GRADIENT ADJUSTMENT IN AN AXISYMMETRIC VORTEX

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by

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ABSTRACT

We consider gradient adjustment in an axisymmetric vortex. From an initially unbalanced state, the final adjusted state is obtained. Initially, the geopotential is flat i.e. $\phi_0(r_0) = 1$ and the velocity field is given by the Rankine vortex

$$r_0 v_0 = \begin{cases} \dfrac{\varepsilon r_0^2}{a^2} & r_0 < a \\ \varepsilon & r_0 \geq a \,. \end{cases}$$

We derive a system of equations governing the final adjusted state. This system of equations is solved numerically using the shooting method. We also present solutions for the geostrophic adjustment problem and compare these to the solutions for the gradient adjustment problem.

In the gradient adjustment case, the results indicate that

(i) the geopotential decrease at the centre is porportional to ε/a.

(ii) the decrease of the maximum tangential wind is dependent on ε for smaller $a (a < 1)$. For $a \geq 1$, the decrease is independent of ε.

(iii) the stretching is proportional to ε/a.

(iv) the mass removed from within a given region tends to ε for large r.

Comparison between the solutions for the geostrophic and gradient adjustment problems shows significant differences for small a.

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1. INTRODUCTION

The problem of understanding the mechanisms by which pressure and wind fields mutually adjust has received much attention since it was first studied by Rossby (1938) and independently by Obukhov (1949). Rossby's study was primarily directed to the relationship between the initially unbalanced state and the final geostrophically adjusted state, while Obukhov's dealt with the transient solution in two dimensions. Mihaljan (1963) obtained the exact solution of the Rossby adjustment problem and Cahn (1945) studied the linear transient adjustment in one dimension. For a general review of the geostrophic adjustment problem, the publication by Blumen (1972) is excellent.

Much emphasis was placed on the geostrophic adjustment problem because it is believed that, for large-scale motions, the non-linear terms describing the accelerations of air particles associated with the curvature of the trajectories are small and may therefore be neglected in the equations of motion. In the present study, the problem of gradient adjustment in an axisymmetric vortex is considered. We consider only the relationship between the initially unbalanced state and the final adjusted state under gradient balance. We also obtain solutions for the linear adjustment problem and compare these to the nonlinear solutions. The solution of the linear problem described here is the one developed by Schubert and Hack (1978).

The initial velocity field is defined by the Rankine vortex

$$r_0 v_0 = \begin{cases} \dfrac{\varepsilon r_0^2}{a^2} & r_0 < a \\ \varepsilon & r_0 \geq a \end{cases}$$

where r_0 is the initial position of a fluid particle, v_0 is the initial tangential velocity, ε is a constant and a is the radius of maximum wind. For large radii, the assumption is made that a fluid particle does not move. This seems reasonable since there must exist a finite region within which the effect of the adjustment process is felt.

We derive a system of equations for the final adjusted state. It is found convenient to use S as the independent variable instead of r where $S = \frac{1}{2}r^2$. The dependent variables are $m_0(S_0(S))$, $\phi(S)$ and $S_0(S)$ where m_0 and ϕ represent the absolute angular momentum and geopotential respectively and S_0 is proportional to the original mass within radius r_0. The system of equations representing the final adjusted state is solved numerically using the shooting method. Finally, we obtain solutions for the tangential wind (v), the geopotential (ϕ), the mass removed (proportional to $\frac{1}{2}(r^2 - r_0^2)$) and the relative vorticity (ζ) as functions of r (radial distance).

2. GOVERNING EQUATIONS

Consider a rotating, homogeneous, incompressible fluid which lies above a flat boundary and has a free surface of mean height H. The equations which govern the motion of this fluid are the shallow water equations. In circular cylindrical coordinates with the axisymmetric assumption the shallow water equations can be written as

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} - \left(f + \frac{v}{r} \right) v + \frac{\partial \phi}{\partial r} = 0, \qquad (2.1)$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial r} + \left(f + \frac{v}{r} \right) u = 0, \qquad (2.2)$$

$$\frac{\partial \phi}{\partial t} + u\,\frac{\partial \phi}{\partial r} + \phi\,\frac{\partial (ru)}{r\partial r} = 0, \qquad (2.3)$$

where u represents the radial component of velocity, v represents the tangential component of velocity, f is the coriolis parameter, ϕ represents the geopotential and r is the radial distance.

Equations (2.1) - (2.3) can be made dimensionless by defining

$t' = ft$: dimensionless time

$r' = rf/c$: dimensionless radial distance

$(u', v') = (u/c,\ v/c)$: dimensionless velocity components

$\phi' = \phi/gH$: dimensionless geopotential

In the above definitions

$c = (gH)^{1/2}$ is the speed of a pure gravity wave

If we drop the primes and let t, r, u, v and ϕ represent dimensionless variables, equations (2.1) - (2.3) become

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} - \left(1 + \frac{v}{r} \right) v + \frac{\partial \phi}{\partial r} = 0, \qquad (2.4)$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial r} + \left(1 + \frac{v}{r}\right) u = 0, \tag{2.5}$$

$$\frac{\partial \phi}{\partial t} + u\,\frac{\partial \phi}{\partial r} + \phi\,\frac{\partial (ru)}{r\partial r} = 0. \tag{2.6}$$

Transient solutions of this system of equations have recently been studied by Schubert and Hack (1978). Here our objective is to find the final adjusted state from an initial unbalanced state. For this purpose it is more convenient to take a Lagrangian point of view as was used by Rossby (1938) and Mihaljan (1963). Thus, let us define $r_o(r)$ as the initial position of a fluid column which ends up at r. Then the mass continuity relation takes the form

$$\phi_o r_o dr_o \;=\; \phi\, r\, dr\,, \tag{2.7}$$

where $\phi_o r_o dr_o$ is proportional to the initial fluid mass between r_o and $r_o + dr_o$. Equation (2.7) states that this mass must be conserved.

We now use (2.5) to obtain an equation for the conservation of angular momentum. Multiplying equation (2.5) by r we get

$$\frac{\partial}{\partial t}\,(rv + \tfrac{1}{2}r^2) + u\,\frac{\partial}{\partial r}\,(rv + \tfrac{1}{2}r^2) = 0. \tag{2.8}$$

Integrating equation (2.8) we obtain

$$rv(r) - r_o(r)\,v_o\,(v_o(r)) \;=\; \tfrac{1}{2}\,[r_o(r)^2 - r^2]\;, \tag{2.9}$$

where $r_o v_o$ is the initial relative angular momentum and rv is the final relative angular momentum.

In the final adjusted state gradient balance exists and the gradient wind equation becomes

$$\frac{d\phi}{dr} = \left(1+\frac{v}{r}\right)v \; . \tag{2.10}$$

This equation can be written as

$$r^3\frac{d\phi}{dr} = (rv + r^2)\, rv \tag{2.11}$$

or

$$r^3\frac{d\phi}{dr} + \tfrac{1}{4}r^4 = \left(rv + \tfrac{1}{2}r^2\right)^2 \; . \tag{2.12}$$

The angular momentum equation (2.9) can be written as

$$\left(rv + \tfrac{1}{2}r^2\right) = \left(r_0v_0 + \tfrac{1}{2}r_0^2\right)$$

or

$$m\,(r) = m_0\,(r_0(r)) \; .$$

The system of equations representing the final adjusted state may now be written as

$$r^3\frac{d\,\phi(r)}{dr} + \tfrac{1}{4}r^4 = m^2(r) \tag{2.13}$$

$$m(r) = m_0\,(r_0\,(r)) \tag{2.14}$$

$$\frac{dr_0(r)}{dr} = \frac{\phi(r)r}{\phi_0(r_0(r))r_0(r)} \tag{2.15}$$

POTENTIAL VORTICITY

From (2.9) we can derive the potential vorticity equation. Differentiating (2.9) with respect to r and dividing by r we get

$$\frac{\partial(rv)}{r\partial r} - \frac{\partial(r_o v_o)}{r_o \partial r_o} \frac{r_o}{r} \frac{dr_o}{dr} = \tfrac{1}{2}\left[\frac{\partial}{r_o \partial r_o}(r_o^2)\frac{r_o}{r}\frac{dr_o}{dr} - \frac{\partial}{r\partial r}(r^2)\right]. \tag{2.16}$$

Using (2.7) we get

$$\frac{\partial}{r\partial r}(rv) - \frac{\phi}{\phi_o}\frac{\partial}{r_o \partial r_o}(r_o v_o) = \frac{\phi}{\phi_o} - 1 \tag{2.17}$$

$$\left(\frac{\partial}{r\partial r}(rv) + 1\right) = \frac{\phi}{\phi_o}\left(\frac{\partial}{r_o \partial r_o}(r_o v_o) + 1\right). \tag{2.18}$$

Let

$$\frac{\partial}{r\partial r}(rv) = \zeta$$

and

$$\frac{\partial}{r_o \partial r_o}(r_o v_o) = \zeta_o .$$

Then (2.18) becomes

$$\frac{(\zeta + 1)}{\phi} = \frac{(\zeta_o + 1)}{\phi_o} \tag{2.19}$$

which expresses the conservation of potential vorticity.

INITIAL CONDITIONS

The initial vortex is the Rankine vortex given by

$$r_o v_o = \begin{cases} \dfrac{\varepsilon r_o^2}{a^2} & r_o < a \\ \varepsilon & r_o \geq a \end{cases} \tag{2.20}$$

The initial geopotential is flat, i.e.,

$$\phi_o\,(r_o(r)) = 1\ . \tag{2.21}$$

We now formulate the problem using a new independent variable S. Let

$$S = \tfrac{1}{2} r^2$$

and

$$S_o = \tfrac{1}{2} r_o^{\,2}\ .$$

Then

$$\frac{d}{dr} = r\,\frac{d}{dS}\ .$$

Equation (2.12) becomes

$$4\,S^2\,\frac{d\,\phi(s)}{dS} = m_o^2\,(S_o(S)) - S^2 \tag{2.22}$$

and (2.15) becomes

$$\frac{dS_o(S)}{dS} = \phi\,(S)\ . \tag{2.23}$$

With S as the independent variable, we have the following system of equations for the final adjusted state.

$$\frac{d\,\phi\,(s)}{dS} = \tfrac{1}{4}\left\{\frac{m_o^{\,2}\,(s_o(s))}{S^2} - 1\right\} \tag{2.24}$$

$$\frac{dS_o(S)}{dS} = \phi\,(S) \tag{2.25}$$

where m_o is a specified function of S_o.

BOUNDARY CONDITIONS

$$S = 0 : \qquad S_o(0) = 0$$

and

$$\lim_{S \to \infty} \frac{S_o(S)}{S} = 1 .$$

3. GEOSTROPHIC ADJUSTMENT

It would be very instructive to compare the linear and non-linear adjustment problems. Therefore, we now obtain solutions for the geostrophic adjustment problem on a resting basic state. Following Schubert and Hack, (2.4) - (2.6) become

$$\frac{\partial u}{\partial t} - v + \frac{\partial \phi}{\partial r} = 0 \tag{3.1}$$

$$\frac{\partial v}{\partial t} + u = 0 \tag{3.2}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{r \partial r}(ru) = 0 . \tag{3.3}$$

It is important to note that in the above equations u, v, and ϕ represent perturbation quantities.

From (3.2) we derive the vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{r\partial r}(rv)\right) + \frac{\partial}{r\partial r}(ru) = 0 . \tag{3.4}$$

Combining (3.4) with (3.3) we obtain the potential vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{r\partial r}(rv) - \phi\right) = 0. \tag{3.5}$$

If we assume that the final adjusted state is one of geostrophic balance, (3.5) may be written as

$$\frac{d}{rdr}(rv(r,\infty)) - \phi(r,\infty) = \frac{d}{rdr}(rv(r,o)) - \phi(r,o) . \tag{3.6}$$

With

$\frac{d\phi}{dr}(r,\infty) = v(r,\infty)$, (3.6) becomes

$$\frac{d^2\phi\ (r,\infty)}{dr^2} + \frac{d\,\phi(r,\infty)}{rdr} - \phi\,(r,\infty) \;=\; \zeta(r,o) - \phi(r,o)\,. \tag{3.7}$$

Where

$$\zeta\,(r,o) \;=\; \frac{d}{rdr}\ (rv(r,o))$$

is the initial vorticity. For the initial conditions

$$rv(r,o) \;=\; \begin{cases} \dfrac{\varepsilon r^2}{a^2} & r < a \\[2ex] \varepsilon & r \geq a \end{cases} \tag{3.8}$$

and

$$\phi(r,o) \;=\; 0\,, \tag{3.9}$$

the initial potential vorticity is given by

$$\zeta(r,o) - \phi(r,o) = \begin{cases} \dfrac{2\varepsilon}{a^2} & r < a \\[2ex] 0 & r \geq a\ . \end{cases} \tag{3.10}$$

The solution of (3.7) and (3.10) which remains bounded at the origin and at infinity and which possesses continuous $\phi\,(r,\infty)$ and $v(r,\infty)$ at $r = a$ is

$$\phi\,(r,\infty) \;=\; \begin{cases} \dfrac{-2\varepsilon}{a}\left[\dfrac{1}{a} - K_1\,(a)\ I_0\,(r)\right] & r < a \\[2ex] \dfrac{-2\varepsilon}{a}\ I_1\,(a)\,K_0\,(r) & r \geq a \end{cases} \tag{3.11}$$

$$v(r,\infty) = \frac{d\phi}{dr}(r,\infty) = \begin{cases} \frac{2\varepsilon}{a} K_1(a)\, I_1(r) & r < a \\ \frac{2\varepsilon}{a} I_1(a)\, K_1(r) & r \geq a . \end{cases} \tag{3.12}$$

The final vorticity is given by

$$\zeta(r,\infty) = \frac{d}{rdr}(rv(r,\infty)) = \begin{cases} \frac{2\varepsilon}{a} K_1(a)\, I_0(r) & r < a \\ \frac{-2\varepsilon}{a} I_1(a)\, K_0(r) & r \geq a . \end{cases} \tag{3.13}$$

In (3.11) - (3.13), $I_\nu(r)$ is the modified Bessel function of the first kind of order ν and $K_\nu(r)$ is the modified Bessel function of the second kind of order ν.

The solutions $\phi(r,\infty)$ and $v(r,\infty)$ are shown by the dashed curves in the figures given in Chapter 4. Figures 18 and 25 show the final vorticity $\zeta(r,\infty)$ for the linear adjustment case.

4. GRADIENT ADJUSTMENT

4.1 Method of Solution

The solution of the system of equations (2.24) - (2.25) can be treated as a boundary-value problem. By using the Shooting Method, we can reduce the boundary-value problem to the iterative solution of an initial value problem.

We select the point S=0 as the initial point. At this point, the value of $\phi(o)$ is assumed since it is unknown. The assumed value of $\phi(o)$ is obtained as follows.

Let $\phi(o)$ (upper) and $\phi(o)$ (lower) be the upper and lower bounds of $\phi(o)$ respectively. We know that $\phi(o)$ lies between one and zero. The initial guess is computed from the formula.

$$\phi(o) = \left[\phi(o)\text{ (upper)} + \phi(o)\text{ (lower)}\right] / 2 \tag{4.1}$$

The initial value problem is now solved using the Fourth-Order Runge-Kutta method and the numerical solution at the outer boundary compared to the outer boundary condition. Since we solve the problem numerically, we have to terminate the integration at some $S=S_{max}$. The outer boundary condition is then somewhat different from

$$\lim_{S\to\infty} \frac{S_o}{S} = 1.$$

The outer boundary condition used is

$$\frac{S_o(S_{max})}{S_{max}} = 1 .$$

If the numerical solution for $S_o(S_{max})$ differs greatly from S_{max}, a new value of $\phi(o)$ is assumed and the process repeated until the computed value of $S_o(S_{max})$ and S_{max} are approximately equal.

