THESIS

A LOCAL CHARACTERIZATION OF DOMINO EVACUATION-SHUFFLING

Submitted by

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ABSTRACT

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We consider linear intersection problems in the Grassmanian (the space of k-dimensional subspaces of $\mathbb{C}^n$), where the dimension of the intersection is 2. These spaces are called Schubert surfaces. We build of the previous work of Speyer [1] and Gillespie and Levinson [2]. Speyer showed there is a combinatorial interpertation for what happens to fibers of Schubert intersections above a "wall crossing", where marked points corresponding to the coordinates of partitions coincide. Building off Speyer's work, Levinson showed there is a combinatorial operation associated with the monodromy operator on Schubert curves, involving rectification, promotion, and shuffling of Littlewood-Richardson Young Tableaux, which overall is christened evacuation-shuffling. Gillespie and Levinson [2] further developed a localization of the evacuation-shuffling algorithm for Schubert curves. We fully develop a local description of the monodromy operator on certain classes of curves embedded inside Schubert surfaces [3].

ACKNOWLEDGEMENTS

I would; first and foremost, like to thank my advisor Maria Gillespie for guiding me in both my research and my writing processes. I would also like to thank Kelsey Brown, who was also invaluable in the research process. I appreciate also like to thank Dongzhou Huang and Christopher Peterson for taking time out of their busy schedules to be on my committee. Many thanks to my office mates, for keeping me sane through the at times trying writing process.

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Chapter 1

Introduction

1.1 Preliminaries

This paper pertains to points, curves, and surfaces living in an important geometric construction; the Grassmanian. The Grassmanian $\mathrm{Gr}(n,k)$ is the set of all k-dimensional sub-spaces of an n-dimensional vector space, and for the purposes of this paper the ambient space will always be $\mathbb{C}^n$. An element in the Grassmanian can be thought of as the span of a $k \times n$ matrix whose rows are linearly independent. Note that row operations on this matrix leave the element in the Grassmanian unchanged, so we think of the Grassmanian as the set of all full-rank $k \times n$ matrices in reduced row echelon form, where the pivots in higher rows are further left than pivots in lower rows. Another important concept is that of a **flag**. A full flag $F_\bullet$ in $\mathbb{C}^n$ is a set of subspaces of $\mathbb{C}^n$

$$\{0\} = F_0 \subset F_1 \cdots \subset F_n = \mathbb{C}^n$$

where $\dim(F_i) = i$.

The Grassmanian $\mathrm{Gr}(n,k)$ can be thought of as a projective variety by embedding it into $\mathbb{P}^{\binom{n}{k}-1}$ in the following manner. Given $V \in \mathrm{Gr}(n,k)$, we order the $\binom{n}{k}$ $k \times k$ submatrices of V in some order and call them $\{M_i\}_{i=1}^{\binom{n}{k}}$, then $\varphi V = (\det(M_1) : \det(M_2) : \cdots : \det M_{\binom{n}{k}}) \in \mathbb{P}^{\binom{n}{k}-1}$. φ is called the Plücker embedding, and it is well-defined since row operations either change all of M_i by a common factor , or leave them all unchanged. φ is also injective so we talk interchangably about $\mathrm{Gr}(n,k)$ and its image, and it so happpens that the image of φ is a variety defined by a set of quadratic equations known as the Plücker relations.

As a topological space, the Grassmanian $\mathrm{Gr}(n,k)$ has a cell structure indexed by partitions fitting inside a $k \times (n-k)$ box, with the cell corresponding to a partition λ having dimension $k(n-k) - |\lambda|$. These cells are called the Schubert cells of the Grassmanian. Schubert cells are usually with respect to a flag, and are denoted $\Omega^\circ_\lambda(F_\bullet)$. These cells are not closed in the Zariski topology (they are not the zero set for some collection of polynomials), but one can obtain a closed **Schubert variety** Ω_λ from Schubert cells by taking the union of all Ω_μ° for all partitions μ containing λ. For sufficiently general flags, intersections of Schubert varieties have the expected dimension; that is to say, $\bigcap_{i=1}^r \Omega_{\lambda_i}(F_{i_\bullet})$ has dimension $n(n-k) - \sum_{i=1}^n |\lambda_i|$.

1.2 Schubert Curves and Osculating Flags

The family of flags $\mathcal{F}_t$ are sufficiently general flags such that Schubert problems with respect to these flags have the expected dimension. To define the flag $\mathcal{F}_t$ for some $t \in \mathbb{P}^1$, first consider the image of the map

$$t \mapsto [1 : t : t^2 : \cdots : t^{n-1}]$$

in $\mathbb{P}^{n-1}$. This is called the **rational normal curve**. $\mathcal{F}_t$ is defined to be the maximally tangent flag to the rational normal curve at $t \in \mathbb{P}^1$. $\mathcal{F}_t$ is an example of an osculating flag, which is a flag that is maximally tangent to some curve. The i-dimensional part of the flag is spanned by the first i rows of the matrix formed by iterated derivatives, shown below.

$$\begin{bmatrix} 1 & t & t^2 & \cdots & \cdots & t^{n-1} \\ 0 & 1 & 2t & \cdots & \cdots & (n-1)t^{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & (n-1)! \end{bmatrix}$$

The case when $\sum_{i=1}^r |\lambda_i| = k(n-k)$ is well-understood. The Shapiro-Shapiro conjecture (proved by Mukhon, Tarasov and Varchenko [4]) states that if all the t_i are real the intersection

$$S = S(\lambda_1, \lambda_2, \cdots, \lambda_r) = \bigcap_{i=1}^r \Omega_{\lambda_i}(\mathcal{F}_{t_i})$$

is a reduced union of real points in the Grassmanian. When $\sum_{i=1}^{r} |\lambda_i| = k(n-k) - 1$, S is a **Schubert curve**. The topology of such curves has been explored through tableaux combinatorics by Gillespie and Levinson [2].

