A UNIFIED CONSOLIDATION THEORY

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Synopsis

A theory is presented which considers time lags in consolidation due to both low permeability and "plastic" resistance to deformation. This eliminates the diversion between primary and secondary consolidation.

Shear stresses present in the consolidation experiment are related to the rate of strain of the consolidation sample through use of a non-linear viscous resistance concept.

A procedure for estimating settlements in thick clay layers is explained.

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Introduction

The consolidation experiment is a well controlled test demonstrating the stress strain relationships in soils. It is possible to deduce much information from this test concerning the plastic nature of the soil and the effect of shear stresses on soil behavior. The use of the Terzaghi consolidation theory however, because of its simplified explanation of the consolidation process, gives information only on the permeability and the settlement characteristics of the soil. Even this information may be inaccurate especially when applied to field situations in which the load increment is different than the standard increment used in the test. These inaccuracies have been attributed to the neglect of a consideration of plastic resistance to deformation in the Terzaghi theory.

The effect of plastic resistance has been investigated by Taylor. He postulated a resistance having a viscous form similar to that of a flowing liquid. The coefficient of viscosity was assumed constant over a test increment and because of this limitation the results only applied to the primary portion of the consolidation. The basic idea behind Taylor's work, however, of combining theretarding effect of low permeability with that of viscous resistance hold throughout the consolidation increment if a more valid form of the relationship between stress and viscous flow were used - i.e. one which would describe the viscous resistance even when stresses are extremely small.

Other investigations have considered secondary consolidation to be completely separated from primary consolidation as far as any interaction of effects is concerned and have produced relationships

showing secondary consolidation as a function of time only. In this form, it is difficult to relate the physical properties of the soil with the consolidation phenomenon and very little information on soil strength can be derived from these methods.

The following theory attempts to combine the explanations of primary and secondary consolidation into one useable comprehensive analysis by considering a non-linear plastic resistance to deformation acting along with low permeability to retard settlement.

Definition of Stress

Consider an element containing soil and water taken from some random point in a consolidometer which drains only from the top. The element may look like that shown in Fig. , with plate shape particles which may have a preferential orientation or may not, depending on the previous history. Assume the element large enough to be homogenous in action with repsect to adjacent elements and the voids to be entirely filled with water. At time t_o, the element is in equilibrium under the action of a vertical pressure p_1. The pore pressure is zero and the vertical intergranular stress, T_u, is equal to p_1. The lateral

intergranular stress, σ_e, is smaller than p_i by an amount equal to (twice the) maximum bond or solid friction. A Mohr circle would be as shown in Fig. .

The radius of the circle in Fig. is the value of solid friction on the plane of maximum shear. Solid friction refers to the component of resistance to flow which does not relax with time. Because of the heterogeneous arrangement of particles in the element it is true that solid friction as here defined is not simple friction, but is merely one half the difference between the two principal stresses at equilibrium, however, it does exhibit the principal characteristic of friction (an increase proportional to normal stress).

At time t_o a pressure increment, Δp, is added vertically to the consolidometer. This increase in pressure produces an excess hydrostatic pressure in the pore water, and the water begins to drain upward. The change in pore pressure due to the flow of water is transferred to the soil structure and produces an increase in T_u. The element exhibits an immediate elastic deformation and a slow plastic deformation. This deformation brings into play a viscous resistance to motion in the water films surrounding the particles.

The viscous resistance takes the form of increased shear resistance and enlarges the size of the Mohr circle. At a time, t_b, early in the consolidation process, the circle will be as shown in Fig. . The radius of the circle is large and is equal to $2(f + \tau)$ on the plane of maximum shear where

f = solid friction

τ = viscous resistance.

From Fig. .

$$P_2 = u + 2(f + \tau) + \sigma_e \qquad (1)$$

As σ_v approaches p_2 the rate of deformation becomes small and τ decreases. The Mohr circle becomes smaller and is as shown in Fig. .

Fig. represents the condition of stress at a new equilibrium point with pore pressure again equal to zero. The radius of the circle is somewhat larger than in because the solid friction has increased due to an increase in normal pressure. τ in Fig. is zero.

Eq. (1) comprises a basis for the analysis of consolidation, and the following sections are concerned with comments on the nature of the terms in this equation.

