DISSERTATION

GAUSSIAN MAPS FOR DOUBLE COVERS OF SMOOTH TORIC SURFACES

Submitted by

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In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY PAMELA L. PETERS ENTITLED GAUSSIAN MAPS FOR DOUBLE COVERS OF SMOOTH TORIC SURFACES BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

GAUSSIAN MAPS FOR DOUBLE COVERS OF SMOOTH TORIC SURFACES

This dissertation will expand on the results in Duflot, *Gaussian Maps for Double Coverings*, (Manuscripta Mathematica 82, 1994), considering both double covers of smooth toric surfaces and Hirzebruch surfaces and going further to consider curves on double covers of Hirzebruch surfaces. Results specifically address surjectivity as well as computations of corank.

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Chapter 1

Introduction

In this paper, we expand on the body of knowledge amassed in the past 20 years concerning Gaussian maps. The definition of the general Gaussian map is given in Wahl, "Introduction to Gaussian Maps on an Algebraic Curve", page 304 [23]. Wahl considers two line bundles $\mathcal{L}$ and $\mathcal{M}$ on a smooth projective curve X and uses the kernel of the natural mutiplication map ($\mu_{\mathcal{L},\mathcal{M}} : H^0(X,\mathcal{L}) \otimes H^0(X,\mathcal{M}) \rightarrow H^0(X,\mathcal{L}\otimes\mathcal{M})$) to construct a Gaussian map $\Phi_{X,\mathcal{L},\mathcal{M}} : ker\mu_{\mathcal{L},M} = \mathcal{R}(\mathcal{L},\mathcal{M}) \rightarrow H^0(X,\Omega^1_X \otimes \mathcal{L} \otimes M)$. In his paper [23], he discusses the criteria for surjectivity of this map, i.e. given the above definitions and if the genus of X is g, if the degree of $\mathcal{L}$ and the degree of $\mathcal{M}$ are greater than or equal to $2g+2$ and the degree of $\mathcal{L}$ plus the degree of M is greater than or equal to $6g+3$, then the Gaussian map is surjective. Finally, as Wahl tells us in his paper "The Jacobian Algebra of a Graded Gorenstein Singularity" [24], if X is a nonhyperelliptic curve of genus g greater than or equal to 3 and K_X is the canonical divisor of X, Φ_{X,K_X} is surjective means the "cone over (the canonical embedding of) X admits only equisingular deformations." Thus, the study of the surjectivity of Φ_{X,K_X} has some interest for studies in deformation theory. However, applications to deformation theory are not considered in this paper.

In 1988, Ciliberto, Harris, and Miranda [3] studied the Gaussian map for curves X. If X has genus g greater than or equal to 10 with g not equal to 11, then the map $\Phi_{X,K_X} : \Lambda^2 H^0(X, K_X) \to H^0(X, K_X^{\otimes 3})$ is surjective. Duflot and Miranda [5] in 1992 followed, considering the Gaussian map on surfaces and curves on surfaces, more specifically smooth curves on the Hirzebruch surface, $\mathbb{F}_k$. The most direct contribution to the constructions presented in this paper begins with this Duflot/Miranda paper.

Glenn Murray, first in his thesis ([17], 1994) then later in his paper ([18], 1998) studied the Duflot/Miranda [5] results, but broadened the perspective to include smooth toric surfaces in general.

In her 1994 paper [4], Duflot took the discussion of the Gaussian map into the area of double covers of complex projective varieties. Using X a double cover of a projective variety and $\mathcal{F}$ a line bundle on X, she studied the Gaussian map $\Phi_{X,\mathcal{F}} : \Lambda^2 H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}^2 \otimes \Omega^1_X)$.

In this thesis, we begin in Chapter 2 with definitions and examples of smooth toric surfaces, in particular the Hirzebruch surfaces defined using a four vector fan, and apply these definitions in Chapter 3 to develop the tools we need to explore the Gaussian map. A summary of the results of the Duflot/Miranda paper [5] is included in Chapter 4. Murray's conclusions [17] are summarized in Chapter 4 as well. Theorem 10.4 is one of the main results of Murray.

Chapter 5 presents some new work on surjectivity of various multiplication maps on Hirzebruch surfaces needed for application of the Duflot results, specifically Theorems 5.2 and 5.4, and Chapter 8 presents new work necessary for analyzing Gaussian maps for double covers of Hirzebruch surfaces using again the methods of Duflot, specifically Proposition 8.2. Duflot's work on double covers [4] is summarized in Chapter 9.

Chapters 10 and 11 are predominantly my work, incorporating the earlier work from Chapters 5 and 8 with the background theory of Duflot [4] and Duflot/Miranda [5] and applying it to specific situations. Chapter 10 computes coranks of Gaussian maps for a canonical divisor on the double cover of a smooth toric surface. For example, one of the results we prove in this thesis is:

Corollary 10.10: If S is a smooth toric surface, $\pi : X \to S$ is a double cover of S branched along a smooth curve D, with the line bundle L such that $2L = D$, K_S the canonical divisor, and $K_S + L$ and $2K_S + L$ are ample, then

$$\text{corank } \Phi_{X,K_X} = 40 - 5n + 7K_S \cdot L + L^2 + h^0(S, \mathcal{O}_S(L - K_S)),$$

where $n + 2$ is the number of vectors in the fan defining S.

Chapter 11, again predominantly original work, begins the study of the relationship between Gaussian maps on double covers of smooth toric surfaces, using the case of Hirzebruch surfaces, and Gaussian maps on curves on these double covers, following the theory initially explored in Duflot/Miranda [5].

Chapter 2

Smooth Toric Surfaces, Introduction and General Notation

In this chapter, we define smooth toric surfaces and in particular, Hirzebruch surfaces. We will also discuss some elementary facts about these surfaces. The chief references for this chapter are Fulton [7], Griffiths and Harris [8], and Harris [9].

2.1 Definitions

By definition from Fulton [7], a fan is a collection of strongly convex rational polyhedral cones in a real vector space, meeting along faces. A strongly complex rational polyhedral cone is a cone with apex at the origin generated by a finite number of rational vectors. We shall restrict our discussion to $\mathbb{Z}^2$ and construct a general fan from $n + 2$ vectors with initial side the vector $\langle 0, 1 \rangle$, terminal side the vector $\langle 1, 0 \rangle$, and intermediate vectors arrayed counterclockwise between these two, labeling the vectors consecutively $\{(a_i, b_i)\}_{i=0}^{n+1}$. For any two consecutive vectors, we also require that the determinant of their 2×2 matrix is 1:

$$\begin{vmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{vmatrix} = 1.$$

Thus, no vector is a scalar multiple of another and any two consecutive vectors form a basis for $\mathbb{Z}^2$. See Figure 2.1.

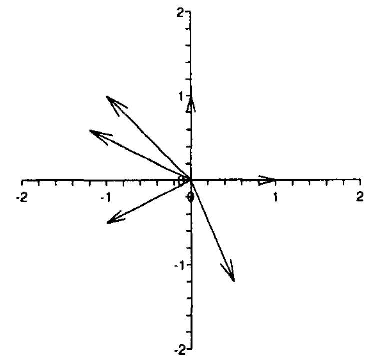

Figure 2.1: *Fan*

2.2 Construction of Complex Manifolds from Fans

From a fan, we can construct a complex manifold M as a quotient space of $n+2$ disjoint copies of $\mathbb{C}^2$, $M = \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \cdots \coprod \mathbb{C}^2 / \sim$. The equivalence relation, $\sim$, is defined as the equivalence relation generated by:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftrightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_0^p \, y_0^q \\ x_0^r \, y_0^s \end{pmatrix}$$

and x_0, y_0 are such that $x_0^p \, y_0^q$ and $x_0^r \, y_0^s$ make sense; p, q, r, s are defined by

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}^{-1} \doteq A_i^{-1}, 0 \leq i \leq n+1.$$

In the above equations, $\binom{x_i}{y_i}$ is a point in the i^{th} $\mathbb{C}^2$ of the disjoint union, called $\mathbb{C}^2_i$, for $0 \le i \le n+1$.

Additionally, the equivalence relation for comparing elements of $\mathbb{C}^2_i$ to $\mathbb{C}^2_j$ is

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim \begin{pmatrix} x_j \\ y_j \end{pmatrix} \leftrightarrow \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i^\alpha \; y_i^\beta \\ x_i^\gamma \; y_i^\delta \end{pmatrix}$$

where $0 \le i, j \le n+1$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_j^{-1} A_i.$$

Note: the matrix A_i^{-1} takes the vector $\langle a_i, b_i \rangle$ of the fan back to $\langle a_0, b_0 \rangle$, then A_j transforms $\langle a_0, b_0 \rangle$ to $\langle a_j, b_j \rangle$.

For convenience of notation, define the equivalence class of $\binom{x_i}{y_i} = \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix} \right] \doteq \begin{bmatrix} x_i \\ y_i \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_i$ is defined as the equivalence class of the origin of the i^{th} $\mathbb{C}^2$.

Again, our intention is to construct a complex manifold M from the above fan, with M as a quotient space of $n+2$ disjoint copies of $\mathbb{C}^2$, $M = \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \cdots \coprod \mathbb{C}^2 / \sim$. The manifold M has the quotient topology.

Let $q: \; \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \cdots \coprod \mathbb{C}^2 \to M$ be the quotient map and $U \subset M$ with $U = \{a \in M \mid q^{-1}(\{a\}) \text{ has exactly } n+2 \text{ elements}\}$.

Lemma 2.1 *Let $\xi \in M$. Then:*

a. if $\xi = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \in q(\mathbb{C}_i^2)$ and $x_i \neq 0,\ y_i \neq 0$, then $q^{-1}(\xi)$ contains exactly n+2 points.

b. if $\xi = \begin{bmatrix} x_i \\ 0 \end{bmatrix} \in q(\mathbb{C}_i^2)$ and $x_i \neq 0$, then $q^{-1}(\xi) = \{\binom{x_i}{0}, \binom{0}{x_i^{-1}}\}$ where $\binom{x_i}{0} \in \mathbb{C}_i^2$ and $\binom{0}{x_i^{-1}} \in \mathbb{C}_{i+1}^2$.

c. if $\xi = \begin{bmatrix} 0 \\ y_{i+1} \end{bmatrix} \in q(\mathbb{C}_{i+1}^2)$ and $y_{i+1} \neq 0$, then $q^{-1}(\xi) = \{\binom{y_{i+1}^{-1}}{0}, \binom{0}{y_{i+1}}\}$ where $\binom{y_{i+1}^{-1}}{0} \in \mathbb{C}_i^2$ and $\binom{0}{y_{i+1}} \in \mathbb{C}_{i+1}^2$.

d. if $\xi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_i$, $q^{-1}(\xi) = \{\binom{0}{0}_i\}$ where $\binom{0}{0}_i \in \mathbb{C}_i^2$.

Note that U is open. Clearly $U \neq M$. To compute $M - U$, we first need the following definition:

Definition 2.2 *Let* $C_i \doteq \{\begin{bmatrix} x_i \\ 0 \end{bmatrix} \mid \binom{x_i}{0} \in \mathbb{C}_i^2\} \cup \{\begin{bmatrix} 0 \\ y_{i+1} \end{bmatrix} \mid \binom{0}{y_{i+1}} \in \mathbb{C}_{i+1}^2\}$.

Theorem 2.3 $M - U = C_0 \cup C_1 \cup \ldots \cup C_n$ *where:*

a. $$C_i \cap C_j = \begin{cases} \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{i+1}\} & \text{if the vectors } \langle a_i, b_i\rangle \text{ and } \langle a_j, b_j\rangle \text{ are adjacent, } j=i+1 \\ \emptyset & \text{if the vectors } \langle a_i, b_i\rangle \text{ and } \langle a_j, b_j\rangle \text{ are not adjacent} \end{cases}$$

b. each C_i is isomorphic to $\mathbb{P}^1$

Note: we may say that $M - U$ is equal to the union of $n+2$ $\mathbb{P}^1$'s arranged in a cycle where $\mathbb{P}_i^1 \cap \mathbb{P}_j^1$ is a single point if and only if i and j are consecutive.

Definition 2.4 $U_i \doteq q(\mathbb{C}_i^2)$.

Lemma 2.5 *(a.) U_i is open in M.*

(b.) let $\varphi_i : \mathbb{C}^2 \to U_i$ be the map defined by $\varphi_i\binom{a}{b} = \begin{bmatrix} a \\ b \end{bmatrix} \epsilon\, U_i$. Then the map $\varphi_j^{-1}\varphi_i$,

$$\varphi_i^{-1}(U_i \cap U_j) \stackrel{\varphi_j^{-1}\varphi_i}{\longrightarrow} \varphi_j^{-1}(U_i \cap U_j),$$

where $\varphi_i^{-1}(U_i \cap U_j) \subseteq \mathbb{C}^2$ and $\varphi_j^{-1}(U_i \cap U_j) \subseteq \mathbb{C}^2$, is biholomorphic.

(c.) M is Hausdorff.

Lemma 2.6 *In the chart $(\mathbb{C}^2, \varphi_i)$, C_i is defined by $y_i = 0$. In the chart $(\mathbb{C}^2, \varphi_{i+1})$, C_i is defined by $x_{i+1} = 0$. Also, we have:*

$$C_i \cap U_j = \begin{cases} \{\begin{bmatrix} x_i \\ 0 \end{bmatrix} \mid x_i \,\epsilon \mathbb{C}\} & j = i \\ \{\begin{bmatrix} 0 \\ y_{i+1} \end{bmatrix} \mid y_{i+1} \,\epsilon \mathbb{C}\} & j = i+1 \\ \emptyset & j \neq i, i+1 \end{cases}$$

2.3 Simple Vector Fan Construction: A Three Vector Fan

Consider $n = 1$, a three vector fan. The fan $\{(a_i, b_i)\}_{i=0}^2$ is composed of initial and terminal vectors, $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$, and one intermediate vector. Using the required determinant, the intermediate vector is computed to be $\langle -1, -1\rangle$. We have the manifold $\mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2 / \sim$, where $\sim$ is defined as follows:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftrightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_0^p\, y_0^q \\ x_0^r\, y_0^s \end{pmatrix}$$

and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}^{-1}.$$

So, if $\binom{x_0}{y_0}$ is in $\mathbb{C}_0^2$,

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_0^{-1}\, y_0^1 \\ x_0^{-1}\, y_0^0 \end{pmatrix} = \begin{pmatrix} x_0^{-1}\, y_0 \\ x_0^{-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ is in } \mathbb{C}_1^2.$$

And

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_0^0\, y_0^{-1} \\ x_0^1\, y_0^{-1} \end{pmatrix} = \begin{pmatrix} y_0^{-1} \\ x_0\, y_0^{-1} \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ is in } \mathbb{C}_2^2.$$

Also, with $i = 1, j = 2$ the equivalence relation for comparing elements of $\mathbb{C}_1^2$ to $\mathbb{C}_2^2$ is

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim \begin{pmatrix} x_j \\ y_j \end{pmatrix} \leftrightarrow \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i^\alpha\, y_i^\beta \\ x_i^\gamma\, y_i^\delta \end{pmatrix}$$

where $0 \leq i, j \leq n+2$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_j^{-1} A_i$ with $A_j = \begin{pmatrix} a_{j-1} & a_j \\ b_{j-1} & b_j \end{pmatrix}$ and $A_i = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}$. So, A_1^{-1} takes the vector $\langle a_1, b_1 \rangle$ of the fan back to $\langle a_0, b_0 \rangle$, then A_2 transforms $\langle a_0, b_0 \rangle$ to $\langle a_2, b_2 \rangle$, yielding

$$\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \sim \begin{pmatrix} x_1^{-1}\, y_1^1 \\ x_1^{-1}\, y_1^0 \end{pmatrix} = \begin{pmatrix} x_1^{-1}\, y_1 \\ x^{-1} \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

This construction yields a manifold that is biholomorphic to $\mathbb{P}^2$.

2.4 Four Vector Fans

When $n = 2$, the fan $\{(a_i, b_i)\}_{i=0}^3$ is composed of four vectors, the initial and terminal vectors, $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$, and two intermediate vectors. Using the required determinant, the intermediate vectors are computed to be $\langle -1, 0 \rangle$ and $\langle k, -1 \rangle$ or alternatively, $\langle -1, k \rangle$ and $\langle 0, -1 \rangle$. Without loss of generality, we shall use the first fan combination, considering $k = 0$ then $k > 0$.

2.4.1 k=0: $\mathbb{P}^1 \times \mathbb{P}^1$

We have the manifold $\mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2 / \sim$, where $\sim$ is defined as follows:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftrightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_0^p \, y_0^q \\ x_0^r \, y_0^s \end{pmatrix}$$

and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}^{-1}.$$

Specifically,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, if $\binom{x_0}{y_0}$ is in $\mathbb{C}_0^2$,

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} y_0 \\ x_0^{-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ is in } \mathbb{C}_1^2,$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_0^{-1} \\ y_0^{-1} \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ is in } \mathbb{C}_2^2,$$

and

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} y_0^{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \text{ is in } \mathbb{C}_3^2.$$

Additionally, the equivalence relation for comparing elements of $\mathbb{C}_i^2$ to $\mathbb{C}_j^2$ is

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim \begin{pmatrix} x_j \\ y_j \end{pmatrix} \leftrightarrow \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i^\alpha \, y_i^\beta \\ x_i^\gamma \, y_i^\delta \end{pmatrix}$$

where $0 \le i,j \le n+2$ $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_j^{-1} A_i$ with $A_j = \begin{pmatrix} a_{j-1} & a_j \\ b_{j-1} & b_j \end{pmatrix}$ and $A_i = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}$. So, A_i^{-1} takes the vector $\langle a_i, b_i \rangle$ of the fan back to $\langle a_0, b_0 \rangle$, then A_j transforms $\langle a_0, b_0 \rangle$ to $\langle a_j, b_j \rangle$.

This construction yields a manifold that is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

2.4.2 Hirzebruch Surfaces, $\mathbb{F}_k$

We define the Hirzebruch surfaces, $\mathbb{F}_k$, by the four vector fan, $\langle 0,1 \rangle, \langle -1,0 \rangle, \langle k,-1 \rangle$ and $\langle 1,0 \rangle$ where $k > 0$. (Note that if $k = 0$, then $\mathbb{F}_0 \doteq \mathbb{P}^1 \times \mathbb{P}^1$.) Therefore, we have the manifold $M = \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2 \coprod \mathbb{C}^2 / \sim$, where $\sim$ is defined as follows:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftrightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_0^p \, y_0^q \\ x_0^r \, y_0^s \end{pmatrix}$$

and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}^{-1}.$$

Specifically,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -k \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}.$$

So if $\binom{x_0}{y_0}$ is in $\mathbb{C}_0^2$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} y_0 \\ x_0^{-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ in } \mathbb{C}_1^2,$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_0^{-1} y_0^{-k} \\ y_0^{-1} \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ in } \mathbb{C}_2^2,$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} y_0^{-1} \\ x_0 y_0^{k} \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \text{ in } \mathbb{C}_3^2.$$

Additionally, the equivalence relation for comparing elements of $\mathbb{C}_i^2$ to $\mathbb{C}_j^2$ is

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim \begin{pmatrix} x_j \\ y_j \end{pmatrix} \leftrightarrow \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i^{\alpha} \; y_i^{\beta} \\ x_i^{\gamma} \; y_i^{\delta} \end{pmatrix}$$

where $0 \leq i, j \leq n+2$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_j^{-1} A_i$ with $A_j = \begin{pmatrix} a_{j-1} & a_j \\ b_{j-1} & b_j \end{pmatrix}$ and $A_i = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}$.

There is a holomorphism, π, describing $\mathbb{F}_k$ as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. We describe π in the following way:

$$\pi : \mathbb{F}_k \to \mathbb{P}^1$$
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \xrightarrow{\pi} [1 : y_0],$$
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \xrightarrow{\pi} [1 : x_1],$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \xrightarrow{\pi} [y_2 : 1],$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \xrightarrow{\pi} [x_3 : 1].$$

Given $[a : b] \in \mathbb{P}^1$, we want to compute $\pi^{-1}(\{[a : b]\})$, the fibre of π over $[a : b]$.

$$\pi^{-1}(\{[a : b]\}) = \{\xi \in \mathbb{F}_n \mid \pi(\xi) = [a : b]\} \tag{2.7}$$

Lemma 2.8 *a. If $a \neq 0$. Then*

$$\pi^{-1}(\{[a : b]\}) = \{\begin{bmatrix} x_0 \\ \frac{b}{a} \end{bmatrix} \mid \begin{pmatrix} x_0 \\ \frac{b}{a} \end{pmatrix} \in \mathbb{C}_0^2\} \cup \{\begin{bmatrix} \frac{b}{a} \\ 0 \end{bmatrix} \mid \begin{pmatrix} \frac{b}{a} \\ 0 \end{pmatrix} \in \mathbb{C}_1^2\},$$

the latter a single point.

b. $\pi^{-1}(\{[0 : 1]\}) = \{\begin{bmatrix} x_2 \\ 0 \end{bmatrix} \mid \binom{x_2}{0} \in \mathbb{C}_2^2\} \cup \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}_3\}.$

Note $\pi^{-1}(\{[1 : 0]\}) = \{\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \mid \binom{x_0}{0} \in \mathbb{C}_0^2\} \cup \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}_1\} \doteq C_0.$

Define $s : \mathbb{P}^1 \to \mathbb{F}_k$ such that $\pi \circ s = id$ by:

$$s([a : b]) = \begin{cases} \begin{bmatrix} 0 \\ \frac{b}{a} \end{bmatrix} & \text{where} \binom{0}{\frac{b}{a}} \in \mathbb{C}_0^2, \text{and } a \neq 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}_3 & \text{if } a = 0 \end{cases} \tag{2.9}$$

Note $s(\mathbb{P}^1) \doteq C_3$.

Verifying that $\pi \circ s = id$, we have $(\pi \circ s)([a : b]) = \pi \begin{bmatrix} 0 \\ \frac{b}{a} \end{bmatrix} = [1 : \frac{b}{a}] = [a : b]$ if $a \neq 0$ and $(\pi \circ s)([a : b]) = \pi \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [0 : 1] = [0 : b]$ if $a = 0$.

2.5 Rational Normal Scrolls

Recall the definition of a rational normal curve from Harris [9]: A rational normal curve $C \subset \mathbb{P}^d$ is the image of the map $v_d : \mathbb{P}^1 \to \mathbb{P}^d$ given by

$$\nu_d : [X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, ..., X_0X_1^{d-1}, X_1^d],$$

up to a change of coordinates. It is the common zero locus of the polynomials $F_{i,j}(Z) = Z_iZ_j - Z_{i-1}Z_{j+1}$ for $1 \leq i \leq j \leq d-1$. We will explore this definition.

Consider the basic case where $d = 2$, so that $C \subset \mathbb{P}^2$. Then

$$\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$$

$$\nu_2 : [X_0, X_1] \mapsto [X_0^2, X_0X_1, X_1^2]$$

C then is the common locus of $F(Z) = Z_1^2 - Z_0Z_2$.

Consider the case where $d = 3$, so that $C \subset \mathbb{P}^3$. Then

$$\nu_3 : \mathbb{P}^1 \to \mathbb{P}^3$$

$$\nu_3 : [X_0, X_1] \mapsto [X_0^3, X_0^2X_1, X_0X_1^2, X_1^3].$$

Then C is the common locus of:

$$F_0(Z) = Z_1^2 - Z_0Z_2$$

$$F_1(Z) = Z_1Z_2 - Z_0Z_3$$

$$F_2(Z) = Z_2^2 - Z_1Z_3.$$

In general, then, we have the following theorem:

Lemma 2.10 *If Q_i is the zero locus of $F_i(Z)$, then the zero locus of $Q_i, Q_j, i \neq j$, is $C \cup l_{ij}$, where l_{ij} is a line. (i.e. The intersection of any two of the above quadrics yields the union of C and a line, l_{ij}.)*

Next we consider the rational normal curve where $d \geq 3$.

Definition 2.11 *A rational normal curve, C of degree d is defined by*

$$C \doteq \{X_0^d, X_0^{d-1}X_1, ..., X_0X_1^{d-1}, X_1^d | [X_0, X_1] \epsilon \mathbb{P}^1\},$$

up to a change of coordinates.

Lemma 2.12 *If $d \geq 3$ and C is a rational normal curve of degree d, with v_d : $\mathbb{P}^1 \to \mathbb{P}^d$ given by*

$$\nu_d : [X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, ..., X_0X_1^{d-1}, X_1^d].$$

Then C is the common locus of

$$F_i(Z) = Z_i^2 - Z_{i-1}Z_{i+1}, 1 \leq i \leq d-1$$

$$F_d(Z) = Z_1Z_{d-1} - Z_0Z_d.$$

Harris [9] also describes rational normal scrolls in the following way. Let $l, k \epsilon \mathbb{Z}^+, k \leq l, n = k+l+1$. Given Λ, Λ' are complimentary linear subspaces in $\mathbb{P}^n$ of dimensions k and l respectively, and that Λ, Λ' are disjoint and span $\mathbb{P}^n$. Chose two rational normal curves, $C \subset \Lambda$ and $C' \subset \Lambda'$ with an isomorphism $\phi : C \to C'$. Given $p \epsilon C$, then $\phi(p) \in C'$. Then we can define

$$S_{k,l} = \bigcup_{p \in C} \overline{p\ \phi(p)},$$

the union of lines joining points of C to C', as the rational normal scroll, $S_{k,l}$. The lines $\overline{p\ \phi(p)}$ are the lines of the ruling of $S_{k,l}$.

Consider a specific rational normal scroll,

$$S_{1,N} = \{[Y_0, Y_1, ..., Y_{N+2}] |\ Y_i \in \mathbb{C},\ \forall i\} \subset \mathbb{P}^{N+2},$$

constructed in the following way. Choose $C = \{[Y_0, Y_1, 0, ..., 0] |\ [Y_0, Y_1] \in \mathbb{P}^1\}$. Let $\Lambda = C$. Choose $\Lambda' = \{[0, 0, Y_2, ..., Y_{N+2}] |\ Y_i$'s not all zero$\}$. Note that there are exactly $N+1$ Y_i's in Λ'. Then choose $C' = \{[0, 0, s^N, s^N t, ..., st^{N-1}, t^N] |\ [s, t] \in \mathbb{P}^1\}$. Clearly, $C' \subseteq \Lambda'$. We can now define an isomorphism $\phi : C \to C'$ by $\phi([s, t, 0, ..., 0]) = [0, 0, s^N, s^N t, ..., st^{N-1}, t^N]$. Let $p \in C,\ \phi(p) \in C'$. Then

$$S_{1,N} = \bigcup_{p \in C} \overline{p\ \phi(p)}.$$

We can explicitly define the lines $\overline{p\ \phi(p)}$ as follows. Let $A, B \in \mathbb{P}^{N+2}$ where $A = [p],\ B = [q],\ p, q \in \mathbb{C}^{N+3}$. Then $\overline{AB} = \{[up + vq] |\ [u, v] \in \mathbb{P}^1\}$. So

$$[Y_0, ... Y_{N+2}] \in S_{1,N} \Leftrightarrow$$

$$\exists\ [u, v] \in \mathbb{P}^1,\ [p] = [X_0, X_1, 0, ..., 0],\ [q] = [0, 0, X_0^N, X_0^{N-1} X_1, ..., X_1^N]$$

such that

$$[Y_0, ..., Y_{N+2}] = [uX_0, uX_1, vX_0^N, vX_0^{N-1} X_1, ..., vX_1^N],\ \text{and}$$

$$S_{1,N} = \{[uX_0, uX_1, vX_0^N, vX_0^{N-1}X_1, ..., vX_1^N] | \, [u,v] \,\epsilon\, \mathbb{P}^1, \; [X_0, X_1] \,\epsilon\, \mathbb{P}^1\}.$$

Note that C is obtained by setting $v = 0$ and C' is obtained by setting $u = 0$.

Consider $p = [1, \lambda, 0, ...0] \,\epsilon\, C$ with $\lambda \,\epsilon\, \mathbb{C}$. Define $\infty \doteq [0, 1, 0, ..., 0] \,\epsilon\, C$. From our definition above, $\phi(p) = [0, 0, 1, \lambda, ..., \lambda^N] \,\epsilon\, C'$ and $\phi(\infty) = [0, 0, ..., 0, 1] \,\epsilon\, C'$. Define $\forall \lambda \,\epsilon\, \mathbb{C}$,

$$C_\lambda = \{[u, u\lambda, v, v\lambda, ..., v\lambda^N] | \, [u, v] \,\epsilon\, \mathbb{P}^1\} \doteq \overline{p \, \phi(p)}$$

So, for example,

$$C_0 = \{[u, 0, v, 0, ..., 0] | \, [u, v] \,\epsilon\, \mathbb{P}^1\} \doteq \overline{0 \, \phi(0)}$$

$$C_\infty = \{[0, u, 0, ..., v] | \, [u, v] \,\epsilon\, \mathbb{P}^1\} \doteq \overline{\infty \, \phi(\infty)}$$

With $0 \leq \lambda \leq \infty$, these are the lines of the ruling of $S_{1,N}$, which compose the rational normal scroll. Note specifically that $\forall \lambda, C \neq C_\lambda, C' \neq C_\lambda$, but $C \cap C_\lambda = \{p\}$ and $C' \cap C_\lambda = \{\phi(p)\}$.

Consider the case $N = 1$,

$$S_{1,1} = \{[uX_0, ux_1, vX_0, vX_1]\} \mid [u, v] \,\epsilon\, \mathbb{P}^1, \; [X_0, X_1] \,\epsilon\, \mathbb{P}^1\}.$$

The Segre embeding, $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, is defined by

$$\sigma([u, v], [X_0, X_1]) = [uX_0, uX_1, vX_0, vX_1].$$

Thus $S_{1,1} = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$.

Similarly,

$$S_{1,2} = \{[uX_0, uX_1, vX_0^2, vX_0X_1, vX_1^2] \mid [u,v] \in \mathbb{P}^1,\ [X_0, X_1] \in \mathbb{P}^1\}.$$

Considering specific restrictions on the components, we have the following theorem:

Theorem 2.13

$$S_{1,2} = \{[Y_0, Y_1, Y_2, Y_3, Y_4] \in \mathbb{P}^5 \mid Y_3^2 - Y_2Y_4 = 0, Y_1Y_2 - Y_0Y_3 = 0, Y_1Y_3 - Y_0Y_4 = 0\}.$$

For values of N greater than 2, we have the following theorem:

Theorem 2.14 *For $N \geq 3$,*

$$S_{1,N} = \{[Y_0, Y_1, Y_2..., Y_{N+2}] \in \mathbb{P}^{N+2}$$

$$\mid Y_{i+2}^2 - Y_{i+1}Y_{i+3} = 0,\ 1 \leq i \leq N-1;\ \ Y_3Y_{N+1} - Y_2Y_{N+2} = 0;\ \ Y_0Y_3 - Y_1Y_2 = 0\}.$$

Theorem 2.15 *$S_{1,N}$ is an algebraic variety.*

Theorem 2.16 *$\mathbb{F}_k$ is biholomorphic to $S_{1,k+1}$.*

The details follow. Define $\tilde{\varphi} : \mathbb{C}_0^2 \coprod ... \coprod \mathbb{C}_3^2 \to \mathbb{P}^{k+3}$ by

$$\tilde{\varphi}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = [1, y_0, x_0, x_0y_0, ..., x_0y_0^{k+1}]$$

$$\tilde{\varphi}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = [y_1, x_1y_1, 1, x_1, ..., x_1^{k+1}]$$

$$\tilde{\varphi}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = [x_2y_2, x_2, y_2^{k+1}, ..., y_2, 1]$$

$$\tilde{\varphi}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = [x_3, 1, x_3^{k+1}y_3, x_3^k y_3, ..., x_3y_3, y_3]$$

Then $\binom{x_i}{y_i} \sim \binom{x_j}{y_j}$ iff $\tilde{\varphi}\binom{x_i}{y_i} = \tilde{\varphi}\binom{x_j}{y_j}$ and there exists a map $\varphi : \mathbb{F}_k \to \mathbb{P}^{k+3}$ defined explicitly as:

$$\varphi(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}) = [1, y_0, x_0, x_0y_0, ..., x_0y_0^{k+1}] = \tilde{\varphi}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\varphi(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) = [y_1, x_1y_1, 1, x_1, ..., x_1^{k+1}] = \tilde{\varphi}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\varphi(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) = [x_2y_2, x_2, y_2^{k+1}, y_2^k, ..., y_2, 1] = \tilde{\varphi}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\varphi(\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}) = [x_3, 1, x_3^{k+1}y_3, x_3^ky_3, ..., x_3y_3, y_3] = \tilde{\varphi}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

Theorem 2.17 *Im* $\varphi = S_{1,k+1}$

PROOF: $\subseteq$: Clearly im $\varphi \subseteq S_{1,k+1}$ since

$$\varphi(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}) = [1, y_0, x_0, x_0y_0, ..., x_0y_0^{k+1}]$$

has $u = 1, v = x_0, X_0 = 1, X_1 = y_0$ in $S_{1,k+1}$. Also

$$\varphi(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) = [y_1, x_1y_1, 1, x_1, ..., x_1^{k+1}]$$

has $u = y_1, v = 1, X_0 = 1, X_1 = x_1$ in $S_{1,k+1}$. Continuing

$$\varphi(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) = [x_2y_2, x_2, y_2^{k+1}, y_2^k, ..., y_2, 1]$$

has $u = x_2, v = 1, X_0 = y_2, X_1 = 1$ in $S_{1,k+1}$. Finally

$$\varphi(\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}) = [x_3, 1, x_3^{k+1}y_3, x_3^k y_3, ..., x_3y_3, y_3]$$

has $u = 1, v = y_3, X_0 = x_3, X_1 = 1$ in $S_{1,k+1}$.