Definition 1.2.1. When $\sum_{i=1}^{r} |\lambda_i| = k(n-k) - 2$, S is a **Schubert surface**.

The topology and combinatorics of certain curves embedded in these surfaces will be the focus of this thesis. In the case of S being a Schubert curve, the real locus of S; denoted $\mathbb{R}(S)$, maps smoothly onto $\mathbb{P}^1$, where the preimage of 0 and ∞ are in bijection with certain chains of Littlewood-Richardson tableaux. The monodromy operator on the curve is given by certain combinatorial operations on the tableaux.

1.3 Overview of results

In the case of S being a Schubert surface, we simplify the problem to understanding the structure of certain one-dimensional curves on the two-dimensional surface. We look at two main classes of curves, those corresponding to the Littlewood-Richardson chain $\mathrm{LR}(\alpha, \begin{array}{|c|}\hline y \\ \hline x \\ \hline\end{array}, \beta, \gamma)$ and those corresponding to the Littlewood-Richardson chain $\mathrm{LR}(\alpha, \begin{array}{|c|c|}\hline x & y \\ \hline\end{array}, \beta, \gamma)$. We completely characterize a local description of the monodromy operator on $\mathrm{LR}(\alpha, \begin{array}{|c|}\hline y \\ \hline x \\ \hline\end{array}, \beta, \gamma)$ curves and on $\mathrm{LR}(\alpha, \begin{array}{|c|c|}\hline x & y \\ \hline\end{array}, \beta, \gamma)$ curves.

Chapter 2

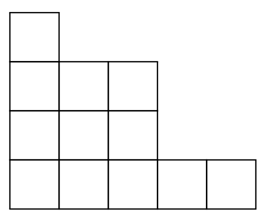
Background and Definitions

2.1 Combinatorics

2.1.1 Young Tableaux

Definition 2.1.1. A **partition** λ of n is a weakly decreasing r-tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. If λ is a partition of n we say λ has size n, or $|\lambda| = n$.

Given a partition λ , its associated **Young diagram** is a left-justified grid of boxes, where the i-th row from the bottom has λ_i boxes. As an example, $(5, 3, 3, 1)$ is a partition of 12 with associated Young diagram



Given a Young diagram Y corresponding to λ, we say Y has **shape** λ.

A Young diagram Y has **skew shape** λ/μ if μ is a partition fitting inside a partition λ, and Y is obtained from the Young diagram corresponding to λ by removing from the bottom left corner the Young diagram corresponding to μ. In this case we say λ is the **outer shape** of Y and μ is the **inner shape** of Y A Young diagram corresponding to $\lambda/\varnothing$ is commonly called a **straight shape**.

Definition 2.1.2. A **semistandard Young tableau** or SSYT is a Young diagram (either of skew shape or straight shape) along with a filling of the boxes by positive integers such that rows increase weakly left to right and columns strictly increase bottom to top.

As an example, below is a SSYT of shape $(5,3,3,1)/(3,1)$.

2				
1	2	4		
	1	2		
			1	1

The **reading word** of an SSYT T is the word formed by reading the entries from T from left to right, top to bottom.

The above SSYT has reading word $(2,1,2,4,1,2,1,1)$. The **content** of a reading word w is the tuple $(a_1, a_2, \ldots, a_j)$ where j is the maximal entry in w, and a_i is the number of i's in w. The above reading word has content $(4,3,0,1)$.

Definition 2.1.3. A word $w = (w_1, w_2, \ldots, w_n)$ is **ballot** if every sub-word $(w_1, \ldots, w_i)$ has partition content.

A word $w = (w_1, w_2, \ldots, w_n)$ is reverse-ballot if $(w_n, w_{n-1}, \ldots, w_1)$ is ballot.

Definition 2.1.4. A Tableau T is **Littlewood-Richardson** or LR if the reading word of T is reverse-ballot.

The previous SSYT is not LR, but

3				
1	2	3		
	1	2		
			1	1

is.

The **standardization** of a tableau T or word w of size (or length) n is the relabeling of the tableau (or word) with the entries 1 through n of the elements in T (or w) from least to greatest, with ties broken by reading order. The resulting tableau (or word) has content $(1,1,\ldots,1) := (1^n)$

and is called a **standard Young Tableau**, or SYT. For example, the above tableau standardizes to

7				
1	5	8		
	2	6		
			3	4

Definition 2.1.5. A **chain of Littlewood-Richardson Tableau** is a collection $\{T_i\}_{i=1}^r$ of Littlewood-Richardson Tableau where T_1 has shape β_1 and T_i has shape $\beta_i/(\text{out}(T_{i-1}))$ (Here out denotes the outer shape). Each successive T_i extends the shape of all of the previous tableau to a new straight shape.

Given a Grassmanian $\text{Gr}(n, k)$. the set consisting of all chains of Littlewood-Richardson Tableau $\{T_i\}_{i=1}^r$, where T_i has content λ_i and $\text{out}(T_r) = (n-k)^k$ is denoted $\text{LR}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. The partition $(n-k)^k$ has special importance in these Grassmanians and is denoted $\boxplus$, and called the **ambient rectangle**.

Definition 2.1.6. The **Schur Coefficient** $c^{\boxplus}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ counts the cardinality of $\text{LR}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Schur Coefficients are independent of the ordering of the partitions.