The new assumed value of $\phi(o)$ is obtained in the following manner. If the computed value of $S_o(S_{max})$ is less than S_{max}, then $\phi(o)$ (lower) is set equal to $\phi(o)$. If $S_o(S_{max})$ (computed) is greater than S_{max}, then $\phi(o)$ (upper) is set equal to $\phi(o)$. Equation (4.1) then gives the new $\phi(o)$.

In using the Fourth-Order Runge-Kutta method to solve a system of first order differential equations, the derivatives have to be evaluated at the initial point. In our problem, one of the derivatives involves the indeterminate quantity S_o^2/S^2 at $S = 0$. To overcome this problem, we employ the concept of limits.

The initial absolute angular momentum is given by

$$m_o = r_o v_o + \tfrac{1}{2} r_o^{\,2}. \tag{4.2}$$

Using (2.20), we get

$$m_o = \begin{cases} \left(\dfrac{2\varepsilon}{a^2} + 1\right) S_o & S_o < a^2/2 \\[2ex] (\varepsilon + S_o) & S_o \geq a^2/2 . \end{cases} \tag{4.3}$$

Hence (2.24) may be written as

$$\frac{d\,\phi(s)}{dS} = \begin{cases} \tfrac{1}{4}\left[\dfrac{\left(\dfrac{2\varepsilon}{a^2} + 1\right)^2 S_o^2}{S^2} - 1\right] & S_o < a^2/2 \\[3ex] \tfrac{1}{4}\left[\dfrac{(\varepsilon + S_o)^2}{S^2} - 1\right] & S_o \geq a^2/2 . \end{cases} \tag{4.4}$$

Now

$$\lim_{S \to 0} \frac{d\phi(s)}{dS} = \lim_{S \to 0} \left[\tfrac{1}{4}\left\{\frac{\left(\dfrac{2\varepsilon}{a^2} + 1\right)^2 S_o^2}{S^2} - 1\right\}\right]$$

$$\lim_{S\to 0} \frac{d\phi(s)}{dS} = \lim_{S\to 0} \left\{ \frac{\left(\frac{2\varepsilon}{a^2} + 1\right)^2 S_o^2}{4S^2} \right\} - \lim_{S\to 0} \tfrac{1}{4} .$$

Using L'Hospital's Rule, we obtain

$$\lim_{S\to 0} \frac{d\phi(s)}{dS} = \left(\frac{2\varepsilon}{a^2} + 1\right)^2 \lim_{S\to 0} \left\{ \frac{S_o \frac{dS_o}{dS}}{4S} \right\} - \tfrac{1}{4} .$$

But

$$\frac{dS_o(o)}{dS} = \phi(o) .$$

Hence

$$\lim_{S\to 0} \frac{d\phi(s)}{dS} = \tfrac{1}{4} \left(\frac{2\varepsilon}{a^2} + 1\right)^2 \phi(o) \lim_{S\to 0} \left(\frac{dS_o}{dS}\right) - \tfrac{1}{4} .$$

Finally, we obtain the relationship

$$\lim_{S\to 0} \frac{d\phi(s)}{dS} = \tfrac{1}{4} \left[\left(\frac{2\varepsilon}{a^2} + 1\right)^2 \phi(o)^2 - 1 \right]. \tag{4.5}$$

4.2 Results

For the Rankine vortex

$$r_o v_o(r_o) = \begin{cases} \dfrac{\varepsilon r_o^2}{a^2} & r_o/a < 1 \\ \varepsilon & r_o/a \geq 1 \end{cases} ,$$

we can write

$$\frac{v_o(r_o)}{v_o(a)} = \begin{cases} \dfrac{r_o}{a} & r_o/a < 1 \\ \dfrac{a}{r_o} & r_o/a \geq 1 \end{cases} .$$

Figure 1 shows a plot of $\frac{v_o(r_o)}{v_o(a)}$ as a function of r_o/a.

In Figures 2-3, we show the final geopotential as a function of r/a for $\varepsilon = 0.05$ and $a = 0.2, 0.5, 1.0$. We find that the geopotential decrease at the centre is inversely proportional to a.

Figures 4-5 show the final adjusted tangential wind for $\varepsilon = 0.05$ and $a = 0.2, 0.5, 1.0$. These graphs indicate that the decrease of the maximum wind is dependent on ε.

Figures 6-8 show the mass removed as a function of r/a for $\varepsilon = 0.05$ and $a = 0.2, 0.5, 1.0$ respectively. These plots show a rather interesting feature of the solution. For large radius, the mass removed tends to ε. This means that for $r \gg 1$, the mass pushed beyond radius r is constant. We shall now show that this result can be expected by considering the relationship between the circulation and the mass removed from within the region bounded by radius r for $r \gg 1$.

The initial circulation, C_o, around radius r_o is given by

$$C_o = \int_0^{2\pi} r_o v_o d\theta \qquad (4.6)$$

$$C_o = 2\pi r_o v_o$$

$$\frac{C_o}{2\pi} = r_o v_o \ . \qquad (4.7)$$

The results show that for large r, $rv << \frac{1}{2}r^2$. Hence (2.9) can be written as

$$\tfrac{1}{2}(r^2 - r_o^2) = r_o v_o \qquad (4.8)$$

or

$$S - S_o = r_o v_o \ . \qquad (4.9)$$

From the mass continuity equation, we can obtain an equation for the mass removed from within any region. From (2.25) we can write

$$\frac{d\,(S - S_o)}{dS} = 1 - \phi(s). \qquad (4.10)$$

Integrating (4.10) we get

$$\int_0^S \frac{d\,(S - S_o)}{dS'} dS' = \int_0^S (1 - \phi(s))\, dS'$$

i.e.,

$$(S - S_o) - (S - S_o)_{S=0} = \int_0^S (1 - \phi(s))\, dS' .$$

But $(S - S_o)_{S=0} = 0.$

Hence the mass removed is given by

$$(S - S_0) = \int_0^S (1 - \phi(s))\, dS' . \tag{4.11}$$

From (3.7) and (3.9) we get

$$\frac{C_0}{2\pi} = \int_0^S (1 - \phi(s))\, dS' . \tag{4.12}$$

For the Rankine vortex, $\frac{C_0}{2\pi} = r_0 v_0 = \varepsilon$ for large r_0. Hence the mass removed from within a given region for large r is constant and equal to ε.

Figure 9 shows ζ/ζ_0 as a function of r/a for $\varepsilon = 0.05$ and $a = 0.2$, 0.5, 1.0 where ζ_0 and ζ are the initial and final vorticity respectively. The initial vorticity field is

$$\zeta_0 = \frac{\partial(r_0 v_0)}{r_0 \partial r_0} = \begin{cases} \frac{2\varepsilon}{a^2} & r_0/a < 1 \\ 0 & r_0/a \geq 1 . \end{cases} \tag{4.13}$$

We note that the final vorticity increases outward from the centre until the discontinuity is reached. Beyond this point, the vorticity is negative and approaches zero for large r/a.

Figures 10-17 show the final adjusted state for the various fields described above, but for $\varepsilon = 0 \cdot 1$. These enable us to further describe the final adjusted state in terms of ε and a. For the geopotential field, we find that the larger value of ε results in a larger decrease of the geopotential at the centre. We can then conclude that the geopotential decrease at the centre is proportional to ε/a.

For the final adjusted tangential wind, we find that the decrease of the maximum wind is dependent on ε for a less than 1. For $a \geq 1$, it appears that the decrease of the maximum tangential wind is independent of ε.

In the case of the vorticity profiles, we note that the stretching is proportional to ε and inversely proportional to a. Hence the stretching is proportional to ε/a. We remark that this is consistent with the decrease of the geopotential at the centre.

We now compare the linear and non-linear solutions. We find that for $\varepsilon = 0.05$, the final adjusted geopotential is significantly different for $a = 0.2$. For $a \geq 0.5$, there is little difference for all practical purposes. The same is true for $\varepsilon = 0.1$.

For the final adjusted tangential wind, we find that for $a = 0.2$, the maximum wind for the geostrophic case is larger than that for the gradient case. This is consistent with the final adjusted geopotential field. For larger values of a, the difference between the two adjusted wind fields is small. Again, the same is true for $\varepsilon = 0.1$.

Comparison of the final vorticity for the two cases shows that for the gradient adjustment problem, the vorticity is smaller in magnitude than that for the geostrophic adjustment problem.

These results indicate that the differences between the gradient and geostrophic adjustment problems are more pronounced for small a. In figures 19-22, we show the final adjusted fields of geopotential and tangential wind for $a = 0.1$, 0.15, 0.25 and $\varepsilon = 0.05$, 0.1. The differences between the linear and non-linear adjustment problems are significant.

Figures 23-24 show the final adjusted vorticity in the gradient case for $a = 0.1$, 0.15, 0.25 and $\varepsilon = 0.05$, 0.1. Figure 25 shows the final adjusted vorticity in the geostrophic case for $a = 0.1$, 0.15, 0.25. Comparison of the final vorticity in these two cases shows that, in general, the geostrophic approximation overestimates the final vorticity.

It should be noted that for small ε and large a, the geostrophic and gradient solutions should be approximately equal. Therefore, agreement between the two solutions for small ε and large a is a good check on the validity of the solution for the gradient adjustment problem.

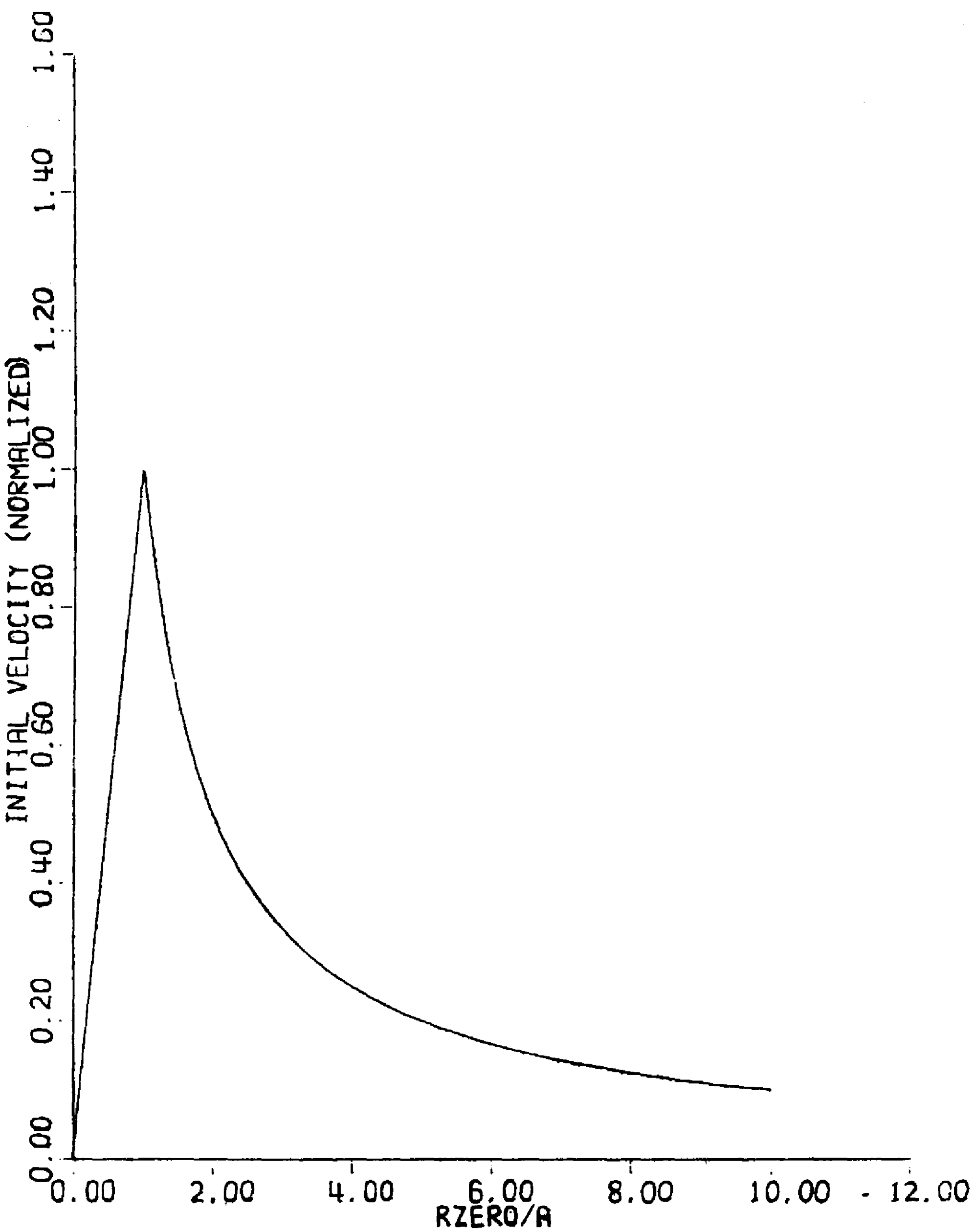

Figure 1. Normalized initial tangential wind ($v_o/v_o(a)$) as a function of r_o/a.

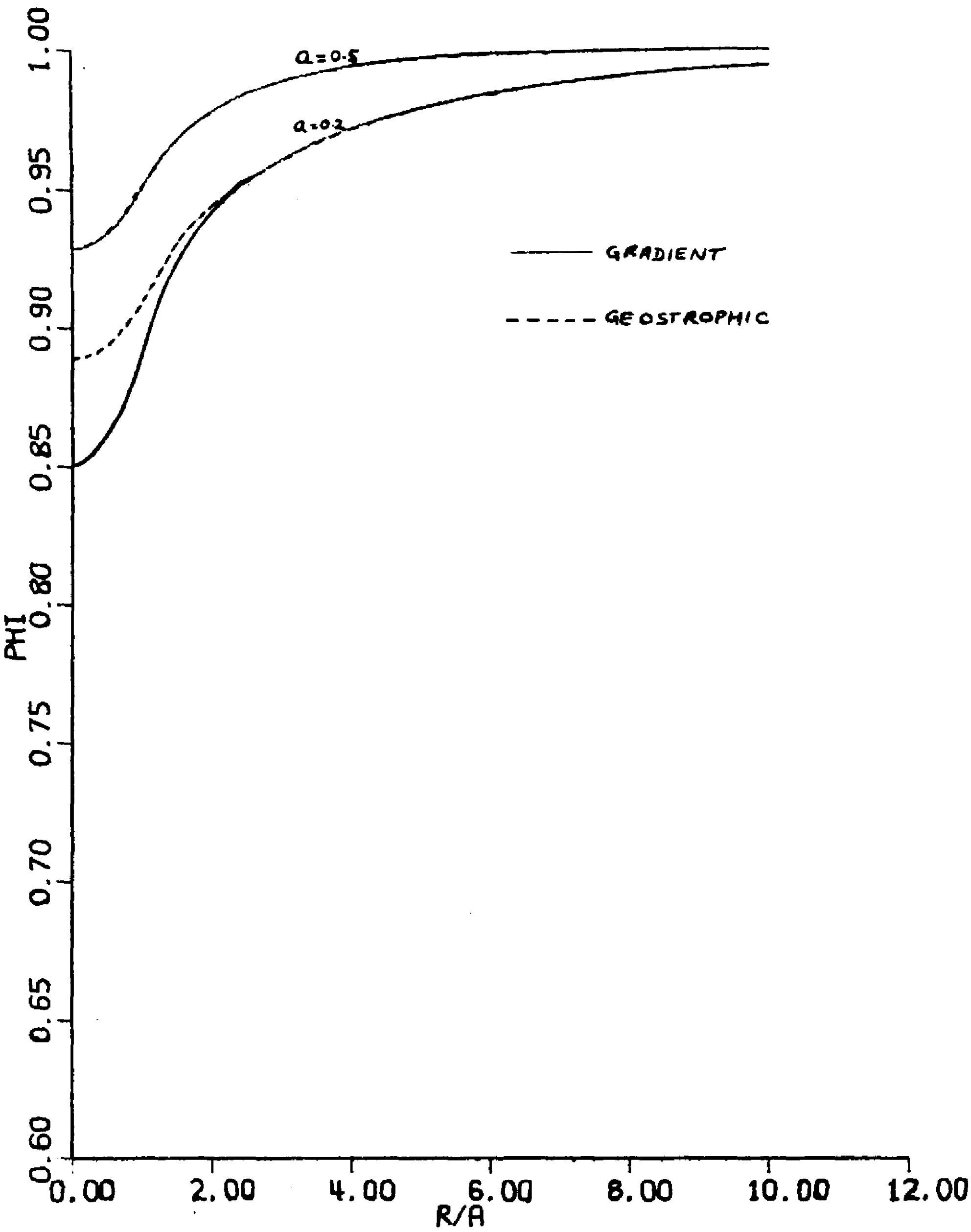

Figure 2. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 0.2$, 0.5.