$\supseteq$: We need to show that in $S_{1,k+1} \subseteq \text{im}\varphi$. Let $p \in S_{1,k+1}$. Then there exists $[u,v], [X_0, X_1] \in \mathbb{P}^1$ such that

$$p = [uX_0, uX_1, vX_0^N, vX_0^{N-1}X_1, ..., vX_1^N].$$

If $uX_0 \neq 0$, then

$$p = [1, \frac{X_1}{X_0}, \frac{vX_0^N}{uX_0}, \frac{vX_0^{N-1}X_1}{uX_0}, ..., \frac{vX_1^N}{X_0}] = \varphi(\begin{bmatrix} \frac{v}{u}X_0^{N-1} \\ \frac{X_1}{X_0} \end{bmatrix}).$$

If $uX_0 = 0$, then either $u = 0$ or $X_0 = 0$. Consider first if $u = 0$. Then

$$p = [0, 0, vX_0^N, vX_0^{N-1}X_1, ..., vX_1^N] = [0, 0, 1, \frac{X_1}{X_0}, ... \frac{X_1^N}{X_0^N}] = \varphi(\begin{bmatrix} \frac{X_1}{X_0} \\ 0 \end{bmatrix}).$$

Instead, if $X_0 = 0$, then

$$p = [0, uX_1, 0, ..., 0, vX_1^N] = [0, 1, 0, ..., 0, \frac{v}{u}x_1^{N-1}] = \varphi(\begin{bmatrix} 0 \\ \frac{u}{v}X_1^{N-1} \end{bmatrix}).$$

If $vX_1 \neq 0$, then

$$p = [\frac{uX_0}{vX_1}, \frac{uX_1}{vX_1}, \frac{vX_0^N}{vX_1}, ..., \frac{vX_1^n}{vX_1}]$$

$$= [\frac{uX_0}{vX_1^N}, \frac{u}{vX_1^{N-1}}, \frac{X_0^N}{X_1^N}, \frac{X_0^{N-1}}{X_1^{N-1}}, ..., \frac{X_0}{X_1}, 1]$$

$$= \varphi(\begin{bmatrix} \frac{u}{vX_1^{N-1}} \\ \frac{X_0}{X_1} \end{bmatrix}).$$

If $vX_1 = 0$, then either $v = 0$ or $X_1 = 0$. Consider first if $v = 0$. Then

$$p = [uX_0, uX_1, 0, ..., 0] = [\frac{X_0}{X_1}, 1, 0, ..., 0] = \varphi(\begin{bmatrix} \frac{X_0}{X_1} \\ 0 \end{bmatrix}).$$

Finally, if $X_1 = 0$, then

$$p = [uX_0, 0, vX_0^N, 0, ..., 0] = [1, 0, \frac{v}{u}X_0^{N-1}, 0, ..., 0] = \varphi(\begin{bmatrix} \frac{v}{u}X_0^{N-1} \\ 0 \end{bmatrix}).$$

Consider $\varphi(C_i) \subseteq S_{1,k+1}$. By definition,

$$C_i \doteq \{\begin{bmatrix} x_i \\ 0 \end{bmatrix} \mid \begin{pmatrix} x_i \\ 0 \end{pmatrix} \epsilon\ \mathbb{C}_i^2\} \cup \{\begin{bmatrix} 0 \\ y_{i+1} \end{bmatrix} \mid \begin{pmatrix} 0 \\ y_{i+1} \end{pmatrix} \epsilon\ \mathbb{C}_{i+1}^2\}.$$

Therefore

$$\varphi(C_0) = \begin{cases} \varphi(\begin{bmatrix} x_0 \\ 0 \end{bmatrix}) = [1, 0, x_0, 0, ..., 0] & \text{with } X_0 = 1, X_1 = 0, u = 1, v = x_0 \\ \varphi(\begin{bmatrix} 0 \\ y_1 \end{bmatrix}) = [y_1, 0, 1, 0, ..., 0] & \text{with } X_0 = 1, X_1 = 0, u = y_1, v = x_1 \end{cases}$$

which is equal to a line of the ruling of $S_{1,k+1}$, specifically, the line $\overline{0\ \phi(0)}$.

We have

$$\varphi(C_1) = \begin{cases} \varphi(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}) = [0, 0, 1, x_1, ..., x_1^{k+1}] & \text{with } X_0 = 1, X_1 = x_1, u = 0, v = 1 \\ \varphi(\begin{bmatrix} 0 \\ y_2 \end{bmatrix}) = [0, 0, y_2^{k+1}, ..., y_2, 1] & \text{with } X_0 = y_2, X_1 = 1, u = 0, v = 1 \end{cases}$$

which is equal to $C' \subseteq S_{1,k+1}$.

Also

$$\varphi(C_2) = \begin{cases} \varphi(\begin{bmatrix} x_2 \\ 0 \end{bmatrix}) = [0, x_2, 0, ..., 0, 1] & \text{with } X_0 = 0, X_1 = 1, u = x_2, v = 1 \\ \varphi(\begin{bmatrix} 0 \\ y_3 \end{bmatrix}) = [0, 1, 0, ..., 0, y_3] & \text{with } X_0 = 0, X_1 = 1, u = 1, v = y_3 \end{cases}$$

also equal to a line of the ruling of $S_{1,k+1}$, specifically the line $\overline{\infty\ \phi(\infty)}$.

Finally

$$\varphi(C_3) = \begin{cases} \varphi(\begin{bmatrix} x_3 \\ 0 \end{bmatrix}) = [x_3, 1, 0, ..., 0] & \text{with } X_0 = x_3, X_1 = 1, u = 1, v = 0 \\ \varphi(\begin{bmatrix} 0 \\ y_0 \end{bmatrix}) = [1, y_0, ..., 0] & \text{with } X_0 = 1, X_1 = y_0, u = 1, v = 0 \end{cases}$$

which is equal to $C \subseteq S_{1,k+1}$. Therefore, $\varphi(C_i) \subseteq S_{1,k+1},\ 0 \leq i \leq 3$

Next we will show that $S_{1,N}$ is rational. First, we need that $S_{1,N}$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. There exists a relation

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{f} S_{1,N}$$

$$([u, v], [X_0, X_1]) \overset{f}{\mapsto} [uX_0, uX_1, vX_0^N, vX_0^{N-1}X_1, ..., vX_1^N].$$

However, this is not well defined, since

$$([\lambda u, \lambda v], [\mu X_0, \mu X_1]) \overset{f}{\mapsto} [\lambda\mu uX_0, \lambda\mu uX_1, \lambda\mu^N vX_0^N, \lambda\mu^N vX_0^{N-1}X_1, ..., \lambda\mu^N vX_1^N],$$

which is not necessarily equal to $[uX_0, uX_1, vX_0^N, vX_0^{N-1}X_1, ..., vX_1^N]$.

Define $V_0 = \{[1, x] \in \mathbb{P}^1 | x \in \mathbb{C}\}$. Define a map

$$\mathbb{P}^1 \times V_0 \xrightarrow{F} S_{1,N}$$

$$F([u, v], [1, x]) = [u, ux, v, vx, vx^2, ..., vx^N].$$

This map is well defined, as $F([\lambda u, \lambda v], [1, x]) = F([u, v], [1, x])$. Additionally, it is 1-1, since if $[u, ux, v, vx, vx^2, ..., vx^N] = [\tilde{u}, \tilde{u}\tilde{x}, \tilde{v}, \tilde{v}\tilde{x}, \tilde{v}\tilde{x}^2, ..., \tilde{v}\tilde{x}^N]$, then $\lambda u = \tilde{u}$, $\lambda v = \tilde{v}$, and there exists $x \neq 0$ such that $\lambda ux = \tilde{u}\tilde{x}$. If $u \neq 0$, then $x = \tilde{x}$, so $[1, x] = [1, \tilde{x}]$, and $[\lambda u, \lambda v] = [\tilde{u}, \tilde{v}]$. If $u = 0$, then $\tilde{u} = 0$ and either $v = 0$ or $v \neq 0$. If $v \neq 0$, then $x = \tilde{x}$, so $[1, x] = [1, \tilde{x}]$ and $[0, v] = [0, \tilde{v}]$. If $v = 0$, then $F([0, 0], [1, x]) = [0, ..., 0]$, not an option. Therefore $[u, v] = [\tilde{u}, \tilde{v}]$ and $[1, x] = [1, \tilde{x}]$.

Define $V_2 = S_{1,k+1} \cap \{[a_0, a_1, ...a_{N+2}] \in \mathbb{P}^{N+3} | a_2 \neq 0\}$. V_2 is open in $S_{1,k+1}$ by definition. Then consider

$$V_2 \xrightarrow{G} \mathbb{P}^1 \times V_0$$

$$G([\psi, \psi x, 1, x, x^2, ..., x^N] = ([\psi, 1], [1, x])$$

where $\psi = \frac{u}{vX_0^{N-1}}$ and $x = \frac{X_1}{X_0}$. This is clearly well defined, and we have already proved that every element of V_2 can be written uniquely in the form

$$[\psi, \psi x, 1, x, x^2, ..., x^N].$$

F and G are inverses since

$$G(F([\psi, 1], [1, x])) = G([u, ux, v, vx, vx^2, ..., vx^N])$$

$$= G([\psi, \psi x, 1, x, x^2, ..., x^N]) = ([\psi, 1], [1, x])$$

and

$$F(G([\psi,\psi x,1,x,x^2,...,x^N]))=F([\psi,1],[1,x])=[u,ux,v,vx,vx^2,...,vx^N]$$

$$=[\psi,\psi x,1,x,x^2,...,x^N].$$

Therefore $\mathbb{P}^1\times\mathbb{P}^1$ is birational to $S_{1,N}$.

Next we will show that $\mathbb{P}^1\times\mathbb{P}^1$ is birational to $\mathbb{P}^2$. We can define a map from $\mathbb{P}^1\times\mathbb{P}^1$ to $\mathbb{P}^2$ using the Segre embedding and projecting from $\mathbb{P}^3$ to $\mathbb{P}^2$ for example with $([u,v],[X_0,X_1])\mapsto[uX_0,uX_1,vX_1]$, but the resulting composite map is not 1-1. Instead, consider a map from the open subset $V_0\times V_0$ of $\mathbb{P}^1\times\mathbb{P}^1$ to $V_0\times V_0$. Then define f by

$$V_0\times V_0\xrightarrow{f}\mathbb{P}^2$$

$$f([1,v],[1,X_1])\overset{f}{\mapsto}[1,v,X_1].$$

This is clearly well defined and one-to-one. Define $V_0^2\subset\mathbb{P}^2$ by

$$V_0^2=\{[1,v,X_1]|(v,X_1)\in\mathbb{C}^2\}.$$

We can now define a map g by

$$V_0^2\xrightarrow{g}V_0\times V_0$$

$$[1,v,X_1]\overset{g}{\mapsto}([1,v],[1,X_1]).$$

Again, g is clearly well defined, and g and f are inverses since

$$f(g([1,v,X_1]))=f([1,v],[1,X_1])=[1,v,X_1]$$

and

$$g(f([1,v],[1,X_1])) = g([1,v,X_1]) = ([1,v],[1,X_1]).$$

So $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$, and therefore $S_{1,N}$ is birational to $\mathbb{P}^2$, which shows that $S_{1,N}$ is rational, by definition.

By Theorem 2.16, we know that $\mathbb{F}_k \overset{\varphi}{\hookrightarrow} \mathbb{P}^{k+3}$, with $\begin{bmatrix} x_i \\ y_i \end{bmatrix} \mapsto [a_0, a_1, ..., a_{k+3}]$. Also, if $k > 2$, there is a projective mapping $\mathbb{P}^{k+3} \overset{\pi}{\to} \mathbb{P}^5$, with $[a_0, a_1, ..., a_{k+2}] \overset{\pi}{\mapsto} [a_0, a_1, a_2, a_3, a_{k+1}, a_{k+2}]$. We can form the composite map, $\phi = \pi \circ \varphi$,

$$\mathbb{F}_k \overset{\phi}{\hookrightarrow} \mathbb{P}^5$$

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} \overset{\phi}{\mapsto} [a_0, a_1, a_2, a_3, a_{k+1}, a_{k+2}].$$

Specifically,

$$\phi(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}) = [1, y_0, x_0, x_0y_0, x_0y_0^k, x_0y_0^{k+1}]$$

$$\phi(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) = [y_1, x_1y_1, 1, x_1, x_1^k, x_1^{k+1}]$$

$$\phi(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) = [x_2y_2, x_2, y_2^{k+1}, y_2^k, y_2, 1],$$

and

$$\phi(\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}) = [x_3, 1, x_3^{k+1}y_3, x_3^k y_3, x_3y_3, y_3].$$

This composite is smooth, algebraic, and one-to-one.

2.6 $\mathbf{F_1}$ as a Blowup of $\mathbf{P^2}$ at a Point

Next, consider $\mathbb{F}_1$. We will show that $\mathbb{F}_1$ is a blowup of $\mathbb{P}^2$ at one point, or equivalently $S_{1,2}$ is a blow up of $\mathbb{P}^2$ at a point. To discuss this further, we need to clarify what we mean by a blowup.

From Harris [9], we can describe blowing up $\mathbb{P}^n$ at a point as follows. Assume $p = [0, ..., 0, 1] \in \mathbb{P}^n$ and $H = \{[x, 0] |\ x \in \mathbb{C}^{n-1} \text{ and } x \neq 0\} = \mathbb{P}^{n-1}$. Consider the function

$$\varphi : \mathbb{P}^n - \{p\} \to H$$

given by projection onto H from p. To explicitly compute a formula for φ, form a line l from p through a second point q with $q \neq p$, i.e. $q = [z_0, ..., z_n]$. The points of the projection will be where the line l intersects H. Then $l(p,q) = \{[vx, vz_n + u] | [u, v] \in \mathbb{P}^1\}$. To be contained in the intersection, the last coordinate from l would have to be zero, hence $vz_n + u = 0$ and $v = -\frac{u}{z_n}$. So $l \cap H = [-\frac{u}{z_n}(z_0, ..., z_{n-1}), 0] = [z_0, ..., z_{n-1}, 0]$. Therefore

$$\varphi([z_0, ..., z_n]) = [z_0, ..., z_{n-1}, 0].$$

The graph of φ is, by definition, $\{(q, \varphi(q)) |\ q \in \mathbb{P}^n - \{p\}\}$. Then the blowup of $\mathbb{P}^n$ at p is, by definition,

$$\widetilde{\mathbb{P}^n} \doteq \{(q, \varphi(q)) |\ q \in \mathbb{P}^n - \{p\}\} \cup \{p\} \times H \subseteq \mathbb{P}^n \times H$$

From there, we have that $\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ is, by definition,

$$\pi(q, \varphi(q)) = q,\ \forall q \in \mathbb{P}^n - \{p\}, \text{ and}$$

$$\pi(p,h) = p, \ \forall h \in H.$$

So, π is the restriction to $\widetilde{\mathbb{P}^n}$ of the projection onto the first coordinate, $\mathbb{P}^n \times H \to \mathbb{P}^n$. Our definition of π also leads to the definition of the exceptional divisor:

$$\pi^{-1}(\{p\}) = \{p\} \times H \doteq E.$$

Lemma 2.18

$\widetilde{\mathbb{P}^n} \cong \{([z_0, z_1, ..., z_n], [w_0, w_1, ...w_{n-1}]) \in \mathbb{P}^n \times \mathbb{P}^{n-1} |\ z_i w_j - w_i z_j = 0, 0 \leq i < j \leq n-1\}$.

In general, for $p \in \mathbb{P}^n$ where $p = [p_0, ..., p_n]$, $a = (a_0, ..., a_n) \neq 0$, and $H_a = \{[z_0, ...z_n] \in \mathbb{P}^n |\ a_0 z_0 + ... + a_n z_n = 0\}$ with $p \notin H_a$, we have a map φ_p, projection from p to H_a, defined as follows:

$$\varphi_p : \mathbb{P}^n - \{p\} \to H_a$$

$$\varphi_p(q) \doteq l(p,q) \cap H_a$$

Consider the line from the point p and through another point $q \neq p$, with $q = [z_o, ..., z_n]$. The point of the projection is where this line intersects H_a. Then $l(p,q) = \{[vp_i + uz_i] | [u,v] \in \mathbb{P}^1, 0 \leq i \leq n\}$. Since $\varphi_p(q) = l(p,q) \cap H_a$, then $a_0(vp_0 + uz_0) + a_1(vp_1 + uz_1) + ... + a_n(vp_n + uz_n) = 0$, by definition of H_a. Or, $v\Sigma_{j=0}^n a_j p_j + u\Sigma_{j=0}^n a_j z_j = 0$. We know that the $a_j \neq 0$ and that p is not an element of H_a, so $\Sigma a_j p_j \neq 0$. Therefore, $v = \frac{-u\Sigma a_j z_j}{\Sigma a_j p_j}$, resulting in

$$\varphi_p(q) \doteq [\frac{-u}{\Sigma a_j p_j} \Sigma a_j z_j + uz_i].$$

Since u is a constant, consider whether $u = 0$. This results in $[0, ..., 0]$, not an option. So $u \neq 0$. The graph of $\varphi_p(q)$ is, by definition, $\{(q, \varphi_p(q)) |\ q \in \mathbb{P}^n - \{p\}\}$.

Then we can define the blowup of $\mathbb{P}^n$ at p by

$$\widetilde{\mathbb{P}^n_{a,p}} = \{(q, \varphi_p(q)) |\; q \in \mathbb{P}^n - \{p\}\} \cup \{p\} \times H_a \subseteq \mathbb{P}^n \times H_a$$

Then, $\pi : \widetilde{\mathbb{P}^n_{a,p}} \to \mathbb{P}^n$, with

$$\pi(q, \varphi_p(q)) = q, \;\forall q \in \mathbb{P}^n - \{p\}, \text{ and}$$

$$\pi(p, h) = p, \;\forall h \in H_a.$$

Also

$$\pi^{-1}(p) = \{p\} \times H_a \doteq E.$$

Lemma 2.19 *Given any $p \in \mathbb{P}^n$ and any hyperplane $H_a \subseteq \mathbb{P}^n$, with $p \notin H_a$, there exists a projective automorphism $T_{a,p} : \mathbb{P}^n \to \mathbb{P}^n$ such that $T(p) = [0, ..., 0, 1]$ and $T(H_a) = H$. $T_{a,p}$ induces an isomorphism between $\widetilde{\mathbb{P}^n_{a,p}}$ and $\widetilde{\mathbb{P}^n}$.*

Let $p \in \mathbb{P}^n$ where $p = [p_0, ..., p_n]$, $a = (a_0, ..., a_n) \neq 0$, and $H_a = \{[z_0, ...z_n] \in \mathbb{P}^n |\; a_0 z_0 + ... + a_n z_n = 0\}$ with $p \notin H_a$, as defined above. Define $\widetilde{H_a} = \{(z_0, ..., z_n) \in \mathbb{C}^{n+1} |\; a_0 z_0 + ... + a_n z_n = 0\}$ and $\tilde{p} = (p_0, ..., p_n)$, such that $\tilde{p} \notin \widetilde{H_a}$. Under the canonical map $\mathbb{C}^{n+1} \to \mathbb{P}^n$, $z \mapsto [z]$ and $\widetilde{H_a} \mapsto H_a$. Let $v_1, ..., v_n$ be a basis for $\widetilde{H_a}$. Then $v_1, ..., v_n, \tilde{p}$ is a basis for $\mathbb{C}^{n+1}$, since $\tilde{p} \notin \widetilde{H_a}$. Define

$$\widetilde{T_{a,p}} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \text{ as}$$

$$\widetilde{T_{a,p}}(v_i) = e_i$$

$$\widetilde{T_{a,p}}(\tilde{p}) = e_{n+1}$$

We will show that the projective automorphism

$$T_{a,p} : \mathbb{P}^n \to \mathbb{P}^n$$

defined by

$$T_{a,p}([z]) = [\widetilde{T_{a,p}}(z)]$$

satisfies the statement of the lemma. Clearly, $T_{a,p}$ is a projective linear isomorphism. Consider $T_{a,p}([p]) = [\widetilde{T_{a,p}}(p)] = [0, ..., 0, 1]$. Let $w \in H_a$, with $w = [w_0, ..., w_n] \in \mathbb{P}^n$ and $(w_0, ..., w_n) \in \mathbb{C}^{n+1}$. Since $a_0 w_0 + ... + a_n w_n = 0$ (as $w \in H_a$), then $(w_0, ..., w_n) \in \widetilde{H_a}$. Therefore $T_{a,p}[w_0, ..., w_n] \doteq [\widetilde{T_{a,p}}(w_0, ..., w_n)] = [\alpha_1 \widetilde{T_{a,p}} v_1 + ... + \alpha_n \widetilde{T_{a,p}} v_n]$ since $v_1, ..., v_n$ is a basis for $\widetilde{H_a}$. And $[\alpha_1 \widetilde{T_{a,p}} v_1 + ... + \alpha_n \widetilde{T_{a,p}} v_n] = [\alpha_1, \alpha_2, ..., \alpha_n, 0]$ from the definition of $\widetilde{T_{a,p}}(v_i)$ above. Also by definition,

$$[\alpha_1, \alpha_2, ..., \alpha_n, 0] \in H.$$

Returning to the discussion of $\widetilde{\mathbb{P}^n}$ from Lemma 2.18, we have that

$$\text{im } \theta = \{([z], [w]) \in \mathbb{P}^n \times \mathbb{P}^{n-1} | \; z_i w_j - w_i z_j = 0, 0 \leq i < j \leq n-1\}.$$

For ease of notation, we will denote the Segre embedding by σ. Then by the Segre embedding (see e.g. Harris, [9]), we can embed $\mathbb{P}^n \times \mathbb{P}^{n-1}$ into $\mathbb{P}^N$, so that $\widetilde{\mathbb{P}^n} \xrightarrow{\sigma \circ \theta} \mathbb{P}^N$ where $N = (n+1)(n) - 1$.

Consider the case where $n = 2$. So

$$\widetilde{\mathbb{P}^2} \xhookrightarrow{\theta} \text{Im } \theta \subseteq \mathbb{P}^2 \times \mathbb{P}^1$$

with

$$\text{Im } \theta = \{([z_0, z_1, z_2], [w_0, w_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_0 w_1 - z_1 w_0 = 0\}.$$

The Segre embedding,

$$\mathbb{P}^2 \times \mathbb{P}^1 \overset{\sigma}{\hookrightarrow} \mathbb{P}^5,$$

is given by

$$\sigma([z_0, z_1, z_2], [w_0, w_1]) = [z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1, z_2 w_0, z_2 w_1].$$

If $([z_0, z_1, z_2], [w_0, w_1]) \in \text{im } \theta$, then $z_0 w_1 = z_1 w_0$, so

$$\sigma(\text{im } \theta) = \text{im } (\sigma \circ \theta) \subseteq \hat{H} \doteq \{[x_0, x_1, x_2, x_3, x_4, x_5] \in \mathbb{P}^5 | x_1 = x_2\} \overset{\pi}{\cong} \mathbb{P}^4$$

where $\pi([x_0, x_1, x_2, x_3, x_4, x_5]) = [x_4, x_5, x_0, x_1, x_3]$.

Next, we will show

$$\text{im}(\pi \circ \sigma \circ \theta) = S_{1,2}$$

$$\doteq \{[Y_0, Y_1, Y_2, Y_3, Y_4] | Y_3^2 - Y_2 Y_4 = 0, Y_1 Y_2 - Y_0 Y_3 = 0, Y_1 Y_3 - Y_0 Y_4 = 0\}.$$

$\supseteq$: Recall we showed previously that

$$S_{1,2} = \{[us, ut, vs^2, vst, vt^2] \in \mathbb{P}^4 \mid [u, v], [s, t] \in \mathbb{P}^1 \times \mathbb{P}^1\}.$$

If $[us, ut, vs^2, vst, vt^2] \in S_{1,2}$, define $z_0 \doteq vs, z_1 \doteq vt, z_2 \doteq u, w_0 \doteq s$, and $w_1 \doteq t$. Then $\pi \circ \sigma \circ \theta([vs, vt, u], [s, t]) \doteq [us, ut, vs^2, vst, vt^2] \in S_{1,2}$.

$\subseteq$: Let $[z_2 w_0, z_2 w_1, z_0 w_0, z_0 w_1, z_1 w_1] \doteq \pi \circ \sigma \circ \theta([z_0, z_1, z_2], [w_0, w_1])$. Recall $z_0 w_1 = z_1 w_0$. If $w_0 \neq 0$, then define $u \doteq z_2, t \doteq w_1, v \doteq \frac{z_0}{w_0}$, and $s = w_0$. Then

$us = z_2 w_0, ut = z_2 w_1, vs^2 = \frac{z_0}{w_0} w_0^2 = z_0 w_0, vst = \frac{z_0}{w_0} w_0 w_1 = z_0 w_1$, and $vt^2 = \frac{z_0}{w_0} w_1^2 = \frac{z_0 w_1}{w_0} w_1 = z_1 w_1$.

If $w_0 = 0$, then $w_1 \neq 0$, so define $u = z_2, t = w_1, v = \frac{z_1}{w_1}$, and $s = 0$. So

$$[z_2 w_0, z_2 w_1, z_0 w_0, z_0 w_1, z_1 w_1] = [0, ut, 0, 0, vt^2] \in S_{1,2}.$$

Therefore, we can finally conclude that $S_{1,2}$ is a blowup of $\mathbb{P}^2$ at a point.

In summary, we have considered rational normal scrolls, resulting in

$$\mathbb{F}_0 \overset{\phi}{\cong} \mathbb{P}^1 \times \mathbb{P}^1 \overset{\sigma}{\cong} S_{1,1} \subseteq \mathbb{P}^3$$

$$\mathbb{F}_1 \overset{\varphi}{\cong} S_{1,2} \subseteq \mathbb{P}^4$$

Also

$$\widetilde{\mathbb{P}^2} \overset{\pi \circ \sigma \circ \theta}{\cong} S_{1,2}.$$

In general, we have that for $k \geq 2$,

$$\mathbb{F}_k \overset{\varphi}{\cong} S_{1,k+1} \subseteq \mathbb{P}^{k+3}.$$

Finally, we have described $S_{1,N}$, for $N \geq 1$, as an algebraic subset of $\mathbb{P}^{N+2}$.

Chapter 3

Divisors on Smooth Toric Surfaces

In this chapter, we collect standard facts we will need about divisors. A general reference is Griffiths and Harris [8]. Also included are some basic facts about forms.

3.1 Divisors on Compact Complex Manifolds

First we shall consider divisors in general, using primarily Griffiths and Harris [8]. Let M be a compact complex manifold of dimension n. We can define a divisor on M as a formal finite $\mathbb{Z}$-linear sum of closed irreducible subvarieties of M of codimension 1, i.e.

$$D = \sum_V a_V V,$$

where V stands for a closed irreducible subvariety of M of codimension 1. The additive group of divisors is denoted by $Div(M)$.

Suppose V is an irreducible hypersurface. Let $p \in V$ be defined by $a = 0$ near p (a an irreducible holomorphic function) and let b be a meromorphic function on M defined near p. We call a the local defining function for V near p. By definition, the order of b along V is

$$\text{ord}_{V,p}(b) = L$$

where L is the largest integer such that $b = a^L h$ near p and h is relatively prime to a near p. Note that this does not depend on the choice of p and $L = 0$ for all but finitely many V.

If $b \in H^0(M, \mathcal{M})$, then we can define the divisor (b) by

$$\text{div}(b) \doteq (b) \doteq \sum_V \text{ord}_V(b) \cdot V,$$

where $\mathcal{M}$ is the sheaf of meromorphic functions on M. Any divisor of this form is called a principal divisor on M, with the set of principal divisors on M denoted as

$$PDiv(M) \doteq \{div(b) | \; b \in H^0(M, \mathcal{M}), b \neq 0\}.$$

Let $\mathcal{M}^*$ be the multiplicative sheaf of meromorphic functions on M not identically zero, with $\mathcal{O}^*$ the subsheaf of nowhere zero holomorphic functions. Then we can identify a divisor D on M with a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. This identification is described as follows. A global section $f \in \mathcal{M}^*/\mathcal{O}^*$ is given by specifying an open cover $\{U_\alpha\}$ of M, and for all α, $f_\alpha \in \mathcal{M}^*(U_\alpha)$ such that f_α is not identically zero in U_α and for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, $\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. For each closed, irreductible subvariety V of codimension 1, choose α such that $U_\alpha \cap U_\beta \neq \emptyset$. Then

$$\sum_V \text{ord}_V(f_\alpha) \cdot V \in Div(M)$$

is a well defined divisor since for any V on M, then $\text{ord}_V(f_\alpha) = \text{ord}_V(f_\beta)$ for all α, β such that $V \cap U_\alpha \neq \emptyset$ and $V \cap U_\beta \neq \emptyset$, and given f_α, $ord_V(f_\alpha) = 0$ for all but finitely many V.

On the other hand, if $D = \sum_V a_V V \in Div(M)$ and there exists an open cover $\{U_\alpha\}$ of M, V has a local defining function $f_{V,\alpha} \in \mathcal{O}(U_\alpha)$. If $f_\alpha = \prod_V f_{V,\alpha}^{a_V} \in \mathcal{M}^*(U_\alpha)$, one obtains a global section of $\mathcal{M}^*/\mathcal{O}^*$. The f_α's are local defining functions on D. Thus, we have the following theorem:

Theorem 3.1 *There is an isomorphism*

$$\nu : \; H^0(M, \mathcal{M}^*/\mathcal{O}^*) \to Div(M)$$

given as follows: If an element of $H^0(M, \mathcal{M}^*/\mathcal{O}^*)$ is defined by the functions f_α on U_α as above, then ν takes this element to $\sum_V ord_V(f_\alpha) \cdot V$.

The cohomology group, $H^1(M, \mathcal{O}^*)$, is defined by

$$H^1(M, \mathcal{O}^*) \doteq \{[g_{\alpha\beta}] \; | g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta), \forall \alpha, \beta \text{ and } \{g_{\alpha\beta}\} \text{ is a cocycle}\}.$$

By definition

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$$

is holomorphic for all α, β and nowhere zero. Also $[g_{\alpha\beta}]$ denotes the cohomology class of the collection of functions $\{g_{\alpha\beta}\}$. In order to be a cocycle, the functions $g_{\alpha\beta}$ satisfy the two conditions:

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1,$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Additionally, two cocycles, $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, are equivalent if there exists an element, $\{k_\gamma\} \in C^0(\mathcal{U}, \mathcal{O}^*)$, such that $d(\{k_\gamma\})\{g'_{\alpha\beta}\} = \{g_{\alpha\beta}\}$. To understand this equation, consider the cochain complex

$$C^0(\mathcal{U},\mathcal{O}^*) \xrightarrow{d} C^1(\mathcal{U},\mathcal{O}^*) \xrightarrow{d} C^2(\mathcal{U},\mathcal{O}^*) \xrightarrow{d} \ldots$$

Now $\{k_\gamma\} \in C^0(\mathcal{U},\mathcal{O}^*)$ means $\forall\gamma, k_\gamma : U_\gamma \to \mathbb{C}$ and k_γ is holomorphic and nowhere zero. By definition, $d(\{k_\gamma\}) \doteq \{k_{\gamma\delta}\}$ where $k_{\gamma\delta} : U_\gamma \cap U_\delta \to \mathbb{C}$ is defined by $k_{\gamma\delta} \doteq \frac{k_\gamma}{k_\delta}$. Note that $\{k_{\gamma\delta}\}$ is a one-cocycle. Verifying, $k_{\gamma\delta}k_{\delta\gamma} = \frac{k_\gamma}{k_\delta}\frac{k_\delta}{k_\gamma} = 1$, and $k_{\gamma\delta}k_{\delta\nu}k_{\nu\gamma} = \frac{k_\gamma}{k_\delta}\frac{k_\delta}{k_\nu}\frac{k_\nu}{k_\gamma} = 1$.

To summarize, $\{g_{\alpha\beta}\} \equiv \{g'_{\alpha\beta}\}$ iff $\forall\gamma, \exists k_\gamma : U_\gamma \to \mathbb{C}$ and k_γ is holomorphic and nowhere zero such that $\forall\alpha,\beta, \frac{k_\alpha}{k_\beta}g'_{\alpha\beta} = g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.

Given a divisor $D = \sum_V a_V V$ on M, together with an open cover $\{U_\alpha\}$ and local defining functions $f_{V,\alpha}$ of D, defined as $f_\alpha \doteq \prod_V f_{V,\alpha}^{a_V}$, then the associated cohomology class $[D]$ is defined as

$$[\{g_{\alpha\beta} \doteq f_\alpha/f_\beta\}] \doteq [D] \in H^1(M,\mathcal{O}^*).$$

This describes a homomorphism

$$[\,]:\ Div(M) \to H^1(M,\mathcal{O}^*),$$

$$[D] = [\sum_V a_V V] = [g_{\alpha\beta}] = [f_\alpha/f_\beta].$$

If the f_α's are the local defining functions on $D = \sum_i a_i V_i$ and the $\hat{f}_\alpha$'s are the local defining functions on $\hat{D} = \sum_i b_i V_i$, then the local defining functions for $D+\hat{D}$ are $f_\alpha\hat{f}_\alpha$. The map $[\,]$ is a homomorphism since $[D+\hat{D}] = [\frac{f_\alpha\hat{f}_\alpha}{f_\beta\hat{f}_\beta}] = [g_{\alpha\beta}][\hat{g}_{\alpha\beta}]$. This gives us

$$\begin{array}{ccc} H^0(M, \mathcal{M}^*/\mathcal{O}^*) & & H^1(M, \mathcal{O}^*) \\ \cong\downarrow \nu & & \| \\ Div(M) & \xrightarrow{[\,]} & H^1(M, \mathcal{O}^*) \end{array}$$

Also we have the long exact sequence in cohomology associated to the short exact sequence $1 \to \mathcal{O}^* \to \mathcal{M}^* \xrightarrow{p} \mathcal{M}^*/\mathcal{O}^* \to 1$:

$$0 \to H^0(M, \mathcal{O}^*) \to H^0(M, \mathcal{M}^*) \xrightarrow{p_*} H^0(M, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta} H^1(M, \mathcal{O}^*) \to \ldots$$

Let $h \in H^0(M, \mathcal{M}^*/\mathcal{O}^*)$. By definition, there exists an open cover U_α of M and meromorphic functions $h_\alpha \in \mathcal{M}^*$, such that $\forall \alpha, \beta, \frac{h_\alpha}{h_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, and $p(\{h_\alpha\}) = h$. Note: $\{h_\alpha\} \in C^0(\mathcal{U}, \mathcal{M}^*)$ Then $\delta(h) \doteq [\frac{h_\alpha}{h_\beta}]$.

