THESIS

THE TIZ-CORRESPONDENCE ADJUSTED FOR SYMMETRY

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ABSTRACT

THE TIZ-CORRESPONDENCE ADJUSTED FOR SYMMETRY

The TIZ-correspondence ([1], Theorem B) is a ternary Galois correspondence between generalized tensor products, polynomial ideals, and affine schemes of tensor operators. We study the TIZ-correspondence under the presence of symmetry. We provide evidence that this correspondence does not have an internal characterization of symmetry, and we propose three definitions of a generalized symmetric tensor product. For each version, we prove a variant of the TIZ-correspondence in this setting. For the last and most general version, we prove that Lie algebras naturally coordinatize these generalized symmetric tensor products. We prove that every symmetric multilinear map t has a universally smallest generalized tensor product space containing t.

We additionally survey the main results in [1]. We give proofs of theorems A-D, demonstrate each theorem with examples, and we provide explicit computations of many of the objects involved.

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Chapter 1

Introduction

Tensors appear throughout mathematics, computer science, and physics as a common generalization of distributive products, multiway arrays, systems of forms, and entangled states. Because of this versatility, tensors serve as natural containers for data across a wide range of applications. However, extracting coordinate-free information from tensors remains a significant challenge. There are simply too many tensors to study them all at once. Indeed, classifying tensors up to a change of coordinates is *wild* in the technical sense of representation theory [2].

In linear algebra, we abstract away the various interpretations of a vector by studying properties of the vector spaces they inhabit. The same perspective is useful for multilinear algebra. Instead of studying tensors through some particular model, it is useful to study tensors as a elements of a tensor space [3]. Studying these spaces of tensors turns out to be equivalent to studying transverse operators on a space and how these operators act. In some cases, these operators form algebras. As such, an existing literature [1, 4–7] has utilized ring and module theory to understand these tensor spaces.

As tensors generalize distributive products in algebra. We may sometimes generalize invariants of algebras to those of tensors. Classical invariants such as centroids, nuclei, and derivations have been long used in nonassociative algebra [8–10], and have recently been adapted to the study of tensor spaces [11, 12]. Despite these developments, there was no general theory to unify and classify the resulting algebras of tensor operators. The TIZ-correspondence introduced in [1] addresses this gap by providing a framework that systematically organizes these algebras and their actions on tensor spaces.

In many applications, the data expresses symmetry. The resulting containers for multilinear data invariant under symmetry is called a *symmetric tensor*; see figure 1 below for examples. A broad literature has studied symmetric tensors in signal processing, quantum information, statistics, and algebraic geometry [3, 13–19]. However, the collection of symmetric tensors in $(\mathbb{C}^n)^{\otimes d}$ forms an $\binom{n+d-1}{d}$-dimensional subspace. So, even after restricting our study to symmetric tensors, there still are too many symmetric tensors to study at once.

(A) GHZ state for $d = n = 3$.

(B) W state for $n = 3$ and $d = 2$.

Figure 1.1: Two examples of symmetric multi-way arrays.

Here we seek to attach new operators to a symmetric tensor and study the resulting symmetric spaces. The methods in [1] give techniques to construct and characterize spaces of tensors, however, these techniques are focused entirely on the heterogeneous case. That is, tensors are viewed as multilinear maps $U_1 \times \cdots \times U_n \rightarrowtail U_0$ of modules U_i over a commutative ring such that U_i is assumed to be isomorphic to U_j when $i \neq j$. We will see that a symmetric multilinear map requires $U_i = U_j$ for all $i, j \in \{1, \cdots, n\}$. As such, we require new tools to repeat the analysis done in [1] in this setting.

In this thesis, we develop a ternary Galois correspondence that mirrors the TIZ-correspondence ([1], Theorem B), but is tailored to symmetric tensor spaces. We present this correspondence in three ways: first, by leveraging the duality between symmetric tensors and homogeneous polynomials, second by symmetrizing each component of the TIZ-correspondence; and third, by generalizing both perspectives to yield a unified theory for symmetric tensor spaces.

1.1 Overview

In chapter 2, we survey some of the main ideas of multilinear algebra which we will be using in the thesis. We then introduce the notation we will be using throughout the rest of the thesis and state some general properties of a Galois correspondence. We conclude by deriving formulas describing an induced group action on tensors, polynomials, and operators and we prove some useful corollaries about how the objects in the TIZ-correspondence change under this action.

In chapter 3, we first show that TIZ does not inherantly capture symmetry for 2×2 matrices over $\mathbb{C}^2$. That is, there are no choices of polynomials and operators such that the resulting tensor space is exactly the space of 2×2 symmetric matrices. This partially justifies our decision to force symmetry externally on the TIZ-correspondence. We then describe our three versions of a symmetric TIZ-correspondence.

In chapter 4, we investigate the algebras in this variant of the TIZ-correspondence. We observe that the centroid and derivation algebras from [1] are still present in version III of this symmetric correspondence. We then that the universal property of the *densor* space still holds in this setting and that Lie algebras are *most* algebras which occur when the operator sets form an algebra. We conclude this chapter by characterizing the associative algebras which can arise as a set of tensor operators under this symmetric correspondence.

We will additionally survey the main results and techniques from [1] which we will be adapting to a symmetric setting. We will provide an overview of theorems A-D, present proofs of each, and supplement with examples when applicable. We do this for two reasons. First, to give an intuitive sketch of many of the results. Second, to emphasize what we would need to modify these results for symmetric tensors. Indeed, theorems A-D are statements about *all* tensors/polynomials/operators and we would like to modify these statements with the appropriate use of the adjective "symmetric".

Chapter 2

Preliminaries

2.1 Tensor Preliminaries

Let k be a field, and let $U_0, \cdots, U_n$ be finite dimensional k-vector spaces with dimensions $d_0, \cdots, d_n$. We will use $\rightarrowtail$ to indicate that a function is multilinear.

Definition 2.1.1. A *tensor* is a multilinear map $\langle t \mid : U_1 \times \cdots \times U_n \rightarrowtail U_0$. For vectors $u_i \in U_i$ let $\langle t \mid u_1, \cdots, u_n \rangle$ denote evaluation of $\langle t \mid$ on the inputs $u_1, \cdots, u_n$. When U_0 is one-dimensional, we call $\langle t \mid$ a multilinear *form*.

Part of the beauty and challenge of studying tensors is that the same tensor can have many different equivalent interpretations. We detail some of the main interpretations we will use here. Given a tensor $\langle t \mid$, we choose a basis $\{e_{i,j} \mid 1 \leq j \leq d_i\}$ for each U_i. Then we may represent t as a $d_1 \times \cdots \times d_n \times d_0$ array with the $(i_1, \cdots, i_n, i_0)$ entry given by the e_{0,i_0} component of $\langle t \mid e_{1,i_1}, \cdots, e_{n,i_n} \rangle$.

Example 2.1.2. Consider the W state from figure 1. Choose the standard basis e_1, e_2 for $\mathbb{C}^2$. As a bilinear map,

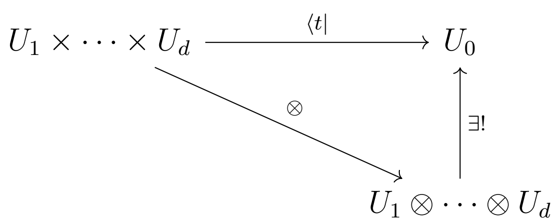
$$
\begin{aligned}
\langle W \mid : \mathbb{C}^2 \times \mathbb{C}^2 &\to \mathbb{C}^2 \\
(e_1, e_1) &\mapsto e_2 \\
(e_1, e_2) &\mapsto e_1 \\
(e_2, e_1) &\mapsto e_1 \\
(e_2, e_2) &\mapsto 0.
\end{aligned}
$$

Then the corresponding multi-way array is a $2 \times 2 \times 2$ array with a 1 in entries $(1, 1, 2)$, $(1, 2, 1)$, and $(2, 1, 1)$. This recovers the depiction of W from figure 1.

Given k-vector spaces U_i as above, define the space $U_1 \otimes \cdots \otimes U_n$, as the k-vector space spanned by vectors of the form $u_1 \otimes \cdots \otimes u_n$ where $u_i \in U_i$ for all $i \in \{1, \cdots, n\}$. Such vectors are called *pure* (or *rank* 1) tensors. This space may also be defined as the image of the multilinear map

$$\otimes : U_1 \times \cdots \times U_n \rightarrowtail U_1 \otimes \cdots \otimes U_n$$
$$(u_1, \cdots, u_n) \mapsto (u_1 \otimes \cdots \otimes u_n).$$

The universal property of the tensor product claims that for any tensor $\langle t \mid : U_1 \times \cdots \times U_n \rightarrowtail U_0$, there exists a unique linear map $U_1 \otimes \cdots \otimes U_n \to U_0$ such that the following diagram commutes.



In fact, we may repeat this construction in the more general setting of each U_i being modules over a commutative ring R. In this setting, the tensor product is called the Whitney tensor product over R and is denoted $\otimes_R$. The symbol $\otimes$ without any subscripts is assumed to be the tensor product over k.

With linear maps $f : U \to V$, we may dualize V to view f as a bilinear map $U \times V^* \rightarrowtail k$. Then this factors through a linear map $U \otimes V^*$ which is an element of the dual space $(U \otimes V^*)^* \cong U^* \otimes V$. Since every finite-dimensional vector space is isomorphic to its dual, we may identify the U^* with U. All in all, we may convert linear maps $U \to V$ into elements of the space $U \otimes V$. Repeating this logic with $U = U_1 \otimes \cdots \otimes U_n$ and $V = U_0$, we conclude that we may write the tensor $\langle t \mid$ as an element of the space $U_1 \otimes \cdots \otimes U_n \otimes U_0 \cong U_0 \otimes \cdots \otimes U_n$. When we are interpreting a tensor this way, we will replace $\langle t \mid$ with t.

Example 2.1.3. Consider again the W state as a bilinear map $\langle W \mid : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ from example 2.1.2. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. By the universal property of the tensor product, we may view $\langle W \mid$ as a linear map

$$\begin{aligned}\langle W \mid : \mathbb{C}^2 \otimes \mathbb{C}^2 &\to \mathbb{C}^2 \\ e_1 \otimes e_1 &\mapsto e_2 \\ e_1 \otimes e_2 &\mapsto e_1 \\ e_2 \otimes e_1 &\mapsto e_1 \\ e_2 \otimes e_2 &\mapsto 0.\end{aligned}$$

As an element of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the W state has the form

$$W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1.$$

Our last interpretation for tensors utilizes the following fact, known as *Tensor-Hom Adjunction*. For vector spaces U and V, let $Hom(U, V)$ denote the k-vector space of linear maps $U \to V$. Then,

Theorem 2.1.4.

$$Hom(U \otimes V, W) \cong Hom(U, Hom(V, W)).$$

In particular, we may interpret a bilinear map $U_1 \times U_2 \rightarrowtail U_0$ as a function $U_1 \to Hom(U_2, U_0)$. Let $U \oslash V \coloneqq Hom(U, V)$. Iterating this process means we may view a tensor $\langle t \mid$ as an element of $U_1 \oslash \cdots \oslash U_n \oslash U_0$.

Example 2.1.5. Considering again the W state from figure 1, the corresponding function becomes,

$$\langle W | : \mathbb{C}^2 \to Hom(\mathbb{C}^2, \mathbb{C}^2)$$
$$e_1 \mapsto \begin{pmatrix} e_1 \mapsto e_2 \\ e_2 \mapsto e_1 \end{pmatrix}$$
$$e_2 \mapsto \begin{pmatrix} e_1 \mapsto e_1 \\ e_2 \mapsto 0 \end{pmatrix}$$

Observation 2.1.6. In conclusion, a tensor may be interpreted as any of

1. A multilinear map $U_1 \times \cdots \times U_n \rightarrowtail U_0$
2. A multi-way array of entries in k
3. A linear map $U_1 \otimes \cdots \otimes U_n \to U_0$
4. An element of $U_0 \otimes \cdots \otimes U_n$
5. An element of $U_0 \oslash \cdots \oslash U_n$

Linear algebra abstracts away the various interpretations of a vector by considering vectors as elements of a vector space. In multilinear algebra, it is often helpful to do the same.

Definition 2.1.7. A *tensor space* is a k-vector space U with an injective k-linear map $U \hookrightarrow U_0 \otimes \cdots \otimes U_n$, called the interpretation map.

Note that the existence of an interpretation map gives a map to any other interpretation we may wish to consider. Since we are focused on algebraic properties of tensor spaces, we will use an interpertation suited for algebraic purposes. For k-vector spaces $U_0, \cdots, U_n$, let $\{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$ denote the space of k-multilinear maps on these spaces. Note that this is a tensor space by definition 2.1.7.

Up to this point, we have made no assumptions on the dimension of any particular U_i. This is typically referred to as a *heterogeneous* setting. If we assume that all spaces U_i have the same dimension (then $U_i \cong U_j \cong V$ for all i, j and for some vector space V), then we are allowed an additional interpretation, namely that of the multplication table for an algebra.

Let A be an algebra and consider momentarily the multiplication map $* : A \times A \rightarrowtail A$. Ring axioms require that $*$ distributes so we conclude that $*$ is a tensor. Writing this tensor as a multi-way array will yield the multiplcation table of A.

Example 2.1.8. Let $A = {}^{\mathbb{C}[x]}\!/\!_{x(x-1)}$ and $B = {}^{\mathbb{C}[x]}\!/\!_{x^2}$. Then the multiplication tables for A and B have the form

A	1	x
1	1	0
x	0	x

B	1	x
1	1	x
x	x	0

Thinking of 1 as the vector $\begin{bmatrix}1\\0\end{bmatrix}$ and x as the vector $\begin{bmatrix}0\\1\end{bmatrix}$, we turn this matrix of linear forms into a 3-way array.

$$A\colon \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) \qquad B : \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)$$

where the left matrix is the front face and the right matrix is the back face of the tensor. We see in this expression that A corresponds to the tensor $GHZ = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$ while B corresponds to the W state with the front and back faces swapped.

Now we describe symmetry in this context. Let $[n] := \{1, \cdots, n\}$ and S_n denote the symmetric group on $[n]$.

Definition 2.1.9. A tensor $\langle t \mid : U_1 \times \cdots \times U_n \rightarrowtail U_0$ is *symmetric* if $\langle t \mid u_1, \cdots, u_n\rangle = \langle t \mid u_{\sigma(1)}, \cdots, u_{\sigma(n)}\rangle$ for all $\sigma \in S_n$.

Example 2.1.10. Both GHZ and W are symmetric.

Observe that the symmetry condition forces the input axis to be isomorphic. Explicitly, in the case of bilinear forms, $\langle t \mid : U \times V \rightarrowtail k$, if $\langle t \mid u, v\rangle = \langle t \mid v, u\rangle$, then the ordering on the inputs forces that $v \in U$. When all input spaces U_i are isomorphic, we call this a *homogeneous* setting. Additionally, observe that a symmetric tensor can be interpreted in any of the ways mentioned in observation 2.1.6. As such, we will let $Sym^n(V)$ denote the subspace of symmetric tensors interpreted as elements of $(V)^{\otimes n}$ and the space of symmetric n-linear forms by $Sym^n(V^*)$.

Note that the fact we are working in a homogeneous setting does not imply that we can interpret a symmetric tensor as a multiplication table for an algebra. This is because the output space U_0 is not required to be isomorphic to any U_i. For example, $\langle t \mid : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}^3 \otimes \mathbb{R}^3$ defined by $(u, v) \mapsto \frac{1}{2} u \otimes v + v \otimes u$ is a symmetric tensor by our definition but $\mathbb{R}^3 \times \mathbb{R}^3 \ncong \mathbb{R}^3 \otimes \mathbb{R}^3$ by dimension considerations.

If $U_0 \cong U_i$ for all input spaces U_i, then we may interpret symmetric tensors as defining the multiplication of an algebra. Let A be a k-vector space with multiplication $* : A \times A \rightarrowtail A$. The property of $*$ being symmetric means that $a * b = b * a$ for all $a, b \in A$. In other words, symmetric tensors viewed as algebras correspond to commutative k-algebras.

2.2 Motivating Examples

Before defining the TIZ-correspondence precisely, we wish to demonstrate some of the key ideas and utility in the TIZ correspondence. We also provide very explicit calculations of some of the objects in this correspondence. For this section, we consider tensors as elements of the tensor space $U_0 \otimes \cdots \otimes U_n$ for finite dimensional k-vector spaces $U_0, \cdots, U_n$.

[1] was motivated by problems involving searching spaces of tensors. Spaces of tensors are usually very large, however, in many search problems, we start with a fixed collection of tensors we wish to study. With these tensors in mind, it becomes easier to throw away bad orbits and reduce to smaller search spaces. For example, the relations defining $U_0 \otimes \cdots \otimes U_n$ identify the actions of scalars on each axis. However, when we are studying specific tensors, we can broaden the notion of scalars which gives us more to quotient out.

Example 2.2.1. Consider the tensor $GHZ = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The definition of the tensor product decrees that for scalars $\lambda \in \mathbb{C}$,

$$\begin{aligned}\lambda e_1 \otimes e_1 \otimes e_1 + \lambda e_2 \otimes e_2 \otimes e_2 &= e_1 \otimes \lambda e_1 \otimes e_1 + e_2 \otimes \lambda e_2 \otimes e_2 \\ &= e_1 \otimes e_1 \otimes \lambda e_1 + e_2 \otimes e_2 \otimes \lambda e_2.\end{aligned}$$

However, as scalars are linear maps $\mathbb{C}^2 \to \mathbb{C}^2$, we may consider other functions $\mathbb{C}^2 \to \mathbb{C}^2$ which *act* as scalars on GHZ. For example, note that the operator $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ acts exactly as scalars do on GHZ. Precisely,

$$\begin{aligned}e_1 \otimes e_1 \otimes e_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_1 \otimes e_1 \otimes e_1 + \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_2 \otimes e_2 \otimes e_2 \\ = e_1 \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_1 \otimes e_1 + e_2 \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_2 \otimes e_2 \\ = e_1 \otimes e_1 \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_1 + e_2 \otimes e_2 \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} e_2\end{aligned}$$

and similarly with the operator $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$. In fact, every operator which acts as a scalar on GHZ is a linear combination of these two matrices. So we would not expect these operators to tell us any thing interesting about GHZ. So then we may wish to study tensors which are equivalent up to this broadened notion of scaling. That is, a space in which a pure tensor $u \otimes v \otimes w$ satisfies

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} u \otimes v \otimes w = u \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} v \otimes w = u \otimes v \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} w.$$

To do so, we quotient the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ by the orbits of $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ under the scaling action. That is, quotient by the space spanned by tensors of the form

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} u \otimes v \otimes w - u \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} v \otimes w, \quad u \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} v \otimes w - u \otimes v \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} w,$$
$$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} u \otimes v \otimes w - u \otimes \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} v \otimes w, \quad u \otimes \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} v \otimes w - u \otimes v \otimes \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} w$$

where $u \otimes v \otimes w$ is a pure tensor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The resulting quotient space is a 4-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ so we have created a smaller tensor space by broadening the notion of scalars. Note that $\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\}$ span an associative commutative unital matrix algebra which acts naturally on $\mathbb{C}^2$. Labelling this algebra $Cen(GHZ)$, we see that the quotient space resulting from this computation is $\mathbb{C}^2 \otimes_{Cen(GHZ)} \mathbb{C}^2 \otimes_{Cen(GHZ)} \mathbb{C}^2$.