2.1.2 Tableaux Operations

In this paper, we use combinatorial operations on tableaux to understand the geometry and topology of certain curves in the Grassmanian. All of these combinatorial operations are a result of iterations of a very simple operation on tableaux called a **jeu de taquin** slide, or JDT slide. "Jeu de taquin" is the French name for the 15 puzzle game, because JDT slides are very similar to the moves you make in that game. Before we define JDT slides, we need some notation to refer to particular cells in a tableau.

An outer corner of a Young diagram of shape $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a cell in the i-th row from the bottom and the λ_i-th column from the left, given that $\lambda_i > \lambda_{i+1}$ or $i = r$. Similarly, an inner corner of a Young diagram of shape λ/μ is an outer corner of μ. The best way to think about inner and outer corners is "legal" squares you could add to the inside or remove from the outside of a Young diagram and have it remain skew shape. Below is a Skew tableau with its inner corners labelled with an X and outer corners labelled in red.

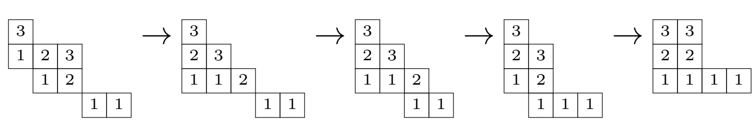
8				
4	6	7		
X	3	5		
		X	1	2

Definition 2.1.7. Given a semistandard Skew tableau T with an inner corner α, a **jeu de taquin slide** into α is detailed in the following steps: Since α is an inner corner, it is either left of or below a cell in T (or both). Switch the empty box with whichever cell will not violate semistandardness. This means if an i is above α and a j is to the right, then switch α with i if $i < j$, else switch with j. Repeat until the empty box moves into an outer corner of T. If T had shape λ/μ, a JDT slide on T has shape λ'/μ' , where $|\mu'| = |\mu| - 1$. JDT slides do not affect the content of a skew tableau.

The whole algorithm can also be reversed to increase the size of the inner and outer shapes by one, this is called a reverse JDT slide. It is a rather remarkable fact that if one performs successive JDT slides on a skew tableau T, the end result (a straight shape) is independent of the order of inner corners chosen. This end result is called the **rectification** of T. Let us rectify the tableau

3				
1	2	3		
	1	2		
			1	1

beginning with the corner in position $(2, 1)$, then the corners $(1, 3)$, then $(1, 2)$ and $(1, 1)$. Here position (i, j) denotes the i-th rowand j-th column.

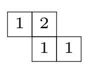
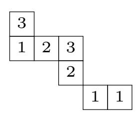


It is known that any LR tableau with content β will rectify to a Young diagram of shape β, with the i-th row consisting of all is. Also, since JDT slides preserve the property of being semistandard, the rectification of any semistandard Young tableau will again be semistandard. It is also known that a JDT slide (or reverse JDT slide) on a Littlewood-Richardson Tableau is again Littlewood-Richardson.

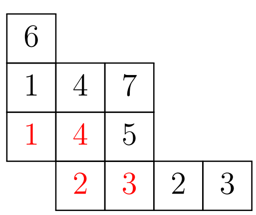
When we are studying the monodromy operator on a Shubert curve, we are looking at what is happening as we change the coordinate corresponding to our partition $\square$. This means we are moving the position of the $\square$ in $\mathrm{LR}(\lambda_1, \square, \lambda_3, \lambda_4)$ cyclically through the list. In tableaux-chain language, this means we are somehow moving one skew tableau past another. There are two ways this happens, but the first, **shuffling**, is a common combinatorial operation on skew tableaux.

Definition 2.1.8. Given two adjacent skew tableaux T and S (this just means the outer shape of T is the inner shape of S), $\mathrm{sh}(T, S)$ (the **shuffle** of T past S) outputs two adjacent skew tableaux S' and T' such that T' is now the outer tableau, the total shape of T' and S' is the same total shape as that of T and S, and the content of T' and S' match those of T and S, respectively.

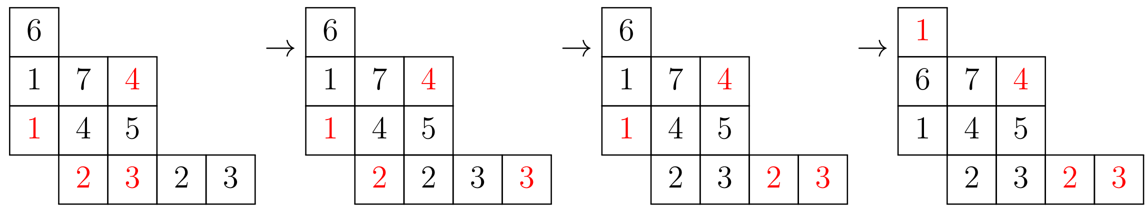
The exact mechanics of this are as follows : First, standardize both S and T. Next, perform JDT slides of S into outer corners of T, from greatest entry to least. Keep track of where the outer corners of T end up after the slides. At this stage, unstandardize the inner skew tableau to have the content of S and the outer skew tableau to have the content of T. This is $\mathrm{sh}(T, S)$.

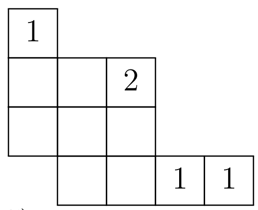
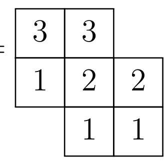
Let us compute $\mathrm{sh}(T, S)$ where $T =$  and $S =$ 

First we standardize obtaining



where the entries of T are in red, and those of S in black. The results of the successive JDT slides are shown below.



