DISSERTATION

EKEDAHL-OORT AND NEWTON STRATIFICATIONS ON UNITARY SHIMURA VARIETIES, AND ON HODGE-NEWTON REDUCIBLE LOCAL SHIMURA DATA OF ABELIAN TYPE

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ABSTRACT

EKEDAHL-OORT AND NEWTON STRATIFICATIONS ON UNITARY SHIMURA VARIETIES, AND ON HODGE-NEWTON REDUCIBLE LOCAL SHIMURA DATA OF ABELIAN TYPE

This thesis consists of two parts. In the first part, we develop techniques to study the interactions between Ekedahl-Oort stratification and BT_m stratifications with Newton stratification on unitary Shimura varieties. We focus on the case of a unitary Shimura variety with signature $(3, 2)$. This work is in collaboration with Emerald Andrews, Deewang Bhamidipati, Maria Fox, Steven R. Groen, and Heidi Goodson.

The second part addresses a new case of the Harris-Viehmann conjecture, which establishes a parabolic induction formula on the cohomology groups associated to non-basic local Shimura data. It follows that all supercuspidal representations on a Shimura variety are concentrated along the basic locus, making the conjecture relevant to the Langlands program. Historically, many cases of the Harris-Viehmann conjecture have been approached with the additional condition of Hodge-Newton reducibility on the underlying local Shimura datum. Building on previous work by E. Mantovan (EL/PEL case) and S. Hong (Hodge case), we extend the proof of the conjecture to unramified non-basic local Shimura data of abelian type under the assumption of Hodge-Newton reducibility. We leverage X. Shen's construction of Rapoport-Zink spaces of abelian type at the hyperspecial level. This is joint work with Xinyu Zhou.

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My collaborator Xinyu Zhou played a vital role in the process of my completing this dissertation - the second half is based on an ongoing collaboration between the two of us, which started as a summer study group.

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I thank my maternal grandmother and my mother for demonstrating what it means to be an intelligent and determined woman who will change the world if needed to achieve her goals. Shout out to my sister, who is just as stubborn. I thank my late maternal grandfather, who advocated for gender equality in education two generations before it was in vogue.

Finally, thanks to my dear friend Quang. It takes nerves of steel to be married to me. Tôi vui vì bạn là chồng của tôi.[1]

[1] Note to any độc giả Việt reading this: the pronouns "tôi" and "bạn" are used purposefully over "em" and "anh".

DEDICATION

I would like to dedicate this thesis to my teachers and to my maternal grandparents.

ഈ പ്രബന്ധം എന്റെ അധ്യാപകർക്കും, അമ്മൂമ്മയ്ക്കും, അപ്പൂപ്പനും വേണ്ടി സമർപ്പിക്കുന്നു.

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Chapter 1

Introduction

Shimura varieties and their local analogues, Rapoport-Zink spaces, are algebraic objects that exist in multiple mathematical worlds; namely, number theory (via automorphic forms), algebraic geometry (as varieties), and representation theory (via Galois representations). This makes them a Rosetta stone, enabling us to provide answers in one field by translating the questions into the language of another. They are central to the Langlands program, which seeks to relate the ℓ-adic cohomology of these arithmetic varieties to the representation theory of reductive groups. This dissertation investigates the geometric and cohomological properties of these spaces in two parts.

The first part consists of Chapters 2-3, which use the lens of Ekedahl-Oort and Newton stratifications to study unitary Shimura varieties. This part is the result of a collaboration with Emerald Andrews, Deewang Bhamidipati, Maria Fox, Steven R. Groen, and Heidi Goodson.

Chapter 2 focuses on the geometry of unitary Shimura varieties of signature $(q-2,2)$, and explores how the Ekedahl-Oort stratification and the Newton stratification interact with each other. We develop techniques, such as p-rank analysis and the forgetful map to Siegel modular varieties, to determine when these strata intersect. We present a complete answer for the case of the unitary Shimura variety of signature $(3,2)$ in Theorem 2.6.7. The contents of this chapter can be found in [1] and [2].

In Chapter 3, we shift focus to p-divisible groups. We attempt to answer the following question: for a positive integer m, and for a unitary Shimura variety of signature (a,b), can we express a bound on m as a function of a and b for which the stratification based on p^m-torsion group schemes of the unitary Shimura variety is a strict refinement of the Newton stratification?

We answer this question for the supersingular locus in terms of $\min(a,b)$ in Corollary 3.5.7. Our technique involves refining the predictions of [3] by exploiting the added symmetry imposed by the unitary structure, captured in the signature (a,b). The contents of this chapter can be found in [4].

The second part of the thesis, consisting of Chapter 4, addresses the cohomology of certain local Shimura varieties. Specifically, it studies how Rapoport-Zink spaces of abelian type satisfy the Harris-Viehmann conjecture under the assumption of Hodge-Newton reducibility. We recall the definition of local Shimura data and present some examples of the same in Section 4.2. In Section 4.3, we summarize Hong's strategy, then present the construction of Rapoport-Zink spaces of Hodge type (originally developed in [5]). This lays out the foundation for the proof of the Harris-Viehmann conjecture for unramified local Shimura data of Hodge type under the assumption of Hodge-Newton reducibility (published in [6]). In Section 4.4, we recount the construction of Rapoport-Zink spaces of abelian type in [7]. In Section 4.5, we verify that the construction of local Shimura varieties of abelian type at the hyperspecial level in [7] is compatible with Weil descent datum. To follow the strategy in [6], we need to establish some auxiliary formal schemes and intermediate results, which is the focus of Sections 4.6 to 4.10. We recover a special case of the Harris-Viehmann conjecture for local Shimura data of adjoint type in Section 4.11, in Theorem 4.11.3. We prove the Harris-Viehmann conjecture in section 4.12. The main result is Theorem 4.12.1. This is an ongoing collaboration with Xinyu Zhou.

Chapter 2

Stratifications on unitary Shimura varieties

Unitary Shimura varieties are moduli spaces of abelian varieties over an algebraically closed field of positive characteristic, equipped with extra structures such as a polarization and an action of a quadratic imaginary field. One way to study these moduli spaces is by stratifying them on the basis of certain algebro-geometric and arithmetic invariants. In this chapter, we discuss some interactions between Ekedahl-Oort and Newton stratifications for certain unitary Shimura varieties[2].

2.1 Background

To define an integral model of such a Shimura variety, we fix a prime $p > 2$, a positive integer q, non-negative integers a and b such that $a + b = q$, and a quadratic imaginary field K. We further assume that the prime p is inert in K, and so we identify $\mathcal{O}_K/(p)$ as $\mathbb{F}_{p^2}$ throughout.

Definition 2.1.1. We use the **PEL datum** $(K, \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}, {}^*, V, (\cdot,\cdot), \Lambda, \boldsymbol{G}, h)$ of Kottwitz [8], defined as follows:

- K is the quadratic imaginary field introduced above, with * being the non-trivial automorphism of K over $\mathbf{Q}$.
- V is a K-vector space of dimension q, equipped with a perfect alternating $\mathbb{Q}$-bilinear pairing $(\cdot,\cdot) : V \times V \to \mathbb{Q}$ such that $(xv, w) = (v, x^*w)$ for all $x \in K$ and $v, w \in V$.
- $\boldsymbol{G}$ is the algebraic group of K-linear symplectic similitudes of $(V, (\cdot,\cdot))$. We assume that $\boldsymbol{G}_{\mathbf{R}}$ is isomorphic to the real algebraic group $\mathsf{GU}(a, b)$.
- Λ is an $\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-invariant lattice of $V \otimes_{\mathbf{Q}} \mathbf{Q}_p$ such that the alternating form induced by $(\cdot,\cdot)$ is a perfect $\mathbf{Z}_p$-form.

[2]This chapter is based on: E. Andrews, D. Bhamidipati, M. Fox, H. Goodson, S.R. Groen, and S. Nair, "Ekedahl-Oort strata and the supersingular locus in the GU(q-2, 2) Shimura variety," arXiv:2405.04464 (2024) [1] and "The Ekedahl-Oort and Newton stratification of the GU(3,2) Shimura variety" arXiv:2510.01090 (2025) [2].

- $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_{m,\mathbb{C}}) \to \boldsymbol{G}_{\mathbb{R}}$ is the homomorphism of real algebraic groups that maps $z \in \mathbb{C}^\times$ to $\mathrm{diag}(z^a, \overline{z}^b)$.

Let L be the reflex field associated to the PEL-datum $(K, \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}, *, V, (\cdot,\cdot), \Lambda, \boldsymbol{G}, h)$; if $a = b$ then $L = \mathbf{Q}$, and $L = K$ otherwise. Let $\mathbb{F}$ be the residue field of L. Let $\mathbf{A}_f^p$ denote the ring of finite adeles with a trivial component at p. Fix a compact open subgroup $C^p \subset \boldsymbol{G}(\mathbf{A}_f^p)$. For C^p small enough, the construction of Kottwitz [8] attaches to this PEL datum a smooth, quasi-projective scheme $\mathbf{M}(a,b)_{C^p}$ over $\mathrm{Spec}(\mathcal{O}_{L,(p)})$ with the following moduli interpretation.

Let S be an $\mathcal{O}_{L,(p)}$-scheme. Then the set $\mathbf{M}(a,b)_{C^p}(S)$ parameterizes isomorphism classes of tuples (A, ι, λ, ξ), where:

- A is an abelian variety over S of dimension q.
- $\iota : \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a nonzero homomorphism of $\mathbb{Z}_{(p)}$-algebras such that the Rosati involution on $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ induces the involution * on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.
- λ is a one-dimensional $\mathbb{Q}$-subspace of $\mathrm{Hom}(A, A^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ that contains a p-principal $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-linear polarization.
- $\xi : \mathrm{H}_1(A, \mathbf{A}_f^p) \to V \otimes_{\mathbb{Q}} \mathbf{A}_f^p \bmod C^p$ is a C^p-level structure.

We also require that (A, ι) meets Kottwitz's determinant condition of signature (a,b). Two tuples (A, ι, λ, ξ) and $(A', \iota', \lambda', \xi')$ are isomorphic if there exists a prime-to-p isogeny from A to A', commuting with the action of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, mapping ξ to ξ' and λ to λ'.

The integral model $\mathbf{M}(a,b)_{C^p}$ has relative dimension of ab. The main object of study for this paper is the characteristic p **unitary Shimura variety**, denoted by $\mathcal{M}(a,b)$, which is the fibre at p of $\mathbf{M}(a,b)_{C^p}$. In particular, $\mathcal{M}(a,b)$ is defined over the residue field $\mathbb{F}$ of L at p and is of dimension ab. Since $\mathbf{M}(a,b)_{C^p} \cong \mathbf{M}(b,a)_{C^p}$, for the rest of this chapter we assume without loss of generality that $0 \le b \le a$. Many important properties of $\mathcal{M}(a,b)$ can be understood in terms of its geometric points. For this reason, we fix an algebraic closure $\Bbbk$ of $\mathbb{F}$.

We recall the definitions of the Ekedahl-Oort and Newton stratifications of $\mathcal{M}(a,b)$. For more details, see [9].

The Ekedahl-Oort stratification is based on the isomorphism class of the p-torsion group schemes. Two field-valued points (A,ι,λ,ξ) and $(A',\iota',\lambda',\xi')$ of $\mathcal{M}(a,b)$ are in the same **Ekedahl-Oort stratum** if and only if the p-torsion group schemes equipped with induced action and polarization, $(A[p],\iota,\lambda)$ and $(A'[p],\iota',\lambda')$, are isomorphic over $\Bbbk$. The Ekedahl-Oort strata are locally closed, and the closure of each Ekedahl-Oort stratum is a union of Ekedahl-Oort strata.

The Newton stratification, on the other hand, is based on the isogeny class of p-divisible groups. Two field-valued points (A,ι,λ,ξ) and $(A',\iota',\lambda',\xi')$ of $\mathcal{M}(a,b)$ are in the same **Newton stratum** if and only if the p-divisible groups equipped with induced action and polarization, $(A[p^\infty],\iota,\lambda)$ and $(A'[p^\infty],\iota',\lambda')$, are isogenous (in a way that respects the actions and polarizations) over $\Bbbk$. Newton strata are locally closed, and the closure of each Newton stratum is a union of lower-dimensional Newton strata. The unique closed Newton stratum of $\mathcal{M}(a,b)$ is the basic locus. When the prime p is inert, the basic locus coincides with the supersingular locus, which parameterizes supersingular abelian varieties with extra structures. We denote it as $\mathcal{M}(a,b)^{ss}$. In particular, a point (A,ι,λ,ξ) of $\mathcal{M}(a,b)$ is contained in the supersingular locus if and only if A is a supersingular abelian variety.

Remark 2.1.2. When the prime p is split, the basic locus is the same as the supersingular locus if and only if the signature of the unitary Shimura variety is of the form (a,a).

In dimension 1, the Ekedahl-Oort and Newton stratifications coincide, distinguishing supersingular elliptic curves from ordinary ones. However, as the dimension increases, the two stratifications diverge from one another, and their interaction can be much more varied. A guiding question is:

Question 2.1.3. When do a Newton stratum and an Ekedahl-Oort stratum of $\mathcal{M}(a,b)$ intersect?

For many applications, the basic locus is of greater interest than an arbitrary Newton stratum. As such, our primary concern is to understand:

Question 2.1.4. Which Ekedahl-Oort strata of $\mathcal{M}(a,b)$ intersect $\mathcal{M}(a,b)^{ss}$?

We note that Question 2.1.4 is especially relevant, as Ekedahl-Oort stratification can be used to give intrinsic explanations for exotic geometric phenomena observed on the basic locus (as in [10] and [11]). In turn, geometric descriptions of the basic locus have applications to the Kudla-Rapoport program and p-adic representation theory: initial results of the Kudla-Rapoport program used concrete descriptions of basic loci (see [12] and [13], which relied on [11]) to study the intersection theory of special cycles on Shimura varieties. Furthermore, according to the Harris-Viehmann conjecture [14] [15], cohomology groups attached to basic loci are important sources of supercuspidal representations, and are therefore of value to the Langlands program.

The answer to Question 2.1.4 is trivial for $\min(a,b) = 0$, and has been answered fully when $\min(a,b) = 1$ (in [11]) and when $(a,b) = (2,2)$ (following from results in [16]). Due to the work of Goertz and He [17], answering Question 2.1.4 for other signatures is expected to be far more complicated, as these Shimura varieties are no longer of "Coxeter type". However, in [18], we provide a complete answer to Question 2.1.3 (and therefore Question 2.1.4) for signature $(3,2)$, in the process developing techniques that apply in the general signature $(q-2,2)$. That will be the focus of the rest of this chapter.

2.2 Weyl Group Cosets

Results of [19] relate the study of the Ekedahl-Oort strata to cosets in a certain Weyl group. This section introduces the relevant Weyl group cosets and their minimal-length coset representatives.

The Weyl group that is relevant for the study of $\mathcal{M}(a,b)$ is $W = \mathfrak{S}_q$, the symmetric group on $q = a+b$ elements. We consider W as a Coxeter group with a set of **simple reflections**

$$S = \{s_1, \ldots, s_{q-1}\}, \text{where } s_i = (i,\ i+1).$$

The **length** of $w \in W$, denoted $\ell(w)$, is the length of a shortest expression for w as a product of simple reflections. It is proved in [20, Proposition 1.5.2] that the length of an element $w \in \mathfrak{S}_q$

can be computed as the number of inversions, i.e., the cardinality of the set

$$\mathrm{Inv}(w) := \{(i, j) \mid i < j \text{ and } w(i) > w(j)\}. \tag{2.1}$$

In particular, W has a unique element w_0 of maximal length, where $w_0(k) = q + 1 - k$.

For the non-empty subset $J_{(a,b)} = \{s_1, \ldots, s_{a+b-1}\} \setminus \{s_b\}$ of S, we let $W_{(a,b)} := W_{J_{(a,b)}}$ denote the subgroup of W generated by $J_{(a,b)}$. Note that $W_{(a,b)}$ is a parabolic subgroup of W. It follows from [21, Proposition 2.4.4] that every coset of $W_{(a,b)} \setminus W$ contains a unique minimal-length coset representative. Let $\mathbf{W}(a,b)$ be the collection of such minimal-length coset representatives for $W_{(a,b)} \setminus W$.The following theorem is paraphrased from Theorem 6.7 of [19]:

Theorem 2.2.1 (Moonen)**.** There is a bijection of sets:

$$\{\text{Ekedahl-Oort Strata of } \mathcal{M}(a,b)\} \longleftrightarrow \mathbf{W}(a,b).$$

We now recall some properties of Moonen's construction. (See also [22].) Let G be the group of $\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$-linear symplectic similitudes of Λ. Then G is a group scheme over $\mathbf{Z}_p$, and we let $\overline{G}$ be its special fibre. Moonen [19] gives an explicit identification of the Ekedahl-Oort strata with $W_X \backslash W_{\overline{G}}$, for a certain subgroup W_X depending on the signature. Concretely, $W_X \backslash W_{\overline{G}}$ can be described as

$$\{(w_1, w_2) \in \mathbf{W}(a,b) \times \mathbf{W}(b,a) \mid w_2 = w_0 w_1 w_0\}.$$

As the map $\pi \mapsto w_0 \pi w_0$ gives an isomorphism $\mathbf{W}(a,b) \cong \mathbf{W}(b,a)$, one has $W_X \backslash W_{\overline{G}} \cong \mathbf{W}(a,b)$.

For any $\gamma \in \mathbf{W}(a,b)$, let $\mathcal{M}(a,b)_\gamma$ denote the corresponding Ekedahl-Oort stratum and let $(G_\gamma, \iota_\gamma, \lambda_\gamma)$ denote the corresponding p-torsion group scheme. One observes that since G_γ is a p-torsion group scheme, the action of $\mathcal{O}_K$ on G_γ, via ι_γ, factors through $\mathcal{O}_K/(p)$. Since p is assumed to be inert, $\mathcal{O}_K/(p) = \mathbb{F}_{p^2}$, and we abuse notation and refer to this induced action of $\mathbb{F}_{p^2}$ as ι_γ as well. Moonen uses (contravariant) Dieudonné theory to describe $(G_\gamma, \iota_\gamma, \lambda_\gamma)$. For each

$\gamma \in \mathbf{W}(a,b)$, he constructs the **standard object** (N_γ, F, V). This is the Dieudonné module of G_γ, consisting of a vector space N_γ of dimension $2q$ over $\Bbbk$, F a $\mathrm{Frob}_{\Bbbk}$-semilinear operator on N_γ, and V a $\mathrm{Frob}_{\Bbbk}^{-1}$-semilinear operator on N_γ, described explicitly on a basis. The action ι_γ is recorded by a splitting $N_\gamma = N_{\gamma,1} \oplus N_{\gamma,2}$. By Theorem 6.7 of [19], λ_γ is uniquely determined by (G_γ, ι_γ), so it is unnecessary to record the corresponding polarization of N_γ.

The primary usefulness of the Weyl group coset perspective of Ekedahl-Oort strata comes from a natural partial order that can be defined on the minimal Weyl coset representatives. This partial order, which we shall refer to as the Closure order, corresponds precisely to topological closure relations among the Ekedahl-Oort strata on the moduli space $\mathcal{M}(a,b)$.

For example, let $\overline{\mathcal{M}(q-2,2)}_{\gamma'}$ denote the topological closure of the Ekedahl-Oort stratum $\mathcal{M}(q-2,2)_{\gamma'}$. Let $\gamma \preccurlyeq \gamma'$ symbolize the Closure order between the two Ekedahl-Oort strata. By [9, Theorem 1.2],

$$\overline{\mathcal{M}(q-2,2)}_{\gamma'} = \bigcup_{\gamma \preccurlyeq \gamma'} \mathcal{M}(q-2,2)_{\gamma}.$$

Thus, to understand closure relations among Ekedahl-Oort strata, it suffices to analyze the Closure order relations in $\mathbf{W}(q-2,2)$. In [18], we fixed an index set $\gamma_{u,v}$ for natural representatives of Ekedahl-Oort strata of $\mathcal{M}(q-2,2)$, using the Weyl group coset representatives. We recount that here:

For $1 \le u < v \le q$, let

$$\gamma_{u,v} := (2,3,\dots,v)(1,2,\dots,u). \tag{2.2}$$

We note that $\gamma_{u,v}$ fixes every element in the set $\{j \mid v < j \le q\}$ and

$$\gamma_{u,v}(j) = \begin{cases} j+2 & \text{if } 1 \le j \le u-1, \\ 1 & \text{if } j = u, \\ j+1 & \text{if } u+1 \le j \le v-1, \\ 2 & \text{if } j = v. \end{cases} \tag{2.3}$$

Then:

Lemma 2.2.2. [18] The Weyl group coset representatives indexing the Ekedahl-Oort strata for the unitary Shimura variety $\mathcal{M}(q-2,2)$ is given by $\mathbf{W}(q-2,2) = \{\gamma_{u,v} \mid 1 \le u < v \le q\}$, where the length of each representative is $\ell(\gamma_{u,v}) = u + v - 3$.

2.3 Newton stratification on unitary Shimura varieties

The Newton stratification partitions the special fibre of Shimura varieties of PEL type on the basis of the isogeny classes of the p-divisible groups of the underlying abelian varieties with extra structures. More precisely, two points (A, ι, λ, ξ) and $(A', \iota', \lambda', \xi')$ are in the same Newton stratum if there exists an isogeny between the p-divisible groups $A[p^\infty]$ and $A'[p^\infty]$ that respects the induced action and polarization.

By the Dieudonné-Manin theorem, any p-divisible group decomposes into simple isoclinic factors up to isogeny. For coprime non-negative integers m and n, let $G_{m,n}$ be an isoclinic p-divisible group of dimension m, codimension n and height $m+n$. Up to isogeny, it decomposes as:

$$A[p^\infty] \sim \prod_{i=1}^{r} G_{m_i,n_i}^{\ell_i}. \tag{2.4}$$

Here ℓ_i is called the multiplicity of G_{m_i,n_i}. For the p-divisible groups on both sides of Equation (2.4) to have the same height and dimension, we must have

$$\sum_{i=1}^{r} \ell_i m_i = \sum_{i=1}^{r} \ell_i n_i = q.$$

One then defines the **slopes** $\alpha_i = \frac{m_i}{m_i+n_i}$. The multiset of slopes of A is written as $\alpha_A = \left[\alpha_1^{\ell_1}, \ldots, \alpha_r^{\ell_r}\right]$. After relabeling factors in Equation (2.4) if necessary, we may assume $0 \le \alpha_1 < \cdots < \alpha_r \le 1$. Since A is principally polarized, G_{m_i,n_i} and G_{n_i,m_i} occur with the same multiplicity. Therefore we have $\alpha_i + \alpha_{r+1-i} = 1$ for all $1 \le i \le r$. Connecting line segments with the slopes in α_A gives a polygon from $(0,0)$ to $(2q, q)$. We shall refer to this polygon as the **Newton polygon** of A.

Given a $\Bbbk$-point (A, ι, λ, ξ) on $\mathcal{M}(a, b)$, there are additional restrictions on the Newton polygon of A (for details, see [23]).

For $\mathcal{M}(3, 2)$, there are four possible Newton polygons:

1. The Newton polygon with slopes $\beta_{\text{ss}} = \left[\frac{1}{2}^5\right]$;
2. The Newton polygon with slopes $\beta_1 = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right]$;
3. The Newton polygon with slopes $\beta_2 = \left[\frac{0}{1}^2, \frac{1}{2}^3, \frac{1}{1}^2\right]$;
4. The Newton polygon with slopes $\beta_{\max} = \left[\frac{0}{1}^4, \frac{1}{2}, \frac{1}{1}^4\right]$.

Given a multiset α of slopes, we denote by $\mathcal{M}(3, 2)^{\alpha}$ the Newton stratum corresponding to α via the Dieudonné-Manin theorem. The unique closed Newton stratum $\mathcal{M}(3, 2)^{\text{ss}} = \mathcal{M}(3, 2)^{\beta_{\text{ss}}}$ is known as the **supersingular locus** or **basic locus**. On the other extreme, the open and dense Newton stratum $\mathcal{M}(3, 2)^{\text{ord}} = \mathcal{M}(3, 2)^{\beta_{\max}}$ is known as the **μ-ordinary locus**.

We will answer the following question: Given a permutation $\gamma_{u,v} \in \mathbf{W}(3, 2)$ and a multiset of slopes α, does the Ekedahl-Oort stratum $\mathcal{M}(3, 2)_{\gamma_{u,v}}$ intersect the Newton stratum $\mathcal{M}(3, 2)^{\alpha}$?

2.4 Arguments based on p-ranks

Our goal is to understand the interaction between the Ekedahl-Oort stratification and the Newton stratification of $\mathcal{M}(3, 2)$. The first tool that we use to achieve this goal is the p-rank. Recall that $\Bbbk$ is an algebraically closed field of characteristic p. The ***p*-rank** of an abelian variety A over $\Bbbk$ is the integer $0 \leq f_A \leq \dim A$ such that

$$A[p](\Bbbk) \cong (\mathbb{Z}/p\mathbb{Z})^{f_A}. \tag{2.5}$$

Two $\Bbbk$-points (A, ι, λ, ξ) and $(A', \iota', \lambda', \xi')$ of $\mathcal{M}(3, 2)$ are in the same ***p*-rank stratum** if A and A' have the same p-rank. For $0 \leq f \leq 5$, we denote by $\mathcal{M}(3, 2)^{(f_A = f)}$ the p-rank f stratum of $\mathcal{M}(3, 2)$. The closure of a p-rank stratum is the union of all smaller p-rank strata.

We note that the Ekedahl-Oort stratification is a refinement of the p-rank stratification. In particular, if M is the mod-p Dieudonné module corresponding to an Ekedahl-Oort stratum, then the p-rank of M equals the $\Bbbk$-dimension of the largest subspace of M on which F acts bijectively.

Since the p-rank is an invariant under isogenies, it follows that the Newton stratification is also a refinement of the p-rank stratification. In particular, f_A equals the multiplicity of the slope 0 in the Newton polygon of A. It follows from the list of Newton polygons that the non-empty p-rank strata of $\mathcal{M}(3,2)$ are $\mathcal{M}(3,2)^{(f_A=0)}$, $\mathcal{M}(3,2)^{(f_A=2)}$ and $\mathcal{M}(3,2)^{(f_A=4)}$. Since the Ekedahl-Oort stratification and the Newton stratification both refine the p-rank stratification, an Ekedahl-Oort stratum and a Newton stratum can only intersect if they are contained in the same p-rank stratum.

