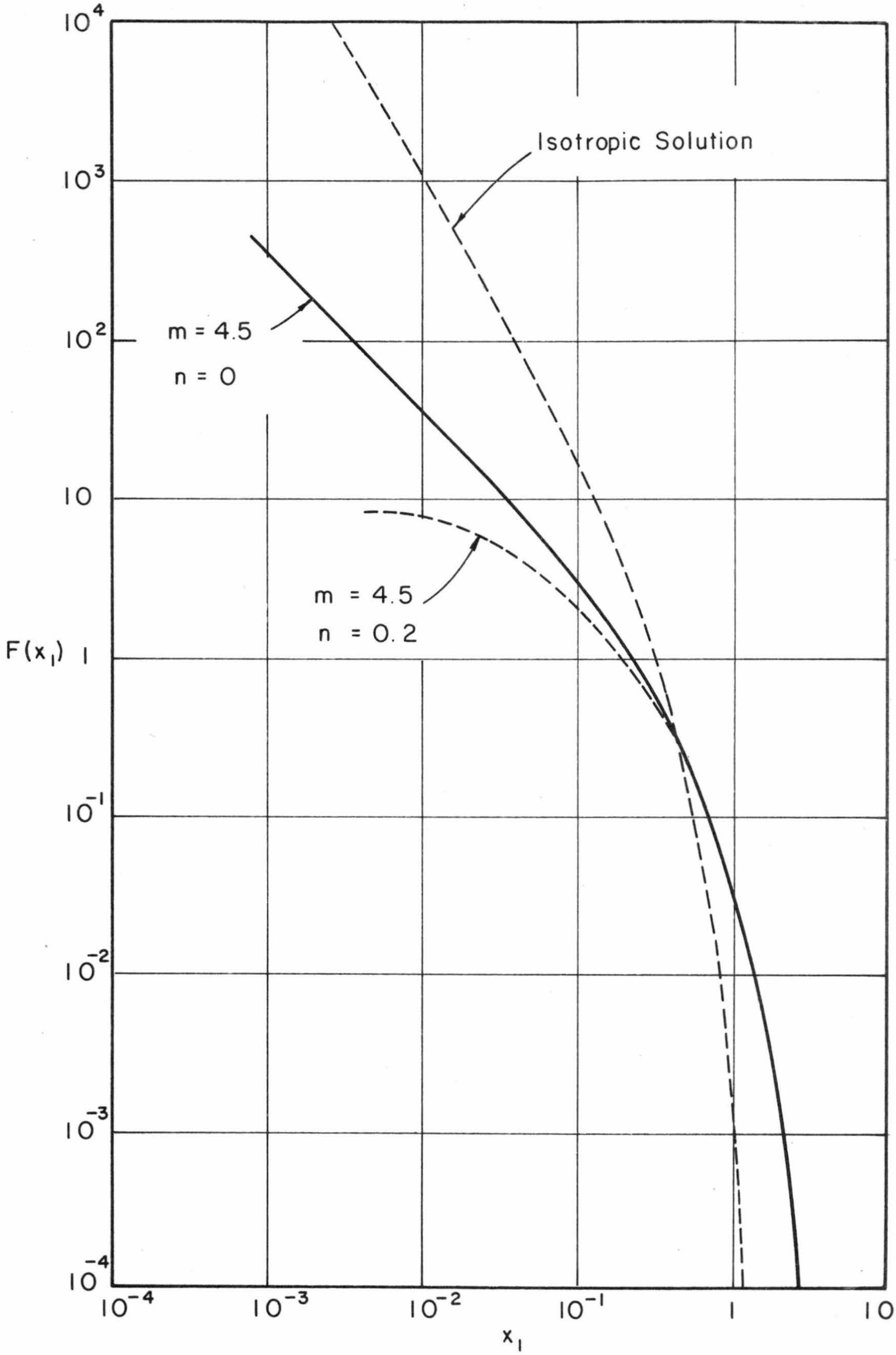

Figure 21. One-dimensional energy spectrum. MOB approximation; $m=4.5$; $n=0,0.2$