Pore Pressure in Saturated Clay Soils -

The excess pore pressures in the consolidation test will be equal to the pressure increment at times t_o and will decrease at a decreasing rate throughout the test. If Darcy's law is assumed valid, two expressions for volume change of the element with time may be equated to produce the following equation:

$$\frac{K}{\gamma_\omega} \frac{\partial^2 u}{\partial z^2} = \frac{1}{He} \frac{\partial e}{\partial t} \qquad (2)$$

where K = coefficient of permeability

γ_ω = unit weight of water

e = void ratio.

The pore pressure is thus related to the rate of change of void ratio and has a value different from zero at any time. The value of u, however, may become very small during the latter part of the test.

Viscous Resistance to Deformation -

This resistance has been considered by many authors to be proportional to the time rate of change of strain, either through extension of the idea of viscous flow in free water or because of the difficulty of handling more complex relationships.

A more realistic visualization of the stress versus rate of strain relationship would be as shown in Fig. .

Fig.

The curve in Fig. may be represented fairly accurately by an equation of the form

$$\tau = C_1 \left(-\frac{\partial e}{\partial t} \right)^{C_2}$$

where C_1 and C_2 are constants.

$$\tau = C_1 \left(-\frac{\partial e}{\partial e}\right)^{C_2} \qquad (3)$$

where C_1 and C_2 are constants.

The constants in this equation are functious of the closeness of soil particles to each other and may change in magnitude in succeeding load increments in the consolidation test.

Solid Friction -

The solid friction, f , which also resists shear deformation in the consolidation smaple is probably very closely related to the void ratio since the void ratio is a measure of the closeness of particles to each other. If this is the case, then f increases during the consolidation process, but may be considered to be a constant equal to its average value for sufficiently small pressure increments.

Lateral Intergranual Stress (σ_e)

Terzaghi assumes intergranular pressure to vary in a linear fashion with void ratio. Taylor suggests also that this relationship is approximately true, but defines intergranular pressure in such a way that it coincides with lateral intergranular pressure, σ_e .

Comparison of e -p curves for σ_v and σ_e is shown in Fig .

σ_e is smaller than σ_v by the value of $2(\tau + f)$ which is the diameter of the Mohr circles at equilibrium shown in Fig . The curves are essentially parallel (Taylor p. 371), and if we represent this relationship by the equation

$$e = C - C_3 \sigma_e$$

we may closely approximate the slope C_3 by the slope of σ_v vs. e .

Equation of Consolidation

Eqs. 1, 2, 3, and 4 state mathematically the various ideas and assumptions connected with this theory.

If we take the second deviation of P_z in eq. 1 with respect to z , we have

$$o = \frac{\partial^2 u}{\partial z^2} + \frac{2\,\partial^2 \tau}{\partial z^2} + \frac{\partial^2 \sigma e}{\partial z^2} \qquad (5)$$

since P_z and f are assumed constant.

The above terms are evaluated from eqs. 2, 3, and 4; and substituted in eq. 5. The resulting equation is

$$o = \frac{-1}{C_3}\frac{\partial^2 e}{\partial z^2} + 2 C_1 C_2 \left(\frac{-\partial e}{\partial t}\right)^{C_2 - 1}\left(\frac{-\partial^3 e}{\partial t\,\partial z^2}\right) + 2 C_1 C_2 (C_2 - 1) \left(\frac{-\partial e}{\partial t}\right)^{C_2 - 2}\left(\frac{-\partial^2 e}{\partial z \partial t}\right)^2 + C_4 \frac{\partial e}{\partial t}$$

where $C_4 = \frac{\partial \omega}{K_e (1+e)}$ which is also considered constant for small increments of pressure.

Testing Program

Consolidation tests were performed on a silty clay. (LL. = 47.5, P.I. = 21). Nine consolidometers with various height-diameter ratios were used to evaluate the effect of height and diameter on the consolidation parameters. Dimensions of the consolidometers are shown in Table .

Table Dimensions of consolidometers.

The consolidometers were plastic lined and drained from the top only.

Duplicate consolidation tests with standard pressure increments were run using each of the consolidometers. Consolidation tests using the non standard pressure increments shown in Table were run with consolidometers 5 and 6.

Table Non standard load increments.

A series of 3 folding head permeability tests were performed on consolidated samples with different void ratios to provide a check on the permeability values obtained from the theory.

Samples were prepared by forming a paste of distilled water and minus AD soil. This saturated paste was allowed to soak for 2 weeks and was then deposited in thin layers in the bottom of a consolidometer which was partially full of distilled water. Each layer was evenly spread by tapping the consolidometer. This technique produced disturbed samples which were practically 100% saturated and also gave duplicate consolidation tests which were reasonably similar. No attempt was made to measure pore pressures in these tests.