Lemma 2.4.1. We have

$$\mathcal{M}(3,2)^{(f_A=4)} = \mathcal{M}(3,2)^{\mathrm{ord}} = \mathcal{M}(3,2)_{\gamma_{4,5}}, \tag{2.6}$$

$$\mathcal{M}(3,2)^{(f_A=2)} = \mathcal{M}(3,2)^{\beta_2} = \mathcal{M}(3,2)_{\gamma_{3,5}} \cup \mathcal{M}(3,2)_{\gamma_{2,5}}, \tag{2.7}$$

$$\mathcal{M}(3,2)^{(f_A=0)} = \mathcal{M}(3,2)^{\beta_1} \cup \mathcal{M}(3,2)^{\mathrm{ss}} = \bigcup_{v<5 \text{ or } u=1} \mathcal{M}(3,2)_{\gamma_{u,v}}. \tag{2.8}$$

Furthermore, the following closure relations hold:

$$\overline{\mathcal{M}(3,2)^{\mathrm{ord}}} = \overline{\mathcal{M}(3,2)_{\gamma_{4,5}}} = \mathcal{M}(3,2), \tag{2.9}$$

$$\overline{\mathcal{M}(3,2)^{\beta_2}} = \overline{\mathcal{M}(3,2)_{\gamma_{3,5}}} = \mathcal{M}(3,2)^{(f_A=2)} \cup \mathcal{M}(3,2)^{(f_A=0)}, \tag{2.10}$$

$$\overline{\mathcal{M}(3,2)^{\beta_1}} = \overline{\mathcal{M}(3,2)_{\gamma_{3,4}}} = \mathcal{M}(3,2)^{(f_A=0)}. \tag{2.11}$$

Proof. Equations (2.6), (2.7) and (2.8) follow immediately from computing the p-ranks of the Ekedahl-Oort strata, for instance by constructing the standard objects introduced in [19, 4.9].

As for the closure relations, observe that the density of the μ-ordinary locus directly implies Equation (2.9). Furthermore, Equation (2.10) follows from the purity of p-rank strata and the fact that $\mathcal{M}(3,2)_{\gamma_{3,5}}$ is the only Ekedahl-Oort stratum in $\mathcal{M}(3,2)^{(f_A=2)}$ that has the same dimension as $\mathcal{M}(3,2)^{(f_A=2)}$. Finally, Equation (2.11) follows using the same argument. □

Lemma 2.4.1 gives a complete answer for all Ekedahl-Oort strata and Newton strata of positive p-rank, since $\mathcal{M}(3,2)^{(f_A=4)}$ and $\mathcal{M}(3,2)^{(f_A=2)}$ contain only one Newton stratum. We henceforth restrict our attention to Ekedahl-Oort strata and Newton strata of p-rank 0.

2.5 Arguments based on the Siegel modular variety

2.5.1 The forgetful map

In this section, we utilize the **forgetful map** from the unitary Shimura variety $\mathcal{M}(a,b)$ to the Siegel modular variety $\mathcal{A}_q$. The forgetful map is defined as

$$\Psi_{a,b} : \mathcal{M}(a,b) \to \mathcal{A}_q$$
$$(A,\iota,\lambda,\xi) \mapsto (A,\lambda,\xi),$$

where $a+b=q$. The forgetful map $\Psi_{a,b}$ induces a map on Ekedahl-Oort strata. More precisely, let $\mathbf{W}_q$ be the set of permutations $\omega \in \mathfrak{S}_{2q}$ satisfying

$$\omega^{-1}(1) < \omega^{-1}(2) < \cdots < \omega^{-1}(q) \quad \text{and} \quad \omega(i) + \omega(2q+1-i) = 2q+1.$$

By [19, 3.6], the Ekedahl-Oort strata of $\mathcal{A}_q$ correspond bijectively to elements of $\mathbf{W}_q$. Thus $\Psi_{a,b}$ induces a map

$$\psi_{a,b} : \mathbf{W}(a,b) \to \mathbf{W}_q.$$

Given $\omega \in \mathbf{W}_q$, let $\mathcal{A}_{q,\omega}$ denote the corresponding Ekedahl-Oort stratum of $\mathcal{A}_q$.

In [18, Theorem 5.2], the map $\psi_{q-2,2}$ is completely described for general $q \geq 2$. We can leverage these results by setting $q=5$. For $\gamma_{u,v} \in \mathbf{W}(a,b)$, let us denote $\omega_{u,v} := \psi_{3,2}(\gamma_{u,v}) \in \mathbf{W}_q$. We first consider the cases when $\mathcal{A}_{5,\omega_{u,v}}$ is contained in the supersingular locus of $\mathcal{A}_5$.

Lemma 2.5.1. The Ekedahl-Oort strata $\mathcal{M}(3,2)_{\gamma_{1,2}}$ and $\mathcal{M}(3,2)_{\gamma_{1,3}}$ are contained in $\mathcal{M}(3,2)^{\mathrm{ss}}$.

Proof. This is the case $q = 5$ of [18, Corollary 5.4]. The explicit description of $\psi_{3,2}$ is used to show that $\omega_{1,2}(3) = \omega_{1,3}(3) = 3$, so $\mathcal{A}_{5,\omega_{1,2}}$ and $\mathcal{A}_{5,\omega_{1,3}}$ are contained in the supersingular locus of $\mathcal{A}_5$ by [24]. This implies that $\mathcal{M}(3,2)_{\gamma_{1,2}}$ and $\mathcal{M}(3,2)_{\gamma_{1,3}}$ are contained in $\mathcal{M}(3,2)^{ss}$. □

2.5.2 Minimal Ekedahl-Oort strata

An Ekedahl-Oort stratum of a PEL-type Shimura variety is called **minimal** if all the parameterized abelian varieties in the Ekedahl-Oort stratum have isomorphic p-divisible groups. A minimal Ekedahl-Oort stratum is completely contained in one Newton stratum. While the existence and uniqueness of minimal Ekedahl-Oort strata in Newton strata of unitary Shimura varieties is in general unknown, every Newton stratum of $\mathcal{A}_q$ contains a unique minimal Ekedahl-Oort stratum [25, 26]. The Dieudonné modules of these minimal Ekedahl-Oort strata are constructed explicitly in [27, 5.3].

Lemma 2.5.2. The Ekedahl-Oort stratum $\mathcal{A}_{5,\omega_{2,4}}$ is the minimal Ekedahl-Oort stratum of $\mathcal{A}_5$ with slope sequence $\beta_1 = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right]$. Thus, the Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{2,4}}$ is completely contained in $\mathcal{M}(3,2)^{\beta_1}$.

Proof. This is an application of [18, Proposition 5.11] with $(n_1, n_2, n_3) = (1,1,3)$. First, [18, Theorem 5.2] gives

$$\omega_{2,4} = \psi_{3,2}(\gamma_{2,4}) = (2,6,8,4)(3,7,9,5).$$

Additionally, we construct the minimal Ekedahl-Oort stratum corresponding to the slope sequence $\alpha = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right]$. In the notation of [27, 5.3], its mod-p Dieudonné module is given by

$$M_\alpha = M_{1,3} \oplus M_{1,1} \oplus M_{3,1}.$$

Finally, one verifies via [19, 3.6] that M_α corresponds to the permutation $\omega_{2,4}$.

Thus we have proved that $\mathcal{A}_{5,\omega_{2,4}}$ is the minimal Ekedahl-Oort stratum of the Newton stratum $\mathcal{A}_5^{\beta_1}$. This implies that $\mathcal{M}(3,2)_{\gamma_{2,4}}$ must be contained in $\mathcal{M}(3,2)^{\beta_1}$. □

2.5.3 Generic slopes

In this section, we use the results of [28] to compute slopes of generic Newton polygons of strata in the Siegel modular variety $\mathcal{A}_q$ to obtain results for Ekedahl-Oort strata for the unitary Shimura variety $\mathcal{M}(q-b,b)$. For our purposes, it suffices to compute the *first slope* of the generic Newton polygon for a given stratum, since it follows from [28, Theorem 4.1] that all other polygons for the Ekedahl-Oort stratum must have first slope greater than or equal to this value. We will use this fact to shed light on some interactions between the Ekedahl-Oort and Newton stratifications in signature $(3,2)$.

Definition 2.5.3. A **final sequence** is a function $\varphi : \{0,1,\ldots,2q\} \to \{0,1,\ldots,q\}$, satisfying the following conditions:

- $\varphi(0) = 0$;
- $\varphi(i-1) \leq \varphi(i) \leq \varphi(i-1)+1$ for $1 \leq i \leq 2q$;
- $\varphi(2q-i) = q-i+\varphi(i)$ for $0 \leq i \leq 2q$.

In the language of [28, Definition 2.2], this is a final sequence *stretched from an elementary sequence*. We will write a final sequence down as an ordered tuple $\varphi = [\varphi(1),\ldots,\varphi(2q)]$.

Given an element $\omega \in \mathfrak{S}_{2q}$, we can construct its corresponding final sequence φ as follows:

- Set $\varphi(0) = 0$;
- For $1 \leq i \leq 2q$, if $\omega(i) > q$, set $\varphi(i) = \varphi(i-1)+1$;
- For $1 \leq i \leq 2q$, if $\omega(i) \leq q$, set $\varphi(i) = \varphi(i-1)$.

Given $\gamma_{u,v} \in \mathbf{W}(3,2)$, let $\varphi_{u,v}$ denote the final sequence corresponding to $\omega_{u,v} = \psi_{3,2}(\gamma_{u,v})$.

We now demonstrate how to compute the first slope of the Newton polygon of a generic point in a given Ekedahl-Oort stratum from the above construction. Note that, for ease of notation, our exposition differs slightly from that of [28]. We first define a map on the set $S = \{1, \dots, 2q\}$ based on the image of the final sequence φ. Let $\tilde{\varphi}$ be the map defined by

$$\tilde{\varphi}(i) = \begin{cases} \varphi(i) & \text{if } \varphi(i) \neq 0, \\ q + i & \text{otherwise.} \end{cases} \tag{2.12}$$

Definition 2.5.4 ([28, Definition 3.1]). Let $\mathcal{D} = \cap_{j=1}^{\infty} \tilde{\varphi}^j(S)$ and let $\mathcal{C} = \mathcal{D} \cap \{q+1, \dots, 2q\}$. The *generic first slope* associated with φ is

$$\lambda_{\varphi} = \frac{\#\mathcal{C}}{\#\mathcal{D}}.$$

We refer to this as the Harashita first slope. With this setup, we can prove the following results.

Lemma 2.5.5. The generic first slope of the Ekedahl-Oort stratum $\mathcal{A}_{q,\omega_{1,4}}$ is $\frac{2}{5}$.

Proof. First, [18, Theorem 5.2] gives $\omega_{1,4} = \psi_{3,2}(\gamma_{1,4}) = (3,6,4)(5,7,8)$ and hence

$$\varphi_{1,4} = [0,0,1,1,2,2,3,3,4,5].$$

It is computed in [28, Example 3.19.(3)] that the generic first slope is $\frac{2}{5}$. □

Lemma 2.5.6. The generic first slope of the Ekedahl-Oort strata $\mathcal{A}_{q,\omega_{1,5}}$ and $\mathcal{A}_{q,\omega_{2,3}}$ is $\frac{1}{3}$.

Proof. From [18, Theorem 5.2] we have that $\omega_{1,5} = \omega_{2,3} = (2,6,4,3)(5,7,8,9)$ and hence

$$\varphi_{1,5} = \varphi_{2,3} = [0,1,1,1,2,2,3,4,4,5].$$

Following Equation (2.12) and Definition 2.5.4, we compute $\mathcal{D} = \{1,2,6\}$ and $\mathcal{C} = \{6\}$. Therefore the generic first slope is $\frac{\#\mathcal{C}}{\#\mathcal{D}} = \frac{1}{3}$. □

Remark 2.5.7. An abstract way to view the phenomenon $\omega_{1,5} = \omega_{2,3}$ is that the Ekedahl-Oort strata $\mathcal{M}(3,2)_{\gamma_{1,5}}$ and $\mathcal{M}(3,2)_{\gamma_{2,3}}$ have the same underlying p-torsion group scheme with polarization, but a different action of $\mathbb{F}_{p^2}$.

Lemma 2.5.8. The generic first slope of the Ekedahl-Oort stratum $\mathcal{A}_{q,\omega_{1,4}}$ is $\frac{2}{5}$.

Proof. First, [18, Theorem 5.2] gives $\omega_{1,4} = \psi_{3,2}(\gamma_{1,4}) = (3,6,4)(5,7,8)$ and hence

$$\varphi_{1,4} = [0,0,1,1,2,2,3,3,4,5].$$

It is computed in [28, Example 3.19.(3)] that the generic first slope is $\frac{2}{5}$. □

Lemma 2.5.9. The generic first slope of the Ekedahl-Oort strata $\mathcal{A}_{q,\omega_{3,4}}$ is $\frac{1}{5}$.

Proof. We note $\omega_{3,4} = \psi_{3,2}(\gamma_{3,4})$ gives rise to

$$\varphi_{3,4} = [0,1,2,3,4,4,4,4,4,5].$$

By Harashita's algorithm in [28], we compute the generic first slope to be $\frac{1}{5}$. □

Corollary 2.5.10. The Ekedahl-Oort strata $\mathcal{M}(3,2)_{\gamma_{1,5}}$, $\mathcal{M}(3,2)_{\gamma_{1,5}}$ and $\mathcal{M}(3,2)_{\gamma_{2,3}}$ are contained in $\mathcal{M}(3,2)^{\mathrm{ss}}$.

Proof. It follows from Lemma 2.5.8 that the first slope of any Newton polygon occurring on the Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{1,4}}$ is at least $\frac{2}{5}$. Similarly, Lemma 2.5.6 posits that the first slope of any Newton polygon occurring on $\mathcal{M}(3,2)_{\gamma_{1,5}}$ or $\mathcal{M}(3,2)_{\gamma_{2,3}}$ is at least $\frac{1}{3}$. On the other hand, the only Newton polygon on $\mathcal{M}(3,2)$ whose first slope is at least $\frac{1}{3}$ is the supersingular Newton polygon $\beta_{\mathrm{ss}} = \left[\frac{1}{2}^{5}\right]$. Thus β_{ss} is the only Newton polygon that can occur on the Ekedahl-Oort strata in question. In other words, these Ekedahl-Oort strata are contained in the supersingular locus $\mathcal{M}(3,2)^{\mathrm{ss}}$. □

Remark 2.5.11. It is insightful to compare Corollary 2.5.10 to Lemma 2.5.1. As opposed to $\mathcal{A}_{5,\omega_{1,2}}$ and $\mathcal{A}_{5,\omega_{1,3}}$, the Ekedahl-Oort strata $\mathcal{A}_{5,\omega_{1,4}}$, $\mathcal{A}_{5,\omega_{1,5}}$ and $\mathcal{A}_{5,\omega_{2,3}}$ are not contained in the supersingular locus of $\mathcal{A}_5$. However, the non-supersingular Newton polygons that occur on these Ekedahl-Oort strata of $\mathcal{A}_5$ are not Newton polygons that occur on $\mathcal{M}(3,2)$. In terms of our moduli interpretation, there are non-supersingular abelian varieties in these Ekedahl-Oort strata of $\mathcal{A}_5$, but these abelian varieties do not admit an action of $\mathcal{O}_K$ of signature $(3,2)$ with compatible polarization.

2.6 Lifting to p-Divisible Groups

By lifting a fixed p-torsion group scheme to a p-divisible group, and applying Rapoport-Zink uniformization [29], one can explicitly produce a point in the intersection of an Ekedahl-Oort stratum and a Newton stratum. In this section, we show that $\mathcal{M}(3,2)_{\gamma_{3,4}}$ intersects the supersingular locus, by explicitly constructing a point in the intersection $\mathcal{M}(3,2)_{\gamma_{3,4}} \cap \mathcal{M}(3,2)^{\mathrm{ss}}$. To do this, we first use p-adic Dieudonné theory to construct the p-divisible groups of these points; we then use the p-divisible groups and Rapoport-Zink uniformization to construct points of $\mathcal{M}(3,2)$.

Let $\breve{\mathbb{Z}}_p = W(\Bbbk)$ be the ring of Witt vectors of $\Bbbk$, denote by $\mathrm{Frob} = W(\mathrm{Frob}_{\Bbbk})$ the lift of $\mathrm{Frob}_{\Bbbk}$ to $\breve{\mathbb{Z}}_p$, and let $\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[\frac{1}{p}]$. Note that $\breve{\mathbb{Q}}_p$ is isomorphic to the completion of the maximal unramified extension of $\mathbb{Q}_p$, with ring of integers isomorphic to $\breve{\mathbb{Z}}_p$, and that the residue field of $\breve{\mathbb{Z}}_p$ is $\Bbbk$. Let φ_1, φ_2 be the two embeddings of $\mathcal{O}_K$ into $\breve{\mathbb{Q}}_p$.

Definition 2.6.1. For any scheme S over $\Bbbk$, a *unitary p-divisible group of signature* $(q-b,b)$ over S is a triple (X,ι_X,λ_X), where

1. X is a p-divisible group over S of height $2q$ and dimension q;
2. $\iota_X : \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(X)$ is an action satisfying the signature $(q-b,b)$ condition

$$\mathrm{charpol}(\iota(a) \mid \mathrm{Lie}(X)) = (T-\varphi_1(a))^{q-b}(T-\varphi_2(a))^b \in \breve{\mathbb{Z}}_p[T],$$

   for all $a \in \mathcal{O}_K$;
3. $\lambda_X : X \to X^\vee$ is a p-principal polarization, meeting the following $\mathcal{O}_K$-linearity condition

$$\lambda_X \circ \iota_X(a) = \iota_X(\overline{a})^\vee \circ \lambda_X,$$

   for all $a \in \mathcal{O}_K$.

Over an algebraically closed field, we may study unitary p-divisible groups linear-algebraically.

Definition 2.6.2. A *unitary p-adic Dieudonné module of signature* $(q-b, b)$ over $\Bbbk$ is a tuple $(M, M = M_1 \oplus M_2, F, V, \langle\cdot,\cdot\rangle)$, where

1. M is a free $\breve{\mathbb{Z}}_p$-module of rank $2q$;
2. $M = M_1 \oplus M_2$ is a decomposition into rank-q summands;
3. $F : M \to M$ is a Frob-semilinear operator, $V : M \to M$ is a Frob^{-1}-semilinear operator, with $F \circ V = V \circ F = p$;
4. $\langle\cdot,\cdot\rangle$ is a perfect alternating $\breve{\mathbb{Z}}_p$-bilinear pairing on M such that $\langle Fx, y\rangle = \langle x, Vy\rangle^{\mathrm{Frob}}$, for all $x, y \in M$;
5. $\dim_{\Bbbk}(M_1/FM_2) = q - b$ and $\dim_{\Bbbk}(M_2/FM_1) = b$;
6. F and V are homogeneous of degree 1 with respect to the decomposition $M = M_1 \oplus M_2$;
7. M_1 and M_2 are each totally isotropic with respect to $\langle\cdot,\cdot\rangle$.

For the rest of this section, we consider signature $(3,2)$ and $q = 5$. By contravariant Dieudonné theory, there is an anti-equivalence of categories between unitary p-divisible groups and unitary p-adic Dieudonné modules, both of signature $(3,2)$ over $\Bbbk$. When no confusion arises, we abbreviate unitary p-divisible groups as X and unitary p-adic Dieudonné modules as M.

Lemma 2.6.3. Let $G_{\gamma_{3,4}}$ be the p-torsion group scheme occurring as the p-torsion subgroup for points in the Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{3,4}}$. There exists a supersingular unitary p-divisible group $X_{3,4}$ of signature $(3,2)$ over $\Bbbk$ such that $X_{3,4}[p] \cong G_{\gamma_{3,4}}$, respecting action and polarization.

Proof. We aim to construct a supersingular unitary p-divisible group $X_{3,4}$ of signature $(3,2)$ such that $X_{3,4}[p] \cong G_{\gamma_{3,4}}$. Using contravariant p-adic Dieudonné theory, it suffices to construct a unitary p-adic Dieudonné module M of signature $(3,2)$ such that all of the slopes of the isocrystal $M[\frac{1}{p}]$ are equal to $\frac{1}{2}$ and such that M/pM is isomorphic to the mod-p Dieudonné module of $G_{\gamma_{3,4}}$.

Let M be the free $\breve{\mathbb{Z}}_p$-module with basis $\{e_i, f_i\}_{1 \le i \le 5}$. Let M_1 be the submodule spanned by $\{e_i\}_{1 \le i \le 5}$ and let M_2 be the submodule spanned by $\{f_i\}_{1 \le i \le 5}$. Define F (resp. V) as the Frob-semilinear (resp. Frob^{-1}-semilinear) operator defined on the basis of M as in Table 2.1 below.

Table 2.1: F and V on M for $\gamma_{3,4}$.

$F(e_1) = f_5$	$V(e_1) = pf_2$	$F(f_1) = -pe_5$	$V(f_1) = e_2$
$F(e_2) = pf_1$	$V(e_2) = f_3$	$F(f_2) = e_1$	$V(f_2) = pe_3$
$F(e_3) = f_2$	$V(e_3) = f_4$	$F(f_3) = pe_2$	$V(f_3) = pe_4$
$F(e_4) = f_3$	$V(e_4) = pf_5$	$F(f_4) = pe_3$	$V(f_4) = e_5$
$F(e_5) = pf_4$	$V(e_5) = -f_1$	$F(f_5) = e_4$	$V(f_5) = pe_1$

We define an alternating pairing $\langle \cdot, \cdot \rangle$ on M by the condition that $\langle e_i, f_j \rangle = (-1)^{i-1}\delta_{ij}$ and claim that $(M, M = M_1 \oplus M_2, F, V, \langle \cdot, \cdot \rangle)$ is a unitary p-adic Dieudonné module of signature $(3, 2)$ since it satisfies each condition of Definition 2.6.2. Indeed:

- M_1 and M_2 both have rank 5, so $M = M_1 \oplus M_2$ has rank $10 = 2q$.

- The operators F and V are homogeneous of degree 1 with respect to the decomposition $M = M_1 \oplus M_2$. Let $\mathbf{A}_F$ and $\mathbf{A}_V$ be the matrices given by the action of F and V, respectively, on the chosen basis. Since $\mathbf{A}_F$ and $\mathbf{A}_V$ have integer entries, to check that $F \circ V = V \circ F = p$, it suffices to verify that $\mathbf{A}_F \mathbf{A}_V = \mathbf{A}_V \mathbf{A}_F = p\mathrm{Id}$, which is true by construction.

- The condition that $\langle e_i, f_j \rangle = (-1)^{i-1}\delta_{ij}$ extends uniquely to a perfect alternating $\breve{\mathbb{Z}}_p$-bilinear pairing on M, and under this pairing M_1 and M_2 are each totally isotropic.

  Let $\mathbf{B}$ be the matrix of this alternating form, and note that $\mathbf{B}$ has integer entries. To check that $\langle Fx, y \rangle = \langle x, Vy \rangle^{\mathrm{Frob}}$, it suffices to verify that $\mathbf{A}_F^T \mathbf{B} = \mathbf{B}\mathbf{A}_V$, which can be verified using the description of F and V in Table 2.1 and the definition of $\langle \cdot, \cdot \rangle$.

- From the definition of F, we have $M_1/FM_2 \cong \mathrm{Span}_{\Bbbk}\{e_2, e_3, e_5\}$ and $M_2/FM_1 \cong \mathrm{Span}_{\Bbbk}\{f_1, f_4\}$. In particular, $\dim_{\Bbbk}(M_1/FM_2) = 3$ and $\dim_{\Bbbk}(M_2/FM_1) = 2$.

By contravariant Dieudonné theory, M defines a unitary p-divisible group $X_{3,4}$ of signature $(3,2)$.

We use [30, Lemma 6.12] to compute the slopes of the isocrystal $M[\frac{1}{p}]$. Note that for any positive integer m, we find that

$$F^{10m}(M) = p^{5m}M,$$

and so $\frac{1}{10m}\max\{k \in \mathbb{Z} : F^{10m}M \subset p^k M\} = \frac{1}{2}$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n}\max\{k \in \mathbb{Z} : F^{n}M \subset p^k M\} = \frac{1}{2},$$

and by [30] all slopes of $M[\frac{1}{p}]$ are equal to $\frac{1}{2}$. Accordingly, $X_{3,4}$ is supersingular.

Finally, we show $X_{3,4}[p] = G_{\gamma_{3,4}}$. Define N as $M/pM \cong \mathrm{Span}_{\Bbbk}\{e_i, f_i\}_1^5$, with splitting $N = N_1 \oplus N_2$ and F and V operators induced from those on M. We will follow the procedure of [19, 3.5] to compute the permutation $\omega \in \mathbf{W}(3,2)$ associated to N. Following from the definition of F and V, the Dieudonné module N has final filtration

$$0 \subset \langle f_3\rangle \subset \langle e_4, f_3\rangle \subset \langle e_4, f_3, f_5\rangle \subset \langle e_1, e_4, f_3, f_5\rangle \subset \langle e_1, e_4, f_2, f_3, f_5\rangle \subset \langle e_1, e_2, e_4, f_2, f_3, f_5\rangle$$
$$\subset \langle e_1, e_2, e_4, f_1, f_2, f_3, f_5\rangle \subset \langle e_1, e_2, e_4, e_5, f_1, f_2, f_3, f_5\rangle \subset \langle e_1, e_2, e_4, e_5, f_1, f_2, f_3, f_4, f_5\rangle \subset N.$$

Intersecting with N_1 gives the filtration $C_{1,\bullet}$

$$0 \subset \langle e_4\rangle \subset \langle e_1, e_4\rangle \subset \langle e_1, e_2, e_4\rangle \subset \langle e_1, e_2, e_4, e_5\rangle \subset N_1.$$

The function $\eta_1(j) = \dim(C_{1,j} \cap N[F])$ is then given by

$$\eta_1(1) = \eta_1(2) = 0, \qquad \eta_1(3) = 1, \qquad \eta_1(4) = \eta_1(5) = 2.$$

The permutation ω corresponding to η is $(1,3)(2,4)$. As $\gamma_{3,4}$ is also equal to $(1,3)(2,4)$, it follows that $X_{3,4}[p] \cong G_{\gamma_{3,4}}$, which finishes the proof. □

Recall that the supersingular locus $\mathcal{M}(3,2)^{\mathrm{ss}}$ is uniformized by a formal scheme called a Rapoport-Zink space. As a framing object, let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be a fixed supersingular unitary p-divisible group of signature $(3,2)$ over $\Bbbk$.