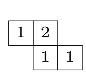
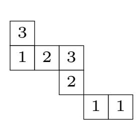
Now we unstandardize to obtain $T' =$  and $S' =$ 

If $C \in \mathrm{LR}(\lambda_1, \ldots, \lambda_r)$, then $\mathrm{sh}_i(C)$ refers to the element of $\mathrm{LR}(\lambda_1, \cdots, \lambda_{i+1}, \lambda_i, \cdots \lambda_r)$ obtained by shuffling the i-th tableau in the chain past the $i+1$-th.

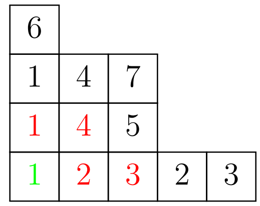
The second way to move one tableau past another that we utilize in this paper is called the **evacuation-shuffle**. While $\mathrm{sh}(\square, \lambda)$ has the interpertation of moving the coordinate for $\square$ past that of λ , $\mathrm{esh}(\square, \lambda)$ has the interpertation of moving the coordinate for $\square$ backwards through $\mathbb{P}^1$ past all the other partitions, until its coordinate is greater than λ's , but less than the next partition in the chain's coordinate. We now define the evacuation-shuffle algorithm for two general adjacent skew tableaux.

Definition 2.1.9. Given two adjacent skew tableaux T and S, with T being the inner tableau, define $\mathrm{esh}(T, S)$ (the evacuation-shuffle of T past S) as follows: Let Y be any standard Young tableau of straight shape μ, where μ is the inner shape of T. Shuffle Y past T. Then shuffle Y past S. Then shuffle T past S, shuffle T past Y, and shuffle S past Y. Finally, delete the entries of Y and destandardize the entries of T' and S'. The result is $\mathrm{esh}(T, S)$

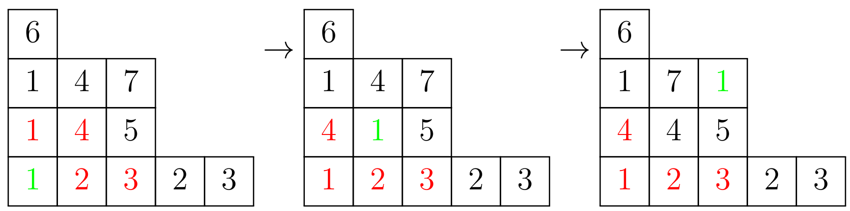
This process is considered non-local because one has to rectify and unrectify into the lower left corner, which involves many more cells than just those of T and S.

Let us compute $\mathrm{esh}(T, S)$ where $T =$  and $S =$ 

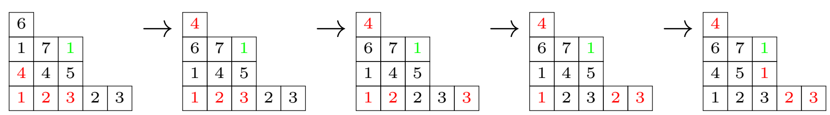
First we standardize obtaining



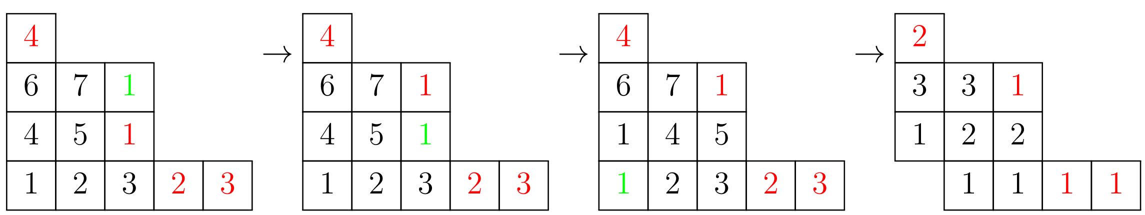
where T has red entries and the inner standard tableau; Y, has green entries. First we shuffle Y past T and S as follows.



We then shuffle T past S.



Finally, we shuffle Y back through T and S and destandardize.



Note how this process was considerably more involved than computing $\mathrm{sh}(T, S)$, and its complexity grows with the size of Y, which in this case was only one box. It is for this reason we are motivated to find a local algorithm for esh that does not involve Y. Another important operation on tableaux is crystal raising and lowering.

Definition 2.1.10. Given a word w , the **i-th crystal raising operator** F_i acts on w in the following manner: replace all instances of i with a right parenthesis and all instances of $i+1$ with a left parenthesis. Match up left and right parentheses in the usual way, and replace the rightmost unpaired right parenthesis with an $i+1$. Then revert the various parentheses back to i's and $i+1$'s. Overall, F_i turns one i into an $i+1$, or leaves the word unchanged.

The **i-th crystal lowering operator** E_i acts on w in the following manner: again replace all instances of i with a right parenthesis and all instances of $i+1$ with a left parenthesis. Match up left and right parentheses in the usual way, but now replace the leftmost unpaired left parenthesis with an i. Then revert the various parentheses back to i's and $i+1$'s. Overall, E_i turns one $i+1$ into an i , or leaves the word unchanged.

These operators can also act on a tableau simply by acting on the reading word of the tableau.