We may also create compact tensor spaces by tensoring over other algebras. These other algebras and their actions on a tensor space highlight different properties than centroids do. We first illustrate this in the case of matrices. The generalization to tensors is proved in 2.5.6 and computed explicitly for GHZ and W in examples 2.3.12 and 2.3.13.

Given a matrix $M \in Mat_{n\times m}$, we interpret M as a bilinear form $\langle M \mid : k^n \times k^m \rightarrowtail k$ via $\langle M \mid u, v\rangle = u^\intercal M v$. Then we may think of a subset $S \subseteq Mat_{n\times m}$ as a system of bilinear forms.

For some subset $\Omega \subseteq Mat_{n\times n} \times Mat_{m\times m}$ of operators, we could create a subspace of $Mat_{n\times m} \cong (k^n \otimes k^m)^*$ by "tensoring" over Ω. That is, $(k^n \otimes_\Omega k^m)^* = \{M \in Mat_{n\times m} : \omega_1^\intercal M = M\omega_2 \, (\forall(\omega_1, \omega_2) \in \Omega)\}$. Observe that additivity is preserved as $\omega_1^\intercal(M + M') = (M + M')\omega_2 = \omega_1^\intercal M + M\omega_2 + \omega_1^\intercal M' + M'\omega_2$. Also note that if $\Omega = R$ for some finite-dimensional commutative k-algebra R, then this is the definition of the Whitney tensor product $k^n \otimes_R k^m$.

Dually, from a system of bilinear forms S, we can construct an algebra called the *adjoint algebra* defined as,

$$Adj(S) \coloneqq \{(\omega_1.\omega_2) \in Mat_{n\times n} \times Mat_{m\times m} : \omega_1^\intercal M = M\omega_2 \;\forall M \in S\}.$$

Note that, for $(\omega_1, \omega_2), (\widetilde{\omega_1}, \widetilde{\omega_2}) \in Adj(S)$ and $M \in S$,

$$(\omega_1\widetilde{\omega_1})^\intercal M = \widetilde{\omega_1}^\intercal \omega_1^\intercal M = \widetilde{\omega_1}^\intercal M\omega_2 = M\widetilde{\omega_2}\omega_2.$$

So the pair $(\omega_1\widetilde{\omega_1}, \omega_2\widetilde{\omega_2}) \notin Adj(S)$ if we view $Adj(S)$ as an associative subalgebra of $End(k^a) \times End(k^b)$. However, $(\omega_1\widetilde{\omega_1}, \omega_2\widetilde{\omega_2}) \in Adj(S)$ if we view $Adj(S)$ as an associative subalgebra of $End(k^a) \times End(k^b)^{op}$. These are related by the following correspondence between sets of tensors and operators [20],

$$S \subseteq (k^n \otimes_\Omega k^m) \iff \Omega \subseteq Adj(S)$$

Example 2.2.2. Let $S = \{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\} \subset Mat_{2\times 2}(\mathbb{C})$. Then we may compute $Adj(S)$ by setting $(X, Y) = (\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix})$ and solving the following system of linear equations,

$$X^\intercal \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Y$$

. After joining X and Y diagonally, this results in the four-dimensional algebra,

$$Adj(S) = \left\langle \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & -1 \\ & & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 0 \end{array}\right] \right\rangle$$

Choosing $\Omega = \{\left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & 0 & 1 \end{array}\right]\}$ so that $\Omega \subseteq Adj(S)$, we compute $\mathbb{C}^2 \otimes_\Omega \mathbb{C}^2$ by setting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and solving following the system of linear equations,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^\intercal M = M \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This forces $a = d = 0$ so we get the tensor space,

$$\mathbb{C}^2 \otimes_\Omega \mathbb{C}^2 = \{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} : b, c \in \mathbb{C}\} = \Big\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Big\rangle.$$

Then we see that $S \subseteq \mathbb{C}^2 \otimes_\Omega \mathbb{C}^2$. Similarly, if we instead started with a system $S \subseteq \mathbb{C}^2 \otimes_\Omega \mathbb{C}^2$ for some Ω, we would see that $\Omega \subseteq Adj(S)$.

Now because we have a Galois connection, composition gives a natural closure operation. In the case of adjoint algebras, the closed sets are,

$$\overline{\Omega} = Adj((k^n \otimes_\Omega k^m)^*) \quad \& \quad \overline{S} = (k^n \otimes_{Adj(S)} k^m)^*.$$

These closed sets can still be computed by solving a system of linear equations.

Example 2.2.3. Continuing example 2.2.2, let us compute the closed sets $\overline{\Omega}$ and $\overline{S}$.

$$\overline{\Omega} = Adj(\mathbb{C}^2 \otimes_\Omega \mathbb{C}^2) = Adj(\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \rangle)$$

$$\overline{S} = \mathbb{C}^2 \otimes_{Adj(S)} \mathbb{C}^2$$

As above, let $(X,Y) = (\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix})$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we compute $\overline{\Omega}$ and $\overline{S}$ by solving the following systems of linear equations,

$$\begin{aligned} \overline{\Omega}: \quad & X^{\intercal} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y, \; X^{\intercal} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} Y \\ \overline{S}: \quad & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{\intercal} M = M \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^{\intercal} M = M \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{\intercal} M = M \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{\intercal} M = M \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

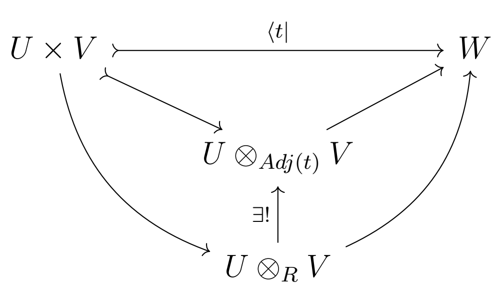
Then $\overline{\Omega}$ is two-dimensional and $\overline{S}$ is one-dimensional. Explicitly, after diagonally joining X and Y, we get the following two expressions for $\overline{\Omega}$ and $\overline{S}$,

$$\overline{\Omega} = \left\langle \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right] \right\rangle$$

$$\overline{S} = \langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle.$$

Indeed, the tensor product satisfies the following universal property.

Theorem 2.2.4. *([5]) For all bilinear maps $\langle t \mid : U \times V \rightarrowtail W$, there exists an algebra $Adj(t)$ such that such that $\langle t \mid$ factors through $U \otimes_{Adj(t)} V$. Furthermore, if U, V are modules over a commutative ring R, then there exists a a unique map $R \to Adj(t)$ and a unique map $U \otimes_R V \to U \otimes_{Adj(t)} V$ making the following diagram commute.*



In this case, $Adj(t)$ is a unital associative (but not necessarily commutative) subalgebra of $End(U) \times End(V)^{op}$ for which we were able to build compressed tensor spaces. We will soon see a way of constructing similar tensor products over algebras which may not be associative, commutative, or unital. Note that if $k = 2k$ and $\langle t \mid : V \times V \rightarrowtail W$ for k-vector spaces U, V, W , $Adj(t)$ is closed to the Jordan product $X \bullet Y = \frac{1}{2}(XY + YX)$. So we may interpret this theorem as "tensoring" over a Jordan subalgebra of $End(V)^2$. Despite the many different kinds of algebras which can appear, we will see that there is a universal choice. That is, we will see that we obtain a universally compressed tensor space by "tensoring" over a Lie algebra instead.

2.3 The TIZ Correspondence

The Galois connection seen in the adjoint algebra is one of many for tensors [4]. The TIZ-correspondence was developed to unify and classify these algebras as well as their Galois connections arising throughout the literature. This is a ternary Galois correspondence describing the relationship between these generalized tensor spaces, *transverse* operators on a tensor space, and polynomial ideals prescribing the way these operators act.

2.3.1 Characterization via the Whitney Tensor Product

Precisely, for k-vector spaces $U_0, \cdots, U_n$, a transverse operator on $U_0 \otimes \cdots \otimes U_n$ is an $(n+1)$-tuple of operators $\omega \in End(U_0) \times \cdots \times End(U_n)$. Choosing a polynomial $p := \sum_e \lambda_e X^e \in k[x_0, \cdots, x_n]$ and a transverse operator $\omega := (X_0, \cdots, X_n)$ prescribes a map $p(\omega) : U_0 \otimes \cdots \otimes U_n \to U_0 \otimes \cdots \otimes U_n$ defined on pure tensors via

$$p(\omega)(u_0 \otimes \cdots \otimes u_n) = \sum_e \lambda_e X_0^{e_0} u_0 \otimes \cdots \otimes X_n^{e_n} u_n$$

and extended linearly to a map on $U_0 \otimes \cdots \otimes U_n$. We will sometimes refer to this as the TIZ action.

Example 2.3.1. Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \end{bmatrix}$, $p = x - y$, and $\omega = (\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix})$. Then,

$$p(\omega)(M) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^{\intercal} M - M \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Given arbitrary subsets $S \subseteq U_0 \otimes \cdots \otimes U_n$, $P \subseteq k[x_0, \cdots, x_n]$, and $\Omega \subseteq End(U_0) \times \cdots \times End(U_n)$, consider three sets defined by the TIZ action, denoted "$T/I/Z$-sets".

Definition 2.3.2.

$$T(P, \Omega) \coloneqq \{t \in U_0 \otimes \cdots \otimes U_n \mid p(\omega)(t) = 0,\ \forall p \in P, \omega \in \Omega\}$$
$$I(S, \Omega) \coloneqq \{p \in k[x_0, \cdots, x_n] \mid p(\omega)(t) = 0,\ \forall t \in S, \omega \in \Omega\}$$
$$Z(S, P) \coloneqq \{\omega \in End(U_0) \times \cdots \times End(U_n) \mid p(\omega)(t) = 0,\ \forall t \in S, p \in P\}$$

Example 2.3.3. In example 2.2.1, we studied scalars on GHZ and claimed these scalars form an associative commutative unital matrix algebra $Cen(GHZ)$. Let us compute the scalars for the tensor $W \coloneqq e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ but now using the language of $T/I/Z$-sets. In this setting, scalars are triples $(X, Y, Z) \in End(\mathbb{C}^2) \times End(\mathbb{C}^2) \times End(\mathbb{C}^2)$ such that

$$\begin{aligned} Xe_1 \otimes e_1 \otimes e_2 + Xe_1 \otimes e_2 \otimes e_1 + Xe_2 \otimes e_1 \otimes e_1 &= e_1 \otimes Ye_1 \otimes e_2 + e_1 \otimes Ye_2 \otimes e_1 + e_2 \otimes Ye_1 \otimes e_1 \\ &= e_1 \otimes e_1 \otimes Ze_2 + e_1 \otimes e_2 \otimes Ze_1 + e_2 \otimes e_1 \otimes Ze_1. \end{aligned}$$

In other words, we are computing the operators $\omega = (X, Y, Z)$ such that $(x - y)(\omega)(W) = (y - z)(\omega)(W) = 0$. This is, we are computing the Z-set, $Z(\{W\}, \{x - y, y - z\})$. Letting

$$(X, Y, Z) \coloneqq (\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix})$$

we have that,

$$
\begin{aligned}
(x-y)(\omega)(W) &= (Xe_1 \otimes e_1 \otimes e_2 + Xe_1 \otimes e_2 \otimes e_1 + Xe_2 \otimes e_1 \otimes e_1) \\
&\quad - (e_1 \otimes Ye_1 \otimes e_2 + e_1 \otimes Ye_2 \otimes e_1 + e_2 \otimes Ye_1 \otimes e_1) \\
&= (x_{11}e_1 + x_{21}e_2) \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) + (x_{12}e_1 + x_{22}e_2) \otimes e_1 \otimes e_1 \\
&\quad - (e_1 \otimes (y_{11}e_1 + y_{21}e_2) \otimes e_2 + e_1 \otimes (y_{12}e_1 + y_{22}e_2) \otimes e_1 + e_2 \otimes (y_{11}e_1 + y_{21}e_2) \otimes e_1) \\
(y-z)(\omega)(W) &= (e_1 \otimes Ye_1 \otimes e_2 + e_1 \otimes Ye_2 \otimes e_1 + e_2 \otimes Ye_1 \otimes e_1) \\
&\quad - (e_1 \otimes e_1 \otimes Ze_2 + e_1 \otimes e_2 \otimes Ze_1 + e_2 \otimes e_1 \otimes Ze_1) \\
&= (e_1 \otimes (y_{11}e_1 + y_{21}e_2) \otimes e_2 + e_1 \otimes (y_{12}e_1 + y_{22}e_2) \otimes e_1 + e_2 \otimes (y_{11}e_1 + y_{21}e_2) \otimes e_1) \\
&\quad - (e_1 \otimes e_1 \otimes (z_{12}e_1 + z_{22}e_2) + e_1 \otimes e_2 \otimes (z_{11}e_1 + z_{21}e_2) + e_2 \otimes e_1 \otimes (z_{11}e_1 + z_{21}e_2)) \\
&= 0.
\end{aligned}
$$

Which yields the linear systems of equations,

$$
\begin{aligned}
&(e_1 \otimes e_1 \otimes e_1): && x_{12} - y_{12} && y_{12} - z_{12} \\
&(e_1 \otimes e_1 \otimes e_2): && x_{11} - y_{11} && y_{11} - z_{22} \\
&(e_1 \otimes e_2 \otimes e_1): && x_{11} - y_{22} && y_{22} - z_{11} \\
&(e_1 \otimes e_2 \otimes e_2): && y_{21} = 0 && y_{21} - z_{21} \\
&(e_2 \otimes e_1 \otimes e_1): && x_{22} - y_{11} && y_{11} - z_{11} \\
&(e_2 \otimes e_1 \otimes e_2): && x_{21} = 0 && z_{21} = 0 \\
&(e_2 \otimes e_2 \otimes e_1): && x_{21} - y_{21} && y_{21} = 0 \\
&(e_2 \otimes e_2 \otimes e_2): && y_{21} = 0 && y_{21} = 0.
\end{aligned}
$$

After reducing this system, we can see that ω has the form,

$$\begin{aligned}\omega &:= (\begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{11} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{11} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{11} \end{bmatrix}) \\ &= (\begin{bmatrix} x_{11} & 0 \\ 0 & x_{11} \end{bmatrix}, \begin{bmatrix} x_{11} & 0 \\ 0 & x_{11} \end{bmatrix}, \begin{bmatrix} x_{11} & 0 \\ 0 & x_{11} \end{bmatrix}) + (\begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}).\end{aligned}$$

We conclude that

$$Cen(W) := Z(\{W\}, \{x - y, y - z\}) = Span\{(I_2, I_2, I_2), (\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})\}.$$

We now make a few observations about $T/I/Z$-sets in general. Observe that if $p, p' \in k[x_0, \cdots, x_n]$,

$$(p + p')(\omega)(u_0 \otimes \cdots \otimes u_n) = (p)(\omega)(u_0 \otimes \cdots \otimes u_n) + (p')(\omega)(u_0 \otimes \cdots \otimes u_n) \tag{2.1}$$

$$pp'(\omega)(u_0 \otimes \cdots \otimes u_n) = p(p'(\omega)(u_0 \otimes \cdots \otimes u_n)) \tag{2.2}$$

so a choice of transverse operator ω gives a ring action of $k[x_0, \cdots, x_n]$ on $U_0 \otimes \cdots \otimes U_n$. This means that we may replace P in the definitions of T and Z-sets with the ideal (P) generated by P.

Similarly, for $t, t' \in U_0 \otimes \cdots \otimes U_n$

$$p(\omega)(t + t') = p(\omega)(t) + p(\omega)(t') \tag{2.3}$$

so we may replace S in the definitions of I and Z-sets with the subspace $\langle S \rangle$ spanned by S.

Unless p is linear, the map $p(\omega)$ will rarely be additive. For example, if $p = x^2 + y^2$, $\omega = (X, Y), \widetilde{\omega} = (\widetilde{X}, \widetilde{Y})$, and $u \otimes v$ is a pure tensor, then

$$\begin{aligned} p(\omega + \widetilde{\omega})(u \otimes v) &= (X + \widetilde{X})^2 u \otimes (Y + \widetilde{Y})^2 v \\ &\neq X^2 u \otimes Y^2 v + \widetilde{X}^2 u \otimes \widetilde{Y}^2 v. \end{aligned}$$

In general, Z-sets will not be closed under any kind of algebraic operations. However, we will see that Z-sets are defined by finitely many polynomial equations. In general, every T-set is a vector subspace of $U_0 \otimes \cdots \otimes U_n$, every I-set is an ideal of $k[x_0, \cdots, x_n]$, and every Z-set is an affine scheme over k ([1], Theorem B). Additionally, if $t \in T(P, \Omega)$ for some subsets P, Ω, then by definition $p(\omega)(t) = 0$ for every $p \in P$ and $\omega \in \Omega$. Then we notice that $P \subseteq I(t, \Omega)$ and $\Omega \subseteq Z(t, P)$. Extending this reasoning to subsets $S \subseteq T(P, \Omega)$ demonstrates that

$$S \subseteq T(P, \Omega) \iff P \subseteq I(S, \Omega) \iff \Omega \subseteq Z(S, P)$$

This is exactly the statement of a *Galois correspondence*. In a Galois correspondence, there are natural closure operations from composition. For a set S, let 2^S denote the power set of S. Then we may view the set $I(S, \Omega)$ as a function $I(-, \Omega) : 2^{U_0 \otimes \cdots \otimes U_n} \to 2^{k[x_0, \cdots, x_n]}$ which sends a subset S to $I(S, \Omega)$. Similarly viewing T-sets as functions $T(-, \Omega)$, the composition $T(-, \Omega) \circ I(S, \Omega) : 2^{U_0 \otimes \cdots \otimes U_n} \to 2^{U_0 \otimes \cdots \otimes U_n}$. We will see in section 2.4that functions like these produce closed sets in any Galois correspondence.

