

EXTENDED GERGONNE SYLLOGISMS

ABSTRACT. Syllogisms with or without negative terms are studied by using Gergonne's ideas. Soundness, completeness, and decidability results are given.

1. BACKGROUND AND MOTIVATION

Gergonne [2] relates the familiar A, E, I, and O sentences without negative terms to five basic sentences that express the "Gergonne relations." These relations are: exclusion, identity, overlap, proper containment, and proper inclusion. What makes these relations especially interesting is that for any pair of non-empty class terms exactly one of them holds.

Faris [1] develops a formal system that takes the Gergonne relations as basic. His system takes advantage of Łukasiewicz's [4], which attempts to formalize the Aristotelian syllogistic. The following paper results from two ideas: 1) If Gergonne had been interested in studying A, E, I, and O sentences *with* negative terms, the count of Gergonne relations would be seven rather than five; and 2) The most Aristotelian way to develop a syllogistic system based on these seven relations is by following Smiley's [5] rather than Łukasiewicz's [4].

After developing the Aristotelian "full syllogistic" based on seven relations, we will discuss a subsystem that is adequate for representing AEIO-syllogisms with or without negative terms.

2. THE SYSTEM

Sentences are defined by referring to:

terms: A, B, C, ...

simple quantifiers: =, =⁻, C⁺⁺, C^{+ -}, C^{- +}, C⁻⁻, Z

comma: ,

Q_1, \dots, Q_n is a *quantifier* provided i) each Q_i ($1 \leq i \leq n$) is a simple quantifier, ii) Q_i precedes Q_j if $i < j$, where precedence among simple

quantifiers is indicated by the above ordering of simple quantifiers, and iii) at least one quantifier is not a Q_i . No expressions are *quantifiers* other than those generated by the above three conditions. So, for example, $=, \subset^{++}$ is a quantifier but $\subset^{++}, =$ is not. Qab is a *sentence* iff Q is a quantifier and a and b are distinct terms. So, for example, $=, \subset^{++}AB$ and $=^-, \subset^{--}, ZAB$ are sentences, but $=, \subset^{++}AA$ is not. Qab is a *simple sentence* iff Qab is a sentence and Q is a simple quantifier. Read simple sentences as follows: $=ab$ as "The a are the b ," $=^-ab$ as "The a are the non- b ," $\subset^{++}ab$ as "The a are properly included in the b ," $\subset^{+-}ab$ as "The a are properly included in the non- b ," $\subset^{-+}ab$ as "The non- a are properly included in the b ," $\subset^{--}ab$ as "The non- a are properly included in the non- b ," and Zab as "Some a are b , some a are non- b , some non- a are b , and some non- a are non- b ." Read Q_1, \dots, Q_nab by putting "or" between sentences that correspond to Q_iab . So, read $=, \subset^{++}ab$ as "The a are the b , or the a are properly included in the b " (or "All a are b .") $=^-, \subset^{+-}, \subset^{-+}, \subset^{--}, Zab$ may be read as "Some a are not b ."

The deducibility relation (\vdash), relating sets of sentences to sentences, is defined recursively. Read " $X \vdash y$ " as " y is deducible from X ." Set brackets are omitted in the statement of the following definition. " X, y " is short for " $X \cup \{y\}$ " and " x, y " is short for " $\{x\} \cup \{y\}$." " a ", " b ", ... range over terms; and " p ", " q ", ... range over "+", and "-". p^* is "+" iff p is "-". $cd(Pab) = Qab$ iff every quantifier that does not occur in P occurs in Q . Read " cd " as "the contradictory of."

$$(B1) \quad =ab \vdash =ba$$

$$(B2) \quad =^-ab \vdash =^-ba$$

$$(B3) \quad \subset^{pq}ab \vdash \subset^{q^*p^*}ba$$

$$(B4) \quad Zab \vdash Zba$$

$$(B5) \quad =ab, Qbc \vdash Qac, \quad \text{where } Q \text{ is } =, =^-, \text{ or } \subset^{pq}$$

$$(B6) \quad =^-ab, =^-bc \vdash =ac$$

$$(B7) \quad =^-ab, \subset^{pq}bc \vdash \subset^{p^*q}ac$$

$$(B8) \quad \subset^{pq}ab, \subset^{qr}bc \vdash \subset^{pr}ac$$

$$(R1) \quad \text{If } X \vdash y \text{ and } y, z \vdash w \text{ then } X, z \vdash w$$

$$(R2) \quad \text{If } X, y \vdash Pab \text{ then } X, Qab \vdash cd(y) \text{ if no quantifier in } P \text{ is a quantifier in } Q$$

(R3) If $X, Pab \vdash y$ and $X, Qab \vdash y$ then $X, Rab \vdash y$ if each quantifier in R is in P or Q

(L1) $X \vdash y$ iff $X \vdash y$ in virtue of B1–R3.

So, for example, $=^-AB, C^{++}BC \vdash C^{-+}AC$ (by B7) and $C^{-+}AC, C^{+-}CD \vdash C^{--}AD$ (by B8). So $=^-AB, C^{++}BC, C^{+-}CD \vdash C^{--}AD$ (by R1). So $=^-AB, C^{++}BC, C^{+-}, ZAD \vdash =, =^-, C^{++}, C^{-+}, C^{--}, ZCD$ (by R2).

THEOREM 1. (D1) *If $X, y \vdash Pab$ then $X, y \vdash cd(Qab)$ if no simple quantifier occurs in both P and Q.* (D2) *If $X, y \vdash cd(Pab)$ and $X, y \vdash cd(Qab)$ then $X, y \vdash cd(Rab)$ if each quantifier in R is in P or Q.* (D3) *If $X, y \vdash z$ and $v, w \vdash y$ then $X, v, w \vdash z$.*

Proof. Begin each proof by assuming the antecedent. (D1) Then $X, Qab \vdash cd(y)$ (by R2). Then $X, y \vdash cd(Qab)$ (by R2). (D2) Then $X, Pab \vdash cd(y)$ and $X, Qab \vdash cd(y)$ (by R2). Then $X, Rab \vdash cd(y)$ (by R3). Then $X, y \vdash cd(Rab)$ (by R2). (D3) Then $X, cd(z) \vdash cd(y)$ and $v, cd(y) \vdash cd(w)$ (by R2). Then $X, v, cd(z) \vdash cd(w)$ (by R1). Then $X, v, w \vdash z$ (by R2). \square