We can now construct the diagram

$$\begin{array}{ccc} H^0(M, \mathcal{M}^*/\mathcal{O}^*) & \xrightarrow{\delta} & H^1(M, \mathcal{O}^*) \\ \cong\downarrow \nu & & \| \\ Div(M) & \xrightarrow{[\,]} & H^1(M, \mathcal{O}^*) \end{array}$$

This diagram commutes since, given $h \in H^0(M, \mathcal{M}^*/\mathcal{O}^*)$, by definition $\nu(h) = \sum_V (\text{ord}_V h_\alpha) V$. Therefore, $[\nu(h)] = [\Sigma_V (\text{ord}_V h_\alpha) V] = [\frac{h_\alpha}{h_\beta}]$ by proof of theorem 3.1, and $[\frac{h_\alpha}{h_\beta}] = \delta(h)$.

Often we will refer to an element of $H^1(M, \mathcal{O}^*)$ as the equivalence class of a line bundle on M using the standard identification of the set of line bundles up to isomorphism with the cocycles defined by the transition functions of the line bundle. We can form a group with the divisors modulo the set of principal divisors, $Div(M)/PDiv(M)$. To summarize,

Theorem 3.2 *There exists an injective homomorphism of groups*

$$\phi : Div(M)/PDiv(M) \hookrightarrow H^1(M, \mathcal{O}^*)$$

given by $\phi(D + PDiv(M)) = [D]$. *If* M *is a projective variety, this is an isomorphism.*

3.2 1-Forms, n-Forms, and Canonical Divisors

By definition, the canonical line bundle, K_M, on a complex n-dimensional manifold, M, is the n^{th} exterior power of the dual bundle to the holomorphic tangent bundle, i.e. $K_M = \Lambda^n T^*(M)$. On a coordinate chart with coordinates $(z_1, ..., z_n)$, it is spanned by the nonvanishing section $dz_1 \wedge ... \wedge dz_n$.

Let's compute the transition functions for this line bundle. Suppose $(\phi_j, \mathcal{O}_j)$ and $(\phi_k, \mathcal{O}_k)$ are two coordinate charts on M, where $\mathcal{O}_j$ and $\mathcal{O}_k$ are open in $\mathbb{C}^n$ and $\phi_j : \mathcal{O}_j \rightarrow M$, $\phi_k : \mathcal{O}_k \rightarrow M$. Then

$$\phi_k^{-1} \circ \phi_j : \mathcal{O}_j \cap \mathcal{O}_k \rightarrow \mathcal{O}_j \cap \mathcal{O}_k \subset \mathbb{C}^n.$$

and the composition $\phi_k^{-1} \circ \phi_j$ is biholomorphic. Then

$$\phi_k^{-1} \circ \phi_j((z_1, ..., z_n))$$

$$= ((\phi_k^{-1} \circ \phi_j)_1(z_1, ..., z_n), (\phi_k^{-1} \circ \phi_j)_2(z_1, ..., z_n), ..., (\phi_k^{-1} \circ \phi_j)_n(z_1, ..., z_n)).$$

The composite $\phi_k^{-1} \circ \phi_j$ gives a coordinate change from $(z_1, ..., z_n)$ to $(w_1, ..., w_n)$ on $\mathcal{O}_j \cap \mathcal{O}_k$

$$(z_1, ..., z_n) \overset{\phi_k^{-1}\circ\phi_j}{\longrightarrow} (w_1, ..., w_n)$$

where $w_i \doteq (\phi_k^{-1} \circ \phi_j)_i(z_1, ..., z_n)$. Then

$$dw_i \doteq \frac{\partial w_i}{\partial z_1}dz_1 + \frac{\partial w_i}{\partial z_2}dz_2 + \cdots + \frac{\partial w_i}{\partial z_n}dz_n$$

So

$$dw_1 \wedge \cdots \wedge dw_n = (\frac{\partial w_1}{\partial z_1}dz_1 + \cdots + \frac{\partial w_1}{\partial z_n}dz_n) \wedge (\frac{\partial w_2}{\partial z_1}dz_1 + \cdots + \frac{\partial w_2}{\partial z_n}dz_n) \wedge \cdots$$

$$\wedge(\frac{\partial w_n}{\partial z_1}dz_1 + \cdots + \frac{\partial w_n}{\partial z_n}dz_n)$$

First, let us consider the $n = 2$ case. Then

$$dw_1 \wedge dw_2 = (\frac{\partial w_1}{\partial z_1}dz_1 + \frac{\partial w_1}{\partial z_2}dz_2) \wedge (\frac{\partial w_2}{\partial z_1}dz_1 + \frac{\partial w_2}{\partial z_2}dz_2)$$

$$= (\frac{\partial w_1}{\partial z_1}dz_1 \wedge \frac{\partial w_2}{\partial z_1}dz_1) + (\frac{\partial w_1}{\partial z_1}dz_1 \wedge \frac{\partial w_2}{\partial z_2}dz_2) + (\frac{\partial w_1}{\partial z_2}dz_2 \wedge \frac{\partial w_2}{\partial z_1}dz_1) + (\frac{\partial w_1}{\partial z_2}dz_2 \wedge \frac{\partial w_2}{\partial z_2}dz_2)$$

$$= (\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_1})(dz_1 \wedge dz_1) + (\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2})(dz_1 \wedge dz_2)$$

$$+(\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_1})(dz_2 \wedge dz_1) + (\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_2})(dz_2 \wedge dz_2)$$

since $f_1 a_1 \wedge ... \wedge f_n a_n = (f_1 \cdots f_n)(a_1 \wedge ... \wedge a_n)$,

$$= (\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2})(dz_1 \wedge dz_2) + (\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_1})(dz_2 \wedge dz_1)$$

since $dz_i \wedge dz_i = 0$,

$$= (\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2})(dz_1 \wedge dz_2) - (\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_1})(dz_1 \wedge dz_2)$$

since $a \wedge b = -b \wedge a$,

$$= (\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2} - \frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_1})(dz_1 \wedge dz_2)$$

$$= J_{wz}(dz_1 \wedge dz_2),$$

where $J_{wz}(z_1, z_2) = \det[\frac{\partial(w_\alpha)}{\partial(z_\beta)}]_{\alpha\beta} = \begin{vmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial z_2} \\ \frac{\partial w_2}{\partial z_1} & \frac{\partial w_2}{\partial z_2} \end{vmatrix}$.

Returning to the general case

$$dw_1 \wedge \cdots \wedge dw_n = (\frac{\partial w_1}{\partial z_1}dz_1 + \cdots + \frac{\partial w_1}{\partial z_{i_1}}dz_{i_1} + \cdots + \frac{\partial w_1}{\partial z_n}dz_n)$$

$$\wedge(\frac{\partial w_2}{\partial z_1}dz_1 + \cdots + \frac{\partial w_2}{\partial z_{i_2}}dz_{i_2} + \cdots + \frac{\partial w_2}{\partial z_n}dz_n) \wedge \cdots \wedge (\frac{\partial w_n}{\partial z_1}dz_1 + \cdots + \frac{\partial w_n}{\partial z_{i_n}}dz_{i_n} + \cdots + \frac{\partial w_n}{\partial z_n}dz_n)$$

$$= [(\frac{\partial w_1}{\partial z_1}dz_1 \wedge \frac{\partial w_2}{\partial z_1}dz_1 \wedge ... \wedge \frac{\partial w_n}{\partial z_1}dz_1)] + [(\frac{\partial w_1}{\partial z_1}dz_1 \wedge \frac{\partial w_2}{\partial z_2}dz_2 \wedge ... \wedge \frac{\partial w_n}{\partial z_2}dz_2)] + ...$$

$$... + [(\frac{\partial w_1}{\partial z_n}dz_n \wedge ... \wedge \frac{\partial w_n}{\partial z_n}dz_n)].$$

Using the distributive law combined with the property

$$f_1 a_1 \wedge ... \wedge f_n a_n = (f_1 \cdots f_n)(a_1 \wedge ... \wedge a_n),$$

we have

$$dw_1 \wedge \cdots \wedge dw_n = \sum_{(i_n,...,i_n)\epsilon\{1,2,...,n\}} (\frac{\partial w_1}{\partial z_{i_1}} \cdots \frac{\partial w_n}{\partial z_{i_n}}) dz_{i_1} \wedge \cdots \wedge dz_{i_n}$$

yielding n^n summands. Since $dz_{i_1} \wedge \cdots \wedge dz_{i_n} = 0$, where $i_j = \sigma(j)$, if $(i_1, ..., i_n)$ is not a permutation of $1, ..., n$ (i.e. there is a repeated index), then

$$dw_1 \wedge \cdots \wedge dw_n = \sum_{\sigma \epsilon S_n} \frac{\partial w_1}{\partial z_{\sigma(1)}} \frac{\partial w_2}{\partial z_{\sigma(2)}} \cdots \frac{\partial w_n}{\partial z_{\sigma(n)}} dz_{\sigma(1)} \wedge \cdots \wedge dz_{\sigma(n)}.$$

Additionally, from the property $a \wedge b = -b \wedge a$, $dz_{\sigma(1)} \wedge \cdots \wedge dz_{\sigma(n)} = (-1)^{sgn\,\sigma} dz_1 \wedge \cdots \wedge dz_n$, so we have

$$dw_1 \wedge \cdots \wedge dw_n = \sum_{\sigma \epsilon S_n} (-1)^{sgn\,\sigma} \frac{\partial w_1}{\partial z_{\sigma(1)}} \frac{\partial w_2}{\partial z_{\sigma(2)}} \cdots \frac{\partial w_n}{\partial z_{\sigma(n)}} dz_1 \wedge \cdots \wedge dz_n.$$

Recall the definition of the determinant of an $n \times n$ matrix, A,

$$detA = \sum_{\sigma \epsilon S_n} (-1)^{sgn\,\sigma} x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)}$$

Let $\frac{\partial w_i}{\partial z_{\sigma(i)}} = x_{i,\sigma(i)}$, and we see that

$$dw_1 \wedge ... \wedge dw_n = J_{wz}(z_1, ..., z_n) dz_1 \wedge ... \wedge dz_n,$$

where

$$J_{wz}(z_1, ..., z_n) = \det \left[\frac{\partial(w_\alpha)}{\partial(z_\beta)} \right]_{\alpha\beta} =$$

$$\begin{vmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial z_2} & \frac{\partial w_1}{\partial z_3} & \cdots & \frac{\partial w_1}{\partial z_n} \\ \frac{\partial w_2}{\partial z_1} & \frac{\partial w_2}{\partial z_2} & \frac{\partial w_2}{\partial z_3} & \cdots & \frac{\partial w_2}{\partial z_n} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \frac{\partial w_n}{\partial z_1} & \frac{\partial w_n}{\partial z_2} & \frac{\partial w_n}{\partial z_3} & \cdots & \frac{\partial w_n}{\partial z_n} \end{vmatrix}. \ [12]$$

Therefore, the coordinate change is effected by the Jacobian, J_{wz},

$$dw_1 \wedge ... \wedge dw_n = J_{wz}(z_1, ..., z_n) dz_1 \wedge ... \wedge dz_n.$$

For an example, consider the line bundle on $\mathbb{P}^n$, $\Lambda^n T^*(\mathbb{P}^n)$. Let $Z_0, ..., Z_n$ be homogeneous coordinates on $\mathbb{P}^n$. Define $U_\alpha = \{[Z_0, ..., Z_n] \,\epsilon\, \mathbb{P}^n | Z_\alpha \neq 0\}$. Recall the map

$$\phi_\alpha^{-1} : U_\alpha \overset{\phi_\alpha^{-1}}{\to} \mathbb{C}^n$$

given by

$$[Z_0, ..., Z_n] \overset{\phi_\alpha^{-1}}{\mapsto} (\frac{Z_0}{Z_\alpha}, ..., \widehat{\frac{Z_\alpha}{Z_\alpha}}, ..., \frac{Z_n}{Z_\alpha}) \doteq (w_{0,\alpha}, ..., \widehat{w_{\alpha,\alpha}}, ..., w_{n,\alpha}),$$

with $w_{i,\alpha} = \frac{Z_i}{Z_\alpha}$, $Z_\alpha \neq 0$. Then consider the map $\phi_\beta^{-1}\phi_\alpha$, given by:

$$\phi_\beta^{-1}\phi_\alpha(w_{0,\alpha}, ..., \widehat{w_{\alpha,\alpha}}, ..., w_{n,\alpha}) = (w_{0,\beta}, ..., \widehat{w_{\beta,\beta}}, ..., w_{n,\beta})$$

where

$$w_{i,\beta} = \begin{cases} \frac{w_{i,\alpha}}{w_{\beta,\alpha}} & i \neq \alpha \\ \frac{1}{w_{\beta,\alpha}} & i = \alpha \end{cases}$$

Therefore

$$dw_{i,\beta} = \begin{cases} \frac{w_{\beta,\alpha} dw_{i,\alpha} - w_{i,\alpha} dw_{\beta,\alpha}}{(w_{\beta,\alpha})^2} & i \neq \alpha \\ \frac{-dw_{\beta,\alpha}}{(w_{\beta,\alpha})^2} & i = \alpha \end{cases}$$

Then define

$$\omega_\beta = dw_{0,\beta} \wedge \cdots \wedge \widehat{dw_{\beta,\beta}} \wedge \cdots \wedge dw_{n,\beta}.$$

Substituting and expanding the wedge product (the $dw_{\beta,\alpha}$ are all eliminated when wedged with the term in the α^{th} position) yields

$$\omega_\beta = \frac{-1}{(w_{\beta,\alpha})^{2n}}(w_{\beta,\alpha} dw_{0,\alpha} \wedge ... \wedge dw_{\beta,\alpha} \wedge ... \wedge w_{\beta,\alpha} dw_{n,\alpha})$$

where $dw_{\beta,\alpha}$ is in the α^{th} position. Using properties of the wedge product and juxtaposing terms to move $dw_{\beta,\alpha}$ to the β^{th} position yields

$$\omega_\beta = \frac{-(w_{\beta,\alpha})^{n-1}}{(w_{\beta,\alpha})^{2n}}(-1)^{\beta-\alpha-1}\, dw_{0,\alpha} \wedge \ldots \wedge dw_{\beta,\alpha} \wedge \ldots \wedge dw_{n,\alpha}$$

with $dw_{\beta,\alpha}$ in the α^{th} position, and $dw_{\alpha,\alpha}$ does not appear. So

$$\omega_\beta = \frac{(-1)^{\beta-\alpha}}{(w_{\beta,\alpha})^{n+1}}\omega_\alpha$$

and

$$\omega_\beta = J_{\alpha\beta}\omega_\alpha$$

where $J_{\alpha\beta} = \frac{(-1)^{\beta-\alpha}}{(w_{\beta,\alpha})^{n+1}}$. So $K_{\mathbb{P}^n}$ is defined by $\{J_{\alpha\beta}\}$ on $U_\alpha \cap U_\beta$.

Following the definitions, we can construct the transition functions for the line bundle corresponding to the divisor $-(n+1)H_0$ with $H_0 \doteq \{[0, Z_1, ..., Z_n]\} \subseteq \mathbb{P}^n$. Consider the divisor is $D = -(n+1)H_0$. In this instance, the H_0 is given in U_j (if $j \neq 0$) by setting $f_{0,j} = w_{0,j}$ equal to 0. Now $H_0 \cap U_0 = \emptyset$, thus set $f_{0,0} = 1$. Then

$$g_{\alpha\beta} = (\frac{f_{0,\alpha}}{f_{0,\beta}})^{-(n+1)} = \begin{cases} (\frac{w_{0,\alpha}}{w_{0,\beta}})^{-(n+1)} & \alpha, \beta \neq 0 \\ (\frac{1}{w_{0,\beta}})^{-(n+1)} & \alpha = 0 \\ (w_{0,\alpha})^{-(n+1)} & \beta = 0 \end{cases}$$

Consider $J_{\alpha\beta}$ as defined above. If $\alpha \neq \beta$, then

$$J_{\alpha\beta} = (-1)^{\beta-\alpha}(w_{\beta,\alpha})^{-n-1} = \begin{cases} (-1)^{\beta-\alpha}(\frac{w_{0,\beta}}{w_{0,\alpha}})^{n+1} = (-1)^{\beta-\alpha}(\frac{w_{0,\alpha}}{w_{0,\beta}})^{-(n+1)} & \beta \neq 0 \\ (-1)^{\beta-\alpha}(\frac{1}{(w_{0,\alpha})^{n+1}} & \beta = 0. \end{cases}$$

If $\alpha = \beta$, then $J_{\alpha\beta} = 1$. So $J_{\alpha\beta} = (-1)^{\beta-\alpha}g_{\alpha\beta}$. Therefore the line bundles defined by $J_{\alpha\beta}$ and $g_{\alpha\beta}$ are isomorphic. Thus, $K_{\mathbb{P}^n} \cong -(n+1)H_0$ as line bundles. We will compute K_S for S a smooth toric surface in section 3.5.

3.3 Properties of Intersection Numbers

An in-depth discussion of intersection numbers is unnecessary for our discussion. However, we will need three key properties of intersection number which characterize intersection numbers in the projective case. Given divisors $D_1, ..., D_n$ on M, M a complex manifold, the intersection number $(D_1 \cdot ... \cdot D_n)$ is an integer. The relevant proposals, from Barth, Peters, and VandeVen ([1], pg. 83), are:

1. The integer $(D_1 \cdot ... \cdot D_n)$ is symmetric and multilinear as a function of its arguments.
2. $(D_1 \cdot ... \cdot D_n)$ depends only on the linear equivalence classes of the D_i.
3. If $D_1, ..., D_n$ are effective divisors that meet transversely at smooth points of M, then

$$(D_1 \cdot ... \cdot D_n) = \#(D_1 \cap ... \cap D_n).$$

3.4 Divisors on Smooth Toric Surfaces

Returning to the discussion of fans, consider the fan constructed in Chapter 2 with vectors $\{(a_i, b_i)\}_{i=0}^{n+1}$ defining a complex manifold, M. We will now look more specifically at a smooth toric surface, S. Using the U_i charts from this fan, we can construct a picture of the toric surface containing a cycle of $n+2$ $\mathbb{P}^1$'s. We shall call these projective lines C_i's, so that, as before, C_i is the x_i axis in U_i (equivalently y_{i+1} axis in U_{i+1}) or $y_i = 0$ in U_i (equivalently $x_{i+1} = 0$ in U_{i+1}).

Theorem 3.3 *(see, e.g., Murray [18], Lemma 2) Any divisor on S is linearly equivalent to a unique integer linear combination of curves $C_1, ..., C_n$. (i.e. $Pic(S)$ is freely generated by the curves $C_1, ...C_n$.)*

Consider the meromorphic function x_i on S. Now

$$\text{div}(x_i) = \sum_{j=0}^{n+1}(a_j b_i - a_i b_j)C_j$$

since

$$\text{ord}_{C_j}(x_i) = a_j b_i - a_i b_j$$

because $x_i = y_j^{a_j b_i - a_i b_j} x_j^{d_{ij}}$ on $U_i \cap U_j$ (for some $d_{ij} \in \mathbb{Z}$) and C_j is defined on U_j by $y_j = 0$. Let $i = 0$. Then $\binom{a_0}{b_0} = \binom{0}{1}$ and

$$\text{div}(x_0) = \sum_{j=0}^{n+1} a_j C_j.$$

Similarly

$$\text{div}(y_i) = \sum_{j=0}^{n+1}(a_{i-1} b_j - a_j b_{i-1})C_j$$

since

$$\text{ord}_{C_j}(y_i) = a_{i-1} b_j - a_j b_{i-1}$$

because $y_i = y_j^{a_{i-1} b_j - a_j b_{i-1}} x_j^{e_{ij}}$ on $U_i \cap U_j$ (for some $e_{ij} \in \mathbb{Z}$) and C_j is defined on U_j by $y_j = 0$. Let $i = 0$, then $\binom{a_{n+1}}{b_{n+1}} = \binom{1}{0}$ and

$$\text{div}(y_0) = \sum_{j=0}^{n+1} b_j C_j.$$

Now, we know that $\text{div}(b) = 0$ in $Pic(S)$, or in other words $\text{div}(b)$ is linearly equivalent to 0. So

$$a_0 C_0 + ... + a_{n+1} C_{n+1} \sim 0.$$

Since $a_0 = 0$ and $a_{n+1} = 1$ from the above, we have

$$a_1C_1 + ... + a_nC_n + C_{n+1} \sim 0, \text{ or}$$

$$C_{n+1} \sim -a_1C_1 - ... - a_nC_n.$$

Also

$$b_0C_0 + ... + b_{n+1}C_{n+1} \sim 0.$$

But $b_0 = 1$ and $b_{n+1} = 0$ from the above, so we have

$$C_0 \sim -b_1C_1 - ... - b_nC_n.$$

By the definition of divisors of meromorphic functions, we know that $\text{div}(x_0^c y_0^d) = c\text{div}(x_0) + d\text{div}(y_0)$. Substituting,

$$\text{div}(x_0^c y_0^d) = c\sum_{j=0}^{n+1} a_jC_j + d\sum_{j=0}^{n+1} b_jC_j = \sum_{j=0}^{n+1}(ca_j + db_j)C_j. \tag{3.4}$$

As an example, consider $\mathbb{F}_k$. We have, by definition, that $\binom{a_0}{b_0} = \binom{0}{1}, \binom{a_1}{b_1} = \binom{-1}{0}, \binom{a_2}{b_2} = \binom{k}{-1}$, and $\binom{a_3}{b_3} = \binom{1}{0}$. This yields $C_0 \sim C_2$ and $C_3 \sim C_1 - kC_2$.

3.5 Computation of the Canonical Divisor for a Toric Surface

Recall initially, we defined fans in terms of $\binom{x_i}{y_i}$. We defined $U_i \doteq q(\mathbb{C}_i^2)$, with $\phi_i^{-1} : U_i \to \mathbb{C}^2$. Specifically, we defined a map $\phi_i : \mathbb{C}^2 \to U_i$ by $\phi_i\binom{x_i}{y_i} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \epsilon U_i$

such that $\phi_j^{-1}\phi_i$ is biholomorphic, with

$$\phi_j^{-1}\phi_i\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i^p y_i^q \\ x_i^r y_i^s \end{pmatrix},$$

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = A_j^{-1}A_i \text{ and } A_j = \begin{pmatrix} a_{j-1} & a_j \\ b_{j-1} & b_j \end{pmatrix},\ A_i = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}. \text{ Then:}$$

$$A_j^{-1}A_i = \begin{pmatrix} b_j & -a_j \\ -b_{j-1} & a_{j-1} \end{pmatrix} \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix} = \begin{pmatrix} a_{i-1}b_j - b_{i-1}a_j & a_ib_j - b_ia_j \\ a_{j-1}b_{i-1} - b_{j-1}a_{i-1} & a_{j-1}b_i - b_{j-1}a_i \end{pmatrix}$$

For ease of notation, we shall use (x_j, y_j) versus $\binom{x_j}{y_j}$ and continue the use of the exponnents p, q, r, s until later in the computations. As all the exponents are dependent on i, j, the subscripts on the exponents will be omitted.

So

$$x_j = x_i^p y_i^q$$

and

$$y_j = x_i^r y_i^s.$$

Then

$$dx_j = x_i^p(qy_i^{q-1})dy_i + y_i^q(px_i^{p-1})dx_i,$$

$$dy_j = x_i^r(sy_i^{s-1})dy_i + y_i^s(rx_i^{r-1})dx_i.$$

Define

$$\omega_j = dx_j \wedge dy_j$$

$$= (x_i^p(qy_i^{q-1})dy_i + y_i^q(px_i^{p-1})dx_i) \wedge (x_i^r(sy_i^{s-1})dy_i + y_i^s(rx_i^{r-1})dx_i)$$

$$= (x_i^p(qy_i^{q-1})dy_i \wedge x_i^r(sy_i^{s-1})dy_i) + (x_i^p(qy_i^{q-1})dy_i \wedge y_i^s(rx_i^{r-1})dx_i)$$

$$+(y_i^q(px_i^{p-1})dx_i \wedge x_i^r(sy_i^{s-1})dy_i) + (y_i^q(px_i^{p-1})dx_i \wedge y_i^s(rx_i^{r-1})dx_i).$$

Since $dx_i \wedge dx_i = 0$, we have

$$= (x_i^p(qy_i^{q-1})dy_i \wedge y_i^s(rx_i^{r-1})dx_i) + (y_i^q(px_i^{p-1})dx_i \wedge x_i^r(sy_i^{s-1})dy_i).$$

Using the properties $f_1a_1 \wedge f_2a_2 = f_1f_2(a_1 \wedge a_2)$ and $a \wedge b = -b \wedge a$ yields

$$= y_i^q(px_i^{p-1})x_i^r(sy_i^{s-1})(dx_i \wedge dy_i) - x_i^p(qy_i^{q-1})y_i^s(rx_i^{r-1})(dx_i \wedge dy_i)$$

$$= (ps)(x_i^{p+r-1}y_i^{q+s-1})(dx_i \wedge dy_i) - (qr)(x_i^{p+r-1}y_i^{q+s-1})(dx_i \wedge dy_i)$$

$$= (ps - qr)(x_i^{p+r-1}y_i^{q+s-1})(dx_i \wedge dy_i).$$

So

$$\omega_j = (ps - qr)(x_i^{p+r-1}y_i^{q+s-1})(dx_i \wedge dy_i)$$

$$J_{ij} = (dx_i \wedge dy_i) = J_{ij}\omega_i$$

where $J_{ij} = (ps - qr)(x_i^{p+r-1}y_i^{q+s-1})$. Note $ps - qr$ is the determinant of the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = A_j^{-1}A_i$. By definition, since the det A_i =det $A_j = 1$, then $ps - qr = 1$. The canonical divisor, K_M, is defined by $\{J_{ij}\}$.

Next we will work out the transition functions for the line bundle corresponding to the divisor $D = -C_0 - C_1 - ... - C_{n+1}$. Recall $U_j \cap C_\nu = \emptyset$ unless $\nu = j$ or $\nu = j - 1$. Consider first $\nu = j$. Then $U_j \cap C_j = \{\begin{bmatrix} x_j \\ y_j \end{bmatrix} \epsilon\, U_j \mid y_j = 0\}$. So C_j is defined by $y_j = 0$ on U_j. If $\nu = j - 1$, then $U_j \cap C_{j-1} = \{\begin{bmatrix} x_j \\ y_j \end{bmatrix} \epsilon\, U_j \mid x_j = 0\}$ and C_{j-1} is defined on U_j by $x_j = 0$. If $U_j \cap C_\nu = \emptyset$, then take as the equation for C_ν

on U_j $f_{\nu,j} = 1$. Then the transition functions for the line bundle corresponding to $-C_0 - C_1 - ... - C_{n+1}$ are, by definition:

$$g_{ij} = [\frac{(1)(1)...(x_i)(y_i)...(1)}{(1)...(x_j)(y_j)...(1)(1)}]^{-1}$$

on $U_i \cap U_j$, where the x_j in the numerator is in the $(j-1)^{st}$ position, the y_j is in the j^{th} position, the x_i in the denominator is in the $(i-1)^{st}$ position, and the y_i is in the i^{th} position, consistent with the equations for the curves C_ν described above, with 1's in the remaining positions (where $U_j \cap C_\nu = \emptyset$). Therefore

$$g_{ij} = [\frac{x_i y_i}{x_j y_j}]^{-1} = \frac{x_j y_j}{x_i y_i} = \frac{x_i^{p+r} y_i^{q+s}}{x_i y_i} = x_i^{p+r-1} y_i^{q+s-1} = J_{ij}$$

on $U_i \cap U_j$. Since the transition functions for the two line bundles are the same, then

$$K_S = -C_0 - C_1 - ... - C_{n+1}. \tag{3.5}$$

As an example, consider $\mathbb{F}_k$. We have

$$K_{\mathbb{F}_k} = -C_0 - C_1 - C_2 - C_3 = -C_2 - C_1 - C_2 - (C_1 - kC_2) = -2C_1 + (k-2)C_2. \tag{3.6}$$

3.6 Computing Intersection Numbers

In this section, we follow the exposition in Murray [15].

We computed the canonical divisor for a toric surface, $K_S = -C_0 - C_1 - ... - C_{n+1} = -C_0 - C_{n+1} - \sum_{i=1}^n C_i$. Also, we have that $C_0 = -\sum_{i=1}^n b_i C_i$ and $C_{n+1} =$

$-\sum_{i=1}^{n} a_i C_i$. Substituting, we get the additional formula

$$K_S = \sum_{i=1}^{n} a_i C_i + \sum_{i=1}^{n} b_i C_i - \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (a_i + b_i - 1) C_i$$

We also know that a divisor for a general toric surface is linearly equivalent to a divisor $D = \sum_{i=0}^{n+1} m_i C_i$. (see Theorem 3.3) Since we have that $m_0 = m_{n+1} = 0$, then $D = \sum_{i=1}^{n} m_i C_i$. We have seen that the curves C_i form a cycle, so their intersection numbers are:

$$C_i \cdot C_j = \begin{cases} C_i^2 & \text{if } j = i \\ 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $D' = \sum_{i=1}^{n} m'_i C_i$ be another divisor. Then

$$D \cdot D' = (\sum_{i=1}^{n} m_i C_i) \cdot (\sum_{i=1}^{n} m'_i C_i)$$

$$= (m_1 C_1 + m_2 C_2 + ... + m_n C_n) \cdot (m'_1 C_1 + m'_2 C_2 + ... + m'_n C_n)$$

$$= m_1 m'_1 C_1^2 + m_1 m'_2 C_1 \cdot C_2 + ... + m_1 m'_n C_1 \cdot C_n + m_2 m'_1 C_2 \cdot C_1 + m_2 m'_2 C_2^2 + ...$$

$$+ m_2 m'_n C_2 \cdot C_n + ... + m_n m'_1 C_n \cdot C_1 + m_n m'_2 C_n \cdot C_2 + ... m_n m'_n C_n^2$$

$$= \sum_{i=1}^{n} m_i m'_i C_i^2 + 2 \sum_{i=1}^{n} m_{i-1} m'_i$$

applying the criteria above. Since $K_S = -C_0 - C_1 - ... - C_{n+1} = \sum_{i=1}^{n} (a_i + b_i - 1) C_i$, substituting, we have

$$D \cdot K_S = -\sum_{i=1}^{n} m_i C_i^2 - 2 \sum_{i=1}^{n} m_i.$$

We next compute the self-intersection numbers of the C_i's. Recall from our earlier discussion of divisors

$$div(x_i) = \sum_{j=0}^{n+1}(a_jb_i - a_ib_j)C_j.$$

Equivalently,

$$\sum_{j=0}^{n+1}(a_jb_i - a_ib_j)C_j = \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j + (a_{i-1}b_i - a_ib_{i-1})C_i$$

$$= \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j + C_i$$

since $a_{i-1}b_i - a_ib_{i-1} = 1$. Therefore

$$C_i = div(x_i) - \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j.$$

Since $K_S = -C_0 - C_1 - ... - C_{n+1}$, we can substitute for C_i, yielding

$$K_S = -C_0 - C_1 - ... - div(x_i) + \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j - ... - C_{n+1}.$$

We can now compute $C_i \cdot K_S$, simplifying using the equation for the intersection numbers:

$$C_i \cdot K_S = C_i \cdot (-C_0 - C_1 - ... - div(x_i) + \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j - ... - C_{n+1})$$

$$= -C_i \cdot C_{i-1} - C_i \cdot div(x_i) + C_i \cdot \sum_{j\neq 1}(a_{i-1}b_j - a_jb_{i-1})C_j - C_i \cdot C_{i+1}$$

$$= -1 - 0 + (a_{i-1}b_{i-1} - a_{i-1}b_{i-1})C_i \cdot C_{i-1} + (a_{i-1}b_{i+1} - a_{i+1}b_{i-1})C_i \cdot C_{i+1} - 1$$

$$= -2 + (a_{i-1}b_{i+1} - a_{i+1}b_{i-1}).$$

Since $C_i \cong \mathbb{P}^1$ and projective lines are rational, we know their genus is zero. Therefore, $g(C_i) = 0$ Also, we have the following theorem:

Theorem 3.7 *(Genus formula for smooth curves on a surface) (see e.g. Griffiths and Harris [8], page 471)*
If S is a smooth surface, C a smooth curve on S, then

$$g(C) = \frac{K_S \cdot C + C \cdot C}{2} + 1.$$

Hence, we have that

$$-2 = C_i^2 + C_i \cdot K_S = C_i^2 + -2 + (a_{i-1}b_{i+1} - a_{i+1}b_{i-1}),$$

thus

$$C_i^2 = -(a_{i-1}b_{i+1} - a_{i+1}b_{i-1}).$$

For example, consider $\mathbb{F}_k$. From our earlier discussion of intersection numbers, we have

$$C_i \cdot C_j = \begin{cases} C_i^2 & \text{if } j = i \\ 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$C_i^2 = -(a_{i-1}b_{i+1} - a_{i+1}b_{i-1}).$$

Then

$$C_1 \cdot C_1 = C_1^2 = k,$$

$$C_1 \cdot C_2 = 1,$$

and

$$C_2 \cdot C_2 = C_2^2 = 0.$$

3.7 Fundamental Theorems

In this section, we will collect some important theorems that we will be using later in the paper: we have already used the first theorem. References include Barth, Peters, and VandeVen [1] and Griffiths and Harris [8].

Definition 3.8 $h^i(M, \mathcal{F}) \doteq \dim H^i(M, \mathcal{F})$, *where* M *is a manifold and* $\mathcal{F}$ *is a coherent sheaf of* $\mathcal{O}_M$*-modules on* M.

Definition 3.9 *Assume* S *is a smooth connected surface, then*

1. *the geometric genus,* $p_g(S) = h^2(S, \mathcal{O}_S)$,
2. $q(S) = h^1(S, \mathcal{O}_S)$,
3. $\chi(S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q(S) + p_g(S)$,
4. *the arithmetic genus,* $p_a(S) = \chi(S) - 1 = p_g(S) - q(S)$.