Example 2.3.4. In our notation, the centroid of a tensor t can be defined as a Z-set by $Cen(t) = Z(\{t\}, \{x - y, y - z\})$. In example 2.2.1, we computed $Cen(GHZ)$ and in example 2.3.3, we computed $Cen(W)$. We now compute the closed T-sets, $T(\{x - y, y -$

$z\}, Cen(GHZ))$ and $T(\{x-y, y-z\}, Cen(W))$. First recall that

$$Cen(GHZ) = Span\{(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})\}$$
$$Cen(W) = Span\{(I_2, I_2, I_2), (\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix})\}.$$

$T(\{x-y, y-z\}, Cen(GHZ))$ is defined to be the set of tensors in $t \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ such that $(x-y)(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix})(t) = 0$ and $(x-y)(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})(t) = 0$ this is exactly the quotient space described at the end of example 2.2.1. Applying the same reasoning to the case of W, we see that the set $T(\{x-y, y-z\}, Cen(W))$ is the space

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \big/ \Xi(\{x-y, y-z\}, Cen(W))$$

where $\Xi(\{x-y, y-z\}, Cen(W))$ is the subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by tensors of the form,

$$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} u \otimes v \otimes w - u \otimes \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} v \otimes w, \quad u \otimes \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} v \otimes w - u \otimes v \otimes \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} w$$

where $u \otimes v \otimes w$ is a pure tensor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. In conjunction with example 2.2.1, we conclude

$$T(\{x-y, y-z\}, Cen(GHZ)) = \mathbb{C}^2 \otimes_{Cen(GHZ)} \mathbb{C}^2 \otimes_{Cen(GHZ)} \mathbb{C}^2$$
$$T(\{x-y, y-z\}, Cen(W)) = \mathbb{C}^2 \otimes_{Cen(W)} \mathbb{C}^2 \otimes_{Cen(W)} \mathbb{C}^2.$$

In general, closed sets will have the form $T(P, Z(S, P))$, $T(I(S, \Omega), \Omega)$, $I(T(P, \Omega), \Omega)$, $I(S, Z(S, P))$, $I(T(P, \Omega), \Omega)$, or $Z(S, I(S, \Omega))$. In conclusion, the TIZ-correspondence takes the following form,

Theorem 2.3.5. *([1], Theorem B) Let $S \subseteq U_0 \otimes \cdots \otimes U_n$, $P \subseteq k[x_0, \cdots, x_n]$, and $\Omega \subseteq End(U_0) \times \cdots \times End(U_n)$. Then,*

1. *$T(P,\Omega)$ is a k-vector subspace of $U_0 \otimes \cdots \otimes U_n$, $I(S,\Omega)$ is an ideal of $k[x_0,\cdots,x_n]$, and $Z(S,P)$ is the zero locus of a finitely generated ideal of polynomials unique to S and P.*

2. *The collection $T/I/Z$-sets form an inclusion-reversing ternary Galois correspondence. Precisely,*

$$S \subseteq T(P,\Omega) \iff P \subseteq I(S,\Omega) \iff \Omega \subseteq Z(S,P).$$

The sets $T(P,\Omega)$ define a generalization of the Whitney tensor product. Indeed, for subsets $P \subseteq k[X]$ and $\Omega \subseteq E_k[U_1,\cdots,U_n]$, define $\Xi(P,\Omega)$ to be the subspace spanned by the orbits of pure tensors under all p,ω combinations. That is,

$$\Xi(P,\Omega) \coloneqq \langle p(\omega)(u_1 \otimes \cdots \otimes u_n) : p \in P, \omega \in \Omega, u_i \in U_i \rangle.$$

Then we may define the following generalized tensor product $◖ \ldots ◗_\Omega^P$,

$$\begin{aligned} ◖ U_1,\cdots,U_n ◗_\Omega^P &= \{t \in U_1 \otimes \cdots \otimes U_n : p(\omega)(t) = 0,\ (\forall p \in k[X], \omega \in \Omega)\} \\ &= U_1 \otimes \cdots \otimes U_n \diagup \Xi(P,\Omega) \end{aligned}$$

The space $◖ U_1,\cdots,U_n ◗_\Omega^P$ comes equipped with a multilinear map $◖ \ldots ◗_\Omega^P : U_1 \times \cdots \times U_n \rightarrowtail ◖ U_1,\cdots,U_n ◗_\Omega^P$ defined by,

$$\begin{aligned} ◖ \ldots ◗_\Omega^P : & U_1 \times \cdots \times U_n \rightarrowtail ◖ U_1,\cdots,U_n ◗_\Omega^P \\ & (u_1,\cdots,u_n) \mapsto u_1 \otimes \cdots \otimes u_n + \Xi(P,\Omega). \end{aligned}$$

In this sense, if $\langle t | : U_1 \times \cdots \times U_n \rightarrowtail U_0$ is a tensor such that $t \in T(P,\Omega)$ for some subsets P,Ω, then we may identify $\langle t |$ with a linear map $◖ U_1,\cdots,U_n ◗_\Omega^P \to U_0$. So then $\langle t |$ factors through $◖ U_1,\cdots,U_n ◗_\Omega^P$. In this way, we identify the T-set $T(P,\Omega)$ with the space of linear maps $\lbag U_1, \cdots, U_n \rbag_\Omega^P \oslash U_0$. Note that in the case of forms, U_0 is one-dimensional and so $T(P,\Omega) = \lbag U_1, \cdots, U_n \rbag_\Omega^P$.

2.3.2 TIZ Correspondence in General

Let $k[X] \coloneqq k[x_0, \cdots, x_n]$ and $E_k[U_0, \cdots, U_n] \coloneqq End(U_0) \times \cdots \times End(U_n)$. As in the previous section, the combination of a polynomial $p \in k[X]$, and a transverse operator $\omega \in E_k[U_0, \cdots, U_n]$, define a linear endomorphism $p(\omega)$ on $\{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$. Let $\langle t \mid p(\omega) \mid$ be the image $\langle t \mid$ under this map. Explicitly, if $p = \sum_e \lambda_e x_0^{e_0} \cdots x_n^{e_n}$ and $\omega = (\omega_0, \cdots, \omega_n)$, then,

$$\langle t \mid p(\omega) \mid u_1, \cdots, u_n\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t \mid \omega_1^{e_1} u_1, \cdots, \omega_n^{e_n} u_n\rangle$$

Example 2.3.6. Consider the W state interperted as a multilinear map as in example 2.1.2. Let $p = xy + z$ and $\omega = (\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$. Then on the input (e_1, e_1),

$$\begin{aligned} \langle W \mid p(\omega) \mid e_1, e_1\rangle &= \langle W \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1\rangle + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid e_1, e_1\rangle \\ &= \langle W \mid e_2, e_2\rangle + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2 \\ &= 0 + e_1. \end{aligned}$$

Throughout this thesis, we study spaces of tensors, polynomials, and transverse operators characterized by the relation $\langle t \mid p(\omega) = 0$. When working with tensors, polynomials, and operators, we only change the objects we are quantifying over. In particular, the $T/I/Z$-sets from definition 2.3.2 become,

Definition 2.3.7. Let subset $S \subseteq \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$, $P \subseteq k[X]$, and $\Omega \subseteq E_k[U_0, \cdots, U_n]$ be subsets.

$$T(P,\Omega) \coloneqq \{\langle t \mid : U_1 \times \cdots \times U_n \rightarrowtail U_0 : \langle t \mid p(w) \mid u_1, \cdots, u_n\rangle = 0,\ (\forall p \in P, \omega \in \Omega, u_i \in U_i)\}$$
$$I(S,\Omega) \coloneqq \{p \in k[X] : \langle t \mid p(w) \mid u_1, \cdots, u_n\rangle = 0,\ (\forall t \in S, \omega \in \Omega, u_i \in U_i)\}$$
$$Z(S,P) \coloneqq \{\omega \in \Omega : \langle t \mid p(w) \mid u_1, \cdots, u_n\rangle = 0,\ (\forall t \in S, p \in k[X], u_i \in U_i)\}.$$

Observation 2.3.8. As in the previous section, note the following formulas,

$$\langle t + t' \mid p(\omega) \mid u_1, \cdots, u_n\rangle = \langle t \mid p(\omega) \mid u_1, \cdots, u_n\rangle + \langle t' \mid p(\omega) \mid u_1, \cdots, u_n\rangle \tag{2.4}$$
$$\langle t \mid p + p'(\omega) \mid u_1, \cdots, u_n\rangle = \langle t \mid p(\omega) \mid u_1, \cdots, u_n\rangle + \langle t \mid p'(\omega) \mid u_1, \cdots, u_n\rangle \tag{2.5}$$
$$\langle t \mid (pp')(\omega) \mid u_1, \cdots, u_n\rangle = \langle t \mid p(p'(\omega)) \mid u_1, \cdots, u_n\rangle \tag{2.6}$$

In particular, we may replace each instance of S in definition 2.3.7 with the vector space $\langle S\rangle$ spanned by S and we may replace P with the ideal (P) of $k[X]$ generated by P. As in the previous section, we can **not** in general replace Ω with the linear subspace of $E_k[U_0, \cdots, U_n]$ spanned by Ω. Additionally, if $P \subseteq P'$, then $T(P,\Omega) \supseteq T(P',\Omega)$ and similarly for S and Ω in T/Z-sets. Finally, as each set is defined by the relation $\langle t \mid p(\omega) = 0$, we have a ternary Galois correspondence. Precisely,

$$S \subseteq T(P,\Omega) \iff P \subseteq I(S,\Omega) \iff \Omega \subseteq Z(S,P).$$

It will become useful later to study $T/I/Z$-sets under scalar extensions to commutative k-algebras L. We change the base ring of a k-vector space U with the map $U \to L \otimes_k U$

defined by sending $u \in U$ to $\mathbb{1}_L \otimes u$. Applying this map componentwise gives a map

$$\begin{aligned} U_0 \otimes \cdots \otimes U_n \to (L \otimes U_0) \otimes \cdots \otimes (L \otimes U_n) &\cong L \otimes (U_0 \otimes \cdots \otimes U_n) \\ &\cong L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\}). \end{aligned}$$

We may similarly change the base ring of the polynomials and operators with the maps $k[X] \to L[X] \coloneqq L \otimes k[X]$ and $E_k[U_0, \cdots, U_n] \to E_L[U_0, \cdots, U_n] \coloneqq End_L(L \otimes U_0) \times \cdots \times End_L(L \otimes U_n)$. We define the sets $T(P, \Omega)_L$, $I(S, \Omega)_L$, and $Z(S, P)_L$ as the images of $T(P, \Omega)$, $I(S, \Omega)$, and $Z(S, P)$ under these maps. With these extensions in mind, we may now state and prove the most general version of the TIZ-correspondence which we will use. We defer the proof until section 2.5 to focus on examples first.

Theorem 2.3.9. *([1], Theorem B) Let $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0, \cdots, U_n]$. Then,*

1. *$T(P, \Omega)_L$ is an L-submodule of $L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $I(S, \Omega)_L$ is an ideal of $L[X]$, and $Z(S, P)_L$ is the zero locus of a finitely generated ideal of polynomials unique to S and P.*

2. *The collection $T/I/Z$-sets form an inclusion-reversing ternary Galois correspondence. Precisely,*

$$S \subseteq T(P, \Omega)_L \iff P \subseteq I(S, \Omega)_L \iff \Omega \subseteq Z(S, P)_L.$$

2.3.3 Important Examples: Derivations of GHZ and W

For a distributive product $* : A \times A \rightarrowtail A$ on a k-algebra A, a derivation of $*$ is a k-linear map $D : A \to A$ such that, for all $a, b \in A$,

$$D(a * b) = D(a) * b + a * D(b).$$

For tensors $\langle t \mid : U \times V \rightarrowtail W$, define a derivation of t as a triple of k-linear maps $(X, Y, Z) \in End(U) \times End(V) \times End(W)$ such that, for all $(u, v) \in U \times V$,

$$Z(\langle t \mid u, v\rangle) = \langle t \mid Xu, v\rangle + \langle t \mid u, Yv\rangle.$$

This is the Z-set $Z(t, (x + y - z)) =: Der(t)$. $Der(t)$ is a Lie subalgebra of $\mathfrak{gl}(U) \times \mathfrak{gl}(V) \times \mathfrak{gl}(W)$. The T-set $T((x + y - z), Der(t))$ captures the notion of "tensoring" over the Lie algebra $Der(t)$. This space is referred to as the *densor* (short for derivation tensor) space and is denoted ◖ t ◗. In ([1], Theorem A), the authors show that ◖ t ◗ is a universally compressed tensor space containing t. That is, if there are any other P, Ω such that P is defined by linear homogeneous polynomials and $t \in T(P, \Omega)$, then ◖ t ◗ $\subseteq T(P, \Omega)$.

In this subsection, we will compute both the derivation algebra and the densor space for the GHZ and W states.

Example 2.3.10. We first compute $Der(GHZ)$. On a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\langle GHZ \mid$ is defined as follows.

$$\begin{aligned} \langle GHZ \mid : &\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \\ &(e_1, e_1) \mapsto e_1 \\ &(e_1, e_2) \mapsto 0 \\ &(e_2, e_1) \mapsto 0 \\ &(e_2, e_2) \mapsto e_2. \end{aligned}$$

Let $d := x + y - z \in \mathbb{C}[x, y, z]$. The derivation algebra of GHZ is the Z-set

$$Der(GHZ) := \{(X, Y, Z) \in End(\mathbb{C}^2)^{\times 3} : \langle GHZ \mid Xu, v\rangle + \langle GHZ \mid u, Yv\rangle = Z\langle GHZ \mid u, v\rangle,\ u, v \in \mathbb{C}^2\}$$

Let

$$(X, Y, Z) := (\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}).$$

Then we compute the defining equations for $Der(GHZ)$ from the definition.

$$\begin{aligned}
(e_1, e_1): \quad & \langle GHZ \mid Xe_1, e_1\rangle + \langle GHZ \mid e_1, Ye_1\rangle - Z\langle GHZ \mid e_1, e_1\rangle \\
& = \langle GHZ \mid (x_{11}e_1 + x_{21}e_2), e_1\rangle + \langle GHZ \mid e_1, (y_{11}e_1 + y_{21}e_2)\rangle - (z_{11}e_1 + z_{21}e_2) \\
& = (x_{11} + y_{11} - z_{11})e_1 + z_{21}e_2 \\
(e_1, e_2): \quad & \langle GHZ \mid Xe_1, e_2\rangle + \langle GHZ \mid e_1, Ye_2\rangle - Z\langle GHZ \mid e_1, e_2\rangle \\
& = \langle GHZ \mid (x_{11}e_1 + x_{21}e_2), e_2\rangle + \langle GHZ \mid e_1, (y_{12}e_1 + y_{22}e_2)\rangle - 0 \\
& = y_{12}e_1 + x_{21}e_2 \\
(e_2, e_1): \quad & \langle GHZ \mid Xe_2, e_1\rangle + \langle GHZ \mid e_2, Ye_1\rangle - Z\langle GHZ \mid e_2, e_1\rangle \\
& = \langle GHZ \mid (x_{12}e_1 + x_{22}e_2), e_1\rangle + \langle GHZ \mid e_2, (y_{11}e_1 + y_{21}e_2)\rangle - 0 \\
& = x_{12}e_1 + y_{21}e_2 \\
(e_2, e_2): \quad & \langle GHZ \mid Xe_2, e_2\rangle + \langle GHZ \mid e_2, Ye_2\rangle - Z\langle GHZ \mid e_2, e_2\rangle \\
& = \langle GHZ \mid (x_{12}e_1 + x_{22}e_2), e_2\rangle + \langle GHZ \mid e_2, (y_{12}e_1 + y_{22}e_2)\rangle - (z_{12}e_1 + z_{22}e_2) \\
& = -z_{12}e_1 + (x_{22} + y_{22} - z_{22})e_2
\end{aligned}$$

All eight of these linear equations are independent. As such, $Der(GHZ)$ is a four dimensional space. Additionally, we may represent $Der(GHZ)$ with the following basis.

$$\begin{aligned}
Der(GHZ) = \Big\langle & (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), \\
& (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})\Big\rangle
\end{aligned}$$

Joining each tuple diagonally leads to the following nicer presentation.

$$Der(GHZ) = \left\langle \left[\begin{array}{cc|c|c} 1 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & \\ \hline & & & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{array}\right], \left[\begin{array}{cc|c|c} 0 & 0 & & \\ 0 & 1 & & \\ \hline & & 0 & \\ \hline & & & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \end{array}\right], \left[\begin{array}{c|c|c} 0 & & \\ \hline & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & \\ \hline & & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{array}\right], \left[\begin{array}{c|c|c} 0 & & \\ \hline & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & \\ \hline & & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \end{array}\right] \right\rangle$$

We now repeat the same calculation for the W state.

Example 2.3.11. On a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\langle W \mid$ is defined as follows.

$$\langle W \mid : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$$

$$(e_1, e_1) \mapsto e_2$$

$$(e_1, e_2) \mapsto e_1$$

$$(e_2, e_1) \mapsto e_1$$

$$(e_2, e_2) \mapsto 0.$$

The derivation algebra of W is the Z-set

$$Der(W) \coloneqq \{(X, Y, Z) \in E_{\mathbb{C}}[\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2] : \langle W \mid Xu, v\rangle + \langle W \mid u, Yv\rangle = Z\langle W \mid u, v\rangle,\ u, v \in \mathbb{C}^2\}$$

As above, let

$$(X, Y, Z) \coloneqq (\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}).$$

Then we compute the defining equations for $Der(W)$ from the definition.

$$
\begin{aligned}
(e_1, e_1): \quad & \langle W \mid Xe_1, e_1\rangle + \langle W \mid e_1, Ye_1\rangle - Z\langle W \mid e_1, e_1\rangle \\
&= \langle W \mid (x_{11}e_1 + x_{21}e_2), e_1\rangle + \langle W \mid e_1, (y_{11}e_1 + y_{21}e_2)\rangle - (z_{12}e_1 + z_{22}e_2) \\
&= (x_{21} + y_{21} - z_{12})e_1 + (x_{11} + y_{11} - z_{22})e_2 \\
(e_1, e_2): \quad & \langle W \mid Xe_1, e_2\rangle + \langle W \mid e_1, Ye_2\rangle - Z\langle W \mid e_1, e_2\rangle \\
&= \langle W \mid (x_{11}e_1 + x_{21}e_2), e_2\rangle + \langle W \mid e_1, (y_{12}e_1 + y_{22}e_2)\rangle - (z_{11}e_1 + z_{21}e_2) \\
&= (x_{11} + y_{22} - z_{11})e_1 + (y_{12} - z_{21})e_2 \\
(e_2, e_1): \quad & \langle W \mid Xe_2, e_1\rangle + \langle W \mid e_2, Ye_1\rangle - Z\langle W \mid e_2, e_1\rangle \\
&= \langle W \mid (x_{12}e_1 + x_{22}e_2), e_1\rangle + \langle W \mid e_2, (y_{11}e_1 + y_{21}e_2)\rangle - (z_{11}e_1 + z_{21}e_2) \\
&= (x_{22} + y_{11} - z_{11})e_1 + (x_{12} - z_{21})e_2 \\
(e_2, e_2): \quad & \langle W \mid Xe_2, e_2\rangle + \langle W \mid e_2, Ye_2\rangle - Z\langle W \mid e_2, e_2\rangle \\
&= \langle W \mid (x_{12}e_1 + x_{22}e_2), e_2\rangle + \langle W \mid e_2, (y_{12}e_1 + y_{22}e_2)\rangle - 0 \\
&= (x_{12} + y_{12})e_1
\end{aligned}
$$

All seven of these linear equations are independent. As such, $Der(W)$ is a five dimensional space. Additionally, we may represent $Der(W)$ with the following basis.

$$
\begin{aligned}
Der(W) = \Big\langle &\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right), \left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right), \left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right), \\
&\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right), \left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\Big\rangle
\end{aligned}
$$

Joining each tuple diagonally leads to the following nicer presentation.

$$Der(W) = \left\langle \left[\begin{array}{c|c|c} \begin{matrix}1 & 0\\ 0 & 0\end{matrix} & & \\ \hline & \begin{matrix}0 & 0\\ 0 & -1\end{matrix} & \\ \hline & & \begin{matrix}0 & 0\\ 0 & 1\end{matrix}\end{array}\right], \left[\begin{array}{c|c|c} \begin{matrix}0 & 0\\ 1 & 0\end{matrix} & & \\ \hline & 0 & \\ \hline & & \begin{matrix}0 & 1\\ 0 & 0\end{matrix}\end{array}\right], \left[\begin{array}{c|c|c} \begin{matrix}0 & 0\\ 0 & 1\end{matrix} & & \\ \hline & \begin{matrix}0 & 0\\ 0 & 1\end{matrix} & \\ \hline & & \begin{matrix}1 & 0\\ 0 & 0\end{matrix}\end{array}\right], \left[\begin{array}{c|c|c} 0 & & \\ \hline & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & \\ \hline & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix}\end{array}\right], \left[\begin{array}{c|c|c} 0 & & \\ \hline & \begin{matrix}0 & 0\\ 1 & 0\end{matrix} & \\ \hline & & \begin{matrix}0 & 1\\ 0 & 0\end{matrix}\end{array}\right] \right\rangle$$

Now we turn to computing the densor space of the GHZ and W states, denoted ◖ GHZ ◗ and ◖ W ◗ respectively. We may write an arbitrary tensor $\langle t | : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrowtail \mathbb{C}^2$ symbolically as

$$\begin{aligned} \langle t | :& \mathbb{C}^2 \times \mathbb{C}^2 \rightarrowtail \mathbb{C}^2 \\ & (e_1, e_1) \mapsto ae_1 + be_2 \\ & (e_1, e_1) \mapsto ce_1 + de_2 \\ & (e_1, e_1) \mapsto ee_1 + fe_2 \\ & (e_1, e_1) \mapsto ge_1 + he_2 \end{aligned}$$

In coordinates, this is the cube with faces given by

$$t = (\begin{bmatrix} a & c \\ e & g \end{bmatrix}, \begin{bmatrix} b & d \\ f & h \end{bmatrix})$$

We compute how the monomials x, y, z act on $\langle t \mid$ via each matrix in a basis of $End(\mathbb{C}^2) \cong Mat_{2\times 2}(\mathbb{C})$. The action of the identity matrix is included for simplicity. The data is organized into the following tables.

