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DISSERTATION

THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF A NON-SYMMETRIC CANTOR FUNCTION

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In partial fulfillment of the requirements of
for the Degree of Doctor of Philosophy
Colorado State University
Fort Collins, Colorado
Spring 1999

UMI Number: 9941550

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February 5, 1999

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ABSTRACT

THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF A NON-SYMMETRIC CANTOR FUNCTION

Each choice of numbers a and c in the segment $(0, \frac{1}{2})$ produces a Cantor set C_{ac} by recursively removing segments from the interior of the interval $[0, 1]$ so that intervals of relative length a and c remain on the left and right sides of the removed segment, respectively. A Cantor function Φ_{ac} is obtained from C_{ac} in much the same way that the standard Cantor function, Φ, is obtained from the Cantor ternary set. When $a = c = \frac{1}{3}$, C_{ac} is the Cantor ternary set, C, and Φ_{ac} is the standard Cantor function, Φ. The derivative of Φ is zero off C, and the upper derivative is infinite on C; the set $N = \{x \in C \mid \text{the lower derivative of } \Phi \text{ is finite}\}$ has Hausdorff dimension $[\ln 2/\ln 3]^2$. In this paper, similar results are established for N^*, the nondifferentiability set of Φ_{ac}. The Hausdorff dimension of N^* is the maximum of the real numbers satisfying the following equations: $x(\ln(1/c))^2 = \ln((a+c)/c)\ln((a/c)^x + 1)$, and $x(\ln(1/a))^2 = \ln((a+c)/a)\ln((c/a)^x + 1)$.

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Contents

1 Introduction

The goal of this thesis is to compute the Hausdorff dimension of the nondifferentiability set of a non-symmetric Cantor function. This Cantor function is obtained from a non-symmetric Cantor set in much the same way that the standard Cantor function is obtained from the standard Cantor ternary set. In 1993, Richard Darst computed the Hausdorff dimension of the nondifferentiability set of the standard Cantor function, and the methods used in this thesis are essentially generalizations of those used by Darst.

One purpose for our goal in this thesis is to get a feel for the size of the relevant nondifferentiability set within the set of all real numbers. We will therefore introduce the topic of Lebesgue measure, which is essentially a generalization of the concept of length to more complicated subsets of the real numbers. The Lebesgue measure of a disjoint collection of intervals, for example, is simply the sum of the lengths of all the intervals. By generalizing the method of adding up the lengths of intervals contained in a set, Lebesgue measure provides a way to compare the sizes of various subsets of the real numbers.

One shortcoming of ordinary Lebesgue measure, however, is its inability to distinguish various sets of Lebesgue measure zero. A set of Lebesgue measure zero is one which contains no intervals and therefore has zero length in the generalized sense. A variety of interesting subsets of the real numbers have this property, including the set of all rational numbers and the standard Cantor ternary set. Though these sets have the same Lebesgue measure, they are indeed very different sets. One big difference, for example, is the fact that the set of all rational numbers is countable and the Cantor ternary set is uncountable. If we limited our measurement of size to Lebesgue measure, we would run into similar problems, for we shall see that all of the

Cantor sets of interest in this thesis have measure zero, as do any of their subsets. We therefore need another way to compare their sizes.

One such way to successfully compare sets of measure zero is to compare their Hausdorff dimensions. Hausdorff dimension is a concept that generalizes the more intuitive concept of dimension which we apply to simple geometrical shapes. For example, we think of a line as having dimension one, a square as having dimension two, and a solid cube as having dimension three. While one might think that sets of Lebesgue measure zero will always have dimension zero, we will see that this is often not the case. Many fractal sets such as the standard Cantor ternary set, for example, have positive Hausdorff dimension. Certain proper subsets, such as the nondifferentiability set of the associated Cantor function, will have positive Hausdorff dimension strictly less than the fractal sets that contain them. This is the method we will use to measure and compare sizes of the sets of interest.

This thesis will be organized into four sections. In the first section following the introduction, we begin by presenting some of the basic definitions and theorems from measure theory, which lead to the intuitive concept of Lebesgue measure described above. We will then introduce the topic of Hausdorff measure, which leads to the precise definition of the Hausdorff dimension of a given set. To conclude the section, we will present some tools which allow us to more easily compute the Hausdorff dimension of sets, and we will finish with some simple examples.

In the second and third sections, we first define the non-symmetric Cantor function of interest and discuss some of its important properties. The main property of concern is differentiability of the function, so we will state and prove a theorem which characterizes the points of differentiability of the Cantor function. This theorem will give us enough insight into the structure of the nondifferentiability set of the function to arrive at a candidate for its Hausdorff dimension, and we will conclude the third section with the main result of the thesis: a verification that this candidate really is the correct Hausdorff dimension of the nondifferentiability set. This nondifferentiability set will be referred to as N^* throughout the thesis.

The final section of the paper will be a brief summary of the results presented. It will also include a few ideas for further research.

2 Preliminaries

To begin our discussion, assume that we are working in some underlying set X. For subsets A and B of X, we will use $A \backslash B$ to denote the set of all elements that are contained in A but not in B, and A^c to denote the set $X \backslash A$, the **complement** of X in A. We will use $P(A)$ to denote the **power set** of A, that is, the collection of all subsets of A. The following definition describes a special subcollection of sets contained in $P(X)$. The following is a summary of the material found in [4].

2.1 Algebras and Sigma Algebras

Definition 2.1 *Let $\mathcal{M}$ be a subset of the collection $P(X)$. We say that $\mathcal{M}$ is an* **algebra** *of subsets of X if the following three conditions hold:*

1. *The empty set is contained in $\mathcal{M}$.*
2. *If A is contained in $\mathcal{M}$, then A^c is also contained in $\mathcal{M}$.*
3. *If A and B are contained in $\mathcal{M}$, then $A \cup B$ is contained in $\mathcal{M}$.*

Note that by induction, property two in the above definition implies that any finite union of sets in the collection $\mathcal{M}$ will also be in $\mathcal{M}$. Also, by DeMorgan's Law, $A \cap B = X \backslash (A^c \cup B^c)$, so finite intersections of sets in $\mathcal{M}$ are also in $\mathcal{M}$. In fact, any finite number of unions, intersections, and complementation of sets in $\mathcal{M}$ will always produce a set in $\mathcal{M}$.

We say that an algebra $\mathcal{M}$ of subsets of X is a **sigma algebra** if it has the following additional property: If $A_1, A_2, A_3, \ldots$ is a countable collection of sets contained in $\mathcal{M}$, then the union $\cup_{i\geq 1} A_i$ is also contained in $\mathcal{M}$. Using the same logic as above, we see that countable intersections of sets in a sigma algebra are also contained in the sigma algebra.

Example. The collection $\{\emptyset, X\}$ is a sigma algebra of subsets of X. The collection $P(X)$ is also sigma algebra of subsets of X. ■

Example. Let $\mathcal{M}$ be the collection of all subsets of $\mathbb{N}$ (the set of nonnegative integers) that are either finite or have finite complements. Then $\mathcal{M}$ is an algebra of subsets of $\mathbb{R}$ but not a sigma algebra of subsets of $\mathbb{R}$, since the even integers can be obtained as a countable union of singleton sets of integers. On the other hand, the collection $\mathcal{N}$ of subsets of $\mathbb{R}$ that are either countable or have countable complements is a sigma algebra of subsets of $\mathbb{R}$ and therefore an algebra. (Here, countable means either finite or countably infinite.) ■

Example. The collection of all open subsets of $\mathbb{R}$ is neither a sigma algebra nor an algebra of subsets of $\mathbb{R}$ since complements of open sets are not always open. ■

While the above example shows that not every collection of subsets forms an algebra or a sigma algebra, every collection of subsets can be extended to give an algebra or a sigma algebra. Specifically, consider any collection $\mathcal{M}$ of subsets of X. Define $\mathcal{N}$ to be the intersection of all sigma algebras containing the collection $\mathcal{M}$. (Note that $\mathcal{N}$ is nonempty since $P(X)$ is a sigma algebra that clearly contains $\mathcal{M}$.) It is easy to show that $\mathcal{N}$ is itself a sigma algebra, and it is clear that $\mathcal{N}$ is therefore the smallest sigma algebra containing $\mathcal{M}$. We will refer to $\mathcal{N}$ as the **sigma algebra generated by $\mathcal{M}$**.

Example. The sigma algebra generated by the open subsets of $\mathbb{R}$ is called the **Borel sigma algebra**. The sets contained in this sigma algebra are called **Borel sets**. ■

2.2 Measure

In this section, we seek a way to assign a "size" to a subset of $\mathbb{R}^n$. We seek a function that takes subsets of $\mathbb{R}^n$ and somehow assigns nonnegative real numbers (or possibly positive infinity) to them according to how "big" they are. There are some obvious properties that such a function should have. For example, if A is a subset of B, then B should be assigned at least as big a number as A. It also seems logical that the empty set should be assigned a size of zero. Finally, the size of the union of a collection of sets should be no bigger than the sums of all their sizes individually. All of these ideas are summarized in the following definition.

Definition 2.2 *Let X be a set, and let μ be a function defined on all subsets of X that takes values in $[0, \infty]$. We say that μ is an* **outer measure** *on X if the following properties hold.*

1. $\mu(\emptyset) = 0$.
2. *If A and B are subsets of X with $A \subset B$, then $\mu(A) \leq \mu(B)$.*
3. *For every countable collection $\{A_i\}_{i\geq 1}$ of subsets of X, we have*

$$\mu\left(\bigcup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1} \mu(A_i).$$

The second and the third properties listed in the above definition are called **monotonicity** and **countable subadditivity**, respectively.

The most intuitive example of an outer measure is given by **Lebesgue outer measure** on $\mathbb{R}$. There are several ways to construct this outer measure, but we will focus on the method that is discussed in [4]. The reader is referred there for justifications of the facts contained in the following discussion.

Let λ denote Lebesgue outer measure on $\mathbb{R}$, the function to be constructed. We first define $\lambda([a, b]) = b - a$ for any closed interval $[a, b]$ so that the outer measure of an interval is simply its length. We then extend the definition of λ in a natural way so that it can be used to compute the measure of unions of closed intervals, and then of any open set in general. Finally, for any subset A of $\mathbb{R}$, we define the Lebesgue outer measure of A as follows:

$$\lambda(A) = \inf\{\lambda(G) \mid A \subset G,\ G \text{ an open set}\}$$

We note that Lebesgue outer measure on $\mathbb{R}^n$ for $n \geq 2$ is obtained in the same way, except that we start by defining the outer measure of an n-cell as the product of the lengths of its sides, and then proceed from there. We also note that Lebesgue outer measure on $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are simply extensions of the concepts of length, area, and volume, respectively. For example, the outer measures of familiar objects in the plane like triangles and circles will be their areas, and the outer measure of a solid sphere in $\mathbb{R}^3$ will be its volume.

One other intuitive property that we would like Lebesgue outer measure (and indeed any outer measure) to have is that of additivity when considering a collection of disjoint sets. In other words, if $\{A_1, A_2, A_3, \dots\}$ is a collection of *disjoint* subsets of $\mathbb{R}^n$, it seems logical to assume that

$$\lambda\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} \lambda(A_i)$$

holds. Unfortunately, the above relation does *not* hold for an arbitrary collection of disjoint subsets of $\mathbb{R}^n$. It does, however, hold for a class of subsets called the

Lebesgue measurable subsets of $\mathbb{R}^n$. This collection of subsets can be shown to be a sigma algebra of subsets of $\mathbb{R}^n$ that contains the Borel sets, and we say that a set is **Lebesgue measurable** if it belongs to this class of sets. The Lebesgue measurable subsets are characterized by the following Theorem (called Carathéodory's Theorem), whose proof can be found in [4].

Theorem 2.3 *Let A be a subset of $\mathbb{R}^n$, and let λ denote Lebesgue outer measure on $\mathbb{R}^n$. Then A is Lebesgue measurable if and only if, for every subset E of $\mathbb{R}^n$, we have*

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap A^c).$$

Though outer measures will be sufficient for the purposes of this paper, we will now give a formal definition of the concept of measure for the sake of completeness.

Definition 2.4 *Let X be a set, and let μ be a function defined on a sigma algebra $\mathcal{M}$ of subsets of X and taking values in $[0, \infty]$. We say that μ is a* **measure** *on X if the following properties hold.*

1. $\mu(\emptyset) = 0$.
2. *If A and B are contained in $\mathcal{M}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.*
3. *For every countable collection $\{A_i\}_{i\geq 1}$ of disjoint sets contained in $\mathcal{M}$, we have*

$$\mu\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} \mu(A_i).$$

Note that the above properties are nearly identical to those of an outer measure, except for the third property. This strengthened property is called **countable additivity**. It is also worth mentioning that for *any* outer measure μ on a set X, we can obtain a measure by restricting μ to a sigma algebra of subsets of X analagous to the Lebesgue measurable subsets above. This class of sets is called the μ-measurable subsets of X. (See [3] for details.)

To conclude this section, we recall that one shortcoming of Lebesgue measure is its failure to distinguish the sizes of various sets of measure zero. For example, consider the standard Cantor ternary set C obtained by starting with the unit interval $[0, 1]$ and recursively removing middle thirds of the resulting intervals. This set is uncountable and can be shown to have Lebesgue measure zero. But the countable set $Q \cap [0, 1]$ of all rational numbers between zero and one can also be shown to have Lebesgue measure zero. Thus, despite the fact that C is clearly "bigger" than $Q \cap [0, 1]$ in the sense of cardinality, Lebesgue measure has failed to distinguish the two sets. How can we measure two such sets in a way that distinguishes them? The key lies in choosing the right measure, which is discussed in the following subsection.

2.3 Hausdorff Measure and Dimension

The concept of dimension is very common in the everyday world of $\mathbb{R}^3$. We think of simple sets of points, such as a line, a plane, and a solid cube as being one, two, and three dimensional, respectively. If we were asked to describe the difference between these three sets of points, we might even appeal to the concept of dimension in our answer. As we will see, the concept of dimension can be used to distinguish a vast class of sets and to compare their sizes. To begin our discussion, we will need a variety of definitions and theorems. We note that all information in this subsection is taken from [3], and the reader is referred there for proofs of many of the theorems.

Definition 2.5 *Let E be a nonempty subset of $\mathbb{R}^n$. The* **diameter** *of E, denoted $|E|$, is defined by*

$$|E| = \sup\{\|x - y\| : x, y \in E\},$$

where $\|\cdot\|$ *denotes the standard Euclidean norm in* $\mathbb{R}^n$. *By convention, we define* $|\emptyset| = 0$.

Definition 2.6 *Let* E *be a subset of* $\mathbb{R}^n$, *let* δ *be a positive real number, and let* $\{A_i\}_{i\geq 1}$ *be an at most countable collection of subsets of* $\mathbb{R}^n$. *We say that the collection* $\{A_i\}_{i\geq 1}$ *is a* **δ-cover** *of* E *if*

$$E \subset \bigcup_{i\geq 1} A_i \quad \text{and} \quad |A_i| \leq \delta \text{ for all } i \geq 1.$$

The key to finding the dimension of a set E contained in $\mathbb{R}^n$ is examining the values of so-called "Hausdorff sums" associated with various δ-covers of E, and then determining what happens as δ approaches zero. Specifically, given a positive real number s and a δ-cover $\{A_i\}_{i\geq 1}$ of E, the associated Hausdorff sum has the form

$$\sum_{i\geq 1} |A_i|^s.$$

The infimum over all such Hausdorff sums is an important quantity, and is denoted

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i\geq 1} |A_i|^s,$$

where the infimum is taken over all δ-covers $\{A_i\}_{i\geq 1}$ of E. Note that $\mathcal{H}^s_\delta$ can be viewed as a function which assigns a number in the set $[0, \infty]$ to every subset of $\mathbb{R}^n$. As the next lemma shows, $\mathcal{H}^s_\delta$ is in fact an outer measure on $\mathbb{R}^n$.

