DISSERTATION

APPLICATIONS OF GENERALIZED INFERENCE

Submitted by

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In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY AMANY HASSAN ABDEL-KARIM ENTITLED APPLICATIONS OF GENERALIZED INFERENCE BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

APPLICATIONS OF GENERALIZED INFERENCE

There are many statistical inference problems in mixed linear models for which exact solutions are not available. In these cases, standard statistical software packages such as SAS and S-PLUS either use asymptotic procedures or methods based on the Satterthwaite-type approximations.

In this dissertation, it is shown that generalized inference (generalized pivotal quantity, generalized confidence interval, generalized test variable and generalized P-value) may be used as an alternative to asymptotic approximations or other small sample approximations. After presenting the definitions pertaining to generalized inference, we show how to extend the concept of generalized confidence intervals to simultaneous generalized confidence intervals. We also show how to extend generalized tests of hypotheses to multiparameter problems. Specifically, the following problems were addressed.

1. All pairwise comparisons of cell-means in unbalanced heterogeneous one-way ANOVA (Extended Tukey)

2. Pairwise comparisons of treatment means to a control mean in unbalanced heterogeneous one-way ANOVA (Extended Dunnett)

3. All cell-means in balanced two-factor crossed mixed linear model.

4. All pairwise comparisons of cell-means in balanced three-factor nested factorial mixed linear model.

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Other applications considered in this dissertation include the use of generalized confidence intervals to compare two non-nested linear models when the response and the predictors have jointly a multivariate normal distribution.

Generalized tests were developed for the following problems.

1. Testing the equality of the cell means in unbalanced heterogeneous one-way ANOVA

2. Testing the equality of the cell means in balanced three-factor crossed mixed linear model with interactions.

Simulation studies were used to estimate the error rates for each problem considered. Comparisons to other existing procedures were carried out when appropriate.

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Chapter 1

INTRODUCTION

Statistical inference plays a key role in many practical applications involving decision making in the presence of partial or uncertain information. Point estimation, confidence intervals, and hypotheses testing, are the three main modes of statistical inference. Exact confidence intervals and tests can be constructed if there exist appropriate pivotal quantities for parameters of interest. If appropriate pivotal quantities do not exist, approximate methods are used to construct confidence intervals and hypotheses tests.

Generalized likelihood ratio tests are a class of tests that can be developed in very general situations. Except in simple situations, they are not exact tests but are usually asymptotically efficient. The distribution of the generalized likihood ratio statistic is typically intractable, and hence asymptotic arguments are used to provide a justification for their use in large samples.

In the context of mixed linear models, Satterthwaite (1941, 1946) introduced approximate methods that can be used for testing and constructing confidence intervals when the variance of the parameter estimate can be expressed as a linear combination of expected mean squares. However, Satterthwaite's method is not satisfactory if the coefficients of the linear combination of expected mean squares are not all of the same sign. Several studies have been conducted to address such situations but still many practitioners are using Satterthwaite and other unsatisfactory approximate methods because there are no standard software packages that can be used in other procedures.

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In 1989, Tsui and Weerahandi introduced the concept of a generalized test variable (GTV) and generalized P-value (GPV) for testing one-sided hypotheses. They applied these concepts to compare means of two exponential distributions and to test the mean of the truncated exponential distribution. Weerahandi (1993) extended the idea of GPV and introduced the concepts of generalized pivotal quantity (GPQ) and generalized confidence interval (GCI). He used the GCI to construct confidence intervals for the variance of random effect in a random effect model. Since then several studies have been conducted to solve various problems using generalized inference.

In 1994, Zhou and Mathew used the concept of GPV for testing hypotheses in balanced and unbalanced cases (unequal cell frequencies) to test the significance of variance component in mixed linear model and to compare the random effects variance components in two independent mixed linear models. In 2000, Chang and Huang used the concepts of GCIs for largest mean, largest quantile, and largest signal-to-noise ratio when there are at least two normal populations. Chiang (2001) introduced the method of surrogate variables which is a systematic approach for constructing confidence intervals for functions of variance components in balanced mixed effects linear models. The intervals are exactly the same as generalized confidence intervals.

In 2002, Iyer and Patterson introduced a simple, but general, recipe for constructing GPQs which can be used to obtain GCI, GTV and GPV of a single unknown parameter for large class of practical problems. They applied the recipe to many problems and demonstrated that all previously published GCIs could be obtained using their recipe. Witkovský (2002) extended the concepts of GCI and GPV to construct approximate simultaneous confidence intervals and to test simultaneously all contrasts of means, in a one-way classification model (generalized Scheffé). He also constructed conservative GCIs for all pairwise mean differences simultaneously and tested all pairwise comparisons of means based on GPV.

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In 2003, Krishnamoorthy and Mathew used the ideas of GPV and GCI to construct exact confidence intervals and tests for a single lognormal mean of small and large samples. They also constructed intervals and tests for the ratio (or the difference) of two lognormal means. In 2004, they developed one-sided tolerance limits in balanced and unbalanced one-way random models, for observable random variable X, $X \sim N(\mu, \sigma_\tau^2 + \sigma_e^2)$ and unobservable random effect $\mu + \tau$, $\tau \sim N(0, \sigma_\tau^2)$.

In 2004, Gamage et al. used the concept of GPV to test the equality of mean vectors for two multivariate normal populations with unequal covariance matrices. Lee and Lin (2004) constructed GCI and GPV for the ratio of means of two normal populations.

Hannig et al. (2004) extended the recipe of Iyer and Patterson (2002) and provided a general construction procedure for a special class of Generalized Pivotal Quantities which they called Fiducial Generalized Pivotal Quantities (FGPQ). They further established a connection between GCIs and Fisher's fiducial inference. In addition they proved a theorem which establishes the asymptotic exactness of GCIs under fairly general conditions.

Most of the previous works dealt with applying the concepts of generalized inference to solve problems concerned one parameter. This dissertation is aimed at developing a systematic way for constructing simultaneous GCIs based on FGPQs for more than one parameter and apply it to unbalanced heterogeneous one-way ANOVA and balanced mixed linear models. The GCI was also used to compare two non-nested linear models. The GTV was constructed to test more than one parameter simultaneously in unbalanced heterogeneous one-way ANOVA and balanced mixed linear models.

The dissertation is organized as follows. Basic definitions of mathematical statistics and terminology involving generalized inference are given in Chapter 2. Chapter 3 contains basic statistical results that are used throughout the dissertation.

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Chapter 4 concerns one-way ANOVA. Simultaneous generalized confidence intervals were developed for all pairwise mean differences (extension of Tukey's method). The performance of the suggested method was assessed by computer simulation and the results were compared with Dunnett's (1980) results using other methods rather GCIs. Simultaneous confidence intervals for comparing all treatment means with a single control (extension of Dunnett's procedure) were also introduced. The performance of the suggested method was tested by computer simulation. Data examples were used for each problem to illustrate the computations.

In Chapter 5, GCIs in balanced mixed linear models are introduced. We dealt with two problems. The first problem is construction of simultaneous GCIs for all means in two-factor crossed balanced mixed linear model. Computer simulation was performed to assess the performance of the suggested method. Second problem is constructing simultaneous GCIs for all pairwise mean differences in three-factor nested factorial balanced mixed linear model. Computer simulation was performed to assess the performance of the suggested method and to compare the results with SAS. Data example was used for each problem to illustrate the procedures.

Chapter 6 deals with using GCIs to compare two non-nested linear models by developing a GCI for the ratio of residual variances of the two models. Computer simulation was conducted to assess the performance of the suggested method and to compare the results with Ahlbrandt (1988) using other methods rather GCIs. A data example was used to show how the procedure works in applications.

In Chapter 7 we consider development of a GTV for testing vector of parameters and applied it to two problems. The first problem was testing the equality of all means in an unbalanced heterogeneous one-way ANOVA. Computer simulation was carried out to assess the performance of the suggested method. A data example was used to show how the procedure works and to compare the result with Weerahandi's (1995) result for the same data example. The second problem was testing the

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equality of all means in three-factor crossed balanced mixed linear model. Computer simulation was carried out to assess the performance of the suggested method and to compare the results with Satterthwaite's method for solving this problem using SAS. A data example was used to show how the procedure works. Finally, Chapter 8 provides some concluding remarks.

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Chapter 2

GENERALIZED INFERENCE DEFINITIONS AND NOTATIONS

In this chapter we provide the basic definitions related to generalized pivotal quantity (GPQ), generalized confidence interval (GCI), generalized test variable (GTV) and generalized P value (GPV). These concepts are generalizations of the established concepts such as pivotal quantity, confidence interval, test statistics and P value. Section 1 and Section 2 of this chapter present the basic definitions. Section 3 outlines a recipe for a systematic way to construct generalized inference for one parameter as presented by Iyer and Patterson (2002).

2.1 Basic Definitions

Basic definitions related to statistical inference are presented according to Lehmann (1986) and Casella and Berger (2002) as the sources.

Let $\boldsymbol{X}$ be a vector of observable data with realizations $\boldsymbol{x}$ where the distribution of $\boldsymbol{X}$ depends on vector of unknown parameters ξ. Suppose $\xi = (\theta, \eta)$ where θ is a scalar parameter of interest and η is a nuisance parameter (possibly vector).

Pivotal Quantity

Suppose $Q(\boldsymbol{X}; \theta)$ has a distribution that does not depend on the unknown parameters. Then $Q(\boldsymbol{X}; \theta)$ is said to be a pivotal quantity for θ.

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Confidence Interval

Suppose $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$ are functions of $\boldsymbol{X}$ such that $P_\theta[L(\boldsymbol{X}) \leq \theta \leq U(\boldsymbol{X})] = 1-\alpha$. The random interval $[L(\boldsymbol{X}), U(\boldsymbol{X}))]$ is called an interval estimator for θ where $1-\alpha$ is the confidence coefficient.

Test Statistic

Suppose H_0 is a null hypothesis. A test statistic for testing H_0 is a statistic used to define the rejection region associated with the test.

***P*-value**

The P-value associated with a hypothesis test is the smallest significance level at which the hypothesis would be rejected with probability 1 for given observations.

2.2 Generalized Inference

Statistical inference based on generalized pivotal quantity, generalized confidence interval, generalized test variable, or generalized P value is called generalized inference. Define $\boldsymbol{\xi} = (\theta, \eta)$ where θ is a scalar parameter of interest and η is the nuisance parameter (possibly a vector). Let $\boldsymbol{X}$ to be the vector of observable data with observed values $\boldsymbol{x}$.

The following definitions are obtained from Weerahandi (1989, 1991, 1993, 1995), Chang and Huang (2000), Iyer and Patterson (2002) and Hannig et.al. (2004).

Generalized Pivotal Quantity (GPQ)

Consider $R = r(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ be a function of $(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$. Then R is said to be a generalized pivotal quantity for θ if it satisfies the following two properties:

Property A: The probability distribution of R is free of unknown parameters.

Property B: The observed value of the generalized pivotal quantity, defined as $r_{obs} = r(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{\xi})$, does not depend on the nuisance parameters η.

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Fiducial Generalized Pivotal Quantity (FGPQ)

In property B above, suppose $r(\boldsymbol{x};\boldsymbol{x},\boldsymbol{\xi}) = \theta$. Then $r(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi})$ is called a Fiducial generalized pivotal quantity.

Generalized Confidence Interval

Suppose $R(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi})$ is a FGPQ for θ. An equal-tailed two sided GCI for θ is given by $R_{\frac{\alpha}{2}} \leq \theta \leq R_{1-\frac{\alpha}{2}}$ where R_γ is the $100\gamma\%$ of the distribution of R.

Remark: A generalized confidence interval is an interval obtained using a generalized pivotal quantity, so it is not necessarily an exact confidence interval, i.e., the confidence coefficient is not necessarily $1-\alpha$.

Generalized Confidence Region (GCR)

Let Θ be the parameter space of θ. Define $C_{1-\alpha}$ to be the region satisfying $Pr[R \in C_{1-\alpha}] = 1-\alpha$. Then the $100(1-\alpha)\%$ generalized confidence region for θ is the subset of the parameter space given by $\{\theta \in \Theta | r_{obs} \in C_{1-\alpha}\}$.

Generalized Test Variable (GTV).

Weerahandi (1989, 1993) considered the problem of testing one sided null hypotheses of the form $H_0 : \theta \leq \theta_0$ against $H_a : \theta > \theta_0$. Let $\boldsymbol{X}$ be a random quantity having a density function $f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\xi})$. A generalized test variable is defined to be a function of the form $T(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi})$ satisfying the following three requirements:

- $t_{obs} = T(\boldsymbol{x};\boldsymbol{x},\boldsymbol{\xi})$ is free of η.
- For fixed $\boldsymbol{x}$ and $\boldsymbol{\xi}_0 = (\theta_0, \eta)$, the distribution of $T(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi}_0)$ is free of the nuisance parameters η.
- For fixed $\boldsymbol{x}$ and $\boldsymbol{\eta}$, $Pr[T(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi}) \geq t|\theta]$ is nondecreasing function of θ that is $T(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi})$ said to be stochastically increasing in θ.

Weerahandi (1995) considered the problem of testing two sided null hypotheses. Let $\boldsymbol{X}$ be a random vector having a density function $f_{\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{\xi})$. A generalized test variable is a function of the form $T(\boldsymbol{X};\boldsymbol{x},\boldsymbol{\xi})$ that satisfies the following requirements:

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- For fixed $\boldsymbol{x}$ and $\boldsymbol{\xi}_0 = (\theta_0, \eta)$, the distribution of $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}_0)$ does not depend on the nuisance parameters η.

- $t_{obs} = T(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{\xi})$ does not depend on nuisance parameters η.

- $Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}) \geq t\right] \geq Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}_0) \geq t\right]$ for all values of θ and given any fixed t,$\boldsymbol{x}$, and η.

Generalized P-value (GPV)

Suppose $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is a generalized test variable for testing the null hypothesis $H_0 : \theta \leq \theta_0$ against $H_a : \theta > \theta_0$. The generalized $P-$value can be calculated as follows.

- If $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is stochastically increasing in θ, then $GPV = Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}) \geq t_{obs} | \theta = \theta_0\right]$.

- if $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is stochastically decreasing in θ, then $GPV = Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}) \leq t_{obs} | \theta = \theta_0\right]$.

Suppose $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is a generalized test variable for testing the null hypothesis $H_0 : \theta \geq \theta_0$ against $H_a : \theta < \theta_0$. The generalized $P-$value can be calculated as follows.

- If $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is stochastically increasing in θ, then $GPV = Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}) \leq t_{obs} | \theta = \theta_0\right]$.

- If $T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi})$ is stochastically decreasing in θ, then $GPV = Pr\left[T(\boldsymbol{X}; \boldsymbol{x}, \boldsymbol{\xi}) \geq t_{obs} | \theta = \theta_0\right]$.

Remark

If the problem of testing two-sided null hypotheses, then $GPV = Pr\left[T \geq t_{obs} | \theta = \theta_0\right]$ as long as the GTV tends to take large values when H_0 is not true.

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2.3 Iyer and Patterson Recipe (2002)

Iyer and Patterson (2002) introduced a simple recipe for calculating generalized pivotal quantity, generalized confidence intervals, generalized test and generalized P-value. This recipe can be used to solve many problems. The recipe says:

Define $\boldsymbol{D}$ to be the observable data vector with observed values $\boldsymbol{d}$, $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_k) \in \Omega \subseteq R^k$, and $\vartheta = h(\xi_1, \cdots, \xi_k)$ be a scalar function of $\boldsymbol{\xi}$ for which a confidence interval is required.

Suppose that the conditions (a) and (b) given below hold:

(a) There exist functions $f_1, \cdots, f_k$, with $f_j : R^k \times R^k \to R$; such that, if you define $U_1, \cdots, U_k$ by $U_i = f_i(\boldsymbol{D}; \boldsymbol{\xi})$, $i = 1, \cdots, k$. Then $\boldsymbol{U} = (U_1, \cdots, U_k)'$ has a joint distribution that is free of $\boldsymbol{\xi}$.

(b) For each $\boldsymbol{D}$, there exists a mapping $g(\boldsymbol{D}, .) \; : \; R^k \to R^k$ such that: $\boldsymbol{g}(\boldsymbol{D}; f(\boldsymbol{D}, \boldsymbol{\xi})) = \boldsymbol{\xi}$ where $\boldsymbol{f} = (f_1, \cdots, f_k)$ i.e., $(g_1(\boldsymbol{D}; f(\boldsymbol{D}; \boldsymbol{\xi})), \cdots, g_k(\boldsymbol{D}; f(\boldsymbol{D}; \boldsymbol{\xi}))) = \boldsymbol{\xi}$. Define:

$$\begin{aligned} R &= R(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) \\ &= h(g_1(\boldsymbol{d}; f(\boldsymbol{D}, \boldsymbol{\xi})), \cdots, g_k(\boldsymbol{d}; f(\boldsymbol{D}, \boldsymbol{\xi}))) \\ &= h(g_1(\boldsymbol{d}; \boldsymbol{U}), \cdots, g_k(\boldsymbol{d}; \boldsymbol{U})) \end{aligned} \tag{2.1}$$

then; the following statements are true:

(1) R is a generalized pivotal quantity for $\vartheta = h(\boldsymbol{\xi})$.

(2) $R_{\alpha/2} \leq \vartheta \leq R_{1-\alpha/2}$ is an equal tailed two sided generalized confidence interval for ϑ (one sided generalized confidence bounds are obtained in an obvious manner).

(3) The generalized test variable for testing $H_0 : \vartheta \leq \vartheta_0$ against $H_a : \vartheta > \vartheta_0$, and the generalized p-value are as follows.

$$\begin{aligned} T &= T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) \\ &= h(\boldsymbol{\xi}) - R \end{aligned}$$

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$$
\begin{aligned}
&= \vartheta - R \qquad (2.2)\\
GPV &= pr\left\{T(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) \geq 0 | \vartheta = \vartheta_0\right\} \quad (t=0) \qquad (2.3)
\end{aligned}
$$

Remark (1)

The required percentiles of R can be estimated using Monte-Carlo methods or analytical approximations.

Remark (2)

The previous procedure for obtaining generalized confidence intervals or tests is valid when $(U_1, \cdots, U_k)$ are mutually independent or if they are not independent.

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Chapter 3

KEY RESULTS FROM LINEAR MODELS AND MATHEMATICAL STATISTICS

This chapter contains a collection of basic key results that are used in this dissertation. The first section states some important matrix results. The second section outlines some basic results on sufficiency, completeness and uniformly minimum variance unbiased estimator (UMVUE). A summary of important results concerning linear and quadratic forms in normal random variables is given in Section 3. Section 4 gives an overview of the Tukey and Dunnett simultaneous confidence interval procedures for the balanced one-way classification model. Section 5 outlines the notation and some main results about balanced mixed linear models. Section 6 deals with some basic results for multivariate normal distribution.

3.1 Matrix Results

3.1.1 Kronecker Products of Matrices

The discussion of balanced linear models are facilitated by using Kronecker products of matrices. The following definition and properties are according to Hocking (1996). The Kronecker (direct) product of the $p \times q$ matrix $\boldsymbol{A}$ and $m \times n$ matrix $\boldsymbol{B}$ is the matrix of size $pm \times qn$ denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$ and defined by

$$\boldsymbol{A} \otimes \boldsymbol{B} = (a_{ij}\boldsymbol{B}). \tag{3.1}$$

Thus $\boldsymbol{A} \otimes \boldsymbol{B}$ is obtained by replacing each element of the matrix $\boldsymbol{A}$ by the matrix $\boldsymbol{B}$ multiplied by that element. The essential properties of Kronecker products are given below.

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- $(\boldsymbol{A} \otimes \boldsymbol{B})' = \boldsymbol{A}' \otimes \boldsymbol{B}'$.

- $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$.

- $\boldsymbol{A} \otimes (\boldsymbol{B} \otimes \boldsymbol{C}) = (\boldsymbol{A} \otimes \boldsymbol{B}) \otimes \boldsymbol{C}$.

- $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{AC} \otimes \boldsymbol{BD}$.

- $(\boldsymbol{A} + \boldsymbol{B}) \otimes (\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \otimes \boldsymbol{C} + \boldsymbol{A} \otimes \boldsymbol{D} + \boldsymbol{B} \otimes \boldsymbol{C} + \boldsymbol{B} \otimes \boldsymbol{D}$.

3.1.2 Some Matrix Identities

1. Let $\boldsymbol{U}_n = \boldsymbol{J}_n \boldsymbol{J}_n'$ denote a matrix of ones of size $n \times n$, $\boldsymbol{J}_n$ is a column vector of n ones and $\boldsymbol{S}_n = \boldsymbol{I}_n - \dfrac{\boldsymbol{U}_n}{n}$. Then

$$\begin{aligned} \boldsymbol{U}_n \boldsymbol{S}_n &= 0 \\ \boldsymbol{S}_n \boldsymbol{J}_n &= 0 \end{aligned}$$

(**Cholesky decomposition**) Assume that $\boldsymbol{A}$ is a $n \times n$ symmetric and positive definite matrix. Then $\boldsymbol{A} = \boldsymbol{S}'\boldsymbol{S}$ where $\boldsymbol{S}$ is an upper triangular matrix. $\boldsymbol{S}$ can be defined as

$$\begin{aligned} S_{11} &= \sqrt{a_{11}} \\ S_{1j} &= \frac{a_{1j}}{S_{11}}, \quad j = 2, \cdots, n \\ S_{ii} &= \sqrt{a_{11} - \sum_{k=1}^{i-1} S_{ki}^2}, \quad i > 1 \\ S_{ij} &= \frac{1}{S_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} S_{ki} S_{kj} \right), \quad j > i \\ S_{ij} &= 0, \quad i > j. \end{aligned} \tag{3.2}$$

In particular, $|\boldsymbol{A}| = \prod_{i=1}^{n} S_{ii}^2$ (Hocking, 1996).

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3.2 Basic Results about Sufficiency, Completeness and Uniformly Minimum Variance Unbiased Estimators

3.2.1 Complete and Sufficient Statistics in the Exponential Family

The following results about sufficient and complete statistics were obtained from Lehmann and Casella (1998).

Exponential Family: A family P_θ of distributions is said to form an s-dimensional exponential family if the distributions P_θ have densities of the form

$$P_\theta(x) \quad = \quad exp\left(\sum_{i=1}^{s} \eta_i(\theta) T_i(x) - \beta(\theta)\right) h(x) \tag{3.3}$$

with respect to some common measure μ. Here, η_i and β are real-valued functions of the parameters and T_i is real-valued statistic, and x is a point in the sample space χ, the support of the density. Use η_i as the parameter, the density can be expressed in the canonical form as follows.

$$p(x|\eta) \quad = \quad exp\left(\sum_{i=1}^{s} \eta_i T_i(x) - A(\eta)\right) h(x) \tag{3.4}$$

where $\eta_1, \cdots, \eta_s$ are called the natural parameters and $e^{A(\eta)} = \int e^{\sum_{i=1}^{s} \eta_i T_i(x)} h(x) d\mu(x)$.

Full rank exponential family: If the exponential family has the form of Equation (3.4) is minimal in the sense that neither T's nor the η's satisfy a linear constraint and the parameter space contains an s-dimensional rectangle, then the exponential family is said to be a full rank.

Sufficient statistics and the Factorization Theorem: A necessary and sufficient condition for a statistic T to be sufficient for a family $P = \{P_\theta, \theta \in \Omega\}$ of x dominated by a σ-finite measure μ is that there exist non-negative functions g_θ and h such that the densities p_θ of P_θ satisfy,

$$p_\theta(x) \quad = \quad g_\theta\left[T(x)\right] h(x) \ (a.e.\mu)$$

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Definition of complete statistic: Let $f(t|\theta)$ be a family of pdfs for a statistic $T(\boldsymbol{X})$. The family of probability distributions is called complete if $E_\theta g(T) = 0$ for all θ implies $p_\theta(g(T) = 0) = 1$ for all θ. Which equivalent to $T(\boldsymbol{X})$ is called a complete statistic.

Theorem on Complete statistics: If X is distributed with density in Equation (3.4), and the family is full rank, then $T = (T_1, \cdots, T_s)$ is complete.

Proposition on Minimal sufficient statistics: If X is distributed with density Equation (3.4), then $T = (T_1, \cdots, T_s)$ is minimal sufficient provided that the family satisfies one of the following conditions:

(a) It is of full rank.

(b) The parameter space contains $s+1$ points η^j $(j = 0, \cdots, s)$. Which span E_s in the sense that they do not belong to a proper affine subspace of E_s where E_s is s-dimensional Euclidean space.

Procedure for calculating minimal sufficient statistics: Let P be a finite family with densities p_i, $i = 0, 1, 2, \cdots, k$, all having the same support. Then, the statistic

$$T(X) = \left(\frac{p_1(X)}{p_0(X)}, \frac{p_2(X)}{p_0(X)}, \cdots, \frac{p_k(X)}{p_0(X)}\right)$$

is minimal sufficient.

3.2.2 Uniformly Minimum Variance Unbiased Estimator (UMVUE)

The following result about UMVUE is obtained from Hocking (1996).

Lehmann-Scheffé theorem: Suppose $\boldsymbol{t}$ is a complete and sufficient vector for the vector of unknown parameters $\boldsymbol{\theta}$. Assume that there exists $\boldsymbol{p}(\boldsymbol{t})$ such that $E[\boldsymbol{p}(\boldsymbol{t})] = \boldsymbol{\theta}$. Then $\boldsymbol{p}(\boldsymbol{t})$ is UMVUE for $\boldsymbol{\theta}$.

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3.3 Results about Linear and Quadratic Forms

Hocking (1996) was used as the reference for the results in this section. Let $\boldsymbol{Y}$ be an N-vector of random variables such that $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$. Let $\boldsymbol{A}$ (symmetric) and $\boldsymbol{B}$ be matrices of known constants having dimension $N \times N$ and $t \times N$, respectively. Then the following results are hold.

1. Let $\boldsymbol{L} = \boldsymbol{BY}$ be a linear form in $\boldsymbol{Y}$. Then $\boldsymbol{L}$ has a normal distribution with

$$\begin{aligned} E(\boldsymbol{L}) &= \boldsymbol{B\mu} \\ Var(\boldsymbol{L}) &= \boldsymbol{BVB}' \end{aligned}$$

   In the case where $\boldsymbol{B}$ does not have full row rank, $Var(\boldsymbol{L})$ is a positive semidefinite matrix whose rank is the rank of $\boldsymbol{B}$.