A *model* is a quadruple $\langle W, \nu_+, \nu_-, \nu \rangle$, where i) W is a non-empty set, ii) ν_+ and ν_- are functions that assign non-empty subsets of W to terms such that $\nu_+(a) \cup \nu_-(a) = W$ and $\nu_+(a) \cap \nu_-(a) = \emptyset$, and iii) ν is a function that assigns t or f to sentences such that the following conditions are met:

- (i) $\nu(=ab) = t$ iff $\nu_+(a) = \nu_+(b)$
- (ii) $\nu(=^-ab) = t$ iff $\nu_+(a) = \nu_-(b)$
- (iii) $\nu(C^{pq}ab) = t$ iff $\nu_p(a) \subset \nu_q(b)$
- (iv) $\nu(Zab) = t$ iff $\nu_p(a) \cap \nu_q(b) \neq \emptyset$ for each p and q
- (v) $\nu(Q_1, \dots, Q_nab) = t$ iff for some i ($1 \leq i \leq n$) $\nu(Q_iab) = t$

y is a *semantic consequence* of X ($X \models y$) iff there is no model $\langle W, \dots, \nu \rangle$ such that ν assigns t to every member of X and ν assigns f to y . X is *consistent* iff there is a model $\langle W, \dots, \nu \rangle$ such that ν assigns t to every member of X . X is *inconsistent* iff X is not consistent.

THEOREM 2 (Soundness). *If $X \vdash y$ then $X \models y$.*

Proof. Straightforward. (For B1, note that for any model $\langle W, \dots, \nu \rangle$, if $\nu_+(a) = \nu_+(b)$ then $\nu_+(b) = \nu_+(a)$. For R2, suppose no quantifier in P is a quantifier in Q, and suppose that $X, Qab \not\models cd(y)$. Then there is a model $\langle W, \dots, \nu \rangle$ in which ν assigns t to every member of X, $\nu(Qab) = t$, and $\nu(cd(y)) = f$. Note that $\nu(cd(y)) = f$ iff $\nu(y) = t$. And note that since no quantifier in P is a quantifier in Q, $\nu(Pab) = f$. So $X, y \not\models Pab$.) \square

A *chain* is a set of sentences whose members can be arranged as a sequence $\langle Q_1[a_1a_2], Q_2[a_2a_3], \dots, Q_n[a_na_1] \rangle$, where $Q_i[a_ia_j]$ is either $Q_ia_ia_j$ or $Q_ia_ja_i$ and where $a_i \neq a_j$ if $i \neq j$. So, for example, $\{=AB, =^-, \subset^{++}CB, ZCA\}$ is a chain. A pair $\langle X, y \rangle$ is a *syillogism* iff $X \cup \{y\}$ is a chain. So $\{\{=AB, =^-, \subset^{++}CB\}, ZCA\}$ is a syllogism.

A *normal chain* is a set of sentences whose members can be arranged as a sequence $\langle Q_1a_1a_2, Q_2a_2a_3, \dots, Q_na_na_1 \rangle$, where $a_i \neq a_j$ if $i \neq j$. A *simple normal chain* is a normal chain in which each quantifier is simple. So, for example, $\{=, =^-AB, =BA\}$ is a normal chain. And $\{=AB, =BA\}$ is a simple normal chain.

By definition, $e(=ab)$ is $=ba$, $e(=^-ab)$ is $=^-ba$, $e(\subset^{pq}ab)$ is $\subset^{q^*p^*}ba$, and $e(Zab)$ is Zba .

$\{Q_1ab, Q_2bc\}$ *a-reduces to* Q_3ac iff the triple $\langle Q_1ab, Q_2bc, Q_3ac \rangle$ is recorded on the following Table of Reductions:

		Q_2bc			
		=	=^-	\subset^{qr}	
Q_1ab	=	=	=^-	\subset^{qr}	Q_3ac
	=^-	=^-	=	\subset^{q^*r}	
	\subset^{pq}	\subset^{pq}	\subset^{pq^*}	\subset^{pr}	

So, for example, $\{=AB, =BC\}$ a-reduces to $=AC$, and $\{\subset^{++}AB, \subset^{+-}BC\}$ a-reduces to $\subset^{+-}AC$.

If X_1 is a simple chain then a sequence of chains $X_1, \dots, X_m (=Y_1), \dots, Y_n$ is a *full reduction of X_1 to Y_n* iff: i) X_m is a normal chain and if $m > 1$ then, for $1 \leq i < m$, if X_i has form $\{Qab\} \cup Z$ then X_{i+1} has form $\{e(Qab)\} \cup Z$, and ii) there is no pair in Y_n that a-reduces to a sentence and if $n > 1$ then, for $1 \leq i < n$, if Y_i has form $\{Q_1ab, Q_2bc\} \cup Z$ then Y_{i+1} has form $\{Q_3ac\} \cup Z$. X *fully reduces to* Y iff there is a full reduction of X to Y .

THEOREM 3. *Every simple chain fully reduces to a simple normal chain.*

Proof. Assume X_1 is a simple chain. We construct a sequence of chains that is a full reduction of X_1 to Y_n . Step 1: If X_1 is a simple

normal chain let $X_1 = Y_1$ and go to Step 2. If X_1 is not a simple normal chain find the alphabetically first pair of sentences in X_1 of form $\langle Qab, Qcb \rangle$ and replace Qcb with $e(Qcb)$, forming X_2 . Repeat Step 1 (with " X_j " in place of " X_1 "). Step 2: If no pair of sentences in Y_1 a-reduces to a sentence, then X_1 fully reduces to Y_1 . If a pair of sentences in Y_1 a-reduces to a sentence x find the alphabetically first pair that a-reduces to x and form Y_2 by replacing this pair with x . Repeat Step 2 (with " Y_j " in place of " Y_1 "). \square

So, for example, given the sequence $\langle \{=AB\}, \{=AB, =BA\} \rangle$, $\{=AB\}$ fully reduces to $\{=AB, =BA\}$. And, given the sequence $\langle \{C^{++}AB, C^{--}CB, C^{++}CA\}, \{C^{++}AB, C^{++}BC, C^{++}CA\}, \{C^{++}AC, C^{++}CA\} \rangle$, $\{C^{++}AB, C^{--}CB, C^{++}CA\}$ fully reduces to $\{C^{++}AC, C^{++}CA\}$. Some chains fully reduce to themselves. $\{C^{++}AB, C^{--}BC, ZCA\}$ is an example.