Lemma 2.7 *Fix any two positive real numbers* s *and* δ. *Then* $\mathcal{H}^s_\delta$ *is an outer measure on* $\mathbb{R}^n$.

Proof: We must show that $\mathcal{H}^s_\delta(\emptyset) = 0$, and that monotonicity and countable subadditivity hold. As a δ-cover of the empty set, we can simply take the empty set, whose diameter is zero, and the result is immediate. The monotonicity condition is equally trivial since if $A \subset B$, then every δ-cover of B is also a δ-cover of A. Thus, all that remains is to verify that countable subadditivity holds.

Take any countable collection $\{A_1, A_2, \dots\}$ of subsets of $\mathbb{R}^n$. We may assume that $\mathcal{H}^s_\delta(A_i)$ is finite for all i, for otherwise the sum $\sum \mathcal{H}^s_\delta(A_i)$ would be infinite and the subadditivity condition would clearly be satisfied. Let $\epsilon > 0$ be given. Then by the definition of $\mathcal{H}^s_\delta$, we can find, for each i, a δ-cover $\{U_{ij}\}_{j\geq 1}$ of A_i such that

$$\sum_{j\geq 1} |U_{ij}|^s < \mathcal{H}^s_\delta(A_i) + \frac{\epsilon}{2^i}.$$

Therefore, the collection $\{U_{ij}\}_{i,j\geq 1}$ is a δ-cover of the union of all the A_is, so we obtain

$$\mathcal{H}^s_\delta\left(\bigcup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1}\sum_{j\geq 1} |U_{ij}|^s < \epsilon + \sum_{i\geq 1} \mathcal{H}^s_\delta(A_i).$$

Since $\epsilon > 0$ is arbitrary, the lemma is proved. ■

Now, consider the effect of letting δ approach zero in the quantity $\mathcal{H}^s_\delta(E)$. For a fixed set E, the quantity $\mathcal{H}^s_\delta(E)$ increases monotonically as δ approaches zero, so a limiting value exists in the nonnegative extended real numbers and is denoted by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E).$$

Note that $\mathcal{H}^s(E)$ is defined for every set E contained in $\mathbb{R}^n$, so $\mathcal{H}^s$ can also be viewed as a function which takes any subset of $\mathbb{R}^n$ as an argument and assigns it a number in the interval $[0, \infty]$. We will refer to the function $\mathcal{H}^s$ as **s-dimensional Hausdorff outer measure** on $\mathbb{R}^n$, as is suggested by the following theorem. We note that from here until the end of the section, all sets are assumed to be subsets of $\mathbb{R}^n$.

Theorem 2.8 *The function $\mathcal{H}^s$ described above is an outer measure on $\mathbb{R}^n$.*

Proof: Again, we must verify the three defining properties for an outer measure. First, since the collection which contains only the empty set is a δ-cover of the empty set for all $\delta > 0$, it is clear that $\mathcal{H}^s(\emptyset) = 0$. The monotonicity condition is also easily verified since if $A \subset B$, we have $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s_\delta(B)$ by Lemma 2.7, and letting δ approach zero yields the desired inequality. Thus, we need only verify countable subadditivity.

As before, take any countable collection $\{A_1, A_2, \dots\}$ of subsets of $\mathbb{R}^n$, and consider any positive real number δ. By Lemma 2.7, we have

$$\mathcal{H}^s_\delta\left(\bigcup_{i\geq 1} A_i\right) \leq \sum_{i\geq 1} \mathcal{H}^s_\delta(A_i) \leq \sum_{i\geq 1} \mathcal{H}^s(A_i).$$

But the above relation holds for all $\delta > 0$, so letting δ approach zero yields the subadditivity condition. ■

We will now describe some critical properties of $s-$dimensional Hausdorff outer measure that lead to the definition of dimension, or more specifically, Hausdorff dimension. For any fixed set E, we can view $\mathcal{H}^s(E)$ as a function of the positive variable s. It is easy to show that this function is nonincreasing as the value of s increases, but a much stronger deduction can be made about $\mathcal{H}^s(E)$. It turns out that $\mathcal{H}^s(E)$ assumes either the value zero or the value infinity for all positive real numbers, except for at most one real number. This special real number is intimately related to the set E and will constitute our definition of the dimension of E. The preceding discussion is summarized in Theorem 2.10, but first we need a lemma.

Lemma 2.9 *Let n be a positive integer, and let d be any real number strictly greater than n. Then $\mathcal{H}^d(\mathbb{R}^n) = 0$.*

Proof: Let $I_n = [0, 1] \times [0, 1] \times \cdots \times [0, 1]$ denote the standard unit cube in $\mathbb{R}^n$. For each positive integer m, let $\mathcal{N}_m$ denote the regular cover of I_n consisting of m^n closed cubes each having side length $1/m$. Thus, two cubes in $\mathcal{N}_m$ will intersect at most on their boundaries, and each cube will have diameter $\sqrt{n}/m$.

Next, fix any positive real number δ, and choose m large enough so that the diameter of each cube in the collection $\mathcal{N}_m$ is less than δ. Then we have

$$\mathcal{H}^d_\delta(I_n) \leq \sum_{A \in \mathcal{N}_m} |A|^d = n^{d/2} m^{n-d}.$$

Since m approaches infinity as δ approaches zero, we see that the above inequality implies that $\mathcal{H}^d(I_n) = 0$. But all of $\mathbb{R}^n$ can be expressed as a countable union of closed unit cubes, all of whose d-dimensional outer measures are zero by an argument analagous to that above. Therefore, it follows from countable subadditivity that $\mathcal{H}^d(\mathbb{R}^n) = 0$. ∎

Theorem 2.10 *Fix any subset E of $\mathbb{R}^n$. Then there exists a unique nonnegative real number γ such that both of the following conditions hold.*

1. $\mathcal{H}^s(E) = \infty$ *for all s satisfying* $0 < s < \gamma$.

2. $\mathcal{H}^s(E) = 0$ *for all s satisfying* $s > \gamma$.

This number γ is called the **Hausdorff dimension** *of the set E. We write*

$$\gamma = \dim_H(E).$$

Proof: Define $X = \{s > 0 \mid \mathcal{H}^s(E) = 0\}$. Our first observation is that by the above Lemma, X is nonempty and bounded below and therefore has an infimum, call it γ. If $\gamma = 0$, then $\mathcal{H}^s(E) = 0$ for all positive real numbers s, and γ is clearly the only number which satisfies both conditions one and two in the theorem. Assume, on the other hand, that $\gamma > 0$ holds. Clearly, for any real number s strictly greater than gamma, we have $\mathcal{H}^s(E) = 0$, for otherwise, γ would not be the infimum of the set X. Thus, condition two is verified.

Choose any positive real number s satisfying $s < \gamma$, and note that we can then choose a real number t such that $s < t < \gamma$. Our next claim is that for sufficiently small δ, we have $\mathcal{H}^t_\delta(E) > 0$. Indeed, if this were not the case, it would follow that $\mathcal{H}^t(E) = 0$, contradicting the fact that $\gamma = \inf X$. Fix a positive real number $\delta < 1$ which is small enough so that $\mathcal{H}^t_\delta(E) > 0$ holds, and let $\{U_i\}_{i\geq 1}$ be any δ-cover of E. Then we have

$$\sum_{i\geq 1} |U_i|^s = \sum_{i\geq 1} |U_i|^{s-t}|U_i|^t \geq \delta^{s-t} \sum_{i\geq 1} |U_i|^t \geq \delta^{s-t}\mathcal{H}^t_\delta(E).$$

Since the above holds for all δ-covers of E, we take the infimum over all such covers to obtain $\mathcal{H}^s_\delta(E) \geq \delta^{s-t}\mathcal{H}^t_\delta(E)$. Letting δ approach zero therefore shows that $\mathcal{H}^s_\delta(E) = \infty$, which verifies condition one.

Since γ satisfies both conditions one and two, it is immediate that there cannot exist another real number which satisfies both conditions simultaneously. ■

The above theorem shows the existence of a number which we have chosen to call the dimension of a set, but we have yet to decide whether this number really does allow us to compare the sizes of two different sets. In other words, if we have a set A that is contained in another set B, we would expect the dimension of A to be less than or equal to that of B. The following theorem confirms that Hausdorff dimension does indeed have this property.

Theorem 2.11 *Suppose that A is a subset of B. Then* $\dim_H(A) \leq \dim_H(B)$.

Proof: Let γ be the Hausdorff dimension of the set B, and choose any real number d strictly greater than γ. Then by the definition of Hausdorff dimension, we have $\mathcal{H}^d(B) = 0$. But A is a subset of B, so by the monotonicity property of outer measures, we have

$$\mathcal{H}^d(A) \leq \mathcal{H}^d(B) = 0.$$

We claim that this implies that $\dim_H(A) \leq d$, for if we had $\dim_H(A) > d$, the definition of Hausdorff dimension would imply that $\mathcal{H}^d(A) = \infty$, a contradiction.

Thus, since $d > \gamma$ is arbitrary, it follows that $\dim_H(A) \leq \gamma = \dim_H(B)$. ■

As it turns out, the Hausdorff dimension of simple subsets of $\mathbb{R}^3$, such as points, lines, planes, and solid cubes, coincides with our intuitive definition of dimension. But Hausdorff dimension is much more general in the sense that it can be applied to *any* subset of $\mathbb{R}^n$. In particular, we shall see that sets can indeed have non-integer Hausdorff dimensions.

2.4 Computational Tools of Hausdorff Dimension

In this subsection, we explore techniques for computing the Hausdorff dimension of some familiar sets. To begin, suppose that we have some set E contained in $\mathbb{R}^n$ and a candidate value γ for the Hausdorff dimension of E. The usual procedure is to show that γ is both an upper bound and a lower bound on the Hausdorff dimension of E. In other words, we would first show that $\dim_H(E) \leq \gamma$ and then show that $\dim_H(E) \geq \gamma$. In most cases, proving that a number is an upper bound is much easier than proving that it is a lower bound, as is illustrated by the following example and discussion.

Example. Consider the unit interval $I = [0, 1]$. According to our intuition, this set is one-dimensional, so we take one as our candidate for its Hausdorff dimension. To proceed, let $\mathcal{M}_n$ be the following collection of intervals: $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, 1]$. Thus, for each positive integer n, $\mathcal{M}_n$ is a cover of I having n intervals of equal length $\delta_n = n^{-1}$.

Choose any real number $d > 1$. We will now form the Hausdorff sum associated with the real number d and the cover $\mathcal{M}_n$. Since $\mathcal{M}_n$ is a δ_n-cover of I, the infimum over all such covers is bounded above by the previously mentioned Hausdorff sum, so we obtain

$$\mathcal{H}^d_{\delta_n}(I) \leq \sum_{A\in\mathcal{M}_n} |A|^d = n^{1-d}.$$

Letting n approach infinity in the above inequality, we see that $\mathcal{H}^d(I) = 0$. It follows that the Hausdorff dimension of I is bounded above by d, and since $d > 1$ is arbitrary, it is bounded above by one. ■

In the previous example, we showed that $\dim_H(I) \leq 1$, but we have yet to show equality. The computation above was aided by the fact that to show $\mathcal{H}^d_\delta(I) \leq M$ for some real number M, we need only find *one specific δ-cover* of I whose associated Hausdorff sum is less than or equal to M. To get a lower bound, we need to somehow turn the inequality around, which requires us to show that *every δ-cover* of I has its associated Hausdorff sum greater than or equal to M. This can be extremely difficult when using the direct approach, so we will appeal to some theorems.

First, let us deal with the difficulty of having to prove a statement about *every* cover of the set in question. Is there a way we could narrow the class of sets we need to consider in computing Hausdorff sums? The next theorem addresses this question.

Theorem 2.12 *Let A be a subset of $\mathbb{R}^n$. In computing the Hausdorff outer measure of A, it is sufficient to consider covers consisting only of*

1. *closed subsets of $\mathbb{R}^n$, or*
2. *open subsets of $\mathbb{R}^n$.*

Proof: The proof of part one is trivial, since the diameter of any set is the same as the diameter of its closure. Thus, in any Hausdorff sum, we can replace each set in the cover with its closure and not change the value of the sum.

To prove part two, fix two positive real numbers s and δ, let $\epsilon > 0$ be given, and take any δ-cover $U_1, U_2, \dots$ of A. For each i, define

$$W_i = \bigcup_{x \in U_i} B(x, \delta_i),$$

where δ_i is a positive real number chosen smaller than δ so that $(|U_i| + 2\delta_i)^s < |U_i|^s + \epsilon/2^i$, and $B(x, \delta_i)$ is the open ball in $\mathbb{R}^n$ of radius δ_i and centered at x. Clearly, W_i is an open set containing U_i with diameter no bigger than $|U_i| + 2\delta_i$. Thus, since δ_i was chosen less than δ, the collection $\{W_1, W_2, \dots\}$ is a 2δ-cover of A consisting of only open subsets of $\mathbb{R}^n$, and we have

$$\sum_{i \geq 1} |W_i|^s \leq \sum_{i \geq 1} (|U_i|^s + \epsilon/2^i) \leq \epsilon + \sum_{i \geq 1} |U_i|^s.$$

Now, since $\epsilon > 0$ is arbitrary, and since $\{U_i\}_{i \geq 1}$ is an arbitrary δ-cover of A, the above inequality yields

$$\sum_{i \geq 1} |W_i|^s \leq \mathcal{H}^s_\delta(A).$$

Let $\mathcal{G}^s_\delta$ and $\mathcal{G}^s$ represent the analogues of $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ which result from using only open subsets of $\mathbb{R}^n$ in covers. Then taking the infimum of the left hand side of the above inequality yields

$$\mathcal{G}^s_{2\delta}(A) \leq \mathcal{H}^s_\delta(A).$$

Letting δ approach zero yields $\mathcal{G}^s(A) \leq \mathcal{H}^s(A)$.

To finish the proof, we must show that $\mathcal{H}^s(A) \leq \mathcal{G}^s(A)$. But this is trivial since every cover of A using open sets only is also a cover of A with no restrictions. This completes the proof. ■

A second tool for simplifying lower bound calculations provides a way of relating Hausdorff outer measure to another known, "easier" outer measure. The idea is contained in the following Theorem.

Theorem 2.13 *Let A be a subset of $\mathbb{R}^n$, let μ be an outer measure on $\mathbb{R}^n$ such that $\mu(A) > 0$, and let s be a positive real number. If there exist positive real numbers c and ϵ such that*

$$\mu(U) \leq c|U|^s$$

for all open (or all closed) sets U with $|U| \leq \epsilon$, then the Hausdorff dimension of A is bounded above by s.