   Further, if $\boldsymbol{L}_1 = \boldsymbol{B}_1\boldsymbol{Y}$ and $\boldsymbol{L}_2 = \boldsymbol{B}_2\boldsymbol{Y}$ are two linear forms, the matrix of covariances of elements of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ is given by

$$Cov(\boldsymbol{L}_1, \boldsymbol{L}_2) = \boldsymbol{B}_1\boldsymbol{V}\boldsymbol{B}_2'$$

2. Let $\boldsymbol{Q}_1 = \boldsymbol{Y}'\boldsymbol{A}_1\boldsymbol{Y}$ and $\boldsymbol{Q}_2 = \boldsymbol{Y}'\boldsymbol{A}_2\boldsymbol{Y}$ be quadratic forms with matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, then

$$Cov(\boldsymbol{Q}_1, \boldsymbol{Q}_2) = 2tr(\boldsymbol{A}_1\boldsymbol{V}\boldsymbol{A}_2\boldsymbol{V}) + 4\boldsymbol{\mu}'\boldsymbol{A}_1\boldsymbol{V}\boldsymbol{A}_2\boldsymbol{\mu}$$

   If $\boldsymbol{L} = \boldsymbol{BY}$ is a linear form and $Q = \boldsymbol{Y}'\boldsymbol{AY}$ is a quadratic form, then

$$Cov(\boldsymbol{L}, \boldsymbol{Q}) = 2\boldsymbol{BVA\mu}$$

3. If $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$ and $\boldsymbol{Q} = \boldsymbol{Y}'\boldsymbol{AY}$, then $\boldsymbol{Q} \sim \chi^2_{r,\lambda}$ with $r = r(\boldsymbol{A})$ and $\lambda = \frac{\boldsymbol{\mu}'\boldsymbol{A\mu}}{2}$ if and only if $\boldsymbol{AV}$ is idempotent.

4. If $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$, $\boldsymbol{Q}_1 = \boldsymbol{Y}'\boldsymbol{A}_1\boldsymbol{Y}$, and $\boldsymbol{Q}_2 = \boldsymbol{Y}'\boldsymbol{A}_2\boldsymbol{Y}$, then $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are independent if and only if $\boldsymbol{A}_1\boldsymbol{V}\boldsymbol{A}_2 = 0$.

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5. If $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$, $\boldsymbol{Q} = \boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ and $\boldsymbol{L} = \boldsymbol{B}\boldsymbol{Y}$, then $\boldsymbol{Q}$ and $\boldsymbol{L}$ are independent if and only if $\boldsymbol{B}\boldsymbol{V}\boldsymbol{A} = 0$.

   The following result follows from the general results above.

6. Let Y_i, $i = 1, 2, \cdots, N$ be a random sample from a normal distribution with mean μ and variance σ^2. The sample mean and the sample variance are given by

$$\begin{aligned} \bar{Y} &= \frac{1}{N}\sum_{i=1}^{N} Y_i \\ S^2 &= \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2. \end{aligned} \tag{3.5}$$

   where $E(\bar{Y}) = \mu$ and $Var(\bar{Y}) = \frac{\sigma^2}{N}$. The statistics $\bar{Y}$ and S^2 are independently distributed as $\bar{Y} \sim N(\mu, \frac{\sigma^2}{N})$ and $S^2 \sim \frac{\sigma^2}{N-1}\chi^2_{N-1}$.

7. Joint independence of linear and quadratic forms: Assume the vector of observations $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$. Consider the linear forms $\boldsymbol{W}_j = \boldsymbol{C}_j\boldsymbol{Y}$, $j = 1, 2, \cdots, l$ and quadratic forms $\boldsymbol{Q}_i = \boldsymbol{Y}'\boldsymbol{A}_i\boldsymbol{Y}$ $i = 1, 2, \cdots, k$. Suppose the quadratic forms are pairwise independent, the linear forms are pairwise independent, and the linear forms and quadratic forms are pairwise independent. Then $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\cdots$, $\boldsymbol{Q}_k$, $\boldsymbol{W}_1$, $\boldsymbol{W}_2$, $\cdots$, $\boldsymbol{W}_l$ are jointly independent.

3.4 A Summary of Tukey and Dunnett Procedures

The following is a summary of Tukey and Dunnett procedures for pairwise comparisons of means in the balanced one-way ANOVA. These procedures were originally developed for balanced, homoscedastic one-way classification models. We will extend these procedures in Chapter 4 to unbalanced and heteroscedastic situations.

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3.4.1 Tukey's Simultaneous CIs

Consider an experiment with data Y_{ij} satisfying one-way, fixed effects analysis of variance model

$$Y_{ij} = \mu_i + e_{ij}, \quad i = 1, \cdots, a, \quad j = 1, \cdots, n \tag{3.6}$$

where $e_{ij} \sim N(0, \sigma^2)$ and μ_i and σ^2 are unknown parameters. Dunnett (1980) mentioned that Tukey's multiple range test (1953) provides Tukey's simultaneous confidence intervals for $\frac{a(a-1)}{2}$ pairwise differences $\mu_i - \mu_j$ which have the form

$$(\bar{y}_{i.} - \bar{y}_{j.}) \pm c\, SE(\bar{y}_{i.} - \bar{y}_{j.})$$

where

$$SE(\bar{y}_{i.} - \bar{y}_{j.}) = \sqrt{\widehat{var}(\bar{y}_{i.} - \bar{y}_{j.})} = \sqrt{\frac{2s^2}{n}},$$

where $\bar{y}_i$ is the observed value of i^{th} sample mean, $\bar{Y}_i$, s_i^2 is the observed value of ith sample variance S_i^2, and $c = \frac{1}{\sqrt{2}} q(\alpha, a, (N-a))$, and $q(\alpha, a, (N-a))$ denotes the critical value from the studentized range table of a normal populations.

In 1956, Kramer extended Tukey's multiple range test to the case of unequal sample sizes. which provided the simultaneous confidence intervals for all pairwise comparisons of a population means.

$$(\bar{y}_{i.} - \bar{y}_{j.}) \pm q(\alpha; a, d) \frac{s}{\sqrt{2}} \left(\frac{1}{n_i} + \frac{1}{n_j} \right)^{\frac{1}{2}} \tag{3.7}$$

where $q(\alpha; a, d)$ is the upper $100(1-\alpha)$ percentile point of the distribution of the studentized range of a normal populations, d is the degrees of freedom and n_i and n_j are the sample sizes associated with groups i and j respectively.

3.4.2 Dunnett's CIs for Comparing all Means to a Control

Dunnett (1955) derived a method for constructing simultaneous confidence intervals for comparing the means of a set of treatments with the mean of a

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control treatment. Consider the observations X_{ij}, $i = 0, \cdots, p$; $j = 1, \cdots, N_i$ are independent and normally distributed with mean μ_i and common variance σ^2. The sample mean and sample variance are respectively $\bar{X}_{i.} = \sum_{j=1}^{N_i} \frac{X_{ij}}{N_i}$ and $S^2 = \sum_{i=0}^{p} \sum_{j=1}^{N_i} \frac{(X_{ij} - \bar{X}_{i.})^2}{n}$ with observed values $\bar{x}_{i.}$ and s^2 respectively where $n = (\sum_{i=0}^{p} N_i) - (p+1)$ and $\bar{X}_0$ is the sample mean for the control treatment. Define

$$Z_i = \frac{(\bar{X}_{i.} - \bar{X}_0) - (\mu_i - \mu_0)}{\sqrt{\frac{1}{N_i} + \frac{1}{N_0}}} \tag{3.8}$$

Define $T_i = \frac{Z_i}{S}$, $i = 1, \cdots, p$. The simultaneous confidence intervals for comparing all the means to a control with joint confidence coefficient $1 - \alpha$ are.

$$\bar{x}_{i.} - \bar{x}_0 \pm d_i'' s \sqrt{\frac{1}{N_i} + \frac{1}{N_0}}, \; i = 1, \cdots, p$$

where d_i'' can be calculated by

$$pr\left\{|T_1| < d_1'', \cdots, |T_p| < d_p''\right\} = 1 - \alpha \tag{3.9}$$

Dunnett designed the tabulated values based on equal sample sizes for all the treatments including the control one. The tables can be found in his paper for $1-\alpha = 95\%$ and $p \leq 9$.

3.5 Balanced Mixed Linear Models

The following is a summary of definitions and important results about balanced mixed linear models from Satterthwaite (1941, 1946) and Hocking (1996).
The balanced mixed linear models contain fixed and random factors with same number of replications to each factor combination. These models used to construct exact confidence intervals and exact tests for the fixed effects, variance components and functions of variance components.

If there are no exact confidence intervals nor tests, suggested procedure as Satterthwaite approximate method can be used. The following is a summary of concepts

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and theorems of balanced mixed linear models which will be used in chapters 5 and 7.

Define T to be the set contains all subsets of the factors in the model such that $T = \{T_F, T_R\}$ with elements $t = t_2\,(t_1)$ where T_F contains the subsets of fixed factors and T_R contains the subsets of random factors. Consider the mixed linear model has the form,

$$\boldsymbol{Y} \;=\; \boldsymbol{X\alpha} + \boldsymbol{Z\beta} + \boldsymbol{e} \tag{3.10}$$

where: $\boldsymbol{Y}$ is an observable random vector, $\boldsymbol{X}$ is a non-random design matrix; $\boldsymbol{X} = \left[\boldsymbol{X}_0|\cdots|\boldsymbol{X}_t|\cdots\right]$ where $\boldsymbol{X}_0 = 1\boldsymbol{j}$ and each matrix $\boldsymbol{X}_t$ is of size N by a_t, $a_t = \prod_{i\in t} a_i$ such that $\boldsymbol{X}_t = \boldsymbol{C}_1 \otimes \cdots \boldsymbol{C}_{k+1}$ where $\boldsymbol{C}_r = \boldsymbol{I}_{a_r}$ if $r \in t$ and $\boldsymbol{C}_r = \boldsymbol{J}_{a_r}$ if $r \notin t$. The vector $\boldsymbol{\alpha}$ is an unobservable parameters and it consists of all fixed effects $\boldsymbol{\alpha}_t$. The vector $\boldsymbol{Z}$ is a non random-design matrix, $\boldsymbol{\beta}$ is an unobservable random vector of parameters; $\boldsymbol{Z} = \left[\cdots|\cdots|\boldsymbol{Z}_t|\cdots|\cdots\right]$ where each matrix $\boldsymbol{Z}_t$ is of size N by a_t such that $\boldsymbol{Z}_t = \boldsymbol{C}_1 \otimes \cdots \boldsymbol{C}_{k+1}$ where $\boldsymbol{C}_r = \boldsymbol{I}_{a_r}$ if $r \in t$, $\boldsymbol{C}_r = \boldsymbol{J}_{a_r}$ if $r \notin t$ and $\boldsymbol{e}$ is an unobservable random vector with independent random variables which have normal distributions with zero means and a common variance $\phi_{\boldsymbol{e}}$. It is assumed that $\boldsymbol{\beta}_t \sim N\left(0, \phi_t \boldsymbol{I}\right)$ where ϕ_t is the variance component for factor t . All the random variables are assumed to be mutually independent. The expectation and the variance of $\boldsymbol{Y}$ have the following forms:

$$\begin{aligned}
E(\boldsymbol{Y}) &= \boldsymbol{X\alpha} \\
\boldsymbol{V} &= \sum_{t\in T_R} \phi_t V_t \\
\boldsymbol{V}_t &= \boldsymbol{L}_1 \otimes \cdots \otimes \boldsymbol{L}_{k+1} \\
\boldsymbol{L}_r &= \boldsymbol{I}_{a_r} \quad \text{if } r \in t \\
&= \boldsymbol{U}_{a_r} \quad \text{if } r \notin t
\end{aligned}$$

where $\boldsymbol{L}_{k+1} = \boldsymbol{U}_n$.

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Fundamental Results

ANOVA table is used to test fixed effects and variance components. Assume $\boldsymbol{Y} \sim N(\boldsymbol{X}\boldsymbol{\alpha}, \boldsymbol{V})$, define the sum square of factor t to have the form.

$$SS(t) \quad = \quad \boldsymbol{Y}^t \boldsymbol{A}_t \boldsymbol{Y} \tag{3.11}$$

where $\boldsymbol{A}_t = \boldsymbol{G}_1 \otimes \cdots \boldsymbol{G}_{k+1}$; $\boldsymbol{G}_r = \boldsymbol{S}_{a_r}$ if $r \in t_2$, $\boldsymbol{G}_r = \boldsymbol{I}_{a_r}$ if $r \in t_1$, and $\boldsymbol{G}_r = \frac{1}{a_r}\bar{\boldsymbol{U}}_{a_r}$ if $r \notin t$. The rank of $\boldsymbol{A}_t$ is $r_t = r(\boldsymbol{A}_t) = \left(\prod_{i \in t_1} a_i\right)\left(\prod_{j \in t_2}(a_j - 1)\right)$. Then,

$$\begin{aligned} t &= t_2(t_1) \in T_R, \ \frac{SS(t)}{\lambda_t} \sim \chi^2_{(r_t,0)} \ \ t \in T_R \\ t &= t_2(\boldsymbol{t}_1) \in T_F, \ \frac{SS(t)}{\lambda_t} \sim \chi^2_{(r_t, \frac{1}{2\lambda_t}, \boldsymbol{\alpha}_t' \boldsymbol{X}_t' \boldsymbol{X}_t \boldsymbol{\alpha}_t)} \ \ t \in T_F \end{aligned} \tag{3.12}$$

where

$$\lambda_t = \sum_{m \in T_R \text{ and } m \supseteq t} \frac{N}{a_m} \phi_m \tag{3.13}$$

with $\frac{N}{a_m} = \prod_{r \notin m} a_r$, $N = a_1 \cdots a_k n$. The statistics ($\hat{\boldsymbol{\alpha}}_t, t \in T_F$ and $SSm, m \in T_R$) are complete and sufficient statistics and also are mutually independent. The estimates $\hat{\boldsymbol{\alpha}}$, and $\hat{\phi}$ of α and ϕ respectively are minimum variance and unbiased estimates. The expectations of mean square errors for fixed and random effects have the forms:

$$\begin{aligned} EMS(t) &= \lambda_t + \frac{1}{r_t} \boldsymbol{\alpha}_t' \boldsymbol{X}_t' \boldsymbol{X}_t \boldsymbol{\alpha}_t \ \ t \in T_F \\ EMS(m) &= \lambda_m \ \ m \in T_R \end{aligned} \tag{3.14}$$

A solution to the normal equations $(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})\boldsymbol{\alpha} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{Y}$ is obtained by calculating $\hat{\boldsymbol{\alpha}}_t = \boldsymbol{H}_t \bar{\boldsymbol{Y}} \sim N(\boldsymbol{\alpha}_t, \lambda_t(\boldsymbol{X}_t'\boldsymbol{X}_t)^{-1})$, $t \in T_F$, $\bar{\boldsymbol{Y}}$ is the vector of the sample means and $\boldsymbol{H}_t = \boldsymbol{B}_1 \otimes \cdots \otimes \boldsymbol{B}_f$, f is the number of fixed effects, $\boldsymbol{B}_r = \boldsymbol{S}_{a_r}$ if $r \in t_2$,

$\boldsymbol{B}_r = \boldsymbol{I}_{a_r}$ if $r \in t_1$, and $\boldsymbol{B}_r = \frac{1}{a_r}\boldsymbol{J}'_{a_r}$ if $r \notin t$.

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Exact F-tests and approximate F-tests

Suppose we need to test that the fixed effect α_t is zero. Assume the expected mean square for α_t has the form $\lambda_t + Q(t)$ where $Q(t) = \frac{1}{r_t}\boldsymbol{\alpha}_t'\boldsymbol{X}_t'\boldsymbol{X}_t\boldsymbol{\alpha}_t$. Assume there exists a random effect $\boldsymbol{\alpha}_m$ such that $EMS(\boldsymbol{\alpha}_m) = \lambda_m$. Thus, under the null hypothesis,

$$F = \frac{MS(t)}{MS(m)} \sim F_{r_t, r_m} \tag{3.15}$$

Assume that such an m does not exist, then suppose that there is a linear combination of variance components which has the form $\lambda_t = \sum_{m \in T_R} g_m \lambda_m$.
Let $MS^* = \sum_{m \in T_R} g_m MS_m$. Under the null hypothesis, the Satterthwaite approximation method is as follows:

$$F = \frac{MS(t)}{MS^*} \sim F_{r_t, \nu^*} \tag{3.16}$$

where $\nu^* = \frac{(\sum_{m \in T_R} g_m MS_m)^2}{\sum_{m \in T_R} \frac{g_m^2 MS_m^2}{\nu_m}}$ is the Satterthwaite's approximate degrees of freedom for MS^*.

Approximate confidence intervals for fixed effects: Let θ be a function of fixed effects and let $\hat{\theta}$ be the estimate of θ. Assume $Var(\hat{\theta}) = \sum_{m \in T_R} g_m EMS_m = \eta^2$ and the estimate of it is $\widehat{Var}(\hat{\theta}) = \hat{\eta}^2$. Then the approximate $(1-\alpha)$ confidence interval for θ has the form

$$\hat{\theta} \mp t_{1-\alpha/2:\nu^*} \hat{\eta} \tag{3.17}$$

where ν^* is the Satterthwaite's degrees of freedom. It is well known that the approximation does not always work well if η^2 is a linear combination of EMS with both positive and negative coefficients. Otherwise, the approximation works well.

3.6 Results about the Multivariate Normal Distribution

The results of this section were taken from Anderson (1984) and Casella and Berger (2002).

The vector $\boldsymbol{X}$ of size p has a multivariate normal distribution iff

$$\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{AZ} \tag{3.18}$$

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where

$$\begin{aligned}\boldsymbol{\mu} &= E(\boldsymbol{X}) \\ \boldsymbol{\Sigma} &= Var(\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{A}'\end{aligned}$$

where $\boldsymbol{\Sigma}$ is a positive definite matrix, $\boldsymbol{A}$ is a lower triangular matrix and $\boldsymbol{Z}$ is a vector of independent standard normal random variables. The density of $\boldsymbol{X}$ has the form

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}\left[det(\boldsymbol{\Sigma})\right]^{\frac{p}{2}}} exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \tag{3.19}$$

Notice that $\boldsymbol{Z}$ can be expressed as $\boldsymbol{Z} = \boldsymbol{A}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})$.

Theorem: Consider $\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_{N-1}$ $(N > p)$ to be independent such that $\boldsymbol{Z}_i \sim N(\boldsymbol{0}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ and $\boldsymbol{A}$ is a lower triangular matrix. Define M to be the sample covariance matrix and has the form

$$\begin{aligned}\boldsymbol{M} &= \frac{1}{N-1}\sum_{i=1}^{N}\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)' \\ &= \frac{1}{N-1}\boldsymbol{S} \\ &= \frac{1}{N-1}\sum_{i=1}^{N-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\end{aligned}$$

where $\boldsymbol{X}_i \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The density of $\boldsymbol{S}$ has the form

$$\frac{|S|^{\frac{1}{2}((N-1)-p-1)}\, exp\left(-\frac{1}{2}tr\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)}{2^{\frac{1}{2}(N-1)p}\pi^{\frac{p(1-p)}{4}}\,|\boldsymbol{\Sigma}|^{\frac{1}{2}(N-1)}\prod_{i=1}^{p}\Gamma\frac{1}{2}\left[(N-1)+1-i\right]} \tag{3.20}$$

The distribution associated with the density in Equation (3.20) is called $W_p(\boldsymbol{\Sigma}, N-1)$. The matrix $\boldsymbol{S}$ can be expressed as $\boldsymbol{S} = \boldsymbol{B}\boldsymbol{B}'$ where $b_{ij} = 0,\ i < j$. The distribution of $M = \frac{1}{N-1}\boldsymbol{S}$ is $W_p\left(\frac{1}{N-1}\boldsymbol{\Sigma}, N-1\right)$.

Bartlett Decomposition: Let $\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_{N-1}$ be independent random variables such that $\boldsymbol{Z}_i \sim N(\boldsymbol{0}, \boldsymbol{I})$. Then $\boldsymbol{S} = \sum_{i=1}^{N-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'$ may be written as $\boldsymbol{U}\boldsymbol{U}'$ where $u_{11}, u_{21}, \cdots, u_{pp}$ are independent such that $u_{ij} = 0,\ i < j$; $u_{ij} \sim N(0,1),\ i > j$ and $u_{ii}^2 \sim \chi^2_{(N-1)-i+1}$ degrees of freedom. Note that $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{U}$.

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Chapter 4

SIMULTANEOUS GCIS IN ONE-WAY CLASSIFICATION MODEL

In this chapter we constructed simultaneous CIs for all pairwise mean differences and comparing all treatment means to a control in unbalanced one-way ANOVA with unequal variances. Section 1 is a review of previous work concerned the simultaneous CIs for all pairwise mean differences or multiple pairwise comparisons with a control in One-way classification model. Section 2 discusses an extension of Tukey's simultaneous confidence intervals for all pairwise differences of treatment means using generalized inference. Section 3 concerns a simulation study. Section 4 presents an example. Section 5 discusses an extension, using generalized inference, of the Dunnett simultaneous intervals for comparing treatment means to a control. Section 6 deals with a simulation study. Section 7 presents an example.

4.1 A Review for Multiple Comparisons of the Means in One-way Classification Model

Several studies have concerned the construction of simultaneous CIs for pairwise differences of means in the one-way ANOVA model under homogeneous variances.

In 1980, Dunnett constructed conservative simultaneous CIs for all pairwise mean differences using Tukey-Kramer method in unbalanced homogeneous one-way ANOVA model. Dunnett (1980) introduced simultaneous CIs for all pairwise mean differences in unbalanced heterogeneous one-way ANOVA model. The confidence intervals' estimates have the form

$$(\bar{y}_i - \bar{y}_j) \pm A_{ij,\alpha,k} \left(\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j} \right)^{\frac{1}{2}} \tag{4.1}$$

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where $\bar{y}_i$ and s_i^2 are the observed values of i^{th} sample mean, $\bar{Y}_i$ and i^{th} sample variance, S_i^2 respectively, and $A_{ij,\alpha,k}$ was chosen to achieve, if possible, the desired joint confidence coefficient to be $1-\alpha$. The methods considered for choosing $A_{ij,\alpha,k}$ were GH (Games and Howell, 1976), C(Cochran, 1964) , T_2 (Tamhane, 1977, 1979) and T_3 (Sidák, 1967). Dunnett (1980) carried out a simulation study to compare the error rates resulted from applying the four procedures for different sample sizes and variances. The number of simulation was 10,000. Dunnet found that the performance of the TK procedure is not acceptable for unequal variance situations. The GH intervals are always shorter than the C intervals and the T_2 intervals are always longer than T_3 intervals. The procedures T_2 and T_3 are both conservative and they become identical in the known variance case (degrees of freedom $\longrightarrow \infty$), whereas for finite degrees of freedom the T_3 procedure is less conservative than T_2 procedure. The estimated error rates for each of the four procedures are shown in Table 4.1.1.

Stoline (1981) concerned homogeneous unbalanced one-way classification model. He compared different procedures used to solve the problem of constructing simultaneous CIs for all pairwise mean differences for k populations with same variance and different sample sizes and means. These procedures were Tukey (T, 1953), Scheffé (S, 1953), Tukey-Kramer (TK, 1956), Benferroni (B, 1966), Dunn (D, 1974), Spjϕtvoll-Stoline (T', 1973), Hochberg ($GT2$, 1974), Hunter's (H, 1976), Gabriel's (G, 1978) and Genizi- Hochberg(GH, 1978). It was concluded that B, H, GH, D, T' and $GT2$ are conservative. The TK simultaneous intervals are narrower than T', $GT2$ and B methods. The TK procedure is superior to the GH method when there are two distinct sample sizes and also generally preferred to the H method except for some imbalanced cases.

Richmond (1982) considered multiple mean comparisons in unbalanced homogeneous one-way classification model where the observable data has $N_n\left(\boldsymbol{\mu}, \sigma^2\boldsymbol{\Omega}\right)$

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Table 4.1.1: Estimated Error Rates for all Pairwise Comparisons (Nominal Value is $\alpha = 0.05$) (Dunnett, 1980)

M	C	Simulation (1) $n_i = (7,7,7,7)$				Simulation (2) $n_i = (7,9,11,13)$				Simulation (3) $n_i = (7,7,7,7,7,7,7,7)$			
		x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞
GH	0.5	0.0531	0.0473	0.0466	0.0453	0.0534	0.0482	0.0458	0.0441	0.0622	0.0512	0.0457	0.0443
	1.0	0.0503	0.0504	0.0495	0.0493	0.0527	0.0526	0.0507	0.0503	0.0593	0.0528	0.0495	0.0491
	2.0	0.0506	0.0472	0.0456	0.0453	0.0488	0.0474	0.0479	0.0485	0.0603	0.0499	0.0440	0.0430
	4.0	0.0519	0.0440	0.0402	0.0406	0.0475	0.0467	0.0430	0.0419	0.0570	0.0444	0.0374	0.0355
	10.0	0.0493	0.0418	0.0356	0.0363	0.0422	0.0411	0.0396	0.0395	0.0521	0.0391	0.0310	0.0286
C	0.5	0.0277	0.0354	0.0430	0.0453	0.0349	0.0404	0.0435	0.0441	0.0260	0.0311	0.0400	0.0443
	1.0	0.0246	0.0322	0.0442	0.0493	0.0306	0.0377	0.0469	0.0503	0.0225	0.0296	0.0424	0.0491
	2.0	0.0281	0.0348	0.0425	0.0453	0.0293	0.0374	0.0447	0.0485	0.0258	0.0302	0.0380	0.0430
	4.0	0.0339	0.0366	0.0387	0.0406	0.0328	0.0383	0.0415	0.0419	0.0298	0.0306	0.0340	0.0355
	10.0	0.0380	0.0369	0.0349	0.0363	0.0353	0.0373	0.0384	0.0395	0.0319	0.0292	0.0290	0.0286
T_2	0.5	0.0376	0.0381	0.0383	0.0378	0.0402	0.0384	0.0369	0.0384	0.0328	0.0333	0.0333	0.0346
	1.0	0.0366	0.0387	0.0405	0.0417	0.0407	0.0419	0.0421	0.0425	0.0323	0.0345	0.0375	0.0383
	2.0	0.0367	0.0377	0.0383	0.0379	0.0364	0.0389	0.0397	0.0404	0.0325	0.0322	0.0319	0.0330
	4.0	0.0372	0.0352	0.0328	0.0350	0.0354	0.0364	0.0354	0.0354	0.0310	0.0277	0.0271	0.0261
	10.0	0.0353	0.0320	0.0296	0.0308	0.0317	0.0315	0.0322	0.0328	0.0267	0.0230	0.0221	0.0223
T_3	0.5	0.0434	0.0392	0.0387	0.0378	0.0434	0.0398	0.0374	0.0384	0.0433	0.0372	0.0340	0.0346
	1.0	0.0392	0.0397	0.0409	0.0417	0.0438	0.0436	0.0423	0.0425	0.0424	0.0381	0.0378	0.0383
	2.0	0.0403	0.0395	0.0383	0.0379	0.0393	0.0397	0.0401	0.0404	0.0436	0.0351	0.0325	0.0330
	4.0	0.0421	0.0365	0.0333	0.0350	0.0373	0.0373	0.0358	0.0354	0.0419	0.0322	0.0275	0.0261
	10.0	0.0400	0.0343	0.0298	0.0308	0.0344	0.0332	0.0324	0.0328	0.0367	0.0273	0.0225	0.0223
T_k	0.5	0.0662	0.0662	0.0648	0.0646	0.1163	0.1140	0.1109	0.1121	0.0937	0.0928	0.0898	0.0905
	1.0	0.0453	0.0455	0.0466	0.0493	0.0492	0.0488	0.0493	0.0503	0.0485	0.0494	0.0497	0.0491
	2.0	0.0713	0.0708	0.0654	0.0638	0.0369	0.0375	0.0376	0.0385	0.0949	0.0954	0.0941	0.0940
	4.0	0.0944	0.0851	0.0789	0.0762	0.0412	0.0395	0.0393	0.0392	0.1402	0.1360	0.1340	0.1341
	10.0	0.1046	0.0907	0.0815	0.0780	0.0411	0.0380	0.0365	0.0374	0.1645	0.1566	0.1522	0.1509

M is the method used. C is the variance multiplier.

distribution where $\boldsymbol{\mu}$ and σ^2 are unknown parameters and $\boldsymbol{\Omega}$ is known positive definite matrix . He constructed conservative simultaneous CIs among treatment means to a control in unbalanced case. The simultaneous CIs became exact when sample sizes are equal. He also constructed simultaneous CIs for all mean pairwise comparisons in balanced case. These simultaneous CIs are conservative or exact depending on the critical value used.