$\{P_1[a_1a_2], \dots, P_n[a_na_1]\}$ is a strand of $\{Q_1[a_1a_2], \dots, Q_n[a_na_1]\}$ iff each P_i is a simple quantifier in Q_i and a_i is the first term in $P_i[a_ia_{i+1}]$ iff a_i is the first term in $Q_i[a_ia_{i+1}]$, where $P[ab]$ is Pab or Pba . So, for example, $\{=AB, =^-AB\}$ is a strand of $\{=, C^{++}AB, =^-, C^{++}AB\}$.

A simple normal chain is a *cd-pair* iff it has one of the following forms:

$$\{=ab, =^-ba \text{ (or } C^{pq}ba \text{ or } Zba)\}, \{=^-ab, C^{pq}ba \text{ (or } Zba)\},$$

$$\text{or } \{C^{pq}ab, C^{qr}ba \text{ (or } Zba)\}.$$

THEOREM 4 (Syntactic decision procedure). *If $\langle X, y \rangle$ is a syllogism then $X \vDash y$ iff every strand of $X \cup \{cd(y)\}$ fully reduces to a cd-pair.*

Proof. Assume $\langle X, y \rangle$ is a syllogism. We use Lemmas 1–3, below. (If) Suppose every strand of X , $cd(y)$ fully reduces to a cd-pair. Then by Lemmas 1 and 2, $X, cd(y)$ is inconsistent. Then $X \vDash y$. (Only if) Suppose some strand of X , $cd(y)$ does not fully reduce to a cd-pair. Then, by Theorem 3, some strand of X , $cd(y)$ fully reduces to a simple normal chain that is not a cd-pair. Then, by Lemmas 1 and 3, $X, cd(y)$ is consistent. Then $X \not\vDash y$. \square

LEMMA 1. *A chain is inconsistent iff each of its strands is inconsistent.*

Proof. Note that a model satisfies $\{Q_1ab\} \cup X$ and $\{Q_2ab\} \cup X$ iff it satisfies $\{Q_3ab\} \cup X$, where the quantifiers in Q_3 are the quantifiers in Q_1 and Q_2 . \square

LEMMA 2. *If a simple chain X fully reduces to a cd-pair, then X is inconsistent.*

Proof. Use the following three lemmas, whose proofs will be omitted since they are easily given. \square

LEMMA 2.1. *Each cd-pair is inconsistent.*

LEMMA 2.2. *If a simple normal chain $\{Q_3ac\} \cup X$ is inconsistent and $\{Q_1ab, Q_2bc\}$ a-reduces to Q_3ac , then $\{Q_1ab, Q_2bc\} \cup X$ is inconsistent.*

LEMMA 2.3. *If a simple chain $\{Qab\} \cup X$ is inconsistent, then $\{e(Qab)\} \cup X$ is inconsistent.*

LEMMA 3. *If a simple chain X fully reduces to a simple normal chain that is not a cd-pair, then X is satisfied in an m -model, where $m \leq n + 2$ and n is the number of terms in X .*

Proof. Use the following three lemmas. \square

LEMMA 3.1. *If a simple chain fully reduces to a simple normal chain X that is not a cd-pair, then X is satisfied in an m -model, where $m \leq n + 2$ and n is the number of terms in X .*

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of “Z” in X .

Case 1: “Z” does not occur in X . If either “=” or “=−” occurs in X then X has form $\{=ab, =ba\}$ or $\{=^-ab, =^-ba\}$. Use $\langle\{1, 2\}, \dots, \nu\rangle$, where, for each term x , $\nu_+(x) = \{1\}$. If neither “=” or “=−” occurs in X then X has form $\{C^{p_1p_2}a_1a_2, \dots, C^{p_{2i-1}p_{2i}}a_i a_{i+1}, \dots, C^{p_{2n-1}p_{2n}}a_n a_1\}$, where $p_{2i} = p_{2i+1}^*$, for $1 \leq i < n$, and $p_{2n} = p_1^*$. We use induction on the number n of terms in X to show that X is satisfied in a 3-model. Basis step: $n = 2$. X has form $\{C^{p_1p_2}a_1a_2, C^{p_2^*p_1^*}a_2a_1\}$. Use $\langle\{1, 2, 3\}, \dots, \nu\rangle$, where $\nu_{p_1}(a) = \{1\}$, and, for terms x other than a , $\nu_q(x) = \{1, 2\}$. Induction step: $n > 2$. By the induction hypothesis $\{C^{p_1p_4}a_1a_3, \dots, C^{p_{2i-1}p_{2i}}a_i a_{i+1}, \dots, C^{p_{2n-1}p_{2n}}a_n a_1\}$ is satisfied in a 3-model $\langle W, \dots, \nu\rangle$, where $p_{2i} = p_{2i+1}^*$, for $2 \leq i < n$, and $p_{2n} = p_1^*$. Construct a model $\langle W, \dots, \nu'\rangle$, $\nu'_{p_2}(a_2) = \nu_{p_1}(a_1) \cup \nu_{p_4^*}(a_3)$, and, for other terms x , $\nu'_+(x) = \nu_+(x)$. Then $\nu'(C^{p_1p_2}a_1a_2) = t$. $\nu'_{p_2^*}(a_2) = \nu_{p_4}(a_3) - \nu_{p_1}(a_1)$ and $p_2^* = p_3$. So $\nu'(C^{p_3p_4}a_2a_3) = t$.

Case 2: “Z” occurs exactly once in X . Then X has at least three members and has form $\{Zab\} \cup \{C^{pq}bc, \dots, C^{rs}da\}$. We use induction on the number of terms in X to show that X is satisfied in a 4-model. Basis step: $n = 3$. X has form $\{Zab\} \cup \{C^{pq}bc, C^{q^*r}ca\}$. Construct a model $\langle\{1, 2, 3, 4\}, \dots, \nu\rangle$, where $\nu_r(a) = \{1, 2\}$, $\nu_p(b) = \{1, 3\}$, and, for other terms x , $\nu_q(x) = \{1, 3, 4\}$. Induction step: $n > 3$. Follow the model construction in the induction step in Case 1.