Proof: Let $\{U_i\}_{i\geq 1}$ be any δ-cover of A using all open (or all closed) sets in which $\delta \leq \epsilon$. Then by the above inequality we have

$$0 < \mu(A) \leq \mu\left(\bigcup_{i\geq 1} U_i\right) \leq \sum_{i\geq 1} \mu(U_i) \leq c\sum_{i\geq 1} |U_i|^s.$$

Thus, taking the infimum over all such δ-covers, using Theorem 2.12, and letting δ approach zero, we obtain

$$\mathcal{H}^s(A) \geq c^{-1}\mu(A) > 0.$$

It follows that for any real number t less than s, we have $\mathcal{H}^t(A) = \infty$, which implies that the Hausdorff dimension of A is bounded above by t. Since $t > s$ was arbitrary, the theorem follows. ■

2.5 Some Examples

In this subsection, we will use the computational tools from the preceding example to help us compute the Hausdorff dimension of some familiar sets. We will begin by showing that the Hausdorff dimension of the unit interval is bounded below by one.

Example. Let I denote the unit interval $[0, 1]$ in $\mathbb{R}$. In Example 2.4, we showed that $\dim_H([0, 1]) \leq 1$. We will now show that $\dim_H([0, 1]) \geq 1$, which will complete the proof that the unit interval has Hausdorff dimension one.

Let λ denote Lebesgue outer measure on $\mathbb{R}$. Let U be any subset of $\mathbb{R}$ that has diameter less than or equal to one, and choose a positive integer k such that $2^{-(k+1)} < |U| \leq 2^{-k}$. (We will use Theorem 2.13.) In the same notation as in Example 2.4, consider the cover $\mathcal{M}_{2^k}$ of I, which consists of 2^k closed intervals of length 2^{-k}. Then the set U can intersect at most three of the intervals in the collection $\mathcal{M}_{2^k}$, so it follows that

$$\lambda(U) \leq 3 \cdot 2^{-k} = 6 \cdot 2^{-(k+1)} < 6|U|.$$

Since the above equation holds for any subset U, Theorem 2.13 implies that $\dim_H(I) \geq 1$. ∎

Example. Let A be any countable subset of $\mathbb{R}^n$. Then we can enumerate the points of A in a (perhaps finite) sequence $\{a_i\}_{i\geq 1}$. Now, consider the cover $\mathcal{M}$ of A which consists of all the singleton sets $\{a_1\}, \{a_2\}, \ldots$. Since every set in $\mathcal{M}$ has diameter zero, $\mathcal{M}$ is a δ-cover of A for any positive real number δ. Thus, for any positive real numbers s and δ, we have $\mathcal{H}^s_\delta(A) = 0$, and it follows that the Hausdorff dimension of A is zero. ∎

As our next example, we will compute the Hausdorff dimension of the non-symmetric Cantor set mentioned in the introduction; we begin by constructing this set. Choose any positive real numbers a and c satisfying $0 < a, c < \frac{1}{2}$, and set $b = 1 - (a + c)$. The Cantor set in question will be denoted by C_{ac} and will be described as the intersection of a recursively defined sequence of sets $C^0_{ac}, C^1_{ac}, C^2_{ac}, \ldots$, where C^0_{ac} is the entire closed interval $[0, 1]$. In general, C^k_{ac} will consist of the union of 2^k disjoint closed intervals, which we will call **stage k black intervals.** For $k \geq 1$, we obtain C^k_{ac} by taking the "left ath" and "right cth" of each stage $k-1$ black interval. Specifically, if $[x, x+\delta]$ is a stage $k-1$ black interval, we remove the segment $(x+a\delta, x+(1-c)\delta)$ to obtain two corresponding stage k black intervals. We will refer to the closure of such a removed segment as a **stage k complementary interval.** Thus, we have defined

$$C_{ac} = \bigcap_{n=1}^{\infty} C_{ac}^{n}.$$

To begin our dimension calculation, note that

$$\mathcal{H}^d(C_{ac}) = \sup_{\delta>0} \inf \sum_{i\geq 1} |U_i|^d,$$

where the infimum is taken over all δ-covers $\{U_i\}_{i\geq 1}$ of C_{ac}. We know that a number γ is the Hausdorff dimension of C_{ac} if it is the unique nonnegative number such that $\mathcal{H}^d(C_{ac}) = 0$ for all $d > \gamma$ and $\mathcal{H}^d(C_{ac}) = \infty$ for all $d < \gamma$. Thus, the Hausdorff dimension is the "cut-off" value of d that is neither too large nor too small. To find this special number, we will consider, for each j, a cover $\{U_{ij}\}_{i\geq 1}$ of C_{ac}, and we will insist that the diameters of these sets shrink to zero as j approaches infinity. A natural choice in the case of our Cantor set is to let $\{U_{ij}\}_{i\geq 1}$ be the collection of the 2^j distinct black intervals at stage j in the construction of C_{ac}; we will denote this collection of stage j black intervals by $\mathcal{M}_j$.

Now, let us examine the Hausdorff sum associated with the cover $\mathcal{M}_j$. There are 2^j total intervals in the collection: one interval of length a^j, j distinct intervals of length $a^{j-1}c$, $C(j,2)$ (j choose 2) distinct intervals of length $a^{j-2}c^2, \ldots$, and $C(j,j) = 1$ interval of length c^j. By the Binomial Theorem, the combined length of these intervals is $(a+c)^j$, and the associated Hausdorff sum is given by

$$\sum_{I\in\mathcal{M}_j} |I|^d = \sum_{i=0}^{j} C(j,i)(a^{j-i}c^i)^d = (a^d + c^d)^j.$$

Thus, the "cutoff" value for d appears to be the number γ that satisfies $a^\gamma + c^\gamma = 1$ because any strictly larger value of d produces Hausdorff sums tending to zero as j approaches infinity, and any strictly smaller value of d produces Hausdorff sums tending to infinity as j approaches infinity. We therefore choose the number γ satisfying

$a^\gamma + c^\gamma = 1$ as our candidate for the Hausdorff dimension of C_{ac}, and we note that its existence and uniqueness follow from the Intermediate Value Theorem and the Mean Value Theorem applied to the function $f(x) = a^x + c^x$.

Theorem 2.14 *Fix a and c in the interval* $(0, \frac{1}{2})$, *and let* γ *be the unique real number satisfying* $a^\gamma + c^\gamma = 1$. *Then the Hausdorff dimension of* C_{ac} *is bounded above by* γ.

Proof: To prove this result, we will restrict our attention to the collection $\mathcal{M}_j$ of stage j black intervals mentioned above. Setting $\delta_j = \max\{a^j, c^j\}$, we see that $\mathcal{M}_j$ is a δ_j-cover of C_{ac} for each positive integer j, and we have

$$\mathcal{H}^d_{\delta_j}(C_{ac}) \leq \sum_{I \in \mathcal{M}_j} |I|^d = (a^d + c^d)^j.$$

Thus, if d is any real number greater than γ, we see that $\mathcal{H}^d(C_{ac}) = 0$ by letting j approach infinity in the above relation. ∎

To show that the Hausdorff dimension of C_{ac} is also bounded below by γ requires more work since we have to make a statement about *every* cover of C_{ac} instead of specific covers like $\mathcal{M}_j$. Our first goal, therefore, is to narrow the class of covers that we need to examine. The following Lemma gives a relationship between arbitrary covers and covers which involve only black intervals in the construction of C_{ac}.

Lemma 2.15 *Let* C_{ac} *denote the Cantor set constructed above, and let* $\mathcal{T}$ *be the collection of all black intervals in the construction of* C_{ac}, *together with the empty set. Define, for each subset* A *of* C_{ac} *and each* $z > 0$,

$$\mu^z_\delta(A) = \inf \sum_{i \geq 1} |U_i|^z,$$

where the infimum is taken over all at most countable δ*-covers* $\{U_i\}_{i \geq 1}$ *of* A, *where* $U_i \in \mathcal{T}$ *for all* i. *Similarly, define*

$$\mu^z(A) = \lim_{\delta \to 0^+} \mu^z_\delta(A).$$

Then μ^z is an outer measure on the collection of subsets of C_{ac}. Furthermore, there exists a constant $R > 0$ (depending on z) such that

$$\mathcal{H}^z \geq R\mu^z. \tag{2.16}$$

Proof: Routine checks show that μ^z is an outer measure on the collection of subsets of C_{ac}. Now, choose a positive integer N large enough so that

$$L := \max\{a^N, c^N\} < b,$$

and let $R = 2^{-N}$. (Recall that $b = 1 - (a + c)$ is the relative length of the removed segments at each stage in the construction of C_{ac}.) Next, take any $A \subset C_{ac}$, choose $\delta > 0$, and consider any δ-cover $\{U_i\}$ of A, where $U_i \subset C_{ac}$ for all i. (We note here that in computing the Hausdorff dimension of A, it is sufficient to use only δ-covers involving subsets of C_{ac}.) Next, we note that we may assume that A has uncountably many elements, for if A were at most countable, it would follow that $\mu^z(A) = \mathcal{H}^z(A) = 0$, so that any choice of R would satisfy (2.16). We may also assume that U_i contains at least two points for each i, since otherwise we would have $|U_i| = 0$, so the set would have no affect on the associated sums.

As a result of our assumptions, we see that for each i, there exists a largest nonnegative integer $k(i)$ such that U_i is a subset of one of the $2^{k(i)}$ black intervals at the $k(i)$th stage in the construction of C_{ac}; suppose that this particular black interval has length L_i. Since $k(i)$ is maximal, U_i cannot be contained in either of the two stage $k(i) + 1$ black intervals which are subsets of the interval of length L_i, so it follows that $|U_i| \geq bL_i$, and we have

$$LL_i < bL_i \leq |U_i|.$$

Now, consider the collection of stage $k(i) + N$ black intervals which are subsets of the original interval of length L_i. Clearly, there are 2^N intervals in this collection, each having length less than or equal to LL_i. Let $\{S_{ij}\}_{j=1}^{2^N}$ denote this covering collection, and note that for each i, the associated collection forms a δ-cover of U_i. It follows that $\{S_{ij}\}_{i,j}$ is a δ-cover of $\mathcal{A}$, and we have

$$\mu_\delta^z(\mathcal{A}) \le \sum_{i,j} |S_{ij}|^z = \sum_{i\ge 1} \sum_{j=1}^{2^N} |S_{ij}|^z \le 2^N \sum_{i\ge 1} |U_i|^z.$$

Finally, if we take the infimum over all δ-covers $\{U_i\}$ and let $\delta \to 0^+$ in the above equation, we obtain (2.16). ■

Note that the above Lemma implies that if we can show that $\mu^d(C_{ac}) > 0$, then $\mathcal{H}^d(C_{ac}) > 0$ so that d is a lower bound on the Hausdorff dimension of C_{ac}. It is easier to show that $\mu^d(C_{ac}) > 0$ since we only need to consider black intervals instead of arbitrary sets in our covers. However, we must still consider the possibility that a cover of C_{ac} contains black intervals from a variety of different stages in the construction of C_{ac}. The next Lemma shows that any cover involving only black intervals can be refined to a cover involving black intervals at the same stage.

Lemma 2.17 *Let j and k be positive integers, let L be any black interval at stage j in the construction of C_{ac}, and let $\mathcal{A}$ denote the collection of stage $j+k$ black intervals contained in L. Then if γ is the unique real number such that $a^\gamma + c^\gamma = 1$, we have*

$$|L|^\gamma = \sum_{I\in\mathcal{A}} |I|^\gamma.$$

Proof: First, we note that the collection $\mathcal{A}$ contains 2^k distinct black intervals. More specifically, if i is any integer chosen so that $0 \le i \le k$, there are $C(k,i)$ distinct black intervals, each of length $|L|a^i c^{k-i}$, which are contained in $\mathcal{A}$. (As before, $C(k,i)$ denotes the binomial coefficient "k choose i.") Thus, we have

$$\sum_{I\in\mathcal{A}} |I|^\gamma = |L|^\gamma \sum_{i=0}^{k} C(k,i) a^{\gamma i} c^{\gamma(k-i)} = |L|^\gamma (a^\gamma + c^\gamma)^k = |L|^\gamma,$$

which completes the proof. ■

We will now use the two lemmas above to show that the Hausdorff dimension of C_{ac} is bounded below by the number γ.

Theorem 2.18 *Fix a and c in the interval* $(0, \frac{1}{2})$, *and let* γ *be the unique real number satisfying* $a^\gamma + c^\gamma = 1$. *Then the Hausdorff dimension of* C_{ac} *is bounded below by* γ.

Proof: To begin, let $\{I_j\}_{j\geq 1}$ be any countable cover of C_{ac} consisting only of black intervals in the construction of C_{ac}. By the geometry of these black intervals, we can expand each of our black intervals I_j slightly to obtain a corresponding open interval $G_j = I_j \cup K_j$ of diameter less than or equal to two, where $K_j \cap C_{ac} = \emptyset$. Thus, the collection $\{G_j\}_{j\geq 1}$ is an open cover of the compact set C_{ac}, so we can extract (by relabeling if necessary) a finite subcover $\{G_1, G_2, G_3, \ldots, G_n\}$. But for each j, we have $G_j \cap C_{ac} \subset I_j$, so it follows that the finite collection $\mathcal{N} = \{I_1, I_2, I_3, \ldots, I_n\}$ of black intervals is also a cover of C_{ac}.

Next, note that each of the black intervals I_j in the collection $\mathcal{M}$ occurs at some stage, say k_j, in the construction of C_{ac}. Let m be the maximum of all the k_js, and let $\mathcal{M}_j$ denote the collection of all stage m black intervals contained in I_j for each $j = 1, 2, \ldots, n$. Clearly, the collection $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_n$ is a cover of C_{ac}, and we claim that $\mathcal{M}$ is precisely the collection of the 2^m distinct stage m black intervals. Indeed, if this were not the case, there would be at least one stage m black interval I that was not a subset of a black interval in the collection $\mathcal{M}$, so all the points in $C_{ac} \cap I$ would not be contained in any of the intervals in $\mathcal{M}$, contradicting the fact that $\mathcal{M}$ is a cover of C_{ac}. Fixing $\delta = 2$ and applying the previous lemma to I_j for each $j = 1, 2, \ldots, n$, we obtain

$$\sum_{j\geq 1} |I_j|^\gamma \geq \sum_{j=1}^{n} |I_j|^\gamma = \sum_{j=1}^{n} \sum_{I\in\mathcal{M}_j} |I|^\gamma \geq (a^\gamma + c^\gamma)^m = 1.$$

Recalling from Lemma 2.15 the definition of the outer measure μ, we now take the infimum over all δ-covers of C_{ac} involving only black intervals to obtain $\mu^\gamma(C_{ac}) \geq \mu_\delta^\gamma(C_{ac}) \geq 1$. Therefore, by Lemma 2.15, there exists a real number R such that $\mathcal{H}^\gamma(C_{ac}) > R > 0$. so that $\dim_H(C_{ac}) \geq \gamma$. The Theorem follows. ∎

As it turns out, the Hausdorff dimension of simple subsets of $\mathbb{R}^3$, such as points, lines, planes, and solid cubes, coincides with our intuitive definition of dimension. But Hausdorff dimension is much more general in the sense that it can be applied to *any* subset of $\mathbb{R}^n$. Sets can even have non-integer dimensions, as the set C_{ac} demonstrates.

In the next section, we continue our work with the set C_{ac} by defining its associated Cantor function. Our ultimate goal will be to find the Hausdorff dimension of the set of points at which this function is not differentiable.