Uusipaikka (1985) constructed exact simultaneous CIs for multiple comparisons among three and four mean values when the estimated p mean values distributed as $normal_p\{\boldsymbol{\theta}, \sigma^2\boldsymbol{V}\}$ where $\boldsymbol{\theta}$ and σ^2 are unknown parameters and $\boldsymbol{V} = diag\left(\frac{1}{n_1} \cdots, \frac{1}{n_p}\right)$. He mentioned that, Tukey-Kramer is practically as good as the exact intervals unless the imbalance is extreme of the design. Spurrier and Isham (1985) developed exact simultaneous CIs for pairwise comparisons of three normal means assuming that there are k independent normal populations with means μ_i, sample sizes n_i and common unknown variance σ^2. This procedure produced simultaneous CIs for the pairwise differences of three normal means which are uniformly shorter than the Tukey-Kramer intervals for unequal sample sizes. Both methods are equivalent for equal sample sizes case. If there are four populations, then computations become intractable.

Hayter (1989) provided simultaneous CIs for pairwise comparisons assuming the cell means have $N_p(\boldsymbol{\mu}, \sigma^2\boldsymbol{V})$ distribution where $\boldsymbol{\mu}$ and σ^2 are unknown parameters and $\boldsymbol{V}$ is known positive-definite and symmetric matrix. The simultaneous CIs are conservative when the studentized range distribution used for calculating the critical value. The simultaneous CIs are exact using the critical value he derived.

Witkovský (2002) constructed simultaneous GCIs for all pairwise mean differences in unbalanced heterogeneous one-way ANOVA assuming k random samples from normal populations with mean μ_i, sample size n_i and variance σ_i^2 for

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$i = 1, \cdots, k$. The following are $100(1-\alpha)\%$ simultaneous CIs (generalized Scheffé)

$$(\bar{y}_i - \bar{y}_j) \pm \gamma^{\kappa}_{ij,1-\frac{\alpha}{2}} \sqrt{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}} \quad i = 1, \cdots, k, \; j = i+1, \cdots, k$$

$$\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{F^{-1^{(F^{\kappa}_{ij})}}_{[\kappa, f_i, f_j, \varphi_{ij}]}(1-\alpha)}$$

where $F^{F^{\kappa}_{ij}}_{[\kappa, f_i, f_j, \varphi_{ij}]}$ is the *cdf* of the random variable $F^{\kappa}_{ij} \sim \chi^2_{\kappa}\left(\frac{f_i s^2_{\varphi_{ij}}}{\chi^2_{f_i}} + \frac{f_j c^2_{\varphi_{ij}}}{\chi^2_{f_j}}\right)$, $\kappa = k-1$, $s^2_{\varphi_{ij}} = \sin^2(\varphi_{ij}) = \frac{\frac{s_i^2}{n_i}}{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}}$, $c^2_{\varphi_{ij}} = \cos^2(\varphi_{ij}) = \frac{\frac{s_j^2}{n_j}}{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}}$, $\varphi_{ij} = \arctan\sqrt{\frac{\frac{s_i^2}{n_i}}{\frac{s_j^2}{n_j}}}$, α is the chosen nominal significance level, $f_i = n_i - 1$ and $f_j = n_j - 1$. He developed simulation runs to construct tables of the critical values $\gamma^{\kappa}_{ij,1-\alpha}$ for $\alpha = 0.05$, $\kappa = 1, \cdots, 5$, $f_i = 1, \cdots, 10$, $f_j = f_i, \cdots, 10$ and $\varphi_{ij} = [0^o : 10^o : 90^o]$. From the simulation runs, he concluded that the simultaneous GCIs are quite conservative.

For the unbalanced one-way classification model with unequal variances, none of the previous authors appear to have developed simultaneous generalized confidence intervals for pairwise differences among means that extend the traditional Tukey or Dunnett intervals which are exact simultaneous interval procedures in balanced, homoscedastic cases. This is the topic for the rest of this chapter.

4.2 Extension of the Tukey Method for Heterogeneous, Unbalanced One-Way ANOVA Using GCIs

Consider a independent normal populations such that the i^{th} population has sample size n_i, mean μ_i and variance σ_i^2 , $i = 1, 2, \cdots, a$ where μ_i and σ_i^2 are unknown parameters. The model has the form

$$Y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, \cdots, a, \; j = 1, \cdots, n_i \tag{4.2}$$

where Y_{ij} denotes the j^{th} response in the i^{th} population, μ_i is the expected response corresponding the i^{th} population and $\epsilon_{ij} \sim N(0, \sigma_i^2)$ are independent, let $\boldsymbol{Y}$ be the vector of responses arranged in lexicographic order. Thus $\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$ where

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$\mathbf{\Sigma}^*$ is a diagonal matrix with σ_i^2 on diagonal. Define $\bar{Y}_i$ to be the sample mean and S_i^2 to be the sample variance with observed values $\bar{y}_i$ and s_i^2 respectively.

Thus $\bar{\boldsymbol{Y}} \sim MVN(\boldsymbol{\mu}, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is a diagonal matrix with $\frac{\sigma_i^2}{n_i}$ on diagonal. The aim was to construct simultaneous GCIs for all pairwise mean differences $\mu_i - \mu_j,\ i \neq j$.

Let

$$U_i = \frac{SS_i}{\sigma_i^2} = \frac{(n_i - 1)S_i^2}{\sigma_i^2}, \qquad V_i = \frac{\bar{Y}_i - \mu_i}{\frac{\sigma_i}{\sqrt{n_i}}}. \tag{4.3}$$

We note that $U_i \sim \chi^2_{n_i - 1}$, $V_i \sim N(0, 1)$ and U_i and V_i are mutually independent random variables. Thus

$$\sigma_i^2 = \frac{(n_i - 1)S_i^2}{U_i}, \qquad \mu_i = \bar{Y}_i - V_i \frac{\sigma_i}{\sqrt{n_i}}.$$

The FGPQs for σ_i and μ_i, respectively, are as follows.

$$R_{\sigma_i} = \sqrt{\frac{(n_i - 1)s_i^2}{U_i}}, \qquad R_{\mu_i} = \bar{y}_i - V_i \sqrt{\frac{(n_i - 1)s_i^2}{n_i U_i}} \tag{4.4}$$

Proposition 4.2.1

Consider the one-way layout with unequal sample sizes and heterogeneous variances as described in Equation (4.2). Define $\bar{Y}_i$, S_i^2, to be the sample mean and the sample variance, respectively, for group i, with observed values $\bar{x}_i$ and s_i^2. Then $\bar{\boldsymbol{Y}} \sim MVN(\boldsymbol{\mu}, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is a diagonal matrix with $\frac{\sigma_i^2}{n_i}$ on diagonal. Let R_{σ_i} and R_{μ_i} be the FGPQs for σ_i and μ_i, respectively, as defined in Equation (4.3). Then the two-sided simultaneous GCIs for all pairwise mean differences $\mu_i - \mu_j,\ i \neq j$ are

$$(\hat{\mu}_i - \hat{\mu}_j) \pm d_{1-\alpha} \sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_j)}, \quad i \neq j \tag{4.5}$$

where $d_{1-\alpha}$ is the $1 - \alpha$ percentile point of the distribution of

$$D = \max_{all\ i \neq j} \left| \frac{\frac{s_i}{\sqrt{n_i}} T_{n_i - 1} - \frac{s_j}{\sqrt{n_j}} T_{n_j - 1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_j)}} \right| \tag{4.6}$$

with

$$\widehat{Var}(\hat{\mu}_i - \hat{\mu}_j) = \sqrt{\frac{S_i^2}{n_i} + \frac{S_j^2}{n_j}} \tag{4.7}$$

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and its realized value given by

$$\widehat{var}(\hat{\mu}_i - \hat{\mu}_j) = \sqrt{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}}, \tag{4.8}$$

where T_{n_i-1} and T_{n_j-1} are independent t random variables with degrees of freedom $n_i - 1$ and $n_j - 1$ respectively.

Proof:

Using the FGPQs for μ_i and μ_j, we get the following FGPQs for $\mu_i - \mu_j$, $i \neq j$.

$$\begin{aligned}
R_{\mu_i-\mu_j} &= (\bar{y}_i - \bar{y}_j) - \left(V_i\sqrt{\frac{(n_i-1)s_i^2}{n_i U_i}} - V_j\sqrt{\frac{(n_j-1)s_j^2}{n_j U_j}} \right) \\
&= (\bar{y}_i - \bar{y}_j) - \left(\sqrt{\frac{s_i^2}{n_i}} \frac{\frac{\bar{Y}_i-\mu_i}{\frac{\sigma_i}{\sqrt{n_i}}}}{\sqrt{\frac{(n_i-1)S_i^2}{(n_i-1)\sigma_i^2}}} - \sqrt{\frac{s_j^2}{n_j}} \frac{\frac{\bar{Y}_j-\mu_j}{\frac{\sigma_j}{\sqrt{n_j}}}}{\sqrt{\frac{(n_j-1)S_j^2}{(n_j-1)\sigma_j^2}}} \right) \\
&= (\bar{y}_i - \bar{y}_j) - \left(\sqrt{\frac{s_i^2}{n_i}} T_{n_i-1} - \sqrt{\frac{s_j^2}{n_j}} T_{n_j-1} \right)
\end{aligned} \tag{4.9}$$

The FGPQ for $\mu_i - \mu_j$ is a linear combination of independent t distributions. So the distribution of $R_{\mu_i-\mu_j}$ is free of all parameters. The observed value $r_{\mu_i-\mu_j}$ of $R_{\mu_i-\mu_j}$ is $r_{\mu_i-\mu_j} = (\bar{y}_i - \bar{y}_j) - ((\bar{y}_i - \mu_i) - (\bar{y}_j - \mu_j)) = \mu_i - \mu_j$ which does not depend on the nuisance parameters. So $R_{\mu_i-\mu_j}$ is a generalized pivotal quantity for $\mu_i - \mu_j$. Notice that

$$D = \max_{all\ i \neq j} \left| \frac{\frac{s_i}{\sqrt{n_i}} T_{n_i-1} - \frac{s_j}{\sqrt{n_j}} T_{n_j-1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_j)}} \right|$$

The distribution of D is free of unknown parameters. The $1-\alpha$ percentile $d_{1-\alpha}$ of the random variable D may be obtained by computer simulation.

4.3 Simulation Study

A simulation study was conducted to assess the performance of the simultaneous GCIs for all pairwise comparisons of the cell means $\mu_i - \mu_j$, $i \neq j$ and to compare the results with those obtained using the methods discussed in Dunnett (1980).

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Without loss of generality, it was assumed that $\boldsymbol{\mu} = \mathbf{0}$. The simulation settings were chosen to exactly match those used by Dunnett (1980). The simulations were used to estimate the coverage error rates. The sample sizes, variance multipliers and multiplication factors used in the simulations are summarized in Table 4.3.1.

Table: 4.3.1: Parameters and Sample Sizes for the Simulation Study

Sample size	(7,7,7,7), (7,9,11,13), (7,7,7,7,7,7,7,7)
Variance Multiplier	0.5, 1, 2, 4, 10
Multiplication factor	1, 2, 8, ∞

All combinations of Sample sizes, Variance Multipliers and Multiplication factors were used in the simulation study.

4.3.1 Simulation Details

The simulation was carried out using the following steps.

Step 1. Set $\mu_i = 0,\ i = 1, \cdots, a$. Select one of the settings for sample size, variance multiplier and multiplication factor.

Step 2. Generate $\bar{Y}_i$ independent of S_i^2 as follows

$$\begin{aligned} \bar{Y}_i &= \mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_i}}, \qquad i = 1, \cdots, a \\ S_i^2 &= \frac{\sigma_i^2 U_i}{n_i - 1}, \qquad i = 1, \cdots, a \end{aligned}$$

where V_i are independent $N(0,1)$ distributions and U_i are independent $\chi^2_{n_i-1}$ distributions and all variables are jointly independent. The value σ_i^2 is the desired value for the variance of the i^{th} group. The observed values of $\bar{Y}_i$ and S_i^2 are $\bar{y}_i$ and s_i^2 respectively.

Step 3. For $q = 1, \ldots, Q$, generate independent random vectors

$$\left(U_1^{(q)}, \ldots, U_a^{(q)}, V_1^{(q)}, \ldots, V_a^{(q)}\right),$$

where $U_i^{(q)} \sim \chi^2_{n_i-1}$ and $V_i^{(q)} \sim N(0,1)$, for $i = 1, \ldots, a$ and all random variables are jointly independent. We used $Q = 10000$. Define

$$R_{\mu_i}^{(q)} = \bar{y}_i - V_i^{(q)}\sqrt{\frac{(n_i - 1)s_i^2}{n_i U_i^{(q)}}}.$$

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Thus

$$R^{(q)}_{\mu_i-\mu_j} = (\bar{y}_i - \bar{y}_j) - \left(\sqrt{\frac{s_i^2}{n_i}}T^{(q)}_{n_i-1} - \sqrt{\frac{s_j^2}{n_j}}T^{(q)}_{n_j-1}\right)$$

where $T^{(q)}_{n_i-1}$ and $T^{(q)}_{n_j-1}$ are independent t distributions with degrees of freedom $n_i - 1$ and $n_j - 1$ respectively. There are $\frac{a(a-1)}{2}$ of $R^{(q)}_{\mu_i-\mu_j}$ of $\mu_i - \mu_j$.

Step 4. For $q = 1, \cdots, Q$, calculate

$$\begin{aligned} D_q &= \max_{all\ i \neq j} \left| \frac{(\bar{y}_i - \bar{y}_j) - R^{(q)}_{\mu_i-\mu_j}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_j)}} \right| \\ &= \max_{all\ i \neq j} \left| \frac{\frac{s_i}{\sqrt{n_i}}T^{(q)}_{n_i-1} - \frac{s_j}{\sqrt{n_j}}T^{(q)}_{n_j-1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_j)}} \right| \end{aligned} \tag{4.10}$$

where $\widehat{var}(\hat{\mu}_i - \hat{\mu}_j) = \frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}$. Order D_q such that $D_1 < \cdots < D_Q$. For $\alpha = 0.05$, the required percentile $d_{0.95}$ of the distribution of D is estimated by D_{9500} in (4.10). If the true $\mu_i - \mu_j$, $i \neq j$ are all covered simultaneously by the respective GCIs, then record this case as a *success*.

Step 5. Repeat steps $1 - 4$ for M times. We used M equal to 10000. Let p be the proportion of successes out of the M trials. Then $1 - p$ is the estimated familywise coverage error rate. The estimated familywise error rates for all the combinations in table 4.3.1 are shown in Table 4.3.2.

4.3.2 Discussion of Simulation Results

The results in Table 4.3.2 show that, the simultaneous GCIs are conservative when sample sizes are small. when the sample sizes n_i increase, the estimated familywise coverage error rates approach the nominal error rate. When variances are known, the estimated familywise cover error rates are very close to the nominal error rate.

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Table 4.3.2: Estimated Familywise Coverage Error Rates for the Differences $\mu_i - \mu_j$ where $i =\neq j = 1, \cdots, a$ Using Simultaneous GCIs (nominal $\alpha = 0.05$)

Variance multiplier	Simulation (1) $n_i = (7,7,7,7)$				Simulation (2) $n_i = (7,9,11,13)$				Simulation (3) $n_i = (7,7,7,7,7,7,7,7)$			
	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞
0.5	0.0339	0.0445	0.0502	0.0507	0.0406	0.0428	0.0485	0.0498	0.0323	0.0425	0.0496	0.0522
1.0	0.028	0.0407	0.0508	0.0498	0.0345	0.0419	0.0473	0.051	0.028	0.0407	0.0502	0.0512
2.0	0.0343	0.0416	0.0508	0.0495	0.0341	0.0446	0.0480	0.0492	0.035	0.0421	0.0525	0.0495
4.0	0.0404	0.0474	0.0524	0.05	0.0397	0.0463	0.0475	0.0492	0.0403	0.0438	0.0541	0.0497
10.0	0.0468	0.0491	0.0516	0.0512	0.0436	0.0478	0.0503	0.0512	0.0444	0.047	0.0537	0.0494

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4.3.3 Comparison of Simultaneous GCIs with Dunnett (1980) Simultaneous CIs

When sample sizes are small, the estimated familywise error rates using GH procedure is closer to 0.05 than using simultaneous GCIs. When sample sizes are moderate and variances are very divergent, the estimated familywise error rates using simultaneous GCIs are closer to 0.05 compared to procedures mentioned by Dunnett (1980).

4.4 Example

An example is presented to illustrate the computations of simultaneous GCIs for all pairwise mean differences in one-way ANOVA model. The data were taken from Ott (1993). Consider data collected to compare the means of three different population groups (treatments). These data are shown in Table 4.4.1.

Table: 4.4.1: Data from Ott (1993) for Three Different Treatment Groups

Trt 1	35.8	29.8	38.9	33.3	28.1	27.1	27.1	27.9
	38.4	25.8	28.2	28.2	22.7	36.0	25.9	
Trt 2	24.8	20.1	19.5	19.2	23.2	17.8	14.9	17.4
	25.5	25.6	17.9	19.3	16.6	18.6	20.1	24.7
	23.3	19.2	22.1	21.1				
Trt 3	19.7	17.4	18.4	18.8	17.0	22.3	20.7	22.0
(Control)	21.9	16.6	19.5	19.5	21.2	16.6	22.3	15.3
	23.1	19.5	11.7	22.1	21.9	17.6	19.8	16.0
	20.4							

The following summary statistics were computed.

$$\begin{aligned}
\bar{x}_1 &= 30.213, \quad \bar{x}_2 = 20.545, \quad \bar{x}_3 = 19.252 \\
var1 &= 34.57, \quad var2 = 8.24, \quad var3 = 10.37 \\
std1 &= 5.88, \quad std2 = 2.87, \quad std3 = 3.22
\end{aligned}$$

The simultaneous GCIs for all pairwise mean differences were constructed for the three groups. The simultaneous GCIs are

$$5.9 \leq \mu_1 - \mu_2 \leq 13.5$$

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$$\begin{aligned} 7.6 \leq \mu_1 - \mu_3 \leq 14.4 \\ -1 \leq \mu_2 - \mu_3 \leq 3.6 \end{aligned} \tag{4.11}$$

The difference of the variances of populations 1 and 3 is larger than the difference of the variances of populations 2 and 3, this should lead to a CI for $\mu_2 - \mu_3$ that is narrower than for $\mu_1 - \mu_3$. This was found when using simultaneous GCIs procedure. Using SAS (Tukey option), the simultaneous CIs are

$$\begin{aligned} 6.7 &\leq \mu_1 - \mu_2 \leq 12.6 \\ 8.2 &\leq \mu_1 - \mu_3 \leq 13.7 \\ -1.3 &\leq \mu_2 - \mu_3 \leq 3.8. \end{aligned} \tag{4.12}$$

4.5 Extended Dunnett Method for Heterogeneous Unbalanced One-Way ANOVA

Consider the one-way model described in Equation (4.2) with a independent normal populations such that the i^{th} group has sample size n_i, mean μ_i and variance σ_i^2, $i = 1, 2, \cdots, a$. Y_{ij} denotes the j^{th} response in the i^{th} group, Let $\boldsymbol{Y}$ be the vector of responses arranged in lexicographic order. Thus $\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$ where $\boldsymbol{\Sigma}^*$ is a diagonal matrix with σ_i^2 on diagonal. Define $\bar{Y}_i$ to be the sample mean for the i^{th} group and S_i^2 to be the sample variance, with observed values $\bar{y}_i$ and s_i^2 respectively. Let $\bar{\boldsymbol{Y}}$ denote the vector of sample means. Then $\bar{\boldsymbol{Y}} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}$ is a diagonal matrix with $\frac{\sigma_i^2}{n_i}$ on diagonal. The aim is to construct simultaneous GCIs for all differences $\mu_i - \mu_a,\ i = 1, \cdots, a-1$. In the following proposition, we will use the FGPQs for μ_i and σ_i given in Equation (4.4). We will also use the notation already established in this chapter.

Proposition 4.5.1

Consider the one-way layout model given in Equation (4.2). The two sided simultaneous GCIs for all mean differences $\mu_i - \mu_a,\ i = 1, \cdots, a-1$ are given by

$$(\hat{\mu}_i - \hat{\mu}_a) \pm d_{1-\alpha}\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_a)}, \quad i = 1, \cdots, a-1 \tag{4.13}$$

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where $d_{1-\alpha}$ is $1-\alpha$ percentile point of the distribution of

$$D = \max_{all\ i \neq j} \left| \frac{\frac{s_i}{\sqrt{n_i}} T_{n_i-1} - \frac{s_a}{\sqrt{n_a}} T_{n_a-1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_a)}} \right| \tag{4.14}$$

$$\widehat{Var}(\hat{\mu}_i - \hat{\mu}_a) = \sqrt{\frac{S_i^2}{n_i} + \frac{S_a^2}{n_a}} \tag{4.15}$$

$$\widehat{var}(\hat{\mu}_i - \hat{\mu}_a) = \sqrt{\frac{s_i^2}{n_i} + \frac{s_a^2}{n_a}} \tag{4.16}$$

where $\widehat{var}(\hat{\mu}_i - \hat{\mu}_a)$ is the observed value of $\widehat{Var}(\hat{\mu}_i - \hat{\mu}_a)$, T_{n_i-1} and T_{n_a-1} are independent t random variables with degrees of freedom $n_i - 1$ and $n_a - 1$, respectively.

Proof

Using the FGPQs for μ_i and μ_a, thus the FGPQs for $\mu_i - \mu_a$ $i = 1, \cdots, a-1$ are

$$\begin{aligned} R_{\mu_i - \mu_a} &= (\bar{y}_i - \bar{y}_a) - \left(V_i \sqrt{\frac{(n_i - 1)s_i^2}{n_i U_i}} - V_a \sqrt{\frac{(n_a - 1)s_a^2}{n_a U_a}} \right) \\ &= (\bar{y}_i - \bar{y}_a) - \left(\sqrt{\frac{s_i^2}{n_i}} \frac{\frac{\bar{Y}_i - \mu_i}{\frac{\sigma_i}{\sqrt{n_i}}}}{\sqrt{\frac{(n_i-1)S_i^2}{(n_i-1)\sigma_i^2}}} - \sqrt{\frac{s_a^2}{n_a}} \frac{\frac{\bar{Y}_a - \mu_a}{\frac{\sigma_a}{\sqrt{n_a}}}}{\sqrt{\frac{(n_a-1)S_a^2}{(n_a-1)\sigma_a^2}}} \right) \\ &= (\bar{y}_i - \bar{y}_a) - \left(\sqrt{\frac{s_i^2}{n_i}} T_{n_i-1} - \sqrt{\frac{s_a^2}{n_a}} T_{n_a-1} \right) \end{aligned} \tag{4.17}$$

The FGPQ for $\mu_i - \mu_a$ is a linear combination of independent t distributions. So the distribution of $R_{\mu_i - \mu_a}$ is free of all parameters. The observed value $r_{\mu_i - \mu_a}$ of $R_{\mu_i - \mu_a}$ is $r_{\mu_i - \mu_a} = (\bar{y}_i - \bar{y}_a) - ((\bar{y}_i - \mu_i) - (\bar{y}_a - \mu_a)) = \mu_i - \mu_a$ which does not depend on the nuisance parameters. So $R_{\mu_i - \mu_a}$ is a generalized pivotal quantity for $\mu_i - \mu_a$. Notice that

$$D = \max_{i=1,\cdots,a-1} \left| \frac{\frac{s_i}{\sqrt{n_i}} T_{n_i-1} - \frac{s_a}{\sqrt{n_a}} T_{n_a-1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_a)}} \right| \tag{4.18}$$

The distribution of D is free of unknown parameters. The $1-\alpha$ percentile $d_{1-\alpha}$ of the random variable D may be obtained by computer simulation.

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4.6 Simulation Study

A simulation study was conducted to assess the performance of the simultaneous GCIs for all pairwise comparisons of cell-means of the form $\mu_i - \mu_a$, $i = 1, \cdots, a-1$. It was assumed without loss of generality that $\boldsymbol{\mu} = \mathbf{0}$. The number of samples used were $a = 4$ and $a = 8$. The sample sizes are $7,7,7,7$ and $7,9,11,13$ for $a = 4$ and $7,7,7,7,7,7,7,7$ for $a = 8$. The variances are $(1, c, c^2, c^3)$ for $a = 4$ and $(1, 1, c, c, c^2, c^2, c^3, c^3)$ for $a = 8$ where $c = \{.5, 1, 4, 10\}$. Different sample sizes and variances were used to investigate the effects of these changes on the familywise coverage error rates. Both sample sizes and variances were also multiplied by $2, 8, \infty$ such that the degrees of freedom associated with the sample variance cover a wide range. The simulation settings match exactly those used by Dunnett (1980). Table 4.3.1 gives the parameters and their values used in the simulation study.

4.6.1 Simulation Details

The simulation was carried out using the following steps.

Step 1. Set $\mu_i = 0$, $i = 1, \cdots, a$. Select one of the settings for sample size, variance multiplier and multiplication factor.