Determination of the Consolidation Constants

The con stant C_3 in equation is defined as the slope of the curve $e = f(\sigma_1)$ for the pressure increment. Since $e = f(\sigma_v)$ is almost parallel to this curve, the slope of $e = f(\sigma_v)$, which is the regular e - P curve from the consolidation test, was used to compute C_3.

The other onstants - C_1, C_2, C_4, are determined from the e - t curves for each of the pressure increments, and are assumed to be constant in each increment.

It is very difficult to solve eq. completely, but a great deal of information on the nature of the equation can be deduced by examining the variation in e as we change only one of the independent variables z to t.

The surface $e = f(z, t)$ should be similar to that shown in Fig. .

It is assumed that after the first few minutes of consolidation (during which the value of e at $z = o$ is changing rapidly) the slope of the curve $e = Z(z)$ changes very little. That is to say, if $e = \sin z$ could be used to define curve b (Fig.), then $e = c \sin z$ could be used to define curve c. This statement is merely an example and does not imply that $\sin z$ defines either curve. This assumption is equivalemt to saying that

$$e = Z(z) \cdot T(t)$$

Substituting this expression into Eq. we obtain,

$$\frac{-1}{C_3} Z'' T + 2C_1 C_2 (-ZT')^{C_{2-1}} (-Z''T') + 2C_1C_2 (C_{2-1})(-ZT')^{C_{2-2}}(-Z'T')^2 + C_4 ZT' = 0$$

Now consider the trace of the surface in the vicinity of $z = H$, where $Z' = \frac{de}{dz} = 0$ and $e = em$.

In this vicinity:

$$-\frac{1}{C_3} Z''T + 2 C_1C_2 (-ZT')^{C_{2-1}} (-Z''T') + C_4 ZT' = 0$$

or

$$-\frac{1}{C_3} \frac{T}{T'} - 2 C_1C_2 (-ZT')^{C_{2-1}} + C_4 \frac{Z}{Z'} = 0$$

The third term of Eq. is independent of t and must be constant when t varies, so for a variation in t only,

$$+ \frac{1}{C_3} \frac{T}{T'} + 2 C_1 C_2 (-ZT')^{C_2-1} = + C_4 \frac{Z}{Z''} = -a_1$$

Let $Z(z)_{z=H} = 1$. The variation in e_m with time is then included in T.

The left member of Eq. now becomes:

$$\frac{em}{C_3} - 2 C_1 C_2 \left(-\frac{dem}{dt}\right)^{C_2} - a_1 \left(-\frac{dem}{dt}\right) = 0$$

Since $\bar{e} = re_{max}$ (another consequence of the assumption $e = Z \cdot T$ we may write this equation in terms of the average void ratio at any time:

$$\frac{\bar{e}}{rC_3} - \frac{2C_1C_2}{r^{C_2}} \left(-\frac{de}{dt}\right)^{C_2} - \frac{a_1}{r} \left(-\frac{d\bar{e}}{dt}\right) = 0$$

or

$$\frac{\bar{e}}{C_3} - 2C_1C_2 \, r^{1-C_2} \left(-\frac{d\bar{e}}{dt}\right)^{C_2} - a_1 \left(-\frac{d\bar{e}}{dt}\right) = 0$$

A solution of the right hand member of Eq. follows

$C_4 Z + a_1 Z'' = 0$ Let $Z = \sin \frac{\pi z}{ZH}$

then $C_4 \sin \frac{\pi z}{2H} - a_1 \frac{\pi^2}{TH^2} \sin \frac{\pi z}{2H} = 0$ and $C_4 = \frac{a_1 \pi^2}{-4H^2}$

This solution is only valid near $z = H$, but it is sufficient to relate a_1 and C_4.