As an example, we compute $F_3 = \big(1,3,3,5,7,4,3,4\big)$. First we replace the 3s and 4s with)s and (s , respectively : $\big(1,),),5,7,\color{red}{(,)},(\big)$, where the red parentheses are paired. We turn the rightmost) into a 4 and revert all the parentheses to obtain $(1,3,4,5,7,4,3,4)$

2.2 Geometry

2.2.1 Speyer covering of $\overline{M}_{0,4}$

Let U_4 be the space consisting of all sets of 4 distinct marked points in $\mathbb{P}^1$. This is also known as $M_{0,4}$, the space of all genus 0 curves with 4 marked points. Given a Grassmanian $\mathrm{Gr}(n,k)$ and partitions $\{\lambda_i\}_{i=1}^4$ such that $\sum_{i=1}^4 |\lambda_i| = k(n-k)$, there is a natural family of curves to study called $S(\lambda_\bullet) \subset U_4 \times \mathrm{Gr}(n,k)$. $S(\lambda_\bullet)$ is defined such that its fiber over $(z_1,z_2,z_3,z_4) \in U_4$ is precisely $\bigcap_{i=1}^4 \Omega_{\lambda_i}(F_{z_i})$. It is a result of the Shapiro-Shapiro conjecture, proved by Mukhin, Tarasov, and Varchenko [4]; that each fiber over points of the real points of U_4, denoted $U(\mathbb{R})$, is a reduced union of real points. Moreover, these points are enumerated by Schur coefficients $c^{\square}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$.

Speyer extended this covering to a covering of $\overline{M}_{0,4}$, which as the compactification of $U_4 = M_{0,4}$ allows the marked points to collide. In so doing, he gave a combinatorial interpertation for what happens when one of the λ_i "slides past" an adjacent one, or formally when the relative ordering of the t_i is changed. Since Schur coefficients are unaffected by reorderings, the number of points before and after such a "wall crossing" is the same. The points in the fiber are indexed by chains of Littlewood-Richardson tableaux filling the ambient rectangle. Changing the relative order of the t_i changes the order of the Littlewood-Richardson tableaux in the chain.

2.2.2 Schubert Curves

This is where Schubert curves enter the picture. The fibers of $S(\lambda_1, \square, \lambda_3, \lambda_4)$ over points in $\overline{M}_{0,4}$ are zero-dimensional when $|\lambda_1| + |\lambda_3| + |\lambda_4| = k(n-k) - 1$. $\overline{M}_{0,4}$ is itself one-dimensional because three of the four marked points are needed to determine $\mathbb{P}^1$ up to isomorphism. This follows from the fact that any linear transformation of $\mathbb{P}^1$ is determined by where it sends any three points. Hence $S(\lambda_1, \square, \lambda_3, \lambda_4)$ is overall one-dimensional.

The space $X = \Omega_{\lambda_1}(F_{t_1}) \cap \Omega_{\lambda_3}(F_{t_3}) \cap \Omega_{\lambda_4}(F_{t_4})$ is also one-dimensional. In fact, Levinson [5] showed these spaces are isomorphic. This isomorphism; α, induces a natural map

$$\phi : \Omega_{\lambda_1}(F_{t_1}) \cap \Omega_{\lambda_3}(F_{t_3}) \cap \Omega_{\lambda_4}(F_{t_4}) \to \mathbb{P}^1$$

in the following manner: Given $x \in X$, we have $\alpha(x) \in S(\lambda_1, \square, \lambda_3, \lambda_4)$, which means $\alpha(x)$ is part of some fiber over a point $(z_1, z_2, z_3, z_4) \in \overline{M}_{0,4}$. Let $\phi(x) = z_2$. The idea is that ϕ maps points in X to the coordinate of the box partition of the corresponding point in $S(\lambda_1, \square, \lambda_3, \lambda_4)$.

2.2.3 Monodromy Operator

Levinson further noticed that the monodromy operator for moving the t_i corresponding to $\square$ around $\mathbb{P}^1$ was given by a particular combinatorial operation on the chains of tableaux. To compute the monodromy on a point corresponding to a $C \in \mathrm{LR}(\lambda_1, \square, \lambda_3, \lambda_4)$, first evacuation shuffle $\square$ past λ_3, obtaining a chain C' in $\mathrm{LR}(\lambda_1, \lambda_3, \square, \lambda_4)$, then compute $\mathrm{sh}_2(C')$. Hence, if ω is the monodromy operator, $\omega = \mathrm{sh}_2 \circ \mathrm{esh}_2$. Levinson and Gillespie [2] then proved that the monodromy operator ω for μ in $S(\lambda_1, \mu, \lambda_3, \lambda_4)$ (when $|\lambda_1| + |\mu| + |\lambda_3| + |\lambda_4| = k(n-k)$ is also given by $\omega = \mathrm{sh}_2 \circ \mathrm{esh}_2$.

2.3 Local Monodromy Operator

The Monodromy Operator can tell us many interesting properties of the real component of the curve $S(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. For example, we know the fibers over 0 are given by chains of Littlewood-Richardson skew tableaux C with content $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (in that order, from bottom left to top right). Evacuation Shuffle performed on λ_2 and λ_3 , denoted $\mathrm{esh}_2 = \mathrm{esh}(\lambda_2, \lambda_3)$ (esh_2 because it acts on the second and third positions in the chain) outputs a chain of Littlewood-Richardson skew tableaux with content $\lambda_1, \lambda_3, \lambda_2, \lambda_4$, which corresponds to the point in the fiber over ∞ that you would be at if you moved t_3 around $\mathbb{P}^1$ in the positive direction (going past ∞ and stopping between t_1 and t_2). To be technically correct, it corresponds to the continuous lift of the previous action on $\mathbb{P}^1$ to $S(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ starting at the point corresponding to your original chain of tableaux C.

In turn, performing the Shuffle operation on λ_3 and λ_2, denoted $\mathrm{sh}(\lambda_3, \lambda_2)$ (or, similarly sh_2) corresponds to the continuous lift of moving t_3 past t_2, starting with the point corresponding to the chain of tableaux $\mathrm{esh}(\lambda_2, \lambda_3)$. Iterating monodromy, which is defined to be $\mathrm{sh}_2 \circ \mathrm{esh}_2$, tells us about the number of connected components in $S(\mathbb{R})$ and about the size of those components in terms of their winding numbers when mapped to $\mathbb{P}^1$ via the natural map. This motivates us to ask about the existence of a local version of the monodromy algorithm, and indeed Levinson and Gillespie [2] defined a local algorithm when one of the partitions has shape $\square$.