3 The Non-symmetric Cantor Function

In this section, we will construct a Cantor function Φ_{ac} based on the Cantor set C_{ac} which was described in the preceding section. We will then discuss some of its main properties, particularly those related to its differentiability. As we will see, the nondifferentiability set of Φ_{ac} is a subset of the underlying Cantor set C_{ac}, so determining the differentiability of Φ_{ac} at any particular number x depends on locating its position using black intervals in the construction of C_{ac}. We therefore begin by describing a system of **locator sequences** analagous to the ternary expansions of numbers used to locate points in the Cantor ternary set. To simplify the notation, we will use Φ to denote the Cantor function Φ_{ac} from this point until the end of the thesis.

First, all stage k black intervals will be represented by finite sequences of zeroes and twos of length k, where zero means "left" and two means "right." More precisely, let I represent any stage k black interval. Then the first digit in the locator sequence of I is zero if I is a subset of the left-hand stage one black interval and two if I is a subset of the right-hand stage one black interval. Assuming that the first j digits of the locator sequence have been determined, where $j < k$, there will exist one and only one stage j black interval of which I is a subset; call it I_j. Note that there are exactly two stage $(j+1)$ black intervals contained in I_j; call the one on the left I_{j1} and the one on the right I_{j2}. Then the $(j+1)$st digit in the locator sequence of I is zero if I is a subset of I_{j1} and two if I is a subset of I_{j2}. It is convenient to also represent complementary intervals by locator sequences. The locator sequence for the complementary interval between I_{j1} and I_{j2} will be identical to that of I in the first $k-1$ digits, with a one in the kth position.

It is also convenient to represent specific points in the Cantor set C_{ac} using locator sequences. Each point x in C_{ac} can be uniquely represented by an infinite sequence of zeroes and twos extending naturally from the sequences corresponding to black intervals. Specifically, for any positive integer k, x is an element of exactly one stage k black interval. Thus, we may define the locator sequence of x so that the first k digits of the sequence agree with the locator sequence of the appropriate stage k black interval. We then identify x with its locator sequence as follows:

$$x = (x(1), x(2), x(3), \dots)$$

Next, we will define the Cantor function Φ associated with the Cantor set C_{ac}. We will first define Φ on the complementary set K, which will denote the union of all of the complementary intervals in the construction of C_{ac}. For any $x \in K$, let $\mathcal{A}_{x,n}$ denote the collection of stage n black intervals "to the left" of x. Specifically, $I \in \mathcal{A}_{x,n}$ if and only if all points in I are less than or equal to x. Note that for each x, there is a smallest nonnegative integer n such that $\mathcal{A}_{x,n}$ is nonempty; call this integer $n(x)$.

Now, define

$$\Phi(x) = \sum_{I \in \mathcal{A}_{x,n}} \frac{|I|}{(a+c)^n}, \qquad \text{for all } x \in K',$$

where n is any integer greater than or equal to $n(x)$. A simple induction argument shows that Φ is well-defined, and it is clear from its definition that Φ is increasing on the complementary set K'.

Lemma 3.1 *The function Φ is uniformly continuous on the complementary set K.*

Proof: Let $\epsilon > 0$ be given. Choose a positive integer N large enough so that

$$\max\left\{\left(\frac{a}{a+c}\right)^N, \left(\frac{c}{a+c}\right)^N\right\} < \epsilon,$$

choose $\delta = \min\{a^N, c^N\}$, and take any $x, y \in K$ satisfying $|x - y| < \delta$. Also suppose without loss of generality that $x \leq y$. Now, consider the collection of all endpoints of stage N complementary intervals I, such that each point of I is greater than or equal to y. Out of this finite list of endpoints, let y' denote the minimum. Similarly, let x' be the maximum of all the stage N complementary interval endpoints less than or equal to x. Then, since δ was chosen to be the minimum length of all stage N black intervals, it follows that there can be at most one black interval between x' and y'. Thus, since the maximum length of this interval is $\max\{a^N, c^N\}$, we have

$$\begin{aligned}
\Phi(y) - \Phi(x) &\leq \Phi(y') - \Phi(x') \\
&\leq \max\left\{\left(\frac{a}{a+c}\right)^N, \left[\frac{c}{a+c}\right]^N\right\} \\
&< \epsilon,
\end{aligned}$$

which completes the proof. ■

Combining the above lemma with Problem 7 on page 88 of [4], it is easy to see that there is a unique continuous extension of Φ (which we will also call Φ) to the entire unit interval $[0,1]$. It is easy to show that this continuous extension is also increasing, and that $\Phi(0) = 0$ and $\Phi(1) = 1$.

Next, we turn our attention to the nondifferentiability set of the function Φ, which we will denote by N^*. We recall that a function is differentiable at a point if and only if its right and left derivatives exist and are equal at that point. In addition, each one-sided derivative exists if and only if the corresponding upper and lower derivates are equal. We will therefore describe the set N^* by comparing upper and lower derivates from the right and from the left at various points in $[0,1]$. Upper derivates of our function will be denoted by Φ^+ and Φ^- from the right and from the left, respectively; and lower derivates will be denoted by Φ_+ and Φ_-.

Our first observation concerns points contained in complementary intervals in the construction of C_{ac}. Clearly, N^* is contained in the Cantor set C_{ac} since Φ is constant and therefore differentiable on the open set $[0,1] \setminus C_{ac}$. At the endpoints of complementary intervals, however, the derivatives from the right and from the left must be examined separately. As the proof of the next theorem demonstrates, the upper left and upper right derivates at endpoints of complementary intervals are never equal.

Theorem 3.2 *Let t be the endpoint of a complementary interval in C_{ac}. Then Φ is not differentiable at t.*

Proof: Since t is the endpoint of a complementary interval, that means that there exists a positive integer n_0 such that for all $n \geq n_0$, there is a unique stage n black interval I_n of which t is an endpoint. For each $n \geq n_0$, let $t(n)$ denote the other endpoint of the interval I_n. Then by the definition of Φ, we have

$$\frac{\Phi(t(n)) - \Phi(t)}{t(n) - t} = \frac{|I_n|}{|I_n|(a+c)^n} = \frac{1}{(a+c)^n}.$$

Thus, if t is the right endpoint of a complementary interval, we see that $\Phi^+(t) = \infty$, and since it is obvious that $\Phi^-(t) = 0$, Φ is not differentiable at t. Similarly, if t is the left endpoint of a complementary interval, we have $\Phi^-(t) = \infty \neq 0 = \Phi^+(t)$, so again Φ is not differentiable at t. ■

To continue our characterization of the points in N^*, we must consider points in C_{ac} that are not endpoints of complementary intervals. Let N^+ denote the set of all nonendpoints in C_{ac} such that Φ is not differentiable from the right, and let N^- denote the set of all nonendpoints in C_{ac} such that Φ is not differentiable from the left. Clearly N^+ and N^- are contained in N^*, but it is still unclear if there are nonendpoints in N^* that don't belong to either set. The next Theorem helps us answer this question.

Theorem 3.3 *For any $x \in C_{ac}$ that is not the endpoint of a complementary interval, we have $\Phi^+(x) = \Phi^-(x) = \infty$.*

Proof: Consider any nonendpoint $x \in C_{ac}$, with locator sequence

$$x = (x(1), x(2), x(3), \ldots).$$

For each positive integer n let $z(n)$ denote the position of the nth zero in the locator sequence of x. Also note that the length of the stage $z(n)$ black interval containing x is $a^n c^{z(n)-n}$.

Next, define a sequence x_n of complementary interval endpoints decreasing to x as follows:

$$x_n = (x(1), x(2), \ldots, x(z(n)-1), 0, 2, 2, 2, \ldots) \qquad \text{for } n = 1, 2, \ldots.$$

Note that there is one stage $z(n+1)$ black interval of length $a^n c^{z(n+1)-n}$ between x_{n+1}

and x_n. Thus, we have

$$\begin{aligned}\Phi(x_n) - \Phi(x) &\geq \Phi(x_n) - \Phi(x_{n+1}) \\ &= \frac{a^n c^{z(n+1)-n}}{(a+c)^{z(n+1)}}.\end{aligned}$$

We also note that x and x_n share the same black interval until stage $z(n+1) - 1$, so

$$x_n - x \leq a^n c^{z(n+1)-(n+1)}.$$

Combining these observations, we see that

$$\begin{aligned}\frac{\Phi(x_n) - \Phi(x)}{x_n - x} &\geq \frac{a^n c^{z(n+1)-n}}{a^n c^{z(n+1)-(n+1)}(a+c)^{z(n+1)}} \\ &= \frac{c}{(a+c)^{z(n+1)}},\end{aligned}$$

which approaches infinity as n approaches infinity. It follows that $\Phi^+(x) = \infty$.

To show that $\Phi^-(x) = \infty$, we let $t(n)$ denote the position of the nth two in the locator sequence of x. The proof follows by reversing the roles of a and c in the above argument and replacing every $z(n)$ with $t(n)$. ∎

As a result of the above theorem, the upper derivates of any nonendpoint in C_{ac} are equal. Thus, if t is a nonendpoint in C_{ac}, then t is contained in N^* if and only if either $\Phi_+(t)$ is finite or $\Phi_-(t)$ is finite. In other words, t is contained in N^* if and only if t is contained in either N^+ or N^-. We therefore arrive at the following decomposition of the nondifferentiability set N^*.

$$N^* = N^+ \cup N^- \cup \{\text{complementary interval endpoints in } C_{ac}\} \tag{3.4}$$

In the next section, we will compute the Hausdorff dimension of the set N^* by making use of the above decomposition. We will see that the countable set of complementary interval endpoints can be more or less ignored, so our efforts will focus on the sets N^+ and N^-. To obtain the appropriate coverings of these sets, our main tool will be the following theorem, which characterizes the points of N^+ in terms of their locator sequences.

Theorem 3.5 *Let Φ denote the Cantor function associated with the Cantor set C_{ac}. Let N^+ denote the set of nonendpoints in C_{ac} at which Φ is not differentiable from the right, and set $r = \ln((a+c)/c)/\ln(1/c)$. Then*

1. *If $t \in N^+$, then $\limsup_{n\to\infty} \dfrac{z(n+1)}{z(n)} \geq r^{-1}$.*
2. *If $\limsup_{n\to\infty} \dfrac{z(n+1)}{z(n)} > r^{-1}$, then $t \in N^+$,*

where $z(n)$ denotes the position of the nth zero in the locator sequence of t.

Proof: 1. Take any nonendpoint $t \in C_{ac}$, and suppose that

$$\limsup_{n\to\infty} \frac{z(n+1)}{z(n)} < r^{-1}. \tag{3.6}$$

We will show that $\Phi_+'(t) = \infty$. Now by (3.6), there exists a real number $q > 0$ and a positive integer m_0 such that

$$r^{-1} - \frac{z(n+1)}{z(n)} \geq q > 0 \qquad \text{for all } n \geq m_0. \tag{3.7}$$

So, consider any positive integer $n \geq m_0$, and choose a real number u_n satisfying

$$0 < u_n < \text{ distance}(t, K_n),$$

where K_n is the complement in $[0,1]$ of the $z(n)$th stage in the construction of C_{ac}.

Next, take any point $x \in (t, t+u_n)$. Then the locator sequences for x and t will agree out to the $z(n)$th positions, and there exists a positive integer $n_0 > n$ such that $z(n_0)$ is the first position in which the sequences disagree. So we have the situation described below:

$$t = (t(1), t(2), \ldots, t(z(n_0) - 1), 0, ?, ?, \ldots)$$

$$x = (t(1), t(2), \ldots, t(z(n_0) - 1), 2, ?, ?, \ldots)$$

Thus, since x and t share the same stage $z(n_0) - 1$ black interval, we have

$$x - t \leq a^{n_0 - 1} c^{z(n_0) - n_0}. \tag{3.8}$$

Now, choose points $x' \leq x$ and $t' \geq t$ which have the following locator sequences:

$$\begin{aligned} t' &= (t(1), t(2), \ldots, t(z(n_0) - 1), 2, 0, 0, \ldots) \\ x' &= (t(1), t(2), \ldots, \ldots, t(z(n_0 + 1) - 1), 0, 2, 2, \ldots) \end{aligned}$$

Clearly, $t' < x'$, and at stage $z(n_0 + 1)$, there is one black interval of length $a^{n_0} c^{z(n_0+1) - n_0}$ between t' and x'. Thus, we have

$$\begin{aligned} \Phi(x) - \Phi(t) &\geq \Phi(x') - \Phi(t') \\ &= \frac{a^{n_0} c^{z(n_0+1) - n_0}}{(a+c)^{z(n_0+1)}}. \end{aligned} \tag{3.9}$$

Combining Equations (3.8) and (3.9), we obtain

$$\begin{aligned} \frac{\Phi(x) - \Phi(t)}{x - t} &\geq ac^{-z(n_0)} \left(\frac{a+c}{c}\right)^{-z(n_0+1)} \\ &= a \left(\frac{a+c}{c}\right)^{r^{-1} z(n_0) - z(n_0+1)} \\ &\geq a \left(\frac{a+c}{c}\right)^{qz(n_0)} \qquad \text{by (3.7)} \\ &\geq a \left(\frac{a+c}{c}\right)^{qz(n)} \qquad \text{for all } t < x < t + u_n. \end{aligned}$$

But the above holds for all $n \geq m_0$, so we see that $\Phi_+(t) = \infty$, which completes the proof of part one.

2. Take any nonendpoint $t \in C_{uc}$, and suppose that

$$\limsup_{n\to\infty} \frac{z(n+1)}{z(n)} > r^{-1}. \tag{3.10}$$

Then by (3.10) there exists a subsequence $\{n_k\}$ of the positive integers, a real number $q > 0$, and a positive integer m_0 such that

$$r^{-1} - \frac{z(n_k+1)}{z(n_k)} \leq -q < 0 \qquad \text{for all } k \geq m_0. \tag{3.11}$$

Next, we define two sequences of points, $u(n)$ and $w(n)$, such that $u(n)$ decreases to t and $w(n)$ increases to t. The points $u(n)$ and $w(n)$ are defined below in terms of their locator sequences:

$$\begin{aligned} u(n) = &\ (t(1), t(2), \ldots, t(z(n)-1), 2, 0, 0, \ldots) \\ w(n) = &\ (t(1), t(2), \ldots, \ldots, t(z(n+1)-1), 0, 0, 0, \ldots) \end{aligned}$$

Clearly, $w(n) < u(n)$. Now, note that there is one stage $z(n)$ complementary interval of length $ba^{n-1}c^{z(n)-n}$ between t and $u(n)$, so that

$$u(n) - t \geq ba^{n-1}c^{z(n)-n}. \tag{3.12}$$

Similarly, there is one stage $z(n+1)-1$ black interval of length $a^n c^{z(n+1)-n-1}$ between $w(n)$ and $u(n)$. Thus, we have

$$\begin{aligned} \Phi(u(n)) - \Phi(t) &\leq \Phi(u(n)) - \Phi(w(n)) \\ &= \frac{a^n c^{z(n+1)-n-1}}{(a+c)^{z(n+1)-1}}. \end{aligned} \tag{3.13}$$

Thus, combining Equations (3.12) and (3.13) we obtain, for all $k \geq m_0$,

$$\begin{aligned} \frac{\Phi(u(n_k)) - \Phi(t)}{u(n_k) - t} &\leq \frac{ac^{-z(n_k)}}{b} \left(\frac{a+c}{c}\right)^{1-z(n_k+1)} \\ &= \left(\frac{a+c}{c}\right) \frac{a}{b} \left(\frac{a+c}{c}\right)^{r^{-1}z(n_k)-z(n_k+1)} \\ &\leq \frac{a}{b} \left(\frac{a+c}{c}\right) \left(\frac{a+c}{c}\right)^{-qz(n_k)} && \text{by (3.11)} \\ &\leq \frac{a}{b} \left(\frac{a+c}{c}\right). \end{aligned}$$

It follows that $\Phi_+(t) \leq 2a/b < \infty$, so that $t \in \mathcal{N}^+$. ■

To handle the set $\mathcal{N}^-$, we introduce a "complementary" Cantor function Ψ. This second function is simply the Cantor function of the related Cantor set C_{ca}, which is constructed in the same way as C_{ac} except that the roles of a and c are reversed. In considering the way that Φ and Ψ are defined, it is clear that there is a relationship between right differentiability of Ψ and left differentiability of Φ. The details are summarized in the following lemma, which will allow us (later in the thesis) to easily compute the Hausdorff dimension of $\mathcal{N}^-$ once we have determined the dimension of $\mathcal{N}^+$.