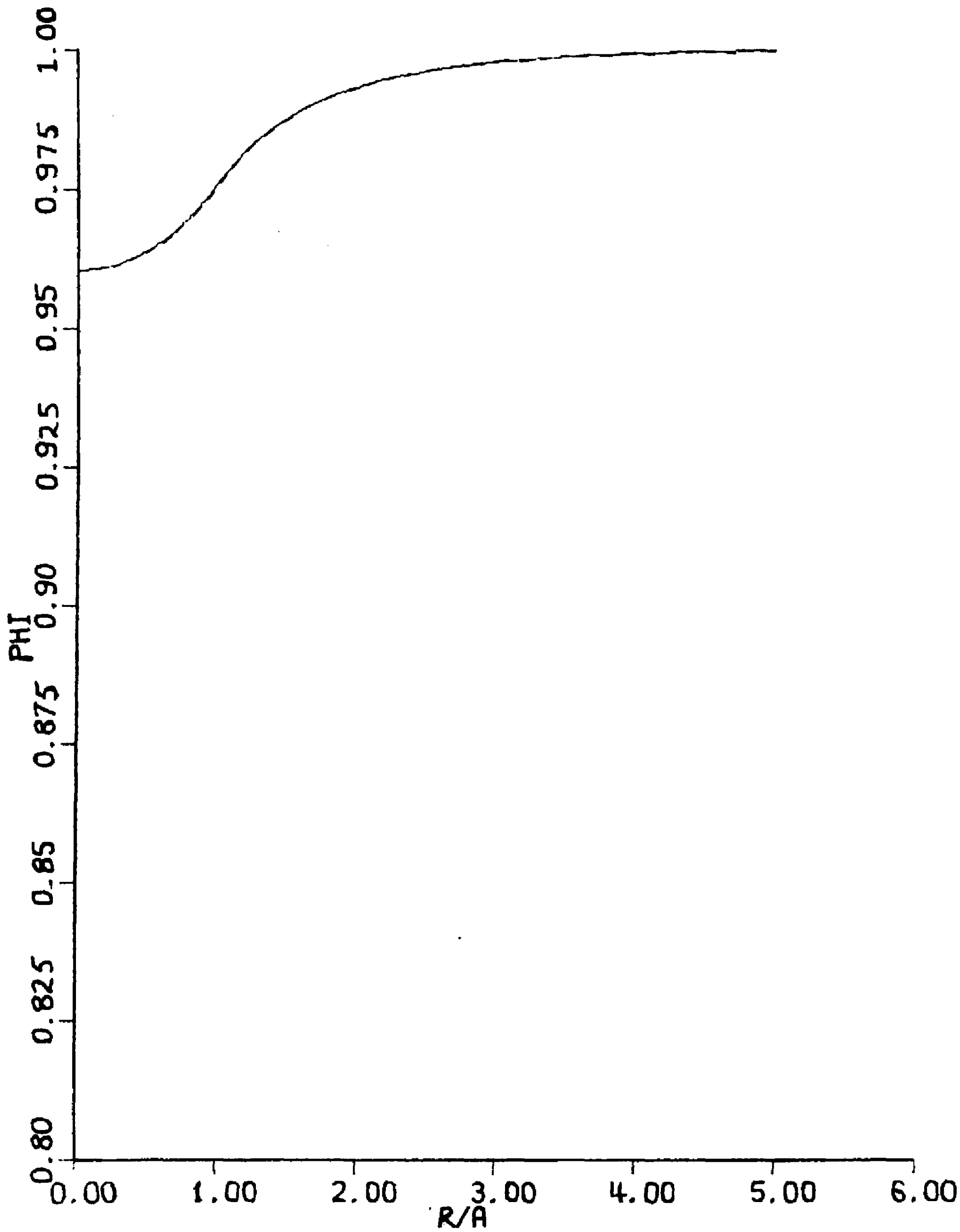

Figure 3. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 1.0$.

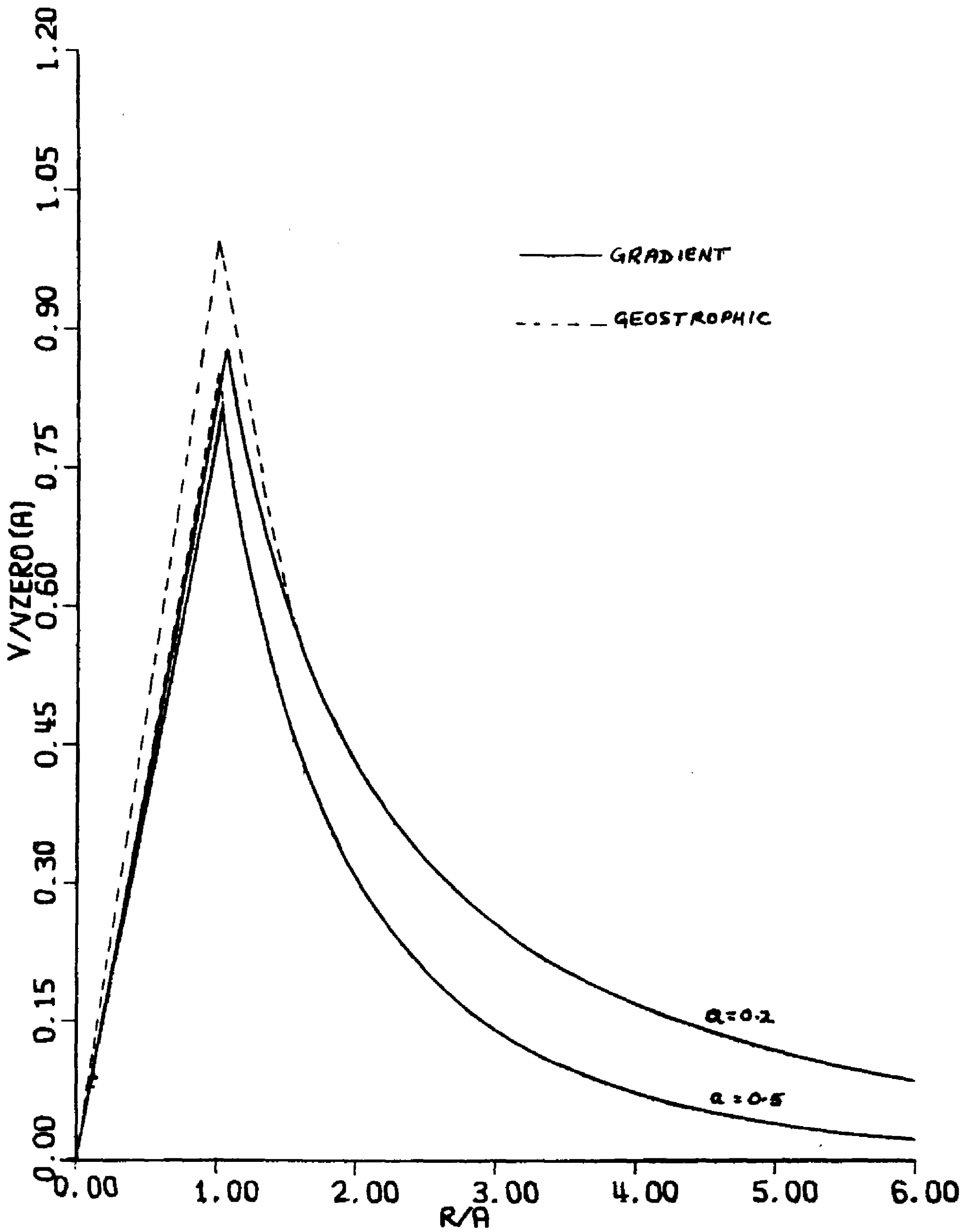

Figure 4. Normalized final tangential wind ($v/v_0(a)$) for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 0.2,\ 0.5$.

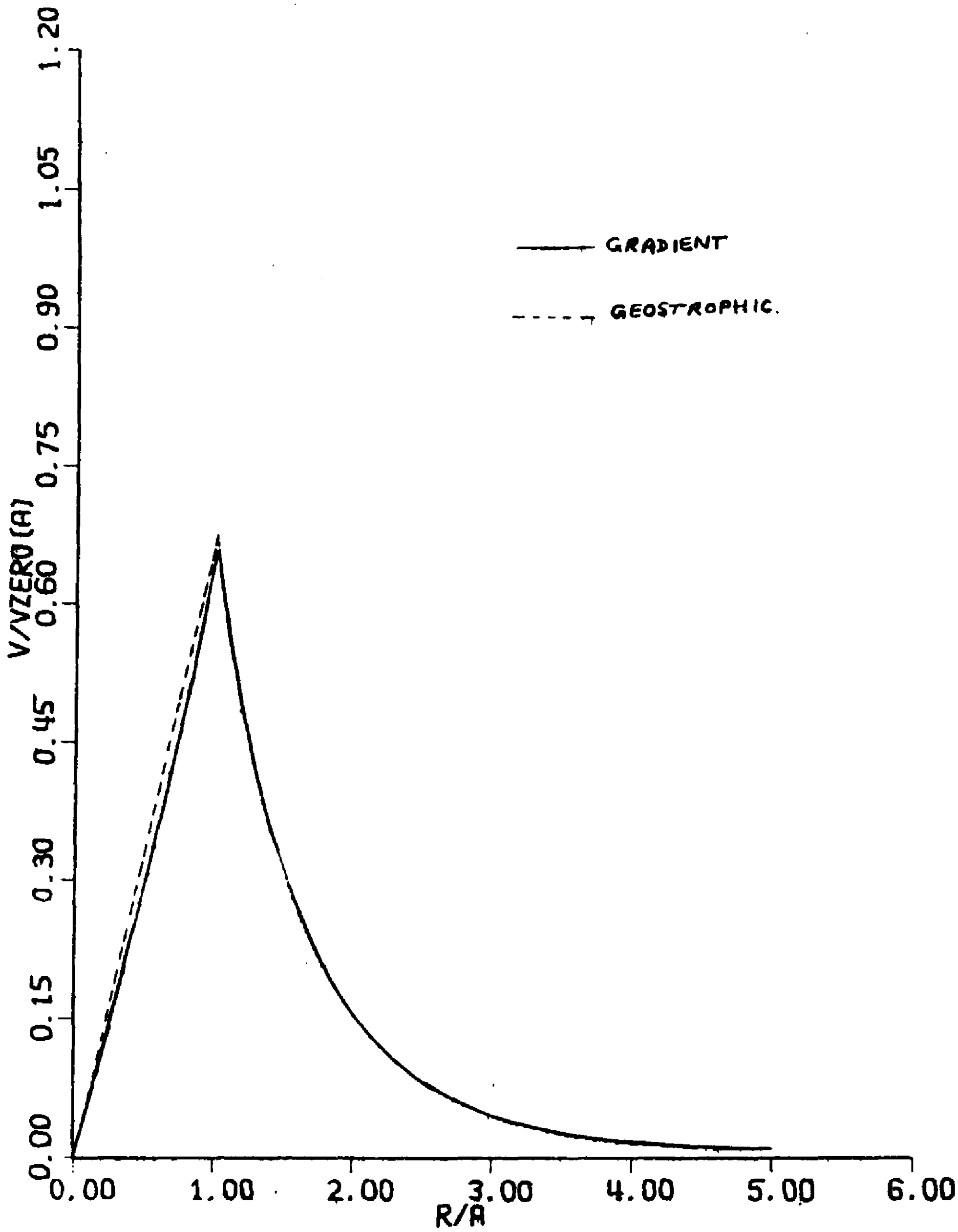

Figure 5. Normalized final tangential wind ($v/v_0(a)$) for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 1.0$.

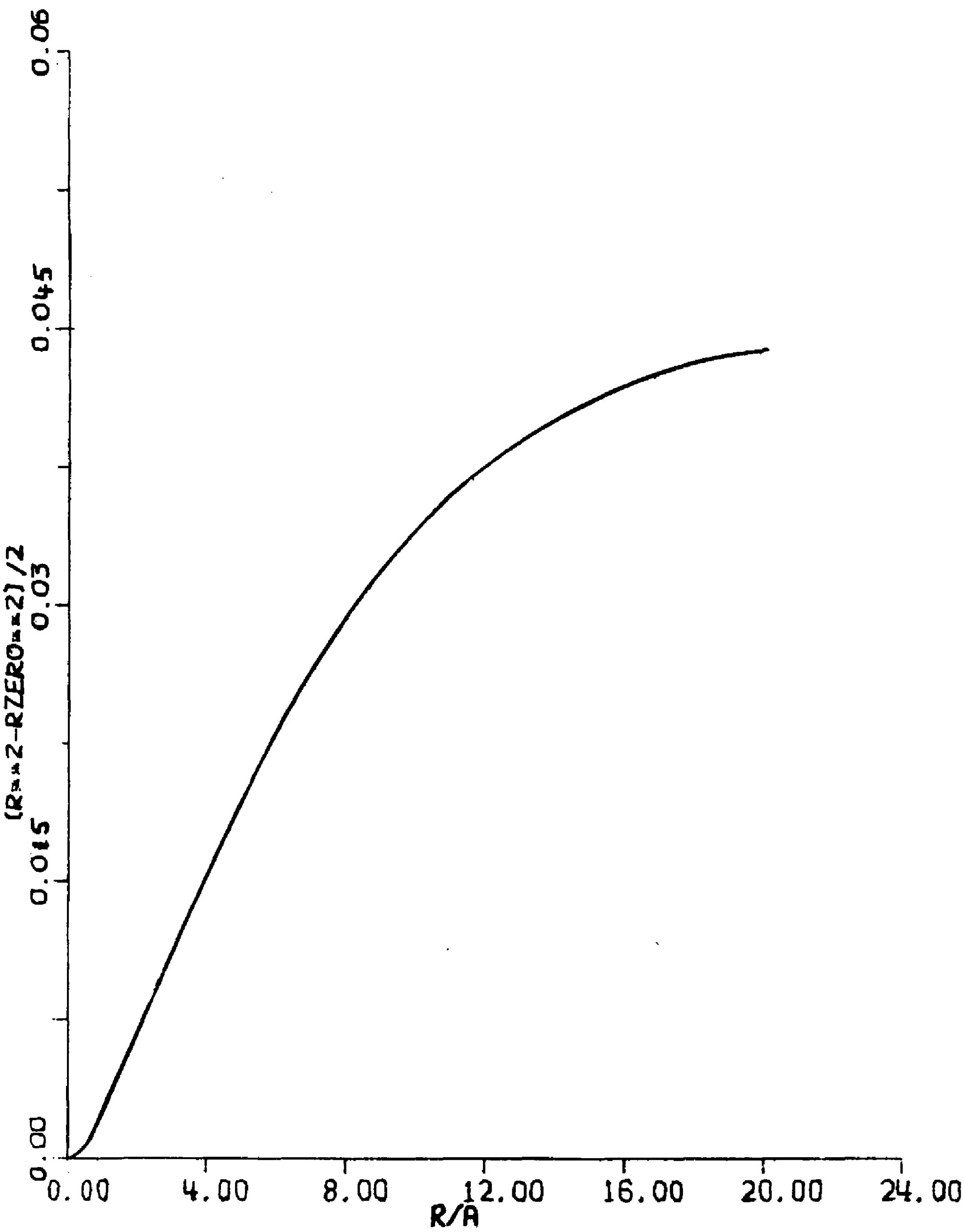

Figure 6. Mass removed as a function of r/a. $\varepsilon = 0.05$ and $a = 0.2$.