X	1 0	1 0	0 1	0 0	0 0
	0 1	0 0	0 0	1 0	0 1
(e_1, e_1)	[a b]	[a b]	[0 0]	[e f]	[0 0]
(e_1, e_2)	[c d]	[c d]	[0 0]	[g h]	[0 0]
(e_2, e_1)	[e f]	[0 0]	[a b]	[0 0]	[e f]
(e_2, e_2)	[g h]	[0 0]	[c d]	[0 0]	[g h]

Y	1 0	1 0	0 1	0 0	0 0
	0 1	0 0	0 0	1 0	0 1
(e_1, e_1)	[a b]	[a b]	[0 0]	[c d]	[0 0]
(e_1, e_2)	[c d]	[0 0]	[a b]	[0 0]	[c d]
(e_2, e_1)	[e f]	[e f]	[0 0]	[g h]	[0 0]
(e_2, e_2)	[g h]	[0 0]	[e f]	[0 0]	[g h]

Z	1 0	1 0	0 1	0 0	0 0
	0 1	0 0	0 0	1 0	0 1
(e_1, e_1)	[a b]	[a 0]	[b 0]	[0 a]	[0 b]
(e_1, e_2)	[c d]	[c 0]	[d 0]	[0 c]	[0 d]
(e_2, e_1)	[e f]	[e 0]	[f 0]	[0 e]	[0 f]
(e_2, e_2)	[g h]	[g 0]	[h 0]	[0 g]	[0 h]

The tables allow us to calculate ◖ GHZ ◗ and ◖ W ◗ directly from the definitions. That is,

$$\text{◖ } GHZ \text{ ◗} = \{\langle t \mid : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrowtail \mathbb{C}^2 : \langle t \mid Xu, v\rangle + \langle t \mid u, Yv\rangle = Z\langle t \mid u, v\rangle,$$

$$\forall\, (X, Y, Z) \in Der(GHZ), u, v \in \mathbb{C}^2\}$$

$$\text{◖ } W \text{ ◗} = \{\langle t \mid : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrowtail \mathbb{C}^2 : \langle t \mid Xu, v\rangle + \langle t \mid u, Yv\rangle = Z\langle t \mid u, v\rangle,$$

$$\forall\, (X, Y, Z) \in Der(W), u, v \in \mathbb{C}^2\}.$$

Now we compute ◖ GHZ ◗ and ◖ W ◗ from the definitions using the tables above and replacing (X, Y, Z) with the basis of $Der(GHZ)$ and $Der(W)$ determined in examples 2.3.10 and 2.3.11 respectively.

Example 2.3.12. The linear equations for GHZ are as follows.

$$
\begin{aligned}
&(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}): && \begin{bmatrix}a\\b\end{bmatrix}-\begin{bmatrix}a\\0\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}-\begin{bmatrix}c\\0\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}-\begin{bmatrix}e\\0\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}g\\0\end{bmatrix}\\
&(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}): && \begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}0\\b\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}0\\d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}-\begin{bmatrix}0\\f\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}0\\h\end{bmatrix}\\
&(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}): && \begin{bmatrix}a\\b\end{bmatrix}-\begin{bmatrix}a\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}c\\0\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}-\begin{bmatrix}e\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}g\\0\end{bmatrix}\\
&(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}): && \begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}0\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}-\begin{bmatrix}0\\d\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}0\\f\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}0\\h\end{bmatrix}
\end{aligned}
$$

Which simplifies to the linear equations $b = c = d = e = f = g = 0$. In other words, we have the following basis for $◖\, GHZ\, ◗$.

$$
◖\, GHZ\, ◗ = \langle(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}),(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})\rangle = \langle e_1\otimes e_1\otimes e_1, e_2\otimes e_2\otimes e_2\rangle.
$$

Example 2.3.13. We conclude this subsection by computing the densor space of $◖\, W\, ◗$. Specifically, we compute the linear equations for W in a manner analogous to that of GHZ.

$$
\begin{aligned}
&(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&-1\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}): && \begin{bmatrix}a\\b\end{bmatrix}-\begin{bmatrix}0\\b\end{bmatrix},-\begin{bmatrix}0\\d\end{bmatrix},-\begin{bmatrix}0\\f\end{bmatrix},-\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}0\\h\end{bmatrix}\\
&(\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix}): && \begin{bmatrix}e\\f\end{bmatrix}-\begin{bmatrix}b\\0\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}d\\0\end{bmatrix},-\begin{bmatrix}f\\0\end{bmatrix},-\begin{bmatrix}h\\0\end{bmatrix}\\
&(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}): && -\begin{bmatrix}a\\0\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}-\begin{bmatrix}c\\0\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}-\begin{bmatrix}e\\0\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}+\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}g\\0\end{bmatrix}\\
&(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}): && \\
&(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix}): && \begin{bmatrix}c\\d\end{bmatrix}-\begin{bmatrix}b\\0\end{bmatrix},-\begin{bmatrix}d\\0\end{bmatrix},\begin{bmatrix}g\\h\end{bmatrix}-\begin{bmatrix}f\\0\end{bmatrix},-\begin{bmatrix}h\\0\end{bmatrix}
\end{aligned}
$$

Which simplifies to the linear equations $b = c = e, a = d = f = g = h = 0$. This implies that $◖ W ◗$ is one-dimensional and spanned by the tensor,

$$◖ W ◗ = \langle (\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) \rangle = \langle W \rangle.$$

2.4 Properties of a Galois Correspondence

We detour momentarily to describe a few key properties of a Galois correspondence. We aim to demonstrate the utility of having such a correspondence, namely, that a Galois correspondence naturally gives rise to closed sets.

Definition 2.4.1. A *preorder* is a set X with a relation $\leq$ on X such that

- $x \leq x$ for all $x \in X$
- $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$.

In this case, we will denote the preorder as $(X, \leq)$.

Definition 2.4.2. Let $(X, \leq)$ and $(Y, \preceq)$ be preorders. Let $x, x' \in X$ and $y \in Y$.

- A function $F : X \to Y$ is *monotone* if $x \leq x' \implies F(x) \leq F(x')$.
- A *Galois connection* (or a Galois correspondence) is a pair of monotone functions $F : X \to Y$, $G : Y \to X$ such that $F(x) \preceq y \iff x \leq G(y)$.

Galois connections give rise to natural closure operations. We detail this construction now.

Definition 2.4.3. An operator $\Delta : X \to X$ on a preorder $(X, \leq)$ is a *closure operator* if

- (Dilating) $x \leq \Delta(x)$ for all $x \in X$.
- (Idempotent) $\Delta^2(x) = \Delta(x)$

Let $F : X \to Y$ and $G : Y \to X$ be a Galois connection. Consider the compositions $F \circ G : Y \to Y$ and $G \circ F : X \to X$. We claim that these are closure operations.

Proposition 2.4.4. *Let $(X, \leq)$ and $(Y, \preceq)$ be preorders. Let $F : X \to Y$ and $G : Y \to X$ be a Galois connection. Then the compositions $F \circ G$ and $G \circ F$ are closure operators.*

Proof. We will show $G \circ F : X \to X$ is a closure operator. The proof for $F \circ G$ will follow identically.

We first show $G \circ F$ is dilating. Let $x \in X$. By the Galois connection, $x \leq G(F(x)) \iff F(x) \preceq F(x)$, which is always true in a preorder.

Now we show $G \circ F$ is idempotent. By the dilating property, $(G \circ F)(x) \leq (G \circ F)^2(x)$. Since G is monotone, it suffices to show $(F \circ G \circ F)(x) \preceq F(x)$. By the Galois connection, this is true if and only if $(G \circ F)(x) \leq (G \circ F)(x)$ which is always true in a preorder. □

So far, we have only discussed Galois correspondences between two preorders. With the goal of the TIZ-correspondence, we now turn our attention to Galois connections between three preorders.

Definition 2.4.5. Let $(X, \leq)$, $(Y, \preceq)$, and $(Z, \subseteq)$ be preorders. A (monotone/antitone) ternary Galois connection is a triple of (monotone/anitone) functions (F, G, H) such that $F : X \times Y \to Z$, $G : X \times Z \to Y$, and $H : Y \times Z \to X$, and

$$x \leq H(y, z) \iff y \preceq G(x, z) \iff z \subseteq F(x, y).$$

Let L be a commutative k-algebra. Define the preorders $(\mathcal{T}, \leq_T), (\mathcal{I}, \leq_I), (\mathcal{Z}, \leq_Z)$ on the sets $\mathcal{T} := 2^{(L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\}))}$, $\mathcal{P} := 2^{(L[X])}$, and $\mathcal{Z} := 2^{(E_L[U_0, \cdots, U_n])}$ where the relation on each power set is subset inclusion. Fixing a subset $P \in \mathcal{P}$ gives a pair of maps

$$\begin{aligned} T(P, -)_L : \mathcal{Z} \to \mathcal{T} \qquad\qquad & Z(-, P)_L : \mathcal{T} \to \mathcal{Z} \\ \Omega \mapsto T(P, \Omega)_L \qquad\qquad & S \mapsto Z(S, P)_L \end{aligned}$$

By the defining equation in definition 2.3.7, $\Omega \leq_Z \Omega' \implies T(P,\Omega)_L \supseteq T(P,\Omega')_L$, and $S \leq_T S' \implies Z(S,P)_L \supset Z(S',P)_L$ so the maps $T(P,-)_L$ and $Z(-,P)_L$ are inclusion-reversing. As it is often more convinient to work with a monotone Galois connection, we instead work in the category $(\mathcal{T}^{op}, \leq_X^{op})$ so now $\Omega \leq_Z \Omega' \implies T(P,\Omega)_L \geq_X T(P,\Omega')_L \implies T(P,\Omega) \leq_X^{op} T(P,\Omega')$ and analagously $S \leq_X^{op} S' \implies Z(S,P) \leq_Z Z(S',P)$ so $T(P,-)$ and $Z(-,P)$ are a pair of monotone functions on preorders $\mathcal{T}^{op}$ and $\mathcal{Z}$.

Theorem 2.3.9 implies that $T(P,-)$ and $Z(-,P)$ form a Galois connection. Then by proposition 2.4.4, the compositions $T(P,-)\circ Z(-,P)$ and $Z(-,P)\circ T(P,-)$ are closure operators. We may repeat this computation to find two closure operators on any pair of preorders chosen from $\mathcal{T}, \mathcal{I}, \mathcal{Z}$. This yields a total of six different closure operators. Written explicitly, for a fixed subsets $S \subseteq L\otimes(\{U_1\times\cdots\times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0,\cdots,U_n]$, closed sets will have the form $T(P, Z(S,P)_L)_L$, $T(I(S,\Omega)_L,\Omega)_L$, $I(T(P,\Omega)_L,\Omega)_L$, $I(S, Z(S,P)_L)_L$, $I(T(P,\Omega)_L,\Omega)_L$, and $Z(S, I(S,\Omega)_L)_L$.

Example 2.4.6. We computed the algebras $Cen(GHZ)$ and $Cen(W)$ in examples 2.2.1 and 2.3.3. In general, the centroid of a tensor t is the collection of scalars of t. In the case of bilinear maps $\langle t : U \times V \rightarrowtail W$, the centroid $Cen(t)$ consists of triples $(\omega,\tau,\gamma) \in End(U) \times End(V) \times End(W)$ such that for all $u \in U, v \in V$,

$$\langle t \mid \omega(u), v\rangle = \langle t \mid u, \tau(v)\rangle = \gamma\langle t \mid u, v\rangle.$$

This is the Z-set $Z(\{t\}, (x_0 - x_1, x_1 - x_2))$. The closure over the centroid is the T-set $T((x_0 - x_1, x_1 - x_2), Cen(t))$ which are all the tensors $\langle s : U \times V \rightarrowtail W$ such that $\gamma(\langle s \mid u, v\rangle) = \langle s \mid \omega(u), v\rangle = \langle s \mid u, \tau(v)\rangle$ for all $(\omega,\tau,\gamma) \in Cen(t)$. In other words, this is the Whitney tensor product over the centroid of t.

Example 2.4.7. We computed the derivation algebras $Der(GHZ)$ and $Der(W)$ in examples 2.3.10 and 2.3.11. In general, if $d \coloneqq x_1 + \cdots + x_n - x_0$ and $t \in \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$ is a tensor, then the Z-set

$$Z(t,d) \coloneqq \{(X_0,\cdots,X_n) \in E_k[U_0,\cdots,U_n] : (\forall u_i \in U_i),$$
$$\langle t \mid X_1u_1,u_2,\cdots,u_n\rangle + \cdots + \langle t \mid u_1,\cdots,X_nu_n\rangle = X_0\langle t \mid u_1,\cdots,u_n\rangle\}$$

is referred to as the *Derivation algebra* $Der(t)$ in [1] . Then the T-set

$$T(d,Der(t)) \coloneqq \{\langle t \mid \in \{U_1 \times \cdots \times U_n \rightarrowtail U_0\} : (\forall u_i \in U_i, (X_0,\cdots,X_n) \in Der(t)),$$
$$\langle t \mid X_1u_1,u_2,\cdots,u_n\rangle + \cdots + \langle t \mid u_1,\cdots,X_nu_n\rangle = X_0\langle t \mid u_1,\cdots,u_n\rangle\}$$

is the densor space ◖ t ◗ containing t. In ([1], Theorem A) , the authors show that the densor is the universally smallest tensor space containing t. Note that $Der(t)$ is a Lie algebra. So ([1], Theorem A) implies that we can create universally compressed tensor spaces by "tensoring" over a Lie algebra. In section 4, we show an analogous statement for symmetric tensors.

2.5 Main Theorems of the TIZ Correspondence

In this section, we wish to survey the main results and techniques from [1] which we will be adapting to a symmetric setting. Note that [1] works in a more general setting than we are. Each U_i is assumed to be a finitely generated projective K-module over a commutative (but not necessarily unital) ring K. We are assuming that each U_i is a finite-dimensional vector space over a field k. As such, the proofs we give here will sometimes differ from those in [1].

Let L be a commutative k-algebra and recall the $T/I/Z$-sets $T(P,\Omega)_L, I(S,\Omega)_L$, and $Z(S,P)_L$. We now prove the TIZ-correspondence stated in theorem 2.3.9.

Theorem 2.5.1. *([1], Theorem B) Let* $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, *and* $\Omega \subseteq E_L[U_0,\cdots,U_n]$. *Then,*

1. *$T(P,\Omega)_L$ is an L-submodule of $L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $I(S,\Omega)_L$ is an ideal of $L[X]$, and $Z(S,P)_L$ is the zero locus of a finitely generated ideal of polynomials unique to S and P.*

2. *The collection $T/I/Z$-sets form an inclusion-reversing ternary Galois correspondence. Precisely,*

$$S \subseteq T(P,\Omega)_L \iff P \subseteq I(S,\Omega)_L \iff \Omega \subseteq Z(S,P)_L.$$

Proof. As sets, the statement of (2) is immediate since each $T/I/Z$-set is defined by the relation $\langle t \mid p(\omega) = 0$.

To show (1), first note that $T(P,\Omega)_L$ is closed under scaling by L by definition. The additivity of $T(P,\Omega)_L$ comes from a calculuation. Let $p = \sum_e \lambda_e X^e \in P$ and $\omega \in \Omega$. Then for all $t, t' \in L \otimes U_1 \times \cdots \times U_n \rightarrowtail U_0$ and $u \in U_0 \times \cdots \times U_n$,

$$\begin{aligned}
\langle t + t' \mid p(\omega) \mid u\rangle &= \sum_e \lambda_e \omega_0 \langle t + t' \mid \omega_1^{e_1} u_1, \cdots, \omega_n^{e_n} u_n\rangle. \\
&= \sum_e \lambda_e \omega_0 \langle t \mid \omega_1^{e_1} u_1, \cdots, \omega_n^{e_n} u_n\rangle + \sum_e \lambda_e \omega_0 \langle t' \mid \omega_1^{e_1} u_1, \cdots, \omega_n^{e_n} u_n\rangle. \\
&= \langle t \mid p(\omega) \mid u\rangle + \langle t' \mid p(\omega) \mid u\rangle
\end{aligned}$$

So $t, t' \in T(P,\Omega)_L \implies t + t' \in T(P,\Omega)_L$.

To show that $I(S,\Omega)_L$ is an ideal of $L[X]$, we recall that fixing some $\omega \in \Omega$ gives a ring action of $L[X]$ on $L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$. Then $I(S,\omega)_L = I(\langle S\rangle, \omega)_L$ is precisely the annihilator of the subspace $\langle S\rangle$. Annihilators of ring actions are always ideals so we may conclude that $I(S,\omega)_L$ is an ideal of $L[X]$. The rest follows as $I(S,\Omega)_L = \bigcap_{\omega\in\Omega} I(S,\omega)_L$ and ideals are closed under intersection.

To show $Z(S,P)_L$ is a subscheme of $E_L[U_0, \cdots, U_n]$, we first note that $E_k[U_0, \cdots, U_n]$ is the spectrum of the polynomial ring $A = k[x_{0,1}, \cdots, x_{0,dim(U_0)}, \cdots, x_{n,1}, \cdots, x_{n,dim(U_n)}]$ in $\sum_i dim(U_i)^2$ variables. $Z(S,P)_k$ is carved out by finitely many polynomials in A. Let I

be the ideal generated by these polynomials. We may now view $Z(S,P)_k = Spec(A/I)$. Since L-points correspond to elements of the fiber product $Z(S,P)_k \times_k Spec(L)$ and closed immersions are stable under base change ([21], Exercise II.3.11.a), then we may identify $Z(S,P)_L$ with $Spec(A/I \otimes L) \subseteq E_L[U_0, \cdots, U_n]$. □

Remark 2.5.2. Our goal is to prove a similar correspondence with the adjective "symmetric" appended to $T/I/Z$-sets in a natural way. Note that the proof of theorem 2.3.9 is independent of symmetry. This means that we can freely utilize the proofs of parts (1) and (2) when considering restricted families of $T/I/Z$-sets. What is less immediate is that a symmetric analogue of the TIZ-correspondence is closed under symmetry. That is, we will need to show that when S, P, Ω are each "symmetric", then the $T/I/Z$-sets built from these satisfy the same symmetry.

Observe that operator schemes $Z(S,P)_L$ are the zero sets of polynomials in upwards of $d_0^2+d_1^2+\cdots+d_n^2$ variables. However, the defining equations of $Z(S,P)_L$ arise from polynomials in $n+1$ variables. This in particular allows for efficient algorithms to compute the equations defining Z-sets. We demonstrate these algorithms in the case that $L=k$, however, the proof ([1],Theorem C) holds in the more general setting where each U_i being a finitely generated projective L-module. For the following, let $D \coloneqq d_0 + \cdots + d_n$ and $E \coloneqq d_0^2 + d_1^2 + \cdots + d_n^2$ be the dimensions of the ambient space of tensors and operators respectively.