Step 2. Generate $\bar{Y}_i$ independent of S_i^2 as follows

$$\begin{aligned} \bar{Y}_i &= \mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_i}}, \qquad i = 1, \cdots, a \\ S_i^2 &= \frac{\sigma_i^2 U_i}{n_i - 1}, \qquad i = 1, \cdots, a \end{aligned}$$

where V_i are independent $N(0,1)$ distributions and U_i are independent $\chi^2_{n_i-1}$ distributions and all variables are jointly independent. The value σ_i^2 is the desired value for the variance of the i^{th} group. The observed values of $\bar{Y}_i$ and S_i^2 are $\bar{y}_i$ and s_i^2 respectively.

Step 3. For $q = 1, \ldots, Q$, generate independent random vectors

$$\left(U_1^{(q)}, \ldots, U_a^{(q)}, V_1^{(q)}, \ldots, V_a^{(q)}\right),$$

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where the component random variables are jointly independent, $U_i^{(q)} \sim \chi^2_{n_i-1}$, and $V_i^{(q)} \sim N(0,1)$, for $i = 1, \ldots, a$. We used $Q = 10000$. Define

$$R_{\mu_i}^{(q)} = \bar{y}_i - V_i^{(q)} \sqrt{\frac{(n_i - 1)s_i^2}{n_i U_i^{(q)}}}.$$

Thus

$$R_{\mu_i-\mu_a}^{(q)} = (\bar{y}_i - \bar{y}_a) - \left(\sqrt{\frac{s_i^2}{n_i}} T_{n_i-1}^{(q)} - \sqrt{\frac{s_a^2}{n_a}} T_{n_a-1}^{(q)} \right)$$

where $T_{n_i-1}^{(q)}$ and $T_{n_a-1}^{(q)}$ are independent t distributions with degrees of freedom $n_i - 1$ and $n_a - 1$ respectively.

Step 4. For $q = 1, \cdots, Q$, calculate

$$D_q = \max_{all\ i \neq a} \left| \frac{\frac{s_i}{\sqrt{n_i}} T_{n_i-1} - \frac{s_a}{\sqrt{n_a}} T_{n_a-1}}{\sqrt{\widehat{var}(\hat{\mu}_i - \hat{\mu}_a)}} \right| \tag{4.19}$$

where $\widehat{var}(\hat{\mu}_i - \hat{\mu}_a) = \frac{s_i^2}{n_i} + \frac{s_a^2}{n_a}$. Order D_q such that $D_1 < \cdots < D_{10000}$. If $\alpha = 0.05$ is used, then $d_{0.95}$ is estimated by D_{9500} and is used to construct simultaneous GCIs for all pairwise differences $\mu_i - \mu_a$, $i = 1, \cdots, a-1$. If the true $\mu_i - \mu_a$, $i = 1, \cdots, a-1$ are covered simultaneously by the GCIs, then is recorded as a *success*.

Step 5. Repeat the steps $1 - 4$ M times. We used M equal to 1000. Let p be the proportion of successes in the M trials. Then $1 - p$ is the estimated familywise coverage error rate. The estimated familywise error rates for all the parameter combinations considered are shown in Table 4.6.1.

4.6.2 Discussion of Simulation Results

The simulation results in Table 4.6.1 show that when the sample sizes n_i increase, the estimated familywise coverage error rates are close to the nominal error rate of 0.05 except for $a = 4$ (unequal sample sizes) and $a = 8$ when the variances are moderate or very divergent. In all cases, the performance of the simultaneous GCIs appears to be acceptable.

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Table 4.6.1

Estimated Familywise Coverage Error Rates for the Differences
$\mu_i - \mu_a$ where $i = 1, \cdots, a-1$ Using Simultaneous GCIs (nominal $\alpha = 0.05$)

Variance multiplier	Simulation (1) $n_i = (7,7,7,7)$				Simulation (2) $n_i = (7,9,11,13)$				Simulation (3) $n_i = (7,7,7,7,7,7,7,7)$			
	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞
0.5	0.03	0.051	0.055	0.049	0.037	0.055	0.050	0.031	0.026	0.024	0.054	0.063
1.0	0.031	0.042	0.049	0.051	0.034	0.053	0.053	0.030	0.018	0.026	0.049	0.060
2.0	0.037	0.042	0.054	0.048	0.034	0.064	0.046	0.035	0.024	0.031	0.043	0.049
4.0	0.042	0.049	0.059	0.041	0.042	0.078	0.039	0.046	0.029	0.035	0.042	0.045
10.0	0.048	0.052	0.060	0.043	0.047	0.080	0.042	0.042	0.034	0.038	0.042	0.045

4.7 Example

Using the same data example as in Section 4.4, simultaneous generalized confidence intervals for $\mu_i - \mu_a$, $i = 1, \cdots, a-1$ are

$$\begin{aligned} 7.7 \leq \mu_1 - \mu_3 \leq 14.3 \\ -.9 \leq \mu_2 - \mu_3 \leq 3.5 \end{aligned} \tag{4.20}$$

The difference of the variances of populations 1 and 3 is larger than the difference of the variances of populations 2 and 3, this should lead to a CI for $\mu_2 - \mu_3$ that is narrower than for $\mu_1 - \mu_3$ and this what we found using simultaneous GCIs procedure. SAS was used to calculate simultaneous 95% confidence limits (Dunnett option), we obtained the following results

$$\begin{aligned} 8.3 \leq \mu_1 - \mu_3 \leq 13.6 \\ -1.1 \leq \mu_2 - \mu_3 \leq 3.7. \end{aligned} \tag{4.21}$$

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Chapter 5

APPLICATIONS OF GENERALIZED INFERENCE IN BALANCED MIXED LINEAR MODELS

In this chapter we developed simultaneous generalized confidence intervals for all cell-means in balanced two-factor crossed mixed linear model with interaction. We also developed simultaneous generalized confidence intervals for all pairwise differences among cell-means in balanced three-factor nested factorial mixed linear model. To our knowledge, there are no exact simultaneous confidence intervals in these cases. The methods we propose can be generalized to other simultaneous confidence interval problems in balanced mixed models in a straightforward manner.

Section 1 contains a review of literature related to simultaneous intervals in mixed linear models. Section 2 discusses construction of simultaneous generalized confidence intervals for all cell-means in balanced two-factor crossed mixed linear model with interaction. Section 3 concerns a simulation study. Section 4 presents an example. Section 5 deals with simultaneous generalized confidence intervals for all pairwise differences of cell-means for balanced three-factor nested factorial mixed linear model. Section 6 contains a simulation study. Section 7 presents an example. Section 8 contains an appendix.

5.1 Review of Literature

The following is a brief review of some previous works on the topic of simultaneous inference in mixed models. In 1987, Edwards and Berry considered the problem of constructing simultaneous confidence intervals for p linear combinations

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of the fixed effects parameters $\boldsymbol{\beta}$ in mixed linear models. The method is based on the assumption that $\hat{\boldsymbol{\beta}} \sim N_k(\boldsymbol{\beta}, \sigma^2 \boldsymbol{V})$ where $\boldsymbol{V}$ is a known symmetric, positive-definite matrix and σ^2 is unknown parameter. Simultaneous confidence intervals were developed for p specified linear combinations of $\boldsymbol{\beta}$, denoted by $\theta_j = \boldsymbol{c}_j'\boldsymbol{\beta}$ where $\boldsymbol{c}_j' = (c_{j1}, \cdots, c_{jk})$, $j = 1, \ldots, p$. Exact simultaneous confidence intervals for $\theta_1, \cdots, \theta_p$ have the form

$$\boldsymbol{c}_j'\hat{\boldsymbol{\beta}} \pm w_\alpha \hat{\sigma} (\boldsymbol{c}_j' \boldsymbol{V} \boldsymbol{c}_j)^{\frac{1}{2}}, \; j = 1, \cdots, p$$

where w_α is the upper-α percentile point of the distribution of $W = \max_{1 \le j \le p} \left\{ \frac{|\boldsymbol{c}_j'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})|}{\hat{\sigma}(\boldsymbol{c}_j' \boldsymbol{V} \boldsymbol{c}_j)^{\frac{1}{2}}} \right\}$. The critical value w_α may be estimated using Monte Carlo methods.

Cheung and Chan (1996) considered the problem of constructing simultaneous pairwise CIs in unbalanced two-way fixed effects model, the model has the form

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk} \quad i = 1, \cdots, r, \quad j = 1, \cdots, c, \quad k = 1, \cdots, n_{ij} \tag{5.1}$$

where μ_{ij} are the means of the j^{th} level of factor B at the i^{th} level of factor A and ϵ_{ijk} are independent $N(0, \sigma^2)$ random variables. Cheung and Chan (1996) constructed exact and approximate simultaneous pairwise CIs for all mean differences. The exact $100(1-\alpha)\%$ level simultaneous pairwise CIs for $\mu_{iu} - \mu_{iv}$ are given by

$$\left(\bar{y}_{iu} - \bar{y}_{iv} - d_\alpha S \sqrt{\left(\frac{1}{n_{iu}} + \frac{1}{n_{iv}} \right)}, \bar{y}_{iu} - \bar{y}_{iv} + d_\alpha S \sqrt{\left(\frac{1}{n_{iu}} + \frac{1}{n_{iv}} \right)} \right) \tag{5.2}$$

where $\bar{y}_{iu}$ and $\bar{y}_{iv}$, $i = 1, \cdots, r$, $1 \le u \ne v \le c$ are the observed values of iu^{th} and iv^{th} sample means, S^2 is minimum variance unbiased estimator of σ^2 and d_α is the critical value. The approximate simultaneous pairwise CIs were used when the sample sizes are not extremely divergent. The only difference between exact and approximate simultaneous pairwise CIs is the critical value. A copy of the

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FORTRAN program for computing the critical values is available from STATLIB at `http:\\www.statlib.cmu.`

Cheung (1998) extended the work of Cheung and Chan (1996) by developing simultaneous one-sided intervals for all pairwise differences of the cell-means in the two factor model of Equation (5.1).

Cheung et al. (2003) constructed exact $100(1-\alpha)\%$ level simultaneous pairwise CIs for $\mu_{iu} - \mu_{iv}$ for the model (5.1) where ϵ_{ijk} are independent $N(0, \sigma_i^2)$. The simultaneous pairwise CIs have same construction as in Equation (5.2) but with different critical values and $\hat{\sigma}_i$ instead of S. Using Tukey-Kramer critical values leads to conservative intervals.

5.2 Simultaneous GCIs for all Means in Balanced Two-factor Crossed Mixed Linear Model with Interaction

This section deals with constructing simultaneous generalized confidence intervals for all means in balanced two-factor crossed mixed linear model with factor A being fixed and factor B being random. Specifically, let

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \quad i = 1, \cdots, a, \quad j = 1, \cdots, b, \quad k = 1, \cdots, n \tag{5.3}$$

where Y_{ijk} denotes the k^{th} response for the i^{th} level of A and the j^{th} level of B, α_i is effect of i^{th} level of main fixed effect of factor A, $\beta_j \sim N(0, \sigma_\beta^2)$ is effect of j^{th} level of random main effect of factor B, $\gamma_{ij} \sim N(0, \sigma_\gamma^2)$ is effect of ij^{th} level of the random interaction effect, and $e_{ijk} \sim N(0, \sigma_e^2)$ are the random effect associated with replicates. All random variables are mutually independent. Thus $\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$ where $\boldsymbol{\Sigma}^* = \sigma_\beta^2 \boldsymbol{V}_2 + \sigma_\gamma^2 \boldsymbol{V}_{12} + \sigma_e^2 \boldsymbol{I}_{abn}$, $\boldsymbol{V}_2 = \boldsymbol{U}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{U}_n$, $\boldsymbol{V}_{12} = \boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{U}_n$, $\boldsymbol{I}_a$ is the $a \times a$ identity matrix and $\boldsymbol{U}_a$ is the matrix of size $a \times a$ with all elements equal to one . Define the vector $\bar{\boldsymbol{Y}}$ by $\bar{\boldsymbol{Y}} = (\bar{Y}_{1..}, \ldots, \bar{Y}_{a..})'$. Then $\bar{\boldsymbol{Y}}$ has the distribution

$$\bar{\boldsymbol{Y}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

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where

$$\boldsymbol{\Sigma} = \frac{n\sigma_\gamma^2 + \sigma_e^2}{bn}\boldsymbol{I}_a + \frac{\sigma_\beta^2}{b}\boldsymbol{U}_a.$$

The ANOVA table for this model is as follows.

Table 5.2.1: ANOVA Table for Balanced Two-factor Crossed Mixed Linear Model with Interaction

Source	df	SS	EMS
A	$a-1$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma_\gamma^2 + \sigma_e^2 + nbQ(\mu_i)$
B	$b-1$	$\boldsymbol{Y}'\left(\frac{\boldsymbol{U}_a}{a} \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma_\gamma^2 + \sigma_e^2 + na\sigma_\beta^2$
AB	$(a-1)(b-1)$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma_\gamma^2 + \sigma_e^2$
Residual	$ab(n-1)$	$\boldsymbol{Y}'(\boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{S}_n)\boldsymbol{Y}$	σ_e^2

The term $Q(\mu_i)$ in the ANOVA table is a function of the expected responses $\mu_i = \mu + \alpha_i$ corresponding to level i of factor A. We observe that

$$Var(\bar{Y}_{i..}) = \frac{n(\sigma_\beta^2 + \sigma_\gamma^2) + \sigma_e^2}{bn} = \frac{1}{abn}\left[EMS(\beta) + (a-1)EMS(\gamma)\right]$$

where EMS represents *expected mean square.* Thus $EMS(\beta)$ represents the expected mean square corresponding to β, etc. An estimator of this variance is given by

$$\widehat{Var}(\bar{Y}_{i..}) = \frac{1}{abn}\left[MS(\beta) + (a-1)MS(\gamma)\right]$$

where MS stands for the *mean square* statistic for the effect under consideration. This variance estimator does not have a scaled chi-square distribution and exact simultaneous confidence intervals for the collection of $\mu_i, i = 1, \ldots, a$ are not available. In this situation, simultaneous generalized confidence intervals can be constructed. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_a)'$ be the vector of expected responses corresponding to levels of factor A. Generalized pivotal quantities for the parameters σ_β^2, σ_γ^2, σ_e^2 and $\boldsymbol{\mu}$

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are needed to construct the simultaneous generalized confidence intervals for the collection of $\mu_i, i = 1, \ldots, a$. To this end, we define the following quantities.

$$\begin{aligned} U_1 &= \frac{SS(\beta)}{\psi_\beta} \sim \chi^2_{b-1}, \qquad \psi_\beta = an\sigma^2_\beta + n\sigma^2_\gamma + \sigma^2_e \\ U_2 &= \frac{SS(\gamma)}{\psi_\gamma} \sim \chi^2_{(a-1)(b-1)}, \qquad \psi_\gamma = n\sigma^2_\gamma + \sigma^2_e \\ U_3 &= \frac{SS(e)}{\psi_e} \sim \chi^2_{ab(n-1)}, \qquad \psi_e = \sigma^2_e \end{aligned} \tag{5.4}$$

We also observed that

$$\begin{aligned} \sigma^2_\beta &= \frac{SS(\beta)}{anU_1} - \frac{SS(\gamma)}{anU_2} \\ \sigma^2_\gamma &= \frac{SS(\gamma)}{nU_2} - \frac{SS(e)}{nU_3} \\ \sigma^2_e &= \frac{SS(e)}{U_3} \end{aligned}$$

Hence we obtained the following generalized pivotal quantities $R_{\sigma^2_\beta}$, $R_{\sigma^2_\gamma}$, and $R_{\sigma^2_e}$ for σ^2_β, σ^2_γ, and σ^2_e, respectively.

$$\begin{aligned} R_{\sigma^2_\beta} &= \frac{ss(\beta)}{anU_1} - \frac{ss(\gamma)}{anU_2} \\ &= (an\sigma^2_\beta + n\sigma^2_\gamma + \sigma^2_e)\left(\frac{ss(\beta)}{anSS(\beta)}\right) - (n\sigma^2_\gamma + \sigma^2_e)\left(\frac{ss(\gamma)}{anSS(\gamma)}\right) \end{aligned} \tag{5.5}$$

$$\begin{aligned} R_{\sigma^2_\gamma} &= \frac{ss(\gamma)}{nU_2} - \frac{ss(e)}{nU_3} \\ &= (n\sigma^2_\gamma + \sigma^2_e)\frac{ss(\gamma)}{nSS(\gamma)} - (\sigma^2_e)\left(\frac{ss(e)}{nSS(e)}\right) \end{aligned} \tag{5.6}$$

$$\begin{aligned} R_{\sigma^2_e} &= \frac{ss(e)}{U_3} \\ &= \sigma^2_e \frac{ss(e)}{SS(e)} \end{aligned} \tag{5.7}$$

Let

$$R_{\mathbf{\Sigma}} = \frac{1}{bn}\left(\frac{ss(\gamma)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{ss(\beta)}{U_1} - \frac{ss(\gamma)}{U_2}\right)\boldsymbol{U}_a.$$

It can be easily verified that $R_{\mathbf{\Sigma}}$ is a FGPQ for the variance matrix $\mathbf{\Sigma}$ of $\bar{\boldsymbol{Y}}$. The observed sums of squares for γ, β,and e are $ss(\gamma)$, $ss(\beta)$, $ss(e)$ respectively.

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Proposition 5.2.1

Consider a balanced two-factor crossed mixed linear model with fixed factor A and random factor B defined in Equation (5.3). Define $\boldsymbol{A}$ to be the lower-triangular cholesky factor of $\boldsymbol{\Sigma}$ so that

$$\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}' \tag{5.8}$$

Let

$$R_{\boldsymbol{\Sigma}} = \frac{1}{bn}\left(\frac{ss(\gamma)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{ss(\beta)}{U_1} - \frac{ss(\gamma)}{U_2}\right)\boldsymbol{U}_a \tag{5.9}$$

where $U_1 \sim \chi^2_{b-1}$ is independent of $U_2 \sim \chi^2_{(a-1)(b-1)}$ and $\boldsymbol{Z}$ is a vector of independent $N(0,1)$ distributions. Let $R_{\boldsymbol{A}}$ be the lower-triangular cholesky factor of $R_{\boldsymbol{\Sigma}}$ so that

$$R_{\boldsymbol{\Sigma}} = R_{\boldsymbol{A}}R'_{\boldsymbol{A}}. \tag{5.10}$$

Then $R_{\boldsymbol{\mu}}$, FGPQ for $\boldsymbol{\mu}$, defined by

$$R_{\boldsymbol{\mu}} = \bar{\boldsymbol{y}} - R_{\boldsymbol{A}}\boldsymbol{Z} \tag{5.11}$$

The two-sided simultaneous generalized confidence intervals for the collection of $\mu_i, i = 1, \ldots, a$, are given by

$$\hat{\mu}_i \pm d_{1-\alpha}\sqrt{\widehat{var}(\hat{\mu}_i)} \tag{5.12}$$

where $d_{1-\alpha}$ is the lower $1-\alpha$ percentile point of the distribution of

$$D = \max_{1\le i\le a}\left\{\frac{|\bar{y}_i - R_{\mu_i}|}{\sqrt{\widehat{var}(\hat{\mu}_i)}}\right\}, \tag{5.13}$$

$$\widehat{Var}(\hat{\mu}_i) = \frac{1}{abn}(MS(\beta) + (a-1)MS(\gamma)), \tag{5.14}$$

$$\widehat{var}(\hat{\mu}_i) = \frac{1}{abn}(ms(\beta) + (a-1)ms(\gamma)) \tag{5.15}$$

is the observed value of $\widehat{Var}(\hat{\mu}_i)$, $MS(\beta) = SS(\beta)/(b-1)$, $MS(\gamma) = SS(\gamma)/(a-1)(b-1)$, $ms(\beta) = ss(\beta)/(b-1)$, $ms(\gamma) = ss(\gamma)/(a-1)(b-1)$ and R_{μ_i} is the i^{th} component of $R_{\boldsymbol{\mu}}$.

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Proof:

The elements of the matrix R_{Σ} are functions of independent χ^2 random variables U_1 and U_2. Thus $R_{\boldsymbol{\mu}}$ is a function of independent $N(0,1)$ and χ^2 random variables. So the distribution of $R_{\boldsymbol{\mu}}$ is free of all parameters. The observed value $r_{\boldsymbol{\mu}}$ of $R_{\boldsymbol{\mu}}$ is given by $r_{\boldsymbol{\mu}} = \bar{\boldsymbol{y}} - \boldsymbol{A}\boldsymbol{A}^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{\mu}) = \boldsymbol{\mu}$ which does not depend on nuisance parameters. Thus $R_{\boldsymbol{\mu}}$ is a generalized pivotal quantity for $\boldsymbol{\mu}$. Notice also that

$$D = \max_{1 \le i \le a} \left\{ \frac{|\bar{y}_i - R_{\mu_i}|}{\sqrt{\widehat{var}(\hat{\mu}_i)}} \right\} = \max \left\{ \frac{|\bar{\boldsymbol{y}} - R_{\boldsymbol{\mu}}|}{\sqrt{\widehat{var}(\hat{\mu}_i)}} \right\} = \max \left\{ \frac{|R_{\boldsymbol{A}}\boldsymbol{Z}|}{\sqrt{\widehat{var}(\hat{\mu}_i)}} \right\}$$

and hence the distribution of D does not depend on unknown parameters. Here, given a vector $\boldsymbol{y} = (y_1, \ldots, y_n)'$, we use the notation $\max\{|\boldsymbol{y}|\}$ to mean $\max_{1 \le i \le n}\{|y_i|\}$. The $1-\alpha$ percentile $d_{1-\alpha}$ of the random variable D may be obtained by computer simulation.

5.3 Simulation Study

A simulation study was conducted to assess the performance of the simultaneous GCIs for the collection $\{\mu_i, i = 1, \ldots, a\}$. Without loss of generality we can assume that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\sigma_e^2 = 1$. A proof of this fact is given in the Appendix to this chapter. Several different combinations of sample sizes and values of σ_β^2 and σ_γ^2 were considered. The familywise coverage error rates were estimated via simulation. The parameter and sample size settings that were used in the simulation study are summarized in Table 5.3.1.

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Table 5.3.1: Parameter and Sample Size Settings for the Simulation Study

Quantity	Values
(a, b, n)	(3, 5, 5), (3, 5, 10), (3, 10, 5), (3, 10, 10) (5, 5, 5), (5, 5, 10), (5, 10, 5), (5, 10, 10) (10, 10, 10), (10, 100, 10)
$\sigma 2_\beta$	0.25, 0.50, 1.00, 4.00, 16.00
σ^2_γ	0.25, 0.50, 1.00, 4.00, 16.00

All combinations of values of (a, b, n), σ^2_β, and σ^2_γ were considered in the simulation study

5.3.1 Simulation Details

The simulation study is conducted using the following steps.

Step 1. Set $\mu_i = 0, i = 1, \ldots, a$ and $\sigma^2_e = 1$. Select one of the settings for sample sizes and variance parameters from Table 5.3.1. Calculate Σ, the variance matrix for the vector of factor A marginal sample means. Calculate $\boldsymbol{A}$ as in Equation (5.8).

Step 2. Generate the object $\left(\bar{\boldsymbol{Y}}, SS(\beta), SS(\gamma), SS(e)\right)$ with independent components as follows.

$$\begin{aligned} \bar{\boldsymbol{Y}} &= \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Z}, \\ SS(\beta) &= (an\sigma^2_\beta + n\sigma^2_\gamma + \sigma^2_e)U_1, \\ SS(\gamma) &= (n\sigma^2_\gamma + \sigma^2_e)U_2, \\ SS(e) &= \sigma^2_e U_3 \end{aligned}$$

where $\boldsymbol{Z}$ is a $a \times 1$ vector of independent $N(0, 1)$ random variables and U_1, U_2, U_3 are independent $\chi 2$ random variables, independent of $\boldsymbol{Z}$, and with degrees of freedom equal to $b - 1$, $(a - 1)(b - 1)$, and $ab(n - 1)$, respectively. Denote these realizations by $(\bar{\boldsymbol{y}}, ss(\beta), ss(\gamma), ss(e))$.

Step 3. Generate Q independent copies of

$$R_{\boldsymbol{\Sigma}} = \frac{1}{bn}\left(\frac{ss(\gamma)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{ss(\beta)}{U_1} - \frac{ss(\gamma)}{U_2}\right)\boldsymbol{U}_a$$

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where U_1 is a chi-squared random variable with $(b-1)$ degrees of freedom and U_2 is a chi-squared random variable with $(a-1)(b-1)$ degrees of freedom and U_1, U_2 are independent. Correspondingly, calculate Q independent copies of $R_{\boldsymbol{A}}$ using Equation (5.9). Denote these by $R_{\boldsymbol{\Sigma},s}$ and $R_{\boldsymbol{A},s}$, respectively, for $s = 1, \ldots, Q$. We used $Q = 10000$. Further calculate Q independent copies of the random vector $\boldsymbol{Z}$ whose components are independent $N(0,1)$ random variables. Denote these by $\boldsymbol{Z}_s$, $s = 1, \ldots, Q$.

Step 4. For $s = 1, \ldots, Q$, calculate

$$D_s = \max \frac{\left| R_{\boldsymbol{A},s} \boldsymbol{Z}_s \right|}{\sqrt{\widehat{var}(\hat{\mu}_i)}}$$

where

$$\widehat{var}(\hat{\mu}_i) = \frac{1}{abn}(ms(\beta) + (a-1)ms(\gamma))$$

with $ms(\beta) = ss(\beta)/(b-1)$ and $ms(\gamma) = ss(\gamma)/[(a-1)(b-1)]$.

Step 5. Sort the values $D_1, \ldots, D_Q$ to obtain the sample order statistics $D_{(1)}, \ldots, D_{(Q)}$. Set $d_{1-\alpha} = D_{\lceil Q(1-\alpha) \rceil}$. When $\alpha = 0.05$ and $Q = 10000$, we have $d_{1-\alpha} = D_{9500}$.

Step 6. Construct the simultaneous confidence intervals as

$$\bar{x}_i \pm d_{1-\alpha} \sqrt{\widehat{var}(\hat{\mu}_i)}$$

and test if the true μ_i fall in the constructed confidence intervals simultaneously. If they do, then count this as a *success*. Otherwise, count it as a *failure*.

Perform M independent repetitions of steps 1-6 above. We used $M = 1000$. Calculate the proportion of these M trials that result in a *success*, i.e., the proportion of times the GCIs simultaneously cover their respective parameters. Denote this proportion by p. This is the estimated familywise coverage rate. The corresponding familywise error rate is $q = 1 - p$.

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The simulations were run for different number of replications, levels of fixed and random factors, and different combinations of σ_β^2 and σ_γ^2 ($\sigma_e^2 = 1$) and the familywise coverage error rates associated with 95% simultaneous GCIs were estimated. Table 5.3.2 summarizes the results of the simulation runs.