Lemma 3.14 *As above, let Φ be the Cantor function associated with C_{ac}, and let Ψ be the Cantor function associated with C_{ca}, and consider any $x \in [0, 1]$. Then*

1. *The number x is contained in the Cantor set C_{ac} if and only if the number $1 - x$ is contained in the Cantor set C_{ca}. Moreover, if the condition is satisfied, then the numbers x and $1 - x$ have complementary locator sequences in their respective Cantor sets.*

2. *The number x is a nonendpoint in C_{ac} if and only if $1 - x$ is a nonendpoint in C_{ca}.*

3. $\Phi(x) = 1 - \Psi(1 - x)$.

4 The Hausdorff Dimension of N^*

In this section, we are at last ready to compute the Hausdorff dimension of N^*, the nondifferentiability set of the Cantor function Φ described in the preceding section. Recall that by (3.4), the nondifferentiability set N^* can be written as the union of three sets: N^+ (the set of all points at which Φ is not differentiable from the right), N^- (the set of all points at which Φ is not differentiable from the left), and the set of all endpoints of complementary intervals in the construction of C_{ac}, which we will denote by F. We will begin by focusing our attention entirely on the set N^+; after arriving at a candidate value for its Hausdorff dimension, we will give a detailed proof that our candidate value is indeed the Hausdorff dimension of N^+. At the end of this section, we will at last turn our attention to the sets N^- and F and arrive at a value for the Hausdorff dimension of the entire nondifferentiability set N^*.

Throughout this section, the symbol $C(m, n)$ will denote the binomial coefficient "m choose n."

4.1 A Candidate for the Hausdorff Dimension

As before, let C_{ac} denote our non-symmetric Cantor set, let Φ be the associated Cantor function, and let N^+ be the set of all nonendpoints of C_{ac} at which Φ is not differentiable from the right. We seek a candidate value for the Hausdorff dimension of the set N^+.

We proceed by analogy to the standard Cantor ternary set, which we denote by C. We will first describe a procedure for "guessing" its Hausdorff dimension, and then apply this procedure to $\mathcal{N}^+$. First, recall that

$$\mathcal{H}^d(C) = \sup_{\delta>0} \inf \sum_{i\geq 1} |U_i|^d,$$

where the infimum is taken over all δ-covers $\{U_i\}$ of C. We know that a number γ is the Hausdorff dimension of C if it is the unique positive number such that $\mathcal{H}^d(C) = 0$ for all $d > \gamma$ and $\mathcal{H}^d(C) = \infty$ for all $d < \gamma$. Thus, to find the "cut-off point," that is, the value of d that's neither too large nor too small, we first consider a sequence $\{\{U_{ij}\}\}_{j=1}^{\infty}$ of covers of C, whose diameters shrink to zero, and examine the associated Hausdorff sums

$$\sum_{i\geq 1} |U_{ij}|^d.$$

In the case of the Cantor ternary set C, a natural choice for the jth cover is the collection of 2^j black intervals at the jth stage in the construction of C. For a real number $d > 0$, the Hausdorff sum associated with d is then given by

$$\sum_{i=1}^{2^j} (3^{-j})^d = 2^j 3^{-jd} = (2/3^d)^j.$$

The "cut-off" value for d is the number such that $2/3^d = 1$, since any larger value produces sums that tend to zero as j approaches ∞, and any smaller value produces sums that tend to infinity as j approaches ∞. This value of d is $\ln 2/\ln 3$, which can be shown to be the Hausdorff dimension of C.

We will now use the same procedure to find a candidate for the Hausdorff dimension of $\mathcal{N}^+$. By our characterization of the points in $\mathcal{N}^+$ (Theorem 3.5), we know that in order for a point t to be in $\mathcal{N}^+$, there has to be "sufficient space" between successive zeroes in the locator sequence of t. In particular, we must have

$$\limsup_{n\to\infty} \frac{z(n+1)}{z(n)} \geq \frac{\ln(1/c)}{\ln((a+c)/c)} = r^{-1}. \tag{4.1}$$

where $z(n)$ is the position of the nth zero in the locator sequence of t. Now, consider a collection E_i, consisting of 2^{i-1} stage u_i black intervals. Specifically, a stage u_i black interval is in E_i if and only if all points in the interval have locator sequences with a zero in the ith position and twos from the $(i+1)$st position up to the u_ith position. Note that each E_j consists of the following collection of intervals:

$$\begin{aligned}
&\mathrm{C}(j-1,0) \text{ intervals of length } a^j c^{u_j - j}\\
&\mathrm{C}(j-1,1) \text{ intervals of length } a^{j-1} c^{u_j-(j-1)}\\
&\vdots\\
&\mathrm{C}(j-1,j-1) \text{ intervals of length } a c^{u_j-1}
\end{aligned}$$

Next, consider for some positive integer j the collection $\{E_j, E_{j+1}, E_{j+2}, \dots\}$. Roughly speaking, Equation (4.1) says that the above collection will cover N^+ if u_j is chosen "sufficiently larger" than j. We therefore set $u_j/j = r^{-1}$. Using the binomial theorem and performing several simplifications, the Hausdorff sum associated with this collection becomes

$$\begin{aligned}
\sum_{k\geq j} |E_k|^d &= \sum_{k\geq j}\sum_{i=0}^{k-1} \mathrm{C}(k-1,i)\left[a^{k-i}c^{u_k-(k-i)}\right]^{\gamma}\\
&= \frac{a^{\gamma}}{a^{\gamma}+c^{\gamma}}\sum_{k\geq j}\left[1+\left(\frac{a}{c}\right)^{\gamma}\right]^k c^{\gamma u_k}\\
&= \frac{a^{\gamma}}{a^{\gamma}+c^{\gamma}}\sum_{k\geq j}\left[\left(1+\left(\frac{a}{c}\right)^{\gamma}\right)c^{\gamma r^{-1}}\right]^k
\end{aligned}$$

Observe that the kth term in the last summation above is that of a geometric series. If the common ratio is chosen to be greater than one, the series will diverge; if it is chosen to be less than one, the series will converge, and the above sum will therefore approach zero as j approaches infinity. The "cutoff" therefore appears to be a common ratio of one, so we will insist that

$$\left[1+\left(\frac{a}{c}\right)^{\gamma}\right]c^{\gamma r^{-1}}=1.$$

Taking logs and rearranging this equation reveals that our candidate for the Hausdorff dimension of X^{+} is a real number satisfying the equation

$$\gamma=\frac{r\ln\left[1+\left(\frac{a}{c}\right)^{\gamma}\right]}{\ln(1/c)}. \tag{4.2}$$

We close this section by showing that the number γ defined above exists in the interval $(0, 1)$ and is unique.

Lemma 4.3 *Let* $f(x)=x[\ln(1/c)]^{2}-\ln[(a+c)/c]\ln[1+(a/c)^{x}]$ *for each real number* x. *Then the equation* $f(x)=0$ *has a unique solution in the segment* $(0,1)$.

Proof: Before we begin the proof, we note that Equation (4.2) is equivalent to the equation $f(\gamma)=0$, so proving this lemma will indeed show that the number γ defined by (4.2) exists and is unique.

To begin, we first observe that $f(x)$ is continuous and differentiable on the interval $[0,1]$, and that by our choices of a and c, we have

$$f(0)=-\ln[(a+c)/c]\ln 2<0, \text{ and}$$

$$f(1)=[\ln(1/c)]^{2}-[\ln((a+c)/c)]^{2}>0.$$

Thus, by the Intermediate Value Theorem, $f(x)=0$ has a solution in the interval $(0,1)$. To show that the solution is unique, we use the Mean Value Theorem. For any real number x, we have

$$f'(x) = \left[\ln\left(\frac{1}{c}\right)\right]^2 - \ln\left(\frac{a+c}{c}\right)\ln\left(\frac{a}{c}\right)\cdot\frac{(a/c)^x}{1+(a/c)^x}.$$

Now, if $a < c$, we clearly have $f'(x) > 0$ since the above equation shows that $f'(x)$ is the sum of two positive numbers. If, on the other hand, $a \geq c$, we still have $f'(x) > 0$ because $\ln[(a+c)/c]$ and $\ln(a/c)$ are both nonnegative, and each is less than $\ln(1/c)$. This shows that the solution to $f(x) = 0$ is indeed unique. ■

4.2 Computation of an Upper Bound

In this subsection, we will show that the number γ described above is an upper bound on the Hausdorff dimension of N^+, the set of all points at which the Cantor function Φ is not differentiable from the right. We recall that

$$\text{If } t \in N^+, \text{ then } \limsup_{n\to\infty}\frac{z(n+1)}{z(n)} \geq r^{-1}. \tag{4.4}$$

where $r = \ln\left(\frac{a+c}{c}\right)/\ln\left(\frac{1}{c}\right)$. Recall also that the Hausdorff dimension of the set N^+ is the unique real number α such that $\mathcal{H}^d(N^+) = 0$ for all real numbers d strictly greater than α, where $\mathcal{H}^d$ denotes d-dimensional Hausdorff outer measure. To verify that γ is an upper bound on the Hausdorff dimension of N^+, it is therefore sufficient to prove the following theorem.

Theorem 4.5 *Let γ denote the unique real number satisfying $0 < \gamma < 1$ such that*

$$\gamma = \frac{r\ln\left[\left(\frac{a}{c}\right)^\gamma + 1\right]}{\ln(1/c)}.$$

Then for any $d > \gamma$ we have $\mathcal{H}^d(N^+) = 0$.

Proof: We first use algebra and properties of logarithms to obtain

$$\frac{\ln(a^\gamma + c^\gamma)}{\gamma\ln(1/c)} = r^{-1} - 1. \tag{4.6}$$

Define

$$\eta(x) = \frac{\ln(a^{\gamma+x} + c^{\gamma+x})}{\ln(a^{\gamma} + c^{\gamma})}.$$

Clearly, η is a continuous, decreasing function of x with $\eta(0) = 1$. Choose any $\epsilon > 0$ small enough so that

$$r^{-1} - \delta > 1, \tag{4.7}$$

where $\delta = (1 - \eta(\epsilon))(r^{-1} - 1)$, and set $d = \gamma + \epsilon$. Define a sequence of sets

$$E_k = \{t \in C_{ac} \mid t_k = 0 \text{ and } t_i = 2 \text{ for } k < i \leq u_k\}, \quad k = 1, 2, \ldots,$$

where the values of u_k will be specified later. Also define

$$E^\infty = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k = \limsup_{k \to \infty} E_k.$$

We will later choose $u_k > k$ so that

$$\frac{u_k}{k} \leq r^{-1} - \frac{\delta}{2} \tag{4.8}$$

holds for k sufficiently large. Note that if (4.8) holds, then by (4.4) it follows that $N^+ \subset E^\infty$.

Next, note that the $(k-1)$st stage in the construction of C_{ac} consists of 2^{k-1} black intervals, with $C(k-1, i)$ intervals of length $a^{k-1-i}c^i$ for $i = 0, 1, \ldots, k-1$. Thus, E_k can be covered by the following collection of intervals:

$C(k-1, 0)$ intervals of length $a^k c^{u_k - k}$

$C(k-1, 1)$ intervals of length $a^{k-1} c^{u_k - (k-1)}$

$\vdots$

$C(k-1, k-1)$ intervals of length $a c^{u_k - 1}$

Thus, we can guarantee that $\mathcal{H}^d(E^\infty) = 0$ by choosing u_k so that

$$a^d c^{(u_k - k)d}(a^d + c^d)^{k-1} = \sum_{i=0}^{k-1} \mathrm{C}(k-1, i) \left[a^{k-i} c^{i+(u_k - k)}\right]^d \leq k^{-2} \tag{4.9}$$

holds for k sufficiently large. Taking logs and rearranging (See Lemma 4.14 below), Equation (4.9) becomes

$$r^{-1} + \delta_k - \delta + \frac{\epsilon}{\gamma}\left(1 - \frac{u_k}{k}\right) \leq \frac{u_k}{k}, \tag{4.10}$$

where $\delta_k = [d \ln a + 2 \ln k - \ln(a^d + c^d)]/[\gamma k \ln(1/c)]$. Then if we eventually choose $u_k > k$, we see that (4.10) will follow from the truth of

$$r^{-1} + \delta_k - \delta < r^{-1} + \frac{1}{k} + \delta_k - \delta \leq \frac{u_k}{k}. \tag{4.11}$$

Now, choose a positive integer N large enough so that the following equations hold:

$$|\delta_k| + \frac{1}{k} < \frac{\delta}{4} \quad \text{for } k \geq N \tag{4.12}$$

$$\delta_k + r^{-1} - \delta > 1 \quad \text{for } k \geq N \tag{4.13}$$

Such a choice for N is possible by (4.7) and the fact that $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

At last, it's time to choose values for u_k in such a way that (4.8) and (4.9) both hold for k sufficiently large. Choose $u_k = k + 1$ for $k < N$. If $k \geq N$, then choose u_k to be the smallest integer satisfying (4.11). Note that $u_k > k$ then holds by (4.13). Thus, since $u_k > k$ is the smallest integer satisfying (4.11), we have

$$\frac{u_k - 1}{k} < r^{-1} + \frac{1}{k} + \delta_k - \delta \quad \text{for } k \geq N.$$

But this implies that

$$\begin{aligned}
\frac{u_k}{k} &< r^{-1} - \left(\delta - \delta_k - \frac{2}{k}\right) \\
&= r^{-1} - \delta + \left(\delta_k + \frac{1}{k}\right) + \frac{1}{k} \\
&\le r^{-1} - \delta + \frac{\delta}{4} + \frac{\delta}{4} \qquad \text{by (4.12)} \\
&= r^{-1} - \frac{\delta}{2}
\end{aligned}$$

for all $k \ge N$. Thus, (4.8) holds, and since (4.11) also holds, we see that (4.10) and therefore (4.9) also hold. It follows that $\mathcal{H}^d(N^+) \le \mathcal{H}^d(E^\infty) = 0$. ■