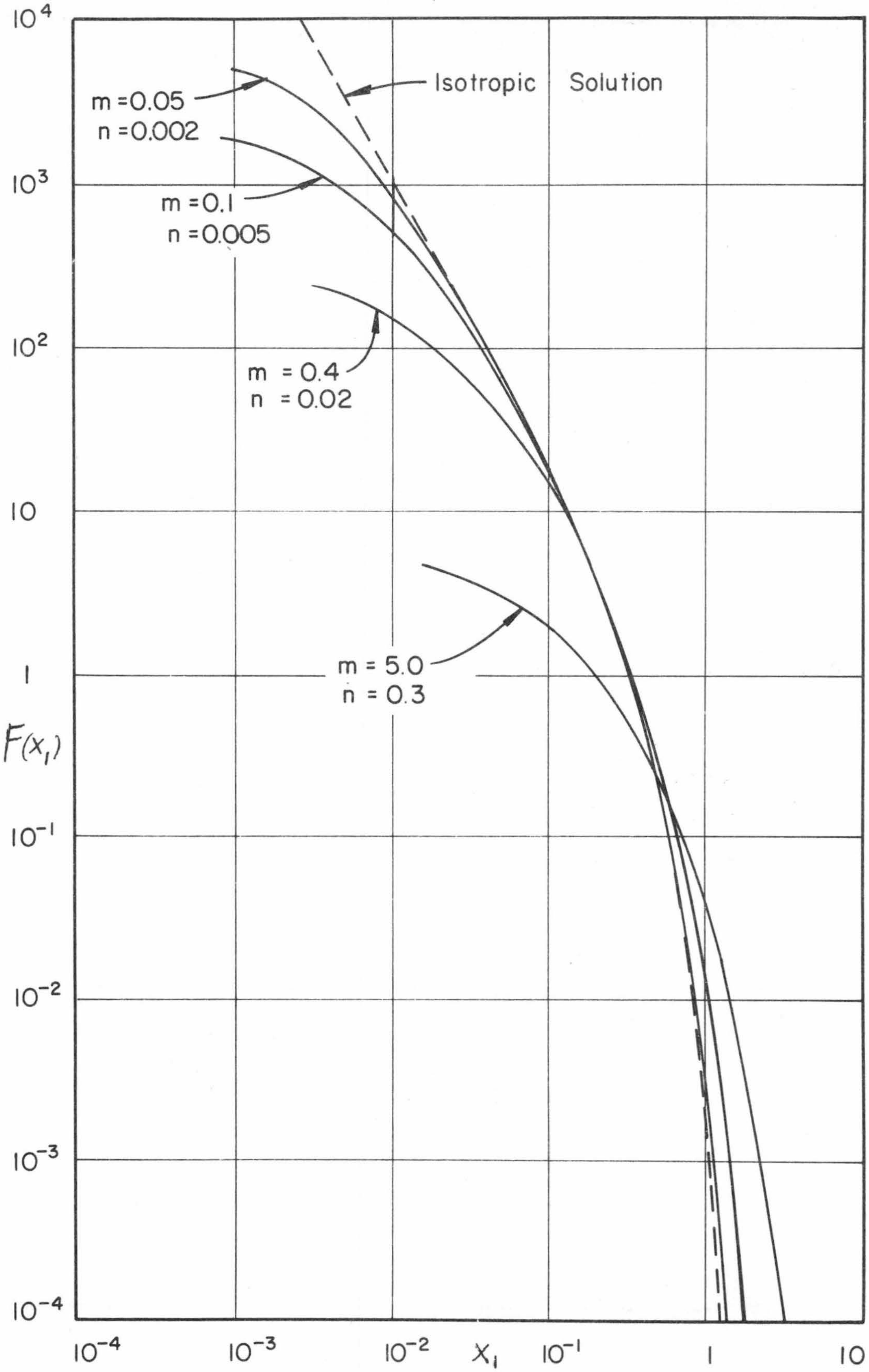

Figure 22. One-dimensional energy spectrum. MOB approx.; m=0.05,0.1,0.4,5.0; n=0.002,0.005,0.02,0.3