Definition 2.3.1. Given a semistandard Young tableau T and an inner corner of T; $\boxed{x}$, we define local-evacuation shuffling via the following algorithm.

- **Phase 0** Let $i = 1$. Construct a skew tableau T' by adding $\boxed{x}$ to T and labelling it $\boxed{x}$.

- **Phase 1** If the $\boxed{x}$ does not precede all the i's in reading order, switch $\boxed{x}$ with the nearest i prior to $\boxed{x}$ in reading order. Increment i by 1. Repeat Phase 1.

  If $\boxed{x}$ does precede all the i's in reading order, label $\boxed{x}$ with an i and move to Phase 2.

- **Phase 2** Apply the crystal operators $F_i, F_{i+1}, \cdots, F_{|T|}$ to the reading word of the resulting skew tableau.

  Label the cell containing $|T| + 1$ as $\boxed{x}$.

The resulting skew tableau and $\boxed{x}$ is the output of local-esh. If $C \in \mathrm{LR}(\lambda_1, \square, \lambda_3, \lambda_4)$, denote by local-esh$_2(C)$ the element in $\mathrm{LR}(\lambda_1, \lambda_3, \square, \lambda_4)$ by performing local-esh on $\square$ and C_3. Gillespie and Levinson [2] showed that local-esh$_2(C) = \mathrm{esh}_2(C)$, thereby providing a local algorithm for evacuation-shuffling a single box past an arbitrary tableau.

Chapter 3

Results and Proofs

The results in this section are due to Brown, Gillespie, and McCann [3], in a yet unpublished manuscript. The main result of this paper is defining a local algorithm for evacuation-shuffling a vertical domino or horizontal domino past an arbitray tableau, and proving it agrees with the standard algorithm.

3.1 Esh of a domino past an arbitrary skew LR tableau

We now define a local esh algorithm for a domino past an arbitrary tableau, and show that it matches the output of esh.

Given a Littlewood-Richardson chain $C \in \mathrm{LR}(\alpha, \gamma, \beta, \mu)$, with γ being a partition of size two (either $\begin{array}{|c|}\hline y \\ \hline x \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline x & y \\ \hline\end{array}$), we compute $\mathrm{esh}_2(C)$ as follows. First create a single tableau T by appending to C_3 an inner corner labelled 0 in the position of the first (in reading order) entry in C_2, and let $i = 0$.

- **Phase 1** If the cell containing i precedes all the cells containing $i+1$, move to Phase 2. If not, apply E_i to T $\beta_i - 1$ times. Increment i by one and repeat.

- **Phase 2** Rename every cell in T with an entry $k \leq i-1$ with $k+1$. Then append a 0 to T in the same position of the last entry of C_2, and let $j = 0$

- **Phase 3** If the cell containing j precedes all the cells containing $j+1$, move to Phase 4. If not, apply E_j to T β_j times. Increment j by one and repeat.

- **Phase 4** Rename every cell in T with an entry $k \leq j$ with $k+1$. Apply $F_{|\beta|+1} \circ F_{|\beta|} \circ \cdots \circ F_{j+1}$ to T, then apply $F_{|\beta|} \circ F_{|\beta|-1} \circ \cdots \circ F_{i+1}$ to T. Label the cell containing an $|\beta|+2$ as y and the cell containing $|\beta|+1$ as x. The resulting chain of skew tableaux is $\mathrm{local}-\mathrm{esh}_2(C)$

While this algorithm at first glance appears not to be very similar to the local algorithm for a single box, the lowering operations are essentially the same as switching a box containing i with the immediately preceding box containing an $i+1$. The lowering operators will turn every $i+1$ into an i, except for the $i+1$ immediately preceding i in the reading word, because the $i+1$ before it will be a left parenthesis that is paired with the single right parenthesisn from the i, and no other pairs will be created because you turn $i+1$'s into i's from left to right. As an example, consider the word $(4,4,3,4,4)$. $E_3(4,4,3,4,4) = (3,4,3,4,4)$, and $E_3(3,4,3,4,4) = (3,4,3,3,4)$, and finally $E_3(3,4,3,3,4) = (3,4,3,3,3)$. The "special cell" we care about (the cell labelled i) switches location with the 4 prior to it in reading order. Adding 1 back to entries later fixes the fact that the 4's were turned into 3's. Phrasing things in terms of crystal operators and relabelling simplifies the proof.

First we prove the algorithms agree when $\alpha = \varnothing$.

Lemma 3.1.1. *The algorithms agree when $\alpha = \varnothing$, for γ a vertical domino and β a partition of n.*

Proof. In this case, we have the domino $\begin{array}{|c|}\hline y\\\hline x\\\hline\end{array}$ in the inner corner, extended by a Littlewood-Richardson tableau T of content β. Since T rectifies to the unique Littlewood-Richardson tableau T_0 of content (and shape) β, T can be obtained by starting with T_0, adding x and y as a two outer corners extending it (necessarily in different rows), and doing outer slides into the x and then y boxes.

An outer slide into the outer corner marked by x in row a starting from T_0 will consist of all a's moving to the right and then all entries below a in the first column moving upwards. Since y is in a higher row than x, say b, the outer slide into y will then also move all b's to the right and then all entries in the first column upwards. This forms T, and hence T has at most two entries a, b not appearing in the first column, and the remaining entries to the right of the first column are labeled by their row index.