Lemma 4.14 *Equations (4.9) and (4.10) are equivalent.*

Proof: We begin by taking logs of both sides of (4.9) and dividing both sides of the resulting equation by $\gamma k \ln(1/c)$ to obtain

$$\frac{d \ln a}{\gamma k \ln\left(\frac{1}{c}\right)} + \frac{dk \ln\left(\frac{1}{c}\right)}{\gamma k \ln\left(\frac{1}{c}\right)} - \frac{du_k \ln\left(\frac{1}{c}\right)}{\gamma k \ln\left(\frac{1}{c}\right)} + \frac{(k-1)}{k} \cdot \frac{\ln(a^d + c^d)}{\gamma \ln\left(\frac{1}{c}\right)} \le -\frac{2 \ln k}{\gamma k \ln\left(\frac{1}{c}\right)}. \tag{4.15}$$

Canceling like terms, rearranging, and using our definitions of η and δ_k, we see that the above equation becomes

$$\delta_k + \frac{d}{\alpha} + \eta(\epsilon)\frac{\ln(a^\gamma + c^\gamma)}{\gamma \ln\left(\frac{1}{c}\right)} \le \frac{d}{\gamma}\left(\frac{u_k}{k}\right). \tag{4.16}$$

Next, since $d = \gamma + \epsilon$, we rearrange and use (4.6) to rewrite the above equation as

$$\delta_k + 1 + \eta(\epsilon)(r^{-1} - 1) + \frac{\epsilon}{\gamma}\left(1 - \frac{u_k}{k}\right) \le \frac{u_k}{k}. \tag{4.17}$$

Finally, we add and subtract a factor of r^{-1} above to obtain (4.10). ■

4.3 Computation of a Lower Bound

In this subsection, we will complete the proof that the number γ defined in Equation (4.2) is the Hausdorff dimension of the nondifferentiability set N^+ by showing that γ is also a lower bound on the Hausdorff dimension of N^+. We begin by making the following observation: if the d-dimensional Hausdorff outer measure of N^+ is strictly greater than zero for some positive real number d chosen strictly less than γ, then the Hausdorff dimension of N^+ must be bounded below by d. To see that this is true, suppose we have chosen a number d strictly less than γ such that the d-dimensional Hausdorff outer measure of N^+ is positive. Then if d were **not** a lower bound on the Hausdorff dimension of N^+, that would mean that its Hausdorff dimension were strictly less than d, which, by the definition of Hausdorff dimension, would indicate that the d-dimensional Hausdorff outer measure of N^+ were positive infinity. But this would then imply that d is an upper bound on the Hausdorff dimension of N^+, which would clearly contradict the result of the previous section. Therefore, to show that γ is also a lower bound on the Hausdorff dimension of N^+, it is sufficient to prove the following theorem.

We note that lemmas referenced in the proof of the following theorem follow its proof.

Theorem 4.18 *Let γ denote the unique real number satisfying $0 < \gamma < 1$ such that*

$$\gamma = \frac{r \ln\left[\left(\frac{a}{c}\right)^{\gamma} + 1\right]}{\ln(1/c)}.$$

Then for any $d < \gamma$ we have $\mathcal{H}^d(N^+) > 0$.

Proof: Define

$$\eta(x) = \frac{\ln\left[\left(\frac{a}{c}\right)^{\gamma} + 1\right]}{\ln\left[\left(\frac{a}{c}\right)^{\gamma x} + 1\right]} \quad \text{and} \quad q = \frac{\ln\left(\frac{a}{c}\right)}{\ln\left(\frac{1}{c}\right)}.$$

Recall also that

$$\gamma = \frac{r \ln\left[\left(\frac{a}{c}\right)^{\gamma} + 1\right]}{\ln(1/c)}.$$

By Lemma 4.36, we can choose a real number t_0, $0 < t_0 < 1$, such that

$$0 < \max\left\{\frac{2}{2+r}, \frac{1}{2}\right\} < t_0\eta(t_0) \leq t\eta(t) < 1 \text{ whenever } t_0 \leq t < 1.$$

Now, choose any real number t such that $t_0 \leq t < 1$, and define

$$d = \gamma t, \qquad \nu = rs, \quad \text{and} \quad \xi = s^{-1}t\eta(t),$$

where s is chosen so that

$$0 < \max\left\{\frac{2}{2+r}, \frac{1}{2}\right\} < t\eta(t) < s < 1. \tag{4.19}$$

Since $t\eta(t) > 2/(2+r)$ and $s^{-1} > 1$, it follows immediately that

$$1 - \xi < \frac{1}{2r^{-1} + 1}. \tag{4.20}$$

To prove the theorem, we will construct a subset E_t of $\mathcal{N}^+$ with

$$\mathcal{H}^d(E_t) > 0. \tag{4.21}$$

It will then follow that $\mathcal{H}^d(\mathcal{N}^+) > 0$. We will again apply Lemma 2.15 to narrow the class of covers we need to consider to sets in the collection $\mathcal{T}$, where

$$\mathcal{T} = \{\text{black intervals in the construction of } C_{ac}\} \cup \{\emptyset\}.$$

Lemma 2.15 shows that there exists $R > 0$ such that

$$\mathcal{H}^d \geq R\mu^d, \tag{4.22}$$

where μ is the outer measure similar to Hausdorff outer measure, except that only sets in the collection $\mathcal{T}$ are used in covers (see the statement of Lemma 2.15). We will establish Equation (4.21) by finding positive constants P and Q such that

$$\mu^d(N^+) \geq PQ. \tag{4.23}$$

Now, let us describe the set E_t. It will correspond to a sequence $0 < k_1 < u_1 < k_2 < u_2 < \ldots$ of positive integers as follows:

$$E_t = \{x = (x(1), x(2), x(3), \ldots) \mid x(k_i) = 0 \text{ and } x(m) = 2 \text{ for } k_i < m \leq u_i, i \geq 1\},$$

where $x = (x(1), x(2), x(3), \ldots)$ is the relevant code expansion of $x \in C_{ac}$. It is easy to show that C_{ac} is closed and contains no endpoints of complementary intervals in the construction of C_{ac}. Now, when $k_i \leq m \leq u_i$, we call m a **fixed choice** for E_t; otherwise, m is a **free choice**. Let $F(p, q)$ denote the number of free choices m with $p < m \leq q$, and note the trivial equality

$$F(0, k_i) = (k_i - u_{i-1} - 1) + F(0, k_{i-1}), \quad i \geq 2. \tag{4.24}$$

We are now ready to specify our definition of u_i and k_i (which of course depend on t). First, we choose $k_1 = 1$. Next, generate successive values of u_i and k_{i+1} using the recursive formulas below:

$$u_i = \nu^{-1}k_i + r_i, \quad i \geq 1 \tag{4.25}$$

$$F(0, k_i) - u_i - 1 + (1 - \xi)k_{i+1} + q\nu\xi(i+1) = s_{i+1}, \quad i \geq 1 \tag{4.26}$$

In particular, we choose u_i so that $r_i = \lceil \nu^{-1}k_i \rceil - \nu^{-1}k_i$, and we choose k_{i+1} to be the largest integer such that $s_{i+1} < 1$. Clearly, $0 \leq r_i < 1$ holds for $i \geq 1$, and Lemma 4.38 shows that $u_i > k_i$ and $0 \leq s_i < 1$ for all $i \geq 1$.

Next, we note that we can rewrite (4.26) as

$$F(0, k_{i-1}) - u_{i-1} - 1 + (1-\xi)k_i + qv\xi i = s_i, \quad i \geq 2.$$

Solving the above equation for $F(0, k_{i-1})$ and substituting into Equation (4.24), we obtain

$$F(0, k_i) = \xi k_i - q\nu\xi i + s_i, \quad i \geq 2, \tag{4.27}$$

which will be needed for future computations. Finally, to assure that E_t is well-defined and does what we want it to do, we must have $k_{i+1} > u_i$ for all i and $E_t \subset N^+$. These facts are verified as Lemmas 4.39 and 4.42, respectively.

Next, let $\{[a_j, b_j]\}_{j\geq 1}$ be any at most countable cover of E_t using only intervals in $\mathcal{T}$. Since E_t contains no endpoints of C_{ac}, $\{(a_j, b_j)\}_{j\geq 1}$ is an open cover of E_t, and since E_t is closed and bounded (and thus compact), we can extract (by re-indexing if necessary) a finite subcover $\{(a_j, b_j)\}_{j=1}^{n}$ of E_t such that $[a_j, b_j] \cap E_t \neq \emptyset$ for $1 \leq j \leq n$. Let the term **m-interval** describe any black interval among the 2^m black intervals at the mth stage in the construction of C_{ac}, and let w be the largest value of m for which one of the covering intervals $\{(a_j, b_j)\}_{j=1}^{n}$ is an m-interval. We wish to verify that

$$\sum_{j\geq 1} (b_j - a_j)^d \geq PQ, \tag{4.28}$$

where $P = (1 + (a/c)^d)^{-2}$ and $Q = (1 + (a/c)^d)^{-r}$.

To begin, choose any positive integer i such that $u_i, k_i > w$, and consider any value of j such that $1 \leq j \leq n$. Let $\mathcal{A}_{u_i}$ be the collection of all u_i-intervals intersecting E_t, and let $\mathcal{B}_{ju_i}$ be the collection of all u_i-intervals intersecting E_t and contained in $[a_j, b_j]$. We claim that Equation (4.28) will follow if (4.29) and (4.30) below both hold (See

Lemma 4.43).

$$\sum_{I \in \mathcal{A}_{u_i}} |I|^d \geq Q \tag{4.29}$$

$$(b_j - a_j)^d \geq P \sum_{I \in \mathcal{B}_{j u_i}} |I|^d \tag{4.30}$$

Next, we seek a more definite way of expressing the sums appearing in (4.29) and (4.30). First, to each positive integer g we will assign a nonnegative integer $p(g)$ as follows: $p(g)$ is the largest positive integer such that $k_{p(g)} \leq g$, or, if no such positive integer exists, we set $p(g) = 0$. Let m denote the nonnegative integer such that the interval $[a_j, b_j]$ appears at the mth stage in the construction of C_{ac}. Then (4.29) and (4.30) are equivalent to (4.31) and (4.32) below, respectively (See Lemma 4.45).

$$\left[1 + \left(\frac{a}{c}\right)^d\right]^{F(0,k_i)} \left(\frac{a}{c}\right)^{di} \left(\frac{1}{c}\right)^{-du_i} \geq Q \tag{4.31}$$

$$1 \geq P\left[1 + \left(\frac{a}{c}\right)^d\right]^{F(m,k_i)} \left(\frac{a}{c}\right)^{d(i-p(m))} \left(\frac{1}{c}\right)^{-d(u_i-m)} \tag{4.32}$$

Define

$$B(g.h) = F(g.k_h) + q\nu\xi(h - p(g)) - \nu\xi(u_h - g) \quad \text{for integers } g, h \text{ with } \; 0 \leq g \leq k_h.$$

Then by Lemma 4.49, Equations (4.31) and (4.32) are equivalent to Equations (4.33) and (4.34) below, respectively.

$$\left[1 + \left(\frac{a}{c}\right)^d\right]^{(s_i - \nu\xi r_i)} \geq Q \tag{4.33}$$

$$\left[1 + \left(\frac{a}{c}\right)^d\right]^{B(m,i)} \leq P^{-1} \tag{4.34}$$

Since (4.19) implies that $\xi < 1$, we see that $s_i - \nu\xi r_i \geq -r\xi r_i \geq -r$, so we have

$$\left[1+\left(\frac{a}{c}\right)^d\right]^{(s_i-\nu\xi r_i)} \geq \left[1+\left(\frac{a}{c}\right)^d\right]^{-r} = Q,$$

which verifies (4.33). To verify (4.34), we will first bound the quantity $B(m,i)$. Note that by Lemma 4.50, we have

$$B(m,i) \leq B(u_{p(m)}, i), \tag{4.35}$$

so since $p(m) < i$, it suffices to bound $B(u_j, i)$ for all $j < i$. So choose any $j < i$. Then by (4.25) and (4.26) we obtain

$$\begin{aligned}
B(u_j, i) &= F(u_j, k_i) + q\nu\xi(i - p(u_j)) - \nu\xi(u_i - u_j) \\
&= F(0, k_i) - F(0, u_j) + q\nu\xi(i-j) - \nu\xi[(\nu^{-1}k_i + r_i) - (\nu^{-1}k_j + r_j)] \\
&= F(0, k_i) - F(0, k_j) + q\nu\xi(i-j) - \xi(k_i - k_j) - \nu\xi(r_i - r_j) \\
&= \xi(k_i - k_j) - q\nu\xi(i-j) + (s_i - s_j) + q\nu\xi(i-j) - \xi(k_i - k_j) - \nu\xi(r_i - r_j) \\
&= (s_i - s_j) - \nu\xi(r_i - r_j) \\
&< 2,
\end{aligned}$$

so that

$$\left[1+\left(\frac{a}{c}\right)^d\right]^{B(m,i)} \leq \left[1+\left(\frac{a}{c}\right)^d\right]^{2} = P^{-1},$$

which verifies (4.34).

Finally, since (4.33) and (4.34) hold, we see from the discussion above that (4.28) also holds, and since the positive constants P and Q are independent of our cover, (4.28) holds for any cover of E_t using intervals from $\mathcal{T}$. Taking the infimum over all such covers, Equations (4.22) and (4.28) yield

$$\mu^d(E_t) \geq PQ \quad \Longrightarrow \quad \mathcal{H}^d(N^+) \geq \mathcal{H}^d(E_t) \geq RPQ > 0,$$

which proves the theorem. ■

Lemma 4.36 *There exists a number* t_0, $0 < t_0 < 1$, *such that* $0 < \max\left\{\frac{1}{2}, \frac{2}{2+r}\right\} < t_0\eta(t_0) \leq t\eta(t) < 1$ *for* $t_0 \leq t < 1$.

Proof: First, let us consider the case where $a \leq c$. Then $t\eta(t)$ is clearly an increasing function of t which approaches one as t approaches one. Thus, the lemma is trivial.

Now, suppose that $a > c$. For ease of notation, set $M = a/c$. We note that η is a decreasing, continuous function of t, with $\eta(1) = 1$. Therefore, since $q, r < 1$, we can choose a number t_0', $0 < t_0' < 1$, such that

$$1 - rtq\eta(t) > 0 \qquad \text{for } t_0' \leq t < 1. \tag{4.37}$$

Consider the function $f(t) = t\eta(t)$. Using the definition of γ and ordinary differentiation rules, we have, for $t_0' \leq t < 1$,

$$\begin{aligned} f'(t) &= \eta(t)\left[1 - \frac{\gamma t M^{\gamma t}\ln M}{(M^{\gamma t}+1)\ln(M^{\gamma t}+1)}\right] \\ &= \eta(t)\left[1 - \frac{M^{\gamma t}}{M^{\gamma t}+1}\cdot rtq\eta(t)\right] \\ &> \eta(t)\left[1 - rtq\eta(t)\right] \\ &> 0, \end{aligned}$$

where the last inequality follows from (4.37). It follows that f is strictly increasing on the interval $[t_0', 1]$. Thus, since $f(1) = 1$, we can choose a number t_0, $t_0' \leq t_0 < 1$, such that

$$0 < \max\left\{\frac{1}{2}, \frac{2}{2+r}\right\} < t_0\eta(t_0) \leq t\eta(t) < 1$$

for all t such that $t_0 \leq t < 1$. ■

Lemma 4.38 *In Equation (4.25), u_i will always be an integer greater than k_i, and in Equation (4.26), such a choice for k_{i+1} is always possible.*

Proof: Since $u_i = \lceil \nu^{-1} k_i \rceil$, it is clear that u_i is always an integer. Also, since $\nu = rs < 1$, it follows that $\nu^{-1} k_i > k_i$, so that $u_i > k_i$ as claimed.