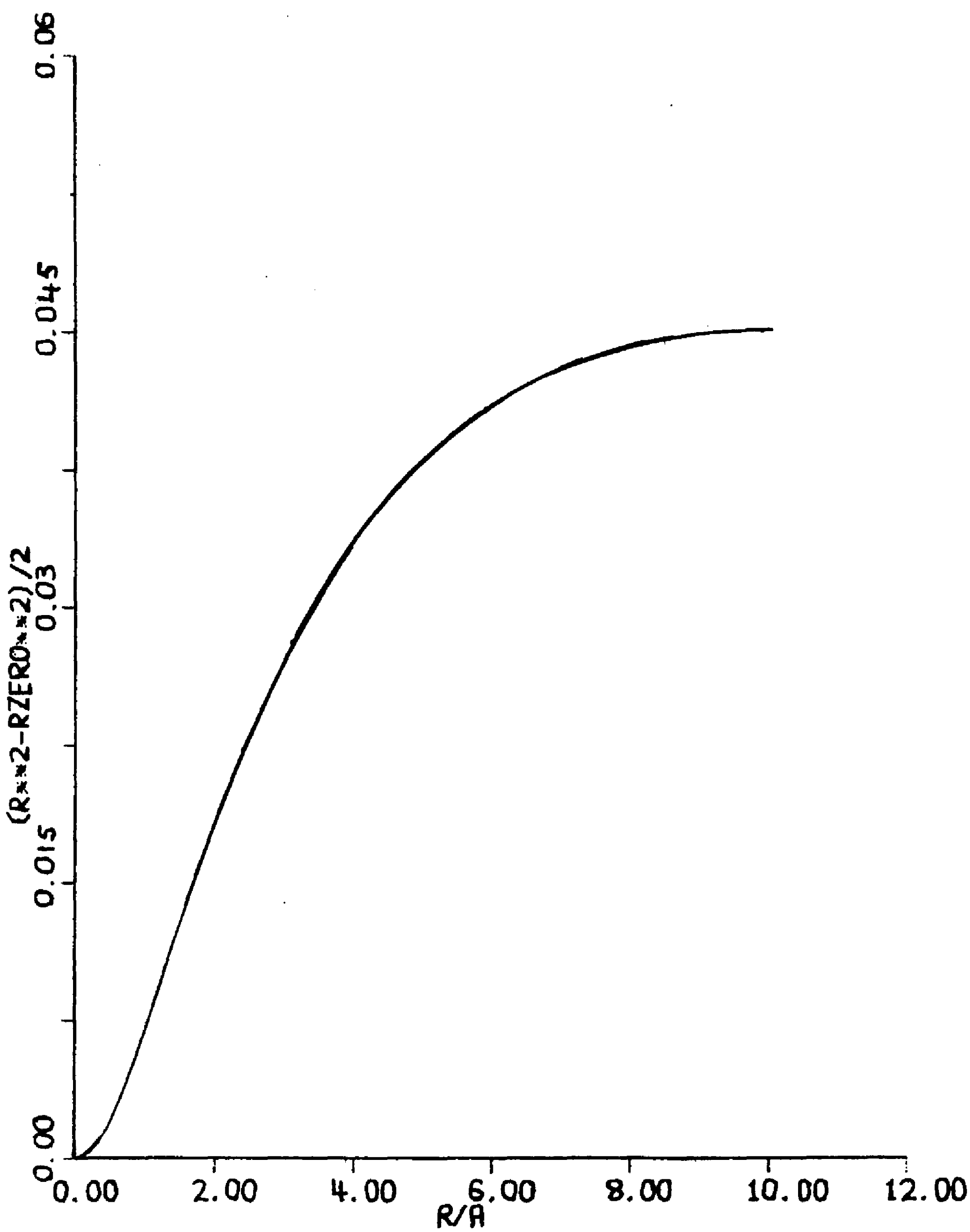

Figure 7. Mass removed as a function of r/a. $\varepsilon = 0.05$ and $a = 0.5$.

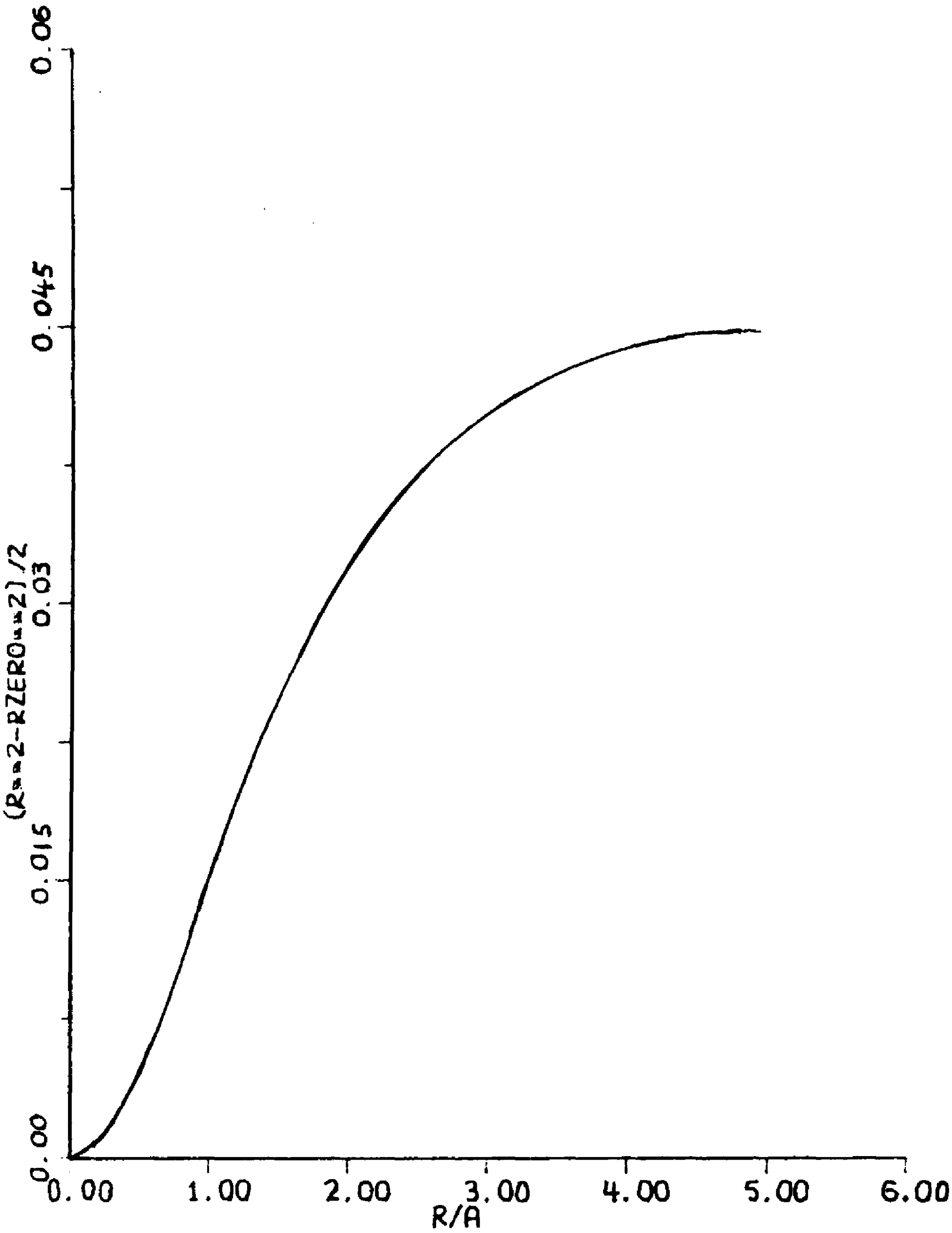

Figure 8. Mass removed as a function of r/a. $\varepsilon = 0.05$ and $a = 1.0$.

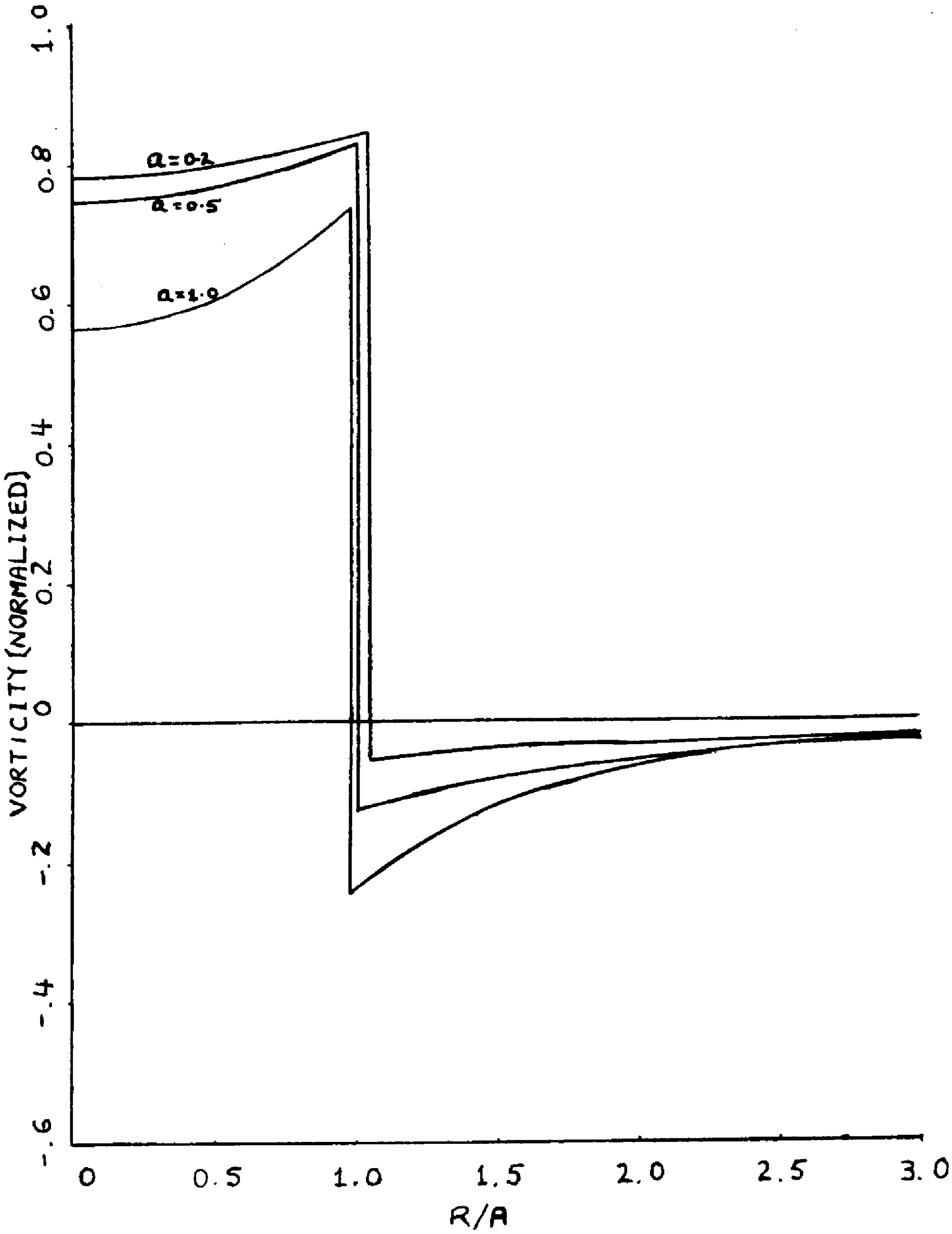

Figure 9. Normalized final gradient vorticity ($\zeta/\zeta_0(o)$). $\varepsilon = 0.05$ and $a = 0.2$, 0.5, 1.0.

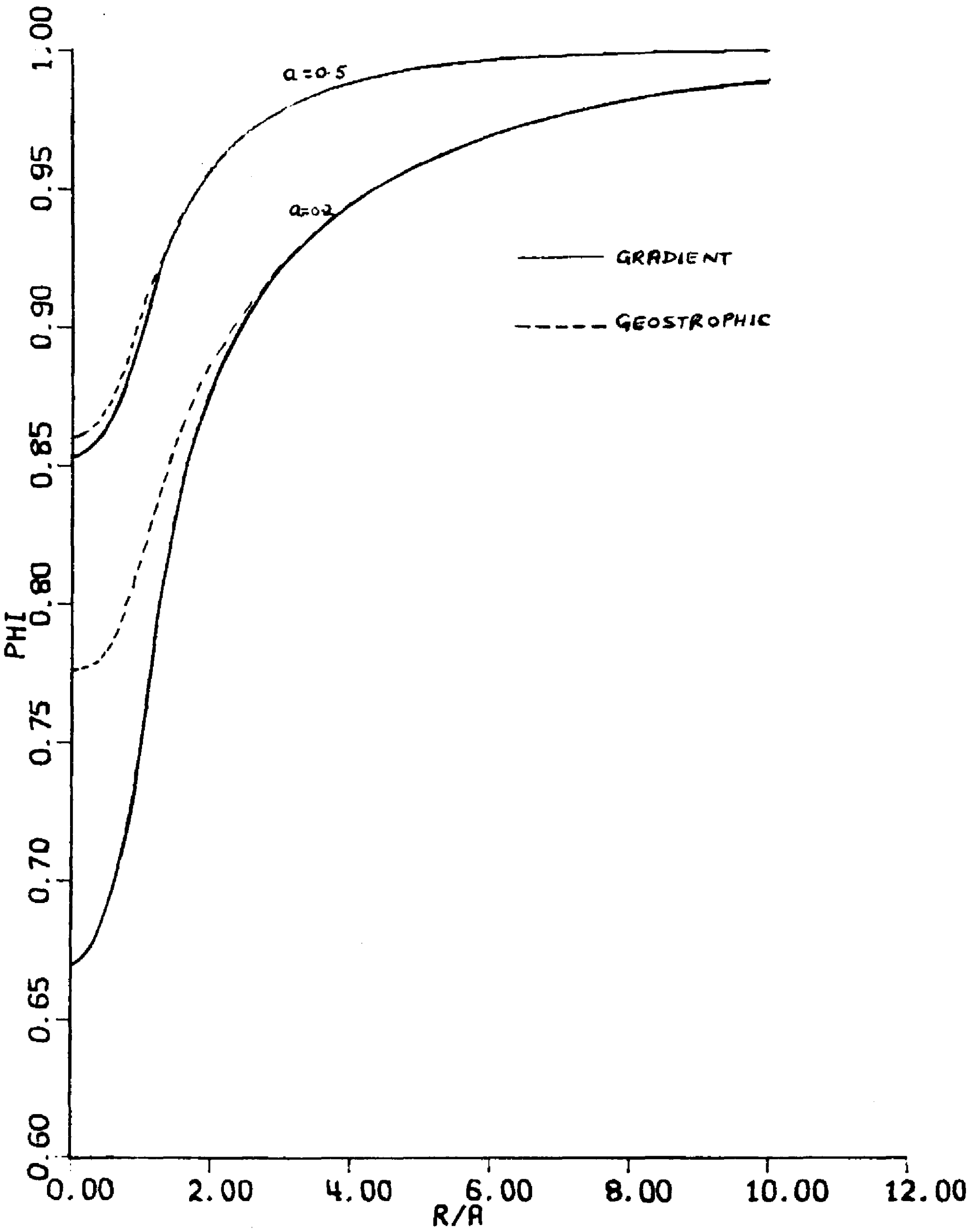

Figure 10. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.1$ and $a = 0.2, 0.5$.