Case 3: "Z" occurs at least twice in X. Then X has form $\{Zab, \dots, Zcd, \dots\}$. We use induction on the number n of terms in X. Basis step: $n = 2$. X has form $\{Zab, Zba\}$. Use $\langle \{1, 2, 3, 4\}, \dots, \nu \rangle$, where $\nu_+(a) = \{1, 2\}$ and, for other terms x , $\nu_+(x) = \{1, 3\}$. Induction step: $n > 2$. X has form $\{Zab, Qbc, \dots, Zde, \dots\}$. By the induction hypothesis, $\{Zac, \dots, Zde\}$ is satisfied in an m -model, where $m \leq n + 2$ and n is the number of terms in X. Suppose Q is "=". Construct model $\langle W, \dots, \nu' \rangle$, where $\nu'_+(b) = \nu_+(c)$ and, for terms x other than b , $\nu'_+(x) = \nu_+(x)$. Suppose Q is "=-". Construct model $\langle W, \dots, \nu' \rangle$, where $\nu'_+(b) = \nu_-(c)$ and, for terms x other than b , $\nu'_+(x) = \nu_+(x)$. Suppose Q is "Z". Construct a model $\langle W, \dots, \nu' \rangle$, where $\nu'_+(b) = (\nu_+(a) \cap \nu_+(c)) \cup (\nu_-(a) \cap \nu_-(c))$, and, for other terms x , $\nu'_+(x) = \nu_+(x)$. Finally, suppose that Q is " \subset^{pq} ". The strategy is to construct a model $\langle W', \dots, \nu' \rangle$ such that X is satisfied in it, where $W' = W \cup \{M\}$, and $\nu'_+(a) \cap \nu'_q(c)$ has at least two members, including M. Then we construct a second model $\langle W', \dots, \nu'' \rangle$, such that X is satisfied in it by letting $\nu''(b) = \nu'_q(c) - \{M\}$, and, for terms x other than b , $\nu''(x) = \nu'_+(x)$. Then $\nu''(Zab) = t$ and $\nu''(\subset^{pq}bc) = t$.

We construct $\langle W', \dots, \nu' \rangle$. If a and c are the only terms in X, let $\alpha = \nu_+(a) \cap \nu_q(c)$ (and, thus, α has at least one member). If terms d_1, \dots, d_n occur in X, where these terms are other than "a" or "c", pick $p_1 - p_n$ such that α has at least one member, where $\alpha = \nu_+(a) \cap \nu_q(c) \cap \nu_{p_1}(d_1) \cap \dots \cap \nu_{p_n}(d_n)$. Let $W' = W \cup \{M\}$, where $M \notin W$. Let $\nu'_+(x) = \nu_+(x) \cup \{M\}$ if $\alpha \subseteq \nu_+(x)$; otherwise, let $\nu'_+(x) = \nu_+(x)$. Then $\nu'_-(x) = \nu_-(x) \cup \{M\}$ if $\alpha \subseteq \nu_-(x)$; otherwise, $\nu'_-(x) = \nu_-(x)$. Note that $\nu'_+(a) \cap \nu'_q(c)$ has at least two members and $M \in \nu'_+(a) \cap \nu'_q(c)$. We show that X is satisfied in $\langle W', \dots, \nu' \rangle$. Suppose $\nu(Qde) = t$. Suppose Q is "=". Then $\nu'_+(d) = \nu_+(d) \cup \{M\}$ and $\nu'_+(e) = \nu_+(e) \cup \{M\}$ or $\nu'_+(d) = \nu_+(d)$ and $\nu'_+(e) = \nu_+(e)$. Then $\nu'(\subseteq de) = t$. Suppose Q is "=-". Then $\nu'_+(d) = \nu_+(d) \cup \{M\}$ and $\nu'_-(e) = \nu_-(e)$ or $\nu'_+(d) = \nu_+(d)$ and $\nu'_-(e) = \nu_-(e) \cup \{M\}$. Then $\nu'(\subseteq^- de) = t$. Suppose Q is " \subset^{pq} ". If $\alpha \subseteq \nu_p(d)$ then $\nu'_p(d) = \nu_p(d) \cup \{M\}$ and $\nu'_q(e) = \nu_q(e) \cup \{M\}$. If $\alpha \not\subseteq \nu_p(d)$ then $\nu'_p(d) = \nu_p(d)$ and either $\nu'_q(e) = \nu_q(e)$ or $\nu'_q(e) = \nu_q(e) \cup \{M\}$. Then $\nu'(\subset^{pq}de) = t$. Finally, suppose Q is "Z". Then, for any p and q , $\nu_p(d) \cap \nu_q(e) \subseteq \nu'_p(d) \cap \nu'_q(e)$. Then $\nu'(Zde) = t$. \square

LEMMA 3.2. *If a simple chain $\{Q_3ac\} \cup X$ is satisfied in an n -model $\langle W, \dots, \nu \rangle$, where n is the number of terms in $\{Q_3ac\} \cup X$, and if $\{Q_1ab, Q_2bc\}$ a -reduces to Q_3ac , then $\{Q_1ab, Q_2bc\} \cup X$ is satisfied in an m -model, where $m \leq n$ and n is the number of terms in $\{Q_1ab, Q_2bc\} \cup X$.*

Proof. Assume the antecedent. Suppose Q_1 is "=". Construct $\langle W, \dots, \nu' \rangle$, where $\nu'_+(b) = \nu_+(a)$, and, for terms x other than b , $\nu'_+(x) = \nu_+(x)$. Suppose Q_1 is "=-". Construct $\langle W, \dots, \nu' \rangle$, where $\nu'_+(b) = \nu_-(a)$, and,

for terms x other than b , $\nu'_+(x) = \nu_+(x)$. Use similar constructions if Q_2 is “=” or “=-”. So, the only a-reduction left is this: $\{C^{pq}ab, C^{qr}bc\}$ a-reduces to $C^{pr}ac$. Construct a model $\langle W', \dots, \nu' \rangle$ such that $W' = W \cup \{M\}$, $M \notin W$, and $\nu'_{p^*}(a) \cap \nu'_r(c)$ has at least two members, including M . To do this follow the procedure in Case 3 of Lemma 3.1. Then construct a model $\langle W', \dots, \nu'' \rangle$ such that $\nu''(b) = \nu'_p(a) \cup \{M\}$ and, for other terms x , $\nu''_+(x) = \nu'_+(x)$. \square