To verify the claim for k_{i+1}, note that by Equation (4.19), we have $t\eta(t) < s \implies \xi < 1$, so that

$$0 < 1 - \xi < 1.$$

Thus, if we choose k_{i+1} to be the largest integer making the left hand side of (4.26) strictly less than one, it is clear that such a choice will make $0 \leq s_{i+1} < 1$ true. ∎

Lemma 4.39 *With our choice of k_1, the successive values of k_i and u_i generated by (4.25) and (4.26) will satisfy $k_{i+1} > u_i$ for $i \geq 1$.*

Proof: We first verify the claim for $i = 1$. Since $u_1 = \nu^{-1} + r_1$ and $F(0, k_1) = 0$, Equation (4.26) becomes

$$(1 - u_1) + (1 - \xi)k_2 + (2q\nu\xi - 2) = s_2 \tag{4.40}$$

Also, combining (4.19) and (4.20) yields

$$0 < 1 - \xi < \frac{1}{2r^{-1} + 1} < \frac{1}{s^{-1}r^{-1} + 1} = \frac{1}{\nu^{-1} + r_1} \tag{4.41}$$

Next, since $2q\nu\xi - 2 < 0$ and $s_2 \geq 0$, Equation (4.40) implies that

$$(1 - u_1) + (1 - \xi)k_2 > 0.$$

Therefore,

$$\begin{aligned} k_2 &> (u_1 - 1)(1-\xi)^{-1} \quad \text{(since by (4.19), } 1-\xi > 0\text{)} \\ &\geq (1-\xi)^{-1} \quad \text{(since } u_1 > k_1 = 1\text{)} \\ &> \nu^{-1} + r_1 \quad \text{by (4.41)} \\ &= u_1. \end{aligned}$$

We now verify the claim for $i \geq 2$. By Equations (4.26) and (4.27), we have, for all $i \geq 2$,

$$\begin{aligned} (1-\xi)(k_{i+1} - u_i) &= (1-\xi)k_{i+1} - (1-\xi)u_i \\ &= s_{i+1} + u_i + 1 - F(0, k_i) - q\nu\xi(i+1) - (1-\xi)u_i \\ &= s_{i+1} + \xi u_i + 1 - \xi k_i + q\nu\xi i - s_i - q\nu\xi(i+1) \\ &\geq \xi(u_i - k_i) - q\nu\xi \\ &\geq \xi(1 - q\nu). \quad \text{(since } u_i > k_i\text{)} \end{aligned}$$

But $1 - \xi > 0$ and $1 - q\nu > 0$, so it follows from the previous string of inequalities that $k_{i+1} > u_i$ holds for all $i \geq 2$. ■

Lemma 4.42 *The set E_t is contained in the nondifferentiability set N^+.*

Proof: Take any point $x = (x(1), x(2), x(3), \ldots) \in E_t$. Let $z(n)$ denote the position of the nth zero in the above code expansion of x. Choose a subsequence $\{n_i\}_{i=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that

$$z(n_i) = k_i \quad \text{for } i = 1, 2, \ldots.$$

This is always possible since $x(k_i) = 0$ for all i. By the definition of E_t, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{z(n+1)}{z(n)} \geq \limsup_{i\to\infty} \frac{z(n_i+1)}{z(n_i)} &\geq \limsup_{i\to\infty} \frac{u_i}{k_i} \\ &\geq \nu^{-1},\end{aligned}$$

the last inequality following from (4.25). Since $\nu^{-1} > r^{-1}$, it follows from our characterization of the nondifferentiable points of Φ that $x \in N^+$. ■

Lemma 4.43 *Equation (4.28) follows from Equations (4.29) and (4.30).*

Proof: Assume that (4.29) and (4.30) hold, and note that the sums in both cases are finite. Our first claim is that

$$\sum_{j=1}^{m} \sum_{I\in\mathcal{B}_{ju_i}} |I|^d \geq \sum_{I\in\mathcal{A}_{u_i}} |I|^d. \tag{4.44}$$

To see this, note that every u_i-interval, $u_i > w$, that intersects E must intersect one of the covering intervals $[a_j, b_j]$ since they cover E. Also, this u_i-interval can only intersect one of the covering intervals by the geometry of the intervals in $\mathcal{T}$. Thus,

$$\mathcal{A}_{u_i} \subset \bigcup_{j=1}^{m} \mathcal{B}_{ju_i},$$

where the above union is disjoint. This establishes (4.44), and we therefore have

$$\begin{aligned}\sum_{j=1}^{\infty} (b_j - a_j)^d \geq \sum_{j=1}^{n} (b_j - a_j)^d &\geq \sum_{j=1}^{n} P \sum_{I\in\mathcal{B}_{ju_i}} |I|^d \quad \text{(by (4.30))} \\ &\geq P \sum_{I\in\mathcal{A}_{u_i}} |I|^d \quad \text{(by (4.44))} \\ &\geq PQ, \quad \text{(by (4.29))}\end{aligned}$$

which verifies the claim. ■

Lemma 4.45 *Equations (4.31) and (4.32) imply Equations (4.29) and (4.30).*

Proof: This will take some work. We first seek a more definite expression for the sum appearing in Equation (4.29). Suppose (more generally than in (4.29)) that for a nonnegative integer g (not necessarily equal to any u_i) we let $\mathcal{A}_g$ denote the collection of all g-intervals that intersect E_t. We seek a definite expression for the associated "Hausdorff sum"

$$\sum_{I\in\mathcal{A}_g} |I|^d.$$

To begin, recall that for each positive integer h, $p(h)$ is the largest nonnegative integer such that $k_{p(h)} \leq h$, or, if no such integer exists $p(h) = 0$. We claim that

$$\sum_{I\in\mathcal{A}_g} |I|^d = (a^d + c^d)^{F(0,g)} a^{dp(g)} c^{d[g-p(g)-F(0,g)]} \tag{4.46}$$

Verification of (4.46). Note first that, by the definition of E_t, there are exactly $2^{F(0,g)}$ different intervals contained in $\mathcal{A}_g$, and that this is also the number of terms in the binomial expansion of the right hand side of (4.46).

We proceed by induction on g. The cases $g = 0$ and $g = 1$ are trivial to verify, so suppose that (4.46) holds for some $g \geq 1$. Since $k_1 = 1$, this yields three cases: $k_{p(g)} \leq g < u_{p(g)}$, $u_{p(g)} \leq g < k_{p(g)+1} - 1$, or $g = k_{p(g)+1} - 1$.

1. If $k_{p(g)} \leq g < u_{p(g)}$ and $x \in E_t$, then $x(g+1) = 2$. Thus, only the "right cth" of the intervals in $\mathcal{A}_g$ will be included in $\mathcal{A}_{g+1}$. So we have

   $$\sum_{I\in\mathcal{A}_{g+1}} |I|^d = c^d \sum_{I\in\mathcal{A}_g} |I|^d.$$

   Also, since in this case $g+1$ is a fixed choice for E_t, we have $F(0,g) = F(0,g+1)$, and it is easy to see that $p(g+1) = p(g)$. Using the inductive hypothesis and the above equality, it is easy to verify that (4.46) holds for $g+1$.

2. If $u_{p(g)} \leq g < k_{p(g)+1} - 1$ and $x \in E_t$, then $x(g+1)$ could be either 0 or 2. So the collection $\mathcal{A}_{g+1}$ has twice as many intervals as $\mathcal{A}_g$, and they consist of the "left *a*ths" and the "right *c*ths" of the intervals in $\mathcal{A}_g$. Thus, we have

$$\sum_{I \in \mathcal{A}_{g+1}} |I|^d = (a^d + c^d) \sum_{I \in \mathcal{A}_g} |I|^d.$$

As in the previous case, it is easy to see that $p(g+1) = p(g)$; however, in this case, $g+1$ is a free choice for E_t this time, so that $F(0, g+1) = F(0, g) + 1$. As before, combining all this information verifies that (4.46) holds for $g+1$.

3. If $g = k_{p(g)+1} - 1$, the verification of (4.46) for $g+1$ follows the exact same pattern as the previous two, this time using the fact that $F(0, g+1) = F(0, g)$, $p(g+1) = p(g) + 1$, and

$$\sum_{I \in \mathcal{A}_{g+1}} |I|^d = a^d \sum_{I \in \mathcal{A}_g} |I|^d.$$

Similarly, we seek a more definite expression for the sum appearing on the right hand side of (4.30). To do this, suppose (more generally than in (4.30)) that we have any two nonnegative integers g and h with $h \leq g$. Choose any h-interval I_h which intersects E_t, and suppose this interval has length L_h. Let $\mathcal{B}_g^{(h)}$ denote the collection of all g-intervals intersecting E_t which are also subsets of I_h. We seek a definite expression this time for the associated "Hausdorff sum"

$$\sum_{I \in \mathcal{B}_g^{(h)}} |I|^d.$$

We claim that

$$\sum_{I \in \mathcal{B}_g^{(h)}} |I|^d = L_h^d (a^d + c^d)^{F(h,g)} a^{d[p(g)-p(h)]} c^{d[(g-h)-(p(g)-p(h))-F(h,g)]} \tag{4.47}$$

Choose an arbitrary nonnegative integer g. It is enough then to show that (4.47) holds for all $h \leq g$. We will proceed by induction on h; specifically, let $P(h)$ be the proposition that for all h-intervals intersecting E_t, Equation (4.47) holds.

Verification of (4.47). Choose an arbitrary nonnegative integer g. It is enough then to show that (4.47) holds for all $h \leq g$. We will proceed by induction on h; specifically, let $P(h)$ be the proposition that for all h-intervals intersecting E_t, Equation (4.47) holds. For $h = 0$, $I_h = [0, 1]$, and (4.47) reduces to (4.46), which has already been verified. So suppose that $P(h)$ holds for some h, $0 \leq h < g$. We wish to verify $P(h+1)$. Choose any $(h+1)$-interval which intersects E_t, and call it I_{h+1}. Once again, we have the following three cases: $k_{p(h)} \leq h < u_{p(h)}$, $h = k_{p(h)+1} - 1$, or $u_{p(h)} \leq h < k_{p(h)+1} - 1$.

1. If $k_{p(h)} \leq h < u_{p(h)}$ and $x \in E_t$, then $x(h+1) = 2$. Thus, I_{h+1} is the "right cth" of some h-interval, call it I_h, that intersects E_t. Also, since $x(h+1) = 2$, it follows that every g-interval intersecting E_t is contained in I_h if and only if it is contained in I_{h+1}. Combining the above observations with the trivial facts that $F(h+1, g) = F(h, g)$, $p(h+1) = p(h)$, and the induction hypothesis, we have

$$\begin{aligned}\sum_{I \in \mathcal{B}_g^{(h+1)}} |I|^d &= \sum_{I \in \mathcal{B}_g^{(h)}} |I|^d \\ &= L_h^d (a^d + c^d)^{F(h,g)} a^{d[p(g)-p(h)]} c^{d[(g-h)-(p(g)-p(h))-F(h,g)]} \\ &= c^d L_h^d (a^d + c^d)^{F(h+1,g)} a^{d[p(g)-p(h+1)]} c^{d[(g-h-1)-(p(g)-p(h+1))-F(h+1,g)]}.\end{aligned}$$

   Thus, since $L_{h+1}^d = c^d L_h^d$, this verifies (4.47) for $h+1$.

2. If $h = k_{p(h)+1} - 1$ and $x \in E_t$, then $x(h+1) = 0$. In this case, $F(h+1, g) = F(h, g)$, but $p(h+1) = p(h) + 1$. A nearly identical analysis to the one above again verifies (4.47) for $h+1$.

3. If $u_{p(h)} \le h < k_{p(h)+1} - 1$ and $x \in E_t$, then $x(h+1)$ could be either 0 or 2. (A quick check shows that, in this case, $F(h+1, g) = F(h, g) - 1$ and $p(h+1) = p(h)$.) Thus, I_{h+1} could be either the left ath or the right cth of an h-interval intersecting E_t; call this h-interval I_h.

Now, by the induction hypothesis, Equation (4.47) holds for our particular h, and note that the collection $\mathcal{B}_g^{(h)}$ can be split into two disjoint parts, the intervals contained in the left ath of I_h (call this subcollection $\mathcal{D}_1$) and the intervals contained in the right cth of I_h (call this subcollection $\mathcal{D}_1$). Thus,

$$\mathcal{B}_g^{(h)} = \mathcal{D}_1 \cup \mathcal{D}_2.$$

Clearly, there exists a finite sequence $x(1), x(2), \ldots, x(h)$ such that

$$\mathcal{D}_1 = \{g\text{-intervals } I \text{ meeting } E_t \mid I \text{ has locator sequence } (x(1), \ldots, x(h), 0)\}$$

$$\mathcal{D}_2 = \{g\text{-intervals } I \text{ meeting } E_t \mid I \text{ has locator sequence } (x(1), \ldots, x(h), 2)\}.$$

It is also clear that the locator sequences of any $\mathcal{D}_1$ or $\mathcal{D}_2$ intervals can be obtained by adding on exactly the same collection of length $g - h + 1$ sequences to the ends of the sequences $(x(1), x(2), \ldots, x(h), 0)$ or $(x(1), x(2), \ldots, x(h), 2)$, respectively. Thus, there exists a number M such that

$$\sum_{I \in \mathcal{D}_1} |I|^d = L_h^d a^d M, \quad \text{and}$$

$$\sum_{I \in \mathcal{D}_2} |I|^d = L_h^d c^d M.$$

Thus, from the above, we see that $\sum_{I \in \mathcal{B}_g^{(h)}} |I|^d = L_h^d M(a^d + c^d)$. Thus, since (4.47) also holds for our particular choice of h, we have

$$M = (a^d + c^d)^{F(h,g)-1} a^{d[p(g)-p(h)]} c^{d[(g-h)-(p(g)-p(h))-F(h,g)]} \tag{4.48}$$

We will now verify that (4.47) holds for $h+1$.

<u>Case A</u>. I_{h+1} is the right cth of I_h. Then $\mathcal{B}_g^{(h+1)} = \mathcal{D}_1$, and combining with (4.48) we have

$$\begin{aligned}\sum_{I \in \mathcal{B}_g^{(h+1)}} |I|^d &= L_h^d c^d (a^d + c^d)^{F(h,g)-1} a^{d[p(g)-p(h)]} c^{d[(g-h)-(p(g)-p(h))-F(h,g)]} \\ &= L_{h+1}^d (a^d + c^d)^{F(h+1,g)} a^{d[p(g)-p(h+1)]} c^{d[(g-h-1)-(p(g)-p(h+1))-F(h+1,g)]},\end{aligned}$$

which verifies (4.47) for $h+1$.

<u>Case B</u>. I_{h+1} is the left ath of I_h. Then $\mathcal{B}_g^{(h+1)} = \mathcal{D}_2$, and a similar set of equalities again verifies (4.47) for $h+1$.