5.3.2 Discussion of Simulation Results

The number of replications n does not have a noticeable effect on the estimated coverage error rates. When the number of levels of both fixed factor and the random factor are small, the estimated coverage error rates are smaller than $\alpha = 0.05$. But when the number of levels of the random factor B becomes larger, the estimated coverage error rates are closer to 0.05. The estimated coverage error rates are very close to the stated value of 0.05 when number of levels of the fixed and random factors are large and the number of levels of the random factor is larger than the number of levels of the fixed factor.

5.4 Example

An example is discussed to illustrate the computation of simultaneous GCIs for all cell-means in the model of Section 5.2. This example was obtained from Hocking (1996, Example 15.9) and detailed data were obtained from Milliken and Johnson (1984). A company is considering replacing the machines used to make a certain component in one of its factories. Three different brands of machines were available. Six employees were randomly selected to participate in the experiment. Each employee was asked to operate each machine three different times. The data are shown in table 5.4.1.

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Table 5.3.2: Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ_β^2 and σ_γ^2.

a	b	n	σ_β^2	σ_γ^2	σ_e^2	Correlation	Estimated coverage error rate
3	5	5	0.25	0.25	1	0.357	0.026
3	5	5	0.25	0.5	1	0.263	0.020
3	5	5	0.25	1	1	0.172	0.021
3	5	5	0.25	4	1	0.056	0.021
3	5	5	0.25	16	1	0.015	0.022
3	5	5	0.5	0.25	1	0.526	0.031
3	5	5	0.5	0.5	1	0.417	0.026
3	5	5	0.5	1	1	0.294	0.025
3	5	5	0.5	4	1	0.106	0.020
3	5	5	0.5	16	1	0.03	0.022
3	5	5	1	0.25	1	0.690	0.032
3	5	5	1	0.5	1	0.588	0.032
3	5	5	1	1	1	0.454	0.026
3	5	5	1	4	1	0.192	0.021
3	5	5	1	16	1	0.058	0.021
3	5	5	4	0.25	1	0.899	0.044
3	5	5	4	0.5	1	0.851	0.042
3	5	5	4	1	1	0.769	0.037
3	5	5	4	4	1	0.488	0.029
3	5	5	4	16	1	0.198	0.022
3	5	5	16	0.25	1	0.973	0.048
3	5	5	16	0.5	1	0.958	0.049
3	5	5	16	1	1	0.930	0.046
3	5	5	16	4	1	0.792	0.038
3	5	5	16	16	1	0.497	0.029

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2:(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
3	5	10	0.25	0.25	1	0.417	0.026
3	5	10	0.25	0.5	1	0.294	0.025
3	5	10	0.25	1	1	0.185	0.021
3	5	10	0.25	4	1	0.057	0.021
3	5	10	0.25	16	1	0.015	0.022
3	5	10	0.5	0.25	1	0.588	0.032
3	5	10	0.5	0.5	1	0.454	0.026
3	5	10	0.5	1	1	0.312	0.026
3	5	10	0.5	4	1	0.109	0.02
3	5	10	0.5	16	1	0.03	0.022
3	5	10	1	0.25	1	0.741	0.036
3	5	10	1	0.5	1	0.625	0.031
3	5	10	1	1	1	0.476	0.029
3	5	10	1	4	1	0.196	0.021
3	5	10	1	16	1	0.058	0.021
3	5	10	4	0.25	1	0.919	0.046
3	5	10	4	0.5	1	0.869	0.043
3	5	10	4	1	1	0.784	0.037
3	5	10	4	4	1	0.494	0.03
3	5	10	4	16	1	0.199	0.022
3	5	10	16	0.25	1	0.978	0.05
3	5	10	16	0.5	1	0.964	0.049
3	5	10	16	1	1	0.936	0.048
3	5	10	16	4	1	0.796	0.038
3	5	10	16	16	1	0.498	0.031

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
3	10	5	0.25	0.25	1	0.357	0.036
3	10	5	0.25	0.5	1	0.263	0.035
3	10	5	0.25	1	1	0.172	0.029
3	10	5	0.25	4	1	0.056	0.033
3	10	5	0.25	16	1	0.015	0.033
3	10	5	0.5	0.25	1	0.526	0.04
3	10	5	0.5	0.5	1	0.417	0.036
3	10	5	0.5	1	1	0.294	0.034
3	10	5	0.5	4	1	0.106	0.031
3	10	5	0.5	16	1	0.03	0.034
3	10	5	1	0.25	1	0.690	0.042
3	10	5	1	0.5	1	0.588	0.04
3	10	5	1	1	1	0.454	0.038
3	10	5	1	4	1	0.192	0.03
3	10	5	1	16	1	0.058	0.033
3	10	5	4	0.25	1	0.899	0.051
3	10	5	4	0.5	1	0.851	0.052
3	10	5	4	1	1	0.769	0.046
3	10	5	4	4	1	0.488	0.04
3	10	5	4	16	1	0.198	0.031
3	10	5	16	0.25	1	0.973	0.051
3	10	5	16	0.5	1	0.958	0.052
3	10	5	16	1	1	0.930	0.051
3	10	5	16	4	1	0.792	0.049
3	10	5	16	16	1	0.497	0.039

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
3	10	10	0.25	0.25	1	0.417	0.036
3	10	10	0.25	0.5	1	0.294	0.034
3	10	10	0.25	1	1	0.185	0.03
3	10	10	0.25	4	1	0.057	0.033
3	10	10	0.25	16	1	0.015	0.033
3	10	10	0.5	0.25	1	0.588	0.04
3	10	10	0.5	0.5	1	0.454	0.038
3	10	10	0.5	1	1	0.312	0.034
3	10	10	0.5	4	1	0.109	0.031
3	10	10	0.5	16	1	0.030	0.034
3	10	10	1	0.25	1	0.741	0.044
3	10	10	1	0.5	1	0.625	0.04
3	10	10	1	1	1	0.476	0.038
3	10	10	1	4	1	0.196	0.031
3	10	10	1	16	1	0.058	0.033
3	10	10	4	0.25	1	0.919	0.051
3	10	10	4	0.5	1	0.869	0.055
3	10	10	4	1	1	0.784	0.048
3	10	10	4	4	1	0.494	0.04
3	10	10	4	16	1	0.199	0.031
3	10	10	16	0.25	1	0.978	0.051
3	10	10	16	0.5	1	0.964	0.051
3	10	10	16	1	1	0.936	0.051
3	10	10	16	4	1	0.796	0.049
3	10	10	16	16	1	0.498	0.039

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
5	5	5	0.25	0.25	1	0.357	0.03
5	5	5	0.25	0.5	1	0.263	0.031
5	5	5	0.25	1	1	0.172	0.033
5	5	5	0.25	4	1	0.056	0.036
5	5	5	0.25	16	1	0.015	0.037
5	5	5	0.5	0.25	1	0.526	0.031
5	5	5	0.5	0.5	1	0.417	0.029
5	5	5	0.5	1	1	0.294	0.031
5	5	5	0.5	4	1	0.106	0.033
5	5	5	0.5	16	1	0.030	0.036
5	5	5	1	0.25	1	0.690	0.04
5	5	5	1	0.5	1	0.588	0.029
5	5	5	1	1	1	0.454	0.029
5	5	5	1	4	1	0.192	0.033
5	5	5	1	16	1	0.058	0.036
5	5	5	4	0.25	1	0.899	0.044
5	5	5	4	0.5	1	0.851	0.042
5	5	5	4	1	1	0.769	0.041
5	5	5	4	4	1	0.488	0.031
5	5	5	4	16	1	0.198	0.034
5	5	5	16	0.25	1	0.973	0.043
5	5	5	16	0.5	1	0.958	0.044
5	5	5	16	1	1	0.930	0.042
5	5	5	16	4	1	0.972	0.042
5	5	5	16	16	1	0.497	0.031

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
5	5	10	0.25	0.25	1	0.417	0.029
5	5	10	0.25	0.5	1	0.294	0.031
5	5	10	0.25	1	1	0.185	0.032
5	5	10	0.25	4	1	0.057	0.036
5	5	10	0.25	16	1	0.015	0.037
5	5	10	0.5	0.25	1	0.588	0.029
5	5	10	0.5	0.5	1	0.454	0.029
5	5	10	0.5	1	1	0.312	0.031
5	5	10	0.5	4	1	0.109	0.033
5	5	10	0.5	16	1	0.03	0.036
5	5	10	1	0.25	1	0.741	0.041
5	5	10	1	0.5	1	0.625	0.034
5	5	10	1	1	1	0.476	0.029
5	5	10	1	4	1	0.196	0.035
5	5	10	1	16	1	0.058	0.036
5	5	10	4	0.25	1	0.919	0.041
5	5	10	4	0.5	1	0.869	0.043
5	5	10	4	1	1	0.784	0.042
5	5	10	4	4	1	0.494	0.031
5	5	10	4	16	1	0.199	0.034
5	5	10	16	0.25	1	0.978	0.043
5	5	10	16	0.5	1	0.964	0.044
5	5	10	16	1	1	0.936	0.042
5	5	10	16	4	1	0.796	0.042
5	5	10	16	16	1	0.498	0.031

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
5	10	5	0.25	0.25	1	0.357	0.047
5	10	5	0.25	0.5	1	0.263	0.05
5	10	5	0.25	1	1	0.172	0.047
5	10	5	0.25	4	1	0.056	0.053
5	10	5	0.25	16	1	0.015	0.059
5	10	5	0.5	0.25	1	0.526	0.047
5	10	5	0.5	0.5	1	0.417	0.045
5	10	5	0.5	1	1	0.294	0.049
5	10	5	0.5	4	1	0.106	0.048
5	10	5	0.5	16	1	0.030	0.057
5	10	5	1	0.25	1	0.690	0.046
5	10	5	1	0.5	1	0.588	0.048
5	10	5	1	1	1	0.454	0.047
5	10	5	1	4	1	0.192	0.046
5	10	5	1	16	1	0.058	0.052
5	10	5	4	0.25	1	0.899	0.051
5	10	5	4	0.5	1	0.851	0.049
5	10	5	4	1	1	0.769	0.046
5	10	5	4	4	1	0.488	0.048
5	10	5	4	16	1	0.198	0.047
5	10	5	16	0.25	1	0.973	0.054
5	10	5	16	0.5	1	0.958	0.053
5	10	5	16	1	1	0.930	0.053
5	10	5	16	4	1	0.792	0.046
5	10	5	16	16	1	0.497	0.048

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ_β^2 and σ_γ^2.

a	b	n	σ_β^2	σ_γ^2	σ_e^2	Correlation	Estimated coverage error rate
5	10	10	0.25	0.25	1	0.417	0.045
5	10	10	0.25	0.5	1	0.294	0.049
5	10	10	0.25	1	1	0.185	0.047
5	10	10	0.25	4	1	0.057	0.052
5	10	10	0.25	16	1	0.015	0.059
5	10	10	0.5	0.25	1	0.588	0.048
5	10	10	0.5	0.5	1	0.454	0.047
5	10	10	0.5	1	1	0.312	0.048
5	10	10	0.5	4	1	0.109	0.048
5	10	10	0.5	16	1	0.030	0.057
5	10	10	1	0.25	1	0.741	0.046
5	10	10	1	0.5	1	0.625	0.049
5	10	10	1	1	1	0.476	0.048
5	10	10	1	4	1	0.196	0.047
5	10	10	1	16	1	0.058	0.052
5	10	10	4	0.25	1	0.919	0.053
5	10	10	4	0.5	1	0.869	0.048
5	10	10	4	1	1	0.784	0.046
5	10	10	4	4	1	0.494	0.048
5	10	10	4	16	1	0.199	0.047
5	10	10	16	0.25	1	0.978	0.055
5	10	10	16	0.5	1	0.964	0.055
5	10	10	16	1	1	0.936	0.053
5	10	10	16	4	1	0.796	0.046
5	10	10	16	16	1	0.498	0.048

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ^2_β and σ^2_γ.

a	b	n	σ^2_β	σ^2_γ	σ^2_e	Correlation	Estimated coverage error rate
10	10	10	0.25	0.25	1	0.417	0.037
10	10	10	0.25	0.5	1	0.294	0.036
10	10	10	0.25	1	1	0.185	0.039
10	10	10	0.25	4	1	0.057	0.044
10	10	10	0.25	16	1	0.015	0.041
10	10	10	0.5	0.25	1	0.588	0.039
10	10	10	0.5	0.5	1	0.454	0.037
10	10	10	0.5	1	1	0.312	0.038
10	10	10	0.5	4	1	0.109	0.04
10	10	10	0.5	16	1	0.030	0.042
10	10	10	1	0.25	1	0.741	0.045
10	10	10	1	0.5	1	0.625	0.039
10	10	10	1	1	1	0.476	0.036
10	10	10	1	4	1	0.196	0.038
10	10	10	1	16	1	0.058	0.044
10	10	10	4	0.25	1	0.919	0.048
10	10	10	4	0.5	1	0.869	0.049
10	10	10	4	1	1	0.784	0.046
10	10	10	4	4	1	0.494	0.036
10	10	10	4	16	1	0.199	0.037
10	10	10	16	0.25	1	0.978	0.051
10	10	10	16	0.5	1	0.964	0.05
10	10	10	16	1	1	0.936	0.048
10	10	10	16	4	1	0.796	0.046
10	10	10	16	16	1	0.498	0.036

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.3.2.(*continued*) Estimated Coverage Error Rates Associated with 95% Simultaneous GCIs for $\{\mu_i, i = 1, \ldots, a\}$ for Different Combinations of a, b, n, σ_β^2 and σ_γ^2.

a	b	n	σ_β^2	σ_γ^2	σ_e^2	Correlation	Estimated coverage error rate
10	100	10	0.25	0.25	1	0.417	0.055
10	100	10	0.25	0.5	1	0.294	0.052
10	100	10	0.25	1	1	0.185	0.045
10	100	10	0.25	4	1	0.057	0.048
10	100	10	0.25	16	1	0.015	0.052
10	100	10	0.5	0.25	1	0.588	0.061
10	100	10	0.5	0.5	1	0.454	0.054
10	100	10	0.5	1	1	0.312	0.050
10	100	10	0.5	4	1	0.109	0.045
10	100	10	0.5	16	1	0.030	0.049
10	100	10	1	0.25	1	0.741	0.060
10	100	10	1	0.5	1	0.625	0.061
10	100	10	1	1	1	0.476	0.053
10	100	10	1	4	1	0.196	0.046
10	100	10	1	16	1	0.058	0.048
10	100	10	4	0.25	1	0.919	0.062
10	100	10	4	0.5	1	0.869	0.062
10	100	10	4	1	1	0.784	0.058
10	100	10	4	4	1	0.494	0.051
10	100	10	4	16	1	0.199	0.046
10	100	10	16	0.25	1	0.978	0.062
10	100	10	16	0.5	1	0.964	0.065
10	100	10	16	1	1	0.936	0.065
10	100	10	16	4	1	0.796	0.058
10	100	10	16	16	1	0.498	0.051

The column labeled 'Correlation' gives the correlation between any pair of marginal means of factor A.

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Table 5.4.1. Productivity Scores for Machine-Person Example

Machine	Person	1	2	3
1	1	52.0	52.8	53.1
1	2	51.8	52.8	53.1
1	3	60.0	60.2	58.4
1	4	51.1	52.3	50.3
1	5	50.9	51.8	51.4
1	6	46.4	44.8	49.2
2	1	62.1	62.6	64.0
2	2	59.7	60.0	59.0
2	3	68.6	65.8	69.7
2	4	63.2	62.8	62.2
2	5	64.8	65.0	65.4
2	6	43.7	44.2	43.0
3	1	67.5	67.2	66.9
3	2	61.5	61.7	62.3
3	3	70.8	70.6	71.0
3	4	64.1	66.2	64.0
3	5	72.1	72.0	71.1
3	6	62.0	61.4	60.5

Here `Machine` is considered a fixed factor and `Worker` is a random factor. The linear model used to analyze the data is a balanced two-factor crossed mixed linear model given below.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \quad i = 1, \cdots, 3, \quad j = 1, \cdots, 6, \quad k = 1, \cdots, 3$$

where Y_{ijk} denotes the k^{th} response for the i^{th} machine and the j^{th} worker, α_i is the i^{th} level of a fixed machine effect, $\beta_j \sim N(0, \sigma_\beta^2)$ is the j^{th} level of a random worker effect, $\gamma_{ij} \sim N(0, \sigma_\gamma^2)$ is the ij^{th} level of the random interaction effect, $e_{ijk} \sim N(0, \sigma_e^2)$ is the error term, and all random variables are mutually independent. The following summary statistics were computed.

$$\begin{array}{lll} \bar{y}_{1..} = 52.35, & \bar{y}_{\cdot\cdot 2} = 60.32, & \bar{y}_{\cdot\cdot 3} = 66.27, \\ ss(\alpha) = 1755.26, & ss(\beta) = 1241.89, & ss(\gamma) = 462.53. \end{array}$$

Generalized simultaneous confidence intervals were constructed for all three marginal means for machines. These are as follows.

$$45.00 \leq \mu_1 \leq 59.70, \quad 52.97 \leq \mu_2 \leq 67.67, \quad 58.92 \leq \mu_3 \leq 73.62$$

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5.5 Simultaneous GCIs for all Pairwise Differences in Three-factor Nested-factorial Model

This section concerns constructing simultaneous GCIs for all pairwise cell-mean differences for three factor nested factorial model. The factors are denoted by A, B, and C where A and B are fixed factors and C is a random factor. Levels of factor C are nested within levels of factor A but crossed with those of B. The mixed linear model we considered has the following form.

$$\begin{aligned} Y_{ijkl} &= \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma_{k(i)} + (\alpha\beta\gamma)_{jk(i)} + e_{ijkl}, \\ & \qquad i = 1, \cdots a, \; j = 1, \cdots b, \; k = 1, \cdots c, \; l = 1, \cdots n. \end{aligned} \tag{5.16}$$

Here Y_{ijkl} denotes the l^{th} response corresponding to the i^{th} level of A, j^{th} level of B and k^{th} level of C, α_i is the effect of i^{th} level of fixed main effect A, β_j is the the effect of j^{th} level of fixed main effect B, $(\alpha\beta)_{ij}$ is the effect of ij^{th} level of the interaction effects between factors A and B, $\gamma_{k(i)} \sim N\left(0, \sigma^2_{C(A)}\right)$ is the effect of $k(i)^{th}$ level of random main effect C (nested within A) , $(\alpha\beta\gamma)_{jk(i)} \sim N\left(0, \sigma^2_{BC(A)}\right)$ denotes the effect of $jk(i)^{th}$ level of interaction effects between B and C (nested within A), $e_{ijkl} \sim N\left(0, \sigma^2_e\right)$ are the error terms, and all random variables are mutually independent. Thus $\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$ where

$$\boldsymbol{\Sigma}^* = \sigma^2_{C(A)} \boldsymbol{V}_{C(A)} + \sigma^2_{BC(A)} \boldsymbol{V}_{BC(A)} + \sigma^2_e \boldsymbol{I}_{abcn}, \tag{5.17}$$

$$\boldsymbol{V}_{C(A)} = \boldsymbol{I}_a \otimes \boldsymbol{U}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{U}_n, \quad \text{and} \quad \boldsymbol{V}_{BC(A)} = \boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{U}_n.$$

Let $\bar{\boldsymbol{\mu}}$ denote the vector of marginal cell-means (arranged in lexicographic order) corresponding to the level combinations of factors A and B and let $\bar{\boldsymbol{Y}}$ denote the corresponding sample cell-means arranged in lexicographic order. We then have $\bar{\boldsymbol{Y}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}$ is a $ab \times ab$ matrix given by

$$\boldsymbol{\Sigma} = \frac{1}{nc} \boldsymbol{D}_{a \times a} \otimes \boldsymbol{I}_b \tag{5.18}$$

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and $\boldsymbol{D}_{a\times a} = (n\sigma^2_{BC(A)} + \sigma^2_e)\boldsymbol{I}_a + n\sigma^2_{C(A)}\boldsymbol{U}_a$. The ANOVA table for the model of this section is given in Table 5.5.1.

Table 5.5.1. ANOVA Table for the Balanced Three-Factor Nested-Factorial Mixed Linear Model

Source	df	SS	EMS
A	$a-1$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nb\sigma^2_{C(A)} +$ $n\sigma^2_{BC(A)} + \sigma^2_e +$ $nbcQ_1(\boldsymbol{\mu})$
B	$b-1$	$\boldsymbol{Y}'\left(\frac{\boldsymbol{U}_a}{a} \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma^2_{BC(A)} + \sigma^2_e +$ $nacQ_2(\boldsymbol{\mu})$
AB	$(a-1)(b-1)$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma^2_{BC(A)} + \sigma^2_e +$ $ncQ_{12}(\boldsymbol{\mu})$
$C(A)$	$a(c-1)$	$\boldsymbol{Y}'\left(\boldsymbol{I}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nb\sigma^2_{C(A)} +$ $n\sigma^2_{BC(A)} + \sigma^2_e$
$BC(A)$	$a(b-1)(c-1)$	$\boldsymbol{Y}'\left(\boldsymbol{I}_a \otimes \boldsymbol{S}_b \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$n\sigma^2_{BC(A)} + \sigma^2_e$
Residual	$abc(n-1)$	$\boldsymbol{Y}'(\boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{S}_n)\boldsymbol{Y}$	σ^2_e

The quantities $Q_1(\boldsymbol{\mu})$, $Q_2(\boldsymbol{\mu})$, and $Q_{12}(\boldsymbol{\mu})$ are functions of the fixed effect parameter $\boldsymbol{\mu}$. The following results are easily verified.

$$
\begin{aligned}
Var(\bar{Y}_{ij..} - \bar{Y}_{is..}) &= \frac{2}{nc}\left(n\sigma^2_{BC(A)} + \sigma^2_e\right) = \frac{2}{nc}\left(EMS(BC(A))\right), \quad j \neq s \\
Var(\bar{Y}_{ij..} - \bar{Y}_{rj..}) &= \frac{2}{nc}\left(n\sigma^2_{C(A)} + n\sigma^2_{BC(A)} + \sigma^2_e\right) \\
&= \frac{2}{bcn}\left(EMS(C(A) + (b-1)EMS(BC(A))\right), \quad i \neq r, \text{ and,} \\
Var(\bar{Y}_{ij..} - \bar{Y}_{rs..}) &= \frac{2}{cn}\left(n\sigma^2_{C(A)} + n\sigma^2_{BC(A)} + \sigma^2_e\right) \\
&= \frac{2}{bcn}\left(EMS(C(A) + (b-1)EMS(BC(A))\right) \quad i \neq r, j \neq s.
\end{aligned}
$$

Minimum variance unbiased estimators of these variances are obtained by substituting *mean squares* in place of *expected mean squares* in the above expressions.

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Note that $\widehat{Var}(\bar{Y}_{ij..} - \bar{Y}_{rj..}), i \neq r$, and $\widehat{Var}(\bar{Y}_{ij..} - \bar{Y}_{rs..}), i \neq r, j \neq s$, do not have scaled chi-square distributions. Thus there are no exact simultaneous CIs for collection of pairwise differences $\mu_{ij} - \mu_{rs}$. So we considered construction of simultaneous generalized confidence intervals for these differences.

Generalized pivotal quantities for the parameters $\sigma^2_{C(A)}, \sigma^2_{BC(A)}, \sigma^2_e$ and also for the vector $\boldsymbol{\mu}$ are needed to construct the simultaneous GCIs for all pairwise differences of cell-means corresponding to the fixed factors.

First define the following quantities.

$$\begin{aligned}
U_1 &= \frac{SS(C(A))}{\psi_{C(A)}}, & \psi_{C(A)} &= bn\sigma^2_{C(A)} + n\sigma^2_{BC(A)} + \sigma^2_e \\
U_2 &= \frac{SS(BC(A))}{\psi_{BC(A)}}, & \psi_{BC(A)} &= n\sigma^2_{BC(A)} + \sigma^2_e \\
U_3 &= \frac{SS(e)}{\psi_e}, & \psi_e &= \sigma^2_e.
\end{aligned}$$

Note that U_1, U_2, U_3 are independent $\chi 2$ random variables with degrees of freedom equal to $a(c-1)$, $a(b-1)(c-1)$, and $abc(n-1)$, respectively.

The FGPQs for $\sigma^2_{C(A)}$, $\sigma^2_{BC(A)}$, and σ^2_e are given below. For ease of notation, they are denoted by $\tilde{\sigma}^2_{C(A)}$, $\tilde{\sigma}^2_{BC(A)}$, and $\tilde{\sigma}^2_e$, respectively.

$$\begin{aligned}
\tilde{\sigma}^2_{C(A)} &= \left(\frac{\psi_{C(A)}\, ss(C(A))}{n\, b\, SS(C(A))}\right) - \left(\frac{\psi_{BC(A)}\, ss(BC(A))}{n\, b\, SS(BC(A))}\right), \\
\tilde{\sigma}^2_{BC(A)} &= \left(\frac{\psi_{BC(A)}\, ss(BC(A))}{n\, SS(BC(A))}\right) - \left(\frac{\psi_e\, ss(e)}{nSS(e)}\right), \\
\tilde{\sigma}^2_e &= \frac{\psi_e\, ss(e)}{SS(e)}.
\end{aligned}$$

As usual we use upper case notation for denoting observable random variables and lower case for the corresponding realized value.

Since $Var\left(\bar{Y}_{ij..}\right) = \left(\frac{1}{nc}(n\sigma^2_{C(A)} + n\sigma^2_{BC(A)} + \sigma^2_e)\right)$ and $Cov\left(\bar{Y}_{ij..}, \bar{Y}_{rs..}\right)) = \left(\frac{\sigma^2_{C(A)}}{c}\right)$, we can conclude that a FGPQ for $(Var(\bar{Y}_{ij..}))$ is given by

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$\frac{1}{bcn}\left(\frac{ss(C(A))}{U_1}+(b-1)\frac{ss(BC(A))}{U_2}\right)$ and a FGPQ for $Cov(\bar{Y}_{ij..},\bar{Y}_{rs..}))$ is given by the quantity $\frac{1}{bcn}\left(\frac{ss(C(A))}{U_1}-\frac{ss(BC(A))}{U_2}\right)$. Then the following proposition.