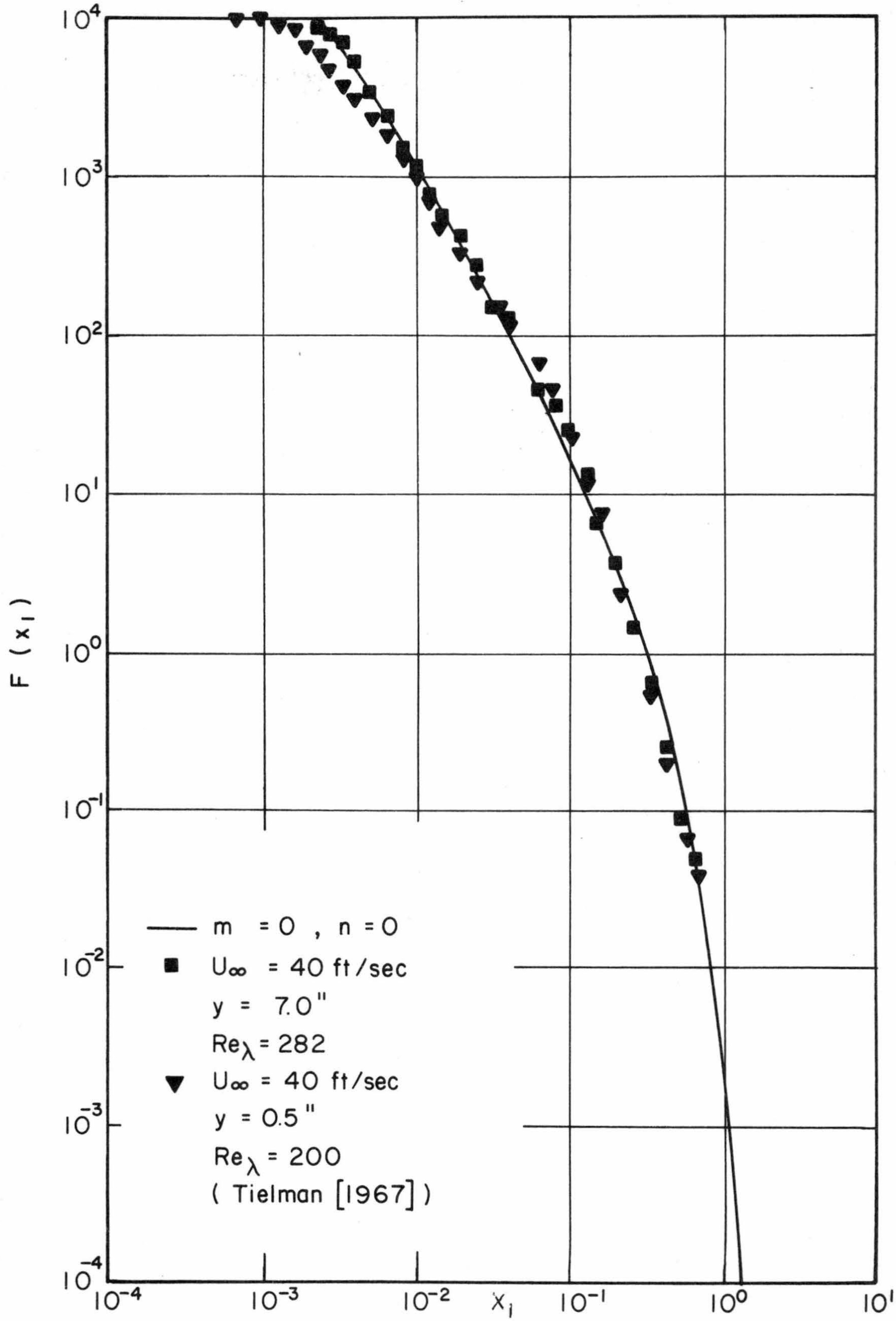

Figure 23. One-dimensional energy spectrum.
MOB approximation; m=0; n=0

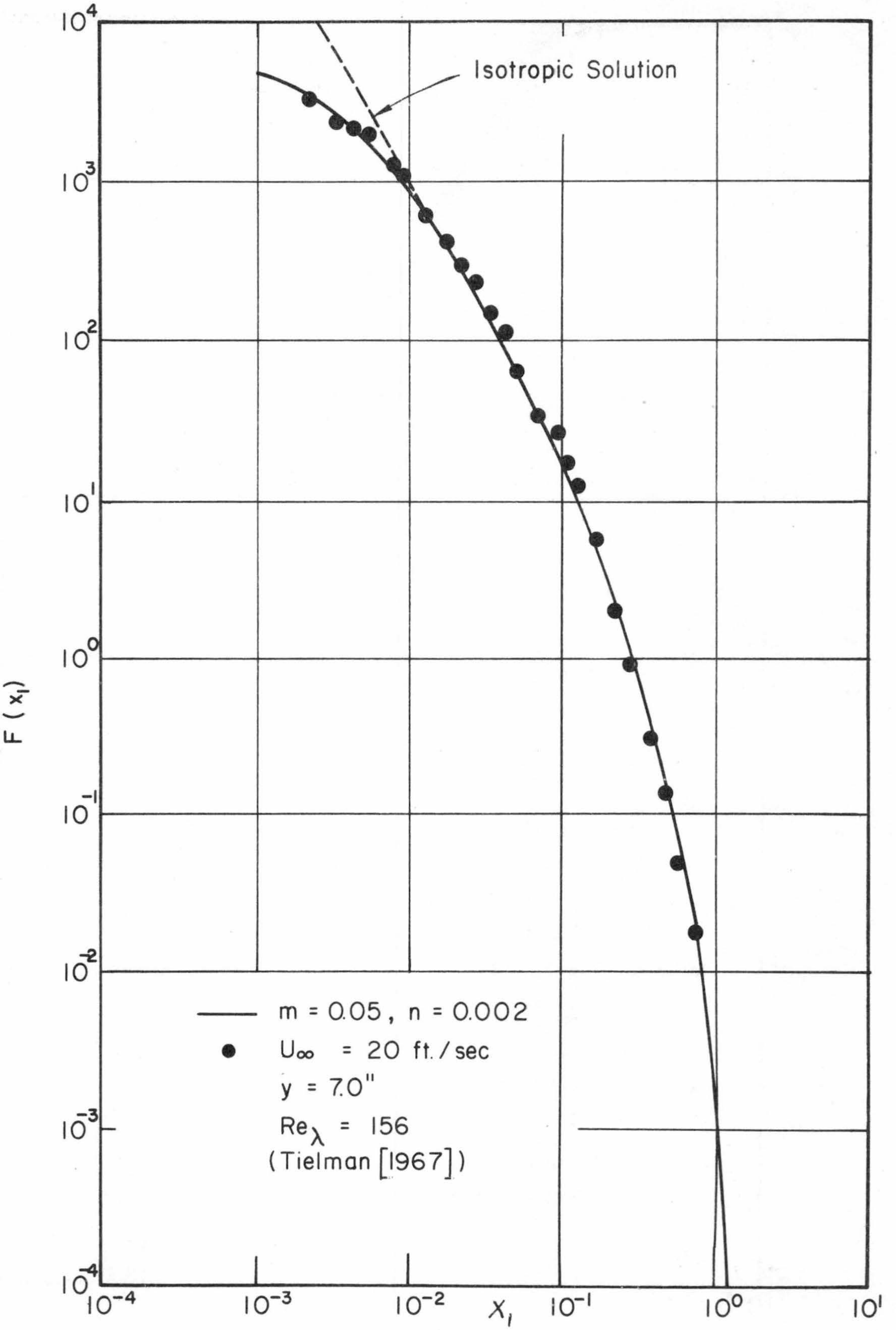

Figure 24. One-dimensional energy spectrum. MOB approximation; m=0.05; n=0.002

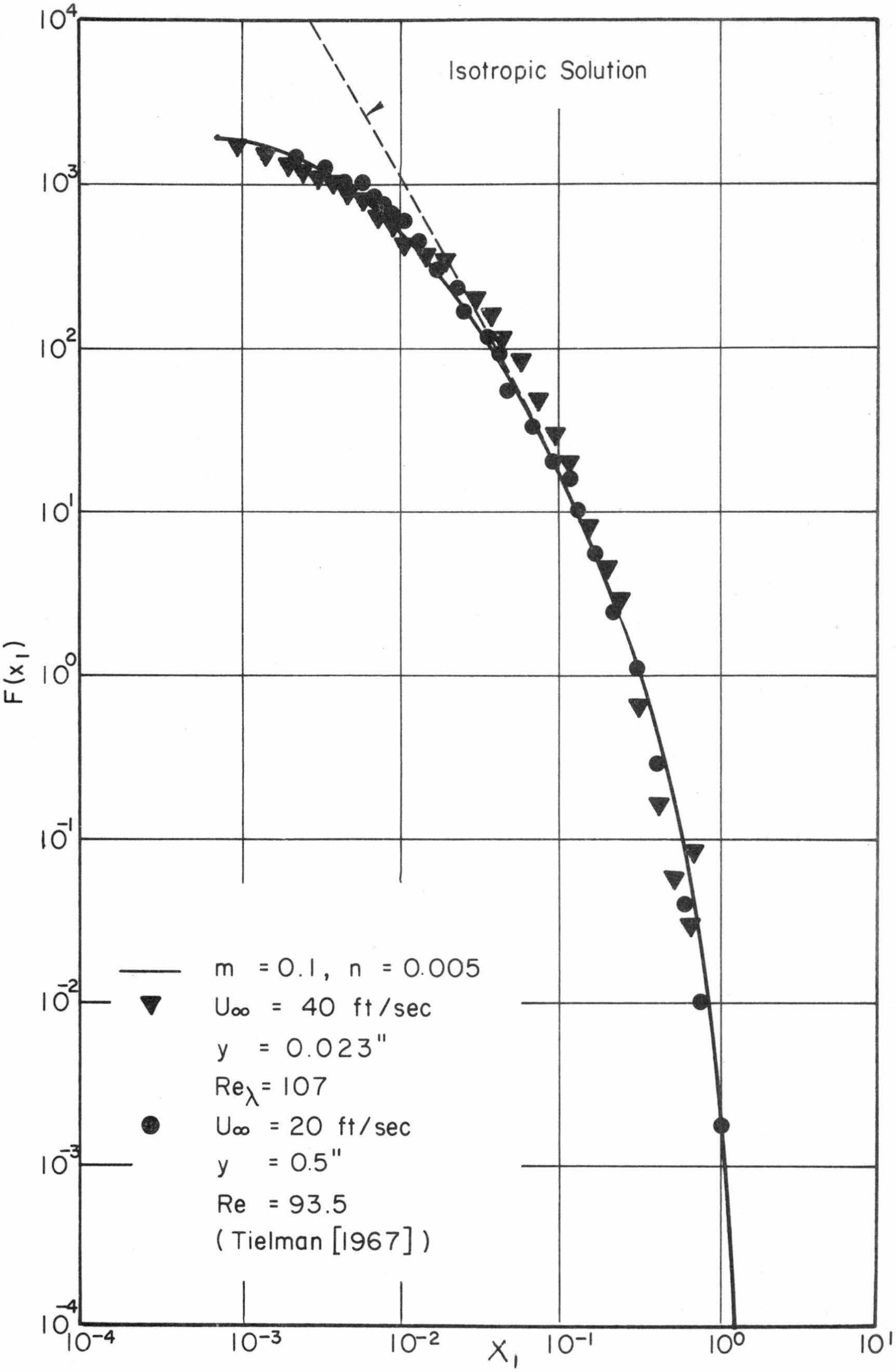