Definition 2.6.4. For any scheme S over $\Bbbk$, denote by $\mathcal{N}(3,2)(S)$ the set of isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$, where:

- (X, ι_X, λ_X) is a unitary p-divisible group of signature $(3,2)$ over S;
- $\rho_X : X \to \mathbb{X}$ is an $\mathcal{O}_K$-linear quasi-isogeny identifying λ_X and $\lambda_{\mathbb{X}}$ up to scaling in $\mathbb{Q}_p^\times$.

By [29], the functor defined above is represented by a formal scheme over $\Bbbk$ which is locally formally of finite type; we will also denote the underlying reduced scheme of this representing object (a "signature $(3,2)$ unitary Rapoport-Zink space") as $\mathcal{N}(3,2)$.

Proposition 2.6.5. The Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{3,4}}$ intersects the supersingular locus and the Newton polygon stratum $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$.

Proof. We first address the supersingular locus.

By the uniformizaton theorem of Rapoport and Zink [29], there exist groups $\{\Gamma_j\}_{j=1}^n$ (arising as subgroups of the $\mathbb{Q}_p$-points of the algebraic group defining the automorphisms of $\mathbb{X}$, and depending on the level structure implicit in the definition of $\mathcal{M}(3,2)$) such that there is an isomorphism of schemes over $\Bbbk$

$$\bigsqcup_{j=1}^n \mathcal{N}(3,2)/\Gamma_j \cong \mathcal{M}(3,2)^{\mathrm{ss}}.$$

In particular, there is a surjection of $\Bbbk$-points

$$\bigsqcup_{j=1}^n \mathcal{N}(3,2)(\Bbbk) \twoheadrightarrow \mathcal{M}(3,2)^{\mathrm{ss}}(\Bbbk).$$

Let $G_{\gamma_{3,4}}$ be the p-torsion group scheme occurring as the p-torsion subgroup for points in the Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{3,4}}$. By Lemma 2.6.3, there exists a supersingular unitary p-divisible group $X_{3,4}$ of signature $(3,2)$ over $\Bbbk$ such that $X_{3,4}[p] \cong G_{\gamma_{3,4}}$. By [11, Lemma 6.1], there exists a quasi-isogeny $\rho_{X_{3,4}} : X_{3,4} \to \mathbb{X}$ that is $\mathcal{O}_K$-linear and identifies the polarizations, up to $\mathbb{Q}_p^\times$-scaling. That is, $(X_{3,4}, \iota_{X_{3,4}}, \lambda_{X_{3,4}}, \rho_{X_{3,4}})$ defines a $\Bbbk$-point of $\mathcal{N}(3,2)$. Let $(A_{3,4}, \iota_{A_{3,4}}, \lambda_{A_{3,4}}, \xi_{A_{3,4}})$ be the image of $(X_{3,4}, \iota_{X_{3,4}}, \lambda_{X_{3,4}}, \rho_{X_{3,4}})$ in $\mathcal{M}(3,2)^{\mathrm{ss}}(\Bbbk)$.

Since $A_{3,4}[p] \cong X_{3,4}[p] \cong G_{\gamma_{3,4}}$, as the p-torsion group schemes equipped with extra structure, the $\Bbbk$-point $(A_{3,4}, \iota_{A_{3,4}}, \lambda_{A_{3,4}}, \xi_{A_{3,4}})$ of $\mathcal{M}(3,2)$ lies in the Ekedahl-Oort stratum indexed by $\gamma_{3,4}$. That is, $(A_{3,4}, \iota_{A_{3,4}}, \lambda_{A_{3,4}}, \xi_{A_{3,4}})$ is an explicit point in the intersection $\mathcal{M}(3,2)_{\gamma_{3,4}} \cap \mathcal{M}(3,2)^{\mathrm{ss}}$.

We now address $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$. By Lemma 2.5.9, the image of $\gamma_{3,4}$ under the forgetful map to the underlying Siegel modular variety is $\omega_{3,4}$ with generic slope $\frac{1}{5}$. The corresponding Newton polygon is completely nested inside the Newton polygon with first slope $\frac{1}{4}$. Thus, the Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{3,4}}$ must generically intersect $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$. □

Remark 2.6.6. By [17, Theorem A], $\mathcal{M}(3,2)$ is not of Coxeter type, so Ekedahl-Oort stratification does not refine the Newton stratification. Every Ekedahl-Oort stratum apart from $\mathcal{M}(3,2)_{\gamma_{3,4}}$ is completely contained in one Newton stratum. Hence, it must be that $\mathcal{M}(3,2)_{\gamma_{3,4}}$ intersects $\mathcal{M}(3,2)^{\mathrm{ss}}$ as well as $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$.

Theorem 2.6.7. The interaction between the Ekedahl-Oort stratification and the Newton stratification of $\mathcal{M}(3,2)$ is as follows:

1. The Ekedahl-Oort strata $\mathcal{M}(3,2)_{\gamma_{1,2}}$, $\mathcal{M}(3,2)_{\gamma_{1,3}}$, $\mathcal{M}(3,2)_{\gamma_{1,4}}$, $\mathcal{M}(3,2)_{\gamma_{1,5}}$ and $\mathcal{M}(3,2)_{\gamma_{2,3}}$ are contained in the Newton stratum $\mathcal{M}(3,2)^{\left[\frac{1}{2}^5\right]}$.

2. The Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{2,4}}$ is contained in the Newton stratum $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$.

3. The Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{3,4}}$ intersects both $\mathcal{M}(3,2)^{\left[\frac{1}{2}^5\right]}$ and $\mathcal{M}(3,2)^{[\frac{1}{4},\frac{1}{2},\frac{3}{4}]}$.

4. The Ekedahl-Oort strata $\mathcal{M}(3,2)_{\gamma_{2,5}}$ and $\mathcal{M}(3,2)_{\gamma_{3,5}}$ are both contained in the Newton stratum $\mathcal{M}(3,2)^{\left[\frac{0}{1}^2,\frac{1}{2}^3,\frac{1}{1}^2\right]}$.

5. The Ekedahl-Oort stratum $\mathcal{M}(3,2)_{\gamma_{4,5}}$ equals the Newton stratum $\mathcal{M}(3,2)^{[\frac{0}{1}^4,\frac{1}{2},\frac{1}{1}^4]}$.

We summarize these results in Table 2.2.

Table 2.2: Newton and Ekedahl-Oort Interactions for $\mathcal{M}(3,2)$

	$[0^4,\ \frac{1}{2},\ 1^4]$	$[0^2,\ \frac{1}{2}^3,\ 1^2]$	$[\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}]$	$[\frac{1}{2}^5]$
$\gamma_{1,2}$	*empty*	*empty*	*empty*	*containment*
$\gamma_{1,3}$	*empty*	*empty*	*empty*	*containment*
$\gamma_{1,4}$	*empty*	*empty*	*empty*	*containment*
$\gamma_{2,3}$	*empty*	*empty*	*empty*	*containment*
$\gamma_{1,5}$	*empty*	*empty*	*empty*	*containment*
$\gamma_{2,4}$	*empty*	*empty*	*containment*	*empty*
$\gamma_{2,5}$	*empty*	*containment*	*empty*	*empty*
$\gamma_{3,4}$	*empty*	*empty*	*generic intersection*	*intersection*
$\gamma_{3,5}$	*empty*	*containment*	*empty*	*empty*
$\gamma_{4,5}$	*equal*	*empty*	*empty*	*empty*

2.7 Generating mod-p Dieudonné modules of a fixed signature

We can attempt to generalize the construction in Lemma 2.6.3 further, by presenting an Ekedahl-Oort stratum in a unitary Shimura variety as a mod-p Dieudonné module with the given signature, then lifting it to a p-adic Dieudonné module. We will focus on the case when the Ekedahl-Oort stratum is indecomposable, i.e., when it does not lie in the image of a product map of smaller unitary Shimura varieties.

The contents of this section are part of an upcoming paper, in collaboration with Emerald Andrews, Deewang Bhamidipati, Maria Fox, Heidi Goodson, and Steven R. Groen.

2.7.1 The case of signature $(q-2,2)$

Suppose $q>2$ is odd. Fix a positive integer $R \leq q$. The following information captures a mod-p Dieudonné module in signature $(q-2,2)$:

$$F(e_i) = \begin{cases} f_{i-1} \text{ for } 2 \leq i \leq q \text{ and } i \neq R \\ 0 \text{ for } i = 1, R \end{cases} \qquad V(e_i) = \begin{cases} f_{i+1} \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R-1 \\ 0 \text{ for } i = R-1, q \end{cases}$$

$$F(f_i) = \begin{cases} (-1)^q e_q = -e_q \text{ for } i = 1 \\ e_{R-1} \text{ for } i = R \\ 0 \text{ for } 2 \leq i \leq q \text{ and } i \neq R \end{cases} \qquad V(f_i) = \begin{cases} e_R \text{ for } i = R-1 \\ e_1 \text{ for } i = q \\ 0 \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R-1 \end{cases}$$

We will denote this object by $B_2(q,R)$, where:

- q is the dimension of the abelian varieties being parameterized.
- The subscript 2 indicates that the signature is of the form $(q-2,2)$.
- R is a parameter controlling all possible $\mathrm{sgn}(q-2,2)$ mod-p Dieudonné modules.

We adhere to the polarization $\langle .,. \rangle$ given in [11], namely:

$$\begin{cases} \langle e_i, f_j \rangle = (-1)^{i+1}\delta_{ij} \\ \langle e_i, e_j \rangle = 0 \\ \langle f_i, f_j \rangle = 0 \\ \langle F(e_i), e_j \rangle = \langle e_i, V(e_j) \rangle \\ \langle F(f_i), f_j \rangle = \langle f_i, V(f_j) \rangle \\ \langle x, y \rangle = -\langle y, x \rangle^{\sigma} \end{cases}$$

It follows from the above that $\langle V(e_i), e_j \rangle = \langle e_i, F(e_j) \rangle$ and $\langle V(f_i), f_j \rangle = \langle f_i, F(f_j) \rangle$. Note that at the level of mod-p Dieudonné modules, we have $F \circ V = 0 = V \circ F$. When we lift $B_2(q, R)$ to a p-adic Dieudonné module $\mathbb{B}_2(q, R)$, the lift identity gives us: $F \circ V = p = V \circ F$. Any successful lift to a p-adic Dieudonné module must obey the polarization in conjunction with this identity.

Let $\{\tilde{e}_1, ..., \tilde{e}_q, \tilde{f}_1, ..., \tilde{f}_q\}$ be the basis of $\mathbb{B}_2(q, R)$, where we send $\tilde{e}_i \mapsto e_i$ and $\tilde{f}_i \mapsto f_i$ on taking mod p. Note that there are infinitely many choices for such a “change of basis"; we are going with the most natural lift. So we must have the polarization:

$$\begin{cases} \langle \tilde{e}_i, \tilde{f}_j \rangle = (-1)^{i+1}\delta_{ij} \\ \langle \tilde{e}_i, \tilde{e}_j \rangle = 0 \\ \langle \tilde{f}_i, \tilde{f}_j \rangle = 0 \\ \langle F(\tilde{e}_i), \tilde{e}_j \rangle = \langle \tilde{e}_i, V(\tilde{e}_j) \rangle \\ \langle F(\tilde{f}_i), \tilde{f}_j \rangle = \langle \tilde{f}_i, V(\tilde{f}_j) \rangle \\ \langle \tilde{x}, \tilde{y} \rangle = -\langle \tilde{y}, \tilde{x} \rangle^{\sigma} \end{cases}$$

Lemma 2.7.1. The following is a p-adic Dieudonné module, denoted by $\mathbb{B}_2(q, R)$, which lifts the mod-p Dieudonné module $B_2(q, R)$:

$$F(\tilde{e}_i) = \begin{cases} p\tilde{f}_q \text{ for } i = 1 \\ \tilde{f}_{i-1} \text{ for } 2 \le i \le q \text{ and } i \ne R \\ p\tilde{f}_{R-1} \text{ for } i = R \end{cases} \quad V(\tilde{e}_i) = \begin{cases} \tilde{f}_{i+1} \text{ for } 1 \le i \le q-1 \text{ and } i \ne R-1 \\ p\tilde{f}_R \text{ for } i = R-1 \\ -p\tilde{f}_1 \text{ for } i = q \end{cases}$$

$$F(\tilde{f}_i) = \begin{cases} (-1)^q\tilde{e}_q = -\tilde{e}_q \text{ for } i = 1 \\ p\tilde{e}_{i-1} \text{ for } 2 \le i \le q \text{ and } i \ne R \\ \tilde{e}_{R-1} \text{ for } i = R \end{cases} \quad V(\tilde{f}_i) = \begin{cases} p\tilde{e}_{i+1} \text{ for } 1 \le i \le q-1 \text{ and } i \ne R-1 \\ \tilde{e}_R \text{ for } i = R-1 \\ \tilde{e}_1 \text{ for } i = q \end{cases}$$

Proof. By construction, we know that $\mathbb{B}_2(q, R)$ agrees with $B_2(q, R)$ upon going mod p. We verify the polarization property holds by direct computation. □

Example 2.7.2. We use this notation to do an explicit example. Consider the case of $B_2(7, 6)$. In the gamma notation introduced earlier, this would be indexed by $\mathcal{M}(5, 2)_{\gamma_{4,6}}$. Its Kraft word would be $V^5(F^{-1}V)(F^{-1})^5VF^{-1}$. It has the following mod-p Dieudonné module structure:

$$F(e_i) = \begin{cases} f_{i-1} \text{ for } 2 \le i \le 7 \text{ and } i \ne 6 \\ 0 \text{ for } i = 1, 6 \end{cases} \quad V(e_i) = \begin{cases} f_{i+1} \text{ for } 1 \le i \le 6 \text{ and } i \ne 5 \\ 0 \text{ for } i = 5, 7 \end{cases}$$

$$F(f_i) = \begin{cases} -e_7 \text{ for } i = 1 \\ e_{R-1} \text{ for } i = 6 \\ 0 \text{ for } 2 \le i \le 7 \text{ and } i \ne 6 \end{cases} \quad V(f_i) = \begin{cases} e_R \text{ for } i = 5 \\ e_1 \text{ for } i = 7 \\ 0 \text{ for } 1 \le i \le 6 \text{ and } i \ne 5 \end{cases}$$

From this, it is fairly straightforward to compute its Ekedahl-Oort type, which we find to be $[0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 6, 7]$. Moreover, Harashita's algorithm [28] reveals the first slope of the generic Newton polygon intersecting the Ekedahl-Oort stratum in the underlying Siegel modular variety to be $\frac{1}{5}$. As such, a priori, one cannot expect the Ekedahl-Oort stratum $\mathcal{M}(5,2)_{\gamma_{4,6}}$ to intersect with the supersingular locus. Yet, we have the following result:

Lemma 2.7.3. The Ekedahl-Oort stratum $\mathcal{M}(5,2)_{\gamma_{4,6}}$ corresponding to the mod-p Dieudonné module $B_2(7,6)$ intersects the supersingular locus.

Proof. The following is a p-adic Dieudonné module, denoted by $\mathbb{B}_2(7,6)$, which lifts the mod-p Dieudonné module $B_2(7,6)$:

$$F(\tilde{e}_i) = \begin{cases} p\tilde{f}_7 \text{ for } i = 1 \\ \tilde{f}_{i-1} \text{ for } 2 \leq i \leq 7 \text{ and } i \neq 6 \\ p\tilde{f}_5 \text{ for } i = 6 \end{cases} \quad V(\tilde{e}_i) = \begin{cases} \tilde{f}_{i+1} \text{ for } 1 \leq i \leq 6 \text{ and } i \neq 5 \\ p\tilde{f}_6 \text{ for } i = 5 \\ -p\tilde{f}_1 \text{ for } i = 7 \end{cases}$$

$$F(\tilde{f}_i) = \begin{cases} -\tilde{e}_7 \text{ for } i = 1 \\ p\tilde{e}_{i-1} \text{ for } 2 \leq i \leq 7 \text{ and } i \neq 6 \\ \tilde{e}_5 \text{ for } i = 6 \end{cases} \quad V(\tilde{f}_i) = \begin{cases} p\tilde{e}_{i+1} \text{ for } 1 \leq i \leq 6 \text{ and } i \neq 5 \\ \tilde{e}_R \text{ for } i = 5 \\ \tilde{e}_1 \text{ for } i = 7 \end{cases}$$

We note that for every $m \in \mathbb{Z}_{>0}$, the following holds:

$$F^{14m}(\mathbb{B}_2(7,6)) = p^{7m}\mathbb{B}_2(7,6).$$

Let $n = n(m) = 14m$ and $k = k(m) = 7m$. By Zink's lemma [30], the slope of the isocrystal obtained from $\mathbb{B}_2(7,6)$ is:

$$\lim_{n\to\infty} \frac{1}{n}\{\max k : F^n(\mathbb{B}_2(7,6)) \subseteq p^k\mathbb{B}_2(7,6)\} = \lim_{m\to\infty} \frac{14m}{7m} = \frac{1}{2}$$

Thus, the slope of the Newton polygon corresponding to the p-divisible group associated to the p-adic Dieudonné module $\mathbb{B}_2(7,6)$ is $\frac{1}{2}$. In other words, we have constructed a point on $\mathcal{M}(5,2)$ that lies on the Ekedahl-Oort stratum corresponding to $B_2(7,6)$ and on the supersingular locus.

□

2.7.2 Potential generalization to higher signatures

Suppose $q > 2$ is odd. Let the mod-p Dieudonné module in $\mathrm{sgn}(q-b,b)$ be denoted by $B_b(q,\underline{R})$, where $\underline{R} = (R_1, ..., R_{b-1})$. This consists of the following information:

$$F(e_i) = \begin{cases} f_{i-1} \text{ for } 2 \leq i \leq q \text{ and } i \neq R_j : 1 \leq j \leq b-1 \\ 0 \text{ for } i = 1, R_j : 1 \leq j \leq b-1 \end{cases}$$

$$V(e_i) = \begin{cases} f_{i+1} \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R_j - 1 : 1 \leq j \leq b-1 \\ 0 \text{ for } i = q, R_j - 1 : 1 \leq j \leq b-1 \end{cases}$$

$$F(f_i) = \begin{cases} (-1)^q e_q = -e_q \text{ for } i = 1 \\ e_{R-1} \text{ for } i = R_j : 1 \leq j \leq b-1 \\ 0 \text{ for } 2 \leq i \leq q \text{ and } i \neq R_j : 1 \leq j \leq b-1 \end{cases}$$

$$V(f_i) = \begin{cases} e_R \text{ for } i = R_j - 1 : 1 \leq j \leq b-1 \\ e_1 \text{ for } i = q \\ 0 \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R_j - 1 : 1 \leq j \leq b-1 \end{cases}$$

Let the p-adic lift be denoted by $\mathbb{B}_b(q,\underline{R})$. We present the tentative structure, once again requiring that the polarization and $F \circ V = p = V \circ F$ be obeyed together:

$$F(\tilde{e}_i) = \begin{cases} p\tilde{f}_q \text{ for } i = 1 \\ \tilde{f}_{i-1} \text{ for } 2 \leq i \leq q \text{ and } i \neq R_j : 1 \leq j \leq b-1 \\ p\tilde{f}_{R_j-1} \text{ for } i = R_j : 1 \leq j \leq b-1 \end{cases}$$

$$V(\tilde{e}_i) = \begin{cases} \tilde{f}_{i+1} \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R_j - 1 : 1 \leq j \leq b-1 \\ p\tilde{f}_{R_j} \text{ for } i = R_j - 1 : 1 \leq j \leq b-1 \\ -p\tilde{f}_1 \text{ for } i = q \end{cases}$$

$$F(\tilde{f}_i) = \begin{cases} (-1)^q \tilde{e}_q = -\tilde{e}_q \text{ for } i = 1 \\ p\tilde{e}_{i-1} \text{ for } 2 \leq i \leq q \text{ and } i \neq R_j : 1 \leq j \leq b-1 \\ \tilde{e}_{R_j-1} \text{ for } i = R_j : 1 \leq j \leq b-1 \end{cases}$$

$$V(\tilde{f}_i) = \begin{cases} p\tilde{e}_{i+1} \text{ for } 1 \leq i \leq q-1 \text{ and } i \neq R_j - 1 : 1 \leq j \leq b-1 \\ \tilde{e}_{R_j} \text{ for } i = R_j - 1 : 1 \leq j \leq b-1 \\ \tilde{e}_1 \text{ for } i = q \end{cases}$$

Chapter 3

Traverso's Isogeny Conjecture

The isogeny cutoff of a p-divisible group X (defined over an algebraically closed field of characteristic p) measures the amount of p-torsion necessary to determine its isogeny class. The minimal height of X measures its distance to the closest minimal p-divisible group (in the sense of Oort). In this chapter, we study these invariants for supersingular unitary p-divisible groups of signature (a, b). The results can be reinterpreted in the language of the stratifications of unitary Shimura varieties based on p-power torsion group schemes. [3].

3.1 Introduction

Given a p-divisible group X over an algebraically closed field of characteristic p, there are two objects that capture partial information about X: its p-torsion subgroup $X[p]$ and its Newton polygon. When X is the p-divisible group of an elliptic curve, the p-torsion group is enough to determine the Newton polygon. In general, Traverso's Isogeny Conjecture [31, Conj 5] predicts how much p-power torsion is needed to determine the Newton polygon. This number is called the *isogeny cutoff* and is expressed in terms of the dimension and height of X. Traverso's conjecture was proven by Nicole and Vasiu [3], with a further refinement in [32].

The papers [3] and [32] both also investigate properties of the *minimal height* of a p-divisible group X. This is a measure of the distance between X and the closest minimal p-divisible group. The minimal height can be used to give an upper bound on the isogeny cutoff, but it is also an interesting geometric invariant of X in its own right. In this chapter, we study both the minimal heights and the isogeny cutoffs of p-divisible groups arising from points in the supersingular locus of unitary Shimura varieties.

[3]The contents of this chapter are found in: E. Andrews, D. Bhamidipati, M. Fox, H. Goodson, S.R. Groen, and S. Nair, "Traverso's Isogeny Conjecture for Some Unitary p-Divisible Groups" arXiv:2507.19708 (2025) [4].

Such "unitary p-divisible groups" come equipped with a signature, which is a pair of integers (a, b). Heuristically, the complexity of X grows with $\min(a, b)$, which we can assume without loss of generality is a.

Unitary p-divisible groups fit into the larger collection of p-divisible groups of dimension $g = a + b$ and height $2g$. It is useful to note how much the extra symmetry imposed by the signature (a, b) action (and compatible polarization) influences the minimal height and isogeny cutoff of X. Indeed, using only the information of the height and dimension, one would expect that the minimal height of a supersingular p-divisible group X could be as much as $\lfloor \frac{g}{2} \rfloor$ [32] and the isogeny cutoff could be as much as $\lceil \frac{g}{2} \rceil$ [3].

3.2 Background

In this section, we give the definition of our main objects of study, unitary p-divisible groups, and their linear algebraic incarnations, unitary Dieudonné modules. We will also recall some useful results from the literature on the structure of these Dieudonné modules.

Throughout this paper, $\Bbbk$ will denote an algebraically closed field of characteristic p. Let $W(\Bbbk)$ denote the ring of Witt vectors over $\Bbbk$, with Frobenius automorphism σ and fraction field $W(\Bbbk)_{\mathbb{Q}}$. For any $W(\Bbbk)$-module M, we'll use $M_{\mathbb{Q}}$ to denote $M \otimes_{W(\Bbbk)} W(\Bbbk)_{\mathbb{Q}}$. Let K be the degree 2 unramified extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$. Let φ_0 and φ_1 denote the two embeddings of K into $W(\Bbbk)_{\mathbb{Q}}$. Given a containment $A \subseteq B$ of two $W(\Bbbk)$-lattices in a $W(\Bbbk)_{\mathbb{Q}}$-vector space, we write $A \subseteq_n B$ to mean that the $W(\Bbbk)$-module B/A has length n.

Throughout the paper, a and b will always be nonnegative integers. Following [11], we make the following two definitions.

Definition 3.2.1. A **unitary p-divisible group of signature (a, b)** over $\Bbbk$ is a tuple (X, ι, λ) where

- X is a p-divisible group over $\Bbbk$, of dimension $a + b$ and height $2(a + b)$;
- $\iota : \mathcal{O}_K \to \mathrm{End}(X)$ is an action satisfying the *signature* (a, b) *condition*: for all $m \in \mathcal{O}_K$,

$$\mathrm{charpol}(\iota(m) \mid \mathrm{Lie}(X)) = (T - \varphi_0(m))^a (T - \varphi_1(m))^b \in W(\Bbbk)[T];$$

- $\lambda : X \to X^\vee$ is a p-principal polarization, meeting the following $\mathcal{O}_K$-linearity condition, for all $m \in \mathcal{O}_K$:

$$\lambda \circ \iota(m) = \iota(\overline{m})^\vee \circ \lambda.$$

When no confusion is possible, we'll refer to this tuple only as X.

Definition 3.2.2. A **unitary Dieudonné module of signature (a, b)** over $\Bbbk$ is a tuple $(M, F, V, \langle\cdot,\cdot\rangle, M = M_0 \oplus M_1)$ where

- M is a free $W(\Bbbk)$-module of rank $2(a+b)$;
- $F : M \to M$ is a σ-semilinear operator, $V : M \to M$ is a σ^{-1}-semilinear operator, with $F \circ V = V \circ F = p$;
- $\langle\cdot,\cdot\rangle : M \times M \to W(\Bbbk)$ is a perfect alternating $W(\Bbbk)$-bilinear pairing on **M** such that $\langle F(x), y\rangle = \langle x, V(y)\rangle^\sigma$ for all $x, y \in M$;
- $M = M_0 \oplus M_1$ is a decomposition of M into rank-$(a+b)$ summands, each totally isotropic with respect to $\langle\cdot,\cdot\rangle$, with the property that F and V are homogenous of degree 1, such that

$$pM_0 \subseteq_b FM_1 \subseteq_a M_0 \quad \text{and} \quad pM_1 \subseteq_a FM_0 \subseteq_b M_1.$$

When no confusion is possible, we'll refer to this tuple only as M.

As discussed in [11], for any algebraically closed field $\Bbbk$, covariant Dieudonné theory gives a bijection between the collection of all unitary p-divisible groups of signature (a, b) over $\Bbbk$ and the collection of all unitary Dieudonné modules of signature (a, b) over $\Bbbk$.