Theorem 3.10 *(Genus formula for smooth curves on a smooth surface)(see e.g. Griffiths and Harris [8], page 471)*
If C *is a smooth curve on a smooth surface* S, *then the genus of the curve,* $g(C)$ *is given by*

$$g(C) = \frac{K_S \cdot C + C \cdot C}{2} + 1.$$

Theorem 3.11 *(Adjunction Formula I, see e.g. Griffiths and Harris [8], page 146)*
Suppose M *is an* n*-dimensional complex manifold,* V *a smooth hypersurface,* $[V]$ *the corresponding line bundle on* M, *then*

$$N_V^* = [-V]|_V.$$

Theorem 3.12 *(Adjunction Formula II, see e.g. Griffiths and Harris [8], page 147)*

Suppose M is an n-dimensional complex manifold, V a smooth hypersurface, $[V]$ the corresponding line bundle on M, then

$$K_V = (K_M \otimes [V])|_V.$$

Theorem 3.13 *(Riemann-Roch for line bundles (or divisors) on a surface S)(see e.g. Griffiths and Harris [8])*

If S is a smooth connected surface, E a line bundle on S, then

$$\chi(E) = \frac{E(E-K_S)}{2} + 1 + p_a(S) = \frac{E(E-K_S)}{2} + \chi(S).$$

Theorem 3.14 *(Noether's formula, see e.g. Barth, Peters, and VandeVen [1], page 26)*

If S is any compact, connected complex surface, then

$$\chi(S) = 1 - q(S) + p_g(S) = \frac{1}{12}(c_1^2(S) + c_2(S)),$$

where $c_1(S)$ is the first Chern class of S, and $c_2(S)$ is the second Chern class of S.

Remark: By definition, $c_i(S) = c_i(\mathbb{T}(S))$ where $\mathbb{T}(S)$ is the holomorphic tangent bundle of S. Thus $c_1^2(S) = K_S^2$ for a surface.

Theorem 3.15 *(Riemann-Roch for a rank 2 bundle over a surface S) (see e.g. Duflot/Miranda [5])*

If S is a smooth surface and $\mathcal{E}$ is a vector bundle over S then

$$\chi(\mathcal{E}) = 2\chi(\mathcal{O}_S) - \frac{K_S \cdot c_1(\mathcal{E})}{2} + \frac{c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})}{2}$$

where the c_i's are the Chern classes of $\mathcal{E}$.

Theorem 3.16 *(Kodaira-Serre Duality) (Griffiths and Harris [8])*

Let S be a smooth surface. Let E be a vector bundle on S, K_S the canonical divisor. Then, if E^* is dual to E, there are non-canonical isomorphisms

$$H^0(S,E) \cong H^2(S,E^* \otimes K_S),$$

$$H^2(S,E) \cong H^0(S,E^* \otimes K_S),$$

$$H^1(S,E) \cong H^1(S,E^* \otimes K_S).$$

Corollary 3.17 *If S is a smooth surface, E a divisor on S, then there are non-canonical isomorphisms*

$$H^0(S,\mathcal{O}(E)) \cong H^2(S,\mathcal{O}(K_S - E)),$$

$$H^2(S,\mathcal{O}(E)) \cong H^0(S,\mathcal{O}(K_S - E)),$$

$$H^1(S,\mathcal{O}(E)) \cong H^1(S,\mathcal{O}(K_S - E)).$$

Corollary 3.18 *If S is a smooth surface and $E = \Omega^1_S \otimes \mathcal{O}(\tilde{E}) \doteq \Omega^1_S(\tilde{E})$ for $\tilde{E}$ a divisor on S, then since $\mathbb{T}(S) \otimes K_S \cong \Omega^1_S$, there are non-canonical isomorphisms*

$$H^0(S,\Omega^1_S(\tilde{E})) \cong H^2(S,(\Omega^1_S \otimes \mathcal{O}(\tilde{E}))^* \otimes K_S)$$

$$= H^2(S,\mathbb{T}(S) \otimes \mathcal{O}(-\tilde{E}) \otimes K_S)$$

$$= H^2(S,\Omega^1_S(-\tilde{E})),$$

$$H^1(S,\Omega^1_S(\tilde{E})) = H^1(S,\Omega^1_S(-\tilde{E})),$$

$$H^2(S,\Omega^1_S(\tilde{E})) = H^0(S,\Omega^1_S(-\tilde{E})).$$

Corollary 3.19

$$p_g(S) \doteq h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K_S)).$$

3.8 Invariants for a Smooth Toric Surface

Invariants for a general smooth toric surface, S, include:

1. $\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) = 1$. Since a toric surface is rational, $h^i(\mathcal{O}_S) = 0$ for all $i \geq 1$, and $h^0(\mathcal{O}_S) = 1$, thus $\chi(\mathcal{O}_S) = 1$.

2. $p_a(S) = \chi(\mathcal{O}_S) - 1 = 0$.

3. $q(S) \doteq h^1(\mathcal{O}_S) = 0$.

4. $p_g(S) = h^2(\mathcal{O}_S) = h^0(K_S) = 0$, where K_S is the canonical divisor, by Kodaira-Serre duality 3.16.

Theorem 3.20 *If S is a smooth toric surface defined by a fan of $n+2$ vectors, then $K_S^2 = 10 - n$.*

PROOF: This is proven in Oda [19].

Thus we can use Noether's formula (Theorem 3.14) to compute $c_2(S)$:

$$\chi(S) = \frac{1}{12}(c_1^2(S) + c_2(S)).$$

Since $\chi(S) = 1$ and $c_1^2 = K_S^2 = 10 - n$, substituting yields:

$$1 = \frac{(10-n) + c_2(S)}{12},$$

so

$$c_2(S) = 2 + n.$$

As an example, consider the invariants for $\mathbb{F}_k$:

$$q(\mathbb{F}_k) \doteq h^1(\mathcal{O}_{\mathrm{F}_k}) = 0,$$

$$p_g(\mathbb{F}_k) \doteq h^2(\mathcal{O}_{\mathrm{F}_k}) = 0,$$

$$p_a(\mathbb{F}_k) = 0,$$

$$\chi(\mathbb{F}_k) \doteq \chi(\mathcal{O}_{\mathrm{F}_k}) \doteq 1 - q(\mathbb{F}_k) + p_g(\mathbb{F}_k) = 1.$$

Furthermore, computing,

$$c_1(\mathbb{F}_k)^2 = K^2_{\mathrm{F}_k} = 4k - 4(k-2) = 8$$

and

$$c_2(\mathbb{F}_k) = 4.$$

Chapter 4

Cohomology Computations with a Smooth Toric Surface

In this chapter, we will compute some cohomology groups, $H^i(S, \mathcal{O}(D))$ and $H^i(S, \Omega^1(D))$, for a divisor D on a smooth toric surface, S, with $i = 0, 1, 2$. We will also define and compute some cases of $H^0(S, \Omega^1(log\ D))$. General references for this section are Oda [19], Duflot/Miranda [5], Saito [20], and Murray [17].

4.1 Hirzebruch Surfaces, $i = 0$

In the case where $S = \mathbb{F}_k$, $k > 0, n = 2$, suppose $D = \sum_{i=0}^{3} m_i C_i$, with $m_0 = m_3 = 0$. Recalling the construction of $\mathbb{F}_k$ using a four vector fan, specifically with vectors $\langle 0, 1\rangle, \langle -1, 0\rangle, \langle k, -1\rangle$, and $\langle 1, 0\rangle$, and the resulting equivalences for coordinates x_0 and y_0,

$$x_0 = \begin{cases} y_1^{-1} & \text{in } U_1 \\ x_2^{-1} y_2^k & \text{in } U_2 \\ x_3^k y_3 & \text{in } U_3 \end{cases} \tag{4.1}$$

$$y_0 = \begin{cases} x_1 & \text{in } U_1 \\ y_2^{-1} & \text{in } U_2 \\ x_3^{-1} & \text{in } U_3. \end{cases} \tag{4.2}$$

So the monomial

$$x_0^c y_0^d = \begin{cases} x_1^d y_1^{-c} & \text{in } U_1 \\ x_2^{-c} y_2^{ck-d} & \text{in } U_2 \\ x_3^{ck-d} y_3^c & \text{in } U_3. \end{cases} \tag{4.3}$$

Recall that $\binom{a_0}{b_0} = \binom{0}{1}$, $\binom{a_1}{b_1} = \binom{-1}{0}$, $\binom{a_2}{b_2} = \binom{k}{-1}$, and $\binom{a_3}{b_3} = \binom{1}{0}$. Combining this with the divisor equation $\text{div}(x_0^c y_0^d) = \sum_{j=0}^{n+1}(ca_j + db_j)C_j$ (3.4), we get

$$div(x_0^c y_0^d) = dC_0 - cC_1 + (ck - d)C_2 + cC_3.$$

To be a section of $\mathcal{O}(D)$, $x_0^c y_0^d$ needs to satisfy

$$D + \text{div}(x_0^c y_0^d) \geq 0, \text{ in } U_i \ \forall i.$$

Since $D = \sum_{i=0}^{3} m_i C_i$, with $m_0 = m_3 = 0$, we have $D = 0C_0 + m_1C_1 + m_2C_2 + 0C_3$. We can combine these to get restrictions on c and d, specifically:

$$(0 + d)C_0 + (m_1 - c)C_1 + (m_2 + ck - d)C_2 + cC_3 \geq 0.$$

This gives us

$$d \geq 0$$

$$m_1 - c \geq 0 \text{ so } m_1 \geq c$$

$$m_2 + ck - d \geq 0$$

$$c \geq 0.$$

Combining these inequalities yields

$$0 \le d \le m_2 + ck$$

$$0 \le c \le m_1.$$

This yields parts 1 and 2 of the following:

Lemma 4.4 *(see Duflot/Miranda [5])*

1. *$H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2))$ is zero if either m_1 or $m_2 + m_1k$ is negative.*

2. *If m_1 and $m_2 + m_1k$ are nonnegative, then $H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2))$ is the linear span of $\{x_0^c y_0^d | 0 \le c \le m_1, 0 \le d \le m_2 + ck\}$. Note that this is a set of independant monomials.*

3. *$H^0(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2))$ is zero if either m_1 or $m_2 + m_1k$ is negative.*

4. *If $m_1 \ge 0$ and $m_2 + m_1k \ge 0$, then $H^0(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}$, where*

$$\mathcal{X}(m_1, m_2) \doteq < \{x_0^i y_0^j dx_0 |\ 0 \le i \le m_1 - 2, 0 \le j \le m_2 + ik + k - 1\} >$$

$$\mathcal{Y}(m_1, m_2) \doteq < \{x_0^\alpha y_0^\beta dy_0 |\ 0 \le \alpha \le m_1, 0 \le \beta \le m_2 + k\alpha - 2\} >$$

and

$$\mathcal{M}(m_1, m_2) \doteq$$

$$< \{x_0^i y_0^{m_2+ki+k-1}(y_0 dx_0 + kx_0 dy_0) |\ 0 \le i \le m_1 - 2, m_2 + ki + k - 1 \ge 0\} > .$$

For the proofs of 3 and 4, refer to Duflot/Miranda [5].

Note that sometimes one of more of the sets $\mathcal{X}(m_1, m_2), \mathcal{Y}(m_1, m_2)$, or $\mathcal{M}(m_1, m_2)$ will be empty. For example, $\mathcal{M}$ is zero if $m_1 < 2$.

Corollary 4.5 *If $m_1 \geq 0$ and $m_2 \geq 0$, then*

$$dim\ H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) = (m_1 + 1)(\frac{k}{2}m_1 + m_2 + 1) = \chi(m_1C_1 + m_2C_2).$$

PROOF: Let $m_1 \geq 0$ and $m_2 \geq 0$. Since $H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) =< \{x_0^c y_0^d | 0 \leq c \leq m_1, 0 \leq d \leq m_2 + ck\} >$, to accurately compute the dimension, we need to consider the different possibilities for c and d. If $c = 0$, then $0 \leq d \leq m_2$, and we have $m_2 + 1$ in the basis count. If $c = 1$, then $0 \leq d \leq m_2 + k$, adding m_2+k+1 to the basis count. The maximum value for c is m_1, yielding an inequality $0 \leq d \leq m_2 + m_1k$ or a dimension of $(m_1 + 1)m_2 + k(\sum_{c=1}^{m_1} c) + (m_1 + 1) = (m_1 + 1)(m_2 + 1) + k\frac{(m_1+1)(m_1)}{2} = (m_1 + 1)(\frac{k}{2}m_1 + m_2 + 1)$. Hence

$$\dim\ H^0(\mathbb{F}_n, \mathcal{O}(m_1C_1 + m_2C_2)) = (m_1 + 1)(\frac{k}{2}m_1 + m_2 + 1).$$

Also

$$\chi(m_1C_1 + m_2C_2) = \frac{(m_1C_1 + m_2C_2)(m_1C_1 + m_2C_2 - K_{\mathbb{F}_k})}{2}) + 1$$

by Riemann-Roch, Theorem 3.13,

$$= \frac{m_1^2k + m_1k + 2m_1 + 2m_2 + 2m_1m_2}{2} + 1$$

$$= \frac{1}{2}km_1(m_1 + 1) + m_1m_2 + m_1 + m_2 + 1 = (m_1 + 1)(\frac{k}{2}m_1 + m_2 + 1).$$

Corollary 4.6 *1. If $m_1 \geq 2$ and $m_2 + k - 1 \geq 0$ then*

$$dim\ \mathcal{X}(m_1, m_2) = m_2(m_1 - 1) + \frac{m_1(m_1 - 1)}{2}k.$$

2. *If* $m_1 \geq 0$ *and* $m_2 \geq 2$, *then*

$$dim\ \mathcal{Y}(m_1, m_2) = (m_2 - 1)(m_1 + 1) + k\frac{m_1(m_1+1)}{2}.$$

3. *If* $m_1 \geq 2$ *and* $m_2 + k - 1 \geq 0$, *then*

$$dim\ \mathcal{M}(m_1, m_2) = m_1 - 1.$$

4. *Therefore, if* $m_1 \geq 2$ *and* $m_2 \geq 2$, *then*

$$dim\ H^0(\Omega^1(m_1C_1 + m_2C_2)) = 2m_1m_2 + km_1^2 - 2.$$

PROOF: Since $\mathcal{X} =< \{x_0^i y_0^j dx_0 |\ 0 \leq i \leq m_1 - 2, 0 \leq j \leq m_2 + k(i+1) - 1\} >$, to have an accurate count for $\mathcal{X}$, we need to consider $m_1 \geq 2$ and $m_2 + k - 1 \geq 0$. Then

$$\dim \mathcal{X} = (m_2+k)+(m_2+2k)+...+(m_2+k(m_1-1)) = m_2(m_1-1)+\frac{m_1(m_1-1)}{2}k.$$

Since $\mathcal{Y} =< \{x_0^\alpha y_0^\beta dy_0 |\ 0 \leq \alpha \leq m_1, 0 \leq \beta \leq m_2 + k\alpha - 2\} >$, consider $m_1 \geq 0$ and $m_2 \geq 2$. Therefore

$$\dim \mathcal{Y} = (m_2 - 1) + (m_2 - 1) + k + (m_2 - 1) + 2k + ... + (m_2 - 1) + m_1 k$$

$$= (m_2 - 1)(m_1 + 1) + k\frac{m_1(m_1+1)}{2}$$

Finally, since $\mathcal{M} =< \{x_0^i y_0^{m_2+ki+k-1}(y_0dx_0 + kx_0dy_0) |\ 0 \leq i \leq m_1 - 2,\ m_2 + ki + k - 1 \geq 0\} >$, to get an accurate count, consider $m_1 \geq 2$ and $m_2 + k - 1 \geq 0$. In this case,

$$\dim \mathcal{M} = m_1 - 1.$$

To conclude, if $m_1 \geq 2$ and $m_2 - 2 \geq 0$, then

$$\dim H^0(\Omega^1(m_1C_1 + m_2C_2))$$

$$= m_2(m_1 - 1) + \frac{m_1(m_1-1)}{2}k + (m_2 - 1)(m_1 + 1) + k\frac{m_1(m_1+1)}{2} + m_1 - 1$$

$$= 2m_1m_2 + km_1^2 - 2.$$

However, we can extend part 4 of the above theorem a bit:

Theorem 4.7 *If $m_1 \geq 1$ and $m_2 \geq 1$, then*

$$dim\ H^0(\Omega^1(m_1C_1 + m_2C_2)) = 2m_1m_2 + km_1^2 - 2.$$

PROOF:

Case 1: Let $m_1 = 1, m_2 \geq 2$. Then dim $\mathcal{X} = 0$ and dim $\mathcal{M} = 0$. Also, dim $\mathcal{Y} = 2(m_2 - 1) + k = 2m_2 + k - 2$. Since $H^0(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}$, we have that dim $H^0(\Omega^1(m_1C_1 + m_2C_2)) = 2m_2 + k - 2 = 2m_1m_2 + km_1^2 - 2$ if $m_1 = 1$.

Case 2: Let $m_2 = 1, m_1 \geq 2$. Then dim $\mathcal{X} = (m_1 - 1) + \frac{m_1(m_1-1)}{2}k$, dim $\mathcal{Y} = k\frac{m_1(m_1+1)}{2}$, and dim $\mathcal{M} = m_1 - 1$. Therefore $\dim(\mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}) = m_1 - 1 + \frac{m_1(m_1-1)}{2}k + k\frac{m_1(m_1+1)}{2} + m_1 - 1 = 2m_1 + km_1^2 - 2 = 2m_1m_2 + km_1^2 - 2$, if $m_2 = 1$.

Case 3: Let $m_1 = 1, m_2 = 1$. Then dim $\mathcal{X} = 0$, dim $\mathcal{Y} = k$, and dim $\mathcal{M} = 0$, yielding dim $(\mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}) = k$, consistent with the equation dim $H^0(\Omega^1(m_1C_1 + m_2C_2)) = 2m_1m_2 + km_1^2 - 2$ and the stated conditions.

4.2 General Toric Surfaces, $i = 0$

We will be using the basis $C_1, ..., C_n$ for the Picard group of a smooth toric surface and the fans defined in chapter 2, $\{(a_i, b_i)\}_{i=0}^{n+1}$. Since $(a_0, b_0) = (0, 1)$ and

$(a_{n+1}, b_{n+1}) = (1, 0)$, $m_0 = m_{n+1} = 0$, every a divisor D is linearly equivalent to $m_1C_1 + ... + m_nC_n$. The basis for this discussion is largely Murray [15].

We will now compute $H^i(M, \mathcal{O}(m_1C_1 + ... + m_nC_n))$ and $H^i(M, \Omega^1(m_1C_1 + ... + m_nC_n))$ for a divisor $D = m_1C_1 + ... + m_nC_n$ on a general toric surface. First, consider creating a polygon from the fan. The lattice enclosed by this polygon defines dimension of $H^0(\mathcal{O}(m_1C_1+...+m_nC_n))$, hence it is called the $H^0(\mathcal{O}(m_1C_1+ ... + m_nC_n))$ polygon or the H^0 polygon associated with the fan and the divisor $D = m_1C_1+...+m_nC_n$. Consider a monomial $x_0^c y_0^d$. Using the equivalence relation from chapter 2,

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sim \begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftrightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_0^p \ y_0^q \\ x_0^r \ y_0^s \end{pmatrix}$$

where $x_0^p \ y_0^q$ and $x_0^r \ y_0^s$ make sense, and

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}^{-1}.$$

Then $x_0 = x_i^A y_i^B, y_0 = x_i^C y_i^D$ where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{pmatrix}.$$

Therefore, we get $x_0^c y_0^d = (x_i^{a_{i-1}} y_i^{a_i})^c (x_i^{b_{i-1}} y_i^{b_i})^d$. Since C_{i-1} meets U_i, defined by $x_i = 0$, and C_i meets U_i, defined by $y_i = 0$, this translates into a requirement that

$$x_i^{m_{i-1}} y_i^{m_i} (x_i^{a_{i-1}} y_i^{a_i})^c (x_i^{b_{i-1}} y_i^{b_i})^d = x_1^{m_{i-1}+a_{i-1}c+b_{i-1}d} y_i^{m_i+a_ic+b_id}$$

must be a polynomial. Or,

$$m_i + a_ic + b_id \geq 0, i = 0, 1, ..., n + 1,$$

where we assume $m_0 = m_{n+1} = 0$. Then for $i = 0, i = n + 1$, $a_ic + b_id \geq 0$, so since $\binom{a_0}{b_0} = \binom{0}{1}$, $d \geq 0$ and since $\binom{a_{n+1}}{b_{n+1}} = \binom{1}{0}$, then $c \geq 0$. Let the axes be $c = 0$ and $d = 0$ (horizontal and vertical axes, respectively.) Thus, the polygon will be in the first quadrant. We will use these inequalities and restrictions on the monomial powers in the fan to define the polygon.

In our $\mathbb{F}_k$ example, $n = 2$. We use the fan $\binom{0}{1}$, $\binom{-1}{0}$, $\binom{k}{-1}$, and $\binom{1}{0}$. The inequalities are:

$$0 \leq c \leq m_1$$

$$0 \leq d \leq m_2 + kc$$

If $m_1 > 0$ and $m_2 > 0$, the polygon is a quadrilateral. The sides are defined by end points $(0,0), (m_1, 0)$ on the c-axis, $(0, m_2)$ on the d-axis, and $(m_1, m_2 + km_1)$. Comparing the polygon to the fan (below), one can see that the sides of the polygon correspond to the duals of the rays of the fan (i.e the inward pointing normal vectors of the sides of the polygon correspond to the rays of the fan, in counterclockwise order). See figure 4.1 below.

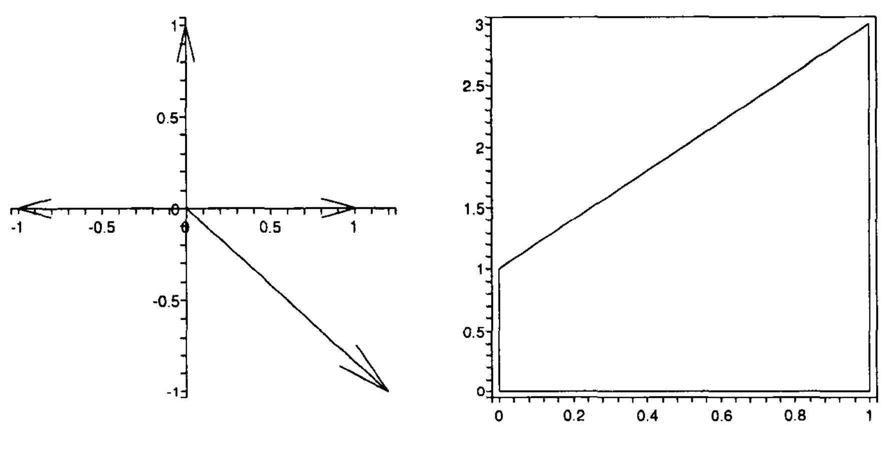

Figure 4.1: The $\mathbb{F}_n$ fan and resulting polygon

Note that all edge vertices of the polygon are integers, since m_1, m_2, and k are integers. Computing the number of lattice points enclosed by the polygon (including edge points), results in a total of $(m_1+1)(m_2+1)+\frac{1}{2}km_1(m_1+1)$ lattice points, exactly the dimension of $H^0(\mathbb{F}_k, \Omega^1(m_1C_1+m_2C_2))$ computed in corollary 4.6.

We also need to consider the cases where $m_1 = 0$ or $m_2 = 0$. If $m_1 = 0$ and $m_2 > 0$, the polygon degenerates into a line on the d-axis, length m_2 and $D = m_2C_2$ and $m_2 + 1$ lattice points. If $m_2 = 0$ and $m_1 > 0$, the polygon degenerates into a triangle with vertices $(0,0), (m_1, 0)$, and $(m_1, m_1 + km_1)$, and $D = m_1C_1$. The number of lattice points enclosed in this case, then, is $\frac{1}{2}km_1(m_1+1)$. (see figure 4.2 below) If both $m_1 = 0$ and $m_2 = 0$, the result is the single point $(0,0)$ and $D = 0$. Therefore, a quadrilateral results specifically when both m_1 and m_2 are strictly positive, with $0 \le c \le m_1, 0 \le d \le m_2$, and $d \le m_2 + kc$.

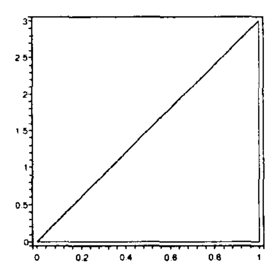

Figure 4.2: Degenerate polygon

As another example, consider a fan constructed with 5 vectors, meeting the criteria set down in the first chapter. Vectors meeting that criteria are: $\binom{a_0}{b_0} = \binom{0}{1}, \binom{a_1}{b_1} = \binom{-1}{0}, \binom{a_2}{b_2} = \binom{k}{-1}, \binom{a_3}{b_3} = \binom{k+1}{-1}$, and $\binom{a_4}{b_4} = \binom{1}{0}$. Our divisor is defined with $D = m_0C_0 + m_1C_1 + m_2C_2 + m_3C_3 + m_4C_4$. However, as before, we have $m_0 = m_4 = 0$, leaving $D = m_1C_1 + m_2C_2 + m_3C_3$. Again, we will use $m_i + a_ic + b_id \ge 0$. We get two sides on the c, d axes from $\binom{a_0}{b_0} = \binom{0}{1}$, $d \ge 0$ and since $\binom{a_4}{b_4} = \binom{1}{0}$, then

$c \geq 0$. Additionally, from $\binom{a_1}{b_1} = \binom{-1}{0}$, we get that $c \leq m_1$. From $i = 2$, we get the requirement that $m_2 + kc - d \geq 0$ and from $i = 3$, we have that $m_3 + (k+1)c - d \geq 0$. Therefore, the constraining inequalities are:

$$0 \leq c \leq m_1,$$

$$0 \leq d \leq m_2 + kc,$$

$$d \leq m_3 + (k+1)c.$$

Assuming $m_2 > m_3$, the polynomial formed has vertices $(0,0)$, $(m_1, 0)$, $(0, m_3)$, $(m_2 - m_3, k(m_2 - m_3) + m_2)$, and $(m_1, km_1 + m_2)$, shown in figure 4.3 below.

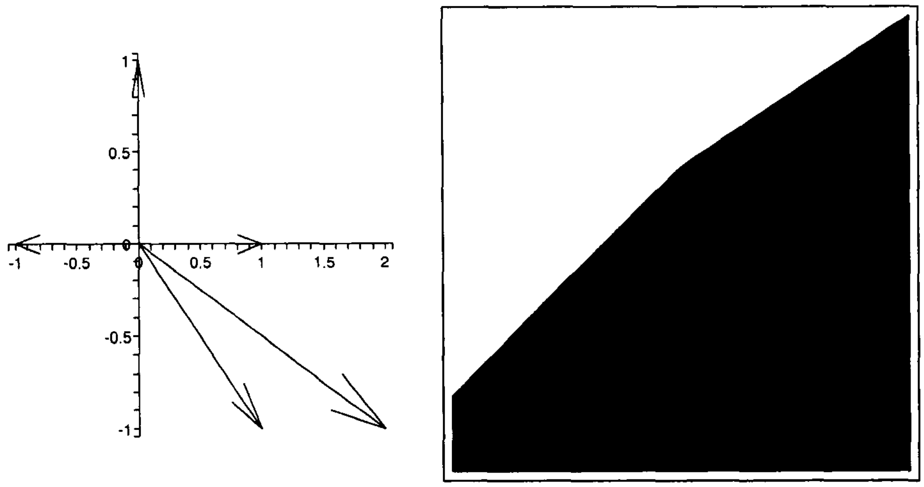

Figure 4.3: The five vector fan and resulting polygon

Again, note that the inward pointing normal vectors to the sides of the polygon correspond to the vectors from the fan, in counterclockwise order. The equations for the edges of the polygon are:

$$c = d = 0$$

$$c = m_1$$

$$d = (k+1)c + m_3$$

$$d = kc + m_2.$$

Note that all edge vertices of the polygon are integers, since m_1, m_2, m_3, and k are integers.

Definition 4.8 *(Lazerfeld [13]) Let M be a smooth surface and D a divisor on M. Then D is very ample if there exists a closed embedding $M \subseteq \mathbb{P}$ of M into some projective space $\mathbb{P} = \mathbb{P}^N$ such that $D = \mathcal{O}_M \doteq \mathcal{O}_{\mathbb{P}^N}(1)|_M$. Also, D is ample if $D^{\otimes m}$ is very ample for some $m > 0$.*

The polygon construction and correlation with the ampleness definition results in the following lemma:

Lemma 4.9 *(see e.g. Murray [18] Lemma 3) Let M be a smooth toric surface given by a fan $\{(a_i, b_i)\}_{i=0}^{n+1}$. Take $m_0 = m_{i+1} = 0$ and let $D = \sum_{i=1}^{n} m_i C_i$ be a divisor on M. Then the following are equivalent:*

1. *The divisor $D = m_1 C_1 + ... + m_n C_n$ is ample.*
2. *The divisor $D = m_1 C_1 + ... + m_n C_n$ is very ample.*
3. *The $H^0(M, \mathcal{O}(m_1 C_1 + ... + m_n C_n))$ polygon has $n+2$ vertices.*
4. *For each $i = 1, ..., n$, we have $0 < m_i C_i^2 + m_{i-1} + m_{i+1}$, $m_1 > 0$ and $m_n > 0$.*

The equivalence of 1 and 4 is called the Toric Nakai Criterion, by Oda [19].

For example, if $n = 3$, taking the 5 vector fan as before, we have $C_1^2 = k, C_2^2 = -1, C_3^2 = -1$. Then we have the criteria $D = m_1 C_1 + m_2 C_2 + m_3 C_3 + m_4 C_4$ is ample if $m_1 > 0$, $m_3 > 0$ and:

$$m_1 k + m_2 > 0,$$

$$-m_2 + m_1 + m_3 > 0,$$

$$-m_3 + m_2 > 0.$$

Note that this correlates with the criteria above required for the polygon to be nondegenerate.

As another example, for the Hirzebruch Surface using the fan $\binom{0}{1}$, $\binom{-1}{0}$, $\binom{k}{-1}$, and $\binom{1}{0}$, we have that $m_0 = m_3 = 0$, so this criteria resolves into:

Corollary 4.10 *A divisor linearly equivalent to* $m_1C_1 + m_2C_2$ *on a Hirzebruch surface is ample if and only if it is very ample if and only if* $m_1 > 0$ *and* $m_2 > 0$.

PROOF: From Lemma 4.9, above, we have that the divisor on a Hirzebruch surface is ample iff it is very ample iff $m_1 > 0$, $m_2 > 0$, $m_1C_1^2 + m_0 + m_2 > 0$ and $m_2C_2^2 + m_1 + m_3 > 0$. From our earlier discussion of intersection numbers (Section 3.6), we have $C_1 \cdot C_1 = C_1^2 = k, C_1 \cdot C_2 = 1$, and $C_2 \cdot C_2 = C_2^2 = 0$. Combining this with $m_0 = m_3 = 0$ yields $m_1k + m_2 > 0$ and $m_1 > 0$. However, if $m_1 > 0$ and $m_2 > 0$, then $m_1k + m_2 > 0$ must be true. Therefore, a Hirzebruch surface is ample if and only if it is very ample if and only if $m_1 > 0$ and $m_2 > 0$.

Glenn Murray [17] computes several 0^{th} cohomology groups, given in the following theorem:

Theorem 4.11 *[17] Let* S *be a smooth toric surface given by a fan* $\{(a_l, b_l)\}_{l=0}^{n+1}$ *and* $D = \sum_{l=1}^{n} m_lC_l$ *be a divisor on* S*. Then*

1. $H^0(S, \mathcal{O}(m_1C_1 + ... + m_nC_n)) =< \{x_0^c y_0^d | m_l + a_lc + b_ld \geq 0, c \geq 0, d \geq 0,$ *for* $l = 1, ..., n\} >$*. Note that this is a set of independant monomials, which may be empty.*

2. $H^0(S, \Omega^1(m_1C_1 + ... + m_nC_n)) = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}$ *where*

$$\mathcal{X} =< \{x_0^i y_0^j dx_0 |\ i,j \geq 0,\ m_l + a_l(i+1) + b_l j \geq 1 \text{ if } a_l \neq 0), l = 1, ..., n\} >,$$

$$\mathcal{Y} =< \{x_0^\alpha y_0^\beta dy_0 |\ \alpha, \beta \geq 0,\ m_l + a_l\alpha \geq 0 \text{ if } b_l = 0,$$

$$m_l + a_l\alpha + b_l(\beta+1) \geq 1 \text{ if } b_l \neq 0, l = 1, ..., n\} >,$$

and

$$\mathcal{M} =< \{\gamma x_0^i y_0^j dx_0 + \delta x_0^{i+1} y_0^{j-1} dy_0 |\ (i,j,\delta,\gamma) \in \mathcal{I}\} >$$

where $\mathcal{I}$ *is the set of* $(i,j,\delta,\gamma) \in \mathbb{Z}^4$ *such that*

(a) $\gamma\delta \neq 0$,

(b) $\gamma a_l + \delta b_l = 0$ *for one or two values of* $l \neq l_0, l_{n+1}$ *call them* l_1, l_2,

(c) $i \geq 0, j \geq 0$,

(d) $m_l + a_l(i+1) + b_l j \geq 1$ *for* $l \neq l_1$, *or* $l \neq l_2$

(e) $m_l + a_l(i+1) + b_l j \geq 0$ *for* $l = l_1$, *or* $l = l_2$,

(f) neither $x_0^i y_0^j dx_0$ *nor* $x_0^{i+1} y_0^{i-1} dy_0$ *is in* $H^0(M, \Omega^1(D))$.

Note that this is a set of independent monomials, which may be empty.