Proposition 5.5.1

Consider the balanced three-factor nested factorial mixed linear model defined in Equation (5.16). Let $\boldsymbol{Y}$ be the vector of responses arranged in lexicographic order and $\bar{\boldsymbol{Y}}$ be the vector of marginal sample cell-means corresponding to level combinations of the fixed factors A and B. Thus

$$\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$$

where $\boldsymbol{\Sigma}^*$ is defined in Equation (5.17) and $\bar{\boldsymbol{Y}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}$ is defined in Equation (5.18). Then a generalized pivotal quantity for $\boldsymbol{\mu}$ is given by

$$R_{\boldsymbol{\mu}} = \bar{\boldsymbol{y}} - R_A \boldsymbol{Z} \tag{5.19}$$

where $\bar{\boldsymbol{y}}$ is the realized value of $\bar{\boldsymbol{Y}}$. A FGPQ for $\boldsymbol{\Sigma}$ is given by

$$R_{\boldsymbol{\Sigma}} = \boldsymbol{E} \otimes \boldsymbol{I}_b$$

where $\boldsymbol{E}$ is the $a \times a$ matrix given by

$$\boldsymbol{E} = \frac{1}{bcn}\left[\left(\frac{ss(C(A))}{U_1} - \frac{ss(BC(A))}{U_2}\right) U_a\right] + \frac{1}{cn}\left[\left(\frac{ss(BC(A))}{U_2}\right) I_a\right]$$

U_1, U_2, $\boldsymbol{Z}$ are independent random variables with $U_1 \sim \chi^2_{a(c-1)}$, $U_2 \sim \chi^2_{a(b-1)(c-1)}$ and $\boldsymbol{Z}$ a vector of length ab consisting of independent standard normal distribution. Let $R_{\boldsymbol{A}}$ be a matrix such that

$$R_{\boldsymbol{\Sigma}} = R_{\boldsymbol{A}} R'_{\boldsymbol{A}}$$

The two-sided simultaneous GCIs for all mean differences $\mu_{ij} - \mu_{rs}$ are given by

$$(\hat{\mu}_{ij} - \hat{\mu}_{rs}) \pm d_{1-\alpha}\sqrt{\widehat{var}(\hat{\mu}_{ij} - \hat{\mu}_{rs})} \tag{5.20}$$

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where $d_{1-\alpha}$ is the lower $1-\alpha$ percentile point of the distribution of the random variable D defined as follows.

$$D = \max_{\substack{1 \le i, r \le a, \\ 1 \le j, s \le b}} \left\{ \frac{|(\bar{y}_{ij..} - (\bar{y}_{rs..}) - R_{\mu_{ij} - \mu_{rs}}|}{\sqrt{\widehat{var}(\bar{Y}_{ij..} - \bar{Y}_{rs..})}} \right\}.$$

Note that

$$\begin{aligned}
\widehat{var}(\bar{Y}_{ij..} - \bar{Y}_{is..}) &= \frac{2\, ms(BC(A))}{nc}, \quad j \neq s \\
\widehat{var}(\bar{Y}_{ij..} - \bar{Y}_{rj..}) &= \frac{2\,(ms(C(A)) + (b-1)ms(BC(A))}{bcn}, \qquad i \neq r, \text{ and,} \\
\widehat{var}(\bar{Y}_{ij..} - \bar{Y}_{rs..}) &= \frac{2\,(ms(C(A)) + (b-1)ms(BC(A))}{bcn}, \qquad i \neq r, j \neq s.
\end{aligned}$$

and ms represents the realized values for the various mean squares in the ANOVA table.

Proof:

The elements of matrix $R_{\boldsymbol{\Sigma}}$ are functions of independent χ^2 random variables. Thus the elements of $R_{\boldsymbol{\mu}}$ are functions of independent $N(0,1)$ and χ^2 random random variables. So the distribution of $R_{\boldsymbol{\mu}}$ is free of all unknown parameters. The realized value of $R_{\boldsymbol{\mu}}$, denoted by $r_{\boldsymbol{\mu}}$, is $r_{\boldsymbol{\mu}} = \bar{\boldsymbol{y}} - A\ A^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{\mu}) = \boldsymbol{\mu}$ which does not depend on nuisance parameters. Thus $R_{\boldsymbol{\mu}}$ is a FGPQ for $\boldsymbol{\mu}$. Furthermore, the distribution of D is easily verified to be free of unknown parameters. The percentile $d_{1-\alpha}$ of the distribution of D may be conveniently obtained by computer simulation.

5.6 Simulation Study

A simulation study was conducted to assess the performance of the simultaneous GCIs for the collection $\{\mu_{ij} - \mu rs, i, r = 1, \ldots, a,\ j, s = 1, \ldots, b\}$. Without loss of generality, we can assume that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\sigma_e^2 = 1$. See **Proposition A2** in the Appendix to this chapter. Several different combinations of numbers of levels for each factor and values of $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$ were considered. The familywise coverage

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error rates were estimated via simulation. The parameter and sample size settings that were used in the simulation study are summarized in Table 5.6.1.

Table 5.6.1. Parameter and Sample Size Settings for the Simulation Study

Quantity	Values
(a, b, c)	(3, 3, 2), (3, 3, 10), (4, 4, 2), (4, 4, 10) (5, 5, 2), (5, 5, 10)
$\sigma^2_{C(A)}$	0.25, 0.50, 1.00, 4.00, 16.00
$\sigma^2_{BC(A)}$	0.25, 0.50, 1.00, 4.00, 16.00

All combinations of values of (a, b, c), $\sigma^2_{C(A)}$, and $\sigma^2_{BC(A)}$ were considered in the simulation study with the exception that $\sigma^2_{C(A)} = 0.5, 4$ and $\sigma^2_{BC(A)} = 0.5, 4$ were not considered for the combinations $(a, b, c) = (5, 5, 2)$ and $(a, b, c) = (5, 5, 10)$. The number of replications n was taken to be 1 in all cases.

5.6.1 Simulation Details

The simulation study was conducted using the following steps.

Step 1. Set $\mu_{ij} = 0$ for all $1 \leq i \leq a$, $1 \leq j \leq b$ and set $\sigma_e^2 = 1$. Select one of the settings from Table 5.6.1 for the number of levels for the factors and variance parameters. Calculate $\mathbf{\Sigma}^*$, the variance matrix of the vector of the observations.

Step 2. Generate the data vector $\boldsymbol{Y}$ from the appropriate multivariate normal distribution. Calculate $SS(C(A))$, $SS(BC(A))$. Here $\boldsymbol{Y}$ is the full data vector and $SS(C(A))$, $SS(BC(A))$ are ANOVA sums of squares corresponding to the effects $C(A)$ and $BC(A)$. The vector $\boldsymbol{Y}$ may be generated as

$$\boldsymbol{Y} = \boldsymbol{A}^* \boldsymbol{Z}^*$$

where $\boldsymbol{Y}$, $\boldsymbol{Z}^*$ are vectors of length $abcn$ and $\boldsymbol{A}^*$ is a matrix such that

$$\boldsymbol{A}^* \boldsymbol{A}^{*\prime} = \mathbf{\Sigma}^*.$$

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Using the raw data vector $\boldsymbol{Y}$, the sample means $\bar{Y}_{ij..}$, the random effects sums of squares $SS(C(A))$ and $SS(BC(A))$, and the corresponding mean squares are calculated as follows:

$$\bar{Y}_{ij..} = \frac{1}{cn}\sum_{k=1}^{c}\sum_{l=1}^{n} Y_{ijkl} \tag{5.21}$$

$$SS_{C(A)} = \boldsymbol{Y}'\left(\boldsymbol{I}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \left(\boldsymbol{I}_c - \frac{\boldsymbol{U}_c}{c}\right) \otimes 1\right)\boldsymbol{Y}$$

$$MS_{C(A)} = \frac{SS_{C(A)}}{a(c-1)} \tag{5.22}$$

$$SS_{BC(A)} = \boldsymbol{Y}'\left(\boldsymbol{I}_a \otimes \left(\boldsymbol{I}_b - \frac{\boldsymbol{U}_b}{b}\right) \otimes \left(\boldsymbol{I}_c - \frac{\boldsymbol{U}_c}{c}\right) \otimes 1\right)\boldsymbol{Y}$$

$$MS_{BC(A)} = \frac{SS_{BC(A)}}{a(b-1)(c-1)}. \tag{5.23}$$

Step 3. Generate Q independent copies of the random vector $\boldsymbol{Z}$ of length ab consisting of independent $N(0,1)$ components. Denote these by $\boldsymbol{Z}_{(1)}, \ldots, \boldsymbol{Z}_{(Q)}$. Independently generate Q independent copies of $U_1 \sim \chi^2_{a(c-1)}$ and $U_2 \sim \chi^2_{a(b-1)(c-1)}$ where U_1 is independent of U_2. Denote these by $U_{1,q}$, $U_{2,q}$, for $q = 1, \ldots, Q$.

Step 4. Generate Q independent copies of the random matrix $R_{\boldsymbol{\Sigma}}$, denoted by $R_{\boldsymbol{\Sigma},q}$, $q = 1, \ldots, Q$, using

$$R_{\boldsymbol{\Sigma},q} = \left\{\frac{1}{bcn}\left[\left(\frac{ss_q(C(A))}{U_{1,q}} - \frac{ss_q(BC(A))}{U_{2,q}}\right)\boldsymbol{U}_a\right] + \frac{1}{cn}\left[\left(\frac{ss(BC(A)))}{U_{2,q}}\right)\boldsymbol{I}_a\right]\right\}\otimes \boldsymbol{I}_b.$$

Calculate $R_{\boldsymbol{A},q}$ such that $R_{\boldsymbol{A},q}R'_{\boldsymbol{A},q} = R_{\boldsymbol{\Sigma}}$.

Step 5. For $q = 1, \ldots, Q$, calculate

$$R_{\boldsymbol{\mu},q} = \bar{\boldsymbol{y}} - R_{\boldsymbol{A}}\boldsymbol{Z}.$$

Step 6. For $q = 1, \ldots, Q$, calculate

$$D_q = \max_{\substack{1 \le i, r \le a \\ 1 \le j, s \le b \\ i, j \ne r, s}} \left\{\frac{|\boldsymbol{l}'_{ij,rs} R_{\boldsymbol{A}}\boldsymbol{Z}|}{\sqrt{\boldsymbol{l}'_{ij,rs}\hat{\boldsymbol{\Sigma}}^*\boldsymbol{l}_{ij,rs}}}\right\}$$

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where $\boldsymbol{l}'$ is a row vector which has a 1 in the position corresponding to μ_{ij}, -1 in the position corresponding to μ_{rs} and zeros elsewhere. Thus $\boldsymbol{l}'\boldsymbol{\mu} = \mu_{ij} - \mu_{rs}$. The quantity $\boldsymbol{l}'_{ij,rs}\hat{\Sigma}^*\boldsymbol{l}_{ij,rs}$ is equal to $\widehat{var}(\bar{Y}_{ij..} - \bar{Y}_{rs..})$. Note that

$$\widehat{var}(\bar{Y}_{ij..}-\bar{Y}_{rs..}) = \begin{cases} \dfrac{2}{nc}\left(n\hat{\sigma}^2_{BC(A)} + \hat{\sigma}^2_e\right) = \dfrac{2}{nc}\left(ms(BC(A))\right) & \text{when } i = r, j \neq s, \text{and} \\ \dfrac{2}{nc}\left(n\hat{\sigma}^2_{C(A)} + n\hat{\sigma}^2_{BC(A)} + \hat{\sigma}^2_e\right) = \dfrac{2}{bcn}\left(ms(C(A) + (b-1)ms(BC(A))\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } i \neq r. \end{cases}$$

In our study we used $Q = 10000$ and $\alpha = 0.05$. So the value of $d_{0.95}$ is estimated by $D_{(9500)}$ where $D_{(1)}, \ldots, D_{(10000)}$ are the realizations of D arranged in ascending order. Using this estimated value of $d_{0.95}$ the full set of simultaneous GCIs are calculated for all pairwise differences $\mu_{ij} - \mu_{rs}$. If the confidence intervals simultaneously cover their respective true values then a *success* is recorded. Otherwise a *failure* is recorded.

The entire process is repeated M times where M is a large number. We used $M = 1000$ in our study. The proportion of successes p out of these M trials is computed. Then $1 - p$ is the estimated familywise coverage error rate. The simulations were run for all parameter settings. The estimated simultaneous coverage error rates are shown in Table 5.6.2.

5.6.2 Discussion of Simulation Results

When the number of levels of the random effect is small then the estimated coverage error is less than 0.05. When the number of levels of the random effect is large then the estimated coverage error is somewhat larger than 0.05. The estimated coverage error is close to 0.05 when both number of levels of fixed factors and random effects are large.

5.6.3 Comparison of GCIs with Intervals Obtained from SAS for all Pairwise Differences $\mu_{ij} - \mu_{rs}$

Approximate simultaneous confidence intervals for all pairwise differences $\mu_{ij} - \mu_{rs}$ may also be obtained by using the statistical software package SAS.

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Table 5.6.2. Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous GCIs for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
3	3	2	.25	.25	0.032	10.082	2.801	11.243	2.828
3	3	2	.25	.5	0.032	11.019	3.059	12.120	3.024
3	3	2	.25	1	0.032	12.682	3.517	13.704	3.385
3	3	2	.25	4	0.032	19.924	5.524	20.811	5.059
3	3	2	.25	16	0.035	36.607	10.146	37.590	9.073
3	3	2	.5	.25	0.035	10.199	2.848	12.245	3.226
3	3	2	.5	.5	0.031	11.134	3.102	13.061	3.389
3	3	2	.5	1	0.031	12.795	3.558	14.551	3.703
3	3	2	.5	4	0.033	20.012	5.550	21.389	5.254
3	3	2	.5	16	0.033	36.666	10.164	37.922	9.179
3	3	2	1	.25	0.04	10.343	2.914	13.975	3.968
3	3	2	1	.5	0.036	11.287	3.171	14.712	4.085
3	3	2	1	1	0.036	12.957	3.626	16.071	4.323
3	3	2	1	4	0.032	20.165	5.602	22.486	5.656
3	3	2	1	16	0.032	36.766	10.195	38.562	9.391
3	3	2	4	.25	0.044	10.585	3.058	21.221	7.248
3	3	2	4	.5	0.042	11.570	3.331	21.769	7.273
3	3	2	4	1	0.041	13.306	3.809	22.802	7.335
3	3	2	4	4	0.04	20.685	5.828	27.951	7.937
3	3	2	4	16	0.031	37.268	10.361	42.145	10.689
3	3	2	16	.25	0.046	10.643	3.142	37.623	14.519
3	3	2	16	.5	0.048	11.663	3.436	38.006	14.499
3	3	2	16	1	0.046	13.465	3.953	38.728	14.476
3	3	2	16	4	0.044	21.170	6.116	42.442	14.495
3	3	2	16	16	0.038	38.266	10.803	53.561	15.523

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.2 (*continued*). Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous GCIs for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
3	3	10	.25	.25	0.056	3.203	0.307	3.522	0.287
3	3	10	.25	.5	0.055	3.509	0.336	3.804	0.308
3	3	10	.25	1	0.057	4.051	0.387	4.313	0.346
3	3	10	.25	4	0.059	6.403	0.611	6.856	0.524
3	3	10	.25	16	0.060	11.804	1.125	11.934	0.945
3	3	10	.5	.25	0.054	3.200	0.307	3.800	0.323
3	3	10	.5	.5	0.058	3.507	0.337	4.064	0.341
3	3	10	.5	1	0.057	4.051	0.388	4.546	0.374
3	3	10	.5	4	0.057	6.405	0.612	6.743	0.539
3	3	10	.5	16	0.059	11.806	1.253	12.022	0.954
3	3	10	1	.25	0.054	3.191	0.309	4.291	0.394
3	3	10	1	.5	0.056	3.499	0.338	4.530	0.406
3	3	10	1	1	0.055	4.046	0.389	4.971	0.431
3	3	10	1	4	0.056	6.406	0.613	7.043	0.574
3	3	10	1	16	0.058	11.808	1.126	12.195	0.970
3	3	10	4	.25	0.054	3.133	0.306	6.414	0.709
3	3	10	4	.5	0.054	3.443	0.336	6.589	0.713
3	3	10	4	1	0.053	3.994	0.389	6.919	0.720
3	3	10	4	4	0.054	6.381	0.617	8.582	0.787
3	3	10	4	16	0.056	11.811	1.131	13.175	1.081
3	3	10	16	.25	0.056	3.061	0.297	11.326	1.416
3	3	10	16	.5	0.057	3.363	0.327	11.442	1.414
3	3	10	16	1	0.056	3.901	0.380	11.664	1.413
3	3	10	16	4	0.054	6.267	0.612	12.829	1.418
3	3	10	16	16	0.052	11.751	1.397	16.407	1.539

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.2 (*continued*). Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous GCIs for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
4	4	2	.25	.25	0.022	9.869	1.883	10.940	2.065
4	4	2	.25	.5	0.022	10.783	2.058	11.785	2.193
4	4	2	.25	1	0.024	12.408	2.370	13.313	2.436
4	4	2	.25	4	0.026	19.473	3.726	20.169	3.582
4	4	2	.25	16	0.029	35.760	6.855	36.390	6.392
4	4	2	.5	.25	0.02	9.988	1.910	11.955	2.441
4	4	2	.5	.5	0.02	10.903	2.082	12.740	2.539
4	4	2	.5	1	0.021	12.526	2.390	14.173	2.731
4	4	2	.5	4	0.026	19.574	3.741	20.762	3.760
4	4	2	.5	16	0.028	35.824	6.862	36.727	6.480
4	4	2	1	.25	0.022	10.128	1.953	13.702	3.133
4	4	2	1	.5	0.021	11.055	2.127	14.408	3.191
4	4	2	1	1	0.02	12.691	2.432	15.713	3.318
4	4	2	1	4	0.022	19.739	3.765	21.881	4.130
4	4	2	1	16	0.025	35.941	6.874	37.386	6.662
4	4	2	4	.25	0.039	10.317	2.057	20.943	6.034
4	4	2	4	.5	0.038	11.289	2.239	21.473	6.031
4	4	2	4	1	0.036	13.003	2.560	22.471	6.040
4	4	2	4	4	0.022	20.257	2.906	27.404	6.267
4	4	2	4	16	0.022	36.484	6.962	41.036	7.860
4	4	2	16	.25	0.047	10.264	2.094	37.112	12.231
4	4	2	16	.5	0.045	11.262	2.296	37.504	12.213
4	4	2	16	1	0.046	13.033	2.647	38.236	12.181
4	4	2	16	4	0.039	20.633	4.114	41.886	12.068
4	4	2	16	16	0.022	37.467	7.243	52.566	12.378

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.2 (*continued*). Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous GCIs for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
4	4	10	.25	.25	0.055	3.503	0.244	3.843	0.246
4	4	10	.25	.5	0.055	3.837	0.267	4.151	0.263
4	4	10	.25	1	0.053	4.431	0.308	4.706	0.295
4	4	10	.25	4	0.055	7.006	0.485	7.186	0.445
4	4	10	.25	16	0.053	12.918	0.894	13.024	0.802
4	4	10	.5	.25	0.057	3.498	0.247	4.147	0.281
4	4	10	.5	.5	0.056	3.834	0.270	4.435	0.295
4	4	10	.5	1	0.056	4.430	0.310	4.961	0.322
4	4	10	.5	4	0.054	7.007	0.486	7.357	0.459
4	4	10	.5	16	0.054	12.918	0.894	13.119	0.809
4	4	10	1	.25	0.053	3.483	0.249	4.683	0.349
4	4	10	1	.5	0.053	3.821	0.272	4.944	0.358
4	4	10	1	1	0.056	4.420	0.313	5.425	0.378
4	4	10	1	4	0.055	7.005	0.489	7.686	0.492
4	4	10	1	16	0.054	12.919	0.895	13.308	0.824
4	4	10	4	.25	0.059	3.405	0.252	6.987	0.658
4	4	10	4	.5	0.055	3.745	0.277	7.180	0.657
4	4	10	4	1	0.054	4.348	0.319	7.545	0.658
4	4	10	4	4	0.053	6.966	0.499	9.366	0.698
4	4	10	4	16	0.056	12.915	0.903	14.378	0.930
4	4	10	16	.25	0.063	3.313	0.244	12.299	1.346
4	4	10	16	.5	0.059	3.641	0.268	12.429	1.342
4	4	10	16	1	0.055	4.226	0.312	12.677	1.336
4	4	10	16	4	0.059	6.811	0.504	13.975	1.316
4	4	10	16	16	0.052	12.823	0.922	17.905	1.370

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.2 (*continued*). Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous GCIs for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average$_1$	Standard deviation$_1$	Average$_2$	Standard deviation$_2$
5	5	2	.25	.25	0.022	9.730	1.463	10.660	1.714
5	5	2	.25	1	0.02	12.225	1.834	12.981	2.005
5	5	2	.25	16	0.02	35.206	5.287	35.530	5.190
5	5	2	1	.25	0.039	10.008	1.539	13.304	2.619
5	5	2	1	1	0.029	12.531	1.904	15.279	2.777
5	5	2	1	16	0.019	35.398	5.313	36.485	5.446
5	5	2	16	.25	0.051	10.049	1.694	35.583	9.935
5	5	2	16	1	0.052	12.791	2.137	36.733	9.889
5	5	2	16	16	0.038	37.029	5.726	51.012	10.325
5	5	10	.25	.25	0.054	3.720	0.200	4.063	0.200
5	5	10	.25	1	0.05	4.705	0.252	4.978	0.240
5	5	10	.25	16	0.055	13.712	0.731	13.789	0.654
5	5	10	1	.25	0.047	3.700	0.203	4.935	0.294
5	5	10	1	1	0.052	4.696	0.255	5.725	0.313
5	5	10	1	16	0.052	13.716	0.733	14.085	0.670
5	5	10	16	.25	0.043	3.499	0.192	12.848	1.206
5	5	10	16	1	0.047	4.467	0.247	13.253	1.188
5	5	10	16	16	0.047	13.619	0.750	18.858	1.160

Average$_1$ and Standard deviation$_1$ are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average$_2$ and Standard deviation$_2$ are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.3 reports the estimated coverage errors for nominally 95% simultaneous confidence intervals obtained using `PROC MIXED` in `SAS`. The parameter settings used for the simulation study are those given in Table 5.6.1.

Comparison of the simulation results for GCIs and SAS intervals

When the number of levels of the random factor is small the estimated coverage errors associated with the simultaneous GCIs are less than the estimated coverage error rates for the SAS intervals. The average lengths of the simultaneous GCIs is larger than the corresponding SAS intervals. When $\sigma^2_{C(A)}$ is large then the estimated coverage errors associated with the SAS intervals are larger than those associated with simultaneous GCIs. When the number of levels of the random factor is large the estimated coverage errors and the average lengths of the intervals for the two methods are very close to one another and the coverage probabilities are close to the nominal value of 95%.

5.7 Example

We now illustrate the computation of simultaneous GCIs for the three way nested factorial model using a data example taken from Hocking (1996, Example 15.10). This example is based on a study of high efficiency particulate air (HEPA) categories. For the study two aerosol types were used with filters from each of two different manufacturers. Each manufacturer provided three different filters. Since the filters were unique to the manufacturer, filters are nested within manufacturer. The data from this example are shown in the Table 5.7.1.

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Table 5.6.3. Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous CIs from SAS PROC MIXED for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
3	3	2	.25	.25	0.054	9.071	2.570	10.395	2.509
3	3	2	.25	.5	.052	9.911	2.793	11.240	2.699
3	3	2	.25	1	0.051	11.402	3.191	12.760	3.053
3	3	2	.25	4	0.050	17.877	4.936	19.527	4.658
3	3	2	.25	16	0.050	32.796	9.000	35.420	8.450
3	3	2	.5	.25	0.050	9.161	2.654	11.158	2.763
3	3	2	.5	.5	0.052	10.012	2.882	11.954	2.931
3	3	2	.5	1	0.052	11.512	3.282	13.400	3.248
3	3	2	.5	4	0.050	17.983	5.011	19.963	4.770
3	3	2	.5	16	0.050	32.868	9.043	35.667	8.506
3	3	2	1	.25	0.053	9.216	2.724	12.489	3.282
3	3	2	1	.5	0.051	10.085	2.968	13.220	3.408
3	3	2	1	1	0.050	11.619	3.390	14.562	3.661
3	3	2	1	4	0.054	18.142	5.140	20.791	5.009
3	3	2	1	16	0.050	33.001	9.128	36.151	8.624
3	3	2	4	.25	0.073	9.164	2.766	18.203	5.841
3	3	2	4	.5	0.071	10.057	3.027	18.763	5.859
3	3	2	4	1	0.066	11.640	3.493	19.812	5.917
3	3	2	4	4	0.053	18.433	5.448	24.979	6.565
3	3	2	4	16	0.054	33.534	9.543	38.856	9.400
3	3	2	16	.25	0.089	8.969	2.707	31.491	11.484
3	3	2	16	.5	0.086	9.854	2.976	31.880	11.813
3	3	2	16	1	0.084	11.430	3.452	32.610	11.756
3	3	2	16	4	0.073	18.329	5.532	36.406	11.683
3	3	2	16	16	0.054	34.005	10.080	47.630	12.765

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.3 (*continued*). Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous CIs from SAS PROC MIXED for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
3	3	10	.25	.25	0.056	3.191	0.302	3.517	0.285
3	3	10	.25	.5	0.057	3.493	0.328	3.800	0.307
3	3	10	.25	1	0.060	4.027	0.374	4.310	0.346
3	3	10	.25	4	0.058	6.328	0.571	6.587	0.524
3	3	10	.25	16	0.059	11.607	1.027	11.940	0.947
3	3	10	.5	.25	0.057	3.191	0.309	3.788	0.319
3	3	10	.5	.5	0.058	3.497	0.337	4.054	0.337
3	3	10	.5	1	0.057	4.039	0.386	4.537	0.372
3	3	10	.5	4	0.060	6.359	0.586	6.740	0.539
3	3	10	.5	16	0.058	11.636	1.038	12.027	0.955
3	3	10	1	.25	0.056	3.174	0.311	4.268	0.387
3	3	10	1	.5	0.056	3.484	0.340	4.509	0.400
3	3	10	1	1	0.056	4.031	0.393	4.952	0.426
3	3	10	1	4	0.056	6.383	0.604	7.034	0.571
3	3	10	1	16	0.060	11.681	1.058	12.196	0.971
3	3	10	4	.25	0.060	3.103	0.305	6.351	0.695
3	3	10	4	.5	0.059	3.411	0.336	6.527	0.698
3	3	10	4	1	0.057	3.960	0.390	6.860	0.705
3	3	10	4	4	0.056	6.349	0.622	8.537	0.774
3	3	10	4	16	0.057	11.774	1.122	13.153	1.075
3	3	10	16	.25	0.063	3.024	0.293	11.187	1.391
3	3	10	16	.5	0.064	3.322	0.323	11.304	1.389
3	3	10	16	1	0.060	3.855	0.376	11.527	1.386
3	3	10	16	4	0.060	6.205	0.610	12.701	1.390
3	3	10	16	16	0.053	11.685	1.146	16.311	1.509