Note that reversing the labeling of the embeddings φ_0 and φ_1, or on the level of Dieudonné modules the labeling of M_0 and M_1, changes the signature from (a, b) to (b, a). So, without loss of generality, we will always assume $a \le b$, so that $\min(a, b) = a$.

Recall that a p-divisible group is called supersingular (resp. superspecial) if it is isogenous (resp. isomorphic) to a product of the p-divisible groups of supersingular elliptic curves. We refer to the corresponding p-adic Dieudonné module supersingular (resp. superspecial).

Let $(M, F, V, \langle\cdot,\cdot\rangle, M = M_0 \oplus M_1)$ be a unitary Dieudonné module over $\Bbbk$ of signature (a, b). Due to the compatibility between the action and the polarization, much of the structure of M is controlled by the submodule M_0 (or, symmetrically, M_1). To see this, for either $i = 0$ or $i = 1$, it is possible to define a new pairing on M_i, as

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle : M_i \times M_i \to W(\Bbbk)$$
$$\langle\!\langle x, y\rangle\!\rangle := \langle x, F(y)\rangle.$$

We will use the same notation to denote the extension of scalars of this pairing to a $W(\Bbbk)_{\mathbb{Q}}$-valued pairing on $(M_i)_{\mathbb{Q}}$. Given a $W(\Bbbk)$ lattice $L \subseteq (M_i)_{\mathbb{Q}}$, we denote

$$L^\vee = \{x \in (M_i)_{\mathbb{Q}} \mid \langle\!\langle x, y\rangle\!\rangle \in W(\Bbbk) \text{ for all } y \in L\}.$$

Let τ be the σ^2-linear operator $\frac{1}{p}F^2 = V^{-1} \circ F$. As F and V interchange M_0 and M_1, note that τ is an operator on each $(M_i)_{\mathbb{Q}}$. Some useful properties of τ and the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, due to Vollaard [33], are recorded below.

Proposition 3.2.3. Let $(M, F, V, \langle\cdot,\cdot\rangle, M = M_0 \oplus M_1)$ be a unitary Dieudonné module over $\Bbbk$ of signature (a, b). For either $i = 0$ or $i = 1$, and any $W(\Bbbk)$ lattice $L \subseteq (M_i)_{\mathbb{Q}}$, we have the following:

- $\tau(L)^\vee = \tau(L^\vee)$;
- $(L^\vee)^\vee = \tau(L)$;
- $M_i = F^{-1}(pM_{i+1}^\vee)$, with indices mod 2.

The relationship between M_0 and M_1 can be expressed in the following four (equivalent) chain conditions:

$$pM_0^\vee \subseteq_a M_0 \subseteq_b M_0^\vee, \quad pM_0^\vee \subseteq_a \tau(M_0) \subseteq_b M_0^\vee,$$
$$pM_1^\vee \subseteq_b M_1 \subseteq_a M_1^\vee, \quad pM_1^\vee \subseteq_b \tau(M_1) \subseteq_a M_1^\vee.$$

Proof. These statements can be found in [33, Subsection 1.11, Proposition 1.12, and Remark 1.13]. Note that the article of Vollaard is concerned only with the Dieudonné modules of *supersingular* p-divisible groups, but that assumption is not relevant to these statements. □

3.3 Isogenies to Superspecial p-Divisible Groups

In this section, for any supersingular unitary p-divisible group X of signature (a,b), we will construct a superspecial p-divisible group $\mathbb{X}$ and an isogeny $\rho : X \to \mathbb{X}$ with the property that $\ker(\rho) \subseteq X[p^a]$ (in fact, contained in $X[p^{a-1}]$ when $a = b$).

Consider a supersingular unitary p-divisible group X over $\Bbbk$ of signature (a,b), with Dieudonné module $(M, F, V, \langle\cdot,\cdot\rangle, M = M_0 \oplus M_1)$. For any $i \geq 0$, we define $W(\Bbbk)$ lattices

$$T_i = M_0 + \tau(M_0) + \cdots + \tau^i(M_0),$$
$$S_i = M_1 + \tau(M_1) + \cdots + \tau^i(M_1).$$

Lemma 3.3.1. Let M be a unitary p-divisible group of signature (a,b) over $\Bbbk$. Assume that M is supersingular. There exist integers m and n such that T_m and S_n are τ-invariant.

Proof. Note that because M is supersingular, the operator F is a slope $\frac{1}{2}$ operator on the isocrystal $M_{\mathbb{Q}}$. So, the operator τ on both $(M_0)_{\mathbb{Q}}$ and $(M_1)_{\mathbb{Q}}$ is slope zero. Then, the existence of m and n follows from [29, Proposition 2.17]. □

We have the following lemmata that will be relevant for bounding the isogeny cut-offs:

Lemma 3.3.2. Let M be a supersingular unitary Dieudonné module over $\Bbbk$ of signature (a,b). Let m be minimal such that T_m is τ-invariant. Then,

$$p^a T_m \subseteq pT_m^\vee \subseteq pM_0^\vee \subseteq_a M_0 \subseteq T_m \subseteq p^{1-a}T_m^\vee.$$

Lemma 3.3.3. Let M be a supersingular unitary Dieudonné module over $\Bbbk$ of signature (a,b). Let n be minimal such that S_n is τ-invariant. Then,

$$p^a S_n \subseteq M_1 \subseteq S_n.$$

When $a = b$, the results in Lemma 3.3.2 and Lemma 3.3.3 can be improved slightly.

Lemma 3.3.4. Let M be a supersingular unitary Dieudonné module over $\Bbbk$ of signature (a,a). Let m be minimal such that T_m is τ-invariant, and let n be minimal such that S_n is τ-invariant. Then,

$$p^{a-1}T_m \subseteq M_0 \subseteq T_m \quad \text{and} \quad p^{a-1}S_n \subseteq M_1 \subseteq S_n.$$

Proposition 3.3.5. Let M be a unitary Dieudonné module of signature (a,b) over an algebraically closed field $\Bbbk$. Define

$$\Lambda = M + \tau(M) + \tau^2(M) + \cdots.$$

Then Λ is the Dieudonné module of a superspecial p-divisible group over $\Bbbk$, and

$$p^a\Lambda \subseteq M \subseteq \Lambda.$$

Further, when $a = b$,

$$p^{a-1}\Lambda \subseteq M \subseteq \Lambda.$$

Proof. Note that $\Lambda = \Lambda_0 + \Lambda_1$, where

$$\Lambda_0 = M_0 + \tau(M_0) + \tau^2(M_0) + \cdots = T_m,$$
$$\Lambda_1 = M_1 + \tau(M_1) + \tau^2(M_1) + \cdots = S_n,$$

and then apply Lemmas 3.3.2 and 3.3.3 to see that

$$p^a\Lambda \subseteq M \subseteq \Lambda.$$

Because the F and V operators commute with τ and M is stable under F and V, Λ is also stable under F and V. So, Λ is the Dieudonné module of a p-divisible group. This p-divisible group is superspecial, because the condition that $\tau(\Lambda) = \Lambda$ is equivalent to $F\Lambda = V\Lambda$.

When $a = b$, by Lemma 3.3.4,

$$p^{a-1}\Lambda \subseteq M \subseteq \Lambda. \qquad \square$$

Corollary 3.3.6. Let X be a supersingular unitary p-divisible group of signature (a, b). When $a < b$ there exists a superspecial p-divisible group $\mathbb{X}$ and an isogeny $\rho : X \to \mathbb{X}$ with $\mathrm{Ker}(\rho) \subseteq X[p^a]$. When $a = b$, there exists an isogeny $\rho : X \to \mathbb{X}$ with $\mathrm{Ker}(\rho) \subseteq X[p^{a-1}]$.

Proof. This follows from Proposition 3.3.5 and covariant Dieudonné theory. $\square$

3.4 Minimal Heights

The goal of this section is to translate the results of the previous section into the language of *minimal heights*, and to give a complete description of minimal heights occurring for supersingular unitary p-divisible groups of signature (a, b). From [32], we have the following definition:

Definition 3.4.1. Let X be a supersingular p-divisible group. The *minimal height* of X is the smallest integer q_X such that there exists an isogeny $\rho : X \to \mathbb{X}$ to a superspecial p-divisible group, with $\ker(\rho) \subseteq X[p^{q_X}]$.

As noted in [32], the minimal height of X can also be thought of as a “minimal distance" to a superspecial p-divisible group.

Lemma 3.4.2. Let X be a unitary p-divisible group of signature (a, b) over an algebraically closed field, with associated Dieudonné module M. Let q_X be the minimal height of X.

With $\Lambda = M + \tau(M) + \tau^2(M) + \cdots$, the integer q_X can also be characterized as the minimal integer r_M such that

$$p^{r_M}\Lambda \subseteq M \subseteq \Lambda.$$

Proof. By covariant Dieudonné theory, the lattice inclusions $p^{r_M}\Lambda \subseteq M \subseteq \Lambda$ define an isogeny $\rho_\Lambda : X \to X_\Lambda$ to the superspecial p-divisible group X_Λ, with the property that $\ker(\rho_\Lambda) \subseteq X[p^{r_M}]$. So, $q_X \le r_M$.

On the other hand, by definition of q_X, there exists a superspecial p-divisible group $\mathbb{X}$ and an isogeny $\rho : X \to \mathbb{X}$ with $\ker(\rho) \subseteq X[p^{q_X}]$. If $M_{\mathbb{X}}$ is the p-adic Dieudonné module of $\mathbb{X}$, this gives an inclusion

$$M \subseteq M_{\mathbb{X}}$$

with the property that $p^{q_X} M_{\mathbb{X}} \subseteq M \subseteq M_{\mathbb{X}}$.

Since $\mathbb{X}$ is superspecial, $FM_{\mathbb{X}} = VM_{\mathbb{X}}$ and so $\tau(M_{\mathbb{X}}) = M_{\mathbb{X}}$. But then,

$$M \subseteq M + \tau(M) + \tau^2(M) + \cdots \subseteq M_{\mathbb{X}},$$

which is simply the notation for the inclusions

$$M \subseteq \Lambda \subseteq M_{\mathbb{X}}.$$

In particular, $p^{q_X}\Lambda \subseteq p^{q_X}M_{\mathbb{X}} \subseteq M \subseteq \Lambda \subseteq M_{\mathbb{X}}$. By the minimality of r_M, we have $q_X \ge r_M$. Thus, $q_X = r_M$. □

It follows immediately from the above lemma that if X_1 and X_2 are two supersingular p-divisible groups over $\Bbbk$, of minimal height q_1 and q_2, then the minimal height of $X_1 \times_{\Bbbk} X_2$ is $\max(q_1, q_2)$.

We will now give a series of examples of specific p-divisible groups and their minimal heights. These examples will be used in Theorem 3.4.6.

Example 3.4.3. Consider a and b with $a + b = g$ odd, with the convention that $a \leq b$. (Though, since $a + b$ is odd, this immediately implies $a < b$.) For notational convenience, set $r = \frac{g-1}{2}$. In this example, we construct a supersingular unitary p-divisible group of signature (a, b) over $\Bbbk$, with a minimal height of a.

To define the unitary Dieudonné module $M_{(a,b)} = (M_{(a,b)}, F, V, \langle \cdot, \cdot \rangle, M_{(a,b)} = M_0 \oplus M_1)$, we set $M_0 = \mathrm{Span}_{W(\Bbbk)}\{e_1, \ldots, e_g\}$ and $M_1 = \mathrm{Span}_{W(\Bbbk)}\{f_1, \ldots, f_g\}$, with $M_{(a,b)} = M_0 \oplus M_1$.

Let the F operator be extended σ-semilinearly from the conditions:

$$\begin{aligned} F(e_i) &= f_i && 1 \leq i \leq a, \\ F(e_j) &= pf_j && a+1 \leq j \leq g, \\ F(f_j) &= pe_{j+1} && r+1 \leq j \leq r+a, \\ F(f_i) &= e_{i+1} && \text{otherwise,} \end{aligned}$$

and V be defined as pF^{-1}.

Note that, by construction,

$$pM_0 \subseteq_b FM_1 \subseteq_a M_0 \quad \text{and} \quad pM_1 \subseteq_a FM_0 \subseteq_b M_1.$$

We define the pairing $\langle \cdot, \cdot \rangle : M_{(a,b)} \times M_{(a,b)} \to W(\Bbbk)$ by choosing $\delta \in W(\Bbbk)^\times$ such that $\delta^\sigma = -\delta$, and setting

$$\langle e_i, f_{r+i} \rangle = \delta$$

for all $1 \leq i \leq g$, and $\langle e_i, f_j \rangle = 0$ otherwise, declaring M_0 and M_1 to both be totally isotropic with respect to $\langle \cdot, \cdot \rangle$, and setting $\langle f_j, e_i \rangle = -\langle e_i, f_j \rangle$.

The fact that this pairing has the property that $\langle F(x), y\rangle = \langle x, V(y)\rangle^{\sigma}$ can be checked as a matrix computation: $A_F^T B = (BA_V)^{\sigma}$, where A_F and A_V are the matrices defining F and V on the basis $\{e_1, \ldots, e_g, f_1, \ldots, f_g\}$, and B is the matrix of the form $\langle\cdot,\cdot\rangle$.

However, note that it is essential that g is odd. To see this clearly, let $k = \lfloor \frac{g}{2} \rfloor - a$, and observe that if we want $\langle\cdot,\cdot\rangle$ to have the F and V compatibility above, then

$$\begin{aligned}
\delta &= \langle e_1, f_{r+1}\rangle \\
&= \langle F^g p^{-k} f_{r+1}, f_{r+1}\rangle \\
&= \langle p^{-k} f_{r+1}, V^g f_{r+1}\rangle^{\sigma^g} \\
&= \langle p^{-k} f_{r+1}, p^k e_1\rangle^{\sigma^g} \\
&= \langle f_{r+1}, e_1\rangle^{\sigma^g} \\
&= (-\delta)^{\sigma^g},
\end{aligned}$$

so g must be odd.

To show that $M_{(a,b)}$ is supersingular, we use [30, Lemma 6.12] to compute the first slope of the isocrystal $(M_{(a,b)})_{\mathbb{Q}}$. For any positive integer m,

$$F^{2gm}(M_{(a,b)}) = p^{gm} M_{(a,b)},$$

and so

$$\lim_{n\to\infty} \frac{1}{n} \max\{k \in \mathbb{Z} : F^n M_{(a,b)} \subseteq p^k M_{(a,b)}\} = \frac{1}{2},$$

and by [30, Lemma 6.12] the first slope of $(M_{(a,b)})_{\mathbb{Q}}$ is $\frac{1}{2}$. But since the Newton polygon has non-decreasing slopes, and goes from $(0,0)$ to $(g, 2g)$, all slopes must be equal to $\frac{1}{2}$.

We've shown that $M_{(a,b)}$ is a supersingular unitary Dieudonné module of signature (a,b). Let $X_{(a,b)}$ be the corresponding p-divisible group.

Finally, we compute the minimal height of $X_{(a,b)}$. Let $\Lambda_{(a,b)} = M_{(a,b)} + \tau(M_{(a,b)}) + \tau^2 M_{(a,b)} + \cdots$. By definition of F and V on $M_{(a,b)}$,

$$\tau^a(e_1) = p^{-a} e_{a+1},$$

and so while $p^a \Lambda \subseteq M_{(a,b)} \subseteq \Lambda$, it is not true that $p^{a-1}\Lambda \subseteq M_{(a,b)}$. Then, by Lemma 3.4.2, the minimal height of $X_{(a,b)}$ is a.

Example 3.4.4. Consider a and b with $a + b$ even, and $a < b$. In this example, we construct a supersingular unitary p-divisible group of signature (a, b) over $\Bbbk$, with a minimal height of a.

Since $a < b$, and $a + b$ is even, it is also true that $a < b - 1$. So, we can consider $M_{(a,b-1)}$ (with its additional structure) and define $M_{(0,1)}$ as in Example 3.4.3. We then define $M_{(a,b)}$ as

$$M_{(a,b)} = M_{(a,b-1)} \oplus M_{(0,1)}$$

with the F and V maps and the polarization defined as a product as well.

Note that $M_{(a,b)}$ is a unitary p-adic Dieudonné module of signature (a, b) because it has been constructed as a product of p-adic Dieudonné modules of signatures $(a, b - 1)$ and $(0, 1)$. Since $M_{(a,b)}$ is a product of supersingular Dieudonné modules, it is also supersingular.

Let $X_{(a,b)}$ be the p-divisible group associated to $M_{(a,b)}$. By construction,

$$X_{(a,b)} \cong X_{(a,b-1)} \times_{\Bbbk} X_{(0,1)}.$$

Note that, by the previous example, $X_{(a,b-1)}$ has minimal height a. Since $FM_{(0,1)} = VM_{(0,1)}$, the p-divisible group $A_{(0,1)}$ is superspecial, so has minimal height 0. Then, $X_{(a,b)}$ has minimal height $\max(a, 0) = a$.

Example 3.4.5. In this example, we construct a supersingular unitary p-divisible group of signature (a, a) over $\Bbbk$, with a minimal height of $a - 1$.

Let $M_{(a-1,a)}$ be the p-adic Dieudonné module of signature $(a-1,a)$ constructed in Example 3.4.3. Define $M_{(1,0)}$ exactly as $M_{(0,1)}$ of Example 3.4.3, but with $(M_{(1,0)})_0 = (M_{(0,1)})_1$ and $(M_{(1,0)})_1 = (M_{(0,1)})_0$. As a result, $M_{(1,0)}$ is a superspecial p-adic unitary Dieudonné module of signature $(1,0)$.

Now, define $M_{(a,a)}$ as the product

$$M_{(a,a)} =: M_{(a-1,a)} \oplus M_{(1,0)},$$

equipped with product F and V operators and product polarization. By construction, $M_{(a,a)}$ is a supersingular unitary Dieudonné module of signature (a,a). Let $X_{(a,a)} \cong X_{(a-1,a)} \times_{\Bbbk} X_{(1,0)}$ be the corresponding p-divisible group. Since $X_{(a-1,a)}$ has minimal height $a-1$ and $X_{(1,0)}$ has minimal height 0, it follows that $X_{(a,a)}$ has minimal height $a-1$.

The three previous examples combine to construct a supersingular p-divisible group $X_{(a,b)}$ of signature (a,b), for any signature (with the convention that $a \leq b$). When $a \neq b$, this p-divisible group has minimal height a. When $a < b$, the minimal height is $a-1$.

Theorem 3.4.6. Let a and b be nonnegative integers, with the convention that $a \leq b$. If $a < b$, then for any supersingular p-divisible group X over $\Bbbk$ of signature (a,b), the minimal height of X is at most a. Furthermore, for any $0 \leq q \leq a$, there exists a supersingular p-divisible group over $\Bbbk$ of signature (a,b) with minimal height exactly q.

If $a = b$, the minimal height of any supersingular p-divisible group over $\Bbbk$ of signature (a,a) is at most $a-1$. Furthermore, for any $0 \leq q \leq a-1$, there exists a supersingular p-divisible group over $\Bbbk$ of signature (a,a) with minimal height exactly q.

Proof. First, consider the case where $a < b$. By the first statement in Corollary 3.3.6, the minimal height of any supersingular p-divisible group X over $\Bbbk$ of signature (a,b) is at most a.

By Examples 3.4.3 and 3.4.4 (depending on the parity of $a+b$), there exists a supersingular p-divisible group $X_{(a,b)}$ of signature (a,b), with minimal height a. Extending these examples, for any $0 \leq q \leq a$,

$$X_{(q,b)} \times_{\Bbbk} \underbrace{X_{(1,0)} \times_{\Bbbk} X_{(1,0)} \times_{\Bbbk} \cdots \times_{\Bbbk} X_{(1,0)}}_{a-q \text{ times}}$$

is a supersingular p-divisible group that has a minimal height of q.

Now consider the case where $a = b$. By the second statement in Corollary 3.3.6, the minimal height of any supersingular p-divisible group X over $\Bbbk$ of signature (a, a) is at most $a - 1$.

In Example 3.4.5, we constructed a supersingular p-divisible group $X_{(a,a)}$ of signature (a, a), with minimal height $a - 1$. Extending this example, for any $0 \leq q \leq a - 1$

$$X_{(q,a)} \times_{\Bbbk} \underbrace{X_{(1,0)} \times_{\Bbbk} X_{(1,0)} \times_{\Bbbk} \cdots \times_{\Bbbk} X_{(1,0)}}_{a-q \text{ times}}$$

is a supersingular p-adic Dieudonné module of signature (a, a) with corresponding p-divisible group of minimal height q. □

3.5 Isogeny Cutoffs

In this section, we apply the results of the previous section to understand *isogeny cutoffs* of supersingular unitary p-divisible groups.

Definition 3.5.1. The *isogeny cutoff* of a p-divisible group X is the minimal non-negative integer b_X such that, for any other p-divisible group X', an isomorphism $X[p^{b_X}] \cong X'[p^{b_X}]$ implies the existence of an isogeny $X \to X'$. In other words, b_X is the smallest level of p-power torsion that determines X up to isogeny.

Definition 3.5.2. Fix nonnegative integers a and b, with the convention that $a \leq b$. We define the *signature* (a, b) *supersingular isogeny bound* to be the least $B_{(a,b)}$ such that $b_X \leq B_{(a,b)}$ for all supersingular unitary p-divisible groups X of signature (a, b), defined over any algebraically closed field $\Bbbk$ of characteristic p.

Remark 3.5.3. If X is a unitary p-divisible group of fixed signature (a, b), it also makes sense to define the *enhanced isogeny cutoff* $\widetilde{b}_X$, as the minimal non-negative integer such that, for any other unitary p-divisible group X' also of signature (a,b), an isomorphism $X[p^{\widetilde{b}_X}] \cong X'[p^{\widetilde{b}_X}]$ *respecting the action and polarization on both* X *and* X', implies the existence of an isogeny $X \to X'$. Note that then $\widetilde{b}_X \leq b_X$. One can similarly define the *enhanced signature* (a,b) *supersingular isogeny bound* $\widetilde{B}_{(a,b)}$. In that case, we find $\widetilde{B}_{(a,b)} \leq B_{(a,b)}$.

However, note that [11, Lemma 5.1], the values of $\widetilde{b}_X$ and $\widetilde{B}_{(a,b)}$ do not depend on whether or not we require the isogeny $X \to X'$ to respect the unitary action and polarization.

We will now construct an explicit example of a unitary p-divisible group with a large isogeny cutoff.

Example 3.5.4. Let $k \geq 1$, and consider (a,b) with $a+b=g$ odd. We will construct a p-divisible group ${}^kX_{(a,b)}$ such that ${}^kX_{(a,b)}[p^k] \cong X_{(a,b)}[p^k]$ (where $X_{(a,b)}$ is the p-divisible group constructed from $(M_{(a,b)}, F, V, \langle\cdot,\cdot\rangle, M_{(a,b)} = M_0 \oplus M_1)$ in Example 3.4.3).

As before, set $r = \frac{g-1}{2}$. Let ${}^kM_{(a,b)} = (M_{(a,b)}, F_k, V_k, \langle\cdot,\cdot\rangle, M_{(a,b)} = M_0 \oplus M_1)$, so all the data of ${}^kM_{(a,b)}$ agrees with that of $M_{(a,b)}$ in Example 3.4.3, except for F_k and V_k, which we alter from F and V by the following four changes:

$$F_k(f_a) = e_{a+1} + p^k e_1,$$

$$F_k(e_{r+1}) = pf_{r+1} - p^{k+1} f_{r+a+1},$$

$$V_k(e_{a+1}) = pf_a - p^{k+1} f_g, \text{ and}$$

$$V_k(f_{r+1}) = e_{r+1} + p^k e_{r+a+1}.$$

The property that $\langle F_k(x), y\rangle = \langle x, V_k(y)\rangle^\sigma$ can again be checked as a matrix computation: $A_{F_k}^T B = (BA_{V_k})^\sigma$. By construction, the action on ${}^kM_{(a,b)}$ is still of signature (a,b), we have $F_k \circ V_k = V_k \circ F_k = p$, and $F \equiv F_k \mod p^k$ and $V \equiv V_k \mod p^k$.

Therefore, for each $k \geq 1$, we've constructed a unitary p-divisible group ${}^kX_{(a,b)}$ of signature (a,b), such that ${}^kX_{(a,b)}[p^k] \cong X_{(a,b)}[p^k]$. In fact, this isomorphism respects the induced action and polarization on both ${}^kX_{(a,b)}[p^k]$ and $X_{(a,b)}[p^k]$, since by construction the splitting of Dieudonné modules into 0 and 1 parts was identical, and the polarization form on both Dieudonné modules was also identical. Note that we do not claim that ${}^{k}X_{(a,b)}$ is supersingular. In fact, we have the following result:

Lemma 3.5.5. Consider (a, b) with $a + b = g$ odd, and let $1 \leq k \leq (a - 1)$. Then the first Newton slope of ${}^{k}X_{(a,b)}$ is at most $\frac{k}{2a}$. In particular ${}^{k}X_{(a,b)}$ is not supersingular.

Now, we can combine the above Example 3.5.4 with the slope computation of Lemma 3.5.5 and the results on minimal height to understand both isogeny cutoffs and the supersingular isogeny bounds.

Theorem 3.5.6. Fix nonnegative integers a and b, with the convention that $a \leq b$. Let X be any supersingular unitary p-divisible group of signature (a, b) over $\Bbbk$.

If $a < b$, then the isogeny cutoff b_X of X is at most $a + 1$. If $a = b$, then the isogeny cutoff b_X is at most a. Further, when $a < b$ we have the following constraints on the supersingular isogeny bounds:

$$a \leq \widetilde{B}_{(a,b)} \leq B_{(a,b)} \leq a + 1$$

and when $a = b$, we have:

$$a - 1 \leq \widetilde{B}_{(a,a)} \leq B_{(a,a)} \leq a.$$

Proof. We showed in Theorem 3.4.6 that any supersingular unitary p-divisible group of X of signature (a, b) has minimal height at most a (when $a < b$) and minimal height at most $a - 1$ (when $a = b$).

Lemma 6.3 of [32] observes that the isogeny cutoff of any p-divisible group is at most one more than the minimal height, so $b_X \leq a + 1$ (when $a < b$) and $b_X \leq a$ (when $a = b$).

Since the above bounds on b_X hold for *any* supersingular unitary p-divisible group of signature (a, b), this gives the upper bounds on supersingular isogeny cutoffs: $B_{(a,b)} \leq a + 1$ (when $a < b$) and $B_{(a,a)} \leq a$.