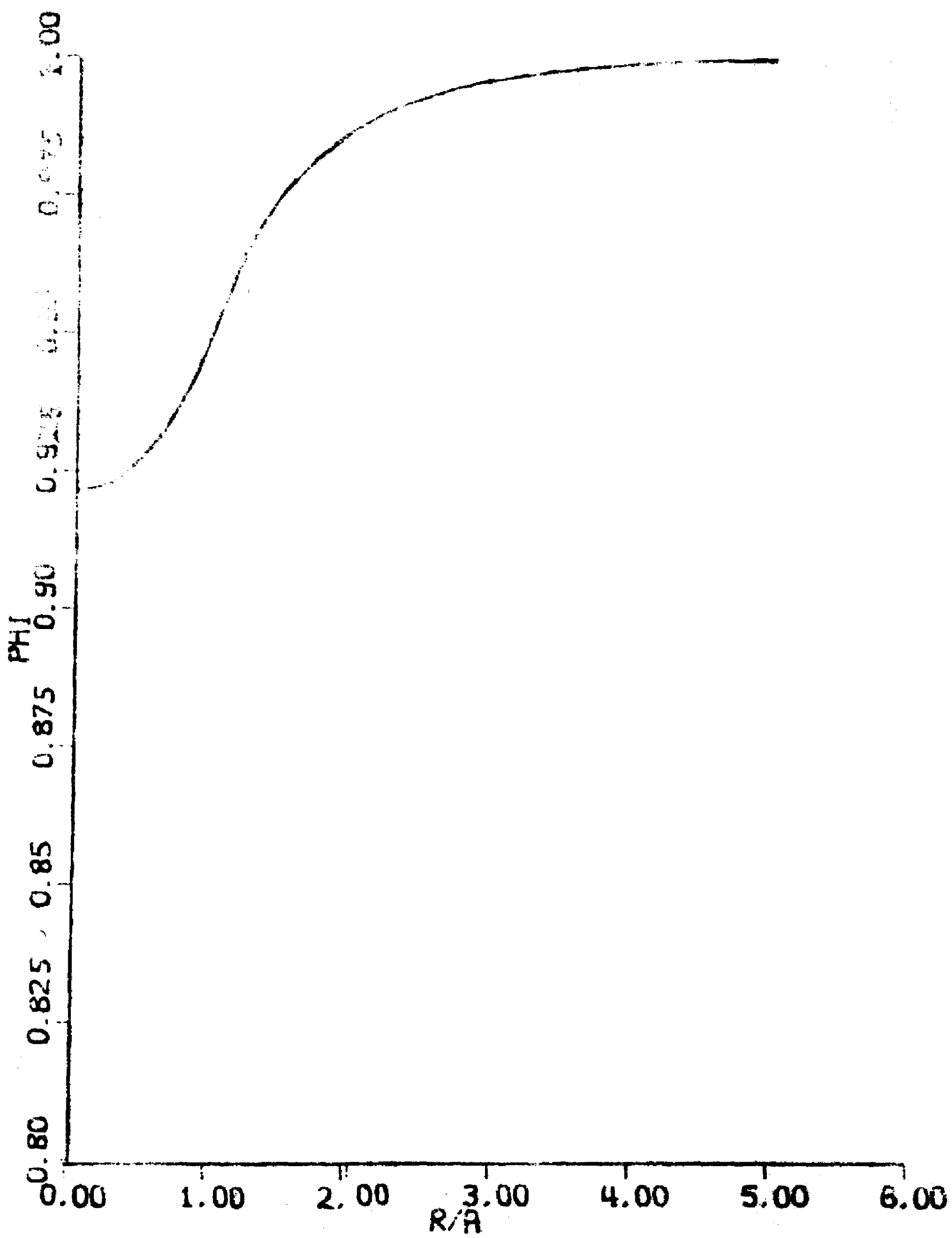

Figure 11. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.1$ and $a = 1.0$.

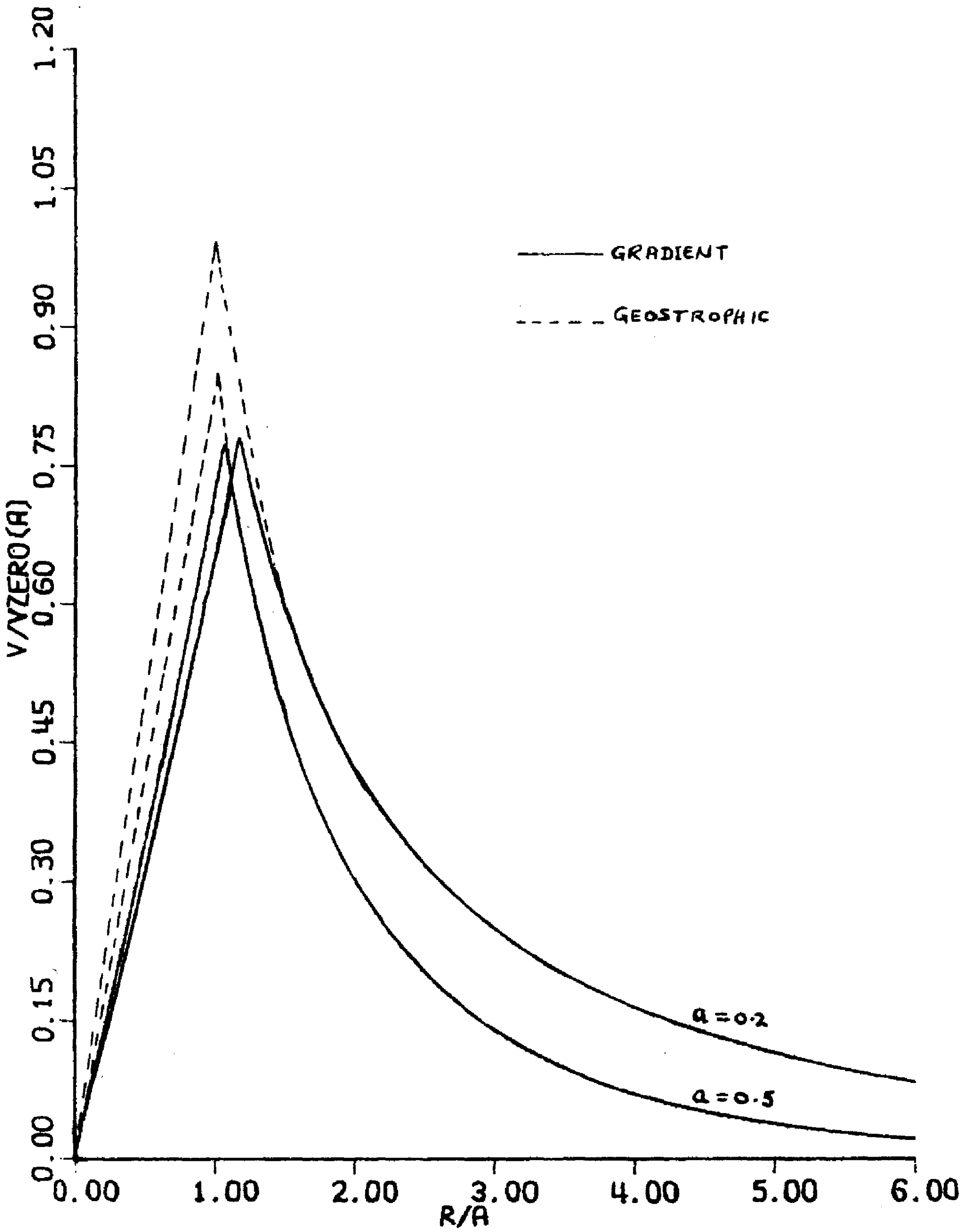

Figure 12. Normalized final tangential wind ($v/v_o(a)$) for both geostrophic and gradient adjustment. $\varepsilon = 0.1$ and $a = 0.2, 0.5$.

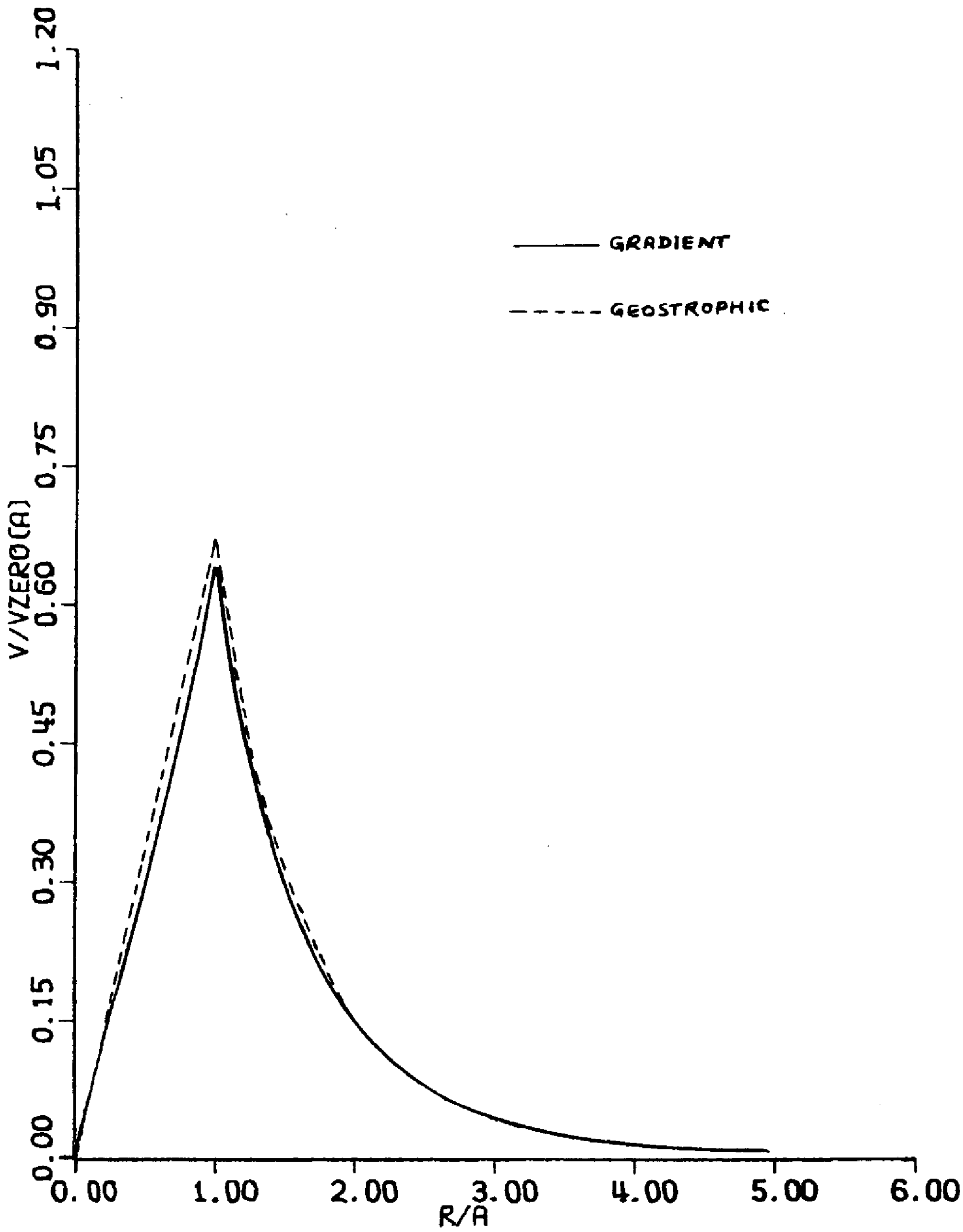

Figure 13. Normalized final tangential wind ($v/v_o(a)$) for both geostrophic and gradient adjustment. $\varepsilon = 0.1$ and $a = 1.0$.

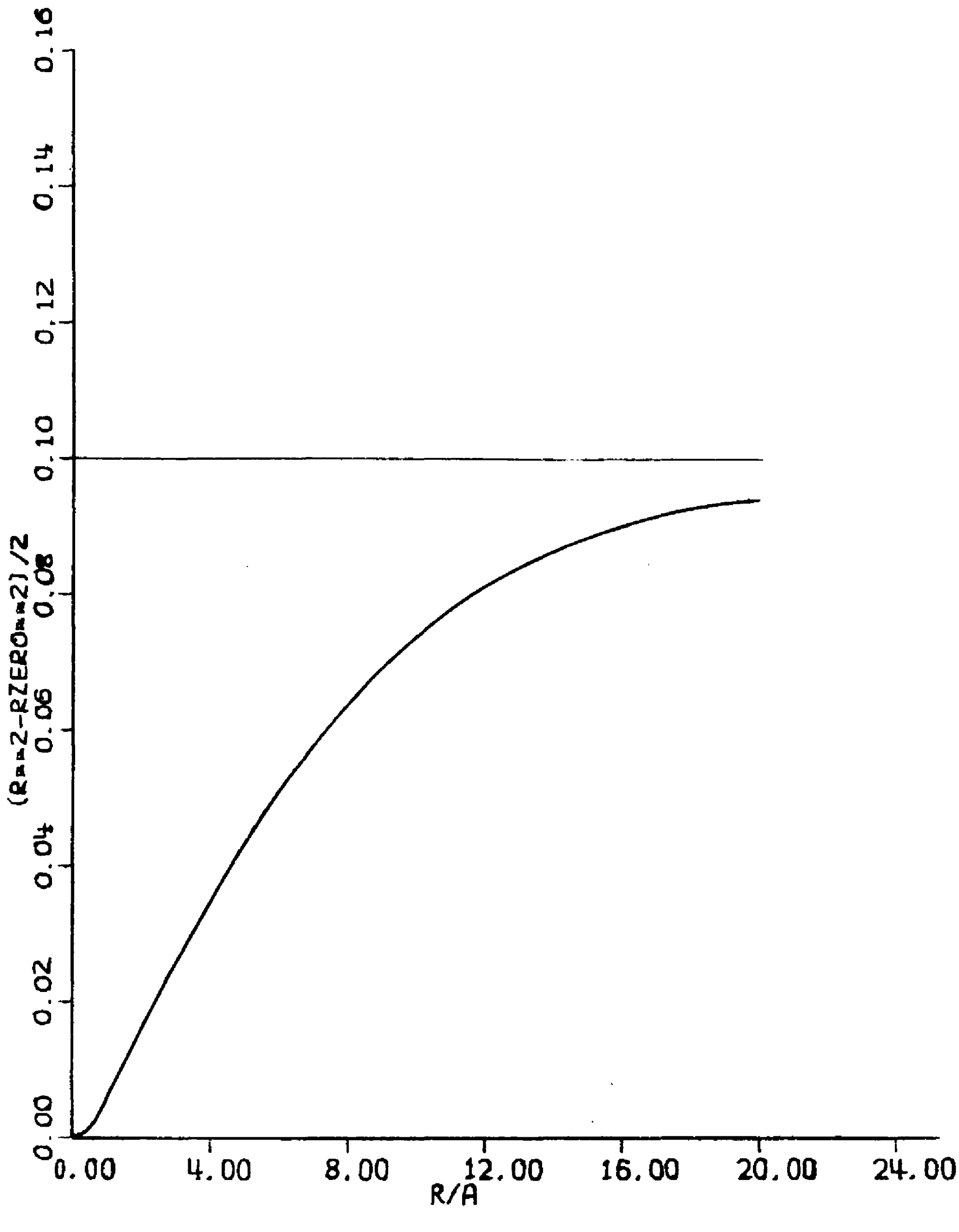

Figure 14. Mass removed as a function of r/a. $\varepsilon = 0.1$ and $a = 0.2$.