Because shuffling is known to an involution, shuffling the domino $\begin{array}{|c|}\hline y\\\hline x\\\hline\end{array}$ past T gives us T_0, with y in the outer corner in row b and x in the outer corner in row a. This is $\mathrm{esh}(\gamma, \beta)$ since the inner shape of $\begin{array}{|c|}\hline y\\\hline x\\\hline\end{array}$ is $\varnothing$. Let a be the smallest number that does not appear above y in the first column, and b the larger number.

Now, starting from $(\begin{smallmatrix} \boxed{y} \\ \boxed{x} \end{smallmatrix}, T)$, we apply the local algorithm for comparison. Phase 1 at step i turns all the i's into $i-1$ except the one prior to the i in the reading word, then does so for $i = 2, \cdots, a-1$ as well. Phase 1 does not terminate for $i < a-1$ since there is an $i+1 < a$ in the first column above i, which precedes it. Phase 1 terminates at step $i = a-1$ since the $a-1$ is in the $a+1$-th row and hence precedes all the a's, which are all in row a. The result is an $a-1$ in the first column and the $a+1$-th row, and an i in the $i+2$-th row and first column for all $j < a-1$ After adding 1 to entries less than or equal to $a-1$ the result is therefore Little-wood-Richardson again, with an entry a in the first column and $a+1$-th row, and entries $i+1$ in the $i+2$-th row and first column for $i < a-1$. Move to Phase 2. We switch x past $1, 2, \cdots, a, a+1, \ldots, b-1$ and replace x by $b-1$, at the begining of the b-th row. We move to Phase 3 here since all the b's are in row b and the $b-1$ precedes them. Once we add one to entries the result is again Littlewood-Richardson, since now the entries in the i-th row are all i, for every i. We are left with a larger Littlewood-Richardson tableau of shape and content $\beta + e_a + e_b$ (Where e_α is the vector or 0's with a 1 in position α). Now, Phase 3 applies $F_b, F_{b+1}, \ldots, F_{n+1}$, which changes the last entry of row b to $b+1, b+2, \ldots$ and finally to $n+2$ (and changes no other entries) and then replaces that box with y. Phase 4 applies $F_a, F_{a+1}, \ldots, F_n$, which changes the last entry of row a to $a+1, a+2, \ldots, n+1$ and finally changes the $n+1$ to x. Hence we are left with T_0 along with y at the end of row b and x at the end of row a, as desired. □

The same is true of a horizontal domino $\boxed{x}\boxed{y}$.

Lemma 3.1.2. *The algorithms agree when $\alpha = \varnothing$, for γ a horizontal domino and β a partition of n.*

Proof. In this case, we have the domino $\boxed{x}\boxed{y}$ in the inner corner, extended by a Littlewood-Richardson tableau T of content β. Since T rectifies to the unique Littlewood-Richardson tableau T_0 of content (and shape) β, T can be obtained by starting with T_0, adding x and y as two successive outer corners extending it (not necessarily in different rows), and doing outer slides into the x and then y boxes.

- **Case 1:** Suppose x and y are in different rows a and b. An outer slide into the outer corner marked by x in row a starting from T_0 will consist of all a's moving to the right and then all entries below a in the first column moving upwards, with x ending in the bottom left corner. Since y is in a lower row than x (as x precedes y in the reading word), the outer slide into y will then also move all b's in the b-th row to the right, which since the b in the first column was moved up one, results in the y in the second column with a b both below and to the left of it. Hence after this point y slides down, moving every entry $i \leq b$ in the second column up one. This forms T, and hence T has one entry a not appearing in the first column, and one entry b not appearing in the second column. The remaining entries to the right of the second column are labeled by their row index.

  Because shuffling is known to an involution, shuffling the domino $\boxed{x \mid y}$ past T gives us T_0, with x in the outer corner in row a and y in the outer corner in row b. This is $\mathrm{esh}(\gamma, \beta)$ since the inner shape of $\boxed{x \mid y}$ is $\varnothing$. Let a be the number that does not appear above x in the first column, and let b the number missing from the second column.

  Now, starting from $(\boxed{x \mid y}, T)$, we apply the local algorithm for comparison. Phase 1 at step j turns all the j's into $j-1$ except the one prior to the j in the reading word, then does so for $j = 2, \cdots, a-1$ as well. The result is an $a-1$ in the first column and the a-th row, and an i in the $i+1$-th row and first column for all $b \leq i < a-1$, and a j in the $j+1$-th row and second column for $j < b$. After adding 1 to entries less than or equal to $a-1$ the result is therefore Little-wood-Richardson. Move to Phase 2. We switch x past $1, 2, \cdots, b-1$ and replace x by $b-1$, at the beginning of the b-th row. Once we add one to entries the result is again Littlewood-Richardson. We are left with a larger Littlewood-Richardson tableau of shape and content $\beta + e_a + e_b$. Now, Phase 3 applies $F_b, F_{b+1}, \ldots, F_{n+1}$, which changes the last entry of row b to $b+1, b+2, \ldots$ and finally to $n+2$ (and changes no other entries) and then replaces that box with y. Phase 4 applies $F_a, F_{a+1}, \ldots, F_n$, which changes the last entry of row a to $a+1, a+2, \ldots, n+1$ and finally changes the $n+1$ to x. Hence we are left with T_0 along with y at the end of row b and x at the end of row a, as desired.