LEMMA 3.3. *If a simple chain $\{Qab\} \cup X$ is satisfied in an n -model, where n is the number of terms in $\{Qab\} \cup X$, then $\{e(Qab)\} \cup X$ is satisfied in an n -model, where n is the number of terms in $\{e(Qab)\} \cup X$.*

Proof. Straightforward. \square

THEOREM 5 (Semantic decision procedure). *If $\langle X, y \rangle$ is a syllogism then $X \models y$ iff $X, cd(y)$ is not satisfied in an m -model, where $m \leq n + 2$ and n is the number of terms in X .*

Proof. Assume $\langle X, y \rangle$ is a syllogism. (Only if) Immediate. (If) Assume $X, cd(y)$ is not satisfied in an m -model, where $m \leq n + 2$ and n is the number of terms in X . Then every strand of $X, cd(y)$ is not satisfied in an m -model where $m \leq n + 2$ and n is the number of terms in $X, cd(y)$. Then every strand of $X, cd(y)$ fully reduces to a cd -pair (by Theorem 3 and Lemma 3 of Theorem 4). Then $X \models y$ (by Theorem 4).

Given Theorem 5, it is natural to ask whether, for any n , there is an n -termed syllogism that requires an $n + 2$ model to show that it is invalid. The answer is Yes. If $n = 2$, use $\{\langle Za_1a_2 \rangle, cd(Za_2a_1)\}$. If $n > 2$, use $\{\langle Za_1a_2, C^{++}a_2a_3, \dots, C^{++}a_{n-1}a_n \rangle, cd(Za_n a_1)\}$. Consider a model $\langle W, \dots, \nu \rangle$ in which $\{Za_1a_2, C^{++}a_2a_3, \dots, C^{++}a_{n-1}a_n, Za_n a_1\}$ is satisfied. Note that $\nu_+(a_1)$ has at least two members, since $\nu(Za_1a_2) = t$. So $\nu_+(a_n)$ has at least n members. $\nu_-(a_n)$ has at least two members since $\nu(Za_n a_1) = t$. \square

THEOREM 6 (Completeness). *If $\langle X, y \rangle$ is a syllogism and $X \models y$ then $X \vdash y$.*

Proof. Assume the antecedent. Then, by Theorem 4, every strand of $X, cd(y)$ fully reduces to a cd -pair. So, by Lemmas 1–4, below, $X \vdash cd(cd(y))$. That is $X \vdash y$. \square

LEMMA 1. *If $\{x, y\}$ is a cd -pair, then $x \vdash cd(y)$.*

Proof. 1) $=ab \vdash =ba$ (by B1). So $=ab \vdash cd(=-ba)$ (and $cd(C^{pq}ba)$ and $cd(Zba)$) (by D1). 2) $=^-ba \vdash =^-ab$ (by B2). So $=^-ba \vdash cd(=ab)$ (by D1). And $=^-ab \vdash =^-ba$ (by B2). So $=^-ab \vdash cd(C^{pq}ba)$ (and

$cd(Zba)$) (by D1). 3) $\subset^{pq}ba \vdash \subset^{q^*p^*}ab$ (by B3). So $\subset^{pq}ba \vdash cd(=ab)$ (and $cd(=\bar{a}b)$) (by D1). $\subset^{pq}ab \vdash \subset^{q^*p^*}ba$ (by B3). So $\subset^{pq}ab \vdash cd(\subset^{qr}ba)$ (and $cd(Zba)$) (by D1). $\subset^{qr}ba \vdash \subset^{r^*q^*}ab$ (by B3). So $\subset^{qr}ba \vdash cd(\subset^{pq}ab)$ (by D1). 4) $Zba \vdash Zab$ (by B4). So $Zba \vdash cd(=ab)$ (and $cd(=\bar{a}b)$ and $cd(\subset^{pq}ab)$) (by D1). \square

LEMMA 2. If $X = \{Q_3ac\} \cup Z$, $Y = \{Q_1ab, Q_2bc\} \cup Z$, $\{Q_1ab, Q_2bc\}$ *a-reduces* to Q_3ac , and $X - \{x\} \vdash cd(x)$, for every x such that $x \in X$, then $Y - \{y\} \vdash cd(y)$, for every y such that $y \in Y$.

Proof. Assume the antecedent. Case 1: $y \in Z$. $\{Q_3ac\} \cup Z - \{y\} \vdash cd(y)$. We use

LEMMA 2.1. If $\{Q_1ab, Q_2bc\}$ *a-reduces* to Q_3ac then $Q_1ab, Q_2bc \vdash Q_3ac$.

Proof. Given B5–B8, we only need to show that: i) $=\bar{a}b, =bc \vdash =\bar{a}c$; ii) $\subset^{pq}ab, =bc \vdash \subset^{pq}ac$; and iii) $\subset^{pq}ab, =\bar{b}c \vdash \subset^{pq^*}ac$. For i), $=bc \vdash =cb$ (by B1) and $=\bar{a}b \vdash =\bar{b}a$ (by B2). $=cb, =\bar{b}a \vdash =\bar{c}a$ (by B5). So $=\bar{a}b, =bc \vdash =\bar{c}a$ (by D3). $=\bar{c}a \vdash =\bar{a}c$ (by B2). So $=\bar{a}b, =bc \vdash =\bar{a}c$ (by R1). Use similar reasoning for ii) and iii).

So $Q_1ab, Q_2bc \vdash Q_3ac$ (by Lemma 2.1). So $\{Q_1ab, Q_2bc\} \cup Z - \{y\} \vdash cd(y)$ (by D3).

Case 2: $y = Q_1ab$. $Z \vdash cd(Q_3ac)$. $Q_2bc, cd(Q_3ac) \vdash cd(Q_1ab)$ (by Lemma 2.1 and R2). So $Z, Q_2bc \vdash cd(Q_1ab)$ (by R1).

Case 3: $y = Q_2bc$. Use reasoning similar to that for Case 2. \square

LEMMA 3. If $X = \{Qab\} \cup Z$, $Y = \{e(Qab)\} \cup Z$, and $X - \{x\} \vdash cd(c)$, for every x such that $x \in X$, then $Y - \{y\} \vdash cd(y)$, for every y such that $y \in Y$.