Theorem 2.5.3. *([1],Theorem C) For subsets $S \subseteq U_0 \otimes \cdots \otimes U_n$, $P \subseteq k[X]$, and $\Omega \subseteq E_k[U_0, \cdots, U_n]$, there exists an algorithm to,*

- *Construct generators for $T(P,\Omega)$ in time polynomial in $|P|, |\Omega|$, and D.*
- *Construct generators $I(S,\Omega)$ in polynomial time if the valence n is bounded. In general, this algorithm returns in quasi-polynomial time.*
- *Construct defining polynomials for $Z(S,P)$ in time polynomial in the encoding size of S and P. In the case where P is linear, there exists an algorithm to construct generators for $Z(S,P)$ as a k-vector space.*

Proof. In each case, we assume an efficient oracle to write down elements of the field k and we assume an efficient black-box model for linear algebra over k. Let $TIME_{LA}(m, n)$ denote the time needed to solve m linear equations with n variables over k. We assume each S, P, Ω is finite. Store each polynomial $p_i = \sum_e \lambda_{i,e} X^e \in P$ as a tuple $p_i = [(e, \lambda_{i,e}) : \lambda_{i,e} \neq 0]$. Choose a basis of each U_i so that $\omega \in E_k[U_0, \cdots, U_n]$ may be represented as a tuple of square matrices.

We begin with computing equations defining Z-sets, write a generic operator $\omega = (\omega_0, \cdots, \omega_n)$ as an $n+1$-tuple in E indeterminates. With the high-school matrix multiplication algorithm, we may compute $\omega_i^{e_i}$ with $O(d_i^3 e_i)$ operations and so we may compute $p(\omega)$ in $O(|p|(\sum_{i=0}^n d_i^3 e_i))$ operations. Then $Z(S, P)$ is defined by the equations $\{p(\omega)(t) = 0 : p \in P, t \in S\}$. In the case where P is linear, then each $p(\omega)(t) = 0$ is linear in the E many variables populating ω. Then the task of finding generators for $Z(S, P)$ can be achieved by solving $|P||S|$ linear equations in E variables. This task requires $TIME_{LA}(|P||S|, E)$ operations.

We now turn to computing T-sets. Recall that in writing tensors as elements of the space $T := U_0 \otimes \cdots \otimes U_n$, $T(P, \Omega)$ is the space $T / \Xi(P, \Omega)$ where $\Xi(P, \Omega)$ is generated by tensors of the form $p(\omega)(t)$ where t is a pure tensor.

First consider the case of a single polynomial and transverse operator (p, ω). We first compute $p(\omega)$. For each pair $(e, \lambda_e) \in p$, we note that the high-school matrix multiplication algorithm computes $\omega_i^{e_i}$ in $O(d_i^3 e_i)$ time. Then as above, we may write down $p(\omega)$ with $O(|p|(\sum_{i=0}^n d_i^3 e_i))$ operations. Note that this expression is polynomial in $|p|$ and $\sum_i d_i$. To compute $\Xi(p, \omega)$, it suffices to compute $p(\omega)(t)$ as t ranges over a basis of T. This requires us to write down D many such tensors. We conclude that, for arbitrary finite P, Ω, a generating set for $\Xi(P, \Omega)$ consists of $D|P||\Omega|$ tensors. Then to compute $\Xi(P, \Omega)$, we write down $p(\omega)(t)$ for each $p \in P, \omega \in \Omega$, and basis tensor $t \in T$.

Now consider the case of I-sets. We claim that $I(S, \Omega)$ is generated by polynomials p such that $deg_{x_i}(p) \leq d_i|\Omega|$ for all $i \in \{0, \cdots, n\}$. Indeed, for all $\omega = (\omega_0, \cdots, \omega_n) \in \Omega$, the minimal polynomial $min(\omega_i) \in I(S, \{\omega\})$ since $min(\omega_i)(\omega)(t) = 0(t) = 0$ for all $t \in T$. Then the least common multiple $lcm(\{min(\omega_i) : \omega \in \Omega\})$ is in $I(S, \Omega)$ with $deg_{x_i}(lcm(\{min(\omega_i) : \omega \in \Omega\})) \leq d_i|\Omega|$. Since $k[x_i]$ is a Euclidean domain, there exists polynomials $r_i(x_i), q_i(x_i) \in k[x_i]$ such that $x_i^{e_i} = r_i(x_i) + q_i(x_i)min(\omega_i)$ with $deg(r_i) \leq deg(min(\omega_i))$. Repeat this argument for each $i \in \{0, \cdots, n\}$ and the result follows.

We encode the data of S and Ω into a matrix M such that $I(S, \Omega)$ is the nullspace of M. Index the columns of M by tuples $e \subseteq \{0, \cdots, d_0|\Omega|\} \times \cdots \times \{0, \cdots, d_n|\Omega|^{n+1}\}$. The upper bound on each set is described by the preceeding paragraph. Observe that there are $D|\Omega|$ many columns. For each $t \in S$, and $\omega \in \Omega$, create a $D \times D|\Omega|^{n+1}$ matrix $M(t, \omega)$ where each column is the tensor $X^e(\omega)(t)$ flattened into a vector. Then stack each $M(t, \omega)$ vertically and observe that the nullspace is a generating set for $I(S, \Omega)$. This method relies on solving $D|S||\Omega|$ linear equations in $D|\Omega|^{n+1}$ variables. □

Remark 2.5.4. The proof of theorem 2.5.3 is independent of symmetry. This means that, as long as the "symmetric" $T/I/Z$-sets we consider satisfy the conditions of theorem 2.3.9, then the algorithms of theorem 2.5.3 can be used analagously in our restricted setting.

2.5.1 Universal Property of the Densor Space

We now turn our attention to the universal property of the derivation tensor product. First consider the examples of the GHZ and W tensors.

Example 2.5.5. Recall from examples 2.2.1 and 2.3.3 that

$$Cen(GHZ) = Span\{(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})\}$$
$$Cen(W) = Span\{(I_2, I_2, I_2), (\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix})\}.$$

Recall from examples 2.3.10 and 2.3.11 that

$$Der(GHZ) = \Big\langle (\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}),$$
$$(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})\Big\rangle$$
$$Der(W) = \Big\langle (\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}), (\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}),$$
$$(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}), (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix})\Big\rangle$$

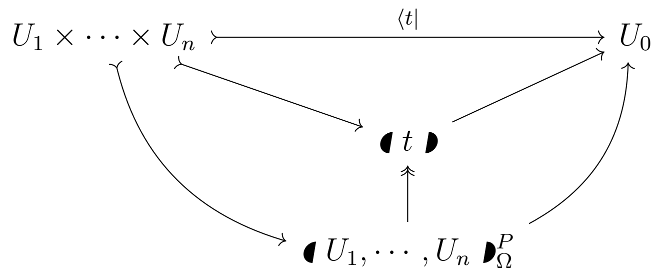
If $k \neq \mathbb{F}_2$, then $Cen(GHZ) \subseteq Der(GHZ)$ and $Cen(W) \subseteq Der(W)$. By the inclusion-reversing property of the TIZ-correspondence, we conclude that $◖ GHZ ◗ \subseteq T((x-y, y-z), Cen(GHZ))$ and $◖ W ◗ \subseteq T((x-y, y-z), Cen(W))$. If we are working over $\mathbb{F}_2$, we can move to a field extension L and then show $Cen(GHZ)_L \subseteq Der(GHZ)_L$ and $Cen(W)_L \subseteq Der(W)_L$. However, even without a field extension, observe that,

$$◖ GHZ ◗ = \langle e_1 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_2 \rangle \subseteq k^2 \otimes_{Cen(GHZ)} k^2 \otimes_{Cen(GHZ)} k^2$$
$$◖ W ◗ = \langle W \rangle \subseteq k^2 \otimes_{Cen(W)} k^2 \otimes_{Cen(W)} k^2.$$

In general, $Cen(t)$ may not always be contained in $Der(t)$, however, there will always exist a field extension for which $Cen(t)_L \subseteq Der(t)_L$. That said, as field extensions are faithfully flat, we get a containment $◖ t ◗ \subseteq T((x-y, y-z), Cen(t))$ without needing a field extension.

For the following theorem, define the support of a linear homogeneous polynomial $p = \alpha_0 x_0 + \cdots + \alpha_n x_n$ as all of the i such that $\alpha_i \neq 0$. Define the support of a linear homogeneous ideal P as the union of the support of all $p \in P$. Finally, let d denote the derivation polynomial $d \coloneqq x_1 + \cdots + x_n - x_0$. Recall the generalized tensor products $◖ U_1, \cdots, U_n ◗_{\Omega}^{P}$ from section 2.3.1.

Theorem 2.5.6. *([1],Theorem A) Let k be a field and $\langle t \mid : U_1 \times \cdots \times U_n \rightarrowtail U_0$. Let $P \subseteq k[x_0, \cdots, x_n]$ be a linear homogeneous ideal of full support and $\Omega \subseteq E_k[U_0, \cdots, U_n]$ be subsets such that $\langle t \mid$ factors through $\LEFTcircle U_1, \cdots, U_n \RIGHTcircle^P_\Omega$. Then there exists a surjective map $\LEFTcircle U_1, \cdots, U_n \RIGHTcircle^P_\Omega \to \LEFTcircle t \RIGHTcircle$ making the following diagram commute.*



Before proving this theorem, we will need to briefly study how a torus action leads to embeddings of T-sets. For some commutative k-algebra L, let $\mathbb{T}(L)$ denote the torus $(L^\times)^{n+1}$. Define the following action of $\mathbb{T}(L)$ on tensors, polynomials, and operators. Let $\tau \in \mathbb{T}(L)$.

Definition 2.5.7.

- For $\langle t \mid \in L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, define $\langle t^\tau \mid u_1, \cdots, u_n\rangle = \tau_0 \langle t \mid \tau_1 u_1, \cdots, \tau_n \tau_n\rangle$.
- For $p \in L[x_0, \cdots, x_n]$, define $p^\tau = p(\tau^{-1} x_0, \cdots, \tau^{-1} x_n)$.
- For $\omega \in E_L[U_0, \cdots, U_n]$, define $\tau\omega = (\tau_0 \omega_0, \cdots, \tau_n \omega_n)$.

Lemma 2.5.8. *For all $\tau \in (k^\times)^{n+1}$, $T(P^\tau, \Omega) = T(P, \tau^{-1}\Omega)$.*

Proof. This follows from the definitions of the torus action on P and Ω. □

We may now prove theorem 2.5.6.

Proof. If k is algebraically closed, then we only need statements from the Galois correspondence. Choose some finite collection $p_i \in P$ such that $p \coloneqq \sum_i p_i$ has full support. Say $p = \sum_i \alpha_i x_i$. Since k is algebraically closed, there exists some tuple of nonzero scalars $\tau \coloneqq \lambda_i : i \in 0, \cdots, n \in (k^\times)^{n+1}$ such that scaling α_i by λ_i transforms p into d. This implies that $d^{\tau^{-1}} \in P$. By the inclusion-reversing property, $Z(t, d^{\tau^{-1}}) \supseteq Z(t, P)$ and

$T(d^{\tau^{-1}}, Z(t, d^{\tau^{-1}})) \subseteq T(P, Z(t, P))$. By lemma 2.5.8, $T(d^{\tau^{-1}}, Z(t, d^{\tau^{-1}})) = T(d, Z(t, d)) =$ $◖ t ◗$ and the result follows.

Note that this argument relied on being able to write p as a single linear homogeneous polynomial of full support. This is not possible in general, as shown by example 2.6.5. In this case, base change to a suitable field extension L such that there exists $\tau \in (L^\times)^{n+1}$ such that $p = d^{\tau^{-1}}$. Then the inclusion-reversing property of a Galois correspondence implies that $L \otimes Z(t, d^{\tau^{-1}}) \supseteq L \otimes Z(t, P)$. This implies that $L \otimes ◖ t ◗ = L \otimes T(d^{\tau^{-1}}, Z(t, d^{\tau^{-1}})) =$ $L \otimes T(d, Z(t, d)) \subseteq L \otimes T(P, Z(t, P))$. Since k is a field, L is faithfully flat over k and so $◖ t ◗ \subseteq T(P, Z(t, P))$. Since $\Omega \subseteq Z(t, P)$, then $T(P, Z(t, P)) \subseteq T(P, \Omega)$. This shows that $t \in T(P, \Omega) \implies ◖ t ◗ \subseteq T(P, \Omega)$. The rest follows easily once we identify a T-set $T(P, \Omega)$ with the space of linear maps $U_0 \oslash ◖ U_1, \cdots, U_n ◗_\Omega^P$. □

2.5.2 Algebras of Operators in TIZ

By this point, we have seen how T-sets are generalizations of tensor products and we have seen how Z-sets generalize examples of algebraic invariants previously seen in the isomorphism literature. We have seen examples of associative algebras such as centroids, Jordan algebras such as the adjoint, as well as Lie algebras such as the derivation algebra. Now it is natural to ask what other algebras can show up in this correspondence. In other words, given an ideal P, which products can we put on $Z(S, P)$ to make this an algebra for all S. In this subsection, we will classify a large family of products and we will see that most of these products give rise to Lie algebras.

Choose tuples $\lambda, \rho \in k^{n+1}$. We endow $E_k[U_0, \cdots, U_n]$ with a product, denoted $\bullet_{(\lambda,\rho)}$, defined by,

$$\omega \bullet_{(\lambda,\rho)} \tau := (\lambda_i \omega_i \tau_i + \rho_i \tau \omega_i)_{i \in \{0, \cdots, n\}}.$$

Example 2.5.9. Say $n = 2$, $\omega = (\omega_0, \omega_1, \omega_2), \tau = (\tau_0, \tau_1, \tau_2) \in End(U_0) \times End(U_1) \times End(U_2)$. Then for $(\lambda_0, \lambda_1, \lambda_2), (\rho_0, \rho_1, \rho_2) \in k^3$, the product $\bullet_{(\lambda,\rho)}$ has the form

$$\omega \bullet_{(\lambda,\rho)} \tau = (\lambda_0\omega_0\tau_0 + \rho_0\tau_0\omega_0, \lambda_1\omega_1\tau_1 + \rho_1\tau_1\omega_1, \lambda_2\omega_2\tau_2 + \rho_2\tau_2\omega_2)$$

Example 2.5.10. For simplicity, say $n = 0$ and let $M, N \in End(U_0)$.

- $(\lambda = 1, \rho = 0)$. $M \bullet_{(\lambda,\rho)} N = MN$ which implies that $(End(U_0), \bullet_{(\lambda,\rho)}) = End(U_0)$ as an associative algebra. Similarly, if $(\lambda = 0, \rho = 1)$, then $M \bullet_{(\lambda,\rho)} N = NM$ and $(End(U_0), \bullet_{(\lambda,\rho)})$ is an associative algebra equal to $End(U_0)^{op}$.

- $(\lambda = \rho = 1)$. Then $M \bullet_{(\lambda,\rho)} N = MN + NM$ which implies that $(End(U_0), \bullet_{(\lambda,\rho)})$ is a Jordan algebra.

- If $(\lambda = 1, \rho = -1)$, then $M \bullet_{(\lambda,\rho)} N = MN - NM$ and $(End(U_0), \bullet_{(\lambda,\rho)})$ is a Lie algebra equal to $\mathfrak{gl}(U_0)$.

We are interested in when $Z(S, P)$ is an algebra under some product $\bullet(\lambda, \rho)$. To guarantee $Z(S, P)$ is closed under addition, we will be restricting to the case when P is generated by linear homogeneous polynomials. For such an ideal P generated by $p_1, \cdots, p_m$, define the rank of P as the rank of the $(m \times (n + 1))$ matrix $\begin{array}{c} (x_0) \quad \cdots \quad (x_n) \\ \left[\begin{array}{c} p_1 \\ \hline \vdots \\ \hline p_r \end{array}\right] \end{array}$.

As we want to study ideals P such that $Z(S, P)$ is an algebra *for all* S, we may assume that there are no inputs $u_i \neq 0$, such that $\langle t \mid U_1, \cdots, U_{i-1}, u_i, U_{i+1}, \cdots, U_n \rangle = 0$ for all $t \in S$. Such a system of tensors is called *nondegenerate*. Additionally, we may consider a nondegenerate system which is surjective onto U_0. Such a system is called *full*. A system of tensors which is full and nondegenerate is called *fully nondegenerate*. Finally, for some subset $A \subseteq \{0, \cdots, n\}$, let $Z_A(S, P)$ denote the restriction of $Z(S, P)$ to A. That is, $Z_A(S, P) \subseteq E_k[U_A] \coloneqq End(U_{a_1}) \times \cdots \times End(U_{a_{|A|}})$.

Theorem 2.5.11. *([1], Theorem D) Let* $S \subseteq \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$ *be a fully nondegenerate system of tensors and suppose* $dim(U_i) > 1$ *for all* i*. Let* $A \subseteq \{0, \cdots, n\}$ *and let* $P = (p_1, \cdots, p_r)$ *be supported on* A *with rank* r*. If* $Z_A(S, P)$ *is closed under some product* $\bullet_{(\lambda,\rho)}$ *for nonzero* λ, ρ*, then for at most* $|A| - 2r$ *indices* $a \in A$*, the products* $\bullet_{(\lambda_a,\rho_a)}$ *are non-Lie.*

Note that if $dim(U_i) = 1$ for some i, then $End(U_i) \cong k$ and there is no distinction between any product $\bullet_{(\lambda_i,\rho_i)}$, so it is reasonable to focus on the case when $dim(U_i) > 1$ for all $i \in \{0, \cdots, n\}$. This theorem demonstrates the prevalence of Lie algebras when considering algebras of transverse operators. For example, if $S \subseteq \{U_1 \times U_2 \rightarrowtail U_0\}$ and $P = (p)$ is principle with full support, then for at least one $i \in \{0, 1, 2\}$, $\bullet_{(\lambda_i,\rho_i)}$ is a Lie product. To get associative algebras, one would either need to add additional polynomials to P or to work with an ideal without full support. Examples of associative algebras exist in both cases, see section 4.3.1.

Proof. We show that when $Z_A(S, P)$ is closed under products $\bullet_{(\lambda,\rho)}$, then $|A| - 2r$ many (λ_a, ρ_a) satisfy $\lambda_a + \rho_a = 0$.

Consider the symmetric k-bilinear form $(|)_k : k^{|A|} \times k^{|A|} \rightarrowtail k$ defined by,

$$\left(\begin{bmatrix} l_{a_1} \\ \vdots \\ l_{a_{|A|}} \end{bmatrix} \mid \begin{bmatrix} l'_{a_1} \\ \vdots \\ l'_{a_{|A|}} \end{bmatrix}\right)_k = (\lambda_{a_1} + \rho_{a_1}) l_{a_1} l'_{a_1} + \cdots + (\lambda_{a_n} + \rho_{a_n}) l_{a_n} l'_{a_n}.$$

Example 2.5.12. Let $n = 2$ and $A = \{0, 1, 2\}$. Then $(|)_k : k^3 \times k^3 \rightarrowtail k$ has the form,

$$\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \mid \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}\right)_k = (\lambda_0 + \rho_0) a_0 b_0 + (\lambda_1 + \rho_1) a_1 b_1 + (\lambda_2 + \rho_2) a_2 b_2.$$

If $\lambda_a + \rho_a = 0$ for some $a \in A$, then $(e_a \mid v)_k = 0$ for all $v \in k^{|A|}$. We conclude that $span\{e_a : \lambda_a + \rho_a = 0\} \subseteq Rad((|)_k)$. If k has torsion, then one can construct inputs v_a such that $\sum_{a \in A} (\lambda_a + \rho_a) v_a u_a \equiv \sum_{a \in A} 0 u_a$ when $\lambda_a + \rho_a \neq 0$. We conclude that the number of Lie products we seek is the dimension of the torsion free part of $Rad((|)_k)$. Denote this by f.