- **Case 2**: Suppose x and y are in the same row a. Then when we unrectify T_0, our first slide is necessarily into the outer corner labelled x. All of the a's slide over, and all entries $i \leq a-1$ in the first row slide up one. Then, an outer slide into the corner labelled y again will cause all the a's to move to the right, but when y is in the second column, there will be an $a-1$ both to the left of and below it. Hence y will move down, moving the $a-1$ up. The end result is all entries $i \leq a-1$ in the second column will move up one as well.This forms T, and hence T has one entry a not appearing in neither the first column nor the second column. The remaining entries to the right of the second column are labeled by their row index.

  Because shuffling is known to an involution, shuffling the domino $\boxed{x \mid y}$ past T gives us T_0, with x in the outer corner in row a and y to the right of x. This is $\mathrm{esh}(\gamma, \beta)$ since the inner shape of $\boxed{x \mid y}$ is $\varnothing$. Let a be the number that does not appear above x in the first column and y in the second column.

  Now, starting from $(\boxed{x \mid y}, T)$, we apply the local algorithm for comparison. Phase 1(a) at step j turns all the j's into $j-1$ except the one prior to the j in the reading word, and does so for $j = 1, 2, \cdots, a-1$. The result is an $a-1$ in the second column and the a-th row, and an i in the $i+1$-th row and second column for all $i < a-1$. After adding 1 to entries less than or equal to $a-1$ the result is therefore Little-wood-Richardson. Move to Phase 2. We switch x past $1, 2, \cdots, a-1$ and replace x by $a-1$, at the beginning of the a-th row. After again adding 1 to entries $i \leq a-1$, we are left with the Littlewodd-Richarson tableau of straight shape and content $\beta + 2e_a$. Again Phases 3 and 4 change the tableau into T_0 with an x and y in the a-th row, and the relative order of x and y is preserved. Hence we are left with T_0 along with x at the end of row a and y to the right of x.

□

Though the last two results were stated as lemmas, they are actually the bulk of the proof. It is a known fact that that crystal operators commute with JDT slides, and hence with rectifying and unrectifying. The process of relabelling all entries below$i$ also commutes with JDT slides, since JDT slides depend only on the relative order of cells, and this relabelling does not change the relative order of cells, since the i changed to an $i+1$ precedes every $i+1$ in reading order, so in the reading order it is still the lesser entry. This leads to the main result of this paper.

Theorem 3.1.3. *Local-esh*(β, T) *agrees with* $\text{esh}(\beta, T)$

Proof. Rectify β and T. Do $\text{local} - \text{esh}$. Because of the lemmas this agrees with doing esh, which is just shuffling since the inner shape is empty. Unrectify. Because every step of $\text{local} - \text{esh}$ commutes with JDT slides, this is the same as doing $\text{local} - \text{esh}(\beta, T)$. But also, this agrees with computing $\text{esh}(\beta, T)$ because rectifying, shuffling, and unrectifying is exactly the esh algorithm. Hence $\text{local} - \text{esh}$ and esh agree.

□

3.2 Example

Let us compute $\text{local} - \text{esh}(T, S)$, with T having content $(1, 1)$ (i.e. being a vertical domino) and $T =$

y	
	x

and $S =$

3				
1	2	3		
		2		
			1	1

where $T \cup x = S =$

3				
1	2	3		
	y	2		
		x	1	1

We replace y with 0

3				
1	2	3		
	0	2		
		x	1	1

and begin in Phase 1 with $i = 0$. The 0 does not precede all the 1's, so we apply $(E_0)^2$, which turns all the 1's except the 1 in the third row into a 0. We obtain

3				
1	2	3		
	0	2		
		x	0	0

and we repeat Phase 1 with $i = 1$. The 1 precedes all the 2's, so we move to Phase 2, renaming the 0's to 1's and the 1's to 2's. Then we add a 0 in the position of the x in T. The result is

3				
2	2	3		
	1	2		
		0	1	1

Now we begin Phase 3 with $j = 0$. The 0 does not precede all the 1's, so we perform $(E_0)^3$ and repeat Phase 1 with $j = 1$ on

3				
2	2	3		
	1	2		
		0	0	0

Again, the 1 does not precede the all the 2's, so we preform $(E_1)^2$ to obtain

3				
1	2	3		
	1	1		
		0	0	0

And finally we do E_2 since the 2 does not precede every 3, and have

3				
1	2	2		
	1	1		
		0	0	0

Then, since $j = 3$ and there is no 4's , the 3 definitionally precedes them so we move to Phase 4 by adding 1 to every 0, 1, 2 , and 3.

4				
2	3	3		
	2	2		
		1	1	1

Then, since $i = 1$, $j = 3$, and the size of S is 7, we perform $F_8 \circ F_7 \circ \cdots F_4$ and $F_7 \circ F_6 \circ \cdots F_2$. F_4 turns the 4 into a 5, then the successive raising operators turn that cell into a 9. Then F_2 turns the first (in reading order) 2 into a 3, and succesive raising operators turn the last 3 into an 8. Then we replace 9 with y and 8 with 8 to obtain

$\mathrm{local} - \mathrm{esh}(T, S) =$

y				
3	3	x		
	2	2		
		1	1	1

Let us compare with the general algorithm for $\mathrm{esh}(T, S)$ to see that they agree. First, we create a tableau T' by taking the union of S and T and Y, a standard Young tableau of straight shape the inner shape of T.

$T' =$

3				
1	2	3		
3	y	2		
1	2	x	1	1

First we shuffle Y past T obtaining

3				
1	2	3		
y	3	2		
x	1	2	1	1

Then we shuffle Y past S:

3				
1	3	3		
y	2	2		
x	1	1	1	2

We shuffle T past S:

y				
3	3	3		
2	2	y		
1	1	1	1	2

Now we shuffle Y back through S and T obtaining

y				
3	3	x		
3	2	2		
1	2	1	1	1

which after deleting Y matches the local algorithm, as desired.

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