Proof. Assume the antecedent. Case 1: $y \in Z$. $\{Qab\} \cup Z - \{y\} \vdash cd(y)$. $e(Qab) \vdash Qab$ (by B1–B4). So $\{e(Qab)\} \cup Z - \{y\} \vdash cd(y)$ (by D3). Case 2: $y = e(Qab)$. $Z \vdash cd(Qab)$. $cd(Qab) \vdash cd(e(Qab))$ (by B1–B4 and R2). So $Z \vdash cd(e(Qab))$ (by R1). \square

LEMMA 4. If each strand $Y \cup \{z\}$ of $X \cup \{y\}$ is such that $Y \vdash cd(z)$, then $X \vdash cd(y)$.

Proof. Use D2 and R3. (The proof is illustrated below.) \square

The proof of the above theorem provides a mechanical procedure for showing that $X \vdash y$ given that $X \vDash y$. We illustrate by showing that $=, \subset^{++}AB, =BC \vdash cd(=\bar{a}, \subset^{+-}AC)$. First, fully reduce the following strands as indicated: i) $\{=AB, =BC, =\bar{A}C\}$ to $\{=AB, =BC, =\bar{C}A\}$ to $\{=AC, =\bar{C}A\}$; ii) $\{=AB, =BC, \subset^{+-}AC\}$ to $\{=AB, =BC, \subset^{+-}CA\}$

to $\{=AC, C^{+-}CA\}$; iii) $\{C^{++}AB, =BC, =-AC\}$ to $\{C^{++}AB, =BC, =-CA\}$ to $\{C^{++}AC, =-CA\}$; and iv) $\{C^{++}AB, =BC, C^{+-}AC\}$ to $\{C^{++}AB, =BC, C^{+-}CA\}$ to $\{C^{++}AC, C^{+-}CA\}$. By the proof of Lemma 1: $=AC \vdash cd(=-CA)$; $=AC \vdash cd(C^{+-}CA)$; $C^{++}AC \vdash cd(=-CA)$; and $C^{++}AC \vdash cd(C^{+-}CA)$. By the proof of Lemma 2: $=AB, =AC \vdash cd(=-CA)$; $=AB, =AC \vdash cd(C^{+-}CA)$; $C^{++}AB, =BC \vdash cd(=-CA)$; and $C^{++}AB, =BC \vdash cd(C^{+-}CA)$. By the proof of Lemma 3: $=AB, =AC \vdash cd(=-AC)$; $=AB, =AC \vdash cd(C^{+-}AC)$; $C^{++}AB, =BC \vdash cd(=-AC)$; and $C^{++}AB, =BC \vdash cd(C^{+-}AC)$. By D2, $=AB, =AC \vdash cd(=-, C^{+-}AC)$ and $C^{++}AB, =BC \vdash cd(=-, C^{+-}AC)$. By R3, $=, C^{++}AB, =AC \vdash cd(=-, C^{+-}AC)$.

3. GERGONNE SYLLOGISMS

Faris [1] is motivated by an interest in providing a decision procedure for Gergonne syllogisms. Faris construes syllogisms as sentences, following Łukasiewicz's [4], rather than as inferences, as in Smiley's [5]. For us, a *Gergonne syllogism* is a syllogism consisting of Gergonne sentences, which are defined as follows, using Gergonne's symbols in [2]. The *Gergonne-quantifiers* are: $H =_{df} =-, C^{+-}$; $X =_{df} C^{-+}, Z$; $| =_{df} =, C =_{df} C^{++}$, and $\supset =_{df} C^{--}$. A *Gergonne-sentence* is any sentence of form $Q_1, \dots, Q_m ab$, where Q_i is a Gergonne-quantifier. So Theorem 4 above gives an alternative solution to the problem that motivated Faris' [1], since every Gergonne syllogism may be expressed in our system. Note, for example, that " H, XAB " is expressed as " $=-, C^{+-}, C^{-+}, ZAB$ ".

4. SYSTEM B

In this section we develop a subsystem B which expresses no sentences other than those that may be expressed by using sentences of form "All... are ---", "No... are ---", "Some... are ---", or "Some... are not ---", where the blanks are filled by expressions of form x or non- x (the "A, E, I, and O sentences, respectively, with or without negative terms.")

The *B-quantifiers* ("B" for "basic") are: $=, C^{++}(A^{++})$; $=-, C^{+-}(A^{+-})$; $=-, C^{-+}(A^{-+})$; $=, C^{--}(A^{--})$; $=-, C^{+-}, C^{-+}, C^{--}, Z(O^{++})$; $=, C^{++}, C^{-+}, C^{--}, Z(O^{+-})$; $=, C^{++}, C^{+-}, C^{--}, Z(O^{-+})$; and $=-, C^{++}, C^{+-}, C^{-+}, Z(O^{--})$. Qab is a *B-sentence* iff Qab is a sentence and Q is a B-quantifier. So, for example, $A^{++}AB$ is a B-sentence. And a *B-syllogism* is a syllogism composed of B-sentences.

We define y is *B-deducible from X* ($X \vdash_B y$), where X, y is a set of B-sentences, and where $ct(A^{pqab}) = A^{q^*p^*ab}$, $cd(A^{pqab}) = O^{pqab}$, and $cd(O^{pqab}) = A^{pqab}$:

- (B₁) $A^{pqab} \vdash_B A^{q^*p^*ba}$
- (B₂) $A^{pqab}, A^{qrbc} \vdash_B A^{prac}$
- (R₁) If $X \vdash_B y$ and $y, z \vdash_B w$ then $X, z \vdash_B w$
- (R₂) If $X, y \vdash_B ct(z)$ or $cd(z)$ then $X, z \vdash_B cd(y)$
- (L₁) If $X \vdash y$, then $X \vdash y$ in virtue of B₁-R₂.

THEOREM 7. (D₁) *If $X, y \vdash_B z$ and $u, v \vdash_B y$ then $X, u, v \vdash_B z$.*

Proof. Use the reasoning for the proof of Theorem 1. □

THEOREM 8 (Soundness). *If $X \vdash_B y$ then $X \vDash y$.*

Proof. Straightforward. □

By definition, $e(A^{pqab})$ is $A^{q^*p^*ba}$ and $e(O^{pqab})$ is $O^{q^*p^*ba}$. And, by definition, a set X of sentences *b-reduces to* a sentence y iff $\langle X, y \rangle$ has form $\langle \{A^{pqab}, A^{qrbc}\}, A^{prac} \rangle$.