Let $Z_A(P)$ denote the closed subscheme of $\mathbb{A}_k^{|A|}$ carved out by P. Precisely, $Z_A(P) = \{z \in k^{|A|} : p(z) = 0, \forall p \in P\}$. Recall that $d_i := dim(U_i)$ and consider the morphism $\varphi : \mathbb{A}_k^{n+1} \to E_k[U_0, \cdots, U_n]$ sending $(k_0, \cdots, k_n) \mapsto (k_0 I_{d_0}, \cdots, k_n I_{d_n})$. $Z_A(P)$ is a subscheme of $\mathbb{A}_k^{n+1}$ and $Z_A(S, P)$ is a subscheme of $E_k[U_0, \cdots, U_n]$. Observe that since $p(k) = 0$, we have that for all $t \in S$, $\langle t \mid p(\varphi(k)) = \langle t \mid p(k_0 I_{d_0}, \cdots, k_n I_{d_n}) = 0$. This implies that φ restricts to an embedding $Z_A(P) \hookrightarrow Z_A(S, P)$.

We show that $|A| - 2r \leq f$. Suppose Z be a subspace of $L^{|A|}$ such that for all $a, b \in Z$, $(a \mid b)_k = 0$. Such a subspace is called a *totally isotropic* subspace of $L^{|A|}$. The rank of a totally isotropic subspace is bounded above by $\frac{|A|-f}{2}$ so it suffices to show show $Z_A(P)$ is totally isotropic and compute it's rank.

Since P has rank $r \leq |A|$, the $m \times (|A|)$ matrix $\begin{array}{c} (x_{a_1}) \quad \cdots \quad (x_{a_{|A|}}) \\ \left[\begin{array}{c} p_1 \\ \hline \vdots \\ \hline p_r \end{array}\right] \end{array}$ defines a surjection $\pi : k^{|A|} \to k^r$ which implies that $k^{|A|} = k^r \oplus ker(\pi)$. However, $ker(\pi) = Z_A(P)$ by definition. Then $dim(Z_A(P)) = |A| - r$.

We now show that $Z_A(P)$ is a totally isotropic subspace of $\mathbb{A}_k^{|A|}$. Let $l = (l_a)_{a \in A}$ and $l' = (l'_a)_{a \in A} \in Z_A(P)$. Then $p(l) = p(l') = 0$ for all $p \in P$. Then since φ restricts to an embedding $Z_A(P) \hookrightarrow Z_A(S, P)$, $\varphi(l)$ and $\varphi(l') \in Z_A(S, P)$. Suppose $p = \sum_{a \in A} \alpha_a x_a$. Since $Z_A(S, P)$ is closed under $\bullet_{(\lambda, \rho)}$,

$$\begin{aligned} 0 = \langle t \mid p(\varphi(l) \bullet_{(\lambda,\rho)} \varphi(l')) &= \langle t \mid p([(\lambda_a + \rho_a) l_a l'_a I_{d_a}]_{a \in A}) \\ &= \sum_{a \in A} \alpha_a (\lambda_a + \rho_a) l_a l'_a (\langle t \mid x_a(I_{d_a})). \end{aligned}$$

Since this is quantified over all $\langle t$ in a fully nondegenerate system, we conclude that $\sum_{a \in A}(\lambda_a + \rho_a) l_a l'_a = 0$. This is exactly the statement that $(l \mid l')_k = 0$ and so $Z_A(P)$ is a totally isotropic subspace of $\mathbb{A}_k^{|A|}$.

Putting these pieces together,

$$dim(Z_A(P)) = dim(ker(\pi)) = |A| - r \leq \frac{|A| - f}{2} \leq f + \frac{|A| - f}{2}$$
$$\implies 2|A| - 2r \leq |A| + f$$
$$\implies |A| - 2r \leq f$$

The result follows as we now have at least $|A| - 2r$ many $a \in A$ for which $\lambda_a + \rho_a = 0$. □

Remark 2.5.13. Note that the Z-sets in theorem 2.5.11 are not required to be symmetric. This means that no modifications are needed to adapt this proof to "symmetric" Z-sets.

2.6 Group actions on T/I/Z-sets

Let G be a group which acts on index set $\{1, \cdots, n\}$. To state a symmetrized TIZ-correspondence, we first define an action of G on tensors, polynomials, and operators.

Definition 2.6.1. Let $g \in G$, $\langle t | \in \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$, $p \in k[X]$, $\omega \in E_k[U_0, \cdots, U_n]$, and $u = (u_1, \cdots, u_n)$ with $u_i \in U_i$.

- $u^g = (u_0, u_{g(1)}, \cdots, u_{g(n)})$.
- $\langle t^g \mid u^g \rangle = \langle t \mid u \rangle$.
- $p^g(x_1, \cdots, x_n) = p(x_0, x_{g(1)}, \cdots, x_{g(n)}) = g^{-1}p(x_1, \cdots, x_n)$.
- $\omega^g = (\omega_0, \omega_{g(1)}, \cdots, \omega_{g(n)})$.

Lemma 2.6.2. *This defines a group action.*

Proof.

- $(u^g)^h = (u_{h(1)}, \cdots, u_{h(n)})^g = (u_{gh(1)}, \cdots, u_{gh(n)})$.
- $\langle (t^g)^h \mid (u^g)^h \rangle = \langle t^g \mid u^g \rangle = \langle t \mid u \rangle$.

- $(p^g)^h(x_1, \cdots, x_n) = p^g(x_{h(1)}, \cdots, x_{h(n)}) = p(x_{gh(1)}, \cdots, x_{gh_(n)})$.
- $(\omega^g)^h = (\omega_{h(1)}, \cdots, \omega_{h(n)})^g = (\omega_{gh(1)}, \cdots, \omega_{gh(n)})$.

$\square$

Example 2.6.3. Let $V = \mathbb{R}^n$. Let $\langle \cdot \mid : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ define left multiplication by a scalar. That is, $\langle \cdot \mid \lambda, v \rangle = \lambda \cdot v$. Applying the permutation (12) will permute the inputs to $\langle \cdot \mid$ which will produce a new tensor $\langle \cdot^{(12)} \mid : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ via $\langle \cdot^{(12)} \mid v, \lambda \rangle = \lambda \cdot v$.

Example 2.6.4. Let $\langle out \mid : \mathbb{R}^2 \times \mathbb{R}^3 \to Mat_{2\times3}(\mathbb{R})$ be the outer product. In coordinates,

$$\langle out \mid \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \\ e \end{bmatrix} \rangle = \begin{bmatrix} ac & ad & ae \\ bc & bd & be \end{bmatrix}$$

Applying the permutation (12) will produce a new tensor $\langle out^{(12)} \mid \mathbb{R}^3 \times \mathbb{R}^2 \to Mat_{2\times3}(\mathbb{R})$ via

$$\langle out^{(12)} \mid \begin{bmatrix} c \\ d \\ e \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \rangle = \begin{bmatrix} ac & ad & ae \\ bc & bd & be \end{bmatrix}$$

Now that we have defined an action of G on tensors, polynomials, and operators, we next want to study how G acts on the tensor $\langle t \mid p(\omega)$. Since $\langle t^g \mid : U_{g(1)} \times \cdots \times U_{g(n)} \rightarrowtail k$, the action of $g \in G$ induces an action on $p(\omega)$. Denote the image as $q(\tau)$. We now relate $q(\tau)$ to $p(\omega)$. We have that

$$\langle t \mid p(\omega) \mid u \rangle = \langle t^g \mid q(\tau) \mid u^g \rangle \iff q = p^{g^{-1}} = gp, \text{ and } \tau = \omega^g.$$

As such, we get the following formula.

Corollary 2.6.5.

$$\langle t \mid p(\omega) \mid u \rangle = \langle t^g \mid (gp)(\omega^g) \mid u^g \rangle \tag{2.7}$$

Example 2.6.6. Let $\langle t \mid: U_1 \times U_2 \times U_3 \rightarrowtail k$ with $dim(U_1) \neq dim(U_2) \neq dim(U_3)$. Let $p = x_1^3 x_2^2 x_3$ and $\omega = (\omega_1, \omega_2, \omega_3) \in End(U_1) \times End(U_2) \times End(U_3)$. then

$$\begin{array}{ccc}
\langle t \mid p(\omega) \mid u_1, u_2, u_3 \rangle & \xrightarrow{g} & \langle t^g \mid q(\tau) \mid u_{g(1)}, u_{g(2)}, u_{g(3)} \rangle \\
\| & & \| \\
\langle t \mid \omega_1^3 u_1, \omega_2^2 u_2, \omega_3 u_3 \rangle & \xrightarrow{g} & \langle t \mid \omega_2^2 u_2, \omega_3 u_3, \omega_1^3 u_1 \rangle \\
 & & \| \\
 & & \langle t \mid (gp)(\omega^g) \mid u^g \rangle
\end{array}$$

From this formula, we can now reason about the action of g on $T/I/Z$ sets. For $\langle t \mid \in \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$, $p \in k[X]$, and $\omega \in E_k[U_0, \cdots, U_n]$, we have the following special cases of corollary 2.6.5.

$$\langle t \mid (gp)(\omega^g) \mid u \rangle = \langle t^{g^{-1}} \mid p(\omega) \mid u^{g^{-1}} \rangle \tag{2.8}$$

$$\langle t^g \mid p(\omega^g) \mid u^g \rangle = \langle t \mid (g^{-1}p)(\omega) \mid u \rangle \tag{2.9}$$

$$\langle t^g \mid (gp)(\omega) \mid u^g \rangle = \langle t \mid p(\omega^{g^{-1}}) \mid u \rangle \tag{2.10}$$

For subsets $S \subseteq \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$, $P \subseteq k[X]$, $\Omega \subseteq E_k[U_0, \cdots, U_n]$, and for some $g \in G$, denote the image of each set under g with S^g, gP, and Ω^g respectively. Then these special cases imply the following corollary.

Corollary 2.6.7. *The following sets are in bijection.*

- $T(gP, \Omega^g)$ *and* $T^{g^{-1}}(P, \Omega)$
- $I(S^g, \Omega^g)$ *and* $g^{-1}I(S, \Omega)$
- $Z(S^g, gP)$ *and* $Z^{g^{-1}}(S, P)$

It will be useful for our investigation into algebras to develop corollary 2.6.7 using the scheme-theoretic formulation of T/I/Z-sets given in section 2.1. Fortunately, since the sets

$T(P,\Omega)_L, I(S,\Omega)_L$, and $Z(S,P)_L$ are the images of $T/I/Z$-sets under scalar extensions, corollary 2.6.7 remains unchanged under extensions.

Corollary 2.6.8. *For any commutative k-algebra L, the following sets are in bijection.*

- $T(gP,\Omega^g)_L$ *and* $T^{g^{-1}}(P,\Omega)_L$
- $I(S^g,\Omega^g)_L$ *and* $g^{-1}I(S,\Omega)_L$
- $Z(S^g,gP)_L$ *and* $Z^{g^{-1}}(S,P)_L$

If $U_i \cong U_j \cong V$ for all pairs (i,j), the above bijections become equalities.

Chapter 3

Tensors, Polynomials, and Operators under Symmetry

3.1 TIZ Does Not Inherantly Capture Symmetry

Since we are studying the effects of the TIZ-correspondence under symmetry, we first want to study whether this correspondence internally captures symmetry. We prove this is false for 2×2 matrices. Precisely,

Proposition 3.1.1. *Let $k = \mathbb{C}$ and $V = \mathbb{C}^2$. There do not exist subsets $P \subseteq k[x, y]$ and $\Omega \subseteq E_k[V, V]$ such that $Sym^2(V) = T(P, \Omega)$.*

Proof. Let $\bigwedge^2(V)$ denote the space of 2×2 skew-symmetric matrices. That is, matrices M such that $M^\intercal = -M$. Then since every matrix can be written as a sum of a symmetric and a skew symmetric matrix, $Sym^2(V) \cong {Mat_{2\times 2}}/{\bigwedge^2(V)} \cong {V \otimes V}/{\bigwedge^2(V)}$. Recall that from the characterization of T-sets given in section 2.3.1, $T(P, \Omega) = {V \otimes V}/{\Xi(P, \Omega)}$. Hence, it suffices to show there are no such P, Ω such that $\Xi(P, \Omega) = \bigwedge^2(V)$.

By definition of Ξ, $\Xi(P, \Omega) = \bigoplus_{p \in P, \omega \in \Omega} \Xi(p, \omega)$. Since $V = \mathbb{C}^2$, then $\bigwedge^2(V)$ is one-dimensional. If there were subsets P, Ω such that $\Xi(P, \Omega) = \bigwedge^2(V)$, then there exists some polynomial p' and transverse operator ω' such that $\Xi(p', \omega') = \bigwedge^2(V)$. As such, it suffices to show there does not exist a single polynomials p and transverse operator ω such that $\Xi(p, \omega) = \bigwedge^2(V)$.

An alternate description of $\bigwedge^2(V)$ is as the image of the *skew-symmetrization* operator $\pi_\wedge$. That is, the linear map

$$\begin{aligned} \pi_\wedge : & V \otimes V \to \bigwedge^2(V) \\ & u \otimes v \mapsto \frac{1}{2}(u \otimes v - v \otimes u) \end{aligned}$$

. On the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ of $V \otimes V$, this is the 4×4 matrix,

$$\pi_\wedge = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So then since $\Xi(p, \omega)$ is the image of $p(\omega)$, it suffices to show that for no single choice of polynomial p and transverse operator ω, the map $p(\omega) = \pi_\wedge$.

By eigentheory, if p has a summand with x-degree or y-degree greater than 1, then $p(\omega) \cdot (u_1 \otimes u_2)$ can be rewritten in terms of some $p'(\omega)$ where the x and y-degrees of every monomial in p' is less than or equal to 1. So we may further assume $p = a + bx + cy + dxy$ for some $a, b, c, d \in k$. Then by setting $\omega_1 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ and $\omega_2 = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$, we can identify each monomial of $p(\omega)$ with a 4×4 matrix via

$$a \mapsto aI_2 \otimes I_2 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

$$bx \mapsto bX \otimes I_2 = \begin{bmatrix} bx_{11} & bx_{12} & 0 & 0 \\ bx_{21} & bx_{22} & 0 & 0 \\ 0 & 0 & bx_{11} & bx_{12} \\ 0 & 0 & bx_{21} & bx_{22} \end{bmatrix}$$

$$cy \mapsto cI_2 \otimes Y = \begin{bmatrix} cy_{11} & 0 & cy_{12} & 0 \\ 0 & cy_{11} & 0 & cy_{12} \\ cy_{21} & 0 & cy_{22} & 0 \\ 0 & cy_{21} & 0 & cy_{22} \end{bmatrix}$$

$$dxy \mapsto dX \otimes Y = \begin{bmatrix} dx_{11}y_{11} & dx_{11}y_{12} & dx_{12}y_{11} & dx_{12}y_{12} \\ dx_{11}y_{21} & dx_{11}y_{22} & dx_{12}y_{21} & dx_{12}y_{22} \\ dx_{21}y_{11} & dx_{21}y_{12} & dx_{22}y_{11} & dx_{22}y_{12} \\ dx_{21}y_{21} & dx_{21}y_{22} & dx_{22}y_{21} & dx_{22}y_{22} \end{bmatrix}$$

where $A \otimes B$ denotes the Kronecker product of A and B. Then we can verify with a computation that $\pi_\wedge$ is not in the span of these four Kronecker products. $\square$

3.2 First Pass: Symmetric TIZ Through Homogeneous Polynomials

Section 3.1 gives evidence that there is no intrinstic characterization of symmetry in the TIZ-correspondence. That is, we cannot just add some additional polynomials and operators to move the normal TIZ-correspondence to a symmetric setting. This means that we must force symmetry externally on the correspondence. In other words, we must assume that our tensors, polynomials, and operators satisfy some kind of symmetry. We now describe some of the ways that symmetry can be enforced on the correspondence.

3.2.1 TIZ Action on Homogeneous Polynomials

We first recall the polarization map. Choose a basis $\{x_1, \cdots, x_n\}$ to identify V with k^n. Let $k[X]_d$ denote the homogeneous degree d component of $k[X]$. Given a symmetric d-linear form $\langle t \mid$, evaluating the tensor on d copies of the same input gives rise to a homogeneous degree d polynomial in variables $\{x_1, \cdots, x_n\}$.

Example 3.2.1. Consider the tensor $GHZ \coloneqq e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$ viewed as a trilinear map $k^2 \times k^2 \times k^2 \rightarrowtail k$. Then

$$\langle GHZ \mid \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle = x^3 + y^3$$

Now if k has sufficiently large characteristic, then we define a map in the other direction. For a subset $I \subseteq [d]$ and d input vectors $u_1, \cdots, u_d$, let u_I be the sum of the u_i for all $i \in I$. Given a homogeneous polynomial Q of degree d in $k[x_1, \cdots, x_n]$, define a d-linear form t via,

$$\langle t \mid u_1, \cdots, u_d \rangle \coloneqq \frac{1}{d!} \sum_{i=1}^{d} (-1)^{d-i} (\sum_{I \subseteq [n] : |I| = d} Q(u_I)).$$

Example 3.2.2. Starting with the polynomial $Q(x, y) = x^3 + y^3$, build the form t as

$$
\begin{aligned}
&\langle t \mid u, v, w\rangle = \langle t \mid \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\rangle \\
&= \frac{1}{6}(Q(u+v+w) - Q(u+v) - Q(u+w) - Q(v+w) + Q(u) + Q(v) + Q(w)) \\
&= \frac{1}{6}((u_1+v_1+w_1)^3 + (u_2+v_2+w_2)^3 \\
&\qquad - (u_1+v_1)^3 - (u_2+v_2)^3) - (u_1+w_1)^3 - (u_2+w_2)^3 - (v_1+w_1)^3 - (v_2+w_2)^3 \\
&\qquad + (u_1^3+u_2^3) + (v_1^3+v_2^3) + (w_1^3+w_2^3) \\
&= u_1v_1w_1 + u_2v_2w_2
\end{aligned}
$$

Now we use this model to define a TIZ action on symmetric tensors. Observe that $Mat_{n\times n}(k)$ acts naturally on k^n which induces an action on $Sym^d(V)$ via,

$$
\begin{aligned}
&Mat_{n\times n}(k) \times k[x_1, \cdots, x_n] \to k[x_1, \cdots, x_n]_d \\
&(M, p(X)) \mapsto p(MX).
\end{aligned}
$$

After passing this through the polarization map, this is equivalent to having $\Delta(End(V)) = \{(\varphi, \cdots, \varphi) : \varphi \in End(V)\} \subseteq End(V)^{\times n}$ act via the symmetric function $e_n = x_1 \cdots x_n$.