For the lower bounds, first consider the case when $a < b$ and $a + b$ is odd, and recall the p-divisible groups $X_{(a,b)}$ and ${}^{a-1}X_{(a,b)}$ constructed in Examples 3.4.3 and 3.5.4, respectively. Note that $X_{(a,b)}$ is supersingular but, by Lemma 3.5.5, the p-divisible group ${}^{a-1}X_{(a,b)}$ is not. However,

$$X_{(a,b)}[p^{a-1}] \cong {}^{a-1}X_{(a,b)}[p^{a-1}]$$

(in a way respecting actions and polarizations on both sides). So, for $X = X_{(a,b)}$,

$$a \leq \widetilde{b}_X \quad \text{and} \quad a \leq b_X.$$

But then, by definition of the supersingular isogeny bounds, we must have

$$a \leq \widetilde{B}_{(a,b)} \quad \text{and} \quad a \leq B_{(a,b)}$$

and so $a \leq \widetilde{B}_{(a,b)} \leq B_{(a,b)} \leq a+1$.

Next, in the case where $a < b$ and $a+b$ is even, note that $a < b-1$, so we may consider $Y = X_{(a,b)} = X_{(a,b-1)} \times_{\Bbbk} X_{(0,1)}$ of Example 3.4.4 and $Y' = {}^{a-1}X_{(a,b-1)} \times_{\Bbbk} X_{(0,1)}$. As in the previous case, Y is supersingular, Y' is not supersingular, and $Y[p^{a-1}] \cong Y'[p^{a-1}]$. Since $a \leq b_Y$,

$$a \leq \widetilde{B}_{(a,b)} \leq B_{(a,b)} \leq a+1.$$

Finally, when $a = b$, we consider $Y = X_{(a,a)} = X_{(a-1,a)} \times_{\Bbbk} X_{(1,0)}$ of Example 3.4.5 and $Y' = {}^{a-2}X_{(a-1,a)} \times_{\Bbbk} X_{(1,0)}$. Since Y is supersingular, Y' is not supersingular, and $Y[p^{a-2}] \cong Y'[p^{a-2}]$, we have

$$a-1 \leq \widetilde{B}_{(a,a)} \leq B_{(a,a)} \leq a. \qquad \square$$

Recall the *Ekedahl-Oort stratification* of $\mathcal{M}(a,b)$, where two $\Bbbk$-points $(A_1, \iota_1, \lambda_1, \eta_1)$ and $(A_2, \iota_2, \lambda_2, \eta_2)$ are in the same Ekedahl-Oort stratum if and only if $A_1[p]$ and $A_2[p]$, equipped with their induced action and polarization, are isomorphic. Generalizing the Ekedahl-Oort stratification is the BT_m *stratification* of $\mathcal{M}(a,b)$: for $m = 1$, this is exactly the Ekedahl-Oort stratification. In general, the BT_m stratification (defined for Shimura varieties of Hodge type by Vasiu in [34]) is defined by the property that two $\Bbbk$-points are in the same BT_m stratum if and only if the group schemes $A_1[p^m]$ and $A_2[p^m]$ are isomorphic, with their induced action and polarization. Note that this is a stratification in the sense of [35, Remark 2.1.1]; in particular, for m large enough, there are infinitely many strata. Given a valid p^m-torsion group scheme G with extra structure, we'll denote the BT_m stratum defined by G as $\mathcal{M}(a,b)_G$.

The next result follows from Theorem 3.5.6.

Corollary 3.5.7. Fix nonnegative integers a and b, with the convention that $a \le b$. Let $m \ge a+1$ if $a < b$ and $m \ge a$ if $a = b$.

For each BT_m stratum $\mathcal{M}(a,b)_G$, and for any algebraically closed field $\Bbbk$, either

$$\mathcal{M}(a,b)_G(\Bbbk) \subseteq \mathcal{M}(a,b)^{ss}(\Bbbk)$$

or

$$\mathcal{M}(a,b)_G(\Bbbk) \cap \mathcal{M}(a,b)^{ss}(\Bbbk) = \varnothing.$$

Proof. We will show that, if $\mathcal{M}(a,b)_G(\Bbbk)$ has a non-empty intersection with $\mathcal{M}(a,b)^{ss}(\Bbbk)$, then $\mathcal{M}(a,b)_G(\Bbbk) \subseteq \mathcal{M}(a,b)^{ss}(\Bbbk)$.

Consider a $\Bbbk$-point $(A_1,\iota_1,\lambda_1,\eta_1)$ in $\mathcal{M}(a,b)_G(\Bbbk) \cap \mathcal{M}(a,b)^{ss}(\Bbbk)$, and let $(A_2,\iota_2,\lambda_2,\eta_2)$ be any other point of $\mathcal{M}(a,b)_G$. Let (X_1,ι_1,λ_1) and (X_2,ι_2,λ_2) be the unitary p-divisible groups of signature (a,b) defined from these points.

By the assumption that these two points both lie in the same BT_m stratum $\mathcal{M}(a,b)_G$, we have that $X_1[p^m]$ and $X_2[p^m]$ are isomorphic (with their induced action and polarization). Theorem 3.5.6 tells us that $\widetilde{B}_{(a,b)} \le m$. So, in particular, we have $\widetilde{b}_{X_1} \le m$.

Since $X_1[p^m] \cong X_2[p^m]$ and $\widetilde{b}_{X_1} \le m$, the p-divisible groups X_1 and X_2 are isogenous, and so the point $(A_2,\iota_2,\lambda_2,\eta_2)$ is also contained in the supersingular locus $\mathcal{M}(a,b)^{ss}$. $\square$

Corollary 3.5.8. Fix nonnegative integers a and b with the convention that $a \le b$. Let $m \ge a+1$ if $a < b$ and $m \ge a$ if $a = b$.

There exists a set $\mathcal{G}$ of p^m-torsion group schemes G (equipped with action and polarization), each with corresponding BT_m stratum $\mathcal{M}(a,b)_G$, such that

$$\bigsqcup_{G \in \mathcal{G}} \mathcal{M}(a,b)_G = \mathcal{M}(a,b)^{ss}.$$

Proof. This follows from Corollary 3.5.7 and the fact that the supersingular locus $\mathcal{M}(a,b)^{ss}$ and the BT_m strata are defined as reduced subschemes of $\mathcal{M}(a,b)$. □

Chapter 4

A new case of the Harris-Viehmann conjecture

Let $p > 2$ and ℓ be distinct primes. *Rapoport-Zink spaces* are formal moduli spaces of p-divisible groups, originally constructed in [29] as a local analogue of PEL-type Shimura varieties. Rapoport-Zink spaces that parameterise dimension 1 p-divisible groups are called Lubin-Tate spaces. Their ℓ-adic cohomology played a crucial role in modern algebraic number theory by realizing the local Langlands correspondence for GL_n [36]. This indicates a more general phenomenon, where the ℓ-adic cohomology of appropriate local analogues of Shimura varieties can materialize the local Langlands correspondence. Rapoport and Viehmann conjectured the existence of *local Shimura varieties* in [15], laying the foundation for a general theory of the local analogues of Shimura varieties. They start with a *local Shimura datum*, and associate a tower of rigid analytic spaces to it. In this chapter, we focus on the *Harris-Viehmann conjecture*, which provides a parabolic inductive formula for the ℓ-adic cohomology of local Shimura varieties when certain criteria are satisfied by the underlying local Shimura data.

4.1 Notations

- Fix algebraic closures $\overline{\mathbb{F}}_p$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{F}_p$ and $\mathbb{Q}_p$ respectively.
- $\breve{\mathbb{Q}}_p$ is the p-adic completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}}$,
- $W = W(\overline{\mathbb{F}}_p)$ is the ring of Witt vectors,
- G is a connected reductive group over $\mathbb{Q}_p$,
- Let $\sigma \in \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the relative Frobenius automorphism. b and b' in $G(\breve{\mathbb{Q}}_p)$ are σ-conjugate if there exists some $g \in G(\breve{\mathbb{Q}}_p)$ such that $b' = gb\sigma(g)^{-1}$,
- $B(G)$ is the Kottwitz set associated to G (c.f. [37] for more details on this),
- $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

4.2 Background

Definition 4.2.1. [15, Definition 5.1] A local Shimura datum over $\mathbb{Q}_p$ is a triple $(G, [b], \{\mu\})$ where

1. G is a connected reductive group over $\mathbb{Q}_p$. E.g. $G = \mathrm{GL}_n(\mathbb{Q}_p)$.
2. $[b] \in B(G)$ is a σ-conjugacy class.
3. $\{\mu\}$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_m \longrightarrow G_{\overline{\mathbb{Q}}_p}$.

such that the following conditions hold:

1. $[b] \in B(G, \{\mu\})$ (the subset of the Kottwitz set bounded by $\{\mu\}$, c.f. [29]).
2. μ is minuscule, i.e. $|\mu| \in \{-1, 0, 1\}$.

Moreover, we have the following quantities associated with a local Shimura datum:

1. The field of definition of $\{\mu\}$, called the *reflex field*, which lives inside $\overline{\mathbb{Q}}_p$. It is denoted by $E = E(G, \{\mu\})$. Let $\mathcal{W}_E$ be the Weil group of the reflex field E. For example, let $\mu : \mathbb{G}_m \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, mapping $z \mapsto \mathrm{diag}(z, 1, ..., 1)$. Then $E = \mathbb{Q}_p$.
2. The flag variety $\mathcal{F}\ell_{G,\mu}$, which is an adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$. $\mathcal{F}\ell_{G,\mu,\breve{E}}$ denotes its base change to $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$.
3. The reductive algebraic group J_b over $\mathbb{Q}_p$, for $b \in [b]$. It acts on $\mathcal{F}\ell_{G,\mu,\breve{E}}$, and has the functor of points:

$$J_b(R) = \{g \in G(R \otimes_R \breve{\mathbb{Q}}_p) \ : \ b = gb\sigma(g)^{-1}\} \tag{4.1}$$

   for any $\mathbb{Q}_p$-algebra R. Note that the isomorphism class of J_b depends only on the σ-conjugacy class of b.

4. The weakly admissible open subspace $\mathcal{F}\ell^{\mathrm{wa}}_{G,\mu} \subseteq \mathcal{F}\ell_{G,\mu,\breve{E}}$. This is stabilized by the action of $J_b(\mathbb{Q}_p)$ on $\mathcal{F}\ell_{G,\mu,\breve{E}}$.

In this thesis, we will be paying attention to *non-basic unramified local Shimura data*.

Definition 4.2.2. A local Shimura datum $(G, [b], \{\mu\})$ is *basic* if b is a basic element of the Kottwitz set $B(G)$; i.e., the Newton map $\mathcal{N}_{b,G}$ associated to $[b]$ factors through the centre $Z(G)$ of the group G. If that is not the case, $(G, [b], \{\mu\})$ is *non-basic*.

Definition 4.2.3. [7, Definition 3.4] G is unramified if it is quasisplit over $\mathbb{Q}_p$ and splits over $\mathbb{Q}_p^{\mathrm{ur}}$. Equivalently, G admits a reductive model over $\mathbb{Z}_p$. A local Shimura datum $(G, [b], \{\mu\})$ of Hodge type is *unramified* if G is unramified.

In [29], the local Shimura data considered came exclusively from PEL data. Like with the global theory of Shimura varieties, there are more general cases than that:

Definition 4.2.4. [15, Remark 5.4(i)] [7, Definition 3.4] A local Shimura datum $(G, [b], \{\mu\})$ is of *Hodge type* if there exists an embedding $G \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}})$ for some $\mathbb{Q}_p$-vector space V, and a local Shimura datum $(\mathrm{GL}(V), [b'], \{\mu'\})$ such that $[b]$ and $\{\mu\}$ are mapped to $[b']$ and $\{\mu'\}$ respectively. Here, $\{\mu'\}$ corresponds to $(1^r, 0^{n-r})$ for some integer $0 \le r \le n = \dim(W_{\mathbb{Q}})$.

Examples of local Shimura datum of Hodge type

1. Any local Shimura datum $(G, [b], \{\mu\})$ of EL/PEL type (coming from an EL/PEL Shimura datum) is also of Hodge type.

2. Let (G, X) be a (global) Shimura datum of Hodge type, i.e. there exists some embedding into the Siegel Shimura datum $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$, where $S^{\pm}$ are the connected components of the Siegel upper half-space. Let μ be the cocharacter associated to X, and take $[b] \in B(G, \{\mu\})$. Then $(G_{\mathbb{Q}_p}, [b], \{\mu\})$ is a local Shimura datum of Hodge type that is not of PEL type.

From any local Shimura datum $(G, [b_G], \{\mu_G\})$, we get an associated "adjoint-type" datum $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\})$, where $G^{\mathrm{ad}} := G/Z(G)$ is the adjoint quotient of G. In fact, the quotient map $G \twoheadrightarrow G^{\mathrm{ad}}$ induces a map of Kottwitz sets $B(G) \to B(G^{\mathrm{ad}})$. We write $b_{G^{\mathrm{ad}}}$ for the image of

$b_G \in B(G)$ under this map. We then write

$$\mu_{G^{\mathrm{ad}}} : \mathbb{G}_m \xrightarrow{\mu_G} G_{\overline{\mathbb{Q}}_p} \to G^{\mathrm{ad}}_{\overline{\mathbb{Q}}_p}$$

for the induced cocharacter of G^{ad}. The morphism $(G, [b], \{\mu\}) \longrightarrow (G^{\mathrm{ad}}, [b^{\mathrm{ad}}_G], \{\mu^{\mathrm{ad}}_G\})$ is a morphism of local Shimura data.

This leads us to the following definition:

Definition 4.2.5. [7, Definition 3.4] A local Shimura datum $(G, [b_G], \{\mu_G\})$ is of *abelian type* if there exists a local Shimura datum $(H, [b_H], \{\mu_H\})$ of Hodge type such that there is an isomorphism of the associated adjoint local Shimura data $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) \cong (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\})$. In particular, a local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type is unramified if G is unramified, and an associated local Shimura datum $(H, [b_H], \{\mu_H\})$ of Hodge type is also unramified.

Remark 4.2.6. If one were to compare the terminology with that of the global theory of Shimura varieties, then Definition 4.2.5 should more accurately label these to be local Shimura data of *pre-abelian type*.

Examples of local Shimura datum of abelian type

1. Any local Shimura datum $(G, [b], \{\mu\})$ of Hodge type is also of abelian type.

2. Let $G = \mathrm{PGL}_n$ for $n \geq 2$. Take a minuscule cocharacter $\mu_H : \mathbb{G}_m \longrightarrow H = \mathrm{GL}_n$ and $[b_H] \in B(\mathrm{GL}_n, \mu_G)$. Let $\mu_G := \mu_{H^{\mathrm{ad}}}$ and $b_G := b_{H^{\mathrm{ad}}}$. Then $(G, [b_G], \{\mu_G\})$ is of abelian type and not of Hodge type.

A major development to the theory of local Shimura varieties came in [38], wherein an explicit construction of local Shimura varieties was done in the category of diamonds. Following the advent of the language of perfectoid spaces and diamonds, the following milestones were reached:

1. W. Kim constructed Rapoport-Zink spaces of Hodge type in [5], coming from local Shimura data of Hodge type. These work as integral models of the corresponding local Shimura varieties.

2. X. Shen generalized this to Rapoport-Zink spaces of abelian type in [7], coming from unramified local Shimura data of abelian type.

There are two classical conjectures concerning the cohomology of local Shimura varieties. The *Kottwitz conjecture* formulates the existence of supercuspidal representations realizing the local Langlands correspondence, arising from local Shimura varieties with an underlying basic local Shimura datum [39]. Our focus will be on what lies outside the basic locus, captured by the *Harris-Viehmann conjecture*, which provides a parabolic inductive formula for the ℓ-adic cohomology of local Shimura varieties. In particular, it asserts that all the supercuspidal representations are concentrated in the basic locus.

Before recounting a brief history of results on the Harris-Viehmann conjecture, we build some motivation.

Definition 4.2.7. Let X be an abelian scheme over a finite field $\mathbb{F}/\mathbb{F}_p$. For some $n \in \mathbb{Z}_{>0}$, the Frobenius induces a linear action

$$\mathrm{Fr}^n \circlearrowright \mathrm{H}^1_{\text{cris}}(X/W(\mathbb{F}))$$

The p-adic valuation of the eigenvalues yields the multiset $\{n\lambda\}$, and the multiset of rational slopes $\{\lambda\}$ of the *Newton polygon*. The Newton polygon is an isogeny invariant.

Definition 4.2.8. Abelian varieties A and B over $\overline{\mathbb{F}}_p$ are in the same *Newton stratum* if $A[p^\infty]$ and $B[p^\infty]$ are isogenous p-divisible groups. Equivalently, the Newton polygons of A and B are the same.

For a fixed dimension g, Newton strata are locally closed in $\mathcal{A}_g$. There is a natural partial ordering on the set of Newton strata induced by the nesting of the Newton polygons. The smallest element in this partial ordering is the *basic locus*, which is the unique closed Newton stratum. In certain PEL settings, it parameterizes supersingular abelian varieties with extra structures, and in these cases also known as the supersingular locus.

For an abelian scheme X, consider the Hodge filtration:

$$\mathrm{H}^1_{\mathrm{cris}}(X/W(\mathbb{F})) \otimes_{W(\mathbb{F})} \mathbb{F}_p = \mathrm{H}^1_{dR}(X/\mathbb{F}_p) = F^0 \supset F^1 \supset 0$$

The *Hodge polygon* of X captures the Hodge filtration. It starts at $(0,0)$, with sides given by line segments of length $h^{j,1-j} = \dim F^j/F^{j+1}$ and slopes j for $0 \le j \le 1$. It is not an isogeny invariant.

By Mazur-Ogus [40], the Hodge polygon of an abelian variety always lies above its Newton polygon.

A Hodge polygon HP_X and Newton polygon NP_X are *Hodge-Newton reducible* if they either have a coincident side, or their contact point is a break point of the Newton polygon.

We can rephrase Hodge-Newton reducibility in the vocabulary of local Shimura data:

Definition 4.2.9. [15, Definition 4.24] Let F be a finite extension of $\mathbb{Q}_p$, and $\breve{F}$ be the completion of its maximal unramified extension. Let $(G, [b], \{\mu\})$ be an (unramified) local Shimura datum. The pair $([b], \{\mu\})$ is called *Hodge-Newton reducible* if there exists a proper parabolic subgroup P with Levi factor L (with both defined over F), a representative $\mu' \in \{\mu\}$ which factors through L and an element $b' \in [b] \cap L(\breve{F})$ such that:

- $[b'_L] \in B(L, \{\mu'_L\})$ where $[b'_L]$ is the L-σ-conjugacy class of b' and $\{\mu'_L\}$ is the L-conjugacy class of $\{\mu'\}$.
- Only non-negative characters occur in the action of μ' and $\nu_{b'}$ on $(\mathrm{Lie}\, R_u(P)) \otimes_F \overline{F}$, where $R_u(P)$ is the unipotent radical of P.

We call the local Shimura datum $(G, [b], \{\mu\})$ Hodge-Newton reducible if $([b], \{\mu\})$ is Hodge-Newton reducible.

Example of Hodge-Newton reducibility [41, Example 3.1.2] Let $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathrm{GL}_n$, where $\mathcal{O}_F$ is the ring of integers of a finite extension F of $\mathbb{Q}_p$. For a partition $n = \sum_{k=1}^r j_k$, the Levi factor L is of the form

$$L = \prod_{k=1}^r \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathrm{GL}_{j_k}$$

Using the Newton map, we can obtain the slopes of the Newton polygon $(\nu_1, ..., \nu_n)$ from $[b]$, and the slopes of the σ-invariant Hodge polygon $\overline{\mu} = (\mu_1, ..., \mu_r)$ from $\{\mu\}$. Let

$$m_c := \sum_{k=1}^{c} j_k \text{ for } 0 \le c \le r$$

Let ν_{m_c} denote the *last slope* of the c^{th}-block, and let ν_{m_c+1} denote the *first slope* of the $(c+1)^{\text{th}}$-block. This distinction is important as a given block might have multiple slopes. The local Shimura datum $(\text{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\text{GL}_n, [b], \{\mu\})$ is Hodge-Newton reducible with respect to L when:

- The Newton polygon meets the Hodge polygon at points specified by the Levi subgroup:

$$\sum_{a=1}^{m_c} \nu_a = \sum_{b=1}^{c} \mu_b \text{ for each } 1 \le c \le r.$$

- These points are break points of the Newton polygon: $\nu_{m_c} < \nu_{m_c+1}$ for each $0 \le c \le r-1$.

For example, for $G = \text{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\text{GL}_4$ and $L = \text{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\text{GL}_1 \times \text{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\text{GL}_1 \times \text{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\text{GL}_2$, the slopes of the Newton polygon $Newt(b)$ are $\nu_1 = 0 < \nu_2 = 1 < \nu_3 = \nu_4 = 2.5$.

Remark 4.2.10. A purely group-theoretic definition of Hodge-Newton reducibility is given in [42, Definition 3.1] and [43, Definition 2.1], which is applicable in a more general setting involving local shtuka spaces. As proved in [42, Lemma 3.3], it reduces to the definition mentioned above when G is quasi-split.

History of results on the Harris-Viehmann conjecture:

1. Boyer proved the Harris-Viehmann conjecture for Drinfeld modular varieties in [44].

2. Mantovan proved the Harris-Viehmann conjecture in the PEL type case for a special kind of Hodge-Newton reducibility in [45], assuming that the Newton polygon coincides with the Hodge polygon up to or from the nontrivial break contact point.

3. Shen established the Harris-Viehmann conjecture for the PEL type case with a more general form of Hodge-Newton reducibility in [46], assuming that the contact point is a break point of the Newton polygon.

4. Hong proved the Harris-Viehmann conjecture in the Hodge type case in [6], assuming Hodge-Newton reducibility.

5. Hansen gave another proof of the conjecture in [47] for $G = \mathrm{GL}_n$ under the Hodge-Newton reducibility assumption. He used the construction of local Shimura varieties by Scholze in [38].

4.3 Introduction to thesis project

Our main goal is to establish the Harris-Viehmann conjecture for unramified local Shimura data of abelian type that are Hodge-Newton reducible. We work with Rapoport-Zink spaces arising from unramified local Shimura data of abelian type as constructed in [7]. Our primary strategy is to follow the outline laid down in Serin Hong's thesis [41].

4.3.1 Hong's strategy:

The following conditions are imposed on the local Shimura data $(G, [b], \{\mu\})$:

1. $(G, [b], \{\mu\})$ is unramified: G admits a reductive model over $\mathbb{Z}_p$.

2. $(G, [b], \{\mu\})$ is of Hodge type: i.e., for some n, there exists the following embedding of local Shimura data:

$$(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$$

3. $(G, [b], \{\mu\})$ is Hodge-Newton reducible with respect to a parabolic subgroup $P \subsetneq G$ with Levi factor L.

The first two conditions ensure that the corresponding local Shimura variety arises from a Rapoport-Zink space of Hodge type constructed by Kim [5], which has a concrete interpretation as a moduli space of p-divisible groups. The third condition helps ensure that $(G, [b], \{\mu\})$ naturally reduces to a local Shimura datum $(L, [b_L], \{\mu_L\})$ coming from the specified Levi subgroup. In particular, $(\mathrm{GL}_n, [b'], \{\mu'\})$ is also Hodge-Newton reducible if and only if G is a split group. As such, in most cases, $(\mathrm{GL}_n, [b'], \{\mu'\})$ is *not* Hodge-Newton reducible.

Hong's primary strategy was to factor this embedding through a Hodge-Newton reducible "EL realization", that inherits the EL interpretation from $(\mathrm{GL}_n, [b'], \{\mu'\})$ while retaining the Hodge-Newton reducibility of $(G, [b], \{\mu\})$.

$$\begin{array}{ccc} & (\widetilde{G}, [\widetilde{b}], \{\widetilde{\mu}\}) & \\ \nearrow & & \searrow \\ (G, [b], \{\mu\}) & \hookrightarrow & (\mathrm{GL}_n, [b'], \{\mu'\}) \end{array}$$

Figure 4.1: EL realization

Explicitly, the way to construct the EL realization is as follows. Fix a Borel $B \subset G$ with a maximal torus T. Suppose G has the associated root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ and the Weyl group Ω. Let E_j be the field of definition for each orbit of $X^*(T)/\Omega$. Then:

$$\widetilde{G} = \prod_j \mathrm{Res}_{\mathcal{O}_{E_j}/\mathbb{Z}_p} \mathrm{GL}_{m_j}$$

The closed embedding $(G, [b], \{\mu\}) \hookrightarrow (\widetilde{G}, [\widetilde{b}], \{\widetilde{\mu}\})$ induces a closed embedding of Rapoport-Zink spaces $\mathrm{RZ}_G \hookrightarrow \mathrm{RZ}_{\widetilde{G}}$. Once in this EL realization setting, the method for proving the Harris-Viehmann conjecture reduces to three main steps:

1. Construct an appropriate Rapoport-Zink space out of the EL realization.

2. Establish the parabolic induction formula in the conjecture for the Rapoport-Zink space built from the EL realization.

3. Pull back along the closed embedding $\mathrm{RZ}_G \hookrightarrow \mathrm{RZ}_{\widetilde{G}}$, and obtain the parabolic induction formula in the conjecture for the original local Shimura datum of Hodge type.

4.3.2 Rapoport-Zink spaces of Hodge type

We briefly recount the theory of Rapoport-Zink spaces of Hodge type and their rigid analytic fibres, following [5].

Fix an unramified local Shimura datum $(G, [b], \{\mu\})$ of Hodge type, along with a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$. The condition of being unramified allows us to pick $b \in [b]$ and $\mu \in \{\mu\}$ such that $b \in G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$. This allows for the construction of an F-isocrystal with G-structure, that gives rise to a p-divisible group with G-structure up to isogeny, denoted by $\underline{X} = (X, (t_i))$, where (t_i) is a collection of crystalline tensors (c.f. [5] for the detailed construction). Let $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ denote the category of $\breve{\mathbb{Z}}_p$-algebras where p is nilpotent.