Note: In the case of $\mathbb{F}_k$, with the usual four vector fan, the above results would be refined. Using the vector definitions $\binom{a_0}{b_0} = \binom{0}{1}$, $\binom{a_1}{b_1} = \binom{-1}{0}$, $\binom{a_2}{b_2} = \binom{k}{-1}$, and $\binom{a_3}{b_3} = \binom{1}{0}$we see that:

1. If m_1 and $(m_2 + m_1k)$ are nonnegative,

$$H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) =< \{x_0^c y_0^d | 0 \leq c \leq m_1, 0 \leq d \leq m_2 + ck\} >$$

since $m_1 + (-1)c + (0)d \geq 0$ implies $m_1 \geq 0$ and $m_2 + kc + (-1)d \geq 0$ implies $m_2 + kc \geq d$ and both $c, d \geq 0$.

2. Again, given the vector fan values above, we see for $\mathcal{X}$ that $m_1+(-1)(i+1)+(0)j \geq 0$ yields $m_1-2 \geq i$ and $m_2+k(i+1)+(-1)j \geq 1$ yields $m_2+ik+k-1 \geq j$. For $\mathcal{Y}$, $m_1+(-1)\alpha \geq 0$ yields $m_1 \geq \alpha$ and $m_2+k\alpha+(-1)(\beta+1) \geq 1$ yields $m_2+k\alpha-2 \geq \beta$. Finally, for $\mathcal{M}$, we see that $m_1+(-1)(i+1)+(0)j \geq 1$ yields $m_1-2 \geq i$ and $m_2+k(i+1)+(-1)j \geq 1$ yields $m_2+ki+k-1 \geq j \geq 0$. Hence if $m_1 \geq 0$ and $m_2 \geq 1-k$, then $H^0(\mathbb{F}_k, \Omega^1(m_1C_1+m_2C_2)) = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}$, where

$$\mathcal{X} =< \{x_0^i y_0^j dx_0 |\ 0 \leq i \leq m_1-2, 0 \leq j \leq m_2+ik+k-1\} >$$

$$\mathcal{Y} =< \{x_0^\alpha y_0^\beta dy_0 |\ 0 \leq \alpha \leq m_1, 0 \leq \beta \leq m_2+k\alpha-2\} >$$

and

$$\mathcal{M} =< \{x_0^i y_0^{m_2+ki+k-1}(y_0dx_0+kx_0dy_0) |\ 0 \leq i \leq m_1-2, m_2+ki+k-1 \geq 0\} > .$$

Of course, these results were stated earlier in Lemma 4.4.

Additionally, we have the following two lemmas, from Murray [17] and [18], the analogs of Corollary 4.5 and Corollary 4.6 for a general toric surface:

Lemma 4.12 *(Murray [17] Theorem 1.21)*
Let S be a smooth toric surface given by a fan $\{(a_i,b_i)\}_{i=0}^{n+1}$. Take $m_0 = m_{l+1} = 0$ and let $D = \sum_{l=1}^n m_lC_l$ be a divisor on S. If D is ample, then

$$h^0(\mathcal{O}(D)) = \frac{1}{2}D\cdot(D-K)+1 = \chi(\mathcal{O}_S(D)).$$

Note: if $n = 2$ and D is ample, $m_1 > 0$ and $m_2 > 0$, then we can compute

$$h^0(\mathcal{O}(m_1C_1+m_2C_2)) = \frac{1}{2}(m_1C_1+m_2C_2)(m_1C_1+m_2C_2-(-C_0-C_1-C_2-C_3))+1$$

$$= \frac{1}{2}(m_1C_1 + m_2C_2)(C_0 + (m_1+1)C_1 + (m_2+1)C_2 + C_3) + 1$$

$$= \frac{1}{2}(m_1C_1C_0 + m_1(m_1+1)C_1^2 + m_1(m_2+1)C_2C_1 + m_1C_1C_3 + m_2C_2C_0$$

$$+m_2(m_1+1)C_2C_1 + m_2(m_2+1)C_2^2 + m_2C_2C_3) + 1.$$

Again, from our discussion of intersection numbers, we have that $C_1^2 = k, C_1 \cdot C_2 = 1$, and $C_2^2 = 0$. Therefore

$$m_1 + m_1m_2 + m_2 + \frac{k}{2}m_1^2 + \frac{k}{2}m_1 + 1 = (m_1+1)(\frac{k}{2}m_1 + m_2 + 1).$$

In fact, $h^0(m_1C_1 + m_2C_2) = (m_1+1)(\frac{k}{2}m_1 + m_2 + 1)$ if m_1 or m_2 equals 0, as we have already seen (Corollary 4.5).

Note: Recalling some of the basic theorems restated in Section 3.7, we have that $\chi(D) = \frac{1}{2}D(D-K)+1+p_a$ by Riemann-Roch. Since $p_a(S) = 0$, then $h^0(\mathcal{O}(D)) = \chi(D)$.

Lemma 4.13 *(Murray [18] Corollary 1) Let S be a smooth toric surface given by a fan $\{(a_i, b_i)\}_{i=0}^{n+1}$. Take $m_0 = m_{l+1} = 0$ and let $D = \sum_{l=1}^{n} m_lC_l$ be an ample divisor on S. Then*

$$h^0(\Omega_S^1(D)) = D^2 - n = \chi(\Omega_S^1(D)).$$

Note: if $n = 2$ and D is ample, then

$$h^0(\Omega^1(m_1C_1 + m_2C_2)) = (m_1C_1 + m_2C_2)^2 - 2$$

$$= m_1^2C_1^2 + 2m_1m_2C_1C_2 + m_2^2C_2^2 - 2 = m_1^2k + 2m_1m_2 - 2,$$

consistent with Corollary 4.6.

4.3 Computations for $i = 1, 2$; Hirzebruch Surfaces

Given the computations for $H^0(\mathbb{F}_k, \mathcal{O}(D))$ and $H^0(\mathbb{F}_k, \Omega^1(D))$, we can now compute H^2. We will use Kodaira-Serre duality, Corollary 3.17. For our purposes, we can restate this theorem as

$$h^2(S, \mathcal{O}(D)) = h^0(S, \mathcal{O}(K_S - D)).$$

For a Hirzebruch surface, $\mathbb{F}_k$, with the usual four vector fan, we use the vector definitions $\binom{a_0}{b_0} = \binom{0}{1}$, $\binom{a_1}{b_1} = \binom{-1}{0}$, $\binom{a_2}{b_2} = \binom{k}{-1}$, and $\binom{a_3}{b_3} = \binom{1}{0}$, and $m_0 = m_3 = 0$. We know from Section 3.5 that $K_{\mathbb{F}_k} = -2C_1 + (k-2)C_2$. Using Kodaira-Serre duality, Corollary 3.17, we get

$$h^2(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) = h^0(\mathbb{F}_k, \mathcal{O}(-(m_1+2)C_1 + (k - m_2 - 2)C_2)).$$

From Lemma 4.4 and the above, we have the following corollary:

Corollary 4.14 *(Duflot/Miranda [5]) If $m_1 \geq -1$ or $m_2 + km_1 \geq -k - 1$,*

$$H^2(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) = 0.$$

PROOF: From Lemma 4.4, item 1, we have that $H^0(\mathbb{F}_k, \mathcal{O}(aC_1 + bC_2))$ is zero if either a or $b + ka$ is negative. From Kodaira-Serre duality, Corollary 3.17, as applied above, we know $H^2(\mathbb{F}_k, \mathcal{O}(D)) \cong H^0(\mathbb{F}_k, \mathcal{O}(K_{\mathbb{F}_k} - D))$. So $H^2(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) \cong H^0(\mathbb{F}_k, \mathcal{O}((-m_1 - 2)C_1 + (k - m_2 - 2)C_2))$. Then $H^0(\mathbb{F}_k, \mathcal{O}((-m_1 - 2)C_1 + (k - m_2 - 2)C_2) = 0$ if $-m_1 - 2 < 0$ or $k - m_2 - 2 + (-m_1 - 2)k < 0$. Therefore, we have that $H^2(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) = 0$ if $m_1 \geq -1$ or $m_2 + km_1 \geq -k - 1$.

Similarly, we can use Kodaira-Serre duality (Corollary 3.18) with the definition of h^i above, with S a surface, yielding

$$h^2(S, \Omega^1(D)) = h^0(S, \Omega^1(-D)).$$

Thus

$$h^2(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = h^0(\mathbb{F}_k, \Omega^1(-m_1C_1 - m_2C_2)).$$

This combined with Lemma 4.4 leads to

Corollary 4.15 *(Duflot/Miranda [5]) If $m_1 \geq 1$ or $m_2 + km_1 \geq 1$, then*

$$H^2(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = 0.$$

PROOF: From Lemma 4.4, item 3, we have that $H^0(\mathbb{F}_k, \Omega^1(aC_1 + bC_2))$ is zero if either a or $b + ka$ is negative. From the above, we have that $h^2(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = h^0(\mathbb{F}_k, \Omega^1(-m_1C_1 - m_2C_2))$. Then $H^0(\mathbb{F}_k, \Omega^1(-m_1C_1 - m_2C_2)) = 0$ if $-m_1 < 0$ or $-(m_2 + m_1k) < 0$. Therefore, we have that $H^2(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = 0$ if $m_1 \geq 1$ or $m_2 + km_1 \geq 1$.

Next we will use Riemann-Roch to determine h^1. Recall the basic intersection numbers computed earlier, $C_1^2 = k, C_1 \cdot C_2 = 1$, and $C_2^2 = 0$. We know that

$$\chi(D) \doteq h^0(\mathbb{F}_k, D) - h^1(\mathbb{F}_k, D) + h^2(\mathbb{F}_k, D).$$

So

$$\chi(m_1C_1+m_2C_2) \doteq h^0(\mathbb{F}_k, m_1C_1+m_2C_2)-h^1(\mathbb{F}_k, m_1C_1+m_2C_2)+h^2(\mathbb{F}_k, m_1C_1+m_2C_2).$$

Using Riemann-Roch for divisors D on a surface ($\chi(D) = D(D-K)/2+1+p_a$), this is equivalent to

$$\chi(m_1C_1+m_2C_2) = \frac{(m_1C_1+m_2C_2)[(m_1+2)C_1+(2+m_2-k)C_2]}{2}+1+p_a$$

$$= \frac{m_1(m_1+2)C_1^2+m_1(2+m_2-k)C_1C_2+m_2(m_1+2)C_2C_1+m_2(2+m_2-k)C_2^2}{2}+1+p_a$$

$$= \frac{m_1(m_1+2)k+2m_1+m_1m_2-km_1+m_2m_1+2m_2}{2}+1+p_a$$

$$= \frac{1}{2}km_1(m_1+1)+(m_1+m_1m_2+m_2)+1+p_a.$$

Since the arithmetic genus of $\mathbb{F}_k$ is zero, $p_a = 0$,

$$\chi(m_1C_1+m_2C_2) = \frac{1}{2}km_1(m_1+1)+(m_1+m_1m_2+m_2)+1.$$

From Corollary 4.5, if $m_1 \geq 0$ and $m_2 \geq 0$, we have that

$$\dim H^0(\mathcal{O}(m_1C_1+m_2C_2)) = (m_1+1)(\frac{k}{2}m_1+m_2+1).$$

Also, from Corollary 4.14, we have that if $m_1 \geq -1$ or $m_2+km_1 \geq -k-1$, then

$$H^2(\mathbb{F}_k, m_1C_1+m_2C_2) = 0.$$

Again, since

$$\chi(\mathcal{O}(m_1C_1+m_2C_2)) \doteq$$

$$h^0(\mathcal{O}(m_1C_1 + m_2C_2)) - h^1(\mathcal{O}(m_1C_1 + m_2C_2)) + h^2(\mathcal{O}(m_1C_1 + m_2C_2)),$$

we have that if $m_1 \geq 0$ and $m_2 \geq 0$,

$$\chi(\mathcal{O}(m_1C_1 + m_2C_2)) = \frac{1}{2}km_1(m_1 + 1) + (m_1 + m_1m_2 + m_2) + 1$$

$$= (m_1 + 1)(\frac{k}{2}m_1 + m_2 + 1) - h^1(\mathcal{O}(m_1C_1 + m_2C_2)) + 0.$$

Therefore $h^1(\mathcal{O}(m_1C_1 + m_2C_2)) = 0$, and we have the following lemma:

Lemma 4.16 *If $m_1 \geq 0$ and $m_2 \geq 0$, then $H^1(\mathcal{O}(m_1C_1 + m_2C_2)) = 0$.*

Recall Riemann-Roch for a rank 2 bundle $\mathcal{E}$ over a surface S, Theorem 3.15. In the case of $\mathbb{F}_k$, we will use

Theorem 4.17 *(see e.g. Duflot/Miranda [5]) Let $S = \mathbb{F}_k$ and $\mathcal{E} = \Omega^1_{\mathbb{F}_k}(D)$, then $c_1(\mathcal{E}) = K + 2D$ and $c_2(\mathcal{E}) = 4 + K \cdot D + D \cdot D$.*

Now $K_{\mathbb{F}_k} = -2C_1 + (k-2)C_2$. If $D = m_1C_1 + m_2C_2$, we see that:

$$K_{\mathbb{F}_k} \cdot c_1(\Omega^1_{\mathbb{F}_k}(D)) = K_{\mathbb{F}_k} \cdot (K_{\mathbb{F}_k} + 2D)$$

$$= (-2C_1 + (k-2)C_2) \cdot (-2C_1 + (k-2)C_2 + 2(m_1C_1 + m_2C_2))$$

$$= (-2C_1 + (k-2)C_2) \cdot ((2m_1 - 2)C_1 + (2m_2 + k - 2)C_2)$$

$$= -2m_1k - 4m_1 - 4m_2 + 8.$$

Also

$$c_1^2(\Omega^1_{\mathbb{F}_k}(D)) = (K_{\mathbb{F}_k} + 2D)^2$$

$$= (-2C_1 + (k-2)C_2 + 2(m_1C_1 + m_2C_2))^2$$

$$= ((2m_1 - 2)C_1 + (2m_2 + k - 2)C_2)^2$$

$$= 4m_1^2k - 4m_1k + 8m_1m_2 - 8m_1 - 8m_2 + 8,$$

and

$$c_2(\Omega^1_{\mathrm{F}_k}(D)) = 4 + K_{\mathrm{F}_k} \cdot D + D \cdot D$$

$$= (-2C_1 + (k-2)C_2) \cdot (m_1C_1 + m_2C_2)) + (m_1C_1 + m_2C_2)^2$$

$$= m_1^2k + 2m_1m_2 - m_1k - 2m_1 - 2m_2.$$

So

$$c_1^2(\Omega^1_{\mathrm{F}_k}(D)) - 2c_2(\Omega^1_{\mathrm{F}_k}(D))$$

$$= (4m_1^2k - 4m_1k + 8m_1m_2 - 8m_1 - 8m_2 + 8) - 2(m_1^2k + 2m_1m_2 - m_1k - 2m_1 - 2m_2)$$

$$= 2m_1^2k + 4m_1m_2 - 2m_1k - 4m_1 - 4m_2 + 8.$$

Substituting into Riemann-Roch from Theorem 3.15, we get

$$\chi(\Omega^1_{\mathrm{F}_k}(D)) = \chi(\Omega^1_{\mathrm{F}_k}(m_1C_1 + m_2C_2))$$

$$= 2(1) - \frac{-2m_1k - 4m_1 - 4m_2 + 8}{2} + \frac{2m_1^2k + 4m_1m_2 - 2m_1k - 4m_1 - 4m_2 + 8}{2}$$

$$= m_1^2k + 2m_1m_2 - 2$$

$$= (m_1C_1 + m_2C_2)^2 - 2$$

From Theorem 4.7, if $m_1 \geq 1$ and $m_2 \geq 1$, we have

$$\dim H^0(\Omega^1(m_1C_1 + m_2C_2)) = 2m_1m_2 + km_1^2 - 2.$$

Also, from Corollary 4.15, we have that if $m_1 \geq 1$ or $m_2 + km_1 \geq 1$, then

$$H^2(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) = 0.$$

Again, from Duflot/Miranda [5], since

$$\chi(\Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2))$$

$$\doteq h^0(\Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2)) - h^1(\Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2)) + h^2(\Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2)),$$

we have that

$$m_1^2k + 2m_1m_2 - 2 = 2m_1m_2 + km_1^2 - 2 - h^1(\Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2)) + 0.$$

Thus, if $m_1 \geq 1$ and $m_2 \geq 1$, $h^1(\Omega^1_{\mathbb{F}_k}(m_1C_1+m_2C_2)) = 0$, and we have the following lemma:

Lemma 4.18 *If $m_1 \geq 1$ and $m_2 \geq 1$, then $H^1(\mathbb{F}_k, \Omega^1_{\mathbb{F}_k}(m_1C_1 + m_2C_2)) = 0$.*

4.4 Theorems for $i = 1, 2$; General Toric Surfaces

We have already extensively used Kodaira-Serre duality, Theorem 3.16, in the last sections computations for Hirzebruch surfaces. In this section, we will state two theorems based on duality.

Theorem 4.19 *(Oda, [19], Corollary 2.9, page 77)*
If S is a smooth toric surface and D is an ample divisor, then

$$h^1(S, \mathcal{O}_S(D)) = h^2(S, \mathcal{O}_S(D)) = 0.$$

We will also be using Murray [17] Lemma 2.10,

Theorem 4.20 *[17] If S is a smooth toric surface and D is ample, then*

$$h^1(S, \Omega_S^1(D)) = 0,$$

$$h^2(S, \Omega_S^1(D)) = 0.$$

Note: From Lemma 4.9, D is ample if $m_i C_i^2 + m_{i-1} + m_{i+1} > 0, 1 \leq i \leq n, m_1 > 0$ and $m_n > 0$.

4.5 $\Omega_S^1(\log\ D)$: Definition and Discussion

The following discussion is excerpted from Saito [20].

Theorem 4.21 *[20] Let M an n-dimensional complex manifold, and $V \subset M$ be a hypersurface of M defined by an equation $h(z) = 0$, where h is holomorphic on M. Let ω be a meromorphic q-form on M, which may have poles only along V. Then the following four conditions for ω are equivalent:*

1. *$h\omega$ and $h\ d\omega$ are holomorphic on M.*

2. *$h\omega$ and $dh \wedge \omega$ are holomorphic on M.*

3. *There exists a holomorphic function $g(z)$ and a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on M such that:*

   (a) $\dim_{C} V \cap \{z \epsilon S : g(z) = 0\} \leq n - 2$

   (b) $g\omega = \frac{dh}{h} \wedge \xi + \eta$.

4. *There exists an $(n-2)$-dimensional analytic set $A \subset V$ such that the germ of ω at any point $p \epsilon V - A$ belongs to $\frac{dh}{h} \wedge \Omega_{M,p}^{q-1} + \Omega_{M,p}^q$, where $\Omega_{M,p}^q$ denotes the module of germs of holomorphic q-foms on M at p.*

The proof is in Saito, [20].

This leads to the following definition:

Definition 4.22 *[20] A meromorphic q-form on M is called a q-form with logarithmic pole along V or logarithmic q-form if it satisfies the equivalent conditions of 4.21. Let $h_p = 0$ be a reduced equation for V, locally at $p \in V$. A meromorphic q-form is logarithmic along V at p if $h_p\omega$ and $h_p d\omega$ are holomorphic.*

We denote

$$\Omega^q_{M,p}(\log\ V) \doteq \{\text{germs of logarithmic } q\text{-forms at } p\},$$

$$\Omega^q_M(\log\ V) \doteq \cup_{p \in M} \Omega^q_{M,p}(\log\ V).$$

Theorem 4.23 *(Saito [20]) If M is a complex manifold, V a smooth hypersurface of M defined by an equation $h(z) = 0$ where h is holomorphic on M, then $\Omega^q_M(\log\ V), q = 0, 1, ..., n$ are coherent $\mathcal{O}_M$-modules.*

Saito defines the residue morphism as follows:

Definition 4.24 *If ω is a meromorphic q-form on a complex manifold M and there exists a holomorphic function $g(z)$, a holomorphic $(q-1)$-form ξ, and a holomorphic q-form η on M such that $g\omega = \frac{dh}{h} \wedge \xi + \eta$, then the residue morphism, res, is a sheaf homomorphism:*

$$res : \Omega^q_M(\log\ V) \to \mathcal{O}_V$$

$$\omega \overset{res}{\mapsto} \frac{1}{g}\xi.$$

This leads us to the residue exact sequence for logarithmic q-forms:

Theorem 4.25 *(see e.g. Saito [20] page 276)*

If M is a complex manifold and V is a smooth hypersurface on M, then the sequence

$$0 \to \Omega^q_M \to \Omega^q_M(\log\ V) \xrightarrow{res} \mathcal{O}_V \to 0$$

is exact.

Applying this short exact sequence to a general surface Y, D a smooth curve on Y, and any divisor E on Y, we get the short exact sequence

$$0 \to \Omega^1_Y(E) \to \Omega^1_Y(log\ D)(E) \to \mathcal{O}(E)|_D \to 0. \qquad (4.26)$$

Recall the standard short exact sequence

$$0 \to \mathcal{O}_Y(-D)(E) \to \mathcal{O}_Y(E) \to \mathcal{O}(E)|_D \to 0. \qquad (4.27)$$

Next we will construct the long exact sequences from the short exact sequences 4.26 and 4.27, respectively:

$$0 \to H^0(Y, \Omega^1_Y(E)) \to H^0(Y, \Omega^1_Y(log\ D)(E))$$

$$\to H^0(Y, \mathcal{O}(E)|_D) \to H^1(Y, \Omega^1_Y(E)) \to \ldots \qquad (4.28)$$

and

$$0 \to H^0(Y, \mathcal{O}_Y(E - D)) \to H^0(Y, \mathcal{O}_Y(E))$$

$$\to H^0(Y, \mathcal{O}_Y(E)|_D) \to H^1(Y, \mathcal{O}_Y(E - D)) \to \ldots \qquad (4.29)$$

This leads us to the following theorems:

Theorem 4.30 *If Y is a smooth surface, E a divisor, D a smooth curve on Y,* *and* $h^1(Y,\Omega^1_Y(E))=0$, *then*

$$h^0(Y,\Omega^1_Y(log\ D)(E)) = h^0(Y,\mathcal{O}_Y(E)|_D) + h^0(Y,\Omega^1_Y(E)).$$

Since $\chi(E-D) = h^0(Y,\mathcal{O}_Y(E-D)) - h^1(Y,\mathcal{O}_Y(E-D)) + h^2(Y,\mathcal{O}_Y(E-D))$, and $h^2(Y,\mathcal{O}_Y(E-D)) = h^0(Y,\mathcal{O}_Y(K_Y+D-E)$ by Kodaira-Serre duality (Corollary 3.17), we have:

Theorem 4.31 *If Y is a smooth surface, E a divisor, D a smooth curve on Y, and $h^1(Y,\mathcal{O}_Y(E))=0$, then*

$$h^0(Y,\mathcal{O}_Y(E)|_D) = h^0(Y,\mathcal{O}_Y(E)) - \chi(E-D) + h^0(Y,\mathcal{O}_Y(K_Y+D-E)).$$

PROOF: If $h^1(Y,\mathcal{O}_Y(E))=0$, then the sequence

$$0\to H^0(\mathcal{O}_Y(E-D))\to H^0(Y,\mathcal{O}_Y(E))\to H^0(Y,\mathcal{O}_Y(E)|_D)\to H^1(Y,\mathcal{O}_Y(E-D))\to 0$$

is exact. Thus

$$h^0(\mathcal{O}_Y(E-D)) - h^0(Y,\mathcal{O}_Y(E)) + h^0(Y,\mathcal{O}_Y(E)|_D - h^1(Y,\mathcal{O}_Y(E-D)) = 0.$$

Therefore

$$h^0(Y,\mathcal{O}_Y(E)|_D = h^1(Y,\mathcal{O}_Y(E-D) - h^0(\mathcal{O}_Y(E-D)) + h^0(Y,\mathcal{O}_Y(E)).$$

Theorem 4.32 *If Y is a smooth surface, E a divisor, D a smooth curve on Y, $h^1(Y,\mathcal{O}_Y(E))=0$ and $h^1(Y,\Omega^1_Y(E))=0$, then*

$$h^0(Y,\Omega^1_Y(\log\ D)(E))$$

$$= h^0(Y,\Omega^1_Y(E)) + h^0(Y,\mathcal{O}_Y(E)) - \chi(E-D) + h^0(Y,\mathcal{O}_Y(K_Y+D-E)).$$

4.5.1 Calculating $\Omega_S^1(\log\ D)$: Smooth Toric Surfaces

Again, we will be using the basis $C_1, ..., C_n$ for the Picard group of a smooth toric surface and the fans defined in Chapter 2, $\{(a_i, b_i)\}_{i=0}^{n+1}$. Since $(a_0, b_0) = (0, 1)$ and $(a_{n+1}, b_{n+1}) = (1, 0)$, $m_0 = m_{n+1} = 0$, every divisor D is linearly equivalent to $m_1 C_1 + ... + m_n C_n$.

Theorem 4.33 *Let S be a smooth toric surface defined by a fan of $n+2$ vectors as above and E an ample divisor, D a smooth curve on S, then*

$$h^0(S, \Omega_S^1(\log\ D)(E)) = E^2 - n + D \cdot E - g(D) + 1 + h^0(S, \mathcal{O}_S(K_S + D - E)),$$

where $g(D)$ is the genus of D.

PROOF: From Theorem 4.32 above, we have that

$$h^0(S, \Omega_S^1(\log\ D)(E))$$

$$= h^0(S, \Omega_S^1(E)) + h^0(S, \mathcal{O}_S(E)) - \chi(E - D) + h^0(S, \mathcal{O}_S(K_S + D - E)).$$

Since E is ample, Theorems 4.12 and 4.13 show that

$$\chi(\Omega_S^1(E)) = h^0(S, \Omega_S^1(E))$$

and

$$\chi(\mathcal{O}_S(E)) = h^0(S, \mathcal{O}_S(E)).$$

Thus

$$h^0(S, \Omega_S^1(\log\ D)(E)) = \chi(\Omega_S^1(E)) + \chi(\mathcal{O}(E)) - \chi(E - D) + h^0(S, \mathcal{O}_S(K_S + D - E)).$$

But $\chi(\Omega_S^1(E)) = E^2 - n$ by Lemma 4.13 and $\chi(\mathcal{O}_S(E)) = \frac{E(E-K_S)}{2} + 1$ by Lemma 4.12. Hence

$$h^0(S, \Omega_S^1(\log\ D)(E))$$

$$= E^2 - n + \frac{E(E-K_S)}{2} + 1 - \frac{(E-D)(E-K_S-D)}{2} - 1 + h^0(Y, \mathcal{O}_Y(K_Y + D - E))$$

$$= E^2 - n + \frac{1}{2}(E^2 - E \cdot K_S) - \frac{E^2 - E\cdot K_S - 2E\cdot D + D \cdot K_S + D^2}{2}$$

$$+ h^0(Y, \mathcal{O}_Y(K_Y + D - E))$$

$$= E^2 - n + E \cdot D - (g(D) - 1) + h^0(Y, \mathcal{O}_Y(K_Y + D - E)).$$

4.5.2 Calculating $\Omega^1_{\mathbb{F}_k}(\log\ D)$: Hirzebruch Surfaces

For a specific example, we will address Hirzebruch surfaces, $\mathbb{F}_k$, with $D \sim \tilde{\alpha}C_1 + \tilde{\beta}C_2, \tilde{\alpha} \geq 0, \tilde{\beta} \geq 0$ and $E \sim m_1C_1 + m_2C_2$. We are using the same basis for Pic $\mathbb{F}_k$ and the same fan to define $\mathbb{F}_k$, as usual.

Theorem 4.34 *Let $\mathbb{F}_k$ be a Hirzebruch surface, with $D \sim \tilde{\alpha}C_1 + \tilde{\beta}C_2, \tilde{\alpha} \geq 0, \tilde{\beta} \geq 0$ and $E \sim m_1C_1 + m_2C_2$.*

1. *If $m_1 \geq 1, m_2 \geq 1$ and $\tilde{\alpha} \geq m_1 + 2, \tilde{\beta} + k \geq m_2 + 2$, then*

$$h^0(\mathbb{F}_k, \Omega^1_{\mathbb{F}_k}(\log\ D)(E)) = E^2 - 1 + \frac{1}{2}E(E - K_{\mathbb{F}_k}).$$

2. *If $m_1 \geq 1, m_2 \geq 1$, and $\tilde{\alpha} < m_1 + 2$ or $k\tilde{\alpha} + \tilde{\beta} < m_2 + km_1 + k + 2$, then*

$$h^0(\mathbb{F}_k, \Omega^1_{\mathbb{F}_k}(\log\ D)(E)) = E^2 + D \cdot E - g(D) - 1,$$

where $g(D)$ is the genus of D.

PROOF: If $E = m_1C_1 + m_2C_2$, then E is ample if $m_1 > 0$ and $m_2 > 0$ by Corollary 4.10. Thus Theorem 4.33 says if $m_1 \geq 1$ and $m_2 \geq 1$,

$$h^0(\mathbb{F}_k, \Omega^1_{\mathrm{F}_k}(\log\ D)(E)) = E^2 - 2 + E \cdot D - (g(D) - 1) + h^0(\mathbb{F}_k, \mathcal{O}_{\mathrm{F}_k}(K_{\mathrm{F}_k} + D - E)).$$

We know that $K_{\mathrm{F}_k} + D - E = (-2 + \tilde{\alpha} - m_1, k - 2 + \tilde{\beta} - m_2)$. If $\tilde{\alpha} \geq m_1 + 2$ or $\tilde{\beta} + k \geq m_2 + 2$, then by Theorem 4.12

$$\begin{aligned} h^0(\mathbb{F}_k, \mathcal{O}_{\mathrm{F}_k}(K_{\mathrm{F}_k} + D - E)) &= \chi(\mathcal{O}_{\mathrm{F}_k}(K_{\mathrm{F}_k} + D - E)) \\ &= \frac{(K_{\mathrm{F}_k} + D - E)(D - E)}{2} + 1 \\ &= \frac{(K_{\mathrm{F}_k} + D)D}{2} - \frac{D \cdot E}{2} - \frac{(K_{\mathrm{F}_k} + D)E}{2} + \frac{E^2}{2} + 1 \\ &= g(D) - D \cdot E + \frac{E(E - K_{\mathrm{F}_k})}{2}. \end{aligned}$$

Adding this to the above yields

$$h^0(\mathbb{F}_k, \Omega^1_{\mathrm{F}_k}(\log\ D)(E)) = E^2 + \frac{1}{2}E(E - K_{\mathrm{F}_k}).$$

If $\tilde{\alpha} < m_1 + 2$ or $k\tilde{\alpha} + \tilde{\beta} < m_2 + km_1 + k + 2$, then $-2 + \tilde{\alpha} - m_1 < 0$ or $k(-2 + \tilde{\alpha} - m_1) + k - 2 + \tilde{\beta} - m_2 < 0$, thus $h^0(\mathbb{F}_k, \mathcal{O}_{\mathrm{F}_k}(K_{\mathrm{F}_k} + D - E)) = 0$, by Lemma 4.4. Thus

$$h^0(\mathbb{F}_k, \Omega^1_{\mathrm{F}_k}(\log\ D)(E)) = E^2 - 2 + D \cdot E - g(D) + 1.$$

Note: The remaining cases can be analyzed using Lemma 4.4 (part 2) to compute $h^0(\mathbb{F}_k, \mathcal{O}_{\mathrm{F}_k}(K_{\mathrm{F}_k} + D - E))$, but we do not do this here.

Chapter 5

Multiplication Maps

In this chapter, we will explore multiplication maps on Hirzebruch surfaces and toric surfaces and surjectivity of these maps. We will be using results about the surjectivity of multiplication maps in our later discussion of Gaussian maps. This chapter also includes the first of our original work. The original theorems of this chapter are Theorems 5.2 and 5.4.

5.1 General Definition

Definition 5.1 *In general, given a smooth projective variety X with line bundles $\mathcal{F}$ and $\hat{\mathcal{F}}$ on X, the multiplication map*

$$\mu : \; H^0(X, \mathcal{F}) \otimes H^0(X, \hat{\mathcal{F}}) \to H^0(X, \mathcal{F} \otimes \hat{\mathcal{F}})$$

is defined by with

$$\mu(s \otimes t) = st$$

where s and t are sections of $\mathcal{F}$ and $\hat{\mathcal{F}}$ respectively.

5.2 Multiplication Map, Hirzebruch Surfaces

Note: We will be using the multiplication map in our proof of the surjectivity of the Gaussian map, Chapter 8.

Consider

$$H^0(\mathbb{F}_k, \mathcal{O}(m_1C_1 + m_2C_2)) \doteq H_k^0(m_1, m_2)$$

with basis

$$\{x_0^c y_0^d | 0 \leq c \leq m_1, \; 0 \leq d \leq m_2 + ck\}$$

if $m_1 \geq 0$ and $m_2 + m_1k \geq 0$, zero otherwise.

The multiplication map, in the above coordinates, is

$$\mu : H_k^0(m_1, m_2) \otimes H_k^0(\hat{m}_1, \hat{m}_2) \to H_k^0(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$$

$$(x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) \mapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}}$$

where $m_1, \hat{m}_1 \geq 0$, $m_2 + m_1k \geq 0, \hat{m}_2 + \hat{m}_1k \geq 0$.

Note: if $m_1 < 0$ or $\hat{m}_1 < 0$ or $m_2 + m_1k < 0$ or $\hat{m}_2 + \hat{m}_1k, 0$, then the domain of the multiplication map is zero.

Proposition 5.2 *If* $m_1, \hat{m}_1 \geq 0$, $m_2, \hat{m}_2 \geq 0$, *then the multiplication map*

$$\mu : H_k^0(m_1, m_2) \otimes H_k^0(\hat{m}_1, \hat{m}_2) \to H_k^0(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$$

is surjective.