Example 3.2.3. After choosing the standard basis for $\mathbb{C}^2$, let W be a bilinear map as in 2.1.2. Let $\omega = (\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) \in End(\mathbb{C}^2)^{\times 3}$. As a homogeneous polynomial, $W = x^2y \in k[x, y]$. Then,

$$
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}
$$

So as a homogeneous polynomial, $M * W = xy^2$. Passing this through the polarization operator gives the bilinear map

$$\begin{aligned} \langle M * W \mid : \mathbb{C}^2 \times \mathbb{C}^2 &\rightarrowtail \mathbb{C}^2 \\ (e_1, e_1) &\mapsto 0 \\ (e_1, e_2) &\mapsto e_2 \\ (e_2, e_1) &\mapsto e_2 \\ (e_2, e_2) &\mapsto e_1. \end{aligned}$$

Note that this is the same bilinear map as $\langle W \mid (xy)(\omega) \mid$.

$$\begin{aligned} &\langle W \mid (xy)(\omega) \mid : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrowtail \mathbb{C}^2 \\ &(e_1, e_1) \mapsto \langle W \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid e_2, e_2 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} 0 = 0 \\ &(e_1, e_2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid e_2, e_1 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1 = e_2 \\ &(e_2, e_1) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid e_2, e_1 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1 = e_2 \\ &(e_2, e_2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \langle W \mid e_1, e_1 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2 = e_1 \end{aligned}$$

By remark 2.5.2, closure under this TIZ-action would give a ternary Galois correspondence between subsets $S \subseteq Sym^d(V)$, ideals $P \subseteq k[e_n]$, and subsets of operators $\Omega \subseteq \Delta(End(V))$. So then we may state the first version of a symmetric TIZ-correspondence.

Theorem 3.2.4. *Let $S \subseteq Sym^n(V^*), P \subseteq k[x]$, and $\Omega \subseteq \Delta End(V) \subseteq E_k[V, \cdots, V]$. Then,*

(a) For all $p \in k[x]$, $\omega \in \Delta(End(V))$, and $\langle t \mid \in Sym^n(V^*)$, the tensor $\langle t \mid p(\omega) \in Sym^d(V^*)$.

(b) *The set* $T(P,\Omega)\cap Sym^n(V^*)$ *forms a vector subspace of* $Sym^n(V^*)$*, the set* $I(S,\Omega)$ *forms an ideal of* $k[x]$*, and the set* $Z(S,P)\cap\Delta End(V)$ *forms an affine subscheme of* $\Delta End(V)$*.*

(c) *There exists a ternary inclusion-reversing Galois correspondence between vector spaces* $T(P,\Omega)\cap Sym^n(V^*)$*, ideals* $I(S,\Omega)$*, and affine schemes* $Z(S,P)\cap\Delta End(V)$*. Precisely,*

$$S\subseteq T(P,\Omega)\iff P\subseteq I(S,\Omega)\iff \Omega\subseteq Z(S,P).$$

Proof. By remark 2.5.2, it suffices to show part (a). We proceed with a computation. Let $p=\sum_{i=1}^m \lambda_i x^i$, $\omega=(X,\cdots,X)$, and $g\in G$. Then,

$$\begin{aligned}\langle t\mid p(\omega)\mid u\rangle &= \langle t\mid p(\omega)\mid u_1,\cdots,u_n\rangle\\ &=\sum_{i=1}^m \lambda_i\langle t\mid X^iu_1,\cdots,X^iu_n\rangle\\ &=\sum_{i=1}^m \lambda_i\langle t\mid X^iu_{g(1)},\cdots,X^iu_{g(n)}\rangle\\ &=\langle t\mid p(\omega)\mid u_{g(1)},\cdots,u_{g(n)}\rangle.\end{aligned}$$

□

Observe that, as the variable x is acting as the degree n symmetric polynomial e_n, the action of a subset Ω will rarely be additive. As such, we are unable to build algebras of tensor operators. With this goal in mind, we may then try to act by a different symmetric polynomial so that we can build Z-sets defined by linear polynomials. The only symmetric polynomial in n variables is the polynomial $e_1 \coloneqq x_1+\cdots+x_n$. So if the expression $\langle t\mid e_1(\omega)$ is symmetric for all $t\in Sym^d(V)$ and $\omega\in\Delta(End(V))$, then we can use remark 2.5.2 to get a TIZ-correspondence which allows the construction of algebras from Z-sets.

Theorem 3.2.5. *(Symmetric* TIZ *version I) Let* $S\subseteq Sym^n(V^*), P\subseteq k[x]$*, and* $\Omega\subseteq \Delta End(V)\subseteq E_k[V,\cdots,V]$*. Then,*

1. *For all* $p \in k[x], \omega \in \Delta(End(V))$ *and* $\langle t \mid \in Sym^n(V^*)$*, the tensor* $\langle t \mid p(\omega) \mid \in Sym^d(V^*)$*.*

2. *The set* $T(P,\Omega) \cap Sym^n(V^*)$ *forms a k-vector subspace of* $Sym^n(V^*)$*, the set* $I(S,\Omega)$ *forms an ideal of* $k[x]$*, and the set* $Z(S,P) \cap \Delta End(V)$ *forms an affine subscheme of* $\Delta End(V)$*.*

3. *There exists a ternary inclusion-reversing Galois correspondence between vector spaces* $T(P,\Omega) \cap Sym^n(V^*)$*, ideals* $I(S,\Omega)$*, and affine schemes* $Z(S,P) \cap \Delta End(V)$*. Precisely,*

$$S \subseteq T(P,\Omega) \iff P \subseteq I(S,\Omega) \iff \Omega \subseteq Z(S,P).$$

Proof. By remark 2.5.2, it suffices to show part (1). We proceed with a computation. Let $p = \sum_{i=1}^{m} \lambda_i x^i$, $\omega = (X, \cdots, X)$, and $g \in S_n$. Then,

$$\begin{aligned}
\langle t \mid p(\omega) \mid u \rangle &= \langle t \mid p(\omega) \mid u_1, \cdots, u_n \rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \langle t \mid u_1, \cdots, X^i U_j, \cdots u_n \rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \langle t \mid u_{g(1)}, \cdots, X^i u_{g(j)}, \cdots, u_{g(n)} \rangle \\
&= \langle t \mid p(\omega) \mid u_{g(1)}, \cdots, u_{g(n)} \rangle
\end{aligned}$$

□

From here, we have a natural generalization to partially symmetric tensors. For a space $Sym^{d_j}(U_j^*)$, let $\Delta_j(End(U_j)) \subseteq End(U_j)^{\times d_j}$ be the diagonally embedded copy of $End(U_j)$.

Theorem 3.2.6. *(Partially-symmetric TIZ version I) Let* $S \subseteq \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}, P \subseteq k[x_0, \cdots, x_n]$*, and* $\Omega \subseteq \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n)) \subseteq E_k[U_1, \cdots, U_1, \cdots, U_n, \cdots, U_n]$*. Then,*

(a) *For all* $p \in k[x_0, \cdots, x_n], \omega \in \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$ *and* $\langle t \mid \in \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$, *the tensor* $\langle t \mid p(\omega) \in \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$.

(b) *The set* $T(P, \Omega) \cap \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$ *forms a* k*-vector subspace of* $\{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$, *the set* $I(S, \Omega)$ *forms an ideal of* $k[x_0, \cdots, x_n]$, *and the set* $Z(S, P) \cap \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$ *forms an affine subscheme of* $\Delta End(V)$.

(c) *There exists a ternary inclusion-reversing Galois correspondence between vector spaces* $T(P, \Omega) \cap \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$, *ideals* $I(S, \Omega)$, *and affine schemes* $Z(S, P) \cap \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$. *Precisely,*

$$S \subseteq T(P, \Omega) \iff P \subseteq I(S, \Omega) \iff \Omega \subseteq Z(S, P).$$

Proof. Combine 2.3.9 with 3.2.5. The rest follows from remark 2.5.2. □

3.3 Second Pass: Symmetry on each axis

From section 3.2.1, we saw that we could develop a TIZ-correspondence for subsets of symmetric tensors by restricting the operators to $\Delta(End(V))$ and acting via some symmetric polynomial. Observe that we are able to prove an analogy of theorem 2.3.9 regardless of which symmetric polynomial we chose. This means that we could avoid choosing a symmetric polynomial altogether by acting via the TIZ-action from [1] and instead requiring P to consist of symmetric polynomials. This model of TIZ gives a correspondence between subspaces of symmetric tensors, ideals of symmetric polynomials, and subsets of diagonally-embedded operators in $End(V)$. In section 2.6, we studied how a group G acting on $\{1, \cdots, n\}$ induces an action on tensors, polynomials, and operators. Define the symmetric tensors, polynomials, and operators to be those invariant under this action. That is, $\langle t^g = \langle t$, $gp = p$, and $\omega^g = \omega$. Then by remark 2.5.2, the most important part of a TIZ-correspondence for symmetric tensors is that a symmetric $T/I/Z$-set is closed under the TIZ-action. We will show this now.

Proposition 3.3.1. *Let $t \in \{U^{\times n} \rightarrowtail V\}$ be a symmetric tensor, $p \in k[X]$ be a symmetric polynomial, and $\omega \in E_k[U, \cdots, U, V]$ be a symmetric operator. Then $\langle t \mid p(\omega)$ is a symmetric tensor.*

Proof. By 2.6.7, $\langle t \mid p(\omega) \mid u\rangle = \langle t^g \mid gp(\omega^g) \mid u^g\rangle$. Since t, p, ω are all symmetric, $\langle t^g \mid gp(\omega^g) \mid u^g\rangle = \langle t \mid p(\omega) \mid u^g\rangle$. □

From this and remark 2.5.2, we get a generalization of theorem 3.2.5.

Theorem 3.3.2. *(Symmetric TIZ-correspondence version II) Let $S \subseteq Sym^n(V^*), P \subseteq \bigwedge_k(x_1, \cdots, x_n)$, and $\Omega \subseteq \Delta End(V)$. Then,*

1. *For all $p \in \bigwedge_k(x_1, \cdots, x_n)$, $\omega \in \Delta(End(V))$, and $s \in Sym^n(V^*)$, the tensor $\langle t \mid p(\omega) \in Sym^n(V^*)$.*
2. *The set $T(P, \Omega) \cap Sym^n(V^*)$ forms a k-vector subspace of $Sym^n(V)$, the set $I(S, \Omega) \cap \bigwedge(x_1, \cdots, x_n)$ forms an ideal of $\bigwedge_k(x_1, \cdots, x_n)$, and the set $Z(S, P) \cap \Delta End(V)$ forms an affine subscheme of $\Delta End(V)$.*
3. *There exists a ternary inclusion-reversing Galois correspondence between vector spaces $T(P, \Omega) \cap Sym^n(V^*)$, ideals $I(S, \Omega) \cap \bigwedge_k(x_1, \cdots, x_n)$, and affine schemes $Z(S, P) \cap \Delta End(V)$. Precisely,*

$$S \subseteq T(P, \Omega) \iff P \subseteq I(S, \Omega) \iff \Omega \subseteq Z(S, P).$$

Proof. Proposition 3.3.1 is the content of part (1). Then part (2) follows from the proof of theorem 2.3.9 given in [1] along with the observation that intersecting vector spaces, ideals, and affine schemes is itself a vector space, ideal, or affine scheme respectively. Part (3) follows immediatly from the definitions of $T(P,\Omega)$, $I(S,\Omega)$, and $Z(S,P)$ as sets. $\square$

Remark 3.3.3. Note that the calculation from proposition 3.3.1 would apply analogously to skew-symmetry or similar forms of symmetry. More generally, letting χ be a multiplicative character of G, we may define χ-symmetry by requiring that $\langle t \mid u^g\rangle = \chi(g)\langle t \mid u\rangle$. Then by enforcing the same conditions on the polynomials and operators, we get a variant of theorem 2.3.9 for χ-symmetric tensors, polynomials, and operators.

This model also provides a natural generalization to partially symmetric tensors.

Theorem 3.3.4. *(Partially-symmetric TIZ version II) Let* $S \subseteq \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}, P \subseteq \bigwedge_k(x_{0,0},\cdots,x_{0,d_1}) \otimes \cdots \otimes \bigwedge_k(x_{n,0},\cdots,x_{n,d_n})$, *and* $\Omega \subseteq \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n)) \subseteq E_k[U_1,\cdots,U_1,\cdots,U_n,\cdots,U_n]$. *Then,*

1. *For all* $p \in \bigwedge_k(x_{0,0},\cdots,x_{0,d_1}) \otimes \cdots \otimes \bigwedge_k(x_{n,0},\cdots,x_{n,d_n}), \omega \in \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$ *and* $\langle t \mid \in \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$, *the tensor* $\langle t \mid p(\omega) \in \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$.

2. *The set* $T(P,\Omega) \cap \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$ *forms a k-vector subspace of* $Sym^n(V^*)$, *the set* $I(S,\Omega) \cap \bigwedge_k(x_{0,0},\cdots,x_{0,d_1}) \otimes \cdots \otimes \bigwedge_k(x_{n,0},\cdots,x_{n,d_n})$ *forms an ideal of* $\bigwedge_k(x_{0,0},\cdots,x_{0,d_1}) \otimes \cdots \otimes \bigwedge_k(x_{n,0},\cdots,x_{n,d_n})$, *and the set* $Z(S,P) \cap \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$ *forms an affine subscheme of* $\Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$.

3. *There exists a ternary inclusion-reversing Galois correspondence between vector spaces* $T(P,\Omega) \cap \{Sym^{d_1}(U_1^*) \times \cdots \times Sym^{d_n}(U_n^*) \rightarrowtail Sym^{d_0}(U_0)\}$, *ideals* $I(S,\Omega) \bigwedge_k(x_{0,0},\cdots,x_{0,d_1}) \otimes \cdots \otimes \bigwedge_k(x_{n,0},\cdots,x_{n,d_n})$, *and affine schemes* $Z(S,P) \cap \Delta_1(End(U_1)) \times \cdots \times \Delta_n(End(U_n))$. *Precisely,*

$$S \subseteq T(P,\Omega) \iff P \subseteq I(S,\Omega) \iff \Omega \subseteq Z(S,P).$$

Proof. Combine 2.3.9 with 3.3.2. The rest follows from remark 2.5.2. □

Remark 3.3.5. Note that having P and Z consisting entirely of symmetric polynomials and operators is not enough to imply that every tensor in $T(P, \Omega)$ is symmetric. For example, consider the polynomial $xy \in k[x, y]$ and the operators $\omega := (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})$. Then for any matrix $M \in T(xy, \omega)$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Any matrix of the form $M = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ is in $T(xy, \omega)$, however, these matrices are not symmetric unless $b = c$.

For an example where the operators have full rank, we can consider the polynomial $x + y$ with the operators $\omega := (\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$. Then $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in T(x+y, \omega)$, yet is not symmetric. However, the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is also in $T(x + y, \omega)$. There exist similar counterexamples for having only T and Z symmetric or T and P symmetric so we really need to have all three axes be symmetric.

While intersecting $T/I/Z$-sets with their symmetric sets gives a well-defined version of the TIZ-correspondence for symmetric tensors, there may be more operators we wish to consider. We show that, from the perspective of isomorphism testing, restricting to operators in $\Delta End(V)$ is a natural choice. Here it will be convinient to interpret symmetric tensors as elements of $V^{\otimes n}$. For tensors $t, t' \in V^{\otimes n}$, an isomorphism $t \cong t'$ is an invertible transverse operator $\omega \in GL(V)^{\times n}$ such that $(x_1 \cdots x_n)(\omega)(t) = t'$.

Theorem 3.3.6. *Let H be a subgroup of $GL(V)^{\times n}$ of the form $H := H_1 \times \cdots \times H_n$ such that $H \supseteq \Delta GL(V)$ and $(x_1 \cdots x_n)(H)(t)$ is symmetric for all $t \in Sym^n(V)$. Then $H_i = \lambda_{i,j} H_j$ for all $1 \le i, j \le n$ and some scalar $\lambda_{i,j}$ depending only on i, j.*

Proof. By assumption, for all rank 1 symmetric tensors $t = v^{\otimes n}$ and all $h = (h_1, \cdots, h_n) \in H$, $(x_1 \cdots x_n)(h_1, \cdots, h_n)(t) = h_1 v \otimes \cdots \otimes h_n v$ is symmetric of rank 1 if and only if $h_i = \lambda_{i,j} h_j$ for all $1 \leq i, j \leq n$ and some scalar $\lambda_{i,j}$ depending only on i, j. The result follows. □

As such, it is natural from the perspective of isomorphism to consider invertible operators in $\Delta GL(V)$ and hence, operators in $\Delta End(V)$.

3.4 Third Pass: Symmetry as Spaces

While restricting to operators in $\Delta(End(V))$ is natural from the perspective of isomorphism testing, restricting to polynomials in $\bigwedge_k(x_1, \cdots, x_n)$ would force us to throw away some natural algebras we could form from Z-sets. For example, the centroid is defined as $Z(\{t\}, (x_1 - x_2, \cdots, x_{n-1} - x_n))$. As each polynomial $x_i - x_j$ is not symmetric, the centroid of a symmetric tensor cannot be constructed as a Z-set under versions I or II of our symmetric TIZ-correspondence. So working exclusively in these versions of the correspondence would force us to throw away some very natural algebras. However, the ideal $(x_1 - x_2, \cdots, x_{n-1} - x_n)$ is symmetric as a set. Here we investigate a symmetric TIZ-correspondence for symmetric *sets* as opposed to sets of symmetric elements.

Definition 3.4.1. Let G be a group acting on $\{1, \cdots, n\}$. Subsets $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0, U_1, \cdots, U_n]$ are *symmetric* if, for all $g \in G$, $S^g = S$, $gP = P$, and $\Omega^g = \Omega$ respectively. That is, if $\langle t \mid \in S$ then $\langle t^g \mid \in S$ and similarly for P and Ω.

By remark 2.5.2, the most important part of a TIZ-correspondence for symmetric sets is that $T/I/Z$-sets built from symmetric S, P, Ω are themselves symmetric. We will show this now.

Lemma 3.4.2. *Let G be a group acting on $\{1, \cdots, n\}$. Let $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0, U_1, \cdots, U_n]$ be symmetric. Then $T(P, \Omega)_L$, $I(S, \Omega)_L$, and $Z(S, P)_L$ are all symmetric.*

Proof. By corollary 2.6.8,

$$T(P,\Omega)_L = T^g(gP,\Omega^g)_L = T^g(P,\Omega)_L$$
$$I(S,\Omega)_L = gI(S^g,\Omega^g)_L = gI(S,\Omega)_L$$
$$Z(S,P)_L = Z^g(S^g,gP)_L = Z^g(S,P)_L.$$

□

Example 3.4.3. Consider the W state, given in coordinates as $(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix})$ and viewed as a bilinear map $k^2 \times k^2 \rightarrowtail k^2$. Let $d \coloneqq x_1 + x_2 - x_0$. Since d is symmetric under permuting x_1 and x_2 and W is a symmetric tensor, we would expect that the corresponding Z-set $Z(w,d)$ is symmetric. Indeed, we saw from example 2.3.11,

$$Der(W) =< (\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}), (\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}), (\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}),$$
$$(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}), (\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}) > .$$

where we see that permuting entries 2 and 3 of any tuple remains in the space.

Then since $T/I/Z$-sets formed from symmetric sets are symmetric, remark 2.5.2 indicates that a TIZ-correspondence for symmetric sets should be easy to prove.