We define the set-valued covariant functor Ξ_b on $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ as follows: for any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, $\Xi_b(R)$ is the set of isomorphism classes of pairs $(\mathbb{X}, \iota)$, where $\mathbb{X}$ is a p-divisible group and the map $\iota : X_{R/p} \longrightarrow \mathbb{X}_{R/p}$ is a quasi-isogeny. Ξ_b is independent of the choice of $b \in [b] \cap G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)$ up to isomorphism. It was shown in [29] that the functor Ξ_b is represented by a formal scheme RZ_b which is locally formally of finite type and formally smooth over $\breve{\mathbb{Z}}_p$. Let $\mathbb{X}_{\mathrm{GL},b}$ be the universal p-divisible group over RZ_G.

For a pair $(\mathbb{X}, \iota)$, we get an isomorphism of F-isocrystals with G-structure induced by ι:

$$\mathbb{D}(\iota) : \mathbb{D}(\mathbb{X}_{R/p})[1/p] \xrightarrow{\cong} \mathbb{D}(X_{R/p})[1/p]$$

Let $(t_{\mathbb{X},i})$ denote the inverse image of the tensors $(t_i)_R$ under this isomorphism.

Let $\mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$ be the collection of formally smooth and formally finitely generated algebras over $\breve{\mathbb{Z}}_p/p^n\breve{\mathbb{Z}}_p$ for some $n \in \mathbb{Z}_{>0}$. This is a full subcategory of $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$. Let (s_i) be a family of crystalline tensors. We define the set-valued covariant functor $\Xi^{(s_i)}_G$ on $\mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$ as follows: take any $R \in \mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$, a morphism $f : \mathrm{Spf}\ (R) \longrightarrow \mathrm{RZ}_b$, and a p-divisible group $\mathbb{X}$ over $\mathrm{Spec}\ (R)$ which pulls back to $f^*\mathbb{X}_{\mathrm{GL},b}$ over $\mathrm{Spf}\ (R)$. Then $f \in \Xi_G^{(s_i)}(R)$ if and only if there exists a (unique) family of tensors $(\mathbf{t}_i)$ on $\mathbb{D}(\mathbb{X})$ such that the following holds:

- For an ideal of definition I of R containing p, the pullback of $(\mathbf{t}_i)$ over R/I is compatible with the pullback of $(t_{\mathbb{X},i})$ over R/I.
- Let $\mathcal{R}$ be a p-adic lift of R that is formally smooth over $\breve{\mathbb{Z}}_p$. Then the following $\mathcal{R}$-scheme is a G-torsor:

$$\mathcal{P}_{\mathcal{R}} \coloneqq \mathrm{Isom}_{\mathcal{R}}((\mathbb{D}(\mathbb{X})_{\mathcal{R}}, (\mathbf{t}_i)_{\mathcal{R}}), (\Lambda^* \otimes_{\mathbb{Z}_p} \mathcal{R}, s_i \otimes 1))$$

- The Hodge filtration of $\mathbb{X}$ is a $\{\sigma^{-1}(\mu^{-1})\}$-filtration with respect to $\mathbf{t}_i$.

By [5, Theorem 4.9.1], there exists a closed formal scheme RZ_G which is formally smooth over $\breve{\mathbb{Z}}_p$ and locally formally of finite type over $\mathrm{Spf}\ \breve{\mathbb{Z}}_p$, representing the functor $\Xi_G^{(s_i)}$ on $\mathrm{Nilp}^{\mathrm{sm}}_{\breve{\mathbb{Z}}_p}$. Its isomorphism class only depends on the local Shimura datum of Hodge type $(G, [b], \{\mu\})$, and is independent of the choice of (s_i). By [48], RZ_G admits a rigid analytic fibre, denoted by $\mathrm{RZ}_G^{\mathrm{rig}}$. By varying the level structure $K_G \subset G(\mathbb{Z}_p)$, we can construct a tower of étale coverings of this rigid analytic fibre, written as $\mathrm{RZ}_{G,b}^{K_G}$. At the infinite level, this construction is the local Shimura variety of Hodge type conjectured by [15] and proven in [5]. The cohomology groups of this object detect the representations of $G(\mathbb{Q}_p)$, $J_b(\mathbb{Q}_p)$, and $\mathcal{W}_E$.

To state the Harris-Viehmann conjecture, we need to set out some notations:

Let $(G, [b], \{\mu\})$ be a non-basic unramified local Shimura datum over $\mathbb{Q}_p$ which is Hodge-Newton reducible with respect to a parabolic subgroup P of G with Levi factor L. Fix a prime $\ell \neq p$. Let RZ_G be the Rapoport-Zink space associated to the local Shimura datum. Let $K_p \subset G(\mathbb{Q}_p)$ be a maximal compact subgroup determining the level structure. We get a directed system via the partial ordering induced by reverse inclusion $K_p \leq K'_p :\iff K_p \supseteq K'_p$. Let ρ be an admissible ℓ-adic representation of $J_b(\mathbb{Q}_p)$.

We first define

$$\mathrm{H}^{i,j}(\mathrm{RZ}_G^{\infty})_\rho \coloneqq \varinjlim_{K_p} \mathrm{Ext}^{\,j}_{J_b(\mathbb{Q}_p)}(\mathrm{H}^i(\mathrm{RZ}_G^{K_p}), \rho)$$

From this, we get a virtual representation:

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho := \sum_{i,j\geq 0}(-1)^{i+j}\mathrm{H}^{i,j}(\mathrm{RZ}_G^\infty)_\rho$$

Theorem 4.3.1. [41, Theorem 1] Let $(G,[b],\{\mu\})$ be a non-basic unramified local Shimura datum of Hodge type. Suppose it is Hodge-Newton reducible with respect to a parabolic subgroup P of G with Levi factor L. Fix a prime $\ell \neq p$. For any admissible $\overline{\mathbb{Q}}_\ell$-representation ρ of $J_b(\mathbb{Q}_p)$, the virtual representation of $G(\mathbb{Q}_p)\times \mathcal{W}_E$ is expressed by the following parabolic inductive formula:

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho = \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{H}^\bullet(\mathrm{RZ}_L^\infty)_\rho$$

It follows that the virtual representation $\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho$ contains no supercuspidal representations of $G(\mathbb{Q}_p)$. On the other hand, suppose $(G,[b],\{\mu\})$ is basic. Then J_b is an inner twist of G, so there is no non-trivial Levi subgroup available for parabolic induction. Thus, all supercuspidal representations are concentrated along the basic locus.

Strategy to prove Harris-Viehmann in this setting

The method (originally devised in [45]) is as follows:

1. By the existence of the EL-realization, we have $\widetilde{G}$ and corresponding Levi subgroup $\widetilde{L}$ coming from a fixed parabolic $\widetilde{P}$. The closed embedding $G\hookrightarrow\widetilde{G}$ induces the closed embedding of Rapoport-Zink spaces of Hodge type $\mathrm{RZ}_G\hookrightarrow\mathrm{RZ}_{\widetilde{G}}$ by [5]. In [45], $\widetilde{P}$ is used to construct the formal scheme $\mathrm{RZ}_{\widetilde{P}}$. By [49], the pullback of $\mathrm{RZ}_{\widetilde{P}}$ over RZ_G gives rise to the space RZ_P, which is the analogue of a Rapoport-Zink space corresponding to a fixed parabolic P of G.

2. It is then shown that the tower of rigid analytic spaces over RZ_G has a fibration by the tower of rigid analytic spaces over RZ_P. This is proved by:

Lemma 4.3.2. [41, Lemma 4.3.1] The rigid analytic fibres of RZ_G, RZ_P and RZ_L fit into the following diagram

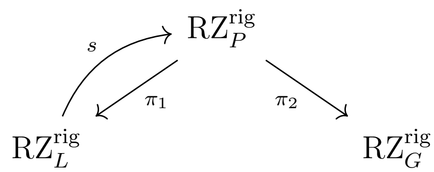

Figure 4.2: Relations between RZ_G, RZ_P and RZ_L.

such that:

(a) s is a closed immersion.

(b) π_1 is a fibration in balls, i.e. the fibres are isomorphic to a formal spectrum of a power series ring over $\breve{\mathbb{Z}}_p$.

(c) π_2 is an isomorphism.

3. The cohomologies of RZ_G, RZ_L with RZ_P are compared via this lemma. We summarize how Lemma 4.3.2 is used to prove the Harris-Viehmann conjecture (from [6]):

Fact: For every integer $m > 0$, there exists a formal scheme $\psi : \mathrm{RZ}_P^{(m)} \longrightarrow \mathrm{RZ}_P$ with the following properties:

1. For any $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$, a morphism $f : \mathrm{Spf}\ \ (R) \longrightarrow \mathrm{RZ}_P$ factors through $\mathrm{RZ}_P^{(m)}$ if and only if the filtration $f^*\mathbb{X}_P^\bullet[p^m]$ splits.

2. The map $\pi_1 : \mathrm{RZ}_P \longrightarrow \mathrm{RZ}_L$ induces an isomorphism between the formal schemes RZ_P and $\mathrm{RZ}_P^{(m)}$ over RZ_L.

Let $K'_{(m)} := \ker(P(\mathbb{Z}_p) \longrightarrow P(\mathbb{Z}_p/p^m\mathbb{Z}_p))$. Let $\mathcal{P}_m$ and $\mathcal{P}'_m$ be two distinct covers of $\mathrm{RZ}_P^{(m)}$, defined by two Cartesian diagrams:

$$\begin{array}{ccc} \mathcal{P}_m & \longrightarrow & \mathrm{RZ}_P^{\mathrm{rig},(m)} \\ \downarrow & & \downarrow \\ \mathrm{RZ}_P^{K'_{(m)}} & \longrightarrow & \mathrm{RZ}_P^{\mathrm{rig}} \end{array} \qquad \begin{array}{ccc} \mathcal{P}'_m & \longrightarrow & \mathrm{RZ}_P^{K'_{(m)}} \\ \downarrow & & \downarrow {\scriptstyle \pi_1,} \\ \mathrm{RZ}_L^{\mathrm{rig},(m)} & \longrightarrow & \mathrm{RZ}_L^{\mathrm{rig}} \end{array}$$

Figure 4.3: Covers of $\mathrm{RZ}_P^{(m)}$.

Let $D = \dim(\mathrm{RZ}_P) - \dim(\mathrm{RZ}_L)$. By [6, Prop 4.3.2], since π_1 is a fibration in balls as per Lemma 4.3.2, we have the following quasi-isomorphism for every integer $m > 0$:

$$R\Gamma_c(\mathrm{RZ}_P^{K'_{(m)}} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell) \cong R\Gamma_c(\mathrm{RZ}_L^{K'_{(m)}} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell(-D))[-2D]$$

This yields

$$\mathrm{H}^\bullet(\mathrm{RZ}_L^\infty)_\rho = \mathrm{H}^\bullet(\mathrm{RZ}_P^\infty)_\rho$$

Since π_2 is an isomorphism between the rigid analytic generic fibres as per Lemma 4.3.2, we obtain:

$$\mathrm{RZ}_G^{K_p} \cong \mathrm{RZ}_G^{K_p} \times_{\mathrm{RZ}_G^{\mathrm{rig}}} \mathrm{RZ}_P^{\mathrm{rig}} \cong \bigsqcup_{K_p \backslash G(\mathbb{Q}_p)/P(\mathbb{Q}_p)} \mathrm{RZ}_P^{K_p \cap P(\mathbb{Q}_p)}$$

In particular, this yields:

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho = \mathrm{Ind}\,_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}_P^\infty)_\rho$$

4.4 Rapoport-Zink spaces of abelian type

In this section, we review Shen's construction of Rapoport-Zink spaces of abelian type in [7].

Definition 4.4.1. A local Shimura datum $(G, [b], \{\mu\})$ is of *abelian type*, if there exists an associated local Shimura datum of Hodge type $(H, [b_H], \{\mu_H\})$ such that there is an isomorphism of the adjoint local Shimura data $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) \cong (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\})$. A local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type is called *unramified* if G is unramified and there is an associated unramified local Shimura datum $(H, [b_H], \{\mu_H\})$ of Hodge type, satisfying $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) \cong (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\})$.

In particular, any local Shimura datum of Hodge type is also of abelian type, and the following example establishes the latter as a strictly larger class of objects:

Example 4.4.2. [15, Example 5.6] Let $G = \mathrm{PGL}_n$ and $H = \mathrm{GL}_n$ for $n \geq 2$. Consider a non-trivial minuscule cocharacter $\mu_H : \mathbb{G}_m \to \mathrm{GL}_n$ and $[b_H] \in B(\mathrm{GL}_n, \mu_H)$. Take $\mu_G = \mu_{H^{\mathrm{ad}}}$ and $[b_G] = [b_{H^{\mathrm{ad}}}]$. Then $(G, [b], \{\mu\})$ is an unramified local Shimura datum of abelian type that is not of Hodge type.

The associated local reflex field $E = E(G, [b_G], \{\mu_G\})$ is an unramified extension of $\mathbb{Q}_p$, giving rise to $\breve{E} = W_{\mathbb{Q}}$ and $\mathcal{O}_{\breve{E}} = W$.

We will recount how Shen used unramified local Shimura data of abelian type to construct formal Rapoport-Zink spaces of abelian type in [7]. To do that, we first need to develop the group theory further.

Let $\pi_1(G)$ be the fundamental group of G. For a local Shimura datum $(G, [b_G], \{\mu_G\})$, consider the Kottwitz map (c.f. [50]):

$$\kappa_G : G(W_{\mathbb{Q}}) \longrightarrow \pi_1(G)$$

sending $g \in G(W)\mu(p)G(W)$ to the class of μ_G. Then there exists $c_{b_G,\mu_G} \in \pi_1(G)$ such that $\kappa_G(b_G) - \mu_G = (1-\sigma)c_{b_G,\mu_G}$, with a unique $\pi_1(G)^\Gamma$-coset, where $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Let $(H, [b_H], \{\mu_H\})$ be an unramified local Shimura datum of Hodge type associated to the unramified local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type.

Consider the map $\omega_H : X^H_{\mu_H}(b_H) \to c_{b_H,\mu_H}\pi_1(H)^\Gamma$. Let $X^H_{\mu_H}(b_H)^+ \subset X^H_{\mu_H}(b_H)$ be its fibre over c_{b_H,μ_H}, and $\mathrm{RZ}^+_H \subset \mathrm{RZ}_H$ to be the corresponding formal subscheme. Similarly, we have $X^G_{\mu_G}(b_G)^+ \subset X^G_{\mu_G}(b_G)$ as the fibre of the map $\omega_G : X^G_{\mu_G}(b_G) \to c_{b_G,\mu_G}\pi_1(G)^\Gamma$ over c_{b_G,μ_G}.

By [7, Corollary 2.6], $X^G_{\mu_G}(b_G)^+ \cong X^H_{\mu_H}(b_H)^+$. This motivates setting $\mathrm{RZ}^+_G := \mathrm{RZ}^+_H$. Moreover, by [7, Theorem 2.1], $X^G_{\mu_G}(b_G) = J_{b_G}(\mathbb{Q}_p) X^G_{\mu_G}(b_G)^+$. We define

$$\mathrm{RZ}_G := J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^+_G = \bigsqcup_{J_{b_G}(\mathbb{Q}_p)/J_{b_G}(\mathbb{Q}_p)^+} \mathrm{RZ}^+_G$$

where $J_{b_G}(\mathbb{Q}_p) \supset J_{b_G}(\mathbb{Q}_p)^+ := \mathrm{Stab}_{J_{b_G}(\mathbb{Q}_p)}(X^G_{\mu_G}(b_G)^+)$. By construction, $\overline{\mathrm{RZ}_G}^{\mathrm{perf}} \cong X^G_{\mu_G}(b_G)$.

This construction is independent of the auxiliary local Shimura data of Hodge type, as there is a canonical isomorphism $\mathrm{RZ}^+_{H_1} \cong \mathrm{RZ}^+_{H_2}$ for two distinct such choices $(H_1, [b_{H_1}], \{\mu_{H_1}\})$ and $(H_2, [b_{H_2}], \{\mu_{H_2}\})$.

By [7, Prop 4.9], we have the following results:

1. $\mathrm{RZ}_{G^{\mathrm{ad}}} = \mathrm{RZ}_H / X_*(Z_H)^\Gamma$.

2. $\mathrm{RZ}_G = \beta^*_G(\mathrm{RZ}_{G^{\mathrm{ad}}})$ where $\beta_G : \pi_1(G)^\Gamma \longrightarrow \pi_1(G^{\mathrm{ad}})^\Gamma$.

Lemma 4.4.3. Let $(G, [b_G], \{\mu_G\})$ be an unramified local Shimura datum of abelian type, and $(H, b_H, \{\mu_H\})$ be the associated local Hodge type Shimura datum. Assume the natural map $\pi_1(H)^\Gamma \twoheadrightarrow \pi_1(H^{\mathrm{ad}})^\Gamma = \pi_1(G^{\mathrm{ad}})^\Gamma$ is surjective. Then we have the following isomorphisms between formal schemes over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$:

$$\mathrm{RZ}^+_G = \mathrm{RZ}^+_H \cong \mathrm{RZ}_H/\pi_1(H)^\Gamma \cong \mathrm{RZ}_{H^{\mathrm{ad}}}/\pi_1(H^{\mathrm{ad}})^\Gamma \cong \mathrm{RZ}_{G^{\mathrm{ad}}}/\pi_1(G^{\mathrm{ad}})^\Gamma$$

Proof. By construction, $\mathrm{RZ}^+_G = \mathrm{RZ}^+_H$. The right-action of $\pi_1(H)^\Gamma$ on $c_{b_H,\mu_H}\pi_1(H)^\Gamma$ induces an action on RZ_H. More precisely, any $g \in \pi_1(H)^\Gamma$ induces an isomorphism

$$R_g : c_{b_H,\mu_H}\pi_1(H)^\Gamma \tilde{\to} c_{b_H,\mu_H}\pi_1(H)^\Gamma.$$

This induces a Cartesian diagram

$$\begin{array}{ccc} \mathrm{RZ}_H & \xrightarrow{R'_g} & \mathrm{RZ}_H \\ \downarrow{\scriptstyle \omega_H} & & \downarrow{\scriptstyle \omega_H} \\ c_{b_H,\mu_H}\pi_1(H)^\Gamma & \xrightarrow{R_g} & c_{b_H,\mu_H}\pi_1(H)^\Gamma \end{array}$$

Figure 4.4: R_g and R'_g.

The isomorphism R'_g is the induced action of g on RZ_H. In particular, all fibres of ω_H are identified under the action of $\pi_1(H)^\Gamma$. Thus, $\mathrm{RZ}_G^+ = \mathrm{RZ}_H^+ \cong \mathrm{RZ}_H/\pi_1(H)^\Gamma$.

By [7, Proposition 4.9], we have

$$\mathrm{RZ}_H/X_*(Z_H)^\Gamma \cong \mathrm{RZ}_{G^{\mathrm{ad}}}.$$

By [51, Lemma 1.5], we have a short exact sequence:

$$0 \longrightarrow X_*(Z_H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(H^{\mathrm{ad}}) \longrightarrow 0$$

Consider the left exact functor $M \mapsto M^\Gamma$. Under the assumption $\pi_1(H)^\Gamma \twoheadrightarrow \pi_1(H^{\mathrm{ad}})^\Gamma$, we get

$$0 \longrightarrow X_*(Z_H)^\Gamma \longrightarrow \pi_1(H)^\Gamma \longrightarrow \pi_1(H^{\mathrm{ad}})^\Gamma \longrightarrow 0$$

In particular, we see that

$$\mathrm{RZ}_G^+ = \mathrm{RZ}_H^+ \cong \mathrm{RZ}_H/\pi_1(H)^\Gamma \cong (\mathrm{RZ}_H/X_*(Z_H)^\Gamma)/\pi_1(H^{\mathrm{ad}})^\Gamma \cong \mathrm{RZ}_{G^{\mathrm{ad}}}/\pi_1(G^{\mathrm{ad}})^\Gamma.$$

□

Remark 4.4.4. The condition that the map $\pi_1(H)^\Gamma \longrightarrow \pi_1(H^{\mathrm{ad}})$ is a surjection is true for most standard unramified algebraic groups considered for the study of (local) Shimura varieties. There is a natural long exact sequence in Galois cohomology:

$$0 \longrightarrow \pi_1(Z_H)^\Gamma \longrightarrow \pi_1(H)^\Gamma \longrightarrow \pi_1(H^{\mathrm{ad}}) \longrightarrow \mathrm{H}^1(\Gamma, \pi_1(Z_H)) \longrightarrow ...$$

The surjectivity we desire occurs if and only if $H^1(\Gamma, \pi_1(Z_H)) = 0$. This happens in almost all cases when H is unramified.

4.5 Weil descent datum

In this subsection, we discuss the Weil descent data for Rapoport-Zink spaces of abelian type. In particular, $\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)$ will be a virtual representation of $G(\mathbb{Q}_p) \times \mathcal{W}_E$, where $\mathcal{W}_E$ is the Weil group associated to the reflex field $E = E(G, \{\mu_G\})$ of the local Shimura datum of abelian type $(G, [b_G], \{\mu_G\})$.

There is a natural way of constructing Weil descent data on RZ_G following Shen's construction of RZ_G. The idea is as follows: A Weil descent datum in the case $G = \mathrm{GL}_n$ is discussed in [29]. We can define a Weil descent datum on a Rapoport-Zink space RZ_H of Hodge type by restriction along a Hodge embedding $H \hookrightarrow \mathrm{GL}_n$. Assuming RZ_H is associated with RZ_G, we can then "transfer" this Weil descent datum from RZ_H to RZ_G. We note that the above construction recovers the case of local Shimura varieties and Rapoport-Zink spaces of Hodge type from the local Shimura datum $(H, [b_H], \{\mu_H\})$.

We first review the Weil descent data in the Hodge type case as constructed in [5, §7.3]. Assume $(H, [b_H], \{\mu_H\})$ is an unramified local Shimura datum of Hodge type, and fix a *Hodge embedding* $\iota : (H, [b_H], \{\mu_H\}) \to (\mathrm{GL}_n, [b'], \{\mu'\})$. This ι is a map of local Shimura data induced by a closed embedding $\iota : H \to \mathrm{GL}_n$ of groups. Then by the construction of RZ_H in [5], we get a closed immersion of formal groups $\iota_{\mathrm{RZ}} : \mathrm{RZ}_H \to \mathrm{RZ}_{\mathrm{GL}_n}$. The latter space $\mathrm{RZ}_{\mathrm{GL}_n}$ has a natural and explicit Weil descent datum in terms of p-divisible groups. Kim in [5] then defines the Weil descent datum on RZ_H to be the restriction of that on $\mathrm{RZ}_{\mathrm{GL}_n}$ along the closed immersion ι_{RZ}. We denote this Weil descent datum as

$$\Phi^{\mathrm{Kim}}_{(H,\iota)} : \mathrm{RZ}_H \xrightarrow{\sim} \mathrm{RZ}_H^{(\tau)},$$

where τ is the relative Frobenius on $\breve{E}$ over E, and $\mathrm{RZ}_H^{(\tau)} := \mathrm{RZ}_H \times_{\mathrm{Spf}\ \mathcal{O}_{\breve{E}},\tau} \mathrm{Spf}\ \ \mathcal{O}_{\breve{E}}$. As the notation suggests, $\Phi^{\mathrm{Kim}}_{(H,\iota)}$ a priori depends on the choice of Hodge embedding ι. We check in the next lemma that it is independent of this choice.

Lemma 4.5.1. The Weil descent datum $\Phi^{\mathrm{Kim}}_{(H,\iota)}$ is independent of the Hodge embedding $H \hookrightarrow \mathrm{GL}_n$.

Proof. Let $(H, [b_H], \{\mu_H\})$ be an unramified local Shimura datum of Hodge type. Consider two Hodge embeddings $\iota : H \hookrightarrow \mathrm{GL}_n$ and $\iota' : H \hookrightarrow \mathrm{GL}_m$. We get a third Hodge embedding by taking the product $\iota'' := \iota \times \iota' : H \hookrightarrow \mathrm{GL}_{n+m}$. It then suffices to verify that the Weil descent data defined by ι and ι'' coincide, which is clear since the induced embedding $f : \mathrm{GL}_n \to \mathrm{GL}_{n+m}$ commutes with Weil descent data on both sides. □

Theorem 4.5.2. There exists a Weil descent datum Φ^{Shen} for a local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type.

Proof. Let $(H, [b_H], \{\mu_H\})$ be a local Shimura datum of Hodge type associated to $(G, [b_G], \{\mu_G\})$. From the compatibility between Φ^{PR} and Φ^{Kim}, the action of the Weil descent datum τ_H must be equivariant with the morphism of v-sheaves $\omega_H : \mathrm{RZ}_H \longrightarrow \pi_1(H)^\Gamma$. Since Shen's Rapoport-Zink spaces are built out from an auxiliary local Shimura datum of Hodge type, the relevant Weil descent datum τ_G must be equivariant with $\omega_G : \mathrm{RZ}_G \longrightarrow \pi_1(G)^\Gamma$.

Let $\tau_G \in \mathrm{Gal}(K_0/E_G)$ where $K_0 = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. Since $\mathcal{W}_E \subset \Gamma$ for $E = E(G, \{\mu_G\})$, the action of the Weil descent datum on $\pi_1(G)^\Gamma$ is trivial. By equivariance, we see that:

$$\omega_G(\tau_G(*)) = \tau_G(\omega_G(*)) = \omega_G(*).$$

In particular, $\tau_G(*) \equiv *$ in the $\pi_1(G)^\Gamma$-coset RZ^+, yielding $\mathrm{RZ}_G^+ \cong \mathrm{RZ}_G^{+\,(\tau)}$. It follows that $\mathrm{RZ}_G = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}_G^+ \cong (J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}_G^+)^{(\tau)} = \mathrm{RZ}_G^{(\tau)}$. Hence, $\varinjlim_K \mathrm{H}^\bullet(\mathrm{RZ}_G^K)$ is endowed with a natural action of $\mathcal{W}_{E_G}$. □

Lemma 4.5.3. The Weil descent datum $\Phi_{(G,\iota)}^{\mathrm{Shen}}$ is independent of the auxiliary choice of Hodge type datum $(H, [b_H], \{\mu_H\})$.

Proof. Let $(H_1, [b_{H_1}], \{\mu_{H_1}\})$ and $(H_2, [b_{H_2}], \{\mu_{H_2}\})$ be two distinct auxiliary local Shimura data of Hodge type associated to the same local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type. Recall that by [7, theorem 4.6], we have a canonical identification $\mathrm{RZ}_{H_1}^+ \cong \mathrm{RZ}_{H_2}^+ =: \mathrm{RZ}_G^+$. The construction of the Weil descent datum in Theorem 4.5.2 follows. □

Remark 4.5.4. The Weil descent datum can alternatively be characterized in the language of p-adic local shtukas. See [52] and [53] for greater detail.