If X_1 is a chain of B-sentences then a sequence of chains X_1, \dots, X_m ($= Y_1, \dots, Y_n$) is a *full B-reduction of X_1 to Y_n* iff: i) X_m is a normal chain and if $m > 1$, then, for $1 \leq i < m$, if X_i has form $\{Qab\} \cup Y$, then X_{i+1} has form $\{e(Qab)\} \cup Y$; and ii) there is no pair of sentences in Y_n that b-reduces to a sentence and if $n > 1$ then, for $1 \leq i < n$, Y_i has form $\{A^{pqab}, A^{qrbc}\} \cup X$ and Y_{i+1} has form $\{A^{prac}\} \cup X$. X *fully B-reduces to* Y iff there is a full B-reduction of X to Y .

THEOREM 9. *Every chain of B-sentences fully B-reduces to a normal chain of B-sentences.*

Proof. Imitate the proof of Theorem 3. □

A normal chain of B-sentences is a cd-B-pair iff it has one of the following forms: $\{A^{pqab}, A^{q^*p^*ba}\}$ or $\{A^{pqab}, O^{q^*p^*ba}\}$.

THEOREM 10 (Syntactic decision procedure). *If $\langle X, y \rangle$ is a B-syllogism then $X \vDash y$ iff $X, cd(y)$ fully B-reduces to a cd-B-pair.*

Proof. Assume $\langle X, y \rangle$ is a B-syllogism. We use Lemmas 1 and 2, below. (If) Suppose $X, cd(y)$ fully B-reduces to a cd-B-pair. Then, by Lemma 1, $X, cd(y)$ is consistent. Then $X \vDash y$. (Only if) Suppose $X \vDash y$. Then $X, cd(y)$ is inconsistent. Then $X, cd(y)$ fully B-reduces to a cd-B-pair (by Lemma 2 and Theorem 9). □

LEMMA 1. *If a chain X of B-sentences fully B-reduces to a cd-B-pair then X is inconsistent.*

Proof. Imitate the proof of Lemma 2 of Theorem 4. \square

LEMMA 2. *If a chain X of B-sentences fully B-reduces to a normal chain of B-sentences that is not a cd-B-pair, then X is satisfied in a 3-model.*

LEMMA 2.1. *If a chain of B-sentences fully B-reduces to a normal chain of B-sentences X that is not a cd-B-pair, then X is satisfied in a 3-model.*

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of "O" in X.

Case 1: "O" does not occur in X. We use induction on the number n of terms in X. Basis step: $n = 2$. Then X has form $\{A^{pq}ab, A^{qp}ba\}$ or form $\{A^{pq}ab, A^{q^*p^*}ba\}$. If $p = q$, use $\langle\{1, 2, 3\}, \dots, \nu\rangle$, where $\nu_+(x) = \{1\}$. If $p \neq q$, use $\langle\{1, 2, 3\}, \dots, \nu\rangle$, where $\nu_+(a) = \{1\}$, and, for terms x other than a , $\nu_+(x) = \{2, 3\}$. Induction step: $n > 2$. Then X has form $\{A^{p_1p_2}a_1a_2, \dots, A^{p_{2i-1}p_{2i}}a_i a_{i+1}, \dots, A^{p_{2n-1}p_{2n}}a_n a_1\}$, where $p_{2i} = p_{2i+1}^*$. By Case 1 of Lemma 3.1 $\{C^{p_1p_2}a_1a_2, \dots, C^{p_{2i-1}p_{2i}}a_i a_{i+1}, \dots, C^{p_{2n-1}p_{2n}}a_n a_1\}$, where $p_{2i} = p_{2i+1}^*$, for $1 \leq i < n$, and $p_{2n} = p_1^*$, is satisfied in a 3-model. So X is satisfied in a 3-model.

Case 2: "O" occurs exactly once in X. Suppose there are exactly two terms in X. Then X has form $A^{pq}ab, O^{q^*p}ba$ (or $O^{qp}ab$ or $O^{q^*p^*}ba$). 3-models are easily constructed to show that X is consistent. Suppose there are more than two terms in X. We use induction on the number n of terms in X to show that X is satisfied in a 3-model. Basis step: $n = 3$. Then X has form $\{O^{pq}ab, A^{rs}bc, A^{s^*u}ca\}$. So there is a strand of X with one of the following forms: $\{C^{pq}ab, C^{rs}bc, C^{s^*u}ca\}$, $\{C^{p^*q}ab, C^{rs}bc, C^{s^*u}ca\}$, and $\{C^{p^*q^*}ab, C^{rs}bc, C^{s^*u}ca\}$. So, by Case 1 of Lemma 3.1 of Theorem 4, X is consistent if $p = u$ or $q = r$. Suppose $p \neq u$ and $q \neq r$. Then X has form $\{O^{pq}ab, A^{q^*s}bc, A^{s^*p^*}ca\}$. If $p = q$, there is a strand of X with form $\{=^-ab, =^-bc, =^-ca\}$ or form $\{=^-ab, =^-bc, =ca\}$. If $p \neq q$, there is a strand of X with form $\{=ab, =bc, =ca\}$ or form $\{=ab, =^-bc, =^-ca\}$. Each of these four chains can easily be shown to be satisfied in a 3-model. Induction step: $n > 3$. X has form $O^{pq}ab, A^{rs}bc, A^{s^*u}cd, \dots$. By the induction hypothesis, $O^{pq}ab, A^{r^*u}bd, \dots$ is satisfied in a 3-model $\langle W, \dots, \nu \rangle$. Construct $\langle W, \dots, \nu' \rangle$, where $\nu'_s(c) = \nu_r(b)$, and, for terms x other than c , $\nu'_+(x) = \nu_+(x)$. Note that $\nu'(A^{rs}bc) = t$, since $\nu'_r(b) = \nu'_s(c)$, and $\nu'(A^{s^*u}cd) = t$, since $\nu'_{s^*}(c) = \nu'_{r^*}(b)$.