Figure 25. One-dimensional energy spectrum. MOB approximation; m=0.1; n=0.005

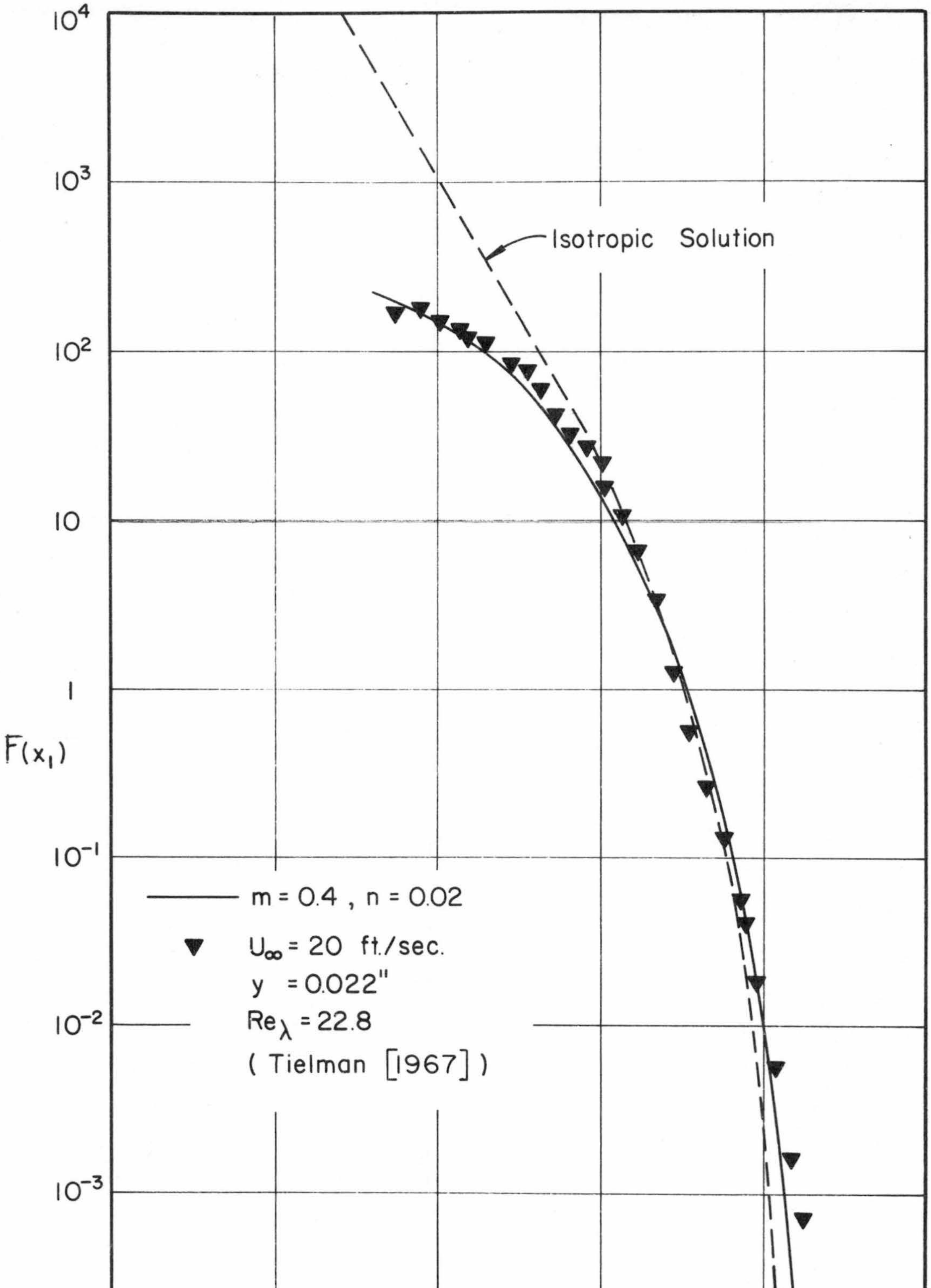

Figure 26. One-dimensional energy spectrum. MOB approximation; m=0.4; n=0.02

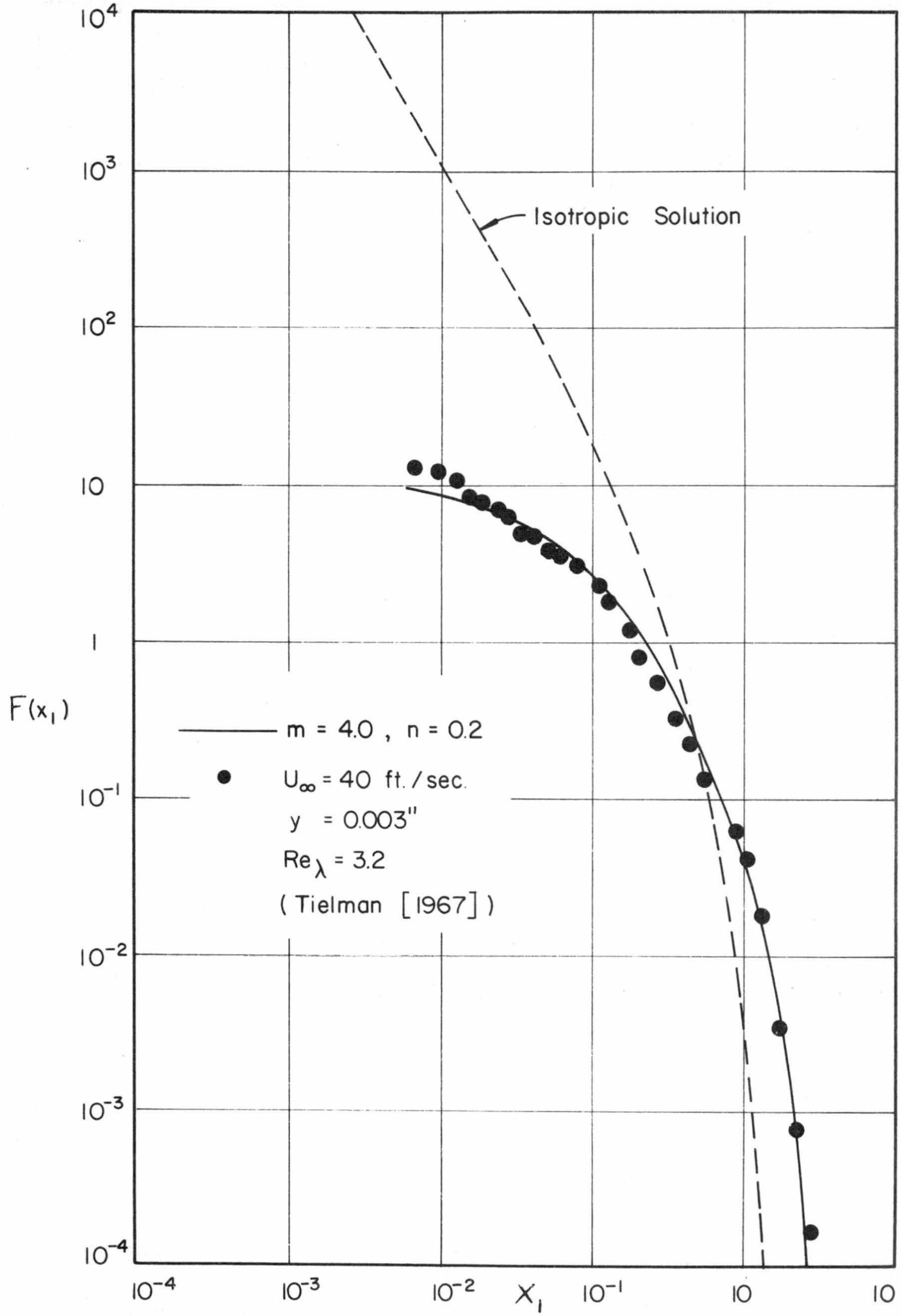

Figure 27. One-dimensional energy spectrum. MOB approximation; m=4.0; n=0.2