4.6 Relations to the Hodge case

Definition 4.6.1 ([6] 4.1.4)**.** Let $(G, [b_G], \{\mu_G\})$ be a local Shimura datum, and fix a parabolic subgroup P_G of G with Levi factor L_G and unipotent radical U_G. The datum $(G, [b_G], \{\mu_G\})$ is called *Hodge-Newton reducible* with respect to P_G and L_G (*Hodge-Newton-reducible*, for short) if there exists $\mu_G \in \{\mu_G\}$ and $b_G \in [b_G] \cap L_G(\breve{\mathbb{Q}}_p)$ with the following properties:

1. $\mu_G \in \{\mu_G\}$ factors through L_G.

2. $b_G \in [b_G] \cap L_G(\breve{\mathbb{Q}}_p)$ such that $[b_G] \cap L_G(\breve{\mathbb{Z}}_p)\mu_G(p)L_G(\breve{\mathbb{Z}}_p)$ is non-empty.

3. the action of μ_G and ν_{b_G} on $\mathrm{Lie}(U_G) \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$ produces only non-negative characters.

Recall our strategy is to use the known results of Harris-Viehmann in the Hodge-Newton-reducible Hodge type case to deduce results for the Hodge-Newton-reducible abelian type case. However, given an unramified local Shimura datum of abelian type that is Hodge-Newton-reducible, it is not true that any associated datum of Hodge type is Hodge-Newton-reducible. But the following lemma shows that we can at least find one that is Hodge-Newton-reducible.

Lemma 4.6.2. If a local abelian type Shimura datum $(G, [b_G], \{\mu_G\})$ is Hodge-Newton-reducible, then there exists an associated local Shimura datum $(H, [b_H], \{\mu_H\})$ of Hodge type that is also Hodge-Newton-reducible.

Proof. Since $(G, [b_G], \{\mu_G\})$ is Hodge-Newton-reducible (with respect to some fixed parabolic subgroup P_G and Levi subgroup L_G), by definition, there exists a corresponding local datum $(L_G, [b_G], \{\mu_G\})$ coming from the Levi group of G with the properties in Definition 4.6.1. This correspondence is maintained upon passing to the adjoint local data. By definition,

$$(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}]) \cong (H^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\}),$$

so we can set $(L_{G^{\mathrm{ad}}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) =: (L_{H^{\mathrm{ad}}}, [b_{H^{ad}}], \{\mu_{H^{\mathrm{ad}}}\})$ as the adjoint for some datum $(L_H, [b_H], \{\mu_H\})$ from $(H, [b], \{\mu_H\})$. We choose any such H and L_H.

We claim there is some b_H and minuscule μ_H such that $(H, [b_H], \{\mu_H\})$ is Hodge-Newton reducible with respect to $(L_H, [b_H], \{\mu_H\})$. To see this, we can pick a lift of $\mu_{H^{\mathrm{ad}}}$ by setting $\mu_H := \mu_H \cap L_H$. The kernel of the map from the lift to $\mu_{H^{\mathrm{ad}}}$ is contained in the cocharacter group of the centre of L_H. For any $\lambda \in X_*(Z(L_H))$, the pairing with any weight coming from μ_H is trivial. Thus, μ_H is minuscule since $\{\mu_{H^{\mathrm{ad}}}\} = \{\mu_{G^{\mathrm{ad}}}\}$ is minuscule.

Let $x \in [b_{G^{\mathrm{ad}}}] \cap L_{G^{\mathrm{ad}}}(\breve{\mathbb{Z}}_p)\mu_{G^{\mathrm{ad}}}(p)L_{G^{\mathrm{ad}}}(\breve{\mathbb{Z}}_p) = [b_{G^{\mathrm{ad}}}] \cap L_{H^{\mathrm{ad}}}(\breve{\mathbb{Z}}_p)\mu_{H^{\mathrm{ad}}}(p)L_{H^{\mathrm{ad}}}(\breve{\mathbb{Z}}_p)$. Let y be the lift of x in $L_H(\breve{\mathbb{Z}}_p)\mu_H(p)L_H(\breve{\mathbb{Z}}_p)$. Then b_H can be set to be the σ-conjugacy class of y.

$\square$

Proposition 4.6.3. Suppose a local Shimura datum $(G, [b_G], \{\mu_G\})$ of abelian type is Hodge-Newton-reducible. Then its adjoint local Shimura datum $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\})$ embeds into $(\mathrm{PGL}_n, [b_{\mathrm{PGL}_n}], \{\mu_{\mathrm{PG}}$ with the embedding factoring through $(\widetilde{H}^{\mathrm{ad}}, [b_{\widetilde{H}}], \{\mu_{\widetilde{H}}\})$ where

$$\widetilde{H}^{\mathrm{ad}} = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{PGL}_n \supsetneq L_{\widetilde{H}^{\mathrm{ad}}} = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{PGL}_{m_1} \times \ldots \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{PGL}_{m_r}$$

Proof. Let $(H, [b_H], \{\mu_H\})$ be an associated Hodge-Newton reducible local Hodge type Shimura datum, whose existence is guaranteed by the previous lemma. It embeds into the local Shimura datum $(\mathrm{GL}_n, [b_{\mathrm{GL}_n}], \{\mu_{\mathrm{GL}_n}\})$, factoring through the EL realization $(\widetilde{G}, [\widetilde{b}], \{\mu_{\widetilde{G}}\})$, where

$$\widetilde{H} = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n \supsetneq L_{\widetilde{H}} = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{m_1} \times \ldots \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{m_r}$$

[6, 4.2.1] Passing to the level of adjoint local data, we find that

$$(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) = (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\}) \hookrightarrow (\mathrm{GL}_n^{\mathrm{ad}}, [b'^{\mathrm{ad}}], \{\mu_{\mathrm{GL}_n^{\mathrm{ad}}}\}),$$

and must factor through $(\widetilde{H}^{\mathrm{ad}}, [b_{\widetilde{H}^{\mathrm{ad}}}]\{\mu_{\widetilde{H}^{\mathrm{ad}}}\})$.

Now,

$$(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) \cong (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\})$$
$$(\widetilde{G}^{\mathrm{ad}}, [b_{\widetilde{G}^{\mathrm{ad}}}], \{\mu_{\widetilde{G}^{\mathrm{ad}}}\}) \cong (\widetilde{H}^{\mathrm{ad}}, [b_{\widetilde{H}^{\mathrm{ad}}}], \{\mu_{\widetilde{H}^{\mathrm{ad}}}\})$$
$$(\mathrm{GL}_n^{\mathrm{ad}}, [b_{\mathrm{GL}_n^{\mathrm{ad}}}], \{\mu_{GL_n^{\mathrm{ad}}}\}) \cong (\mathrm{PGL}_n, [b_{\mathrm{PGL}_n}], \{\mu_{\mathrm{PGL}_n}\})$$

Together, this gives us

$$(H, [b_H], \{\mu_H\}) \to (H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\}) \hookrightarrow (\widetilde{H}^{\mathrm{ad}}, [b_{\widetilde{H}^{\mathrm{ad}}}], \{\mu_{\widetilde{H}^{\mathrm{ad}}}\}) \hookrightarrow (\mathrm{PGL}_n, [b_{\mathrm{PGL}_n}], \{\mu_{\mathrm{PGL}_n}\})$$

□

We record the following group-theoretic lemma for later use.

Lemma 4.6.4. Let G be a connected reductive group over a field k. Let $G^{\mathrm{der}} = [G, G]$ be the derived subgroup of G. Then there is a natural isomorphism

$$G \simeq G^{\mathrm{der}} \times_{Z(G^{\mathrm{der}})} Z(G),$$

where the right-hand side is the contracted product. Furthermore, there is a natural isomorphism for any central subgroup $C \subset Z(G)$:

$$Z(G/C) \simeq Z(G)/C.$$

Proof. The map

$$\pi : G^{\mathrm{der}} \times_{Z(G^{\mathrm{der}})} Z(G) \to G, \quad (g, z) \mapsto gz$$

is a well-defined homomorphism. Suppose $\pi(g, z) = gz = 1$. So $g = z^{-1} \in Z(G^{\mathrm{der}})$. Since the pair (z^{-1}, z) is equivalent to $(1, 1)$ in $G^{\mathrm{der}} \times_{Z(G^{\mathrm{der}})} Z(G)$, we see π is injective. On the other hand, $G = G^{\mathrm{der}} \cdot Z(G)$. Thus, π is also surjective. The equality $Z(G/C) \simeq Z(G)/C$ holds for any central subgroup C. □

Example 4.6.5. Consider $G = \mathrm{GL}_n$. Then $G^{\mathrm{der}} = \mathrm{SL}_n$, $Z(G^{\mathrm{der}}) = \mu_n$, and $Z(G) = \mathbb{G}_m$.

4.7 Functoriality of Rapoport-Zink spaces

We present some basic results concerning the functoriality of local Shimura data. Some of the results may have been known, but we include them for exposition.

Lemma 4.7.1. Let $(G, [b_G], \{\mu_G\})$ and $(G', [b_{G'}], \{\mu_{G'}\})$ be unramified local Shimura data of abelian type. Then $(G \times G', [b_G, b_{G'}], \{\mu, \mu'\})$ is also an unramified local Shimura datum of abelian type. Here $[b_G, b_{G'}]$ is defined to be the σ-conjugacy class of $(b_G, b_{G'}) \in G(\breve{\mathbb{Q}}_p) \times G'(\breve{\mathbb{Q}}_p)$ for any representative b_G of $[b_G]$ (resp. $b_{G'}$ of $[b_{G'}]$). $\{\mu_G, \mu_{G'}\}$ is defined similarly.

Proof. Let $(H, [b], \{\mu_H\})$ and $(H', [b'], \{\mu_{H'}\})$ be the local Shimura data of Hodge type associated with $(G, [b_G], \{\mu_G\})$ and $(G', [b_{G'}], \{\mu_{G'}\})$, respectively. By [41, Lemma 2.4.2(i)], we know that $(H \times H', [b_H, b_{H'}], \{\mu_H, \mu_{H'}\})$ is an unramified local Shimura datum of Hodge type. Since

$$(G \times G')^{\mathrm{ad}} \cong G^{\mathrm{ad}} \times G'^{\mathrm{ad}} \cong H^{\mathrm{ad}} \times H'^{\mathrm{ad}} \cong (H \times H')^{\mathrm{ad}},$$

We see that this is a local Hodge datum associated to the datum $(G \times G', [b_G, b_{G'}], \{\mu_G, \mu_{G'}\})$. Hence, the latter is unramified of abelian type. □

The following proposition provides some basic functorial properties for Rapoport-Zink spaces of abelian type. This is a generalization of [5, Theorem 4.9.1].

Proposition 4.7.2. Let $(G, [b_G], \{\mu_G\})$ and $(G', [b_{G'}], \{\mu_{G'}\})$ be unramified local Shimura data of abelian type. Write RZ_G (resp. $\mathrm{RZ}_{G'}$, $\mathrm{RZ}_{G \times G'}$) for the Rapoport-Zink space associated to the datum $(G, [b_G], \{\mu_G\})$ (resp, $(G', [b_{G'}], \{\mu_{G'}\})$, $(G \times G', [b_G, b_{G'}], \{\mu, \mu'\})$).

1. There is a natural isomorphism

$$\mathrm{RZ}_G \times_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)} \mathrm{RZ}_{G'} \xrightarrow{\sim} \mathrm{RZ}_{G \times G'},$$

2. For any map $f : (G, [b_G], \{\mu_G\}) \longrightarrow (G', [b_{G'}], \{\mu_{G'}\})$ between local Shimura data which are unramified of abelian type, there exists an induced morphism

$$\mathrm{RZ}_G \longrightarrow \mathrm{RZ}_{G'}$$

which is a closed embedding if the underlying group homomorphism $f : G \to G'$ is a closed embedding.

Proof. Let $(H, [b_H], \{\mu_H\})$ (resp. $(H', [b_{H'}], \{\mu_{H'}\})$) be an unramified local Shimura datum of Hodge type associated to $(G, [b_G], \{\mu_G\})$ (resp. $(G', [b_{G'}], \{\mu_{G'}\})$). To simplify notations, we will write RZ_G for the Rapoport-Zink formal scheme associated to $(G, [b_G], \{\mu_G\})$, and similarly for other local Shimura data. We will freely use the notations from [7].

1. Recall for the local Shimura datum $(H, [b_H], \{\mu_H\})$ of Hodge type, we can form a coset $c_{b_H,\mu_H}\pi_1(H)^\Gamma$ in the group $\pi_1(H)^\Gamma$; see [7, §2.2] for the construction. The formation of this coset is functorial in $(H, [b_H], \{\mu_H\})$ by the functoriality of the Kottwitz map and the Galois action. Note that there is an isomorphism

$$\tau : \pi_1(H)^\Gamma \times \pi_1(H')^\Gamma \xrightarrow{\sim} \pi_1(H \times H')^\Gamma.$$

There is a natural isomorphism

$$\varphi : c_{b_H,\mu_H}\pi_1(H)^\Gamma \times c_{b_{H'},\mu_{H'}}\pi_1(H')^\Gamma \xrightarrow{\sim} c_{H\times H'}\pi_1(H \times H')^\Gamma.$$

compatible with the Weil descent datum.

Fix elements $x_0 \in c_{b_H,\mu_H}\pi_1(H)^\Gamma$ and $x_0' \in c_{b_{H'},\mu_{H'}}\pi_1(H')^\Gamma$, and take $y_0 := q\varphi(x_0, x_0')$. There is a natural map of étale sheaves

$$\omega_H : \mathrm{RZ}_H \to c_{b_H,\mu_H}\pi_1(H)^\Gamma$$

and RZ_H^+ is defined to be the fibre

$$\mathrm{RZ}_H^+ := \omega_H^{-1}(x_0).$$

The constructions of $\mathrm{RZ}_{H'}^+$ and $\mathrm{RZ}_{H\times H'}^+$ are similar.

By the construction of the spaces RZ_G and $\mathrm{RZ}_{G'}$, we only need to show there is an isomorphism

$$\mathrm{RZ}_H^+ \times \mathrm{RZ}_{H'}^+ \to \mathrm{RZ}_{H\times H'}^+.$$

This then follows from the natural isomorphism

$$\mathrm{RZ}_H \times \mathrm{RZ}_{H'} \xrightarrow{\sim} \mathrm{RZ}_{H\times H'}$$

shown in [41, Proposition 2.6.6].

2. By the [53, Proposition 1.2.1], if there exists a formal scheme that is flat, normal and locally formally of finite type whose v-theoretic integral model is of the form $\mathcal{M}^{\text{int}}_{(\mathcal{G},[b_G],\{\mu_G\})}$, then it must be unique. Such a formal scheme is explicitly constructed by Shen, which we have been denoting by RZ_G.

   From $f : G \longrightarrow G'$, we get an induced map of local Shimura data:

$$f : (G, [b_G], \{\mu_G\}) \longrightarrow (G', [b_{G'}], \{\mu_{G'}\})$$

   Consider the induced map on the associated Hodge-type groups $f_H : H \to H'$ such that $b_H \mapsto f_H(b_H) = b_{H'}$. By [41, Proposition 2.6.6(ii)], this induces a map

$$\mathrm{RZ}_H \to \mathrm{RZ}_{H'},$$

   of the associated Rapoport-Zink spaces of Hodge type.

   We now use [53, Proposition 3.6.2]: a closed embedding of local Shimura data

$$f : (G, [b_G], \{\mu_G\}) \longrightarrow (G', [b_{G'}], \{\mu_{G'}\})$$

   induces the following closed immersion of v-sheaves:

$$\mathfrak{f} : \mathcal{M}^{\text{int}}_{(\mathcal{G},[b_G],\{\mu_G\})} \longrightarrow \mathcal{M}^{\text{int}}_{(\mathcal{G}',[b_{G'}],\{\mu_{G'}\})}$$

   where $\mathcal{G}$ and $\mathcal{G}'$ are parahoric models of G and G' respectively. The statement now is a special case of Lemma 4.7.3 below.

□

Lemma 4.7.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two flat and normal formal schemes locally formally of finite type over $\mathrm{Spf}\ \mathcal{O}_{\breve{E}}$, and let $\mathfrak{f} : \mathcal{X}^{\diamond} \to \mathcal{Y}^{\diamond}$ be a morphism between the associated v-sheaves over $\mathrm{Spd}\,\mathcal{O}_{\breve{E}}$.

If $\mathfrak{f}$ is a closed immersion, then there exists a unique morphism $f : \mathcal{X} \to \mathcal{Y}$ between formal schemes over $\mathcal{O}_{\breve{E}}$ such that f is a closed immersion and that $f^\diamond = \mathfrak{f}$.

Proof. By [52, Proposition 18.4.1], there exists a unique morphism $f : \mathcal{X} \to \mathcal{Y}$ representing $\mathfrak{f}$, i.e., $f^\diamond = \mathfrak{f}$. We only need to verify f is a closed immersion. Since the problem is Zariski-local on the target, we may assume $\mathcal{X} = \mathrm{Spf}\ A^+$ and $\mathcal{Y} = \mathrm{Spf}\ B^+$. Write $\mathcal{X}_\eta$ (resp. $\mathcal{Y}_\eta$) for the adic generic fibre of $\mathcal{X}$ (resp. $\mathcal{Y}$). As pointed out in [52, Remark 18.4.3.], A^+ (resp. B^+) can be identified as the ring of power-bounded functions on the generic fibre $\mathcal{X}_\eta$ (resp. $\mathcal{Y}_\eta$). The map $\mathfrak{f}$ induces a closed immersion $\mathfrak{f}_\eta : \mathcal{X}_\eta^\diamond \to \mathcal{Y}_\eta^\diamond$, and hence a surjection $\mathcal{O}_{\mathcal{Y}_\eta^\diamond} \twoheadrightarrow \mathfrak{f}_{\eta *}\mathcal{O}_{\mathcal{X}_\eta^\diamond}$. Since $\mathcal{X}_\eta$ is a smooth analytic adic space over $\mathrm{Spa}\,\breve{E}$, we have an identification

$$\Gamma(\mathcal{X}_\eta, \mathcal{O}^+) \simeq \Gamma(\mathcal{X}_\eta^\diamond, \mathcal{O}^+)$$

and similarly for $\mathcal{Y}$. This in particular means we obtain a surjection $B^+ \twoheadrightarrow A^+$, which shows the map f is a closed immersion. □

Remark 4.7.4.

1. We note that the argument in the second part of the proof applies to any local Shimura data, as long as the v-theoretic integral model $\mathcal{M}^{\mathrm{int}}_{(\mathcal{G},[b_G],\{\mu_G\})}$ is represented by a Rapoport-Zink space, i.e. a formal scheme which is formally smooth and formally locally of finite type over $\mathrm{Spf}\ \mathcal{O}_{\breve{E}}$.

2. A rather elementary but crucial point is that the v-sheaf functor $\mathfrak{X} \to \mathfrak{X}^\diamond$ on formal schemes is in general not fully faithful. Thus, the proposition above relies on the nice geometric properties of the Rapoport-Zink spaces.

Corollary 4.7.5. Let $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\}) \to (G'^{\mathrm{ad}}, [b_{G'^{\mathrm{ad}}}], \{\mu_{G'^{\mathrm{ad}}}\})$ be a closed embedding of unramified local Shimura data of *adjoint* abelian type. This induces a closed embedding of Rapoport-Zink spaces $\mathrm{RZ}_{G^{\mathrm{ad}}} \to \mathrm{RZ}_{G'^{\mathrm{ad}}}$.

Proof. This immediately follows, since it is a special case of the abelian type. □

For an unramified local Shimura datum of abelian type $(G, [b_G], \{\mu_G\})$, and a compact open subgroup $K_G \subset G(\mathbb{Q}_p)$, we write $\mathrm{RZ}_G^{K_G}$ for the local Shimura variety with K_G-level structure.

Let $(H, [b_H], \{\mu_H\})$ be an associated local Shimura datum of Hodge type. Let K' for the image of K_G under the map $G \to G^{\mathrm{ad}} \cong H^{\mathrm{ad}}$. Let K_H be preimage of K' along the quotient map $H \to H^{\mathrm{ad}}$. Then $K_H \subset H(\mathbb{Q}_p)$ is an open compact subgroup, and $\mathrm{RZ}_H^{K_H}$ is a local Shimura variety of Hodge type with K_H-level structure, built from the datum $(H, [b_H], \{\mu_H\})$.

When no confusion would arise, we write $K = K_G \subset G(\mathbb{Q}_p)$.

4.8 Constructions of auxiliary spaces at the adjoint level

In this section, we construct a few formal schemes related to the adjoint local Shimura datum $(G^{\mathrm{ad}}, [b_{G^{\mathrm{ad}}}], \{\mu_{G^{\mathrm{ad}}}\})$. We should explain why this adjoint datum needs separate constructions, as we already have the Rapoport-Zink space $\mathrm{RZ}_{G^{ad}}$ from [7]. The subtlety comes from our strategy of using the correspondence diagram as in [6, Lemma 4.3.1] for the datum $(G, [b_G], \{\mu_G\})$ *of abelian type*:

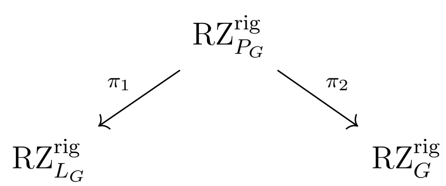

Figure 4.5: Relations between RZ_G, RZ_{P_G} and RZ_{L_G}.

We must relate it to the correspondence diagram for the Hodge type $(H, [b_H], \{\mu_H\})$. It is natural to think the connection between the two diagrams is given by an analogous diagram for the adjoint datum $(H^{\mathrm{ad}}, [b_{H^{\mathrm{ad}}}], \{\mu_{H^{\mathrm{ad}}}\})$. Denote by $P_{H^{\mathrm{ad}}}$ (resp. $L_{H^{\mathrm{ad}}}$) the image of P_H (resp. L_H) under the quotient map $H \twoheadrightarrow H^{\mathrm{ad}}$. But note that in general $P_{H^{\mathrm{ad}}}$ is not the adjoint quotient of P_H (similarly for L_H). Thus, existing results such as [7, Proposition 4.9] do not directly tell us what the Rapoport-Zink space associated to $P_{H^{\mathrm{ad}}}$ or $L_{H^{\mathrm{ad}}}$ is. We will resolve this problem in this subsection.

Lemma 4.8.1. Let $(G, [b_G], \{\mu_G\})$ be an unramified local Shimura datum of abelian type, and let $(L_G, [b_G], \{\mu_G\})$ be the local Shimura datum corresponding to a Levi subgroup $L_G \subsetneq G$. Then $(L_G, [b_G], \{\mu_G\})$ is an unramified local Shimura datum of abelian type.

Proof. Let $(H, [b_H], \{\mu_H\})$ be an unramified local Shimura datum of Hodge type associated to $(G, [b_G], \{\mu_G\})$. Write L' for the image of L under the map $G \to G^{\mathrm{ad}} \simeq H^{\mathrm{ad}}$. Then let L_H be preimage of L' along the quotient map $H \to H^{\mathrm{ad}}$. Thus, L_H is an (unramified) Levi subgroup of H, and $(H, [b_H], \{\mu_H\})$ induces an unramified local Shimura datum $(L_H, [b_H], \{\mu_H\})$ of Hodge type. It is then easy to check $(L_H, [b_H], \{\mu_H\})$ is associated to $(L_G, [b_G], \{\mu_G\})$. □

Lemma 4.8.2. Let $(G, [b_G], \{\mu_G\})$ be an unramified local Shimura datum of abelian type, with a Levi subgroup $L_G \subset G$. Let $I_{b_G, \{\mu_G\}, L_G}$ be the set of $L_G(\breve{\mathbb{Z}}_p)$-conjugacy classes of L_G with cocharacter representative μ' such that

1. $\mu' \in \{\mu_G\}$.

2. $[b_G] \cap L_G(\breve{\mathbb{Z}}_p)\mu'(p)L(\breve{\mathbb{Z}}_p)$ is non-empty.

Then

1. $I_{b_G, \{\mu_G\}, L_G}$ is non-empty.

2. For any $\{\mu'_G\} \in I_{b_G, \{\mu_G\}, L_G}$, the tuple $(L_G, [b_G], \{\mu'_G\})$ is an unramified local Shimura datum of abelian type.

Proof. Let $(H, [b_H], \{\mu_H\})$ be an unramified local Hodge type Shimura datum that is associated to $(G, [b_G], \{\mu_G\})$, with a Levi subgroup $L_H \subset H$.

1. There exists an unramified Levi subgroup $L_H \subsetneq H$, such that $(L_G, [b_G], \{\mu_G\})$ is associated to $(L_H, [b_H], \{\mu_H\})$. Associated to $I_{b_G, \{\mu_G\}, L_G}$, there is $I_{b_H, \{\mu_H\}, L_H}$. The non-emptiness follows from [15].

2. Consider the local Shimura datum $(L_G, [b_G], \{\mu'_G\})$ of abelian type, and its associated local Shimura datum $(L_H, [b_H], \{\mu'_H\})$ of Hodge type. Since H is unramified, so is L_H.

By [Lemma 4.1.2 Hong], we know that $(L_H, [b_H], \{\mu'_H\})$ is unramified. It follows that $(L_G, [b_G], \{\mu'_G\})$ is an unramified local Shimura datum of abelian type.

□

4.9 Constructions of parabolic Rapoport-Zink spaces

Recall that for an unramified local Shimura datum $(H, b_H, \{\mu_H\})$ of Hodge type, we have a map

$$\omega_H : \mathrm{RZ}_H \to c_{b_H,\mu_H}\pi_1(H)^\Gamma,$$

and the fibre over the point c_{b_H,μ_H} is by definition RZ_H^+. The same applies to a Levi subgroup L_H of H: $\mathrm{RZ}_{L_H}^+ := \omega_{L_H}^{-1}(c_{b_H,\mu_H})$ is the fibre of $\omega_{L_H} : \mathrm{RZ}_{L_H} \to c_{b_H,\mu_H}\pi_1(L_H)^\Gamma$.

Since RZ_H is of Hodge type, we adopt the construction of the correspondence diagram from [6, §4].