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.
Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.3 *(continued)*. Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous CIs from SAS PROC MIXED for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
4	4	2	.25	.25	0.040	9.087	1.815	10.195	1.817
4	4	2	.25	.5	0.040	9.940	1.978	11.020	1.950
4	4	2	.25	1	0.041	11.451	2.267	12.507	2.197
4	4	2	.25	4	0.040	18.000	3.520	19.128	3.334
4	4	2	.25	16	0.042	33.050	6.413	34.685	6.041
4	4	2	.5	.25	0.041	9.125	1.851	10.954	2.055
4	4	2	.5	.5	0.042	9.989	2.017	11.732	2.162
4	4	2	.5	1	0.039	11.514	2.310	13.146	2.371
4	4	2	.5	4	0.041	10.076	3.565	19.563	3.425
4	4	2	.5	16	0.043	33.114	6.447	34.932	6.084
4	4	2	1	.25	0.047	9.116	1.883	12.284	2.549
4	4	2	1	.5	0.04	9.996	2.056	12.996	2.614
4	4	2	1	1	0.039	11.549	2.358	14.302	2.757
4	4	2	1	4	0.040	18.174	3.630	20.389	3.634
4	4	2	1	16	0.039	33.128	6.506	35.414	6.175
4	4	2	4	.25	0.088	8.932	1.874	18.007	4.898
4	4	2	4	.5	0.080	9.821	2.060	18.546	4.877
4	4	2	4	1	0.071	11.400	2.387	19.559	4.865
4	4	2	4	4	0.047	18.233	3.765	24.568	5.097
4	4	2	4	16	0.039	33.555	6.724	38.114	6.850
4	4	2	16	.25	0.092	8.685	1.800	31.315	10.199
4	4	2	16	.5	0.091	9.546	1.983	31.682	10.154
4	4	2	16	1	0.096	11.084	2.310	32.377	10.071
4	4	2	16	4	0.088	17.865	3.749	36.014	9.796
4	4	2	16	16	0.047	33.581	6.960	46.874	10.022

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.
Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.3 *(continued).* Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous CIs from SAS PROC MIXED for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average$_1$	Standard deviation$_1$	Average$_2$	Standard deviation$_2$
4	4	10	.25	.25	0.056	3.491	0.244	3.831	0.243
4	4	10	.25	.5	0.056	3.824	0.266	4.139	0.260
4	4	10	.25	1	0.055	4.415	0.304	4.695	0.292
4	4	10	.25	4	0.055	6.958	0.465	7.176	0.441
4	4	10	.25	16	0.054	12.772	0.834	13.011	0.797
4	4	10	.5	.25	0.057	3.482	0.247	4.127	0.276
4	4	10	.5	.5	0.060	3.818	0.270	4.416	0.290
4	4	10	.5	1	0.058	4.414	0.310	4.943	0.318
4	4	10	.5	4	0.054	6.977	0.477	7.343	0.455
4	4	10	.5	16	0.056	12.800	0.845	13.104	0.804
4	4	10	1	.25	0.053	3.458	0.250	4.648	0.342
4	4	10	1	.5	0.056	3.796	0.273	4.911	0.351
4	4	10	1	1	0.056	4.396	0.314	5.395	0.371
4	4	10	1	4	0.056	6.982	0.488	7.662	0.486
4	4	10	1	16	0.055	12.839	0.863	13.288	0.818
4	4	10	4	.25	0.062	3.364	0.251	6.902	0.646
4	4	10	4	.5	0.061	3.701	0.275	7.096	0.646
4	4	10	4	1	0.056	4.301	0.319	7.462	0.647
4	4	10	4	4	0.053	6.915	0.500	9.297	0.685
4	4	10	4	16	0.058	12.870	0.902	14.329	0.918
4	4	10	16	.25	0.066	3.269	0.241	12.136	1.325
4	4	10	16	.5	0.065	3.592	0.266	12.264	1.321
4	4	10	16	1	0.065	4.170	0.310	12.509	1.314
4	4	10	16	4	0.062	6.729	0.501	13.805	1.293
4	4	10	16	16	0.054	12.722	0.925	17.761	1.345

Average$_1$ and Standard deviation$_1$ are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.
Average$_2$ and Standard deviation$_2$ are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.6.3 *(continued).* Estimated Coverage Error Rates Associated with Nominally 95% Simultaneous CIs from SAS PROC MIXED for all Pairwise Comparisons $\mu_{ij} - \mu_{rs}$, $i, r = 1, \ldots, a$, $j, s = 1, \ldots, b$, for Different Combinations of a, b, c, $\sigma^2_{C(A)}$ and $\sigma^2_{BC(A)}$. The Value of σ^2_e Was Taken to be 1 for Every Case.

a	b	c	$\sigma^2_{C(A)}$	$\sigma^2_{BC(A)}$	Estimated Coverage Error Rate	Average_1	Standard deviation_1	Average_2	Standard deviation_2
5	5	2	.25	.25	0.046	9.136	1.437	10.070	1.508
5	5	2	.25	1	0.042	11.520	1.791	12.364	1.811
5	5	2	.25	16	0.036	33.238	5.049	34.334	4.915
5	5	2	1	.25	0.066	9.120	1.498	12.076	2.142
5	5	2	1	1	0.058	11.573	1.873	14.088	2.317
5	5	2	1	16	0.034	33.419	5.131	35.036	5.048
5	5	2	16	.25	0.106	8.625	1.475	30.500	8.453
5	5	2	16	1	0.106	11.016	1.889	31.550	8.337
5	5	2	16	16	0.067	33.573	5.544	46.044	8.413
5	5	10	.25	.25	0.054	3.708	0.200	4.050	0.197
5	5	10	.25	1	0.052	4.693	0.251	4.966	0.237
5	5	10	.25	16	0.058	13.597	0.693	13.771	0.649
5	5	10	1	.25	0.052	3.671	0.203	4.897	0.286
5	5	10	1	1	0.056	4.669	0.256	5.691	0.306
5	5	10	1	16	0.052	13.664	0.717	14.060	0.664
5	5	10	16	.25	0.048	3.455	0.188	12.687	1.190
5	5	10	16	1	0.051	4.410	0.242	13.083	1.170
5	5	10	16	16	0.052	13.506	0.749	18.699	1.131

Average_1 and Standard deviation_1 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i = r$, $j \neq s$.

Average_2 and Standard deviation_2 are, respectively, the average and the standard deviation of lengths, over 1000 simulations, of a confidence interval for the differences $\mu_{ij} - \mu_{rs}$ with $i \neq r$.

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Table 5.7.1. HEPA Filter Example from Hocking (1996).

Manufacturer	Aerosols	Filter	Response
1	1	1	0.787
1	1	2	0.088
1	1	3	0.078
1	2	1	0.717
1	2	2	3.633
1	2	3	1.833
2	1	1	1.080
2	1	2	0.177
2	1	3	0.130
2	2	1	0.897
2	2	2	0.700
2	2	3	1.757

Summary statistics for the above data are shown below.

$\bar{y}_{11} = .318,\quad \bar{y}_{12} = 2.061,\quad \bar{y}_{21} = .462,\quad \bar{y}_{22} = 1.118,$
$SS(A) = 0.478,\quad SS(B) = 4.316,\quad SS(AB) = 0.887,\quad SS(C(A)) = 1.766$
$SS(BC(A)) = 4.099.$

Simultaneous GCIs which were constructed for all pairwise differences of the cell-means corresponding to level combinations of the fixed factors. We obtained the following.

$$-4.967 \leq \mu_{11} - \mu_{12} \leq 1.480$$
$$-3.879 \leq \mu_{21} - \mu_{22} \leq 2.568$$
$$-2.871 \leq \mu_{11} - \mu_{21} \leq 2.581$$
$$-1.783 \leq \mu_{12} - \mu_{22} \leq 3.669$$
$$-3.526 \leq \mu_{11} - \mu_{22} \leq 1.926$$
$$-1.127 \leq \mu_{12} - \mu_{21} \leq 4.325 \tag{5.24}$$

Simultaneous confidence intervals were also constructed for all mean differences of the fixed effects using SAS. These are as follows.

$$-5.003 \leq \mu_{11} - \mu_{12} \leq 1.516$$
$$-3.915 \leq \mu_{21} - \mu_{22} \leq 2.604$$

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$$
\begin{aligned}
-2.901 &\le \mu_{11} - \mu_{21} \le 2.612 \\
-1.814 &\le \mu_{12} - \mu_{22} \le 3.700 \\
-3.557 &\le \mu_{11} - \mu_{22} \le 1.956 \\
-1.158 &\le \mu_{12} - \mu_{21} \le 4.355
\end{aligned}
\tag{5.25}
$$

We observed that simultaneous GCIs are somewhat narrower intervals than the corresponding SAS intervals.

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5.8 Appendix

Proposition A1:

In the simulation study of Section 5.3 we may assume, without loss of generality, that $\boldsymbol{\mu} = 0$ and $\sigma_e^2 = 1$.

Proof:

Using the notation introduced in Section (5.2) the vector of sample means can be represented as

$$\bar{\boldsymbol{Y}} = \boldsymbol{\mu} + \boldsymbol{A}\,\boldsymbol{Z}$$

where $\boldsymbol{A}$ is a matrix such that $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$. Define $\boldsymbol{\ell}_i' = (0, \cdots, 0, 1, 0, \cdots, 0)$, where the ith element is 1 and the other elements are zero. Then the sample marginal mean corresponding to level i of factor A is given by $\bar{x}_i = \boldsymbol{l}_i'\bar{\boldsymbol{y}}$ and a FGPQ R_{μ_i} for μ_i is given by

$$R_{\mu_i} = \boldsymbol{l}_i' R_{\boldsymbol{\mu}}.$$

Also

$$\widehat{Var}(\hat{\mu}_i) = \boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i$$

is the estimated variance of $\bar{Y}_i$. Note that

$$R_{\mu_i} = \boldsymbol{l}_i' R_{\boldsymbol{\mu}} = \boldsymbol{l}_i' \left[\bar{\boldsymbol{y}} - R_{\boldsymbol{A}}\boldsymbol{Z}\right].$$

The matrix $R_{\boldsymbol{A}}$ is a matrix such that

$$R_{\boldsymbol{A}}R_{\boldsymbol{A}}' = R_{\boldsymbol{\Sigma}} = \frac{1}{bn}\left(\frac{ss(\gamma)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{ss(\beta)}{U_1} - \frac{ss(\gamma)}{U_2}\right)\boldsymbol{U}_a,$$

where $U_1 \sim \chi^2_{b-1}$ is independent of $U_2 \sim \chi^2_{(a-1)(b-1)}$. Note that the random variable D in Equation (5.13) is given by

$$D = \max_{1\leq i\leq a} \frac{|\boldsymbol{l}_i'\bar{\boldsymbol{y}} - \boldsymbol{l}_i' R_{\boldsymbol{\mu}}|}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}} = \max_{1\leq i\leq a} \frac{\boldsymbol{l}_i' R_{\boldsymbol{A}}\boldsymbol{Z}}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}}.$$

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Define

$$K = \max_{1 \le i \le a} \frac{|\boldsymbol{l}_i'\bar{\boldsymbol{Y}} - \boldsymbol{l}_i'\boldsymbol{\mu}|}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}}.$$

K may also be expressed as

$$K = \max_{1 \le i \le a} \frac{|\boldsymbol{l}_i'\boldsymbol{A}\boldsymbol{Z}|}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}}.$$

The coverage probability of the simultaneous GCIs is given by $p = Pr(K \le d_{1-\alpha})$ where $d_{1-\alpha}$ is $1-\alpha$ percentile of the distribution of the random variable D. Observe that $K \le d_{1-\alpha}$ if and only if $Pr(D < K) \le 1-\alpha$. Now $D < K$ holds if and only if

$$\max_{1 \le i \le a} \frac{|\boldsymbol{l}_i' R_{\boldsymbol{A}}\boldsymbol{Z}|}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}} \le \max_{1 \le i \le a} \frac{|\boldsymbol{l}_i'\boldsymbol{A}\boldsymbol{Z}|}{\sqrt{\boldsymbol{l}_i'\hat{\boldsymbol{\Sigma}}\boldsymbol{l}_i}}$$

if and only if

$$\max_{1 \le i \le a} |\boldsymbol{l}_i' R_{\boldsymbol{A}}\boldsymbol{Z}| < \max_{1 \le i \le a} |\boldsymbol{l}_i'\boldsymbol{A}\boldsymbol{Z}|$$

The matrix $\boldsymbol{\Sigma}$ can be expressed as

$$\boldsymbol{\Sigma} = \sigma_e^2 \left(\frac{1}{b}\frac{\sigma_\beta^2}{\sigma_e^2}\boldsymbol{U}_a + \left(\frac{1}{b}\frac{\sigma_\gamma^2}{\sigma_e^2} + \frac{1}{bn} \right) \boldsymbol{I}_a \right) = \sigma_e^2 \boldsymbol{\Sigma}^{**}.$$

So $\boldsymbol{A} = \sigma_e \boldsymbol{A}^{**}$ where $\boldsymbol{A}^{**}$ is a matrix such that $\boldsymbol{\Sigma}^{**} = \boldsymbol{A}^{**}\boldsymbol{A}'^{**}$ and it depends on only on the ratios $\dfrac{\sigma_\beta^2}{\sigma_e^2}$ and $\dfrac{\sigma_\gamma^2}{\sigma_e^2}$.

A FGPQ for the matrix $\boldsymbol{\Sigma}$ can be expressed as

$$\begin{aligned}
R_{\boldsymbol{\Sigma}} &= \frac{1}{bn}\left(\frac{ss(\gamma)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{ss(\beta)}{U_1} - \frac{ss(\gamma)}{U_2}\right)\boldsymbol{U}_a \\
&= \frac{1}{bn}\left(\frac{u_2(n\sigma_\gamma^2 + \sigma_e^2)}{U_2}\right)\boldsymbol{I}_a + \frac{1}{abn}\left(\frac{u_1(an\sigma_\beta^2 + n\sigma_\gamma^2 + \sigma_e^2)}{U_1} - \frac{u_2(n\sigma_\gamma^2 + \sigma_e^2)}{U_2}\right)\boldsymbol{U}_a \\
&= \sigma_e^2 \left(\frac{\boldsymbol{I}_a}{bn}\left(\frac{u_2\left(\frac{n\sigma_\gamma^2}{\sigma_e^2} + 1\right)}{U_2} \right) \right. \\
&\qquad \left. + \frac{\boldsymbol{U}_a}{abn}\left(\frac{u_1\left(\frac{an\sigma_\beta^2}{\sigma_e^2} + \frac{n\sigma_\gamma^2}{\sigma_e^2} + 1\right)}{U_1} - \frac{u_2\left(\frac{n\sigma_\gamma^2}{\sigma_e^2} + 1\right)}{U_2} \right) \right) \\
&= \sigma_e^2 R_{\boldsymbol{\Sigma}^{**}}.
\end{aligned}$$

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So $R_{\boldsymbol{A}} = \sigma_e R_{\boldsymbol{A}^{**}}$ where $R_{\boldsymbol{A}^{**}}$ is a matrix such that $R_{\boldsymbol{\Sigma}^{**}} = R_{\boldsymbol{A}^{**}} R'_{\boldsymbol{A}^{**}}$ and it depends only on the ratios $\frac{\sigma_\beta^2}{\sigma_e^2}$ and $\frac{\sigma_\gamma^2}{\sigma_e^2}$. Thus $D < K$ if and only if $\max_{1\leq i\leq a} |\boldsymbol{l}'_i \sigma_e R_{\boldsymbol{A}^{**}} \boldsymbol{Z}| < \max_{1\leq i\leq a} |\boldsymbol{l}'_i \sigma_e \boldsymbol{A}^{**} \boldsymbol{Z}|$ which in turn holds if and only if

$$\sigma_e \max_{1\leq i\leq a} |\boldsymbol{l}'_i R_{\boldsymbol{A}^{**}}) \boldsymbol{Z}| < \sigma_e \max_{1\leq i\leq a} |\boldsymbol{l}'_i \boldsymbol{A}^{**} \boldsymbol{Z}|.$$

So $D < K$ if and only if

$$\max_{1\leq i\leq a} |\boldsymbol{l}'_i R_{\boldsymbol{A}^{**}}) \boldsymbol{Z}| < \max_{1\leq i\leq a} |\boldsymbol{l}'_i \boldsymbol{A}^{**}) \boldsymbol{Z}|$$

which neither depends on μ nor on σ_e^2 but only depends on the ratios $\frac{\sigma_\beta^2}{\sigma_e^2}$ and $\frac{\sigma_\gamma^2}{\sigma_e^2}$. Thus $Pr(D < K)$ is a function only of the ratios $\frac{\sigma_\beta^2}{\sigma_e^2}$ and $\frac{\sigma_\gamma^2}{\sigma_e^2}$. Therefore, without loss of generality we can take $\mu = \boldsymbol{0}$ and $\sigma_e^2 = 1$ in the simulation study of Section 5.3.

Proposition A2:

In the simulation study of Section 5.6 we may assume, without loss of generality, that $\boldsymbol{\mu} = 0$ and $\sigma_e^2 = 1$.

Proof:

The proof of this proposition is similar to that of **Proposition A1**.

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Chapter 6

USING GCIS TO COMPARE NON-NESTED LINEAR MODELS

Prediction is a fundamental application of statistics because the response variable Y of interest may be a future value or a difficult or an expensive quantity to measure. Prediction is typically based on one or more predictor variables or covariates. A question that often arises in practice is "which, if any, of two sets of predictor variables is *better* for predicting Y than the other?" A distinction is often made between the situations where one set of predictors is a subset of the other and where neither set of predictors is a subset of the other. In the former case, one set of predictors is said to be *nested* in the other and in the latter case the two subsets are said to be *non-nested.*

This chapter is primarily concerned a comparison of non-nested sets of predictors although the methods we propose are applicable for comparing nested models as well. Section 1 reviews previous works relating to comparison of of non-nested linear models. Section 2 shows how GCIs can be used to compare non-nested subsets of predictors. Section 3 deals with a simulation study. Section 4 concerns an example. Section 5 contains a small simulation study for constructing GCIs when the number of predictors is large. Section 6 is an appendix.

6.1 Review of Literature

Cox (1961) appears to be the first serious published work to consider a comparison of non-nested linear models. One of the problems he discussed was comparing two regression models. There were two hypotheses: $H_f : E(\boldsymbol{Y}) = \boldsymbol{A\alpha}$ and

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$H_g : E(\boldsymbol{Y}) = \boldsymbol{B\beta}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given matrices and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors of unknown parameters, not necessarily with the same number of components.

Later on, several studies have concerned a comparing non-nested linear models. Davidson and Mackinnon (1981) presented what they called the J-test. Watnik et al. (2001) concluded that the J test over-rejects the true model especially if the rank of the design matrices are large. They also developed a finite sample adjusted J-test.

The above mentioned investigations were concerned with deciding which of the two models is the *correct* model. A different approach to non-nested model comparison was suggested when investigating which subset of predictors leads to smaller prediction errors. In this context, Efron (1984) developed approximate confidence intervals for the difference in mean squared error (MSE) of prediction for two non-nested linear models using bootstrap procedures.

Ahlbrandt (1988), in his Ph.D. dissertation, discussed this problem in considerable detail. The following is the outline of his findings.

Let Y be the dependent variable and $X_1, \cdots, X_{k+m+t}$ be predictors. He predicted the response Y using one of two subsets of predictors, say A and B, where

$$A = S_1 \cup S_3 \qquad \text{and} \qquad B = S_2 \cup S_3,$$

neither S_1 nor S_2 is empty, $S_1 = (X_1, \cdots, X_k)$, $S_2 = (X_{k+1}, \cdots, X_{k+m})$, and $S_3 = (X_{k+m+1}, \cdots, X_{k+m+t})$. It was assumed that k and m are positive integers so that neither S_1 nor S_2 is empty. However S_3 is allowed to be empty, i.e., the two subsets of predictors may have no common predictor variables. The subsets of predictors $A = S_1 \cup S_3$ and $B = S_2 \cup S_3$ are non-nested since neither set is a subset of the other. Denote $m + k + t$ by p. Then $\boldsymbol{T}$ is a $(p+1)$-dimensional random vector. Consider the random vector $\boldsymbol{T} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \\ \boldsymbol{W}_3 \end{pmatrix} \qquad \begin{array}{l} \boldsymbol{W}_1 = (X_1, \cdots, X_k)', \\ \boldsymbol{W}_2 = (X_{k+1}, \cdots, X_{k+m})', \\ \boldsymbol{W}_3 = (X_{k+m+1}, \cdots, X_{k+m+t})'. \end{array} \tag{6.1}$$

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$$\boldsymbol{\mu} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \Sigma_{00} & \Sigma_{01} & \Sigma_{02} & \Sigma_{03} \\ \Sigma_{10} & \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{20} & \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{30} & \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix}. \tag{6.2}$$

Suppose a random sample of size n is available from the distribution of $\boldsymbol{T}$. Denote the matrix of observations by

$$\begin{pmatrix} Y_1 & X_{11} & \dots & X_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ Y_n & X_{n1} & \dots & X_{np} \end{pmatrix} \tag{6.3}$$

Denote the $n \times 1$ vector of responses by $\boldsymbol{Y}$ and the $n \times 1$ columns of data from the p predictor variables by $\boldsymbol{X}_1, \dots, \boldsymbol{X}_p$.

Let $\boldsymbol{m}$ and $\frac{1}{n}\boldsymbol{S}$ denote the maximum likelihood estimators of $\boldsymbol{\mu}$ and Σ, respectively, based on the above multivariate sample of size n. In particular, $\boldsymbol{S}$ is the sample sum of squares and cross-products matrix. Define

$$\boldsymbol{m} = \begin{pmatrix} m_0 \\ \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} \qquad \boldsymbol{S} = \begin{pmatrix} S_{00} & \boldsymbol{S}_{01} & \boldsymbol{S}_{02} & \boldsymbol{S}_{03} \\ \boldsymbol{S}_{10} & \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\ \boldsymbol{S}_{20} & \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{30} & \boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{pmatrix}. \tag{6.4}$$

Ahlbrandt (1988) mentioned that the problem of determining if the two subsets of predictors are equivalent for predicting Y using one of the two subsets of variables, A or B is reduced to a test of the equality of the residual variances from the corresponding two (non-nested) linear models. First define the following quantities.

$$\Sigma_1 = \begin{pmatrix} \Sigma_{01} & \Sigma_{03} \end{pmatrix} \qquad \Sigma_2 = \begin{pmatrix} \Sigma_{02} & \Sigma_{03} \end{pmatrix}$$

$$\Sigma_3 = \begin{pmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{pmatrix} \qquad \Sigma_4 = \begin{pmatrix} \Sigma_{22} & \Sigma_{23} \\ \Sigma_{32} & \Sigma_{33} \end{pmatrix}$$

$$\Sigma_5 = \begin{pmatrix} \Sigma_{12} & \Sigma_{13} \\ \Sigma_{32} & \Sigma_{33} \end{pmatrix} \qquad \Sigma_6 = \begin{pmatrix} \Sigma_{0.13} & \Sigma_{0.13,0.23} \\ \Sigma_{0.13,0.23} & \Sigma_{0.23} \end{pmatrix}$$

$$\boldsymbol{V}_1 = \begin{pmatrix} \boldsymbol{W}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{W}_3 - \boldsymbol{\mu}_3 \end{pmatrix} \qquad \boldsymbol{V}_2 = \begin{pmatrix} \boldsymbol{W}_2 - \boldsymbol{\mu}_2 \\ \boldsymbol{W}_3 - \boldsymbol{\mu}_3 \end{pmatrix}$$

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$$\boldsymbol{S}_1 = \begin{pmatrix} \boldsymbol{S}_{01} & \boldsymbol{S}_{03} \end{pmatrix} \qquad \boldsymbol{S}_2 = \begin{pmatrix} \boldsymbol{S}_{02} & \boldsymbol{S}_{03} \end{pmatrix}$$

$$\boldsymbol{S}_3 = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{13} \\ \boldsymbol{S}_{31} & \boldsymbol{S}_{33} \end{pmatrix} \qquad \boldsymbol{S}_4 = \begin{pmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{pmatrix}. \tag{6.5}$$

If $\epsilon_1 = Y - \mu_{0.13}(\boldsymbol{W}_1, \boldsymbol{W}_3)$ and $\epsilon_2 = Y - \mu_{0.23}(\boldsymbol{W}_2, \boldsymbol{W}_3)$, with

$$\mu_{0.13}(\boldsymbol{W}_1, \boldsymbol{W}_3) = E(Y|\boldsymbol{W}_1, \boldsymbol{W}_3) = \mu_0 + \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_3^{-1}\boldsymbol{V}_1$$
$$\mu_{0.23}(\boldsymbol{W}_2, \boldsymbol{W}_3) = E(Y|\boldsymbol{W}_2, \boldsymbol{W}_3) = \mu_0 + \boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_4^{-1}\boldsymbol{V}_2 \tag{6.6}$$

then $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_1)' \sim MVN_2(\boldsymbol{0}, \Sigma_6)$, where

$$\Sigma_{0.13} = Var(\epsilon_1) = \Sigma_{00} - \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_3^{-1}\boldsymbol{\Sigma}_1'$$
$$\Sigma_{0.23} = Var(\epsilon_2) = \Sigma_{00} - \boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_4^{-1}\boldsymbol{\Sigma}_2' \tag{6.7}$$

and

$$\Sigma_{0.13,0.23} = Cov(\epsilon_1, \epsilon_2) = \Sigma_{00} - \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_3^{-1}\boldsymbol{\Sigma}_1' - \boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_4^{-1}\boldsymbol{\Sigma}_2' + \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_3^{-1}\boldsymbol{\Sigma}_5\boldsymbol{\Sigma}_4^{-1}\boldsymbol{\Sigma}_2'.$$

The special case when $k = m = t = 1$ was considered in detail by Ahlbrandt (1988) to test $H_0 : \Sigma_{0.13} = \Sigma_{0.23}$ versus $H_1 : \Sigma_{0.13} \neq \Sigma_{0.23}$. He used an extension of the well-known Pitman-Morgan test for equality of variances as well as several bootstrap methods. Ahlbrandt (1988) carried out a simulation study to examine the relative performances of the different methods he considered. He recommended Extension Pitman Morgan test to compare non-nested subsets of predictors. The test statistic that he suggested for testing $H_0 : \Sigma_{0.13} = \Sigma_{0.23}$ is

$$r = \frac{\sum_{i=1}^{n}(\hat{\epsilon_{1i}} - \hat{\epsilon_{2i}})(\epsilon_{1i} \hat{+} \epsilon_{2i})}{(\sum_{i=1}^{n}(\epsilon_{1i} - \epsilon_{2i})^2 \sum_{i=1}^{n}(\hat{\epsilon_{1i}} + \hat{\epsilon_{2i}})^2)^{\frac{1}{2}}} \tag{6.8}$$

where $\hat{\epsilon}_1 = Y - \widehat{\mu_{0.13}}$ and $\hat{\epsilon}_2 = Y - \widehat{\mu_{0.23}}$ are vectors of estimated prediction errors. The decision rule is reject H_0 if and only if

$$|r| \left(\frac{n-1}{1-r^2}\right)^{\frac{1}{2}} > t_{n-1,1-\frac{\alpha}{2}}.$$

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Under the null hypothesis, the test statistic has approximately a Student's t-distribution with $n-1$ degrees of freedom.