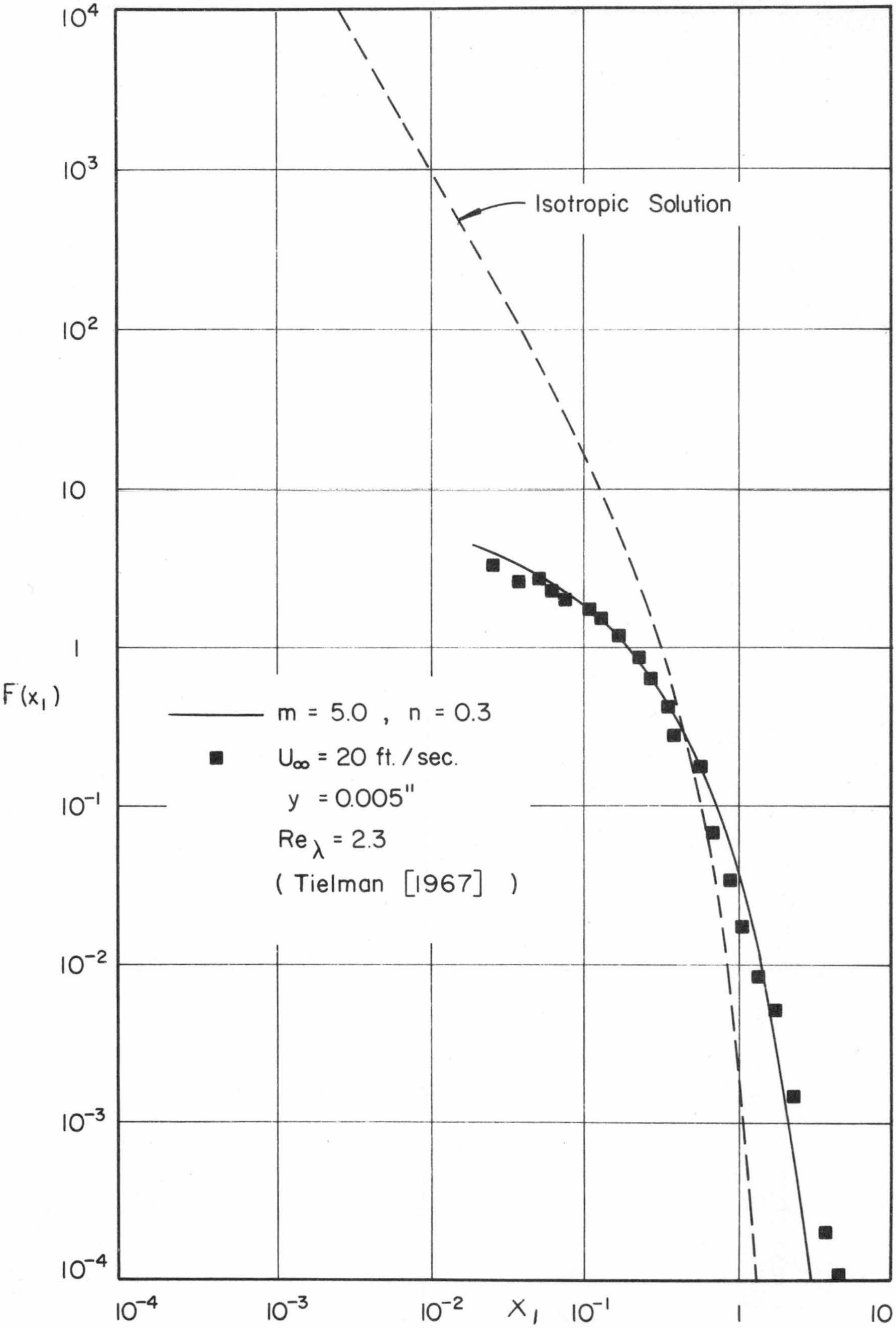

Figure 28. One-dimensional energy spectrum. MOB approximation; m=5.0; n=0.3

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The spectrum of energy in isothermal turbulent shear flow at large wave numbers is studied following the phenomenological approach used by Tchen (1953), Hinze (1959), and Panchev (1968, 1969). The considered spectrum equation consists of the dissipation, the transfer, the production, and the diffusion spectrum function.

Parametric solutions for the three-dimensional energy spectrum function $E(k)$ are obtained firstly by using Heisenberg's type of approximations for the transfer function. Much simpler solutions for $E(k)$ are obtained with the modified Obukhov approximation (Ellison 1962). Some closed form solutions for $E(k)$ are derived by using a vorticity approximation concept.

It is shown that Tchen's k^{-1} law cannot exist if diffusional effects are taken into account. Computed one-dimensional energy spectra (isotropic relations were used) show good agreement with measurements from the viscous region of a turbulent boundary layer.

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