PROOF: Assume $m_1, \hat{m}_1 \geq 0$, $m_2 \geq 0$ and $\hat{m}_2 \geq 0$. Then $m_1 + \hat{m}_1 \geq 0$. Since $m_2 \geq 0, \hat{m}_2 \geq 0$, $m_2 + \hat{m}_2 \geq 0$. Consider M, N, $N \geq kM \geq 0$, such that

$$0 \leq M \leq m_1 + \hat{m}_1,$$

$$0 \leq N - kM \leq m_2 + \hat{m}_2.$$

Since $m_1, \hat{m}_1 \geq 0$, there exists a $c, \hat{c}$ such that $0 \leq c \leq m_1$ and $0 \leq \hat{c} \leq \hat{m}_1$ with $c + \hat{c} = M$. Then

$$0 \leq N - k(c + \hat{c}) \leq m_2 + \hat{m}_2$$

which implies

$$0 \leq N - kc - k\hat{c} \leq m_2 + \hat{m}_2.$$

Thus

$$0 \leq N \leq (m_2 + kc) + (\hat{m}_2 + k\hat{c}).$$

Now since $m_2 \geq 0, \hat{m}_2 \geq 0$, $k \geq 0$, and $c, \hat{c} \geq 0$, both $m_2 + kc \geq 0$ and $\hat{m}_2 + k\hat{c} \geq 0$. Then there exists $d, \hat{d}$ such that

$$0 \leq d \leq m_2 + kc,$$

$$0 \leq \hat{d} \leq \hat{m}_2 + k\hat{c},$$

and

$$0 \leq d + \hat{d} \leq (m_2 + kc) + (\hat{m}_2 + k\hat{c})$$

with $d + \hat{d} = N$. So $x_0^c y_0^d x_0^{\hat{c}} y_0^{\hat{d}} = x_0^{c+\hat{c}} y_0^{d+\hat{d}} = x_0^M y_0^N$ and the multiplication map is surjective.

Next consider the multiplication map with the following definition:

Definition 5.3 *Multiplication maps*

$$\hat{\mu}((m_1, m_2)(\hat{m}_1, \hat{m}_2)) : H^0(\mathbb{F}_k, \Omega^1(m_1C_1 + m_2C_2)) \otimes H^0(\mathbb{F}_k, \mathcal{O}(\hat{m}_1C_1 + \hat{m}_2C_2))$$

$$\rightarrow H^0(\mathbb{F}_k, \Omega^1((m_1 + \hat{m}_1)C_1 + (m_2 + \hat{m}_2)C_2))$$

are defined as

$$\omega \otimes h \mapsto h\omega.$$

Note: Again, this is instrumental in our proof of the surjectivity of the Gaussian map, Chapter 8.

Using this definition, we have the following theorem:

Theorem 5.4 *If* $m_1, m_2 \geq 2$ *and* $\hat{m}_1, \hat{m}_2 \geq 0$*, the multiplication map*

$$\hat{\mu}((m_1, m_2)(\hat{m}_1, \hat{m}_2))$$

is surjective.

PROOF: Recall the definitions of $\mathcal{X}(A, B), \mathcal{Y}(A, B)$, and $\mathcal{M}(A, B)$ (4.4) defining a basis for $H^0(\Omega^1(A, B))$.

Case 1, "pure dx_0":

Consider the basis element in $\mathcal{X}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$. We know $m_1 + \hat{m}_1 \geq 2$. Assume $m_2 + \hat{m}_2 + k \geq 1$. (Otherwise, $\mathcal{X}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) = 0$). Choose M, N such that

$$x_0^M y_0^N dx_0 \in \mathcal{X}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) \subseteq H^0(\mathbb{F}_k, \Omega^1((m_1 + \hat{m}_1)C_1 + (m_2 + \hat{m}_2)C_2))$$

$$\doteq H_k^0(\Omega^1((m_1+\hat{m}_1)C_1+(m_2+\hat{m}_2)C_2)),$$

so that

$$0 \le M \le m_1+\hat{m}_1-2$$

$$0 \le N \le m_2+\hat{m}_2+Mk+k-1.$$

We need to find i, c such that $0 \le i \le m_1-2, 0 \le c \le \hat{m}_1$, and $i+c=M$.

Case 1a: $M \le m_1-2$.

Choose $c=0, i=M$. Since $\hat{m}_1 \ge 0$, $0=c\le \hat{m}_1$. Also, since $i=M$ and $0 \le M \le m_1-2$, $0 \le i \le m_1-2$.

Case 1b: $m_1-2 < M \le m_1+\hat{m}_1-2$

Choose $c=M-m_1+2>0, i=m_1-2$. Then $m_1+\hat{m}_1 \ge M+2$ or $\hat{m}_1 \ge M-(m_1-2)=M-i$ and $\hat{m}_1 \ge c$. Since $i=m_1-2$, and $m_1 \ge 2$, then $0 \le i \le m_1-2$.

In any case, there exists i and c such that $0 \le i \le m_1-2, 0 \le c \le \hat{m}_1$, and $i+c=M$. Next, we focus on finding d and j with $d+j=N$ such that $0 \le j \le m_2+ik+k-1$ and $0 \le d \le \hat{m}_2+ck$.

Case 1c: $N \le m_2+ik+k-1$

Choose $j=N$ and $d=0$. Then $0=d\le \hat{m}_2+ck$, since $\hat{m}_2 \ge 0$, $k \ge 1$, and $c \ge 0$. Also $0 \le j \le m_2+ik+k-1$ since $j=N\le m_2+ik+k-1$ and $m_2+ik+k-1\ge 0$.

Case 1d: $N > m_2+ik+k-1$

Choose $j=m_2+ik+k-1, d=N-(m_2+ik+k-1)$. Then $j \ge 0$ since $m_2 \ge 2, i \ge 0$, and $k \geq 1$, so $0 \leq j = m_2 + ik + k - 1$. Also $N - (m_2 + ik + k - 1) > 0$, so $d > 0$. Since $N \leq m_2 + \hat{m}_2 + Mk + k - 1$, then

$$N - m_2 - ik - k + 1 \leq m_2 + \hat{m}_2 + Mk + k - 1 - m_2 - ik - k + 1$$

$$= \hat{m}_2 + Mk - ik = \hat{m}_2 + (M - i)k = \hat{m}_2 + ck$$

and $0 \leq d \leq ck$. So

$$x_0^c y_0^d \in H^0(\mathcal{O}(\hat{m}_1, \hat{m}_2))$$

$$x_0^i y_0^j dx_0 \in H^0(\Omega^1(m_1, m_2))$$

and

$$\hat{\mu}(x_0^i y_0^j dx_0 \otimes x_0^c y_0^d) = x_0^{i+c} y_0^{j+d} dx_0 = x_0^M y_0^N dx_0.$$

Therefore $\hat{\mu}$ is onto $\mathcal{X}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$.

<u>Case 2, "pure dy_0":</u>

Consider the basis element in $\mathcal{Y}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$. We know $m_1 + \hat{m}_1 \geq 0$. Assume $m_2 + \hat{m}_2 \geq 2$. (Otherwise, $\mathcal{Y}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) = 0$). Choose M, N such that

$$x_0^M y_0^N dy_0 \in \mathcal{Y}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) \subseteq H_k^0(\Omega^1((m_1 + \hat{m}_1)C_1 + (m_2 + \hat{m}_2)C_2)),$$

so that

$$0 \leq M \leq m_1 + \hat{m}_1,$$

$$0 \leq N \leq m_2 + \hat{m}_2 + Mk - 2.$$

We need to find α, c such that $0 \leq \alpha \leq m_1, 0 \leq c \leq \hat{m}_1$, and $\alpha + c = M$.

Case 2a: $M \leq m_1$

Choose $c = 0, \alpha = M$. Since $\hat{m}_1 \geq 0$, $0 = c \leq \hat{m}_1$. Also, since $\alpha = M$ and $0 \leq M \leq m_1$, $0 \leq \alpha \leq m_1$.

Case 2b: $m_1 < M \leq m_1 + \hat{m}_1$

Choose $c = M - m_1, \alpha = m_1$. Since $m_1 + \hat{m}_1 \geq M, \hat{m}_1 \geq M - m_1 = c$. Also $\hat{m}_1 \geq 0$, so $0 \leq c \leq \hat{m}_1$. Since $m_1 \geq 0$ and $\alpha = m_1$, $0 \leq \alpha = m_1$.

These two cases have shown that there exists α and c such that $0 \leq \alpha \leq m_1, 0 \leq c \leq \hat{m}_1$, and $\alpha + c = M$. Next, we find d and β with $d + \beta = N$ such that $0 \leq \beta \leq m_2 + \alpha k - 2$ and $0 \leq d \leq \hat{m}_2 + ck$.

Case 2c: $N \leq m_2 + \alpha k - 2$

Choose $\beta = N$ and $d = 0$. Then $0 = d \leq \hat{m}_2 + ck$, since $\hat{m}_2 \geq 0$, $k \geq 1$, and $c \geq 0$ from the above two cases. Also $\beta = N \leq m_2 + \alpha k - 2$ and $N \geq 0$, so $0 \leq \beta \leq m_2 + \alpha k - 2$.

Case 2d: $N > m_2 + \alpha k - 2$

Choose $\beta = m_2 + \alpha k - 2$ and $d = N - (m_2 + \alpha k - 2)$. Then $\beta \geq 0$ since $m_2 \geq 2, \alpha \geq 0$ from the above cases, and $k \geq 1$, so $0 \leq \beta = m_2 + \alpha k - 2$. Also $N - (m_2 + \alpha k - 2) > 0$, so $d \geq 0$. (If $d = 0$, case 2c applies.) Since $N \leq m_2 + \hat{m}_2 + Mk - 2$, then

$$N - m_2 - \alpha k + 2 \leq m_2 + \hat{m}_2 + Mk - 2 - m_2 - \alpha k + 2$$

$$= \hat{m}_2 + Mk - \alpha k = \hat{m}_2 + (M - \alpha)k = \hat{m}_2 + ck$$

and $0 \leq d \leq ck$. So

$$x_0^c y_0^d \in H^0(\mathcal{O}(\hat{m}_1, \hat{m}_2))$$

$$x_0^\alpha y_0^\beta dy_0 \,\epsilon\, H^0(\Omega^1(m_1, m_2))$$

and

$$\hat{\mu}(x_0^\alpha y_0^\beta dy_0 \otimes x_0^c y_0^d) = x_0^{\alpha+c} y_0^{\beta+d} dy_0 = x_0^M y_0^N dy_0.$$

Therefore $\hat{\mu}$ is onto $\mathcal{Y}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$.

Case 3, mixed case:

Consider the basis element in $\mathcal{M}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$. We know $m_1 + \hat{m}_1 \geq 2$. Assume $m_2 + \hat{m}_2 + k \geq 1$. (Otherwise, $\mathcal{M}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) = 0$). Choose P, Q such that

$$x_0^P y_0^Q (w_0) \,\epsilon\, \mathcal{M}(m_1 + \hat{m}_1, m_2 + \hat{m}_2) \;\subseteq\; H_k^0(\Omega^1((m_1 + \hat{m}_1)C_1 + (m_2 + \hat{m}_2)C_2)),$$

with

$$w_0 = y_0 dx_0 + k x_0 dy_0$$

so that

$$0 \leq P \leq m_1 + \hat{m}_1 - 2,$$

$$Q = m_2 + \hat{m}_2 + kP + k - 1 \geq 0.$$

Case 3a: $P \leq m_1 - 2$.

Choose $c = 0, i = P$. Since $\hat{m}_1 \geq 0$, $0 = c \leq \hat{m}_1$. Also since $i = P$, and $0 \leq P \leq m_1 - 2$, then $0 \leq i \leq m_1 - 2$.

Case 3b: $m_1 - 2 < P \leq m_1 + \hat{m}_1 - 2$.

Choose $i = m_1 - 2, c = P - m_1 + 2$. Since $m_1 \geq 2$, $0 \leq i = m_i - 2$. Also,

$m_1 - 2 < P \leq m_1 + \hat{m}_1 - 2$, so $0 < P - m_1 + 2 \leq \hat{m}_1$ and $0 < c \leq \hat{m}_1$.

In any case, there exists i and c such that $0 \leq i \leq m_1 - 2, 0 \leq c \leq \hat{m}_1$, and $i + c = P$. Next, we focus on finding d and j with $d + j = Q$ such that $0 \leq j \leq m_2 + ik + k - 1$ and $0 \leq d \leq \hat{m}_2 + ck$.

<u>Case 3c:</u> $Q \leq m_2 + ik + k - 1$.

$j = Q, d = 0$. Then $0 = d \leq \hat{m}_2 + ck$ since $\hat{m}_2 \geq 0, k \geq 1, c \geq 0$. Also $0 \leq j \leq m_2 + ki + k - 1$ since $Q = m_2 + \hat{m}_2 + kP + k - 1$.

<u>Case 3d:</u> $Q > m_2 + ik + k - 1$.

Choose $j = m_2 + ik + k - 1, d = Q - (m_2 + ik + k - 1)$. Then $j \geq 0$ since $m_2 \geq 2, i \geq 0$, and $k \geq 1$, so $0 \leq j = m_2 + ik + k - 1$. Also $d \geq 0$ since $Q = m_2 + \hat{m}_2 + kP + k - 1$ and $m_2 + \hat{m}_2 + kP + k - 1 \geq m_2 + ik + k - 1$ since $\hat{m}_2 \geq 0$ and $P \geq i$ (from the above). Since $Q = m_2 + \hat{m}_2 + kP + k - 1 = m_2 + \hat{m}_2 + k(c + i) + k - 1$, then $d = Q - (m_2 + ik + k - 1) = \hat{m}_2 + ck$ and $0 \leq d = \hat{m}_2 + ck$. So

$$x_0^c y_0^d \in H^0(\mathcal{O}(\hat{m}_1, \hat{m}_2)),$$

$$x_0^i y_0^j (w_0) \in H^0(\Omega^1(m_1, m_2)),$$

and

$$\hat{\mu}(x_0^i y_0^j w_0 \otimes x_0^c y_0^d) = (x_0^c y_0^d (x_0^i y_0^{m_2+ki+k-1})(w_0) = x_0^P y_0^Q (w_0).$$

Therefore $\hat{\mu}$ is onto $\mathcal{M}(m_1 + \hat{m}_1, m_2 + \hat{m}_2)$.

5.3 Multiplication Maps, General Toric Surfaces

This section is incomplete. Its completion will be part of further work at a later date.

Let S be a smooth toric surface given by the fan $\{(a_i, b_i)\}_{i=0}^{n+1}$ with $D = \sum_{i=1}^{n} m_i C_i$ a divisor on S, for $i = 1, 2, ..., n$.[17] Consider

$$H^0(S, \mathcal{O}(m_1C_1 + m_2C_2 + ... + m_nC_n)) \doteq H^0(m_1, m_2, ..., m_n)$$

with basis $\{x_0^c y_0^d | m_i + a_i c + b_i d \geq 0, i = 1, ..., n\}$ and $c, d \geq 0$ (Theorem 4.11).

The multiplication map, in terms of the above basis, is defined as

$$\mu_n : H^0(m_1, m_2, ..., m_n) \otimes H^0(\hat{m}_1, \hat{m}_2, ..., \hat{m}_n) \to H^0(m_1+\hat{m}_1, m_2+\hat{m}_2, ..., m_n+\hat{m}_n) \tag{5.5}$$

and is given by

$$(x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) \mapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}}.$$

From an unpublished manuscript by Fakhruddin [6], we have the following theorem:

Theorem 5.6 *[6] Let S be a smooth projective toric surface, $\mathcal{L}$ an ample line bundle on X, and $\mathcal{M}$ a line bundle on S which is generated by global sections. Then the multiplication map $H^0(S, \mathcal{L}) \otimes H^0(S, \mathcal{M}) \to H^0(S, \mathcal{L} \otimes \mathcal{M})$ is surjective.*

The proof is straightforward, along the lines of the proof of Proposition 5.2, but uses the geometry of the H^0-polygons more explicitly.

Thus, we have the following theorem:

Theorem 5.7 *[6] The multiplication map*

$$\mu_n : H^0(m_1, m_2, ..., m_n) \otimes H^0(\hat{m}_1, \hat{m}_2, ..., \hat{m}_n) \to H^0(m_1+\hat{m}_1, m_2+\hat{m}_2, ..., m_n+\hat{m}_n)$$

given by

$$(x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) \mapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}}$$

is surjective, if $m_1C_1 + m_2C_2 + ... + m_nC_n$ *is ample and* $\hat{m}_1C_1 + \hat{m}_2C_2 + ... + \hat{m}_nC_n$ *is generated by global sections.*

Next we will consider the following:

Definition 5.8 *Multiplication maps,* $\hat{\mu_n}((m_1, m_2, ..., m_n)(\hat{m}_1, \hat{m}_2, ..., \hat{m}_n))$,

$$\hat{\mu_n} : H^0(S, \Omega^1(m_1C_1 + m_2C_2 + ... + m_nC_n)) \otimes H^0(S, \mathcal{O}(\hat{m}_1C_1 + \hat{m}_2C_2 + .. + \hat{m}_nC_n))$$

$$\to H^0(S, \Omega^1((m_1 + \hat{m}_1)C_1 + (m_2 + \hat{m}_2)C_2 + ... + (m_n + \hat{m}_n)C_n))$$

are defined as

$$\omega \otimes h \mapsto h\omega$$

We would like to prove an analog of Theorem 5.4 for general toric surfaces. For this purpose, we state the following theorem from Maclagan and Smith [14]:

Theorem 5.9 *[14] Assume* $\mathcal{C} \subseteq \mathcal{K}$ *and* $dim_{\mathbb{R}} pos(\bar{\mathcal{C}}) = r$. *If* $\mathbf{m} = \mathbb{Z}\mathcal{K}$ *and the sheaf* $\mathcal{F}$ *is* $\mathbf{m}$*-regular, then the natural map*

$$H^0(X, \mathcal{F}(\mathbf{p})) \otimes H^0(X, \mathcal{O}_X(\mathbf{q}) \to H^0(X, \mathcal{F}(\mathbf{p} + \mathbf{q}))$$

is surjective for all $\mathbf{p} \in \mathbf{m} + \mathbb{N}\mathcal{C}$ *and all* $\mathbf{q} \in \mathbb{N}\mathcal{C}$. *In particular, the sheaf* $\mathcal{F}(\mathbf{p})$ *is generated by global sections.*

We would like to use this theorem to show that $\hat{\mu_n}$ is surjective and believe it is entirely appropriate, but at this time, I do not fully understand it, hence cannot prove it is applicable to the analysis of the multiplication map, $\hat{\mu}$, in Definition 5.8. Therefore, this is an item for further exploration.

Chapter 6

Double Covers, with Examples

6.1 The Double Cover Construction in Surface Theory

In this section, we will discuss double covers for surfaces, specifically looking at the relationship between line bundles and one-forms on a covering space and the base space of a double cover. We will be using largely as reference Barth, Peters and VandeVen [1].

Let Y be a smooth projective surface and $\mathcal{L}$ be a line bundle on Y such that $\mathcal{L}^{\otimes 2} = \mathcal{L} \otimes \mathcal{L}$ has a section s. Then $D \subseteq Y$ is the divisor corresponding to the zeroes of the section s. Let L be the total space of $\mathcal{L}$ with $p : L \to Y$, the line bundle projection. Then the pullback bundle $p^*\mathcal{L}$ is a line bundle on L.[1]

$$p^*\mathcal{L}\left\{\begin{array}{ccc} p^*L & \longrightarrow & L \\ \downarrow \hat{p} \uparrow t & & \downarrow p \\ L & \xrightarrow{p} & Y \end{array}\right\}\mathcal{L} \tag{6.1}$$

By definition, $p^*L = \{(a,b) \epsilon L \times L \mid p(a) = p(b)\}$. Now, there is a section t of $p^*\mathcal{L}$ defined by $t(e) = (e,e) \epsilon p^*L$, where $e \epsilon L$ and t is a section of $p^*\mathcal{L}$ since $\hat{p}$ is defined by $\hat{p}(a,b) = p(a)$. Define $X = \{z \epsilon L| \ (p^*s - t^2)(z) = 0\}$. Then $X \subseteq L$.

Define $\pi : X \to Y$ as $p|_X$.

Another way of viewing double covers is as follows. If you have L, a locally free rank one $\mathcal{O}_Y$-module and a section s of $\mathcal{L}^{\otimes 2}$, you can form an $\mathcal{O}_Y$-algebra on $\mathcal{O}_Y \oplus L^{-1}$ and define X, the double cover of Y as $\underline{\text{Spec}}(\mathcal{O}_Y \otimes L^{-1})$, up to isomorphism.

Locally, X is defined by an equation $s = t^2$. Assume characteristic not equal to 2, over a point of Y where $s \neq 0$, one has 2 points of X. Over a point of Y where $s = 0$ we have only one point of X. Then $\{s = 0\} \subset Y$ is the branch divisor, D. D is a divisor, non-negative, in the linear system of $\mathcal{L}^{\otimes 2}$.

So double covers of Y are determined by a line bundle $\mathcal{L}$ and an effective divisor D in the linear system determined by $\mathcal{L}^{\otimes 2}$. If the divisor D is locally defined by $s = 0$, then the double cover is locally defined by $t^2 = s$.

Proposition 6.2 *Let Y be a smooth compact surface and $\mathcal{L}$ be a line bundle on Y such that $\mathcal{L}^{\otimes 2}$ has a global section s, not identically zero. Then there exists a surface X and a map $\pi : X \to Y$ such that*

1. $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes 2}$.
2. *D is the divisor on Y corresponding to the section s.*
3. *D is an effective divisor in the linear system of $\mathcal{L}^{\otimes 2}$.*
4. *X is smooth at x_0 if and only if s is smooth at $\pi(x_0)$. Thus X is smooth iff D is smooth.*

So singularities of X correspond to the singularities of the branch locus of D.

We call this together with the map $\pi : X \to Y$ the double covering of Y branched along D, determined by the line bundle $\mathcal{L}$.[1]

Lemma 6.3 *(Barth, Peters, and VandeVen [1]) Let $\pi : X \to Y$ be the double covering of Y branched along a smooth divisor D and determined by the line bundle $\mathcal{L}$, i.e. $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(D)$. Then*

1. $K_X = \pi^*(K_Y \otimes \mathcal{L})$.
2. $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$.

Lemma 6.4 *(see e.g. Duflot [4]) Let $\pi : X \to Y$ be the double covering of Y branched along a smooth divisor D and determined by the line bundle $\mathcal{L}$, i.e. $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(D)$. Then*

$$\pi_* \Omega^1_X = \Omega^1_Y \oplus (\Omega^1_Y(log D) \otimes \mathcal{L}^{-1}).$$

6.2 A Simple Example: $\mathbf{P}^1 \times \mathbf{P}^1$ as a Double Cover of $\mathbf{P}^2$

Consider the map $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$\pi([a,b],[c,d]) = [ac, \frac{ad+bc}{2}, bd].$$

We can see this map is well defined by considering

$$\pi(\lambda[a,b], \mu[c,d]) = \pi([\lambda a, \lambda b],[\mu c, \mu d])$$

$$= [\lambda\mu ac, \frac{\lambda\mu ad + \lambda\mu bc}{2}, \lambda\mu bd] = \lambda\mu[ac, \frac{ad+bc}{2}, bd]$$

We will compute $\pi^{-1}[Z_0, Z_1, Z_2]$ for every $[Z_0, Z_1, Z_2] \in \mathbb{P}^2$. Suppose $([a,b],[c,d]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is such that $\pi([a,b],[c,d]) = [Z_0, Z_1, Z_2]$. Then

Case 1: Let $[Z_0, Z_1, Z_2] = [1, Z_1, Z_2] \in U_0(\mathbb{P}^2)$ be a fixed point.

Case 1a: If $Z_1^2 \neq Z_2$, let D be any fixed complex number such that $D^2 = Z_1^2 - Z_2$. Clearly $Z_1 + D \neq Z_1 - D$ since $D \neq 0$. Therefore

$$\pi([1, Z_1 + D], [1, Z_1 - D]) = [1, Z_1, Z_2],$$

$$\pi([1, Z_1 - D], [1, Z_1 + D]) = [1, Z_1, Z_2].$$

Also, if $\pi([a,b],[c,d]) = [1, Z_1, Z_2]$, then either $([a,b],[c,d]) = ([1, Z_1 + D], [1, Z_1 - D])$ or $([a,b],[c,d]) = ([1, Z_1 - D], [1, Z_1 + D])$.

Case 1b: If $Z_1^2 = Z_2$, then

$$\pi([1, Z_1], [1, Z_1]) = [1, Z_1, Z_1^2] = [1, Z_1, Z_2]$$

and if $\pi([a,b],[c,d]) = [1, Z_1, Z_2]$, then $([a,b],[c,d]) = ([1, Z_1], [1, Z_1])$.

Case 2: If $[0, 1, Z_2] \in (\mathbb{P}^2 - U_0(\mathbb{P}^2)) \cap U_1(\mathbb{P}^2)$, then

$$\pi([2, Z_2], [0, 1]) = [0, 1, Z_2]$$

$$\pi([0, 1], [2, Z_2]) = [0, 1, Z_2]$$

and if $\pi([a,b],[c,d]) = [0, 1, Z_2]$, then

$$([a,b],[c,d]) = ([2, Z_2], [0, 1]) \text{ or } ([a,b],[c,d]) = ([0, 1], [2, Z_2]).$$

Case 3: The only remaining point in $\mathbb{P}^2$ is $[0, 0, 1]$ and

$$\pi^{-1}(\{[0, 0, 1]\}) = \{([0, 1], [0, 1])\}.$$

Therefore, by definition, $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is a covering map, specifically a double cover, branched along the smooth quadric defined by $Z_1^2 = Z_2$ in U_0.

6.3 The Relationships between Cohomology on X and Cohomology on Y

We first state some general theorems about sheaf cohomology without proof. References include Griffiths and Harris [8], Harris [9], and Barth, Peters, and VandeVen [1].

Theorem 6.5 *(An application of the Leray Spectral Sequence)*
Let $\pi : X \to Y$ be a double cover branched along a smooth divisor D and determined by $\mathcal{L}$, with $\mathcal{D} \cong \mathcal{L}^{\otimes 2}$. Let $\mathcal{F}$ be a sheaf on X. Since π has finite fibers, the Leray spectral sequence for π degenerates, and

$$H^i(Y, \pi_* \mathcal{F}) \cong H^i(X, \mathcal{F})$$

for every $i \geq 0$.

Theorem 6.6 *(The Projection Formula, applied to Double Covers)*
Let $\pi : X \to Y$ be a double cover branched along a smooth divisor D and determined by $\mathcal{L}$, with $\mathcal{D} \cong \mathcal{L}^{\otimes 2}$. If $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module of finite rank on Y (i.e. a vector bundle on Y), then

$$\pi_*(\mathcal{F} \otimes \pi^* \mathcal{G}) \cong \pi_* \mathcal{F} \otimes \mathcal{G}$$

as sheaves.

For example, if $\mathcal{G}$ is a locally free sheaf on Y and $\mathcal{F} = \mathcal{O}_X$, then

$$\pi_*\pi^*\mathcal{G} = \pi_*(\mathcal{O}_X \otimes \pi^*\mathcal{G}) \cong \pi_*\mathcal{O}_X \otimes \mathcal{G} = (\mathcal{O}_Y \oplus \mathcal{L}^{-1}) \otimes \mathcal{G} = \mathcal{G} \oplus (\mathcal{L}^{-1} \otimes \mathcal{G}).$$

Now, let $\mathcal{G}$ be a locally free sheaf on Y. By Theorem 6.5 above,

$$H^i(X, \pi^*(\mathcal{G})) \cong H^i(Y, \pi_*\pi^*(\mathcal{G})) \cong H^i(Y, \mathcal{G}) \oplus H^i(Y, \mathcal{L}^{-1} \otimes \mathcal{G}).$$

Hence we have the following theorem:

Theorem 6.7 *If $\pi : X \to Y$ is a double cover branched along a smooth divisor D, $D = 2L$, $\tilde{G}$ a divisor on Y, and $\tilde{E} = \pi^*\tilde{G}$, then*

$$H^i(X, \mathcal{O}_X(\tilde{E})) \cong H^i(Y, \mathcal{O}_Y(\tilde{G})) \oplus H^i(Y, \mathcal{O}_Y(\tilde{G} - L)).$$

Also if $\pi : X \to Y$ is a double cover branched along a smooth divisor D, $D = 2L$, $\tilde{G}$ a divisor on Y, and $\tilde{E} = \pi^*\tilde{G}$, then

$$H^i(X, \Omega^1_X(\hat{E})) \cong H^i(Y, \pi_*(\Omega^1_X(\hat{E})))$$

$$\cong H^i(Y, \pi_*(\Omega^1_X \otimes \mathcal{O}_X(\hat{E})))$$

$$\cong H^i(Y, \pi_*(\Omega^1_X \otimes \pi^*\mathcal{O}_Y(\hat{G})))$$

$$\cong H^i(Y, \pi_*\Omega^1_X \otimes \mathcal{O}_Y(\hat{G}))).$$

Now, by Lemma 6.4,

$$\pi_*\Omega^1_X = \Omega^1_Y \oplus (\Omega^1_Y(\log\ D) \otimes \mathcal{L}^{-1}).$$

Therefore

$$H^i(X,\Omega^1_X(\hat{E})) \cong H^i(Y,\Omega^1_Y(\hat{G})) \oplus H^i(Y,\Omega^1_Y(\log\ D) \otimes \mathcal{O}(\hat{G}-L)),$$

which we state in the following theorem:

Theorem 6.8 *If $\pi : X \to Y$ is a double cover branched along a smooth divisor D, $D = 2L$, $\hat{G}$ a line bundle on Y, $\hat{E} = \pi^*\hat{G}$, then*

$$H^i(X,\Omega^1_X(\hat{E})) \cong H^i(Y,\Omega^1_Y(\hat{G})) \oplus H^i(Y,\mathcal{O}_Y(\hat{G}-L) \otimes \Omega^1_Y(\log(D))).$$

6.4 Double Covers of Hirzebruch Surfaces

Again, let D be a smooth irreducible curve on $\mathbb{F}_k$ and $\pi : X \to \mathbb{F}_k$ be a double cover of $\mathbb{F}_k$, branched along D, X is uniquely determined by D, up to isomorphism, and X is smooth. Let L be a line bundle on $\mathbb{F}_k$, $2L = D$, and L be linearly equivalent to $\alpha C_1 + \beta C_2$, so that D is linearly equivalent to $2\alpha C_1 + 2\beta C_2$. Since D is a smooth curve, $\alpha, \beta \geq 0$.

We know that $K_X = \pi^*(K_{\mathbb{F}_k} + L) = \pi^*((\alpha-2)C_1 + (\beta+k-2)C_2)$.

Lemma 6.9 *(Barth, Peters, and VandeVen [1], page 237; pages 273-274)*

a) $p_g(X) = p_g(\mathbb{F}_k) + h^0(\mathbb{F}_k, K_{\mathbb{F}_k} + L) = 0 + h^0(\mathbb{F}_k, (\alpha-2)C_1, (\beta+k-2)C_2)$.

b) If D_1, D_2 are two divisors on $\mathbb{F}_k$, then

$$\pi^*(D_1) \cdot \pi^*(D_2) = 2(D_1 \cdot D_2).$$

Thus,

c) $c_1(X)^2 = K_X^2 = (\pi^*(K_{\mathbb{F}_k} + L))^2 = 2(K_{\mathbb{F}_k} + L)^2 = 2(K_{\mathbb{F}_k}^2 + 2K_{\mathbb{F}_k} \cdot L + L \cdot L) = 16 + 4K_{\mathbb{F}_k} \cdot L + 2L \cdot L$, *since* $K_{\mathbb{F}_k}^2 = 8$.

Also

d) $c_2(X) = 2c_2(\mathbb{F}_k)+2(L\cdot K_{\mathrm{F}_k})+4(L\cdot L) = 8+2L\cdot K_{\mathrm{F}_k}+4L\cdot L$, *since* $c_2(\mathbb{F}_k) = 4$.

e) $\chi(X) = 2\chi(\mathbb{F}_k) + \frac{1}{2}(L \cdot K_{\mathrm{F}_k}) + \frac{1}{2}(L \cdot L) = 2 + \frac{1}{2}(L \cdot K_{\mathrm{F}_k}) + \frac{1}{2}(L \cdot L)$.

Lemma 6.10 *Let X be defined as above.*

1. *If $\alpha \geq 2$ and $\beta + k \geq 2$, then $q(X) = 0$.*

2. *If $\alpha = 1$, then $q(X) = 0 \; \forall \beta \geq 0$.*

3. *If $\alpha = 0$, then $q(X) = \beta - 1 \; \forall \beta \geq 1$.*

PROOF: By our definitions, we have that $L+K_{\mathrm{F}_k} = (\alpha-2)C_1+(\beta+k-2)C_2$. Then $\chi(X) = 2 + \frac{1}{2}(L \cdot K_{\mathrm{F}_k} + L \cdot L)$ and $p_g(X) = h^0(\mathbb{F}_k, K_{\mathrm{F}_k} + L)$ by Lemma 6.9 above. By Corollary 4.5, if $\alpha-2 \geq 0$ and $\beta+k-2 \geq 0$, then $h^0(\mathbb{F}_k, L+K_{\mathrm{F}_k}) = \chi(L+K_{\mathrm{F}_k})$. By Theorem 3.13, this is equal to $\frac{2}{2}(L + K_{\mathrm{F}_k})(L) + 1$. Hence $p_g(X) = \chi(X) + 1$. Therefore $q(X) = 0$ if $\alpha \geq 2$ and $\beta + k \geq 2$.

Now suppose $\alpha = 1$. Then $p_g(X) = h^0(\mathbb{F}_k, L+K_{\mathrm{F}_k}) = h^0(-1C_1+(\beta+k-2)C_2) = 0$ by Lemma 4.4. We know that $q(X) = 1 - \chi(X) + 0$ and $\chi(X) = 2 + \frac{1}{2}(L \cdot K_{\mathrm{F}_k} + L \cdot L) = 2+\frac{1}{2}(1,\beta)((-2,k-2)+(1,\beta)) == 2+\frac{1}{2}(-k-\beta+k-2+\beta) = 1$. Hence, q(X)=0.

Now consider $\alpha = 0$. Again $p_g(X) = 0$ by Lemma 4.4. Also $\chi(X) = 2 + \frac{1}{2}(0,\beta)(-2,\beta+k-2) = 2-\beta$. Hence, $q(X) = 1-(2-\beta)+0 = \beta - 1$.