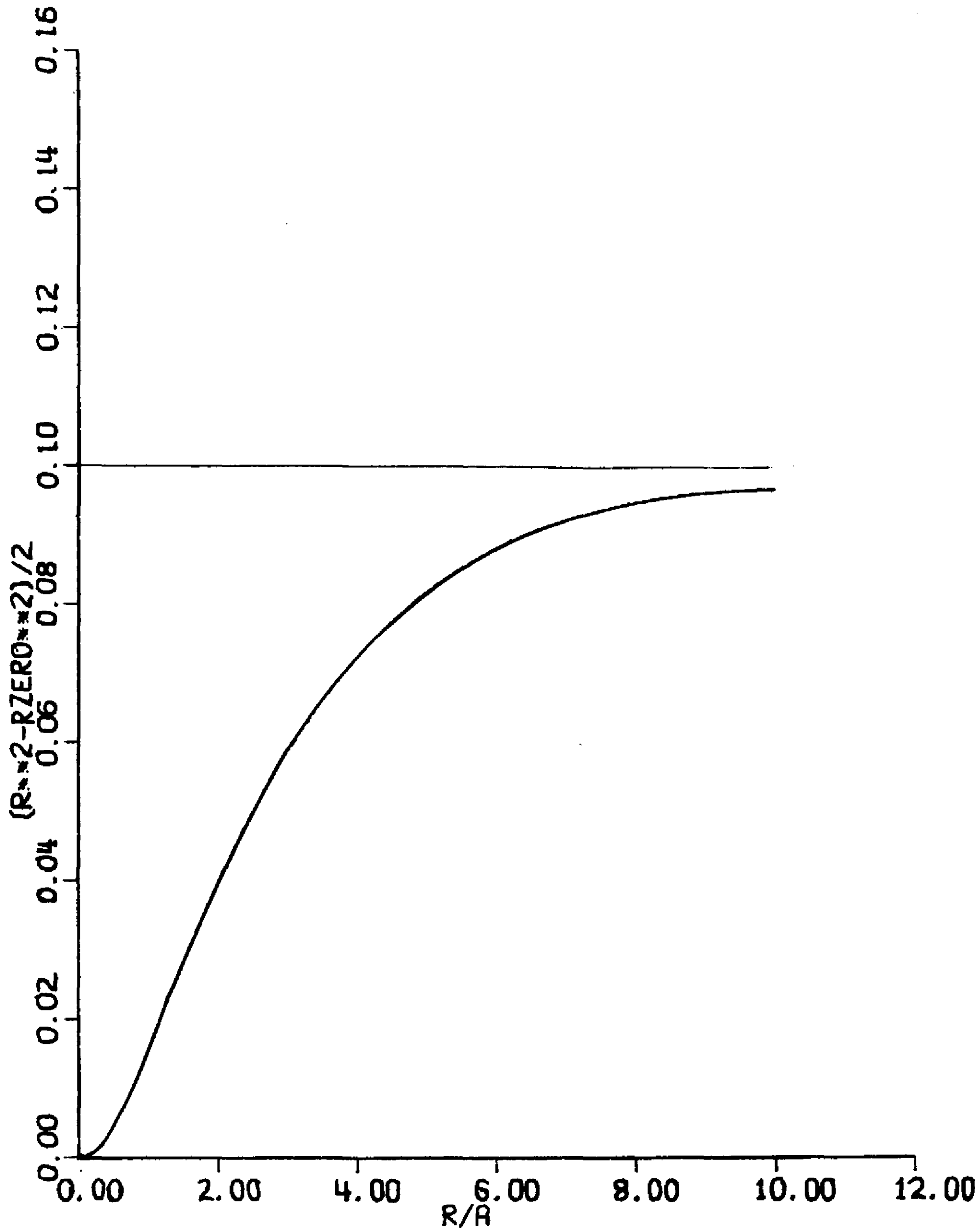

Figure 15. Mass removed as a function of r/a. $\varepsilon = 0.1$ and $a = 0.5$.

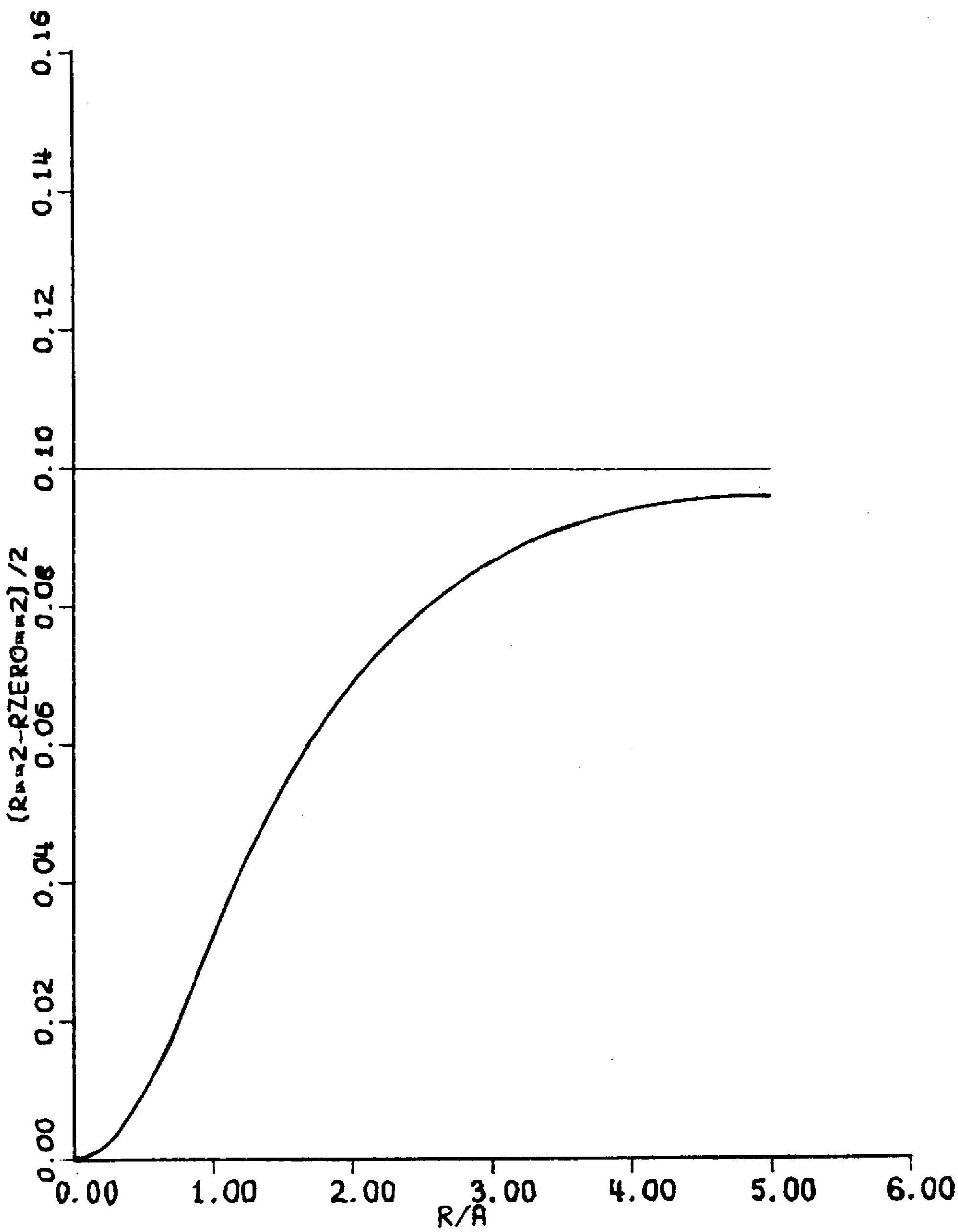

Figure 16. Mass removed as a function of r/a. $\varepsilon = 0.1$ and $a = 1.0$.

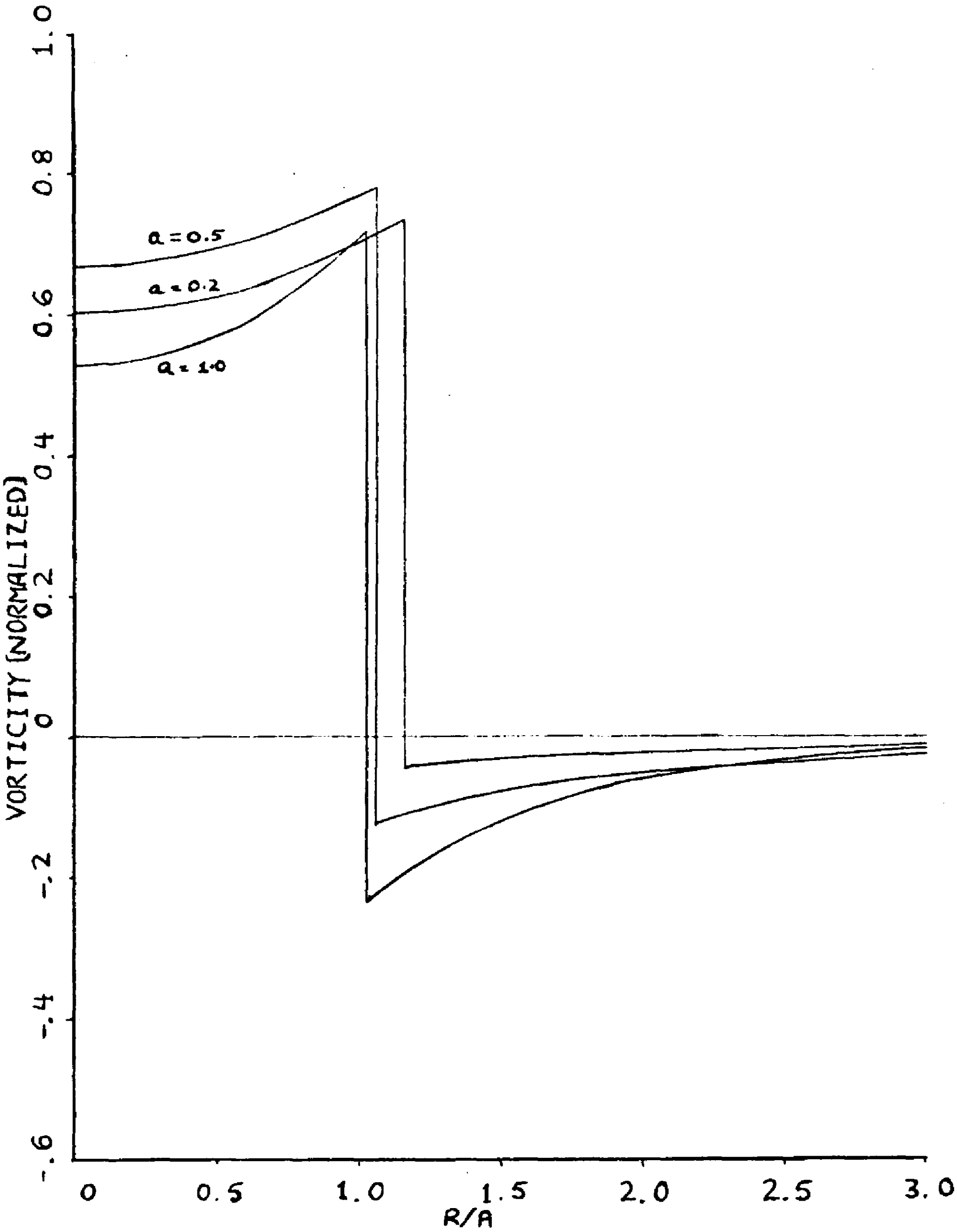

Figure 17. Normalized final gradient vorticity $(\zeta/\zeta_0(o))$. $\varepsilon = 0.1$ and $a = 0.2, 0.5, 1.0$.

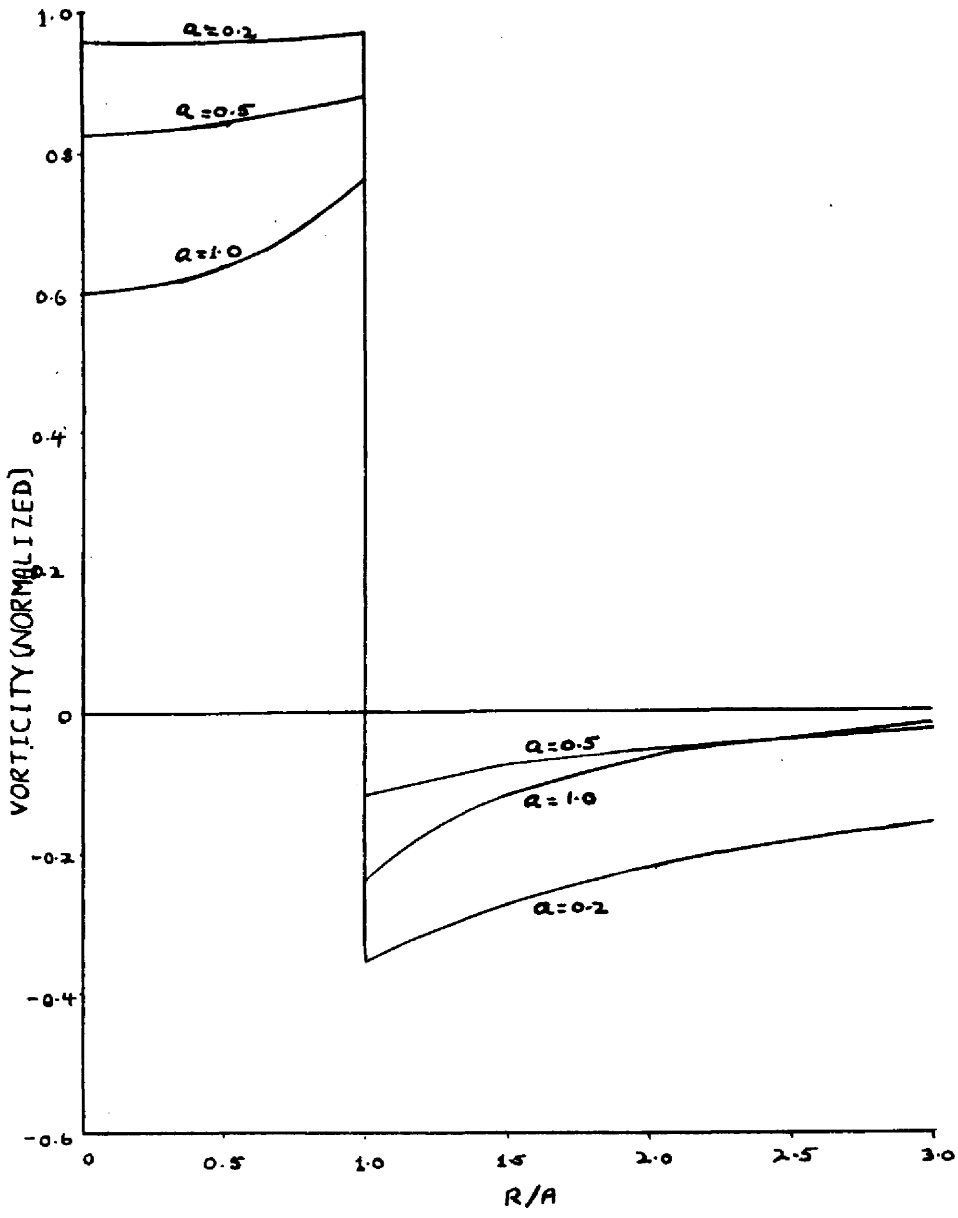

Figure 18. Normalized final geostrophic vorticity for a = 0.2, 0.5, 1.0.