Eq. then becomes

$$\frac{\bar{e}}{C_3} - 2C_1C_2 \, r^{1-C_2} \left(-\frac{d\bar{e}}{dt}\right)^{C_2} - \frac{4H^2C_4}{\pi^2} \left(-\frac{d\bar{e}}{dt}\right) = 0$$

This is the equation of average void ratio against rate of change of void ratio which should hold for the experimental curve. Values of C_2 and C_4 may be found as follows:

Solve for the middle term.

$$-2C_1C_2\, r^{1-C_2}\left(\frac{-d\bar{e}}{dt}\right)^{C_2} = \frac{4H^2}{\pi^2} C_4\left(\frac{-d\bar{e}}{dt}\right) - \frac{\bar{e}}{C_3}$$

Evaluate this equation at $t = 10$, 100, and 1000, and divide to obtain the following pair of equations:

$$\frac{\left(\frac{-d\bar{e}}{dt}\right)_{10}^{C_2}}{\left(\frac{-d\bar{e}}{dt}\right)_{100}^{C_2}} = \frac{\frac{4H^2}{\pi 2} C_4 \left(\frac{-d\bar{e}}{dt}\right)_{10} - \frac{\bar{e}_{10}}{C_3}}{\frac{4H^2}{\pi^2} C_4 \left(\frac{-de}{dt}\right)_{100} - \frac{e_{100}}{C_3}}$$

$$\frac{\left(\frac{-d\bar{e}}{dt}\right)_{100}^{C_2}}{\left(\frac{-d\bar{e}}{dt}\right)_{1000}^{C_2}} = \frac{\frac{4H^2}{\pi 2} C_4 \left(\frac{-d\bar{e}}{dt}\right)_{100} - \frac{\bar{e}_{100}}{C_3}}{\frac{4H^2}{\pi^2} C_4 \left(\frac{-d\bar{e}}{dt}\right)_{1000} - \frac{\bar{e}_{1000}}{C_3}}$$

These equations are solved by trial and error for values of C_2 and C_4. These values are tabulated in Table .

To find C_1, a value for r must be known, and this depends on the shape of the curve $e = Z(z)$. It has already been assumed that the e vs. z curve maintains the same relative shape as time passes, so the rate of change of e is approximately proportional to e. That is,

$$\frac{\partial e}{\partial t} = - \omega e$$

Substituting this into equation we obtain:

$$\frac{1}{C_3}\left(\frac{-\partial^2 e}{\partial z^2}\right) + 2C_1C_2\,\omega^{C_{2-1}}\,e^{C_{2-1}}\left(\frac{\omega\,\partial^2 e}{\partial z^2}\right) + 2C_1C_2\,(C_{2-1})$$

$$\omega^{C_{2-2}}\,e^{C_{2-2}}\left(\frac{\omega \partial e}{\partial z}\right)^2 - C_{4\omega e} = 0$$

Values of C_2, C_3, C_4, and ω were found for consolidometer 5 for a pressure environment 1-2 tons/ft^2, and a value of C_1 was assumed in order to solve equation by a numerical method to produce the curves shown in Fig. .

Fig. Shape of $e = Z(z)$.

The value of γ is 0.88 for 10 min. and 0.99 for 800 min. The contrary of γ does not verify that assumption of small change in the e-z curves, but it gives the feeling that only a small error is introduced by using this assumption.

Equation may therefore be solved for C_1 after C_2 and C_4 are known by using an average value of γ.

Values of C_1 are shown in table .

Significance of Results

Variation of Contents

In a discussion of the results, it should first be pointed out that values of C_4 are inconsistant for consolidometers 1, 2, and 3. This is normal since C_4 is a function of the coefficient of permeability and the samples in these consolidometers were so thin that permeability played a very small part in determining the shape of the consolidation curve. Values of C_4 taken from these curves may therefore show a considerable error: It is felt also that data from consolidometer 7 are erratic. All four of these consolidometers fall outside the limits in dimensions suggested by Enkeboll ( ) and it is felt that they do not give consistant results. For these reasons, the following discussion is limited to consolidometers 4, 5, 6, 8, and 9.

The equation

$$\tau = C_1 \left(-\frac{de}{dt}\right)^{C_2}$$

gives the relationship between the shearing stress and the rate of

strain for the soil sample. An examination of Table shows that there is no variation of C_2 with pressure or with height in the standard tests summarized there. C_1 does not vary with height, but does vary with pressure in the fashion shown in Fig. . In this figure, the values of C_1 have been plotted against average values of σ for each pressure increment. It is apparent that C_1 is proportional to σ. This observation indicates that if we wish to produce the same rate of strain in a sample under 8 Tons/ft^2 normal pressure as we have in a sample under 2 Tons/ft^2 we must apply four times as much shearing stress. This agrees with intuition and may even explain the proportionality between normal stress and shear strength for clays in shear tests under constant strain testing.

Values for the coefficient of permeability may be found through equation once C_4 is known. These values are tabulated in Table and plotted in Fig. .

Table

Fig.