6.4.1 Determining the Kodaira Dimension of X

Let $\pi : X \rightarrow Y$ is a double cover branched along a smooth divisor D, $D = 2L$, K_X the canonical divisor and $K_X = \pi^*(K_{\mathrm{F}_k} + L)$. Also let $t \geq 0$. Then let's compute

$H^0(X, tK_X)$.

$$H^0(X, tK_X) = H^0(X, t\pi^*(K_Y + \mathcal{L})) \text{ by Lemma 6.3}$$

$$= H^0(X, \pi^*(tK_{\mathbb{F}_k} + t\mathcal{L}))$$

$$= H^0(\mathbb{F}_k, \pi_*\pi^*(tK_{\mathbb{F}_k} + t\mathcal{L}))$$

$$= H^0(\mathbb{F}_k, (K_{\mathbb{F}_k} \otimes \mathcal{L})^t \otimes \pi_*\mathcal{O}_X)$$

$$= H^0(\mathbb{F}_k, tK_{\mathbb{F}_k} + t\mathcal{L}) \oplus H^0(\mathbb{F}_k, tK_{\mathbb{F}_k} + (t-1)\mathcal{L}).$$

Since $tK_{\mathbb{F}_n} + t\mathcal{L} = t(\alpha - 2)C_1 + t(\beta + k - 2))C_2$, we can compute exactly the dimension of the summand above corresponding to this divisor, if α and β are large enough. Indeed, if $A \geq 0$ and $B \geq 0$, then

$$h^0(\mathbb{F}_k, \mathcal{O}(AC_1 + BC_2)) = (A+1)(B+1) + \frac{kA(A+1)}{2}.$$

Thus, if $\alpha \geq 2, \beta + k - 2 \geq 0$ (recall $t \geq 0$), then

$$h^0(\mathbb{F}_k, \mathcal{O}(tK_{\mathbb{F}_k} + t\mathcal{L})) = (t(\alpha-2)+1)(t(\beta+k-2)+1) + \frac{kt(\alpha-2)(t(\alpha-2)+1)}{2}.$$

Computing the coefficient of t^2 in this formula, we get

$$(\alpha - 2)(\beta + k - 2) + \frac{k(\alpha-2)^2}{2} = (\alpha - 2)(\beta - 2 + \frac{k}{2}\alpha).$$

This coefficient is zero (given the conditions above) if and only if $\alpha = 2$ or $\beta - 2 + \frac{k}{2}\alpha = 0$. Note that if $\beta - 2 + \frac{k}{2}\alpha = 0$ then $\beta = 2 - \frac{k}{2}\alpha$. But $\beta \geq 2 - k$, so $2 - \frac{k}{2}\alpha \geq 2 - k$ or $\frac{k}{2} \leq k$. So $\frac{\alpha}{2} \leq 1$ or $\alpha \leq 2$. Thus, if $\alpha > 2$, X is of general type.

If $\alpha = 2$ and $\beta + k - 2 \neq 0$, then X has Kodaira dimension 1.

Chapter 7

The Gaussian Map

In this chapter, we will define and discuss the Gaussian map. Primary references are Wahl ([23], [22]), Griffiths and Harris [8], and Duflot/Miranda [5].

7.1 Gaussian Maps on Projective Varieties

Once again, consider the multiplication map where X is a smooth projective variety and $\mathcal{F}$ and $\mathcal{G}$ are line bundles on X, defined as follows:

$$\mu : H^0(X, \mathcal{F}) \otimes H^0(X, \mathcal{G}) \to H^0(X, \mathcal{F} \otimes \mathcal{G})$$

given by

$$\mu(s \otimes t) = st.$$

Definition 7.1 *Given the multiplication map, μ, then $ker \mu = \mathcal{R}(\mathcal{F}, \mathcal{G})$ [23].*

Given an open set $U \subset X$, let T be a generator of $\mathcal{F}|_U$. Let $\alpha = \sum \sigma_i \otimes \tau_i \in \mathcal{R}(\mathcal{F}, \mathcal{G})$. Then we can write $\sigma_i = f_i T$ locally for some $f_i \in H^0(\mathcal{O}_U)$. Given a generator S of $\mathcal{G}|_U$, then we can write $\tau_i = g_i S$ locally for some $g_i \in H^0(\mathcal{O}_U)$. (Note that

$\sum f_i g_i = 0$.) Then we can define the Gaussian map

$$\Phi_{X,\mathcal{F},\mathcal{G}}(\alpha) \doteq \sum(f_i dg_i - g_i df_i) \otimes T \otimes S \in H^0(\Omega^1_X \otimes \mathcal{F} \otimes \mathcal{G}).$$

This is well defined, proof in Wahl [21], page 123.

If $\mathcal{F} = \mathcal{G}$, then $\Lambda^2 H^0(\mathcal{F}) \subset \mathcal{R}(\mathcal{F}, \mathcal{F})$ by identifying $\sigma \wedge \tau$ with $\frac{1}{2}(\sigma \otimes \tau - \tau \otimes \sigma)$. Then we restrict the domain of $\Phi_{X,\mathcal{F},\mathcal{F}}$ to $\Lambda^2 H^0(X, \mathcal{F})$. We will write $\Phi_{X,\mathcal{F},\mathcal{F}}|_{\Lambda^2 H^0(X,\mathcal{F})} \doteq \Phi_{X,\mathcal{F}}$.

$$\Phi_{X,\mathcal{F}} : \Lambda^2 H^0(X, \mathcal{F}) \to H^0(X, \Omega^1_X \otimes \mathcal{F}^2)$$

is given by, if $\sigma = fT$ and $\tau = gT$ locally,

$$\Phi_{X,\mathcal{F}}(\sigma \wedge \tau) = (fdg - gdf) \otimes T \otimes T[23].$$

We can verify this with the following computation:

$$\Phi_{X,\mathcal{F}}(\sigma \wedge \tau) = \Phi_{X,\mathcal{F},\mathcal{F}}(\frac{1}{2}(\sigma \otimes \tau - \tau \otimes \sigma))$$

$$= \frac{1}{2}([(fdg - gdf) \otimes T \otimes T] + [(-gdf - fd(-g)) \otimes T \otimes T])$$

$$= \frac{1}{2}(2fdg - 2gdf) \otimes T \otimes T = (fdg - gdf) \otimes T \otimes T.$$

Hence, we have shown that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}(\mathcal{F},\mathcal{F}) & \stackrel{\Phi_{X,\mathcal{F},\mathcal{F}}}{\to} & H^0(X, \Omega^1_X \otimes \mathcal{F}^2) \\ \nwarrow & & \Phi_{X,\mathcal{F}} \nearrow \\ & \Lambda^2 H^0(X,\mathcal{F}) & \end{array} \tag{7.2}$$

and

$$\text{im } \Phi_{X,\mathcal{F}} = \text{im } \Phi_{X,\mathcal{F},\mathcal{F}}.$$

If $\mathcal{L}, \mathcal{M}$, and $\mathcal{N}$ are line bundles, then there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{R}(\mathcal{L},\mathcal{M}) \otimes H^0(\mathcal{N}) & \overset{\Phi_{X,\mathcal{L},\mathcal{M}} \otimes id}{\longrightarrow} & H^0(\Omega^1 \otimes \mathcal{L} \otimes \mathcal{M}) \otimes H^0(\mathcal{N}) \\ \downarrow a & & \downarrow \hat{\mu} \\ \mathcal{R}(\mathcal{L},\mathcal{M} \otimes \mathcal{N}) & \overset{\Phi_{X,\mathcal{L},\mathcal{M}\otimes\mathcal{N}}}{\rightarrow} & H^0(\Omega^1 \otimes \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}) \end{array} \tag{7.3}$$

where the horizontal maps are defined using Gaussian maps as indicated and the vertical map, $\hat{\mu}$, is a multiplication map, as defined in Section 5.2. The map a is defined by

$$(\sum \sigma_i \otimes \tau_i) \otimes hT' \overset{a}{\mapsto} \sum(\sigma_i \otimes h\tau_i).$$

To see that the diagram is commutative, and using the notation and definitions from earlier in the section, if hT' is a local representation for an element of $H^0(\mathcal{N})$,

$$(\sum \sigma_i \otimes \tau_i) \otimes hT' \overset{\Phi_{X,\mathcal{L},\mathcal{M}} \otimes id}{\mapsto} \sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes hT'.$$

Also

$$\sum(\sigma_i \otimes h\tau_i) \overset{\Phi_{X,\mathcal{L},\mathcal{M}\otimes\mathcal{N}}}{\mapsto} h\sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes T'$$

since

$$\begin{aligned} &\Phi_{X,\mathcal{L},\mathcal{M}\otimes\mathcal{N}}(\sum(\sigma_i \otimes h\tau_i)) \\ &= \sum(f_i d(hg_i) - (hg_i)df_i) \otimes T \otimes S \otimes T' \\ &= \sum(f_i(hdg_i + g_i dh) - hg_i df_i) \otimes T \otimes S \otimes T' \\ &= [h\sum(f_i dg_i - g_i df_i) + \sum(f_i g_i)dh] \otimes T \otimes S \otimes T', \end{aligned}$$

but $\sum(f_i g_i) = 0$.

By definition of the multiplication map, $\hat{\mu}$,

$$\sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes hT' \stackrel{\hat{\mu}}{\mapsto} h\sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes T'.$$

So

$$(\sum \sigma_i \otimes \tau_i) \otimes hT' \stackrel{\Phi_{X,\mathcal{L},\mathcal{M}\otimes\mathcal{N}} \circ a}{\mapsto} h\sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes T'$$

and

$$(\sum \sigma_i \otimes \tau_i) \otimes hT' \stackrel{\hat{\mu}\circ\Phi_{X,\mathcal{L},\mathcal{M}}}{\mapsto} h\sum(f_i dg_i - g_i df_i) \otimes T \otimes S \otimes T'.$$

Note the naturality of the Gaussian maps for $f : X \to Y, f^*\mathcal{G}$, and $f^*\hat{\mathcal{G}}$, i.e. there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{R}(\mathcal{G}, \hat{\mathcal{G}}) & \stackrel{\Phi_{Y,\mathcal{G},\hat{\mathcal{G}}}}{\to} & H^0(\Omega^1_Y \otimes \mathcal{G} \otimes \hat{\mathcal{G}})) \\ \downarrow & & \downarrow \\ \mathcal{R}(f^*\mathcal{G}, f^*\hat{\mathcal{G}}) & \stackrel{\Phi_{X,f^*\mathcal{G},f^*\hat{\mathcal{G}}}}{\to} & H^0(\Omega^1_X \otimes f^*\mathcal{G} \otimes f^*\hat{\mathcal{G}}) \end{array} \tag{7.4}$$

7.2 Gaussian Map, Curves on a Surface

Consider a surface S and any curve C on S, with D defined as a divisor on C, Ω^1_C the line bundle associated to the holomorphic 1-forms on C. This is the canonical line bundle on C. Then the Gaussian map for this data is:

$$\phi_{C,D} : \Lambda^2\, H^0(C, \mathcal{O}(D)) \to H^0(C, \mathcal{O}(2D) \otimes \Omega^1_C).$$

By the Adjunction Formula (Theorem 3.12), if $C \subseteq S$, we have a relationship between the canonical divisor on S and the canonical divisor on C:

$$K_C = (K_S \otimes [C])|_C,$$

or

$$\mathcal{O}_C(K_C) = (\mathcal{O}_S(K_S) \otimes \mathcal{O}_S(C))|_C,$$

or

$$\Omega^1_C = (\Omega^2_S \otimes \mathcal{O}_S(C))|_C.$$

In other words,

$$\Omega^1_C = \mathcal{O}_C(K_C) = \mathcal{O}_S(K_S + C)|_C.$$

There is a commutative diagram (from Duflot/Miranda [5], page 449):

$$\begin{array}{ccc} \Lambda^2\, H^0(S, \mathcal{O}_S(K_S + C)) & \stackrel{\phi_{S,K_S+C}}{\rightarrow} & H^0(S, \mathcal{O}(2K_S + 2C) \otimes \Omega^1_S) \\ & & \downarrow a \\ \downarrow \text{res} & & H^0(C, \Omega^1_S(2K_S + 2C)|_C) \quad (7.5) \\ & & \downarrow b \\ \Lambda^2\, H^0(C, \mathcal{O}_C(K_C)) & \stackrel{\phi_{C,K_C}}{\rightarrow} & H^0(C, \mathcal{O}(2K_C) \otimes \Omega^1_C)[5], \end{array} \tag{7.6}$$

where res, a, and b are defined below.

The restriction map, res, on the exterior product comes directly from the exact sequence of sheaves (Griffiths and Harris [8], pg 139)

$$0 \rightarrow \mathcal{O}_S(E - C) \rightarrow \mathcal{O}_S(E) \rightarrow \mathcal{O}_S(E)|_C \rightarrow 0 \tag{7.7}$$

Setting $E = K_S + C$ and using the Adjunction Formula to identify $\mathcal{O}_C(K_C) \doteq$

$\Omega^1_C = \mathcal{O}_S(K_S + C)|_C$ yields

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + C) \to \Omega^1_C \to 0. \tag{7.8}$$

The associated long exact sequence in cohomology on S is

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}_S(K_S)) \to H^0(S, \mathcal{O}_S(K_S + C)) \xrightarrow{r} H^0(S, \Omega^1_C) \to \\ H^1(S, \mathcal{O}_S(K_S)) \to H^1(S, \mathcal{O}_S(K_S + C)) \to H^1(S, \Omega^1_C) \to \\ H^2(S, \mathcal{O}_S(K_S)) \to \ldots \end{aligned} \tag{7.9}$$

The map, res, is the map on Λ^2 induced by r in the above long exact sequence.

Thus, if $H^1(S, \mathcal{O}_S(K_S)) = 0$, then res is onto. So,

Lemma 7.10 *If S is a smooth surface, C a curve on S and $q(S) = 0$, then res: $\Lambda^2\ H^0(S, \mathcal{O}_S(K_S + C)) \to \Lambda^2\ H^0(C, \mathcal{O}_C(K_C))$ is onto.*

PROOF: $H^1(S, \mathcal{O}_S(K_S)) = H^1(S, \mathcal{O}_S)$ by Kodaira-Serre duality, and $H^1(S, \mathcal{O}_S) \doteq q(S)$. By assumption, $q(S) = 0$, hence the map res is onto.

Note: if $q(S) = 0$, then S is called a regular surface.

The vertical maps a and b are derived as follows: Taking $E = 0$ in the short exact sequence 7.8 yields

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \tag{7.11}$$

with $\mathcal{O}_C = \mathcal{O}_S|_C$.

If we tensor the short exact sequence 7.11 with $\mathcal{O}_S(2K_S+2C)\otimes\Omega^1_S$, recalling that $\Omega^1_S(-) \doteq \mathcal{O}_S(-)\otimes\Omega^1_S$, this results in the sequence

$$0 \to \Omega^1_S(2K_S+C) \to \Omega^1_S(2K_S+2C) \to \Omega^1_S(2K_S+2C)|_C \to O.$$

Since tensor and extension by 0 commute,

$$\mathcal{O}_C \otimes \mathcal{O}_S(2K_S+2C)\otimes\Omega^1_S = \mathcal{O}_C\otimes\Omega^1_S(2K_S+2C) = \Omega^1_S(2K_S+2C)|_C.$$

The associated long exact sequence is

$$\begin{aligned}0 \to H^0(S,\Omega^1_S(2K_S+C)) \to H^0(S,\Omega^1_S(2K_S+2C)) \to H^0(S,\Omega^1_S(2K_S+2C)|_C) \to\\ H^1(S,\Omega^1_S(2K_S+C)) \to H^1(S,\Omega^1_S(2K_S+2C)) \to H^1(S,\Omega^1_S(2K_S+2C)|_C) \to\\ H^2(S,\Omega^1_S(2K_S+C)) \to \ldots\end{aligned} \tag{7.12}$$

Now $H^0(S,\Omega^1_S(2K_S+2C)|_C) = H^0(C,\Omega^1_S(2K_S+2C)|_C)$. This yields the map a:

$$a: \; H^0(S,\Omega^1_S(2K_S+2C)) \to H^0(C,\Omega^1_S(2K_S+2C)|_C).$$

From the above, we have the following:

Lemma 7.13 *The map* $a: \; H^0(S,\Omega^1_S(2K_S+2C)) \to H^0(C,\Omega^1_S(2K_S+2C)|_C)$ *is surjective if* $H^1(S,\Omega^1_S(2K_S+C)) = 0$.

This follows directly from the long exact sequence 7.12.

Let $\mathbb{T}(C)$ be the one dimensional tangent vector bundle for C in S, a sub line bundle of $\mathbb{T}(S)|_C$, the two dimensional tangent vector bundle to S. Additionally, let $N_{C/S}$ be the one dimensional normal bundle for C in S. By the Adjunction Formula, Theorem 3.11, $N^*_{C/S} = \mathcal{O}_S(-C)|_C$.

Then the normal bundle sequence for $N_{C/S}$ is

$$0 \to \mathbb{T}(C) \to \mathbb{T}(S)|_C \to N_{C/S} \to 0,$$

where $\mathbb{T}(s)|_C \cong \mathbb{T}(C) \oplus N_{C/S}$ locally. Taking the dual of the sequence yields

$$0 \to N^*_{C/S} \to (\mathbb{T}(S)|_C)^* \to \mathbb{T}(C)^* \to 0$$

Recall $\Omega^1_S \doteq \mathbb{T}(S)^*$, and $\Omega^1_C \doteq \mathbb{T}(C)^*$. So $\Omega^1_S|_C \doteq (\mathbb{T}(S)^*)|_C$, and the dual and restriction commute, yielding the sequence

$$0 \to N^*_{C/S} \to \Omega^1_S|_C \to \Omega^1_C \to 0.$$

Next, we tensor this sequence with $\mathcal{O}_S(2K_S + 2C)|_C$. Now

$$\mathcal{O}_S(2K_S + 2C)|_C = \mathcal{O}_S(2K_S + C)|_C \otimes \mathcal{O}_S(2K_S + C)|_C = \Omega^1_C \otimes \Omega^1_C = (\Omega^1_C)^{\otimes 2},$$

so the tensoring yields the exact sequence

$$0 \to N^*_{C/S} \otimes (\Omega^1_C)^{\otimes 2} \to \Omega^1_S|_C \otimes (\Omega^1_C)^{\otimes 2} \to \Omega^1_C \otimes (\Omega^1_C)^{\otimes 2} \to 0.$$

Also

$$N^*_{C/S} \otimes (\Omega^1_C)^{\otimes 2} = \mathcal{O}_S(-C)|_C \otimes \mathcal{O}_S(2K_S + 2C)|_C = \mathcal{O}_S(2K_S + C)|_C$$

$$\Omega^1_S|_C \otimes (\Omega^1_C)^{\otimes 2} = (\Omega^1_S \otimes \mathcal{O}_S(2K_S + 2C))|_C = \Omega^1_S(2K_S + 2C)|_C$$

yielding the exact sequence

$$0 \to \mathcal{O}_S(2K_S + C)|_C \to \Omega^1_S(2K_S + 2C)|_C \to (\Omega^1_C)^{\otimes 3} \to 0.$$

The associated long exact sequence on C is

$$0 \to H^0(C, \mathcal{O}_S(2K_S + C)|_C) \to H^0(C, \Omega^1_S(2K_S + 2C)|_C) \xrightarrow{b} H^0(C, (\Omega^1_C)^{\otimes 3}) \to$$
$$H^1(C, \mathcal{O}_S(2K_S + C)|_C) \to H^1(C, \Omega^1_S(2K_S + 2C)|_C) \to H^1(C, (\Omega^1_C)^{\otimes 3}) \to$$
$$H^2(C, \mathcal{O}_S(2K_S + C)|_C) \to ... \tag{7.14}$$

which yields the b map

$$b: \ H^0(C, \Omega^1_C(2K_S + 2C)|_C) \to H^0(C, (\Omega^1_C)^{\otimes 3}).$$

Combining this map with the long exact sequence 7.14 yields

Lemma 7.15 *The map* $b: \ H^0(C, \Omega^1_S(2K_S+2C)|_C) \to H^0(C, (\Omega^1_S)^{\otimes 3})$ *is surjective if* $H^1(S, \mathcal{O}_S(2K_S + C)) = 0$ *and* $H^0(S, \mathcal{O}_S(-K_S)) = 0$.

PROOF: Using the long exact sequence 7.14 above, b is surjective if $H^1(C, \mathcal{O}_S(2K_S + C)|_C) = 0$. In addition, using the exact sequence 7.7 with $E = 2K_S + C$, we see that since the sequence

$$...H^1(S, \mathcal{O}_S(2K_S + C)) \to H^1(C, \mathcal{O}_S(2K_S + C)|_C) \to H^2(S, \mathcal{O}_S(2K_S)) \to ...$$

is exact, $H^1(C, \mathcal{O}_S(2K_S+C)|_C) = 0$ if $H^1(S, \mathcal{O}_S(2K_S+C)) = 0$ and $H^2(S, \mathcal{O}_S(2K_S)) = 0$. Kodaira-Serre duality (Corollary 3.18) says $H^2(S, \mathcal{O}_S(2K_S)) = H^0(S, \mathcal{O}_S(-K_S))$. Hence the map b is surjective if $H^1(S, \mathcal{O}_S(2K_S+C)) = 0$ and $H^0(S, \mathcal{O}_S(-K_S)) = 0$.

Recall the commutative diagram 7.6. We can now state the following lemma:

Lemma 7.16 *Given a surface S and any smooth curve C on S. If*

1. $\phi_{S,K_S+C}: \ \bigwedge^2 H^0(S, \mathcal{O}_S(K_S + C)) \to H^0(S, \mathcal{O}(2K_S + 2C) \otimes \Omega^1_S)$ *is surjective,*

2. $H^1(S, \Omega^1_S(2K_S + C)) = 0$, *and*

3. $H^1(S, \mathcal{O}_S(2K_S + C)) = 0$ *and* $H^0(S, \mathcal{O}_S(-K_S)) = 0$.

then $\phi_{C,K_C} : \bigwedge^2 H^0(C, \mathcal{O}_C(K_C)) \to H^0(C, \mathcal{O}(2K_C) \otimes \Omega^1_C)$ *is surjective.*

PROOF: Item 2 yields surjectivity of the map a and item 3 yields surjectivity of the map b. Since the map ϕ_{S,K_S+C} is surjective by item 1, then the map Φ_{C,K_C} is also surjective.

Chapter 8

Surjectivity of the Gaussian Map

In Duflot/Miranda [5], conditions for the surjectivity of $\Phi_{\mathrm{F}_k,D}$ are given. In this chapter, we use this result to prove conditions for surjectivity of the more general Gaussian map, $\Phi_{\mathrm{F}_k,D_1,D_2}$. To do this, we must first re-examine multiplication maps. Original work in this chapter is Proposition 8.2. The reference for this chapter is predominantly Duflot/Miranda [5].

8.1 Multiplication Maps on Hirzebruch Surfaces

Recall the multiplication map, Definition 5.1,

$$\mu((m_1, m_2), (\hat{m}_1, \hat{m}_2)) : H^0(\mathbb{F}_k, (m_1, m_2)) \otimes H^0(\mathbb{F}_k, (\hat{m}_1, \hat{m}_2))$$

$$\to H^0(\mathbb{F}_k, (m_1 + \hat{m}_1, m_2 + \hat{m}_2)).$$

We have defined the kernel of this map as

$$\ker \mu \doteq \mathcal{R}((m_1, m_2), (\hat{m}_1, \hat{m}_2)).$$

We have the following proposition:

Proposition 8.1 *(Duflot/Miranda, [5])* *If* $m_2 \geq 1$ *and* $m_1 \geq 1$*, then the Gaussian map,*

$$\Phi_{\mathbb{F}_k, m_1 C_1 + m_2 C_2} : \mathcal{R}((m_1, m_2), (m_1, m_2)) \to H^0(\mathbb{F}_k, \Omega^1(2m_1, 2m_2))$$

is surjective.

PROOF: This is a direct translation of Theorem 4.5 of Duflot/Miranda [5], stated using a different basis for $\mathrm{Pic}(\mathbb{F}_k)$, $m_1 C_1 + m_2 C_2$.

8.2 Gaussian Maps for Hirzebruch surfaces

The main result for this section is:

Proposition 8.2 *The Gaussian map,*

$$\Phi_{\mathbb{F}_k, m_1 C_1 + m_2 C_2, \hat{m}_1 C_1 + \hat{m}_2 C_2} : \mathcal{R}((m_1, m_2), (\hat{m}_1, \hat{m}_2)) \to H^0(\mathbb{F}_k, \Omega^1(m_1 + \hat{m}_1, m_2 + \hat{m}_2))$$

is surjective if $m_1, \hat{m}_1 \geq 1$ *and* $m_2, \hat{m}_2 \geq 1$.

PROOF: We will be constructing two commutative diagrams using the Diagram 7.3. For simplicity of notation, the $\mathbb{F}_k$ subscripts will be omitted. Consider the following:

<u>Diagram 1</u>

$$\begin{array}{ccc}
\mathcal{R}((1,1),(m_1,m_2)) & \xrightarrow{\Phi_1} & H^0(\Omega^1(m_1+1, m_2+1)) \\
\uparrow a_1 & & \| \\
\mathcal{R}((1,1),(1,1)) \otimes H^0(m_1-1, m_2-1) & \xrightarrow{\hat{\mu}_1 \circ (\Phi_2 \otimes id)} & H^0(\Omega^1(m_1+1, m_2+1)) \\
\Phi_2 \otimes id \searrow & & \hat{\mu}_1 \nearrow \\
& H^0(\Omega^1(2,2)) \otimes H^0(m_1-1, m_2-1) &
\end{array} \tag{8.3}$$

Diagram 2

$$\begin{array}{ccc}
\mathcal{R}((m_1,m_2),(\hat{m_1},\hat{m_2})) & \xrightarrow{\Phi_3} & H^0(\Omega^1(m_1+\hat{m_1},m_2+\hat{m_2})) \\
\uparrow a_2 & & \| \\
\mathcal{R}((m_1,m_2),(1,1))\otimes H^0(\hat{m_1}-1,\hat{m_2}-1) & \xrightarrow{\hat{\mu_2}\circ(\Phi_1\otimes id)} & H^0(\Omega^1(m_1+\hat{m_1},m_2+\hat{m_2})) \\
\Phi_1\otimes id \searrow & & \hat{\mu_2} \nearrow \\
 & H^0(\Omega^1(m_1+1,m_2+1))\otimes H^0(\hat{m_1}-1,\hat{m_2}-1) &
\end{array} \tag{8.4}$$

Here $\Phi_1 = \Phi_{\mathrm{F}_k,C_1+C_2,m_1C_1+m_2C_2}$, $\Phi_2 = \Phi_{\mathrm{F}_k,C_1+C_2}$, and $\Phi_3 = \Phi_{\mathrm{F}_k,m_1C_1+m_2C_2,\hat{m_1}C_1+\hat{m_2}C_2}$. First, we will show that if $m_1 \geq 1, m_2 \geq 1$, then

$$\mathcal{R}((1,1),(m_1,m_2)) \xrightarrow{\Phi_1} H^0(\Omega^1(m_1+1,m_2+1))$$

(the first line of Diagram 1) is surjective.

Let $m_1 \geq 1, m_2 \geq 1$. In Diagram 1, $\Phi_2 \otimes id$ is surjective by Proposition 8.1. From the supposition, $m_1 \geq 1, m_2 \geq 1$, which meets the criteria for $\hat{\mu_1}$ to be surjective by Proposition 5.4. Therefore, Diagram 1 shows that

$$\mathcal{R}((1,1),(m_1,m_2)) \xrightarrow{\Phi_1} H^0(\Omega^1(m_1+1,m_2+1))$$

is surjective if $m_1 \geq 1, m_2 \geq 1$.

Next let $m_1 \geq 1, m_2 \geq 1$ and $\hat{m_1} \geq 1, \hat{m_2} \geq 1$. From Diagram 2, we have that $\Phi \otimes id$ is surjective by the above argument if $m_1 \geq 1, m_2 \geq 1$. From the supposition, $m_1 \geq 1, m_2 \geq 1$ and $\hat{m_1} \geq 1, \hat{m_2} \geq 1$, , so $\hat{\mu_2}$ is surjective by Proposition 5.4.

Therefore the Gaussian map

$$\Phi_{\mathrm{F}_k, m_1C_1+m_2C_2, \hat{m}_1C_1+\hat{m}_2C_2} : \mathcal{R}((m_1, m_2), (\hat{m}_1, \hat{m}_2)) \to H^0(\Omega^1(m_1+\hat{m}_1, m_2+\hat{m}_2))$$

is surjective if $m_1, \hat{m}_1 \geq 1$ and $m_2, \hat{m}_2 \geq 1$.

8.3 Gaussian Maps of General Toric Surfaces

Recall the multiplication map, 5.5, defined as

$$\mu : H_k^0(m_1, m_2, ..., m_n) \otimes H_k^0(\hat{m}_1, \hat{m}_2, ..., \hat{m}_n) \to H_k^0(m_1+\hat{m}_1, m_2+\hat{m}_2, ..., m_n+\hat{m}_n),$$

We define the kernel of this map as

$$\ker \mu \doteq \mathcal{R}((m_1, m_2, ...m_n), (\hat{m}_1, \hat{m}_2, ..., \hat{m}_n)).$$

We would like to prove the surjectivitiy of the Gaussian map on the kernel of this multiplication map, but first, we need to prove a version of Proposition 8.1 for general toric surfaces using a lemma from Murray [17]:

Lemma 8.5 *If $D = m_1C_1 + m_2C_2 + ... + m_nC_n$ is ample, then*

$$\mathcal{R}((m_1, m_2, ..., m_n), (m_1, m_2, ..., m_n)) \xrightarrow{\Phi} H^0(\Omega^1(2m_1, 2m_2, ..., 2m_n))$$

is surjective.

This would require surjectivity of multiplication maps

$$H^0(S, \Omega_S^1(m_1C_1 + ... + M_nC_n) \otimes H^0(S, \hat{m}_1C_1 + ... + \hat{m}_nC_n)$$

$$\to H^0(S, (m_1+\hat{m}_1)C_1 + ... + (m_n+\hat{m}_n)C_n),$$

and a suitable replacement for Diagram 1 in the last section. We hope Maclagan/Smith [14], restated here as Theorem 5.9, will help solve our first problem. This proof will be left for completion in future work.

Chapter 9

Gaussian Maps for Double Covers

In this chapter, we will explore double covers in general, establishing theorems that we will later use with toric surfaces, Hirzebruch surfaces and curves on Hirzebruch surfaces. The primary reference for this chapter is Duflot [4].

9.1 Gaussian Map with Line Bundles

Consider smooth projective varieties X and Y of the same dimension with X a double cover of Y of degree 2 with smooth branch locus D, and the covering map $\pi : X \to Y$; the line bundle $\mathcal{L} \in Pic(Y)$ is such that $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes 2}$. Recall from Lemmas 6.3 and 6.4 that

1. $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$
2. $\pi^*(K_Y \otimes \mathcal{L}) = K_X$
3. $\pi_* \Omega^1_X = \Omega^1_Y \oplus (\mathcal{L}^{-1} \otimes \Omega^1_Y(\log D))$.

Recall the discussion of Gaussian maps from Section 7.1. Given the map $\pi : X \to Y$. Let $\mathcal{G}$ be a line bundle on Y. Consider the following Gaussian map:

$$\Lambda^2 H^0(X, \pi^* \mathcal{G}) \stackrel{\Phi_{X, \pi^* \mathcal{G}}}{\longrightarrow} H^0(X, (\pi^* \mathcal{G})^2 \otimes \Omega^1_X). \tag{9.1}$$

Using the discussion and isomorphisms of Section 6.3, we may identify $\Phi_{X,\pi^*\mathcal{G}}$ with the bottom row of the following diagram:

$$\begin{array}{ccc}
\Lambda^2 H^0(X,\pi^*\mathcal{G}) & \xrightarrow{\Phi_{X,\pi^*\mathcal{G}}} & H^0(X,(\pi^*\mathcal{G})^2\otimes\Omega^1_X) \\
\downarrow\cong & & \downarrow\cong \\
\Lambda^2 H^0(Y,\mathcal{G}\otimes\pi_*\mathcal{O}_X) & & H^0(Y,\mathcal{G}^2\otimes\pi_*\Omega^1_X) \\
\downarrow\cong & & \downarrow\cong
\end{array}$$

$$\Lambda^2 H^0(Y,\mathcal{G})\oplus\Lambda^2 H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1})\oplus(H^0(Y,\mathcal{G})\otimes H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1})) \xrightarrow{\Phi_{X,\pi^*\mathcal{G}}}$$
$$H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y)\oplus H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y(\log D)\otimes\mathcal{L}^{-1}). \tag{9.2}$$

For ease of reference, we will refer to the various components of the lowest map of this diagram as follows. Let:

$$V_0 = \Lambda^2 H^0(Y,\mathcal{G}),$$

$$W_0 = \Lambda^2 H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1}),$$

$$V_1 = H^0(Y,\mathcal{G})\otimes H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1}),$$

and

$$A_0 = H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y),$$

$$A_1 = H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y(\log D)\otimes\mathcal{L}^{-1}).$$

We have the following theorem from Duflot [4]:

Theorem 9.3 *(Duflot [4]) Given the Gaussian map*

$$\Lambda^2 H^0(Y,\mathcal{G})\oplus\Lambda^2 H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1})\oplus(H^0(Y,\mathcal{G})\otimes H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1}))$$

$$\xrightarrow{\Phi_{X,\pi^*\mathcal{G}}} H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y)\oplus H^0(Y,\mathcal{G}^2\otimes\Omega^1_Y(\log D)\otimes\mathcal{L}^{-1}),$$

then we have:

$$\Phi_{X,\pi^*\mathcal{G}}|_{V_0} : V_0 \to A_0$$

$$\Phi_{X,\pi^*\mathcal{G}}|_{V_1} : V_1 \to A_1$$

$$\Phi_{X,\pi^*\mathcal{G}}|_{W_0} : W_0 \to A_0,$$

and

$$\Phi_{X,\pi^*\mathcal{G}}|_{V_0} = \Phi_{Y,\mathcal{G}}.$$

Consider the map

$$B : \ H^0(Y, \Omega_Y \otimes \mathcal{G}^2 \otimes \mathcal{O}_Y(-D)) \to H^0(Y, \Omega^1 \otimes \mathcal{G}^2)$$

induced by the short exact sequence

$$0 \to \mathcal{O}_Y(-D) \xrightarrow{B} \mathcal{O}_Y \xrightarrow{r} \mathcal{O}_D \to 0. \tag{9.4}$$

Recall that $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes 2}$, so $\mathcal{O}_Y(-D) = \mathcal{L}^{-2}$.