With Equation (4.47) verified, we proceed to prove the Lemma. First, let us show that Equation (4.31) is equivalent to Equation (4.29). To see this, set $g = u_i$. Then have $F(0, g) = F(0, u_i) = F(0, k_i)$, and $p(g) = i$, so that (4.46) becomes

$$\begin{aligned}\sum_{I \in \mathcal{A}_{u_i}} |I|^d &= (a^d + c^d)^{F(0,k_i)} a^{di} c^{d[u_i - i - F(0,k_i)]} \\ &= \left[1 + \left(\frac{a}{c}\right)^d\right]^{F(0,k_i)} \left(\frac{a}{c}\right)^{di} \left(\frac{1}{c}\right)^{-du_i}.\end{aligned}$$

Thus, the left hand sides of (4.29) and (4.31) are equal, which shows that they are equivalent.

Finally, we will show that (4.32) and (4.30) are equivalent. For convenience of notation set

$$M = \left[1 + \left(\frac{a}{c}\right)^d\right]^{F(m,k_i)} \left(\frac{a}{c}\right)^{d(i-p(m))} \left(\frac{1}{c}\right)^{-d(u_i - m)}.$$

For our generic $[a_j, b_j]$, L_h is of course $b_j - a_j$, and $\mathcal{B}_g^{(h)} = \mathcal{B}_{ju_i}$, and we have $F(m, u_i) = F(m, k_i)$. Thus, taking $h = m$ and $g = u_i$ in (4.47) and using (4.32), we have

$$\begin{aligned}\sum_{I \in \mathcal{B}_{ju_i}} |I|^d &= (b_j - a_j)^d (a^d + c^d)^{F(m,k_i)} a^{d[i-p(m)]} c^{d[(u_i-m)-(i-p(m))-F(m,k_i)]} \\ &= (b_j - a_j)^d M,\end{aligned}$$

which establishes the claim. ■

Lemma 4.49 *Equation (4.31) is equivalent to Equation (4.33), and Equation (4.32) is equivalent to Equation (4.34).*

Proof: First, set

$$\kappa = \left[1 + \left(\frac{a}{c}\right)^d\right],$$

and note that

$$\begin{aligned}F(0, k_i) + \frac{di \ln(a/c)}{\ln \kappa} - \frac{du_i \ln(1/c)}{\ln \kappa} &= F(0, k_i) + \frac{\gamma ti \ln(a/c)}{\ln \kappa} - \frac{\gamma tu_i \ln(1/c)}{\ln \kappa} \\ &= (\xi k_i - q\nu\xi i + s_i) + rtq\eta(t) i - rtu_i\eta(t) \\ &= (\xi k_i + s_i) - \nu\xi u_i \\ &= (\xi k_i + s_i) - \nu\xi(\nu^{-1} k_i + r_i) \quad \text{by (4.25)} \\ &= s_i - \nu\xi r_i,\end{aligned}$$

where the second equality follows from (4.27) and the definition of γ. This shows that Equations (4.31) and (4.33) are equivalent. Similarly, we have

$$\begin{aligned}B(m, i) &= F(m, k_i) + rtq\eta(t)(i - p(m)) - rt\eta(t)(u_i - m) \\ &= F(m, k_i) + \frac{d(i - p(m)) \ln(a/c)}{\ln \kappa} - \frac{d(u_i - m) \ln(1/c)}{\ln \kappa},\end{aligned}$$

which shows that Equations (4.32) and (4.34) are equivalent. ■

Lemma 4.50 $B(m, i) \le B(u_{p(m)}, i)$.

Proof: We have two cases to consider; either $k_{p(m)} \le m \le u_{p(m)}$ or $u_{p(m)} < m < k_{p(m)+1}$.

In case $k_{p(m)} \le m \le u_{p(m)}$, we clearly have $F(m, k_i) = F(u_{p(m)}, k_i)$ so we see that

$$\begin{aligned} B(m, i) \le B(u_{p(m)}, i) &\iff -\xi\nu(u_i - m) \le -\xi\nu(u_i - u_{p(m)}) \\ &\iff m \le u_{p(m)}, \end{aligned}$$

which is clearly true.

In case $u_{p(m)} < m < k_{p(m)+1}$, we have $F(u_{p(m)}, k_i) - F(m, k_i) = F(u_{p(m)}, m) = m - u_{p(m)}$, so that

$$\begin{aligned} B(m, i) < B(u_{p(m)}, i) &\iff F(u_{p(m)}, m) - F(m, k_i) - \nu\xi(m - u_{p(m)}) > 0 \\ &\iff (m - u_{p(m)})(1 - \nu\xi) > 0, \end{aligned}$$

which is clearly true since (4.19) implies that $\xi < 1$. Combining the above cases proves the lemma. ■

4.4 The Hausdorff Dimension of N^*

In this subsection, we will compute the Hausdorff dimension of the nondifferentiability set N^*, which is the main result of the thesis. All of the hard work has essentially been done in the preceding subsections, where we showed that γ is the Hausdorff dimension of the set N^+, for we will see that the Hausdorff dimension of N^* is simply the maximum of γ and a closely related number we will call $\tilde{\gamma}$. The number $\tilde{\gamma}$ is defined in the same way as γ except that the roles of a and c are reversed. We will see that $\tilde{\gamma}$ accounts for the set $\mathcal{N}^-$ of points where Φ is not differentiable from the left. The details are summarized in the following theorem.

Theorem 4.51 *Let Φ denote the Cantor function associated with the Cantor set C_{ac}, and let $\mathcal{N}^*$ be the associated nondifferentiability set. Let γ and $\tilde{\gamma}$ be the unique numbers, $0 < \gamma, \tilde{\gamma} < 1$, such that*

$$\gamma = \frac{\ln\left[\frac{a+c}{c}\right]\ln\left[\left(\frac{a}{c}\right)^{\gamma}+1\right]}{\left[\ln\left(\frac{1}{c}\right)\right]^2}, \qquad \tilde{\gamma} = \frac{\ln\left[\frac{a+c}{a}\right]\ln\left[\left(\frac{c}{a}\right)^{\tilde{\gamma}}+1\right]}{\left[\ln\left(\frac{1}{a}\right)\right]^2}.$$

Then $\dim_H(\mathcal{N}^*) = \max\{\gamma, \tilde{\gamma}\}$.

Proof: By Theorem 4.18, we have

$$\mathcal{H}^d(\mathcal{N}^+) = 0 \qquad \text{for all } d > \gamma. \tag{4.52}$$

Now, recall that $\mathcal{N}^-$ is the set of all nonendpoints in C_{ac} where Φ is not differentiable from the left. To compute the Hausdorff measure of $\mathcal{N}^-$, we turn our attention to the Cantor function Ψ associated with the Cantor set C_{ca}. (Recall the definitions of Ψ and C_{ca} from the discussion preceding Lemma 3.14.) Let $\tilde{\mathcal{N}}^+$ be the set of all nonendpoints in C_{ca} at which Ψ is not differentiable from the right. Applying Theorem 4.18 to Ψ and $\tilde{\mathcal{N}}^+$, and remembering that this will reverse the roles of a and c, we see that

$$\mathcal{H}^d(\tilde{\mathcal{N}}^+) = 0 \qquad \text{for all } d > \tilde{\gamma}.$$

But by part three of Lemma 3.14, it is easy to see that $\mathcal{N}^- = 1 - \tilde{\mathcal{N}}^+$, so that

$$\mathcal{H}^d(\mathcal{N}^-) = \mathcal{H}^d(1 - \tilde{\mathcal{N}}^+) = \mathcal{H}^d(\tilde{\mathcal{N}}^+) = 0 \qquad \text{for all } d > \tilde{\gamma}. \tag{4.53}$$

Finally, we let F denote the countable set of interval endpoints in C_{ac} and use the decomposition described in (3.4) to obtain

$$\mathcal{H}^d(N^*) \le \mathcal{H}^d(N^+) + \mathcal{H}^d(N^-) + \mathcal{H}^d(F) = 0$$

for all $d > \max\{\gamma, \tilde{\gamma}\}$. It follows that $\dim_H(N^*) \le \max\{\gamma, \tilde{\gamma}\}$.

To finish the proof, we will show that $\dim_H(N^*) \ge \max\{\gamma, \tilde{\gamma}\}$. First, by Theorem 4.5, we see that

$$\mathcal{H}^d(N^*) \ge \mathcal{H}^d(N^+) > 0 \quad \text{for all } d < \gamma.$$

Similarly, if apply Theorem 4.5 to Ψ and use Equation (4.53), we get

$$\mathcal{H}^d(N^*) \ge \mathcal{H}^d(N^-) = \mathcal{H}^d(\tilde{N}^+) > 0 \quad \text{for all } d < \tilde{\gamma}.$$

Combining these results, we conclude that

$$\mathcal{H}^d(N^*) > 0 \qquad \text{for all } d < \max\{\gamma, \tilde{\gamma}\}.$$

Now, consider any real number $d' < \max\{\gamma, \tilde{\gamma}\}$. Then we can choose a number d satisfying $d' < d < \max\{\gamma, \tilde{\gamma}\}$, and combining with the above inequality, we see that $\mathcal{H}^{d'}(N^*) = \infty$. It follows that $\dim_H(N^*) \ge \max\{\gamma, \tilde{\gamma}\}$. ■

We note that the above result gives us the Hausdorff dimension of N^* as the maximum of two real numbers, γ and $\tilde{\gamma}$. Clearly, these two numbers are equal if $a = c$ so that the Hausdorff dimension of N^* is just their common value. A natural question to ask if a and c are not equal, then, is whether it is possible to determine which of these two numbers is larger. The answer turns out to be simple: If a is strictly greater (less) than c, then γ is strictly greater (less) than $\tilde{\gamma}$. This fact leads to the following theorem, which gives us a more definite way of expressing the Hausdorff dimension of the set N^*.

Theorem 4.54 *Let C_{ac}, Φ, N^*, γ and $\bar{\gamma}$ be as in Theorem 4.51. Then the Hausdorff dimension of N^* is given by the unique real number α satisfying*

$$\alpha = \frac{\ln\left(\frac{a+c}{\min(a,c)}\right)\ln\left(\frac{a^{\alpha}+c^{\alpha}}{(\min(a,c))^{\alpha}}\right)}{\left[\ln\left(\frac{1}{\min(a,c)}\right)\right]^2}$$

Proof: We begin by defining the following functions:

$$f(x) = \left[\ln\left(\frac{1}{c}\right)\right]^2 x - \ln\left(\frac{a+c}{c}\right)\ln\left(1+\left(\frac{a}{c}\right)^x\right)$$

$$g(x) = \left[\ln\left(\frac{1}{a}\right)\right]^2 x - \ln\left(\frac{a+c}{a}\right)\ln\left(1+\left(\frac{c}{a}\right)^x\right)$$

Note that $f(x)$ is the same function that was defined in the statement of Lemma 4.3, and that in the proof of that lemma we showed $f'(x) > 0$ for all real numbers x. Our first observation is that this implies that $g'(x) > 0$ also holds for all real numbers x. This is true because we know that $f'(x) > 0$ for all constant values of a and c such that $0 < a, c < \frac{1}{2}$; in particular, $f'(x) > 0$ still holds if the values of a and c are reversed in the function $f(x)$. But reversing the values of a and c in $f(x)$ gives the function $g(x)$. Next, we note that by using properties of logs and canceling like terms, we can rewrite $f(x)$ and $g(x)$ as follows.

$$f(x) = x\ln(c)\ln(a+c) - \ln(a^x + c^x)\ln((a+c)/c)$$

$$g(x) = x\ln(a)\ln(a+c) - \ln(a^x + c^x)\ln((a+c)/a)$$

Note that by the definition of the number γ (see Theorem 4.51 above), we have $f(\gamma) = 0$, which is equivalent to writing

$$\ln(a^{\gamma} + c^{\gamma}) = \frac{\gamma\ln(c)\ln(a+c)}{\ln((a+c)/c)}.$$

Thus, using properties of logs and canceling like terms, we have

$$
\begin{aligned}
g(\gamma) &= \gamma \ln(a)\ln(a+c) - \ln(a^\gamma + c^\gamma)\ln[(a+c)/a] \\
&= \gamma \ln(a)\ln(a+c) - \frac{\gamma \ln(c)\ln(a+c)\ln[(a+c)/a]}{\ln[(a+c)/c]} \\
&= \gamma\left(\frac{\ln(a)\ln(a+c)\ln[(a+c)/c] - \ln(c)\ln(a+c)\ln[(a+c)/a]}{\ln[(a+c)/c]}\right) \\
&= \frac{\gamma \ln(a+c)}{\ln[(a+c)/c]} \cdot [\ln(a)\ln(a+c) - \ln(a)\ln(c) - \ln(c)\ln(a+c) + \ln(a)\ln(c)] \\
&= \frac{\gamma[\ln(a+c)]^2}{\ln[(a+c)/c]} \cdot \ln\left(\frac{a}{c}\right).
\end{aligned}
$$

Note that the sign of $g(\gamma)$ is entirely determined by the factor of $\ln(a/c)$ that appears in the last equality above. We therefore consider two cases. First, if $a > c$ holds, then $g(\gamma) > 0$, and since we know that $g(\tilde{\gamma}) = 0$ (by the definition of $\tilde{\gamma}$) and that $g'(x) > 0$ for all x, it follows that $\gamma > \tilde{\gamma}$ holds. On the other hand, if $a < c$ holds, a similar argument shows that $\tilde{\gamma} > \gamma$.

To finish the proof, we combine the observations above with Theorem 4.51 to conclude that if $a > c$, then γ is the Hausdorff dimension of N^*, while if $a < c$, then $\tilde{\gamma}$ is the Hausdorff dimension of N^*. Recalling the definitions of γ and $\tilde{\gamma}$ given in the statement of Theorem 4.51, the theorem follows. ■

5 Summary

In this thesis, we have seen that Hausdorff dimension is a useful tool in comparing the sizes of various subsets of the real numbers, particularly those which cannot be distinguished by ordinary Lebesgue measure. We discovered that the class of Cantor sets denoted by C_{ac} have varying non-integral Hausdorff dimensions between zero and one. The Cantor function associated with each Cantor set has a nondifferentiability set which is properly contained in the Cantor set that that generated it. We showed that the Hausdorff dimension of each nondifferentiability set is also a real number between zero and one, but less than or equal to the dimension of the underlying Cantor set. One question that could be asked at this point is whether or not similar observations hold for other Cantor sets and functions.

One Cantor set very similar to those described in this thesis arises from the tent mapping

$$t(x) = \begin{cases} (1/a)x & \text{for } 0 \leq x \leq a/(a+c) \\ (1/c)(1-x) & \text{for } a/(a+c) < x \leq 1. \end{cases}$$

where a and c are positive real numbers strictly less than one-half. Note that $C^1_{ac} = [0, a] \cup [1-c, 1]$ is precisely the set of points which are mapped into the interval $[0, 1]$ by $t(x)$. Thus, we can ask which subset of C^1_{ac} gets mapped into $[0, 1]$ by the iterated map $t(t(x))$. It turns out that this subset consists of the same four intervals comprising C^2_{ac}, except that the positions of the two rightmost intervals are reversed. Letting t^n denote the iterated map $t \circ t \circ \cdots \circ t$ (n times), it can be shown that t^n maps a set similar to C^n_{ac} into the interval $[0, 1]$ for each positive integer n. Continuing this process yields a Cantor set C which is precisely the set of real numbers x such that $t^n(x)$ is contained in the interval $[0, 1]$ for all positive integers n. Perhaps methods similar to those discussed in this thesis could be used to construct a Cantor function corresponding to C and to compute the Hausdorff dimension of its nondifferentiability set.

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