The computed values plot as a straight line on semi log paper. This is indicated by Taylor ( ) as the proper relationship between void ratio and permeability. It is also evident that the computed values and the measured values taken from falling head permeability tests coincide.

A Check of the Theoretical Equation with Computed Constants

It is possible to reproduce the consolidation curve by using Eq. and by noting that a plot of e vs. $\left(\frac{1}{\frac{-de}{dt}}\right)$ has time as its area. See Fig. .

Fig.

The area a , for example is the time to produce a change in void ratio from e_1 to e_z . The curve shown in Fig. may be plotted by using eq. .

As an example of the fit of the theoretical curve to the experiment curve, test from consolidometer 5, press. incr. 4-8 Tons/ft^2, (Fig) has been selected. The constants C_1, C_2, and C_4 are taken from Tables , , and . Fig. is the representation of eq. . e = ____ was chosen as the initial point. It is the value of e for 5 min. for the test. Value of time

against void ratio are found easily by counting the squares on the plot. Results are shown in Fig. .

Non-Standard Tests

Consolidation tests as explained under the section on Testing Programs were run for non standard pressure increments. Results are tabulated in Table and plotted in Fig. . It is apparent that the increment-pressure ratio defined by Taylor has no influence on the values of C_1 and C_2 as found from this theory.

Prediction of Settlements

To predict the settlement of a large clay layer which is fully saturated and which has a uniform pressure increase at all levels, the procedure followed in reproducing the experimental curve may be used. In eq. let H be the height of the clay layer if it is drained on one side, or one-half the height of the clay layer if it is drained on two sides. (In this second case, multiply results of settlement in one-half of layer by 2 to obtain total). Computations for a clay layer 100 inches thick and drained from one side only are as follows:

The same problem was solved by using Terzaghi's method and the curve is shown also in Fig. . There is a fairly large difference in settlements from these two theories at early times, but they approach each other at large values of time.

For pressure increments which are smaller than standard, Taylor has suggested that values of the coefficient of consolidation for Terzaghi's theory be found from standard tests. Using this suggestion for a comparison with Terzaghi's theory resulted in the graph shown as Fig. .

Fig.

It is apparent that for small pressure increments the theory here presented predicts a much smaller settlement for long periods of time than that predicted by Terzaghi. This seems to coincide with observation, (Taylor) ( ) .

Example of Procedure

Following is the complete procedure which could be used to predict a settlement time curve for a given pressure increment for a large saturated clay layer*.

1 - Run consolidation tests on undisturbed samples.
   a - Minimum initial thickness = 0.7 inches if drained on one side, or 1.4 inches if drained on two sides.
   b - Diameter > 2.5 height of sample.
   c - Use random or standard pressure increments, keeping in mind that C_4 can be more accurately determined for larger pressure increments.
2 - Extend the slope of each e - t curve at 10^3 minutes to 10^5 minutes to obtain e_f .
3 - Plot e_f vs. equilibrium pressure on semi log paper. Compute C_3 from the slope of the curve.
4 - Measure $\frac{de}{dt}$ at 10, 100, and 1000 minutes and also pick off values of $(e - e_f)$ for these times.

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* This suggested procedure should be used, of course, only if this theory can be shown to be valid for undisturbed samples.

5 - Compute C_2 and C_4 using eq's . .

6 - Compute C_1 using eq. .

7 - Make a table similar to Table and draw a graph of e vs. $\left(\frac{1}{\frac{-de}{dt}}\right)$.

8 - Measure across and draw a e - t curve.

Conclusions

The theory explained in this paper makes use of the following non-linear description of viscous resistance,

$$\tau = C_1 \left(-\frac{de}{dt}\right)^{C_2}$$

This expression appears to be suitable for the soil tested as evidenced by the manner in which the constants are related to height of sample and consolidation pressure. C_1 shows a linear variation with pressure and is constant with respect to height. C_2 is constant with respect to both variables.

The assumption of small change in the shape of the e-z curve after the first ten minutes seems to be justified since the theoretical curve fits the experimental curve very well. Also the coefficient of permeability computed from the approximate solution checks very well with the measured coefficient.

The consolidometer height should be greater than 0.7 inches (for samples drained on one side) or errors will occur in the computation for the coefficient of permeability. The consolidometer

diameter should be greater than 2.5 times the height to minimize effect of side friction.

The procedure suggested for predicting settlements in thick clay layers is easy to use, requiring only a knowledge of algebra, and seems to give reasonable results as shown by the comparison with Terzaghi's theory.