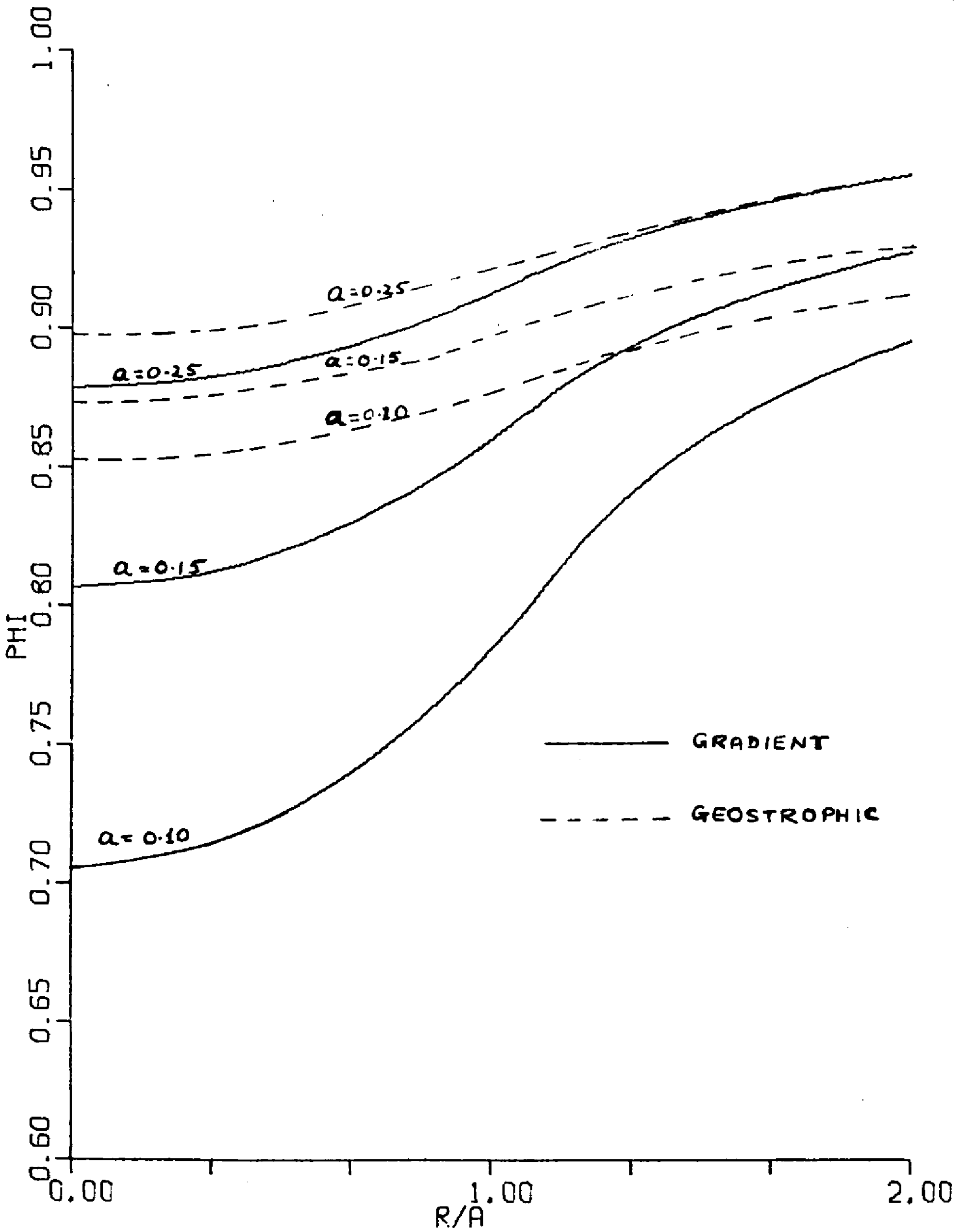

Figure 19. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 0.10$, 0.15, 0.25.

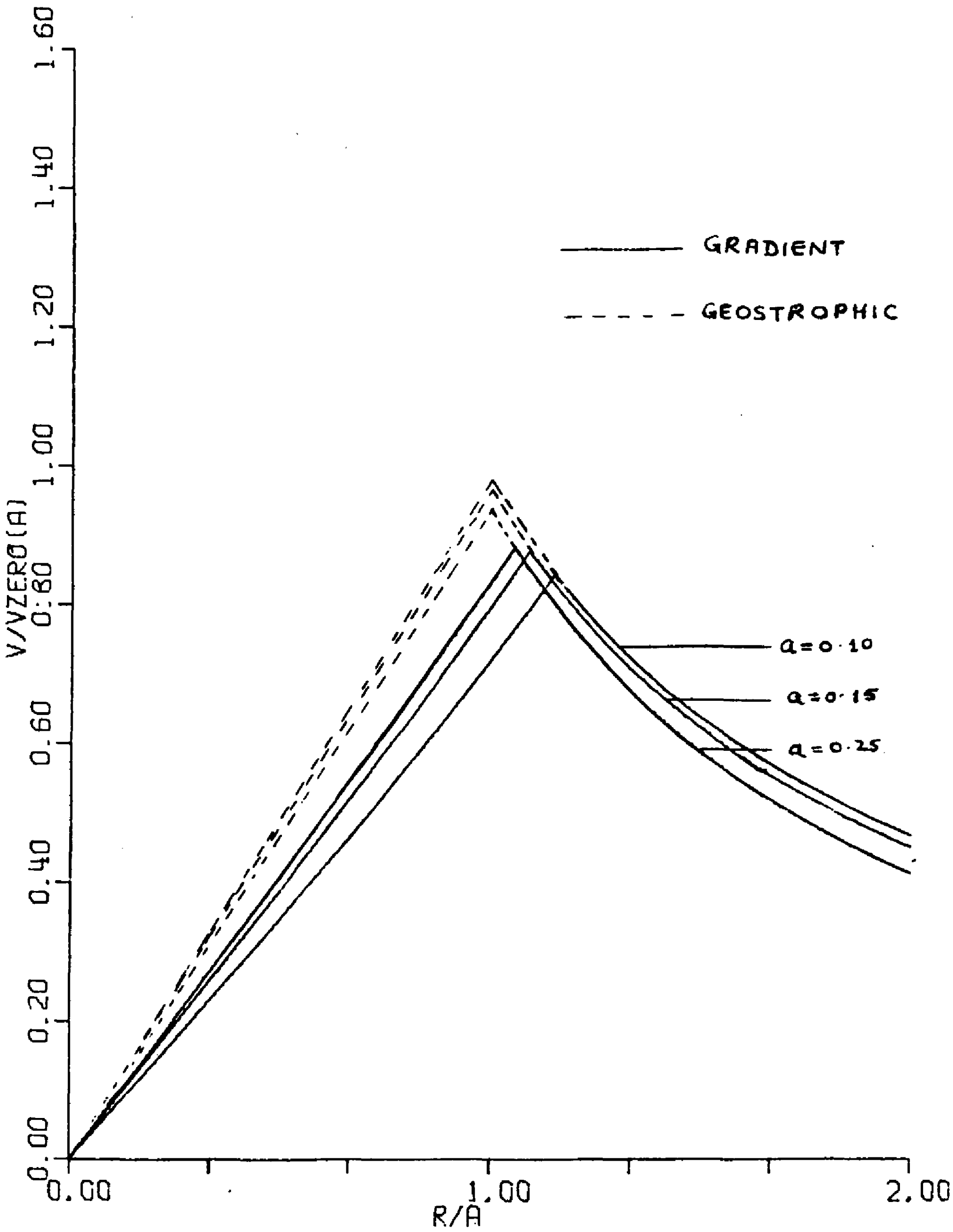

Figure 20. Normalized final tangential wind $(v/v_o(a))$ for both geostrophic and gradient adjustment. $\varepsilon = 0.05$ and $a = 0.10, 0.15, 0.25$.

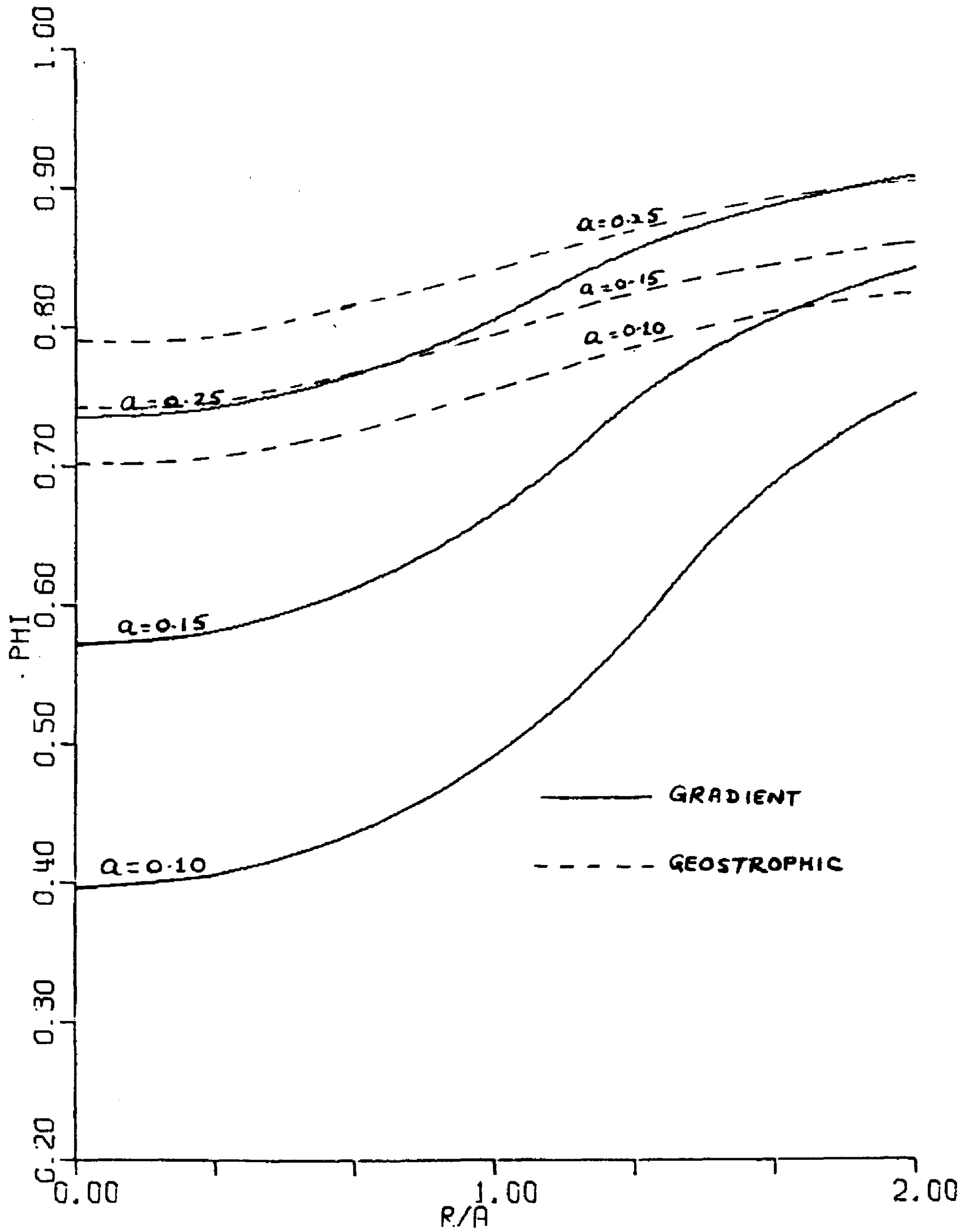

Figure 21. Final adjusted geopotential for both geostrophic and gradient adjustment. $\varepsilon = 0.10$ and $a = 0.10$, 0.15, 0.25.

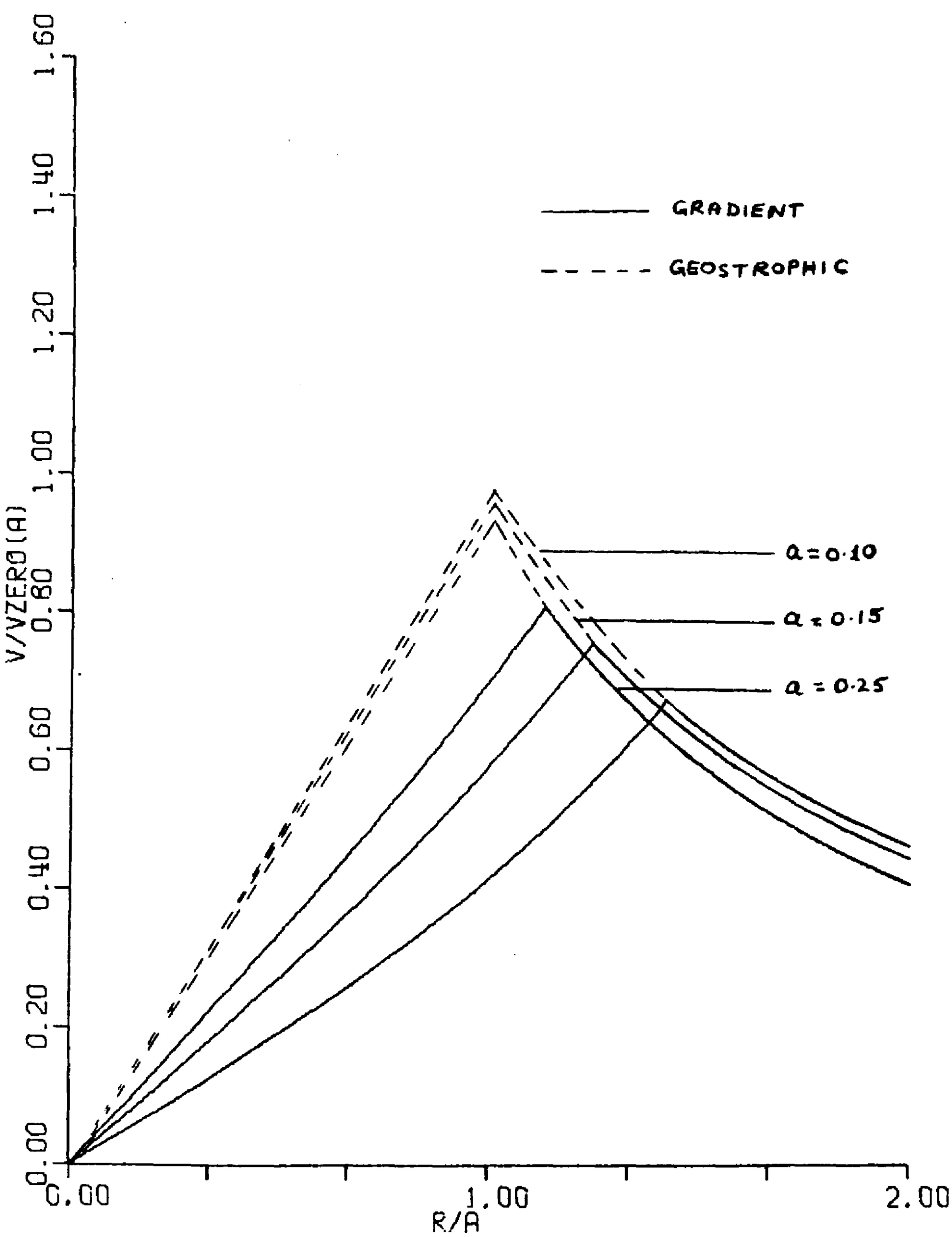

Figure 22. Normalized final tangential wind ($v/v_o(a)$) for both geostrophic and gradient adjustment. $\varepsilon = 0.10$ and $a = 0.10, 0.15, 0.25$.