Theorem 3.4.4. *(Symmetric TIZ-correspondence version III) Let G be a group acting on $\{1,\cdots,n\}$. Let $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0,\cdots,U_n]$ be symmetric. Then,*

1. *$T(P,\Omega)_L$, $I(S,\Omega)_L$, and $Z(S,P)_L$ are all symmetric.*
2. *$T(P,\Omega)_L$ is an L-submodule of $L\otimes(\{U_1\times\cdots\times U_n \rightarrowtail U_0\})$, $I(S,\Omega)_L$ is an ideal of $L[X]$, and $Z(S,P)_L$ is the zero locus of finitely many polynomial equations in $E_L[U_0,\cdots,U_n]$.*
3. *$S \subseteq T(P,\Omega)_L \iff P \subseteq I(S,\Omega)_L \iff \Omega \subseteq Z(S,P)_L$.*

Proof. Part (1) is the content of lemma 3.4.2. Parts (2) and (3) follow from ([1],Theorem B). □

3.5 Algorithms for Computing T/I/Z-Sets

As in the case of [1], equations defining symmetric Z-sets $Z(S, P)$ are in $d_0^2 + \cdots + d_n^2$ variables while equations in P are in only n variables. This allows for efficient algorithms to compute $T/I/Z$-sets.

Theorem 3.5.1. *([1],Theorem C) For symmetric subsets $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $P \subseteq L[X]$, and $\Omega \subseteq E_L[U_0, \cdots, U_n]$, there exists an algorithm to*

- *Construct generators for $T(P, \Omega)_L$ in time polynomial in the encoding size of P and $T(P, \Omega)_L$.*
- *Construct generators $I(S, \Omega)_L$ in polynomial time if the valence n is bounded. In general, this algorithm returns in quasi-polynomial time.*
- *Construct defining polynomials for $Z(S, P)_L$ in polynomial time. In the case where P is linear, there exists an algorithm to construct generators for $Z(S, P)_L$ as an L-module.*

Proof. In section 2.5, we did not analyze the algorithms from ([1], Theorem C) under an arbitrary commutative k-algebra L. Under the assumption that there exists an oracle to efficiently write down elements of L, and that there is an efficient algorithm to solve systems of linear equations over L, then algorithms given in section 2.5 continue to function over these scalars. Since the algorithms from ([1], Theorem C) are independent of symmetry, these algorithms apply in the same manner when S, P, Ω are symmetric. □

Chapter 4

The Algebras in Symmetric TIZ

4.1 Examples of Symmetric Algebras

We pause briefly to recall our two main examples of algebras constructed out of tensor operators. Observe that both algebras can be constructed in the setting of theorem 3.4.4.

Example 4.1.1. (Centroids) The centroid of a trilinear form $\langle t | : V \times V \times V \rightarrowtail k$ is the associative unital algebra $Cen(t) \coloneqq Z(t, x_1 - x_2, x_2 - x_3)$. $Cen(t)$ is additionally commutative if t is fully nondegenerate. If the tensor t is symmetric, then by lemma 3.4.2, $Cen(t)$ is symmetric.

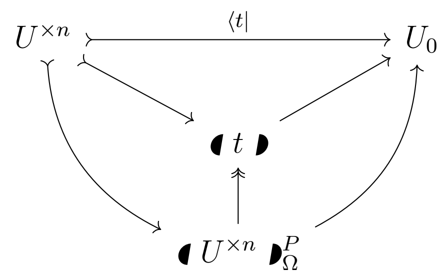
Example 4.1.2. (Derivations) The derivation algebra of a trilinear form $\langle t | : V \times V \times V \rightarrowtail k$ is the Lie algebra $Der(t) \coloneqq Z(t, x_1 + x_2 + x_3)$. If t is symmetric, then by lemma 3.4.2, $Der(t)$ is symmetric.

Note that lemma 3.4.2 applies to closed sets in exactly the same way. That is, $V \otimes_{Cen(t)} V \otimes_{Cen(t)V} = T(x_1 - x_2, x_2 - x_3, Cen(t))$ and $◖\, t\, ◗ = T(x_1 + x_2 + x_3, Der(t))$ are both symmetric as well.

4.2 Universal Property of Symmetric Derivations

Let d denote the polynomial $x_1 + \cdots + x_n$. We defined the Lie algebra of derivations $Der(t) = Z(t, d)$ and observed that $Der(t)$ is closed if t is symmetric. Consider the tensor space $◖\, t\, ◗ \coloneqq T(d, Der(t))$, denoted the *densor* space. Recall the universal property of the densor that we surveyed in theorem 2.5.6.

Theorem 4.2.1. *([1],Theorem A) Let G be a group acting on $\{1, \cdots, n\}$. Let k be a field and $\langle t \in \{U^{\times n} \rightarrowtail V\}$ be a symmetric tensor. Let $P \subseteq L[x_0, \cdots, x_n]$ and $\Omega \subseteq E_L[U_0, \cdots, U_n]$ be symmetric subsets such that P is linear homogeneous with full support and $\langle t \mid$ factors* *through* $◖ U^{\times n} ◗^P_\Omega$. *Then there exists a surjective map* $◖ U^{\times n} ◗^P_\Omega \to ◖ t ◗$ *making the following diagram commute.*



Proof. We remark that the proof of ([1],Theorem A) is independent of symmetry so by the observation that d and $Der(t)$ are symmetric, we may conclude this statement holds when t, P, Ω are symmetric. □

4.3 Most Algebras on Symmetric Spaces are Lie

Recall in section 2.5.2, we showed that a large family of algebras that we could build out of tensor operators were Lie algebras. Specifically, by choosing $\lambda, \rho \in k^{n+1}$ and endowing each index $i \in \{0, \cdots, n\}$ with the product $\omega_i \bullet_{(\lambda,\rho)} \tau_i := \lambda_i \omega_i \tau_i + \rho_i \tau_i \omega_i$, we showed that the product on most indices i were scalar multiples of the Lie product $\omega_i \tau_i - \tau_i \omega_i$. By remark 2.5.13, the same is true for symmetric algebras.

Theorem 4.3.1. *([1],Theorem D) Let* $S \subseteq \{U^{\times n} \rightarrowtail V\}$ *be a fully nondegenerate symmetric susbet of tensors and suppose* $dim(U) > 1$. *Let* $A \subseteq \{0, \cdots, n\}$ *and* $P = (p_1, \cdots, p_r)$ *be supported on* A *with rank* r. *If* $Z_A(S, P)$ *is symmetric and closed under some product* $\bullet_{(\lambda,\rho)}$ *for nonzero* $\lambda, \rho \in k^{n+1}$, *then for at most* $|A| - 2r$ *indices* $a \in A$, *the products* $\bullet_{(\lambda_a,\rho_a)}$ *are non-Lie.*

Proof. As stated in remark 2.5.13, the proof is immediate from the one given in section 2.5. □

4.3.1 Associative Algebras for Symmetric 3-Tensors

Theorem 2.5.11 heavily limits the use of associative algebras which may arise as Z-sets. However, the centroid is an example of an algebra constructed from a Z-set which is closed under an associative product. In this subsection, we are interested in characterizing the ideals P such that $Z_A(S,P)$ is a unital associative algebra. With two exceptions, we will focus on the setting of bilinear forms $\langle t : U \times V \rightarrowtail W$.

For simplicity, consider polynomials in $k[x,y,z]$. As an example of an associative algebra arising as a Z-set, recall the centroid,

$$Cen(t) \coloneqq \{(X,Y,Z) \in E_k[U,V,W] \mid \langle t \mid Xu, v\rangle = \langle t \mid u, Yv\rangle = Z\langle t \mid u,v\rangle \ (\forall u \in U, v \in V)\}.$$

If t is fully nondegenerate, the centroid is commutative and unital, however, not all associative subalgebras of $E_k[U,V,W]$ need to be commutative. Consider the following three associative unital algebras.

$$\mathcal{L}(t) \coloneqq \{(X,Z) \in End(U) \times End(W) \mid \langle t \mid Xu, v\rangle = Z\langle t \mid u,v\rangle \ (\forall u \in U, v \in V)\}$$
$$\mathcal{M}(t) \coloneqq \{(X,Y) \in End(U) \times End(V) \mid \langle t \mid Xu, v\rangle = \langle t \mid u, Yv\rangle \ (\forall u \in U, v \in V)\}$$
$$\mathcal{R}(t) \coloneqq \{(Y,Z) \in End(V) \times End(W) \mid \langle t \mid u, Yv\rangle = Z\langle t \mid u,v\rangle \ (\forall u \in U, v \in V)\}.$$

$\mathcal{L}(t)$ is the Z-set $Z(t,(x-z))$, $\mathcal{M}(t)$ is the Z-set $Z(t,(x-y))$, and $\mathcal{R}(t)$ is the Z-set $Z(t,(y-z))$. For higher-valence tensors, these algebras generalize to be Z-sets of the form $Z(t, x_a - x_b)$ for some choice of $a,b \in \{0,\cdots,n\}$. These Z-sets are referred to as the (a,b)-nucleus $Nuc_{a,b}(t)$ in the literature [1,4,22]. In the case of $\mathcal{L}(t)$, for $(X,Z),(X',Z') \in \mathcal{L}(t)$ and $(u,v) \in U \times V$, we have,

$$\langle t \mid XX'u, v\rangle = Z\langle t \mid X'u, v\rangle = ZZ'\langle t \mid u,v\rangle$$

So $\mathcal{L}(t)$ is an associative subalgebra of $End(U) \times End(W)$. Similarly, $\mathcal{R}(t)$ is an associative subalgebra of $End(V) \times End(W)$. However, something different happens with $\mathcal{M}(t)$. As above, let $(X, Y), (X', Y') \in \mathcal{M}(t)$ and $(u, v) \in U \times V$. Then,

$$\langle t \mid XX'u, v\rangle = \langle t \mid X'u, Yv\rangle = \langle t \mid u, Y'Yv\rangle.$$

So we see that $\mathcal{M}(t)$ is not an associative subalgebra of $End(U) \times End(V)$ but rather $End(U) \times End(V)^{op}$.

Proposition 4.3.2. *([1], Proposition* 6.2.1*) Let $P = (p_1, \cdots, p_r)$ be an ideal supported on A which is generated by linear homogenous polynomials p_i. Then there is a flat extension L/k and $\alpha \in (L^{n+1})_A$ such that $p \coloneqq \sum_a \alpha_a x_a \in P$ and for all $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, $Z_A(S, P)_L \subseteq Z_A(S, p)_L$.*

Proof. In the proof of theorem 2.5.6, we showed that when P is a linear homogeneous ideal of full support, then there exists a field extension L and $\tau \in L^{(n+1)}$ such that $d^{\tau^{-1}} \in P$. Then $Z(S, d^{\tau^{-1}}) \supseteq Z(S, P)$. Let p be $d^{\tau^{-1}}$ and the result follows when $A = \{0, \cdots, n\}$. The argument holds analogously when P is supported on some subset A and we are considering restrictions $\tau \in (L^{(n+1)})_A$. □

Lemma 4.3.3. *([1], Corollary* 6.2.2*) Let k be a field, $A \subseteq \{0, \cdots, n\}$ and let $p \in k[X]$ be linear homogeneous. There exist (λ, ρ) such that $Z_A(S, p)$ is closed under $\bullet_{(\lambda,\rho)}$ for all $S \subseteq \{U_1 \times \cdots \times U_n \rightarrowtail U_0\}$ if and only if $p = \alpha x_a - \beta x_b$ for some $\alpha, \beta \in k$. In this case, $\bullet_{(\lambda,\rho)}$ is an associative product.*

Proof. By theorem 2.5.11, if p were supported on more than two axes, then $Z_A(S, P)$ would be be closed under a Lie product on at least one axis. This shows the "only if" direction. For the other direction, suppose $p = \alpha x_a - \beta x_b$. If $(\alpha, \beta) = (0, 0)$, then $Z_A(S, p) = End(U_a) \times End(U, b)$. If $\alpha \neq 0$ but $\beta = 0$, then for all tensors t and $\omega, \omega' \in Z_A(S, P)$, $\langle t \mid x_a(\omega\omega') = 0$. This is analogous to the case where $\alpha = 0$ and $\beta \neq 0$. If $(\alpha, \beta) \neq (0, 0)$, then there exists a $\tau \in (L^{n+1})_A$ such that $p^\tau = x_a - x_b$. By proposition 4.3.2, it suffices to suppose $p = x_a - x_b$. Then in this case, we may show associativity with a direct computation. Either $a, b \in \{1, \cdots, n\}$

for which $Z_{\{a,b\}}(S,p)$ is a unital associative subalgebra of $End(U_a) \times End(U_b)^{op}$ or one of $a, b = 0$ in which $Z_{\{a,b\}}(S,p)$ is an associative unital subalgebra of $End(U_a) \times End(U_b)$. See the examples of $\mathcal{L}(t), \mathcal{M}(t)$, and $\mathcal{R}(t)$ above for how exactly these two computations will proceed. □

Lemma 4.3.4. *([1], Lemma 6.3.3) Let $P \subseteq k[X]$ be a linear homogeneous ideal. If there exists an $A \subseteq \{0, \cdots, n\}$ such that $(Id_{d_a})_{a\in A} \in Z_A(S,P)$ for all $S \subseteq L \otimes (\{U_1 \times \cdots \times U_n \rightarrowtail U_0\})$, then for every linear homogeneous polynomial $p = \sum_{a\in A} \lambda_a x_a \in P$, $\sum_a \lambda_a = 0$.*

Proof. Consider the multiplication tensor $\mathbf{1}_k : k^n \rightarrowtail k$ which sends $(k_1, \cdots, k_n) \mapsto \prod_{i=1}^n k_i$. If $(Id_{d_a})_{a\in A} \in Z_A(S,P)$ for all subsets S, then in particular, $(1)_{a\in A} \in Z_A(\mathbf{1}_k, P)$. Then for $p = \sum_{a\in A} \lambda_a x_a \in P$, the equation $\langle \mathbf{1}_k \mid p((1)_{a\in A}) \mid k_1, \cdots, k_n\rangle = 0$ becomes $\sum_{a\in A} \lambda_a = 0$. □

We are now ready to prove the classification theorem for bilinear maps $\langle t : U \times V \rightarrowtail W$.

Proposition 4.3.5. *([1], Proposition 6.3.4) Let k be a field, L be a commutative k-algebra, and $P \subseteq L[x_0, x_1, x_2]$ be a linear homogeneous ideal. Then there exists $A \subseteq \{0, 1, 2\}$ such that, for all $S \subseteq \{L \otimes U \times V \rightarrowtail W\}$, $Z_A(S,P)_L$ is a unital associative subalgebra of $E_L[U_A]$ (up to an "op") if and only if P is one of the following:*

1. *$P = 0$. In this case $Z_A(S,P)_L = E_L[U_A]$.*
2. *$P = (x_a - x_b)$ for some indices $a, b \in A$. In this case, $Z_A(S,P)_L$ is one of $\mathcal{L}(S)_L, \mathcal{M}(S)_L$, or $\mathcal{R}(S)_L$.*
3. *$P = (x_0 - x_1, x_1 - x_2)$. In this case, $Z(S,P)_L = \tau Cen(S)_L$ for some $\tau \in (L^\times)^3$.*

Proof. Earlier in this section, we have checked that $\mathcal{L}(S), \mathcal{M}(S)$, and $\mathcal{R}(S)$ are each unital associative subalgebras of $E_k[U_A]$ for some appropriate choice of subset $A \subseteq \{0, 1, 2\}$. $Cen(S)$ embeds into each of $\mathcal{L}, \mathcal{M}$, and $\mathcal{R}$ so in particular, $Cen(S)$ is a unital associative subalgebra of $\mathcal{L}(S)$. Item (1) requires no work to check associativity.

We now show the "if" part. Let $A \subseteq \{0, 1, 2\}$ such that $Z_A(S,P)$ is a unital associative subalgebra of $E_k[U_A]$. First suppose P is principle and generated by a linear homogeneous polynomial $p \in P$. Then $Z_A(S,P)$ is either $\mathcal{L}(S), \mathcal{M}(S)$, or $\mathcal{R}(S)$ by lemma 4.3.4.

If $P = (p_1, p_2)$ is minimally generated by two polynomials, then theorem 2.5.11 implies that both p_i can be written as $\alpha x_a - \beta x_b$ for $i \in \{1,2\}$ and $a, b \in \{0,1,2\}$. Then by proposition 4.3.2, there exists some commutative k-algebra L and $\tau \in (L^{n+1})_A$ such that $(p_1)^{\tau^{-1}} = x_0 - x_1$ and $(p_1)^{\tau^{-1}} = x_1 - x_2$. Then by lemma 2.5.8, $Cen(S)_A = Z_A(S, P^{\tau^{-1}})_L = \tau^{-1} Z_A(S,P)_L$ which means that $\tau Cen(S) = Z_A(S,P)_L$. ☐

With symmetry, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{R}$ coincide.

Proposition 4.3.6. *Let $\langle t \mid : V \times V \times V \rightarrowtail k$ be a symmetric trilinear form. Then $\mathcal{L}(t) = \mathcal{M}(t) = \mathcal{R}(t)$ as algebras of $End(V) \times End(V)^{op}$.*

Proof. Let $(X,Z) \in \mathcal{L}(t)$. We show that $(X,Z) \in \mathcal{R}(t)$. For all $u,v,w \in V$,

$$\langle t \mid Xu, v, w \rangle = \langle t \mid u, v, Zw \rangle$$
$$= \langle t \mid v, Xu, w \rangle = \langle t \mid v, u, Zw \rangle$$

Since this statement is quantified over all $u,v,w \in V$, then $(X,Z) \in \mathcal{R}(t)$.

Now consider (X,Z) and $(X',Z') \in \mathcal{L}(t)$.

$$\langle t \mid XX'u, v, w \rangle = \langle t \mid X'u, v, Zw \rangle = \langle t \mid u, v, Z'Zw \rangle$$
$$= \langle t \mid v, XX'u, w \rangle = \langle t \mid v, X'u, Zw \rangle = \langle t \mid v, u, Z'Zw \rangle.$$

Which similarly implies that $(XX', ZZ') \in \mathcal{R}(t)$. We repeat this argument to show $\mathcal{L}(t) = \mathcal{M}(t)$. ☐

Even in the case where a tensor is not symmetric along every axis, we can still get some identifications between $\mathcal{L}(t), \mathcal{M}(t)$, and $\mathcal{R}(t)$.

Proposition 4.3.7. *Suppose $\langle t \mid : U \times U \rightarrowtail W$ is symmetric. Then $\mathcal{L}(t) = \mathcal{R}(t)$ as subalgebras of $End(U) \times End(W)$*

Proof. We proceed with a computation analogous to that in proposition 4.3.6. Let $(X, Z) \in \mathcal{L}(t)$. Then for all $(u, v) \in U^2$,

$$\langle t \mid Xu, v\rangle = Z\langle t \mid u, v\rangle$$
$$=\langle t \mid v, Xu\rangle = Z\langle t \mid v, u\rangle.$$

Since this statement is quantified over all pairs of inputs (u, v), then $(X, Z) \in \mathcal{R}(t)$. □

Then the characterization of associative algebras from proposition 4.3.5 becomes the following.

Corollary 4.3.8. *Let k be a field, L be a commutative k-algebra, and $P \subseteq L[x_0, x_1, x_2]$ be a linear homogeneous ideal. Then there exists $A \subseteq \{0, 1, 2\}$ such that, for all symmetric $S \subseteq L \otimes (\{U^{\times n} \rightarrowtail V\})$, $Z_A(S, P)_L$ is a unital associative subalgebra of $E_L[U_A]$ (up to an "op") if and only if P is one of the following:*

1. *$P = 0$. In this case $Z(S, P)_L = E_L[U_A]$.*

2. *$P = (x_a - x_b)$ for some indices $a, b \in A$. In this case, $Z_A(S, P)_L$ is isomorphic to either $\mathcal{L}(S)_L$ or $\mathcal{M}(S)_L$.*

3. *$P = (x_0 - x_1, x_1 - x_2)$. In this case, $Z(S, P)_L = \tau Cen(S)_L$ for some $\tau \in (L^{\times})^3$.*

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