From $Q^\vee_{L_H} \subseteq Q^\vee_H$, we obtain a surjective homomorphism

$$\varphi : \pi_1(L_H) := X_*(T_H)/Q^\vee_{L_H} \twoheadrightarrow X_*(T_H)/Q^\vee_H =: \pi_1(H).$$

Note we have $\varphi(c_{b_H,\mu_H}) = c_{b_H,\mu_H}$. Let $\varphi^\Gamma : \pi_1(L_H)^\Gamma \twoheadrightarrow \pi_1(H)^\Gamma$ be the map induced by taking Γ-invariants on the map φ. We get a map of cosets $c_{b_H,\mu_H}\pi_1(L_H)^\Gamma \twoheadrightarrow c_{b_H,\mu_H}\pi_1(H)^\Gamma$.

We have a natural map $\omega_{P_H} := \omega_H \circ \pi_{2,H} : \mathrm{RZ}_{P_H} \longrightarrow c_{b_H,\mu_H}$. We define $\mathrm{RZ}_{P_H}^+ := \omega_{P_H}^{-1}(c_{b_H,\mu_H})$ to be the fibre, and also set $\mathrm{RZ}_{P_G}^+ := \mathrm{RZ}_{P_H}^+$ following Shen.

By construction, $\mathrm{RZ}_G^+ = \mathrm{RZ}_H^+$, $\mathrm{RZ}_{P_G}^+ = \mathrm{RZ}_{P_H}^+$ and $\mathrm{RZ}_{L_G}^+ = \mathrm{RZ}_{L_H}^+$ are closed formal subschemes of RZ_H, RZ_{P_H} and RZ_{L_H} respectively. We extend the construction to the rigid analytic generic fibre to obtain $\mathrm{RZ}_G^{+\mathrm{rig}} = \mathrm{RZ}_H^{+\mathrm{rig}}$, $\mathrm{RZ}_{P_G}^{+\mathrm{rig}} = \mathrm{RZ}_{P_H}^{+\mathrm{rig}}$ and $\mathrm{RZ}_{L_G}^{+\mathrm{rig}} = \mathrm{RZ}_{L_H}^{+\mathrm{rig}}$ respectively.

Let $\pi_{1,G}^+ = \pi_{1,H}^+ := \pi_{1,H}\big|_{\mathrm{RZ}_H^{+\mathrm{rig}}}$ and $\pi_{1,L_G}^+ = \pi_{1,L_H}^+ := \pi_{1,L_H}\big|_{\mathrm{RZ}_{L_H}^{+\mathrm{rig}}}$. Similarly, we can define $\pi_{2,G}^+ = \pi_{2,H}^+ := \pi_{2,H}\big|_{\mathrm{RZ}_H^{+\mathrm{rig}}}$. We note that $L_H \hookrightarrow H$ induces the closed embedding $\psi_H : \mathrm{RZ}_{L_H} \hookrightarrow \mathrm{RZ}_H$ by [5, Theorem 4.9.1].

We define the formal scheme $\mathrm{RZ}_{P_G} := J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^+_{P_G}$. Let $\mathrm{RZ}^{\mathrm{rig}}_{P_G}$ be its rigid analytic generic fibre.

Lemma 4.9.1.

1. There is a map $\pi_{1,G} : \mathrm{RZ}^{\mathrm{rig}}_{P_G} \longrightarrow \mathrm{RZ}^{\mathrm{rig}}_{L_G}$.

2. s_G is a closed immersion

3. $\pi_{1,G}$ is a fibration in balls.

Proof. 1. We first prove this for the +-closed formal subschemes:

$$\pi^+_{1,G}(\mathrm{RZ}^{+\,\mathrm{rig}}_{P_G}) = \pi^+_{1,H}(\mathrm{RZ}^{+\,\mathrm{rig}}_{P_H}) = \pi^+_{1,H}(\omega^{-1}_{P_H}(c_{b_H,\mu_H})) = \pi^+_{1,H} \circ (\omega_H \circ \pi_{2,H})^{-1}(\varphi(c_{b_H,\mu_H}))$$
$$= \pi^+_{1,H} \circ \pi^{-1}_{2,H} \circ \omega^{-1}_H \circ \varphi(c_{b_H,\mu_H}) \subseteq \omega^{-1}_{L_H}(c_{b_H,\mu_H}) = \mathrm{RZ}^{+\,\mathrm{rig}}_{L_H} = \mathrm{RZ}^{+\,\mathrm{rig}}_{L_G}$$

Thus, we get the map $\mathrm{RZ}^{\mathrm{rig}}_{P_G} = J_{b_G}(\mathbb{Q}_p)\pi^+_{1,G}(\mathrm{RZ}^{+\,\mathrm{rig}}_{P_G}) \xrightarrow{\pi_{1,G}} J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^{+\,\mathrm{rig}}_{L_G} = \mathrm{RZ}^{\mathrm{rig}}_{L_G}$.

2. The composition $\pi_{2,G} \circ s_G : \mathrm{RZ}^{\mathrm{rig}}_{L_G} \hookrightarrow \mathrm{RZ}^{\mathrm{rig}}_{G}$ is the closed embedding on the rigid analytic generic fibre that is induced by the closed embedding of local Shimura data

$$(L_G, [b_G], \{\mu_G\}) \hookrightarrow (G, [b_G], \{\mu_G\}).$$

It follows that s_G is a closed immersion.

3. It suffices to show that $\pi^+_{1,G}$ is a fibration in balls. This follows from [6, Lemma 4.3.1] by restricting to $\mathrm{RZ}^{+\,\mathrm{rig}}_{P_H} = \mathrm{RZ}^{+\,\mathrm{rig}}_{P_G}$.

□

4.10 More on Formal subschemes

We revisit the notations from earlier. Recall that we set $\mathrm{RZ}^+_{L_G} := \mathrm{RZ}^+_{L_H}$ and $\mathrm{RZ}^+_{P_G} := \mathrm{RZ}^+_{P_H}$. We write $\mathrm{H}^i_{\text{ét}}(\mathrm{RZ}^{K_G}_G) := \mathrm{H}^i_{\text{ét}}(\mathrm{RZ}^{K_G}_G, \mathbb{Q}_\ell)$ for any compact open subgroup K_G of $G(\mathbb{Q}_p)$. We also write

$$\mathrm{H}^i(\mathrm{RZ}_G) := \varinjlim_{K_G \subset G(\mathbb{Q}_p)} \mathrm{H}^i_{\text{ét}}(\mathrm{RZ}^{K_G}_G).$$

This is a representation of $G(\mathbb{Q}_p) \times \mathcal{W}_E$ on $\mathbb{Q}_\ell$-vector spaces. For an admissible ℓ-adic representation ρ of $J_{b_G}(\mathbb{Q}_p)$, we write

$$\mathrm{H}^{i,j}(\mathrm{RZ}^\infty_G)_\rho := \varinjlim_{K \subset G(\mathbb{Q}_p)} \mathrm{Ext}^j_{J_b(\mathbb{Q}_p)}(\mathrm{H}^i(\mathrm{RZ}^K_G), \rho).$$

Each $\mathrm{H}^{i,j}(\mathrm{RZ}^\infty_G)_\rho$ has a natural action of $G(\mathbb{Q}_p) \times \mathcal{W}_E$. We get a virtual representation:

$$\mathrm{H}^\bullet(\mathrm{RZ}^\infty_G)_\rho := \sum_{i,j \geq 0} (-1)^{i+j} \mathrm{H}^{i,j}(\mathrm{RZ}^\infty_G)_\rho.$$

The following corollary is a consequence of Lemma 4.9.1.

Proposition 4.10.1. There is an isomorphism between the following virtual ℓ-adic representations of $P_G(\mathbb{Q}_p) \times \mathcal{W}_E$:

$$\mathrm{H}^\bullet(\mathrm{RZ}^{+\infty}_{L_G})_\rho = \mathrm{H}^\bullet(\mathrm{RZ}^{+\infty}_{P_G})_\rho$$

Proof. We define a formal subscheme $\mathrm{RZ}^{+,(m)}_{P_G} := \psi_H^{-1} \circ \omega_{P_H}^{-1}(c_{b_H,\mu_H})$, that is equipped with the map $\psi^+_G : \mathrm{RZ}^{+,(m)}_{P_G} \longrightarrow \mathrm{RZ}^+_{P_G}$. We get natural diagrams:

Let $D^+_G = \dim(\mathrm{RZ}^+_{P_G}) - \dim(\mathrm{RZ}^+_{L_G})$. It follows that we have the following quasi-isomorphism for every integer $m > 0$:

$$R\Gamma_c(\mathrm{RZ}^{+K'_{H,(m)}}_{P_G} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell) \cong R\Gamma_c(\mathrm{RZ}^{+K'_{H,(m)}}_{L_G} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell(-D^+_G))[-2D^+_G].$$

$$\begin{CD} \mathcal{P}^+_{G,m} @>>> \mathrm{RZ}^{+\,\mathrm{rig},(m)}_{P_G} \\ @VVV @VVV \\ \mathrm{RZ}^{+K'_{G,(m)}}_{P_G} @>>> \mathrm{RZ}^{+\,\mathrm{rig}}_{P_G} \end{CD} \qquad \begin{CD} \mathcal{P}'^+_{G,m} @>>> \mathrm{RZ}^{+K'_{G,(m)}}_{P_G} \\ @VVV @VV{\pi^+_{1,G}}V \\ \mathrm{RZ}^{+\,\mathrm{rig},(m)}_{L_G} @>>> \mathrm{RZ}^{+\,\mathrm{rig}}_{L_G}. \end{CD}$$

Figure 4.6: Covers revisited for + components.

This yields

$$\mathrm{H}^\bullet(\mathrm{RZ}^{+\infty}_{L_G})_\rho = \mathrm{H}^\bullet(\mathrm{RZ}^{+\infty}_{P_G})_\rho$$

□

Proposition 4.10.2. There is an isomorphism between the following virtual ℓ-adic representations of $P_G(\mathbb{Q}_p) \times \mathcal{W}_E$:

$$\mathrm{H}^\bullet(\mathrm{RZ}^{\infty}_{L_G})_\rho = \mathrm{H}^\bullet(\mathrm{RZ}^{\infty}_{P_G})_\rho.$$

Proof. We use the notation of Proposition 4.10.1, and add a few more:

- $\mathrm{RZ}_{L_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^+_{L_G}$, $\mathrm{RZ}^{\mathrm{rig}}_{L_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^{+\,\mathrm{rig}}_{L_G}$, $\mathrm{RZ}^{\mathrm{rig},(m)}_{L_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^{+\,\mathrm{rig},(m)}_{L_G}$.
- $\mathrm{RZ}_{P_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^+_{P_G}$, $\mathrm{RZ}^{\mathrm{rig},(m)}_{P_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^{+\,\mathrm{rig},(m)}_{P_G}$, $\mathrm{RZ}^{+K'_{G,(m)}}_{P_G} = J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}^{K'_{G,(m)}}_{P_G}$

We use this in the Cartesian diagrams:

$$\begin{CD} J_{b_G}(\mathbb{Q}_p)\,\mathcal{P}^+_{G,m} @>>> \mathrm{RZ}^{\mathrm{rig},(m)}_{P_G} \\ @VVV @VVV \\ \mathrm{RZ}^{K'_{G,(m)}}_{P_G} @>>> \mathrm{RZ}^{\mathrm{rig}}_{P_G} \end{CD} \qquad \begin{CD} J_{b_G}(\mathbb{Q}_p)\,\mathcal{P}'^+_{G,m} @>>> \mathrm{RZ}^{K'_{G,(m)}}_{P_G} \\ @VVV @VV{\pi_{1,G}}V \\ \mathrm{RZ}^{\mathrm{rig},(m)}_{L_G} @>>> \mathrm{RZ}^{\mathrm{rig}}_{L_G} \end{CD}$$

Figure 4.7: Covers revisited for the abelian case.

In particular, we get the following equality:

$$\mathrm{H}^{\bullet}(\mathrm{RZ}^{\infty}_{P_G})_{\rho} = \mathrm{H}^{\bullet}(\mathrm{RZ}^{\infty}_{L_G})_{\rho}$$

□

4.11 Adjoint level

We note that $\mathrm{RZ}_{L_{H^{\mathrm{ad}}}} := \mathrm{RZ}_{L_H}/X_*(Z_H)^{\Gamma}$ is distinct from but related to the formal space $\mathrm{RZ}_{(L_H)^{\mathrm{ad}}} = \mathrm{RZ}_{L_H}/X_*(Z_{L_H})$.

Lemma 4.11.1. There is an identification:

$$\mathrm{RZ}_{L_{G^{\mathrm{ad}}}} \cong \mathrm{RZ}_{L_H^{\mathrm{ad}}} \times_{\pi_1((L_H)^{\mathrm{ad}})} \pi_1(L_{H^{\mathrm{ad}}})^{\Gamma}$$

Proof. Recall that $L_{H^{\mathrm{ad}}}$ is the image of L_H under the projection $H \longrightarrow H^{\mathrm{ad}}$, and can be identified with L_H/Z_H. By the third isomorphism theorem, we see:

$$(L_H)^{\mathrm{ad}} = L_H/Z_{L_H} \cong (L_H/Z_H)/(Z_{L_H}/Z_H) = L_{H^{\mathrm{ad}}}/(Z_{L_H}/Z_H)$$

In other words, there is a short exact sequence:

$$0 \longrightarrow Z_{L_H}/Z_H \longrightarrow L_{H^{\mathrm{ad}}} \longrightarrow (L_H)^{\mathrm{ad}} \longrightarrow 0$$

By [51], it follows that there exists a related short exact sequence:

$$0 \longrightarrow X_*(Z_{L_H})/X_*(Z_H) \longrightarrow \pi_1(L_{H^{\mathrm{ad}}}) \longrightarrow \pi_1((L_H)^{\mathrm{ad}}) \longrightarrow 0$$

Taking the Γ-invariant subgroups is exact, giving us:

$$0 \longrightarrow X_*(Z_{L_H})^{\Gamma}/X_*(Z_H)^{\Gamma} \longrightarrow \pi_1(L_{H^{\mathrm{ad}}})^{\Gamma} \longrightarrow \pi_1((L_H)^{\mathrm{ad}})^{\Gamma} \longrightarrow 0$$

Now we consider a group-theoretic result.

Lemma 4.11.2. Let $\mathscr{X}$ be a formal scheme over W that is locally formally of finite type, with actions of group W-schemes H_1 and H_2. Suppose there is a short exact sequence of group schemes over W

$$0 \longrightarrow H_1/H_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0$$

Then

$$(\mathscr{X}/H_1) \times_{G_2} G_1 \cong \mathscr{X}/H_2$$

Proof. Let G be a group scheme acting on the formal scheme $\mathscr{X}$. By [54, Tag 0AIR], the fppf site of a formal scheme is subcanonical. This implies the étale site is subcanonical. So we may regard $\mathscr{X}$, H_i, and G_i $(i = 1, 2)$ as étale sheaves over $\operatorname{Spf} W$.

Fact: Suppose there is a set S with the action of groups H_1 and H_2. If there is a short exact sequence of groups

$$0 \longrightarrow H_1/H_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0$$

Then

$$(S/H_1) \times_{G_2} G_1 \cong S/H_2$$

□

For us, we have $\mathscr{X} = \mathrm{RZ}_{L_H}$, $H_1 = X_*(Z_{L_H})^\Gamma$, $H_2 = X_*(Z_H)^\Gamma$, $G_1 = \pi_1(L_{H^{\mathrm{ad}}})^\Gamma$ and $G_2 = \pi_1((L_H)^{\mathrm{ad}})^\Gamma$. Thus, we find:

$$\mathrm{RZ}_{L_{G^{\mathrm{ad}}}} = \mathrm{RZ}_{L_{H^{\mathrm{ad}}}} \cong \mathrm{RZ}_{L_H^{\mathrm{ad}}} \times_{\pi_1((L_H)^{\mathrm{ad}})} \pi_1(L_{H^{\mathrm{ad}}})^\Gamma.$$

□

This establishes the following projection:

$$\mathrm{RZ}_{L_{G^{\mathrm{ad}}}} = \mathrm{RZ}_{L_{H^{\mathrm{ad}}}} \cong \mathrm{RZ}_{L_H^{\mathrm{ad}}} \times_{\pi_1((L_H)^{\mathrm{ad}})} \pi_1(L_{H^{\mathrm{ad}}})^{\Gamma} \xrightarrow{p_1} \mathrm{RZ}_{L_H^{\mathrm{ad}}} = \mathrm{RZ}_{L_G^{\mathrm{ad}}}.$$

Let $\mathrm{RZ}_{P_{H^{\mathrm{ad}}}} := \mathrm{RZ}_{P_H}/X_*(Z_H)^{\Gamma}$. This gives rise to the following diagram:

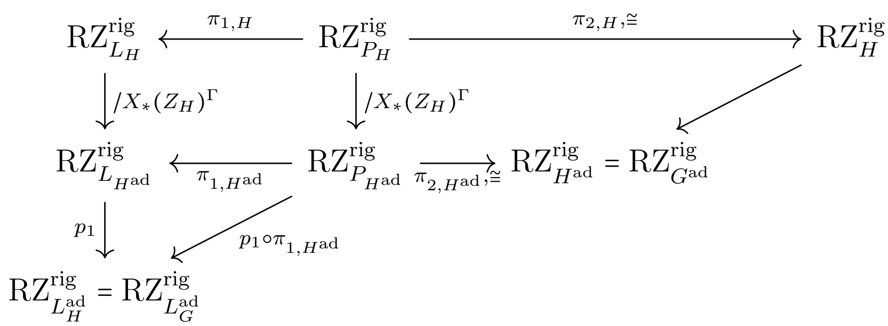

Figure 4.8: Extending to adjoint level.

In particular, since $\pi_{1,H}$ is a fibration of balls, so is $\pi_{1,H^{\mathrm{ad}}}$.

Let K_H^{ad} be the image of K_H under the map $H \longrightarrow H^{\mathrm{ad}}$. Let $K'^{\mathrm{ad}}_H = K_H^{\mathrm{ad}} \cap P_{H^{\mathrm{ad}}}(\mathbb{Z}_p)$. We use this construction to define

$$H^i(\mathrm{RZ}^{K'^{\mathrm{ad}}_H}_{P_{H^{\mathrm{ad}}}}) := H^i(\mathrm{RZ}^{K'_H}_{P_H}/X_*(Z_H)).$$

$$H^i(\mathrm{RZ}^{\infty}_{P_{H^{\mathrm{ad}}}}) := \varinjlim_{K'^{\mathrm{ad}}_H} H^i(\mathrm{RZ}^{K'^{\mathrm{ad}}_H}_{P_{H^{\mathrm{ad}}}})$$

By [6, Prop 4.3.2],

$$\mathrm{H}^{\bullet}(\mathrm{RZ}^{\infty}_{L_H})_{\rho} = \mathrm{H}^{\bullet}(\mathrm{RZ}^{\infty}_{P_H})_{\rho}.$$

Recall that by definition, $H^{\mathrm{ad}} = G^{\mathrm{ad}}$, $L_{H^{\mathrm{ad}}} = L_{G^{\mathrm{ad}}}$ and $P_{H^{\mathrm{ad}}} = P_{G^{\mathrm{ad}}}$. By the functoriality of $\mathrm{H}^\bullet$,

$$\mathrm{H}^\bullet(\mathrm{RZ}^\infty_{L_{G^{\mathrm{ad}}}})_\rho = \mathrm{H}^\bullet((\mathrm{RZ}_{L_H}/X_*(Z_H))^\infty)_\rho = \mathrm{H}^\bullet((\mathrm{RZ}_{P_H}/X_*(Z_H))^\infty)_\rho = \mathrm{H}^\bullet(\mathrm{RZ}^\infty_{P_{G^{\mathrm{ad}}}})_\rho.$$

Now, by [6, Prop 4.3.3], we know that :

$$\mathrm{RZ}_H^{K_H} = \bigsqcup_{K_H\backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p)} \mathrm{RZ}_{P_H}.$$

Observe that by definition, $K_H/Z_H \cong K_G/Z_G$. Thus, by the third isomorphism theorem:

$$K_H\backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p) \cong (K_H/Z_H)\backslash H^{\mathrm{ad}}(\mathbb{Q}_p)/P_{H^{\mathrm{ad}}}(\mathbb{Q}_p)$$
$$\cong (K_G/Z_G)\backslash G^{\mathrm{ad}}(\mathbb{Q}_p)/P_{G^{\mathrm{ad}}}(\mathbb{Q}_p) \cong K_{G^{\mathrm{ad}}}\backslash G^{ad}(\mathbb{Q}_p)/P_{G^{\mathrm{ad}}}(\mathbb{Q}_p).$$

We use this in the following manner:

$$RZ_{G^{\mathrm{ad}}}^{K_{G^{ad}}} = RZ_H^{K_H}/X_*(Z_H)^\Gamma = \bigsqcup_{K_H\backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p)} \mathrm{RZ}_{P_H}^{K_H}/X_*(Z_H)^\Gamma$$
$$= \bigsqcup_{K_{G^{\mathrm{ad}}}\backslash G^{\mathrm{ad}}(\mathbb{Q}_p)/P_{G^{\mathrm{ad}}}(\mathbb{Q}_p)} \mathrm{RZ}_{P_{G^{\mathrm{ad}}}}^{K_{G^{ad}}}.$$

It follows that

$$\mathrm{H}^\bullet(RZ_{G^{\mathrm{ad}}}) = \mathrm{Ind}_{P_{G^{\mathrm{ad}}}(\mathbb{Q}_p)}^{G^{\mathrm{ad}}(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}^\infty_{P_{G^{\mathrm{ad}}}})_\rho.$$

This completes the proof of the following theorem:

Theorem 4.11.3. Let $(G, [b_G], \{\mu_G\})$ be an unramified non-basic local Shimura datum of abelian type, which is Hodge-Newton reducible with respect to a fixed parabolic subgroup P_G and Levi factor L_G. Then for any admissible $\overline{\mathbb{Q}}_\ell$-representation ρ^{ad} of $J_{b_{G^{\mathrm{ad}}}}(\mathbb{Q}_p)$, we have

$$\mathrm{H}^\bullet(\mathrm{RZ}^\infty_{G^{\mathrm{ad}}})_\rho = \mathrm{Ind}_{P_{G^{\mathrm{ad}}}(\mathbb{Q}_p)}^{G^{\mathrm{ad}}(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}^\infty_{L_{G^{\mathrm{ad}}}})_\rho.$$

4.12 The abelian case

We combine our prior results to deduce the main theorem for the abelian case.

Theorem 4.12.1. Let $(G, [b_G], \{\mu_G\})$ be an unramified non-basic local Shimura datum of abelian type, which is Hodge-Newton reducible with respect to a fixed parabolic subgroup P_G and Levi factor L_G. Then for any admissible $\overline{\mathbb{Q}}_\ell$-representation ρ of $J_{b_G}(\mathbb{Q}_p)$, we have:

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho = \mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}_{L_G}^\infty)_\rho$$

as virtual representations of $G(\mathbb{Q}_p) \times \mathcal{W}_E$.

Proof. Let $K_G \subset G(Z_p)$ and $K_H \subset H(\mathbb{Z}_p)$ be corresponding open compact subgroups. Recall from the exposition on Lemma 4.3.2 that we have:

$$\mathrm{RZ}_H^{K_H} \cong \mathrm{RZ}_H^{K_H} \times_{\mathrm{RZ}_H^{\mathrm{rig}}} \mathrm{RZ}_{P_H}^{\mathrm{rig}} \cong \bigsqcup_{K_H \backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p)} \mathrm{RZ}_{P_H}^{K_H}.$$

In particular, by restricting to the + components, we obtain:

$$\mathrm{RZ}_G^{K_G,+} := \mathrm{RZ}_H^{K_H,+} = \bigsqcup_{K_H \backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p)} \mathrm{RZ}_{P_H}^{K_H,+}.$$

Observe that by definition, $K_H/Z_H \cong K_G/Z_G$. Thus, by the third isomorphism theorem:

$$K_H \backslash H(\mathbb{Q}_p)/P_H(\mathbb{Q}_p) \cong (K_H/Z_H) \backslash H^{\mathrm{ad}}(\mathbb{Q}_p)/P_{H^{\mathrm{ad}}}(\mathbb{Q}_p)$$
$$\cong (K_G/Z_G) \backslash G^{\mathrm{ad}}(\mathbb{Q}_p)/P_{G^{\mathrm{ad}}}(\mathbb{Q}_p) \cong K_G \backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p).$$

Hence, we can rewrite the expression as

$$\mathrm{RZ}_G^{K_G,+} = \bigsqcup_{K_G \backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p)} \mathrm{RZ}_{P_H}^{K_G,+}.$$

We now utilize the action of $J_{b_G}(\mathbb{Q}_p)$ to get the full abelian case.

$$\begin{aligned}
\mathrm{RZ}_G^{K_G} &= J_{b_G}(\mathbb{Q}_p)\mathrm{RZ}_G^{K_G,+} \\
&= J_{b_G}(\mathbb{Q}_p) \bigsqcup_{K_G\backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p)} \mathrm{RZ}_{P_G}^{K_G,+} \\
&= \bigsqcup_{J_{b_G}(\mathbb{Q}_p)/J_{b_G}(\mathbb{Q}_p)^+} \bigsqcup_{K_G\backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p)} \mathrm{RZ}_{P_G}^{K_G,+} \\
&= \bigsqcup_{K_G\backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p)} \bigsqcup_{J_{b_G}(\mathbb{Q}_p)/J_{b_G}(\mathbb{Q}_p)^+} \mathrm{RZ}_{P_G}^{K_G,+} \\
&= \bigsqcup_{K_G\backslash G(\mathbb{Q}_p)/P_G(\mathbb{Q}_p)} \mathrm{RZ}_{P_G}^{K_G}.
\end{aligned}$$

Thus, we find:

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho = \mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}_{P_G}^\infty)_\rho.$$

Earlier, we showed in Proposition 4.10.2 that

$$\mathrm{H}^\bullet(\mathrm{RZ}_{P_G}^\infty)_\rho = \mathrm{H}^\bullet(\mathrm{RZ}_{L_G}^\infty)_\rho.$$

We combine these two results to get

$$\mathrm{H}^\bullet(\mathrm{RZ}_G^\infty)_\rho = \mathrm{Ind}_{P_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{H}^\bullet(\mathrm{RZ}_{L_G}^\infty)_\rho.$$

□

Remark 4.12.2. The authors in [55] have recently proved the Harris-Viehmann conjecture in the Hodge–Newton reducible case using very different methods, by studying the moduli stack of parabolic bundles on the Fargues-Fontaine curve.

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