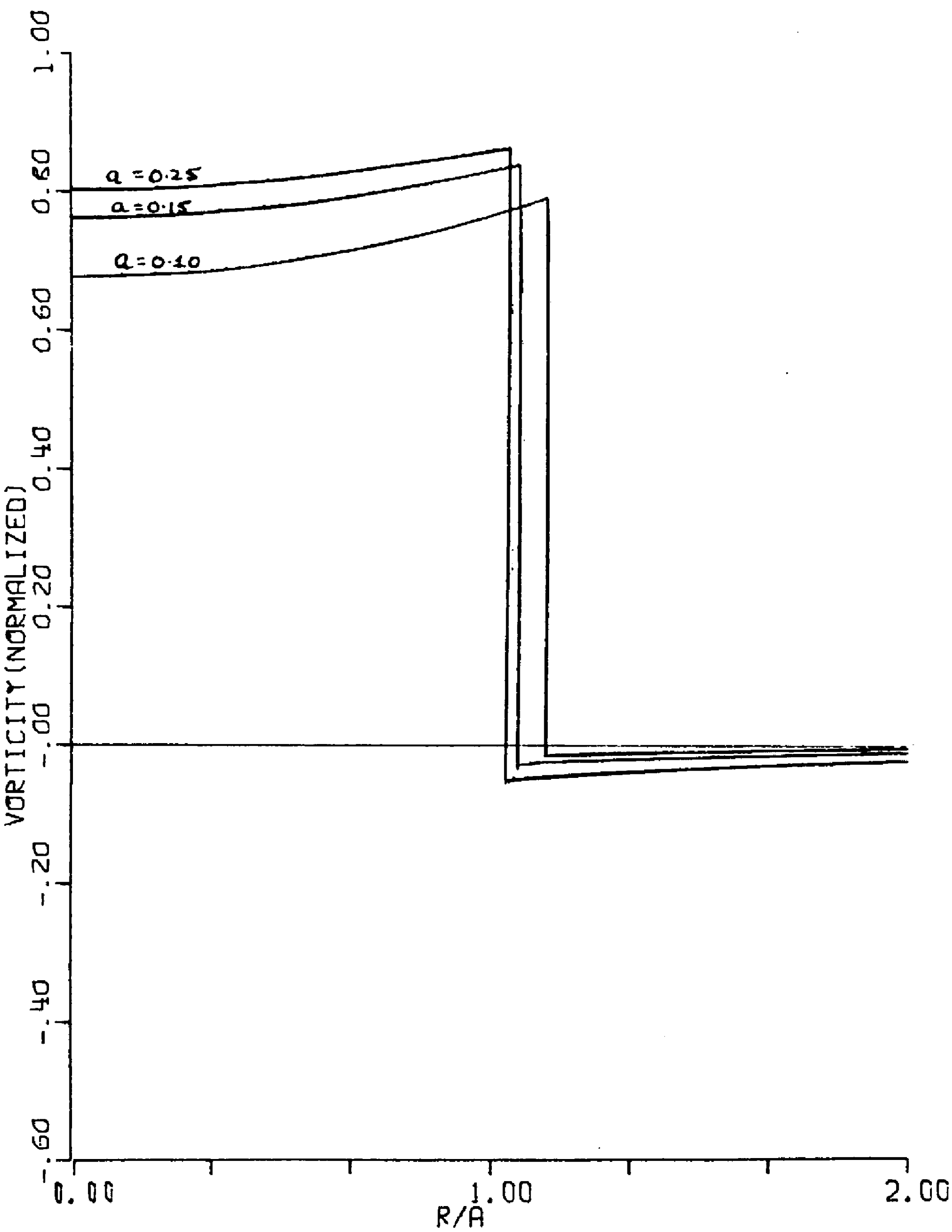

Figure 23. Normalized final gradient vorticity ($\zeta/\zeta_0(o)$). $\varepsilon = 0.05$ and $a = 0.10, 0.15, 0.25$.

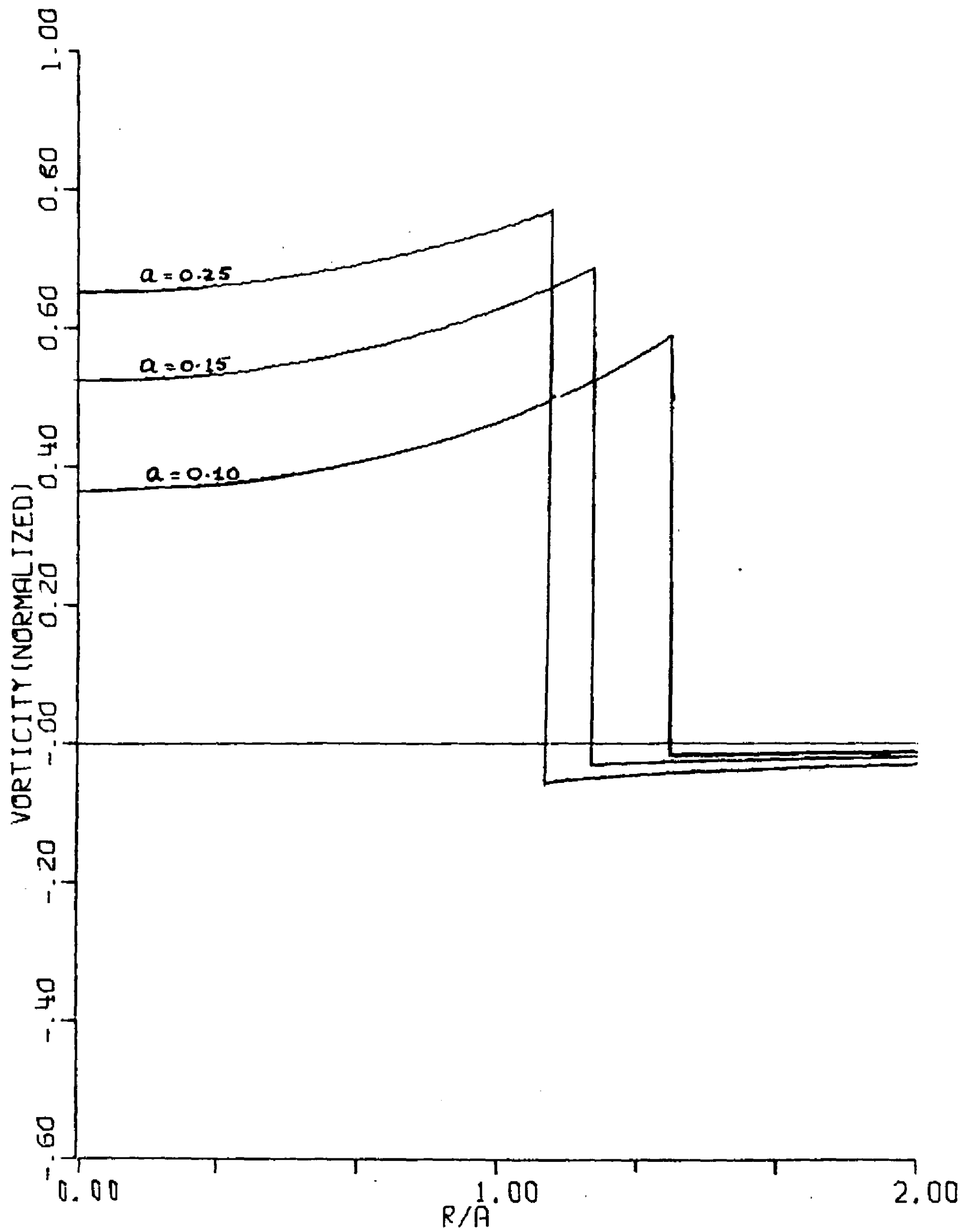

Figure 24. Normalized final gradient vorticity $(\zeta/\zeta_0(o))$. $\varepsilon = 0.10$ and $a = 0.10, 0.15, 0.25$.

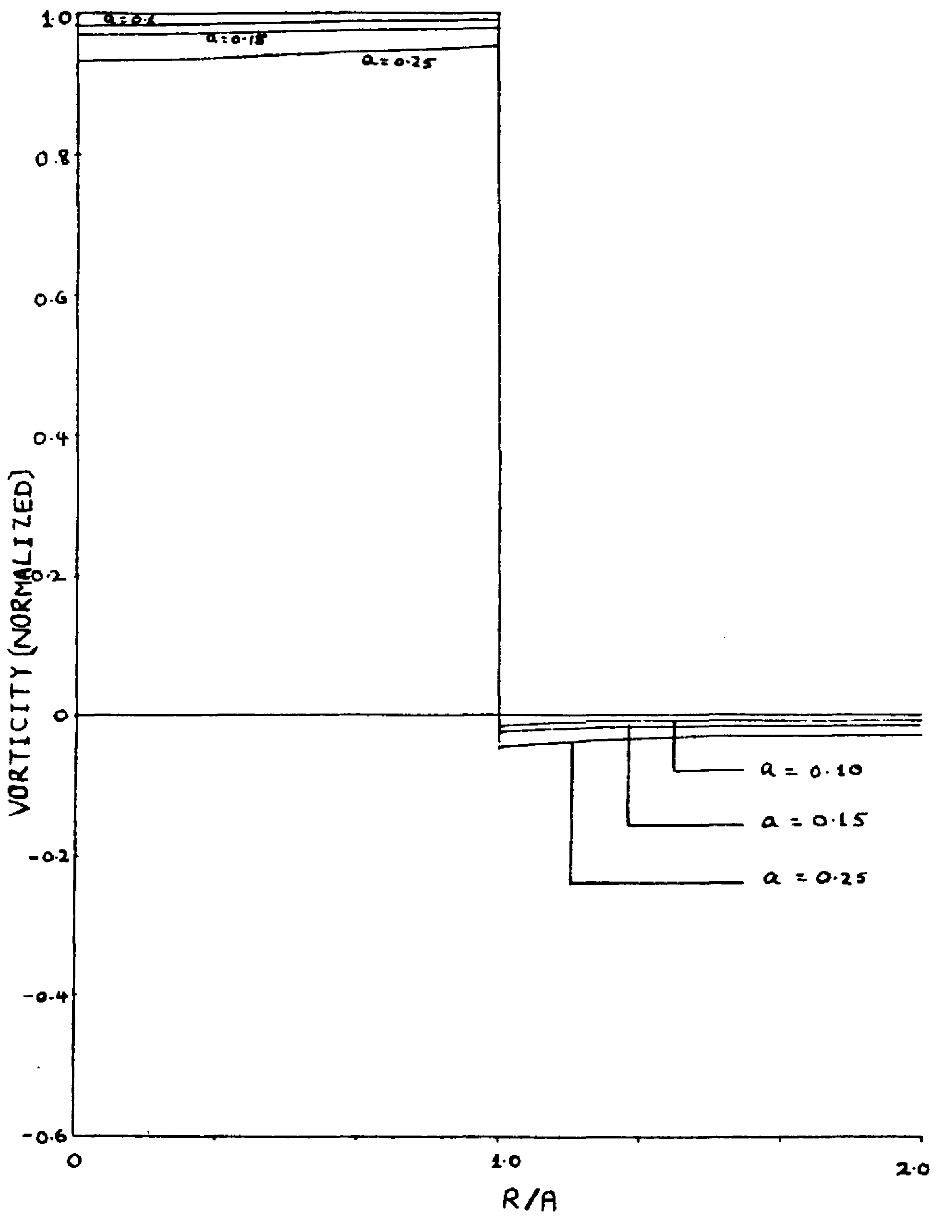

Figure 25. Normalized final geostrophic vorticity for a = 0.10, 0.15, 0.25.

5. SUMMARY AND CONCLUSIONS

We have derived a system of equations for the final adjusted state under gradient balance in an axisymmetric vortex. The initial unbalanced state is one of a flat geopotential and a velocity profile given by the Rankine vortex

$$r_0 v_0 = \begin{cases} \dfrac{\varepsilon r_0^2}{a^2} & , \quad r_0 < a \\ \varepsilon & , \quad r_0 \geq a \, . \end{cases}$$

A redistribution of mass occurs until the pressure and wind fields are in gradient balance. Solutions of the system of equations for the final adjusted state are obtained numerically using the shooting method.

The results indicate that the geopotential decrease at the centre is proportional to ε/a while the decrease in the maximum tangential wind is dependent on ε. The final vorticity increases with radius from the centre until the discontinuity is reached. Beyond this discontinuity, the vorticity is negative and approaches zero for large r. An interesting feature of the solution is the apparent non-conservation of mass. We have shown that for large radius, the mass removed is constant. It is believed that the gravity-inertia waves that develop as a result of the adjustment process cause oscillations of the free surface at large radii. We recall that in the Rossby-Mihaljan problem, energy is not conserved and that conservation of mass is due to the symmetry of the problem. If such symmetry does not occur in our problem then it should not be surprising that within the domain of integration mass is not conserved.

In chapter 3, we have presented solutions for the final adjusted state under geostrophic balance. A comparison between the linear and non-linear solutions shows that significant differences appear only for small a.

It would be appropriate at this point to consider a value of the Rossby radius of deformation. Recall that the dimensionless radial distance r' is given by

$$r' = \frac{rf}{C}$$

where r is the dimensional radial distance. For $r' = 1$, we get

$$r = \frac{C}{f} = \lambda$$

where λ is the Rosby radius of deformation. At 20°N latitude $\frac{1}{f} \approx 5.6$ hours. Following Kasahara (1976), we can obtain a value for C. For a six-layer model atmosphere the equivalent height for the second mode is 823 m. This gives a value of $C \approx 90\ \mathrm{ms}^{-1}$. Hence at 20°N latitude and for $C \approx 90\ \mathrm{ms}^{-1}$, $\lambda \approx 1800$ km.

In conclusion, it appears that for large scale disturbances which are not very intense the geostrophic approximation can be useful for a description of such disturbances. However, for smaller and more intense disturbances the departure from geostrophy may be more pronounced. It would therefore be necessary to consider the non-linear effects in the adjustment process.

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7. Author(s) S. Raghunandan and W. H. Schubert	8. Performing Organization Rept. No. 301
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15. Supplementary Notes

16. Abstracts

We consider gradient adjustment in an axisymmetric vortex. From an initially unbalanced state, the final adjusted state is obtained. Initially, the geopotential is flat i.e. $\phi_o(r_o) = 1$ and the velocity field is given by the Rankine vortex

$$r_o v_o = \begin{cases} \dfrac{\varepsilon r_o^2}{a^2} & r_o < a \\ \varepsilon & r_o \geq a \end{cases}$$

We derive a system of equations governing the final adjusted state. This system of equations is solved numerically using the shooting method. We also present solutions for the goestrophic adjustment problem and compare these to the solutions for the gradient adjustment problem. (continued)

17. Key Words and Document Analysis. 17a. Descriptors

Gradient adjustment
Axisymmetric vortex
Rankine vortex
Geostrophic adjustment
Potential vorticity
Shooting method

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

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16. Abstracts continued

In the gradient adjustment case, the results indicate that

(i) the geopotential decrease at the centre is proportional to ε/a.

(ii) the decrease of the maximum tangential wind is dependent on ε for smaller $a (a < 1)$. For $a \geq 1$, the decrease is independent of ε.

(iii) the stretching is proportional to ε/a.

(iv) the mass removed from within a given region tends to ε for for large r.

Comparison between the solutions for the geostrophic and gradient adjustment problems shows significant differences for small a.