Theorem 9.5 *(Duflot [4]) Given the the map $B : \ H^0(Y, \Omega_Y \otimes \mathcal{G}^2 \otimes \mathcal{O}_Y(-D)) \to H^0(Y, \Omega^1 \otimes \mathcal{G}^2)$, and the map*

$$\Phi_{X,\pi^*\mathcal{G}}|_{W_0} : \Lambda^2 H^0(Y, \mathcal{G} \otimes \mathcal{L}^{-1}) \to H^0(Y, \mathcal{G}^2 \otimes \Omega^1_Y)$$

then

$$\Phi_{X,\pi^*\mathcal{G}}|_{W_0} = B \circ \Phi_{Y,\mathcal{G}\otimes\mathcal{L}^{-1}}.$$

Since the above sequence is exact, we know from the corresponding long exact sequence in cohomology

$$0 \to H^0(Y, \mathcal{L}^{-2} \otimes \Omega^1_Y \otimes \mathcal{G}^2) \xrightarrow{\beta} H^0(Y, \Omega^1_Y \otimes \mathcal{G}^2) \to \ldots$$

that the map B is injective. Combining this with our earlier discussion of Gaussian maps, we find that

$$\text{rank}\Phi_{X,\pi^*\mathcal{G}}|_{W_0} = \text{rank}\Phi_{Y,\mathcal{G}\otimes\mathcal{L}^{-1}}$$

and

$$\text{rank}\Phi_{X,\pi^*\mathcal{G}}|_{V_0} = \text{rank}\Phi_{Y,\mathcal{G}}.$$

Taking the residue short exact sequence 4.25

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log\ D) \to \mathcal{O}_D \to 0,$$

and tensoring it with $\mathcal{G}^2 \otimes \mathcal{L}^{-1}$ yields

$$0 \to \Omega^1_Y \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \to \Omega^1_Y(\log\ D) \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \to \mathcal{O}_D \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \to 0.$$

Similarly, taking the short exact sequence 9.4 and tensoring it with $\mathcal{G}^2 \otimes \mathcal{L}^{-1}$ yields

$$0 \to \mathcal{O}_Y(-D) \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \xrightarrow{\beta} \mathcal{O}_Y \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \xrightarrow{r} \mathcal{O}_D \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1} \to 0.$$

Therefore

$$\begin{aligned} \ldots \to H^0(Y, \mathcal{G}^2 \otimes \mathcal{L}^{-1}) &\xrightarrow{r} H^0(Y, \mathcal{O}_D \otimes \mathcal{G}^2 \otimes \mathcal{L}^{-1}) \\ &\xrightarrow{\delta} H^1(Y, \mathcal{G}^2 \otimes \mathcal{L}^{-3}) \to H^1(Y, \mathcal{G}^2 \otimes \mathcal{L}^{-1}) \ldots \end{aligned} \tag{9.6}$$

is exact.

Let

$$\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}} : H^0(Y, \mathcal{G}) \otimes H^0(Y, \mathcal{G} \otimes \mathcal{L}^{-1}) \to H^0(Y, \mathcal{G}^2 \otimes \mathcal{L}^{-1})$$

be the multiplication map as indicated in the following discussion.

We also have the following proposition:

Proposition 9.7 *(Duflot, [4]) There is a commutative diagram of exact sequences*

$$\begin{array}{ccccc} R(\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}) & \hookrightarrow & H^0(Y,\mathcal{G})\otimes H^0(Y,\mathcal{G}\otimes\mathcal{L}^{-1}) & \stackrel{\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}/2}{\longrightarrow} & H^0(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1}) \\ \downarrow \Phi_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}} & & \downarrow \Phi_{X,\pi*\mathcal{G}}|_{V_1} & & \downarrow r \\ H^0(Y,\Omega^1\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1}) & \hookrightarrow & H^0(Y,\Omega^1(\log D)\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1}) & \to & H^0(Y,\mathcal{O}_D\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1}). \end{array} \tag{9.8}$$

Moreover, if $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, and $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, then this is a commutative diagram of short exact sequences.

The proof of the proposition is in Duflot [4].

This also leads to the following corollary:

Corollary 9.9 *(Duflot [4]) If $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, and $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, then the snake lemma gives an exact sequence*

$$0\to ker\ \Phi_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}\to ker\ \Phi_{X,\pi*\mathcal{G}}|_{V_1}\to ker\ r\to$$

$$cok\ \Phi_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}\to cok\ \Phi_{X,\pi*\mathcal{G}}|_{V_1}\to cok\ r\to 0.$$

A further result is that $\ker r\cong H^0(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$ since 9.6 is exact. And

$$\text{cok } r \doteq H^0(Y,\mathcal{O}_D\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})/\text{im } r = H^0(Y,\mathcal{O}_D\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})/\ker\delta\cong\text{im }\delta\subseteq H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3}). \tag{9.10}$$

Note that $\text{im }\delta = H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$ if $H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$.

Thus

Corollary 9.11 *If $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, and $H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, then coker $r=H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$.*

PROOF: Let $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ be surjective, $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$. Since $H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, then im $\delta=H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$, and equality follows directly from 9.10 above.

From the above corollary and Corollary 9.9, we now have have

$$\ldots\to cok\ \Phi_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}\to cok\ \Phi_{X,\pi*\mathcal{G}}|_{V_1}\to H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})\to 0. \tag{9.12}$$

This leads us to a additional corollary:

Corollary 9.13 *If $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, $H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, and $\Phi_{Y,\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, then cok $\Phi_{X,\pi*\mathcal{G}}|_{V_1}\cong H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$.*

PROOF: From the sequence 9.12 and the given suppositions, we have a short exact sequence. The surjectivity of $\Phi_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ in the sequence yields *cok* $\Phi_{X,\pi*\mathcal{G}}|_{V_1}\cong H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3})$.

Then:

Corollary 9.14 *If $\mu_{\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ is surjective, $H^1(Y,\Omega^1_Y\otimes\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, $H^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-1})=0$, and $\Phi_{Y,\mathcal{G},\mathcal{G}\otimes\mathcal{L}^{-1}}$ and $\Phi_{Y,\mathcal{G}}$ are surjective, then*

$$corank\ \Phi_{X,\pi^*\mathcal{G}}=h^1(Y,\mathcal{G}^2\otimes\mathcal{L}^{-3}).$$

Chapter 10

Gaussian Maps for the Canonical Divisor of a Double Cover

In this chapter, we will expand our discussion of Gaussian maps for double covers to look specifically at the canonical divisor K_X, paralleling some of the theorems in chapter 9. We will also discuss the canonical divisor with our two specific surfaces, smooth toric surfaces and Hirzebruch surfaces. The theorems and corollaries in this chapter are mostly original, except as cited.

10.1 General Discussion

Again consider smooth projective surfaces X and Y with X a double cover of Y of degree 2 with smooth branch locus D, and the covering map $\pi : X \to Y$; the line bundle $\mathcal{L} \in Pic(Y)$ is such that $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes 2}$. Recall the Diagram 9.2. We know $\pi^* G = K_X$, if $G = K_Y + L$.

This yields the diagram of identifications:

$$
\begin{array}{ccc}
\Lambda^2 H^0(X, \mathcal{O}(K_X)) & \overset{\Phi_{X,K_X}}{\longrightarrow} & H^0(X, (\mathcal{O}(K_X))^2 \otimes \Omega^1_X) \\
\downarrow\cong & & \downarrow\cong \\
\Lambda^2 H^0(Y, \mathcal{O}(K_Y + L) \otimes \pi_* \mathcal{O}_X) & & H^0(Y, \mathcal{O}(2K_Y + 2L) \otimes \pi_* \Omega^1_X) \\
\downarrow\cong & & \downarrow\cong \\
\Lambda^2 H^0(Y, \mathcal{O}(K_Y + L)) \oplus \Lambda^2 H^0(Y, \mathcal{O}(K_Y)) & & \\
\oplus (H^0(Y, \mathcal{O}(K_Y + L)) \otimes H^0(Y, \mathcal{O}(K_Y))) & \overset{\Phi_{X,K_X}}{\rightarrow} & H^0(Y, \mathcal{O}(2K_Y + 2L) \otimes \Omega^1_Y) \\
 & & \oplus H^0(Y, \mathcal{O}(2K_Y + L) \otimes \Omega^1_Y(\log D)).
\end{array}
\tag{10.1}
$$

Again, for ease of reference, we will use

$$V_0 = \Lambda^2 H^0(Y, \mathcal{O}(K_Y + L)),$$

$$W_0 = \Lambda^2 H^0(Y, \mathcal{O}(K_Y)),$$

$$V_1 = H^0(Y, \mathcal{O}(K_Y + L)) \otimes H^0(Y, \mathcal{O}(K_Y)),$$

$$A_0 = H^0(Y, \Omega^1_Y(2K_Y + 2L)),$$

$$A_1 = H^0(Y, \mathcal{O}((2K_Y + L) \otimes \Omega^1_Y(\log D))).$$

From Theorem 9.3, we have that, given the Gaussian map Φ_{X,K_X} then we have the following maps:

$$\Phi_{X,K_X}|_{V_0} : V_0 \to A_0$$

$$\Phi_{X,K_X}|_{V_1} : V_1 \to A_1$$

$$\Phi_{X,K_X}|_{W_0} : W_0 \to A_0,$$

and

$$\Phi_{X,K_X}|_{V_0} = \Phi_{Y,K_Y+L}.$$

This leads directly to the following theorem:

Theorem 10.2 *Given smooth surfaces X and Y with X a double cover of Y with smooth branch locus D, and the covering map $\pi : X \to Y$; the line bundle $L \in Pic(Y)$ is such that $\mathcal{O}_Y(D) = 2L$. If $H^0(Y, K_Y) = p_g(Y) = 0$, then*

$$corank\ \Phi_{X,K_X} = corank\ \Phi_{Y,K_Y+L} + h^0(Y, \Omega^1(\log D) \otimes \mathcal{O}(2K_Y + L)).$$

PROOF: Since $H^0(Y, K_Y) = p_g(Y) = 0$, $W_0 = 0$ and $V_1 = 0$. Thus $A_1/\text{im}\Phi_{X,K_X}|_{V_1} = A_1$ and $A_0/\Phi_{X,K_X}|_{W_0} = A_0$.

10.2 Gaussian Map for Double Covers of General Toric Surfaces

Let S be a smooth toric surface, defined by a fan of $n+2$ vectors as usual. Let $\pi : X \to S$ be a double cover of S branched along a smooth curve D, such that $D \sim 2\alpha_1 + \ldots + 2\alpha_n C_n$, α_i are constant, and $\alpha_i \geq 0, i = 1, \ldots, n$. The line bundle L satisfies $2L = D$. Hence, $L \sim \alpha_1 + \ldots + \alpha_n$.

Since $p_g(S) = 0$, we have the following corollary to Theorem 10.2:

Corollary 10.3 *If S is a smooth toric surface, $\pi : X \to S$ be a double cover of S branched along a smooth curve D, with the line bundle L such that $2L = D$, then*

$$corank\ \Phi_{X,K_X} = corank\ \Phi_{S,K_S+L} + h^0(S, \Omega^1(\log D) \otimes \mathcal{O}_S(2K_S + L)).$$

From Murray [15], we have

Theorem 10.4 *If S is a smooth toric surface and $K_S + L$ is ample, then*

$$corank\ \Phi_{S,K_S+L} = 0.$$

Combining this theorem with the previous corollary yields:

Corollary 10.5 *If S is a smooth toric surface, $\pi : X \to S$ is a double cover of S branched along a smooth curve D, with the line bundle L such that $2L = D$, and $K_S + L$ is ample, then*

$$corank\ \Phi_{X,K_X} = h^0(S, \Omega^1(\log D) \otimes \mathcal{O}_S(2K_S + L)).$$

Now we compute $h^0(S, \Omega^1(\log\ D) \otimes \mathcal{O}_S(2K_S + L))$ using Corollary 4.5 results. Applying Theorem 4.32, with $E = 2K_S + L, D = 2L$ yields:

Theorem 10.6 *If S a smooth toric surface and L is as defined above, $h^1(S, \mathcal{O}_S(2K_S + L)) = 0$ and $h^1(S, \Omega^1_S(2K_S + L)) = 0$, then*

$$h^0(S, \Omega^1_S(\log\ 2L)(2K_S + L))$$

$$= h^0(S, \Omega^1_S(2K_S + L)) + h^0(S, \mathcal{O}_S(2K_S + L)) - \chi(2K_S - L) + h^0(S, \mathcal{O}_S(L - K_S)).$$

Combining Corollary 10.3 with Theorem 10.6 yields:

Theorem 10.7 *If S is a smooth toric surface, X, L are as defined above, $h^1(S, \mathcal{O}_S(2K_S + L)) = 0$, and $h^1(S, \Omega^1(2K_S + L)) = 0$, then*

$$corank\ \Phi_{X,K_X}$$

$$= corank\ \Phi_{S,K_S+L} + h^0(S, \Omega^1(2K_S + L)) + h^0(S, \mathcal{O}_S(2K_S + L))$$

$$-\chi(2K_S - L) + h^0(S, \mathcal{O}_S(L - K_S)).$$

Theorem 10.7 combined with Murray's theorem about surjectivity of the Gaussian map (Theorem 10.4) yields:

Theorem 10.8 *If* $h^1(S, \Omega^1(2K_S + L)) = 0$, $h^1(S, \mathcal{O}_S(2K_S + L)) = 0$, *and* $K_S + L$ *is ample, then*

$$corank\ \Phi_{X,K_X}$$

$$= h^0(S, \Omega^1_S(2K_S + L)) + h^0(S, \mathcal{O}_S(2K_S + L)) - \chi(2K_S - L) + h^0(S, \mathcal{O}_S(L - K_S)).$$

Combining Theorem 10.8 with Lemmas 4.12 and 4.13 gives:

Corollary 10.9 *If* $K_S + L$ *is ample and* $2K_S + L$ *is ample, then*

$$corank\ \Phi_{X,K_X}$$

$$= \chi(\mathcal{O}_S(2K_S + L)) + \chi(\Omega^1(S, 2K_S + L)) - \chi(2K_S - L) + h^0(S, \mathcal{O}_S(L - K_S)).$$

We will now use these to compute specific values for the corank. Since we have that $\chi(2K_S+L) = \frac{1}{2}(2K_S+L)(K_S+L)+1$ and $\chi(2K_S-L) = \frac{1}{2}(2K_S-L)(K_S-L)+1$from 3.10 and $\chi(\Omega^1(S, 2K_S + L)) = (2K_S + L)^2 - n$ from 4.13, expanding, we get that

$$\chi(\mathcal{O}_S(2K_S + L)) + \chi(\Omega^1(S, 2K_S + L)) - \chi(\mathcal{O}_S(2K_S - L))$$

$$= \frac{1}{2}(2K_S + L)(K_S + L) + 1 - \frac{1}{2}(2K_S - L)(K_S - L) - 1 + (2K_S + L)^2 - n$$

$$= \frac{1}{2}(2K_S^2 + 3K_S \cdot L + L^2) - \frac{1}{2}(2K_S^2 - 3K_S \cdot L + L^2) + (4K_S^2 + 4K_S \cdot L + L^2) - n$$

$$= 4K_S^2 + 7K_S \cdot L + L^2 - n$$

$$= 4(10 - n) + 7K_S \cdot L + L^2 - n$$

by Theorem 3.20

$$= 40 - 5n + 7K_S \cdot L + L^2.$$

Hence

Corollary 10.10 *If X, S, and L are as above, $K_S + L$ and $2K_S + L$ are ample, then*

$$\textit{corank } \Phi_{X,K_X} = 40 - 5n + 7K_S \cdot L + L^2 + h^0(S, \mathcal{O}_S(L - K_S)).$$

10.3 Gaussian Map for Double Covers of Hirzebruch Surfaces

Let $\pi : X \to \mathbb{F}_k$ be a double cover of $\mathbb{F}_k$ branched along a smooth curve D, such that $D \sim 2\alpha C_1 + 2\beta C_2$, $\alpha \geq 0, \beta \geq 0$. The divisor L satisfies $2L = D$. Hence, $\mathcal{L} \sim \alpha C_1 + \beta C_2$.

We will address specifically Corollary 10.10 for Hirzebruch surfaces:

Corollary 10.11 *If $X, \mathbb{F}_k$, and L are as described above, and $\alpha \geq 5$, $\beta + 2k \geq 5$ and $\beta + 2 - k \geq 0$, then*

$$\textit{corank } \Phi_{X,K_X} = 39 + \frac{11}{2} K_{\mathrm{F}_k} \cdot L + \frac{3}{2} L^2.$$

PROOF: We will use Corollary 10.10. To have $K_{\mathrm{F}_k} + L = (-2+\alpha)C_1 + (k-2+\beta)C_2$ and $2K_{\mathrm{F}_k} + L = (-4+\alpha)C_1 + (2k-4+\beta)C_2$ ample using Theorem 4.10, we need $\alpha \geq 5$ and $\beta + 2k \geq 5$, which we assume. Then Corollary 10.10 says

$$\text{corank } \Phi_{X,K_X} = 30 + 7K_{\mathrm{F}_k} \cdot L + L^2 + h^0(\mathbb{F}_k, \mathcal{O}_{\mathrm{F}_k}(L - K_{\mathrm{F}_k})).$$

We have that $L - K_{\mathrm{F}_k} = (\alpha + 2)C_1 + (\beta + 2 - k)C_2$. Using Lemma 4.5, since

$\alpha \geq 5, \alpha + 2 \geq 0$. By supposition, $\beta + 2 - k \geq 0$, therefore we can compute

$$h^0(\mathbb{F}_k, \mathcal{O}_{\mathbb{F}_k}(L - K_{\mathbb{F}_k})) = \chi(L - K) = \frac{1}{2}(L - K_{\mathbb{F}_k})(L - 2K_{\mathbb{F}_k}) + 1$$

$$= \frac{1}{2}(L^2 - 3L \cdot K_{\mathbb{F}_k} + 2K_{\mathbb{F}_k}^2) + 1.$$

Substituting this into the above yields:

$$\text{corank } \Phi_{X,K_X} = 30 + 7K_{\mathbb{F}_k} \cdot L + L^2 + \frac{1}{2}(L^2 - 3L \cdot K_{\mathbb{F}_k} + 2K_{\mathbb{F}_k}^2) + 1$$

$$= 30 + 7K_{\mathbb{F}_k} \cdot L + L^2 + \frac{1}{2}L^2 - \frac{3}{2}L \cdot K_{\mathbb{F}_k} + (10 - 2) + 1$$

by Theorem 3.20,

$$= 39 + \frac{11}{2}K_{\mathbb{F}_k} \cdot L + \frac{3}{2}L^2.$$

Alternately, we can write this as:

$$\text{corank } \Phi_{X,K_X} = 41 + (\frac{3k}{2}\alpha^2 + 3\alpha\beta - \frac{11}{2}\alpha k - 11\alpha - 11\beta).$$

Remark: even if $\beta + 2 - k < 0$, we can still compute $h^0(\mathbb{F}_k, \mathcal{O}_{\mathbb{F}_k}(L - K_{\mathbb{F}_k}))$ using Lemma 4.4, but we do not do this here.

Chapter 11

Corank of the Gaussian map for "Large" divisors

In this chapter, we consider only double covers of Hirzebruch surfaces. We will consider the Gaussian map for double covers of Hirzebruch surfaces and the curves on double covers, specifically addressing the requirements for surjectivity as well as computing corank of the Gaussian map. The theorems and lemmas in this chapter are original work, though of course based on the earlier work of Duflot/Miranda [5] and Duflot [4].

Let $\pi : X \to \mathbb{F}_k$ be a double cover of $\mathbb{F}_k$ branched along a smooth curve D, such that $D \sim 2\alpha C_1 + 2\beta C_2$, $\alpha, \beta \geq 0$. The line bundle $\mathcal{L}$ satisfies $\mathcal{L}^{\otimes 2} = D$. Hence, $\mathcal{L} \sim \alpha C_1 + \beta C_2$.

11.1 Achieving a Range of Coranks

Consider a general divisor $\mathcal{G} \sim M_1 C_1 + M_2 C_2$ on $\mathbb{F}_k$.

Lemma 11.1 *In the above situation, if $M_1 > \alpha$ and $M_2 > \beta$ (i.e., $\mathcal{G}$ is "large" compared to $\mathcal{L}$), then:*

1. *$\Phi_{X,\pi^*\mathcal{G}}|_{V_0} = \Phi_{\mathrm{F}_k,\mathcal{G}} : V_0 \to A_0$ is surjective. Thus, corank $\Phi_{X,\pi^*\mathcal{G}}$ = corank*

$\Phi_{X,\pi^*\mathcal{G}}|_{V_1}$.

2. $\mu : H^0(M_1, M_2) \otimes H^0(M_1 - \alpha, M_2 - \beta) \to H^0(2M_1 - \alpha, 2M_2 - \beta)$ *is surjective.*

3. $H^1(\mathbb{F}_k, \Omega^1_X(2M_1 - \alpha, 2M_2 - \beta)) = 0$.

4. $H^1(\mathbb{F}_k, \mathcal{O}_{\mathbb{F}_k}(2M_1 - \alpha, 2M_2 - \beta)) = 0$.

5. $\Phi_{\mathbb{F}_k,\mathcal{G},\mathcal{G}-\mathcal{L}}$ *is surjective.*

PROOF: Item 1 follows from Proposition 8.1, since $\Phi_{X,\pi^*\mathcal{G}}|_{V_0} = \Phi_{\mathbb{F}_k,M_1C_1+M_2C_2}$ and $M_1 > \alpha \geq 0, M_2 > \beta \geq 0$, hence $M_1 \geq 1, M_2 \geq 1$. Item 2 follows from Proposition 5.2 since $M_1 > \alpha \geq 0, M_2 > \beta \geq 0$, therefore $M_1 - \alpha \geq 0, M_2 - \beta \geq 0$. Item 3 follows from Lemma 4.18 since $2M_1 - \alpha = M_1 + M_1 - \alpha > 0$ and $2M_2 - \beta = M_2 + M_2 - \beta > 0$. Item 4 follows from Lemma 4.16, since $2M_1 - \alpha = M_1 + M_1 - \alpha > 0$ and $2M_2 - \beta = M_2 + M_2 - \beta > 0$. Finally, item 5 follows from Proposition 8.2 since $M_1, M_2 \geq 1$, $M_1 - \alpha > 0$, and $M_2 - \beta > 0$.

Combining Lemma 11.1 with Corollary 9.14 yields:

Theorem 11.2 *If $X, \mathcal{L}$, and $\mathcal{G}$ are as above, and $M_1 - \alpha > 0$ and $M_2 - \beta > 0$, then*

$$\textit{corank}\ \Phi_{X,\pi^*(M_1,M_2)} = h^1(\mathbb{F}_k, \mathcal{O}(2M_1 - 3\alpha, 2M_2 - 3\beta)).$$

We would like to compute the corank of the above Gaussian map more precisely.

Corollary 11.3 *If $X, \mathcal{L}$, and $\mathcal{G}$ are as above, and $\frac{2M_1}{3} \geq \alpha$ and $\frac{2M_2}{3} \geq \beta$, then*

$$\textit{corank}\ \Phi_{X,\pi^*(M_1,M_2)} = 0.$$

PROOF: Since $M_1 > \frac{2M_1}{3} \geq \alpha$ and $M_2 > \frac{2M_2}{3} \geq \beta$, then by Theorem 11.2

$$\text{corank}\ \Phi_{X,\pi^*(M_1,M_2)} = h^1(\mathbb{F}_k, \mathcal{O}(2M_1 - 3\alpha, 2M_2 - 3\beta))$$

and $h^1(\mathbb{F}_k, \mathcal{O}(2M_1 - 3\alpha, 2M_2 - 3\beta)) = 0$ by Lemma 4.16.

On the other hand, the corank above is not always zero. For example, suppose that X is a double cover of $\mathbb{F}_3$, branched along a smooth curve D linearly equivalent to $14C_1 + 2C_2$; thus the line bundle $\mathcal{L}$ is equal to $7C_1 + C_2$. Now, suppose that $\mathcal{G} \sim 8C_1 + 3C_2$. Then, the conditions of Theorem 11.2 are satisfied.

For simplicity, we will write $h^i(\mathbb{F}_3, \mathcal{O}(aC_1 + bC_2))$ as $h^i(a, b)$ from now on. We compute $h^1(-5, 3)$ as follows:

We know that

$$h^1(-5,3) = -\chi(-5,3) + h^0(-5,3) + h^2(-5,3) = -\chi(-5,3) + 0 + h^0(3,-2),$$

using Kodaira-Serre duality (Theorem 3.17, note $K_{\mathbb{F}_3} = -2C_1 + C_2$) and Lemma 4.4, part 1. Now,

$$\chi(-5,3) = \frac{(-5,3)(-3,2)}{2} + 1 = \frac{45 - 9 - 10}{2} + 1 = 14,$$

and using Lemma 4.4, part 2, we compute

$$h^0(3,-2) = 15.$$

Thus,

$$\text{corank } \Phi_{X,\pi^*(8,3)} = 1.$$

11.2 Curves on Double Covers of Hirzebruch Surfaces

Recall that $\pi : X \to \mathbb{F}_k$ is a double cover of $\mathbb{F}_k$ branched along a smooth curve D, $D \cong 2\alpha C_1 + 2\beta C_2$, with $\alpha, \beta \geq 0$. The line bundle $\mathcal{L}$ satisfies $\mathcal{L}^{\otimes 2} = D$. Hence, $\mathcal{L} \sim \alpha C_1 + \beta C_2$. Let G be a smooth curve on $\mathbb{F}_k$ with $\mathcal{G} \sim m_1 C_1 + m_2 C_2$ and $m_1, m_2 \geq 0$. Assume that $\pi^*\mathcal{G}$ is a smooth curve on X, and let $E = \pi^*\mathcal{G} = \pi^*(m_1 C_1 + m_2 C_2)$.

We refer again to the basic diagram from Chapter 7, 7.6:

$$\begin{array}{ccc} \Lambda^2(H^0(X, \mathcal{O}_X(K_X + E)) & \overset{\Phi_{X,K_X+E}}{\longrightarrow} & H^0(X, \Omega^1(2K_X + 2E)) \\ & & \downarrow a \\ \downarrow res & & H^0(E, \Omega^1_X(E, (2K_X + 2E)\mid_E)) \\ & & \downarrow b \\ \Lambda^2 H^0(E, \Omega^1_E) & \overset{\Phi_{E,K_E}}{\longrightarrow} & H^0(E, (\Omega^1_E)^{\otimes 3}) \end{array} \qquad (11.4)$$

Now, $K_X + E = \pi^*(K_{\mathbb{F}_k} + L + E) = \pi^*(m_1 + \alpha - 2, m_2 + \beta + k - 2)$. Applying 11.3 to $K_X + E = \pi^*(K_{\mathbb{F}_k} + L + E) = \pi^*(m_1 + \alpha - 2, m_2 + \beta + k - 2)$ yields:

Lemma 11.5 *If* $2m_1 \geq \alpha + 4$ *and* $2m_2 \geq \beta + 4 - 2k$, *then* Φ_{X,K_X+E} *is surjective.*

Next, note that Theorems 6.8 and 6.7 say that

$$H^1(X, \Omega^1_X(2K_X + E)) \cong H^1(\mathbb{F}_k, \Omega^1(m_1 + 2\alpha - 4, m_2 + 2\beta + 2k - 4))$$

$$\oplus H^1(\mathbb{F}_k, \Omega^1(\log D) \otimes \mathcal{O}(m_1 + \alpha - 4, m_2 + \beta + 2k - 4))$$

and

$$H^1(X, \mathcal{O}_X(2K_X + E)) \cong H^1(\mathbb{F}_k, \mathcal{O}(m_1 + 2\alpha - 4, m_2 + 2\beta + 2k - 4))$$

$$\oplus H^1(\mathbb{F}_k, \mathcal{O}(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)).$$

Also, the long exact sequences 4.28 and 4.29 say that

$$H^1(\mathbb{F}_k, \Omega^1(\log D) \otimes \mathcal{O}(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)) = 0$$

if

$$H^1(\mathbb{F}_k, \Omega^1(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)) = 0,$$

$$H^1(\mathbb{F}_k, \mathcal{O}(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)) = 0$$

and

$$H^2(\mathbb{F}_k, \mathcal{O}(m_1 - \alpha - 4, m_2 - \beta + 2k - 4)) = 0.$$

Lemma 11.6 *If $m_1 + \alpha - 4 \geq 1$ and $m_2 + \beta + 2k - 4 \geq 1$, and $m_1 - \alpha - 4 \geq -1$ or $m_2 - \beta + 2k - 4 + k(m_1 - \alpha - 4) \geq -k - 1$, then*

$$H^1(X, \Omega^1_X(2K_X + E)) = 0$$

and

$$H^1(X, \mathcal{O}_X(2K_X + E)) = 0.$$

PROOF: The first two inequalities $m_1 + \alpha - 4 \geq 1$ and $m_2 + \beta + 2k - 4 \geq 1$ imply, using Lemmas 4.16, 4.18 and Corollary 3.18, that

$$H^1(\mathbb{F}_k, \Omega^1(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)) = 0$$

$$= H^1(\mathbb{F}_k, \Omega^1(2m_1 + \alpha - 4, 2m_2 + \beta + 2k - 4))$$

and

$$H^1(\mathbb{F}_k, \mathcal{O}(m_1 + 2\alpha - 4, m_2 + 2\beta + 2k - 4)) = 0$$

$$= H^1(\mathbb{F}_k, \mathcal{O}(m_1 + \alpha - 4, m_2 + \beta + 2k - 4)).$$

The second two inequalities, $m_1 - \alpha - 4 \geq -1$ or $m_2 - \beta + 2k - 4 + k(m_1 - \alpha - 4) \geq -k - 1$, imply, using Corollary 4.14, that

$$H^2(\mathbb{F}_k, \mathcal{O}(m_1 - \alpha - 4, m_2 - \beta + 2k - 4)) = 0.$$

Now, $-K_X = \pi^*(2 - \alpha, 2 - k - \beta)$. Also by Theorem 6.7,

$$H^0(X, \mathcal{O}(-K_X)) = H^0(\mathbb{F}_k, \mathcal{O}(2 - \alpha, 2 - k - \beta)) \oplus H^0(\mathbb{F}_k, \mathcal{O}(2 - 2\alpha, 2 - k - 2\beta)).$$

This leads us to our next lemma:

Lemma 11.7 *If $\alpha > 2$ or $\beta + k\alpha > k + 2$, then*

$$H^0(X, \mathcal{O}(-K_X)) = 0.$$

PROOF: If $2 - \alpha < 0$ or $2 - k - \beta + k(2 - \alpha) < 0$, then $2 - 2\alpha < 0$ or $(2 - k - 2\beta + k(2 - 2\alpha) < 0$. Thus, Lemma 4.4 tells us that

$$H^0(\mathbb{F}_k, \mathcal{O}(2 - \alpha, 2 - k - \beta)) \oplus H^0(\mathbb{F}_k, \mathcal{O}(2 - 2\alpha, 2 - k - 2\beta)) = 0.$$

Hence, if $2 - \alpha < 0$ or $(2 - k - \beta) + k(2 - \alpha) < 0$, then $H^0(X, \mathcal{O}(-K_X)) = 0$.

Finally, the above Lemmas 11.5, 11.6, and 11.7 together with Lemma 7.16 yield:

Theorem 11.8 *If*

$$2m_1 \geq \alpha + 4 \text{ and } 2m_2 \geq \beta + 4 - 2k$$

AND

$$m_1 \geq 5 - \alpha \textit{ and } m_2 \geq 5 - \beta - 2k$$

AND

$$m_1 \geq \alpha + 3 \textit{ or } m_2 + km_1 \geq \beta + k\alpha + k + 3,$$

AND

$$\alpha > 2 \textit{ or } \beta + k\alpha > k + 2,$$

then Φ_{E,K_E} *is surjective.*

We will now compute the genus of E, using the formula from equation 3.10:

$$g(E) = \frac{E \cdot E + E \cdot K_X}{2} + 1 = \frac{\pi^*G \cdot \pi^*G + \pi^*G \cdot \pi^*(K_{\mathbf{F}_k} + L)}{2} + 1$$

$$= G \cdot G + G \cdot K_{\mathbf{F}_k} + G \cdot L + 1$$

$$= 2g(G) - 2 + G \cdot L + 1$$

$$= 2g(G) + G \cdot L - 1.$$

One can use the inequalities of Theorem 11.8 and the genus formulas above to get curves of many large genera with surjective Gaussian maps on double covers of Hirzebruch surfaces. Note that the $\alpha > 2$ inequality means the double covers are all of general type.

Chapter 12

Conclusion

Following in the footsteps of Wahl, Ciliberto, Harris, Miranda, Duflot, and Murray, we have explored the Gaussian map on smooth toric surfaces and their double covers. Our new results concern Gaussian maps on double covers of smooth toric surfaces, and more specifically, Hirzebruch surfaces. Duflot's analysis for Gaussian maps of double covers in general showed the need to study multiplication maps and cohomology of sheaves of logarithmic differentials and we needed to do some computations of these latter cohomology groups (Chapter 4). Necessary for our analysis of Gaussian maps, we have shown surjectivity for various multiplication maps on Hirzebruch surfaces (Chapter 5). We have used the results of Chapter 5 to prove surjectivity of many Gaussian maps on Hirzebruch surfaces, more general than those considered in Duflot/Miranda (Chapter 8). Finally, we have gone on to consider further the Gaussian map for double covers of smooth toric surfaces and, again, more specifically, double covers of Hirzebruch surfaces in Chapters 10 and 11.

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