Case 3: "O" occurs at least twice in X. We use induction on the number of terms n in X. Basis step: $n = 2$. X has form $\{O^{pq}ab, O^{rs}ba\}$. It is

easy to show that X is satisfied in a 3-model. Induction step: $n > 2$. X has form $\{O^{pq}ab, Q^{rs}bc, \dots, O^{uv}de, \dots\}$. Suppose Q is “A” and $r = s$ or Q is “O” and $r \neq s$. By the induction hypothesis, $\{O^{pq}ac, \dots, O^{uv}de, \dots\}$ is satisfied in a 3-model $\langle W, \dots, \nu \rangle$. Construct 3-model $\langle W, \dots, \nu' \rangle$, where $\nu'_q(b) = \nu_q(c)$, and, for terms x other than c , $\nu'_+(x) = \nu_+(x)$. Suppose Q is “A” and $r \neq s$ or Q is “O” and $r = s$. By the induction hypothesis, $\{O^{pq^*}ac, \dots, O^{uv}de, \dots\}$ is satisfied in a 3-model $\langle W, \dots, \nu \rangle$. Construct 3-model $\langle W, \dots, \nu' \rangle$, where $\nu'_q(b) = \nu_{q^*}(c)$, and, for terms x other than c , $\nu'_+(x) = \nu_+(x)$. \square

LEMMA 2.2. *If $\{A^{pr}ac\} \cup Y$ is satisfied in a 3-model and if term b does not occur in a member of Y , then $\{A^{pq}ab, A^{qr}bc\} \cup Y$ is satisfied in a 3-model.*

Proof. Assume that $\{A^{pr}ac\} \cup Y$ is satisfied in a 3-model $\langle W, \dots, \nu \rangle$. Construct $\langle W, \dots, \nu' \rangle$, where $\nu'_p(b) = \nu_q(a)$, and, for terms x other than b , $\nu'_+(x) = \nu_+(x)$. \square

LEMMA 2.3. *If $\{Qab\} \cup Y$ is satisfied in a 3-model, then $\{e(Qab)\} \cup Y$ is satisfied in a 3-model.*

Proof. Straightforward. \square

THEOREM 11 (Semantic decision procedure). *If $\langle X, y \rangle$ is a B-syllogism then $X \models y$ iff $X, cd(y)$ is not satisfied in a 3-model.*

Proof. Assume $\langle X, y \rangle$ is a B-syllogism. (Only if) Immediate. (If) Suppose $X, cd(y)$ is not satisfied in a 3-model. Then, by Theorem 9 and Lemma 2 of Theorem 10, $X, cd(y)$ fully B-reduces to a cd -B-pair. So, by Theorem 10, $X \models y$. \square

Theorem 11 extends the result in Johnson’s [3]. There it is shown, in effect, that any invalid syllogism constructed by using B-sentences other than those of form $A^{-+}ab$ or $O^{-+}ab$ is satisfied in a 3-model. There are invalid B-syllogisms that require a domain with at least three members to show their invalidity. This is an example: $\{\{A^{+-}AB, A^{+-}BC\}, O^{+-}AC\}$.

THEOREM 12 (Completeness). *If $\langle X, y \rangle$ is a B-syllogism and $X \models y$ then $X \vdash_B y$.*

Proof. Assume the antecedent. Then, by Theorem 10, $X \cup \{cd(y)\}$ fully B-reduces to a cd -B-pair. Use the following three lemmas. \square

LEMMA 1. *If $\{x, y\}$ is a cd -B-pair, then $x \vdash_B cd(y)$ and $y \vdash_B cd(x)$.*

Proof. (1) $A^{qp^*}ba \vdash_B A^{pq^*}ab$, that is, $ct(A^{pq^*}ab)$ (by B_1). So $A^{pq^*}ab \vdash_B cd(A^{qp^*}ba)$ (by R_2). So $A^{qp^*}ba \vdash_B cd(A^{pq^*}ab)$ (by R_2). (2) $A^{pq^*}ab \vdash_B$

$A^{q^*p^*}ba$, that is, $cd(O^{q^*p^*}ba)$ (by B_1). So $O^{q^*p^*}ba \vdash_B cd(A^{pq}ab)$ (by R_2). \square

LEMMA 2. If $X = \{A^{pr}ac\} \cup Z$, $Y = \{A^{pq}ab, A^{qr}bc\} \cup Z$, and $X - \{x\} \vdash_B cd(x)$, for each sentence x in X , then $Y - \{y\} \vdash_B cd(y)$, for each sentence y in Y .

Proof. Imitate the proof of Lemma 2 of Theorem 6. \square

LEMMA 3. If $X = \{Qab\} \cup Z$, $Y = \{e(Qab)\} \cup Z$, and $X - \{x\} \vdash_B cd(x)$, for each sentence x in X , then $Y - \{y\} \vdash_B cd(y)$, for each sentence y in Y .

Proof. Imitate the proof of Lemma 3 of Theorem 6. \square

5. CONCLUSION

Our interest has been in extending the Aristotelian syllogistic. But, in conclusion, we mention Smiley's classic result in [5] about the Aristotelian syllogistic, which follows from the results obtained above. First, delete sentences of form $A^{-+}ab$ and $O^{-+}ab$ from system B. Let $Aa - b = \emptyset$ if $a = b$; otherwise, let $Aa - b$ be a set of sentences that can be arranged as follows: $\langle A^{++}a_1a_2$ (or $A^{--}a_2a_1$), \dots , $A^{++}a_n a_{n+1}$ (or $A^{--}a_{n+1}a_n$), where $a_1 = a$ and $a_{n+1} = b$. Then, by Theorem 10, a chain of sentences in this subsystem is inconsistent iff it has one of the following forms: i) $Aa - b$, $O^{++}ab$ ($O^{--}ab$); ii) $Aa - b$, $A^{+-}bc$, $Ac - a$; or iii) $Aa - b$, $A^{+-}bc$, $Ad - c$, $O^{+-}da$ (or $O^{+-}ad$). Next, delete sentences of form $A^{--}ab$ and $O^{--}ab$ from this system. The resulting system can express all of the Aristotelian syllogisms. So, as Smiley [5] says, an Aristotelian syllogism $\langle X, y \rangle$ is valid iff $X, cd(y)$ has one of the following forms: i') $Aa - b$, $O^{++}ab$, ii), or iii). (Smiley uses A, E, I, and O instead of our A^{++} , A^{+-} , O^{+-} , O^{++} , respectively.) So, for example, " $A^{++}BC$, $A^{++}BA$; so $O^{+-}AC$ " (Darapti) is valid since " $A^{++}BC$, $A^{++}BA$, $A^{+-}AC$ " has form ii).

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