For his simulation study, Ahlbrandt (1988) noted that, without loss of generality one could assume

$$\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \\ \boldsymbol{W}_3 \end{pmatrix} \sim MVN_{p+1}(\boldsymbol{0}, \boldsymbol{P}) \tag{6.9}$$

where

$$\boldsymbol{P} = \begin{pmatrix} 1 & \boldsymbol{P}_{01} & \boldsymbol{P}_{02} & \boldsymbol{P}_{03} \\ \boldsymbol{P}_{10} & \boldsymbol{I} & \boldsymbol{P}_{12} & \boldsymbol{P}_{13} \\ \boldsymbol{P}_{20} & \boldsymbol{P}_{21} & \boldsymbol{I} & \boldsymbol{P}_{23} \\ \boldsymbol{P}_{30} & \boldsymbol{P}_{31} & \boldsymbol{P}_{32} & \boldsymbol{I} \end{pmatrix} \tag{6.10}$$

with $\boldsymbol{P}_{ij} = \left(\Sigma_{ii}^{-\frac{1}{2}}\right) \Sigma_{ij} \left(\Sigma_{jj}^{-\frac{1}{2}}\right)'$ and $\Sigma_{ii}^{-1} = \left(\Sigma_{ii}^{-\frac{1}{2}}\right) \left(\Sigma_{ii}^{-\frac{1}{2}}\right)'$. He considered various sample sizes ($n = 20, n = 50$, and $n = 100$), and α-levels ($\alpha = .01$, $\alpha = .05$, and $\alpha = .10$). Furthermore, he considered 79 distinct specifications for the matrix $\boldsymbol{P}$.

For each simulation run corresponding to specified α value and n, Ahlbrandt(1988) summarized the results for the 79 covariance matrix specifications by calculating the six quantities listed in Table 6.1.1. If type I error is in fact equal α then 1000 simulations used to estimate error rates are independent Bernolli trials with probability of rejection equal α. Thus $sd(\hat{\alpha}) = \sqrt{\frac{\alpha(1-\alpha)}{1000}}$. Statistics 5 and 6 in Table 6.1.1 were used to determine how many of the estimated type one error rates are significantly different from α. If Statistic 5 is large then the CIs will be liberal. If Statistic 6 is large then the CIs will be conservative. The simulation results will will be shown in Table 6.3.1 for the extended Pitman-Morgan method reported by Ahlbrandt (1988) for the case $k = m = t = 1$ and compares them with simulation results for the performance of GCIs.

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Table 6.1.1: Six Statistics for Summarizing Simulation Results.

1. Mean of the estimated type-I errors

$$\bar{\alpha} = \frac{1}{79} \sum_{k=1}^{79} \hat{\alpha_k}.$$

2. Standard deviation of the estimated type one errors

$$\left(\frac{\sum_{k=1}^{79} (\hat{\alpha_k} - \bar{\alpha})^2}{78} \right)^{\frac{1}{2}},$$

3. Maximum upper deviation of the estimated type one errors

$$\max_{k=1,\cdots,79} (0, (\hat{\alpha_k} - \alpha)),$$

4. Maximum lower deviation of the estimated type one errors

$$\max_{k=1,\cdots,79} (0, (\alpha - \hat{\alpha_k})),$$

5. Percentage of the estimated type one errors above $\alpha + 2\,sd(\hat{\alpha})$

$$\frac{\sum_{k=1}^{79} 1_{(\hat{\alpha_k} > \alpha + 2sd(\hat{\alpha}))}}{79} \times 100,$$

6. Percentage of the estimated type one errors below $\alpha - 2\,sd(\hat{\alpha})$

$$\frac{\sum_{k=1}^{79} 1_{(\hat{\alpha_k} < \alpha - 2sd(\hat{\alpha}))}}{79} \times 100.$$

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6.2 GCIs Method to Compare Non-Nested Subsets of Predictors

We continue to use the notation of Section 6.1. Thus $\boldsymbol{Y}$ denotes the response variable and $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_p$ denote the predictors. Also A and B are two non-nested subsets of predictors with $A = S_1 \cup S_3$, $B = S_2 \cup S_3$, $S_1 = \{X_1, \cdots, X_k\}$, $S_2 = \{X_{k+1}, \cdots, X_{k+m}\}$ and $S_3 = \{X_{k+m+1}, \cdots, X_p\}$. Again S_1 and S_2 are assumed to be nonempty but S_3 is allowed to be empty.

Let $\boldsymbol{T}$, $\boldsymbol{\mu}$, Σ, $\boldsymbol{m}$, and $\boldsymbol{S}$ be the quantities defined in Equations (6.1), (6.2), and (6.4). As before we are interested in testing the null hypothesis $H_0 : \Sigma_{0.13} = \Sigma_{0.23}$ where $\Sigma_{0.13}$ and $\Sigma_{0.23}$ are as defined in Equation (6.7).

Proposition 6.2.1

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be a lower triangular matrices such that

$$\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}' \tag{6.11}$$

$$\text{and} \quad \boldsymbol{S} = \boldsymbol{B}\boldsymbol{B}'. \tag{6.12}$$

Recall that $\boldsymbol{S}$ is the sample sum of squares and cross-products matrix. Let $\boldsymbol{S}_0$ denote the observed value of $\boldsymbol{S}$ and $\boldsymbol{B}_0$ the corresponding observed value of $\boldsymbol{B}$. Define $R_{\boldsymbol{A}}$ by

$$R_{\boldsymbol{A}} = \boldsymbol{B}_0\boldsymbol{U}^{-1} \tag{6.13}$$

where $\boldsymbol{U}$ is a lower triangular matrix of jointly independent random variables U_{ij}, $i \leq j$ such that

$$U_{ij} \sim N(0,1) \quad i > j \tag{6.14}$$

$$U_{ii} \sim \sqrt{\chi^2_{n-i}}. \tag{6.15}$$

Then $R_{\boldsymbol{A}}$ is a FGPQ for $\boldsymbol{A}$ and $R_{\boldsymbol{\Sigma}}$ is a FGPQ for $\boldsymbol{\Sigma}$ where $R_{\boldsymbol{\Sigma}}$ can be calculated as follows.

$$R_{\boldsymbol{\Sigma}} = R_{\boldsymbol{A}}R'_{\boldsymbol{A}} \tag{6.16}$$

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Proof:

The components of $\boldsymbol{U}$ matrix are $N(0,1)$ and $\sqrt{\chi^2_{n-i}}$ distributions and all the elements are independent. Thus the the components of $\boldsymbol{U}^{-1}$ are free of unknown parameters. The components of the matrix $\boldsymbol{B}_0$ are constants. So the probability distribution of $R_{\boldsymbol{A}} = \boldsymbol{B}_0\boldsymbol{U}^{-1}$ is free of unknown parameters which leads to the probability distribution of $R_{\boldsymbol{\Sigma}} = R_{\boldsymbol{A}}R'_{\boldsymbol{A}}$ to be free of unknown parameters.
$R_{\boldsymbol{A}} = \boldsymbol{B}_0\boldsymbol{U}^{-1}$, then $R_{\boldsymbol{A}} = \boldsymbol{B}_0\boldsymbol{B}^{-1}\boldsymbol{A}$. Thus $r_{\boldsymbol{A}} = \boldsymbol{B}_0\boldsymbol{B}_0^{-1}\boldsymbol{A} = \boldsymbol{A}$ which leads to $r_{\boldsymbol{\Sigma}} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$ which does not depend on nuisance parameters. Thus $R_{\boldsymbol{\Sigma}} = R_{\boldsymbol{A}}R'_{\boldsymbol{A}}$ is FGPQ for $\boldsymbol{\Sigma}$.

Corollary 6.2.1

Let $\theta = h(\boldsymbol{\Sigma})$ denotes any function of the elements of $\boldsymbol{\Sigma}$ and $R_\theta = h\left(R_{\boldsymbol{\Sigma}}\right)$ denotes the same function of the elements of $R_{\boldsymbol{\Sigma}}$. Then R_θ is a FGPQ for θ.

Proof.

Using proposition 6.2.1, The probability distribution of $R_{\boldsymbol{\Sigma}}$ are free of unknown parameters which leads to the components of $R_\theta = h\left(R_{\boldsymbol{\Sigma}}\right)$ is free of unknown parameters. Thus the probability distribution of $R_\theta = h\left(R_{\boldsymbol{\Sigma}}\right)$ is free of unknown parameters. From Proposition 6.2.1, $r_{\boldsymbol{\Sigma}} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$ which leads to $r_\theta = h\left(r_{\boldsymbol{\Sigma}}\right) = h(\boldsymbol{\Sigma}) = \theta$ which does not depend on nuisance parameters. Thus R_θ is a FGPQ for θ.

The following proposition is an immediate consequence of Corollary 6.2.1

Proposition 6.2.2

Let $\theta = \dfrac{\Sigma_{0.13}}{\Sigma_{0.23}}$ denote the ratio of variances of the conditional distribution of Y given the predictors in set A and B, respectively. Then a GCI for θ is given by

$$R_{\theta,\alpha/2} \leq \theta \leq R_{\theta,1-\alpha/2} \tag{6.17}$$

where $R_{\theta,\gamma}$ denotes the $100\gamma\%$ percentile of the distribution of R_θ. A test of the null hypothesis $H_0 : \Sigma_{0.13} = \Sigma_{0.23}$ based on the GCI will reject H_0 whenever the value 1 is not contained in the GCI.

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6.3 Simulation Study

A simulation study was developed to assess the performance of the GCI for $\theta = \frac{\Sigma_{0.13}}{\Sigma_{0.23}}$ and to compare the results with those obtained using the methods discussed in Ahlbrandt (1988). Without loss of generality, we may assume that $\boldsymbol{\mu} = \mathbf{0}$ and that $\boldsymbol{\Sigma}$ has ones along its diagonal.

Sample sizes $n = 20$, $n = 50$, and $n = 100$ were used to investigate the effects of different sample sizes on the type-I error rate. Seventy nine different configurations were considered for $\boldsymbol{\Sigma}$. The sample sizes, level of significance and the $\boldsymbol{\Sigma}$ matrices used in the simulation study matched exactly those considered by Ahlbrandt (1988).

Type-I error rates were estimated by simulation and the same set of 6 summary measures were calculated as in Ahlbrandt (1988) for assessing the performance of GCIs and for comparing the performance with Ahlbrandt's (1988) methods.

6.3.1 Simulation Details

The simulation was conducted using the following steps.

Step 0. Select a sample size n and one of the 79 $\boldsymbol{\Sigma}$ matrices. For the chosen $\boldsymbol{\Sigma}$ compute $\boldsymbol{A}$ such that $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ where $\boldsymbol{A}$ is a lower triangular matrix.

Step 1. Simulate a lower-triangular, random matrix $\boldsymbol{U}^\star$ such that

$$\begin{aligned} U^\star_{ij} &\sim N(0,1) \quad i > j, \\ \text{and} \quad U^\star_{ii} &\sim \sqrt{\chi^2_{n-i}} \end{aligned} \tag{6.18}$$

where all the nonzero elements of $\boldsymbol{U}^\star$ are jointly independent.

Step 2. Define $\boldsymbol{B}$ by

$$\boldsymbol{B} = \boldsymbol{A}\boldsymbol{U}^\star.$$

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Step 3. Generate 10000 independent, lower-triangular, random matrices $\boldsymbol{U}_q$, $q = 1, \ldots, 10000$, such that

$$\begin{aligned} & U_{qij} \sim N(0,1) \quad i > j, \\ \text{and} \quad & U_{qii} \sim \sqrt{\chi^2_{n-i}} \end{aligned} \tag{6.19}$$

where U_{qij} denotes the (i,j) element of $\boldsymbol{U}_q$.

Step 4. Calculate the q^{th} realization of the FGPQ for the matrix $\boldsymbol{A}$ as

$$R_{\boldsymbol{A},q} = \boldsymbol{B}_0 \boldsymbol{U}_q^{-1}$$

and the FGPQ for the matrix $\boldsymbol{\Sigma}$ as

$$R_{\boldsymbol{\Sigma},q} = R_{\boldsymbol{A},q} R'_{\boldsymbol{A},q}.$$

Recall that $\boldsymbol{B}_0$ was obtained in Step 2.

Step 5. Calculate the q^{th} realization of the FGPQ R_θ for θ as

$$R_{q,\theta} = \frac{R_{\Sigma_{0.13}}}{R_{\Sigma_{0.23}}}.$$

Step 6. Arrange the values $R_{q,\theta}$, $q = 1, \ldots, 10000$, in increasing order and let $R(50)$, $R(250)$, $R(500)$, $R(9500)$, $R(9750)$, and $R(9950)$ be, respectively, the 50^{th}, the 250^{th}, the 500^{th}, the 9500^{th}, the 9750^{th}, and the 9950^{th} values in the sorted list. Then $100(1-\alpha)\%$ GCIs for θ are given by $[R(50), R(9950)]$, $[R(250), R(9750)]$, and $[R(500), R(9500)]$, respectively, for $\alpha = 0.01, 0.05, 0.1$.

Step 7. Repeat the calculations of Step 1 through Step 6 for M times. We used M equal to 1000. In each instance, record whether or not θ is inside the realized $1-\alpha$ GCI ($\alpha = 0.01, 0.05, 0.10$). Calculate the proportions $p(\alpha)$ ($\alpha = 0.01, 0.05, 0.10$) of times out of the 1000 data simulations the quantity θ is not contained in the calculated GCI. Then $p(\alpha)$ is the estimated type-I error rate for the given sample size n and the selected variance matrix Σ.

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6.3.2 Discussion of Simulation Results

The same six summary quantities were calculated as given in Table 6.1.1. The statistics were calculated for sample sizes $n = 20$, $n = 50$, and $n = 100$ and nominal type-I error rates equal to 0.01, 0.05, and 0.1. The results are shown in Table 6.3.1. The average of the estimated type-I errors is very close to the nominal error rates in each case and the standard deviations are small. The percentage of the estimated type-I errors above $\alpha + 2sd(\hat{\alpha})$ and the percentage of the estimated type-I errors below $\alpha - 2sd(\hat{\alpha})$ are small enough for the GCIs to be useful in practice.

Table 6.3.1: Simulation Results for Assessing the Performance of GCIs in Comparison to Ahlbrandt's (1988) Results for the Extended Pitman-Morgan Method.

		Extended Pitman-Morgan			GCI		
Quantity	n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
Average	20	0.01116	0.05638	0.11341	0.01038	0.05057	0.10072
Average	50	0.01010	0.05210	0.10538	0.01033	0.05205	0.10254
Average	100	0.01049	0.05099	0.10195	0.01011	0.04978	0.09913
st.Dev.	20	0.00546	0.01267	0.01636	0.00336	0.00803	0.01100
st.Dev.	50	0.00334	0.00777	0.00933	0.00322	0.00738	0.01032
st.Dev.	100	0.00335	0.00822	0.00991	0.00326	0.00662	0.00855
Max above	20	0.01300	0.03400	0.05000	0.012	0.017	0.026
Max above	50	0.00700	0.01600	0.02800	0.01	0.018	0.03
Max above	100	0.00900	0.02100	0.02400	0.01	0.013	0.018
Max below	20	0.01000	0.03600	0.05100	0.007	0.019	0.033
Max below	50	0.01000	0.02600	0.01900	0.006	0.017	0.021
Max below	100	0.00600	0.02000	0.01900	0.006	0.013	0.019
% above	20	17.72152	27.84810	37.97468	3.79747	1.26582	2.53164
% above	50	1.26582	3.79747	7.59494	2.53164	8.86076	6.32911
% above	100	2.53165	8.86076	6.32911	2.53164	0	1.26582
% below	20	7.59494	5.06329	2.53165	1.26582	2.53164	3.79747
% below	50	1.26582	3.79747	1.26582	0	1.26582	1.26582
% below	100	0	2.53165	2.53165	0	0	2.53164

The values in the table are summaries for estimated type-I errors calculated over 79 different selections of Σ.

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6.3.3 Comparison of Results of GCIs with Ahlbrandt (1988)

The average of the estimated nominal errors for different sample sizes using Proposition 6.2.2 is less than using Ahlbrandt (1988) except for $n = 50$ and $\alpha = 0.05$. The standard deviation of the estimated type-I error are less than using Ahlbrandt (1988) except for $n = 50$ and $\alpha = 0.1$. The results are better using Proposition 6.2.2 for maximum upper deviation, except for $n = 50$ when $\alpha = 0.01, 0.05, 0.1$ and $n = 100$ when $\alpha = 0.01$ and for maximum lower deviation, except for $n = 50$ when $\alpha = 0.1$. The results are higher for Ahlbrandt, for percentage of the estimated type-I errors above when $n = 20$ and for percentage of the estimated type-I errors below when $n = 20, 50$ and $\alpha = 0.01, 0.05$.

6.4 Example

We used a data example from the book "Regression Analysis concepts and applications" by Graybill and Iyer (1994, Example 4.8.3) to illustrate the method of GCI for comparing non-nested subsets of predictors. An investigator is studying a population of males who have lived in mountain isolation for several generations. The response variable was Y = height of these males at age 18 years. The predictor variables were X_1 = length at birth, X_2 = mother's height at age 18, X_3 = father's height at age 18, X_4 = maternal grand mother's height at age 18, X_5 = maternal grand father's height at age 18, X_6 = paternal grand mother's height at age 18, and X_7 = paternal grand father's height at age 18.

A simple random sample of 20 males of age 18 or more was drawn from the study population. To illustrate the method of this chapter, we choose the first three variables, X_1, X_2, and X_3 and compare the subset of predictors $\{X_1, X_3\}$ with the subset $\{X_2, X_3\}$. The data are given in Table 6.4.1.

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Table 6.4.1

Subject	Y	X_1	X_2	X_3	Subject	Y	X_1	X_2	X_3
1	67.2	19.7	60.5	70.3	11	67.5	18.9	63.3	70.4
2	69.1	19.6	64.9	70.4	12	73.3	20.8	66.2	70.2
3	67.0	19.4	65.4	65.8	13	70.0	20.3	64.9	68.8
4	72.4	19.4	63.4	71.9	14	69.8	19.7	63.5	70.3
5	63.6	19.7	65.1	65.1	15	63.6	19.9	62.0	65.5
6	72.7	19.6	65.2	71.1	16	64.3	19.6	63.5	65.2
7	68.5	19.8	64.3	67.9	17	68.5	21.3	66.1	65.4
8	69.7	19.7	65.3	68.8	18	70.5	20.1	64.8	70.2
9	68.4	19.7	64.5	68.7	19	68.1	20.2	62.6	68.6
10	70.4	19.9	63.4	70.3	20	66.1	19.2	62.2	67.3

The following GCIs were obtained following the procedure outlined in this chapter.

$$\text{99\% GCI is} \qquad 0.4 \le \frac{\sigma^2_{Y|X_1,X_3}}{\sigma^2_{Y|X_2,X_3}} \le 4.2$$

$$\text{95\% GCI is} \qquad 0.6 \le \frac{\sigma^2_{Y|X_1,X_3}}{\sigma^2_{Y|X_2,X_3}} \le 3.1$$

$$\text{90\% GCI is} \qquad 0.7 \le \frac{\sigma^2_{Y|X_1,X_3}}{\sigma^2_{Y|X_2,X_3}} \le 2.7$$

In particular, we cannot reject H_0: $\sigma^2_{Y|X_1,X_3} = \sigma^2_{Y|X_2,X_3}$ at any of the levels considered.

6.5 A Small Simulation Study to Assess the Performance of GCIs When the Number of Predictors is Large

We conducted a small simulation study to evaluate the performance of GCIs for the ratio of two conditional variances when there are a total of nine predictor variables and two non-nested subsets A and B of predictors each have five predictors with one predictor in their intersection. The simulation procedure is identical to that in the case of three predictors as in Section 6.3.

As mentioned earlier, we may assume without loss of generality that the diagonal elements of the variance matrix $\boldsymbol{\Sigma}$ of the 10 dimensional random vector $(Y, X_1, \ldots, X_9)'$ are all equal to 1. For the simulation study, 10 such different $\boldsymbol{\Sigma}$

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matrices of size 10×10 were generated by first generating 10 random upper triangular matrices $\boldsymbol{A}$ whose columns are of unit length and then calculating $\boldsymbol{\Sigma} = \boldsymbol{A}'\boldsymbol{A}$. Let a_{ij} denote the (i,j) element of $\boldsymbol{A}$. For $i > j$ the elements a_{ij} are zero. For $1 \le i \le 10$, define

$$a_{ii} = \left(\prod_{q=1}^{9-i} \cos(\theta_{i,q})\right) \cos(\theta_{i,10-i})$$

and, for $i < j$,

$$a_{ij} = \left(\prod_{q=1}^{10-j} \cos(\theta_{i,q})\right) \sin(\theta_{i,11-j})$$

The angles $\theta_{r,s}$ are generated randomly and independently, using a uniform distribution over the interval $[0, 2\pi]$.

The simulation results are summarized in Table 6.5.1. We noted that the estimated type-I error $\hat{\alpha}$ appears to be closer to the nominal value α for $n = 50$ than for $n = 25$.

Table 6.5.1: Simulation Results of GCIs for the Case $k = m = 4$, $t = 1$.

Case	n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	Case	n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
1	25	0.008	0.04	0.082	6	25	0.008	0.053	0.101
1	50	0.012	0.051	0.109	6	50	0.018	0.056	0.116
2	25	0.016	0.073	0.138	7	25	0.012	0.054	0.114
2	50	0.014	0.062	0.123	7	50	0.013	0.063	0.11
3	25	0.029	0.097	0.18	8	25	0.015	0.052	0.105
3	50	0.015	0.066	0.138	8	50	0.009	0.047	0.103
4	25	0.01	0.067	0.118	9	25	0.006	0.042	0.091
4	50	0.008	0.058	0.111	9	50	0.007	0.039	0.091
5	25	0.017	0.069	0.126	10	25	0.024	0.081	0.153
5	50	0.011	0.063	0.121	10	50	0.018	0.056	0.109

The reported values are estimated type-I error rates for GCIs for the case $k = m = 4$, $t = 1$, based on 1000 simulated data sets, for $n = 25$ and $n = 50$, for 10 different, randomly generated settings for $\boldsymbol{\Sigma}$. Each subset of predictors has 5 variables with one common predictor between the two.

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6.6 Appendix

Let $(X_{i1}, \ldots, X_{ip})'$, $i = 1, \ldots, n$ denote a random sample of size n from a p-dimensional $(n > p)$ normal distribution with mean $\boldsymbol{\mu}$ and variance matrix $\boldsymbol{\Sigma}$. Let $\boldsymbol{S}$ denote the sample sum of squares and cross-products matrix whose (u, v) element is given by

$$S_{uv} = \sum_{q=1}^{n} (X_{qu} - \bar{X}_u)(X_{qv} - \bar{X}_v)$$

where

$$\bar{X}_t = \frac{1}{n} \sum_{q=1}^{n} X_{qt}$$

the sample mean corresponding to the n observations X_{1t}, ..., X_{nt}, on the t^{th} variable X_t.

We list here some well-known facts from the theory of multivariate normal distributions that are used in the construction of a FGPQ for $\boldsymbol{\Sigma}$. Details may be found in T. W. Anderson(1984). book titled "An Introduction to Multivariate Statistical Analysis".

Let $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ be the Cholesky decomposition of $\boldsymbol{\Sigma}$ with $\boldsymbol{A}$ a lower-triangular matrix. Let $\boldsymbol{S} = \boldsymbol{B}\boldsymbol{B}'$ be the Cholesky decomposition of $\boldsymbol{S}$ with $\boldsymbol{B}$ a lower-triangular matrix. Then the joint distribution of the elements of $\boldsymbol{B}$ is the same as the joint distribution of the elements of the matrix $\boldsymbol{A}\boldsymbol{U}$ where $\boldsymbol{U}$ is a lower-triangular matrix whose elements U_{ij} are jointly independent, $U_{ij} \sim N(0, 1)$ for $i > j$ and $U_{ii}^2 \sim \chi^2_{n-i}$ for $i = 1, \ldots, p$.

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Chapter 7

GENERALIZED TEST VARIABLES AND GENERALIZED HYPOTHESES TESTS

This chapter deals with generalized test variables (GTV) and generalized P-value (GPV). Section 1 is a review of literature concerning tests of hypotheses for equality of cell-means in mixed linear models. Section 2 discusses the use of GTV and GPV for testing the equality of cell means simultaneously in unbalanced heterogeneous one-way ANOVA. Section 3 presents a simulation study that was conducted to examine the performance of generalized hypothesis test. Section 4 gives an illustrative example. Section 5 considers generalized tests for testing simultaneously the equality of cell-means in balanced three-factor mixed linear model with interactions. Section 6 outlines the details of a simulation study we conducted to examine the performance of this generalized hypothesis test and provides an overview of Satterthwaite approximation procedures and compares Satterthwaite approximations to generalized tests. An illustrative example is provided in Section 7.

7.1 Review of Literature

In 1989, Tsui and Weerahandi introduced the concept of generalized test variables (GTV) which can be used in situations where no exact test is available. They also introduced the concept of generalized P-value (GPV) for one sided tests and they illustrated these concepts with examples. Weerahandi (1991) used the concepts of (GTV) and (GPV) for one-sided test for testing variance components in one-way

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random classification model and extended the result to certain balanced random and mixed models.

Leping and Mathew (1994) applied the concepts of GPV to test the significance of a variance component and to compare variance components in two independent balanced and unbalanced mixed models when exact F-tests do not exist.

In 1995, Weerahandi introduced the concepts of GTV for two-sided tests. Weerahandi (1995) considered the problem of testing equality of cell- means in unbalanced ANOVA under heterogeneous error variances and proposed a generalized F-test using an appropriately defined generalized P-value (GPV). He considered a linear model of the form

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij},\ i = 1, \cdots, k,\ j = 1, \cdots, n_i \tag{7.1}$$

where Y_{ij} are the random variables representing the observations taken from k populations, n_i is the sample size of the i^{th} population, $\mu + \alpha_i$, which we denote by μ_i, is the mean of the i^{th} population, $\sum_{i=1}^{k} \alpha_i = 0$, and ϵ_{ij} are independent random variables with $\epsilon_{ij} \sim N(0, \sigma_i^2)$. The realized values of Y_{ij} will be denoted by y_{ij}. The sample means and the sample variances (MLEs) of the k treatments are

$$\bar{Y}_i = \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i}, \qquad S_i^2 = \sum_{j=1}^{n_i} \frac{(Y_{ij} - \bar{Y}_i)^2}{n_i} \tag{7.2}$$

with corresponding realized values $\bar{y}_i$ and s_i^2, respectively.

Consider the null hypothesis of equal means written as

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0. \tag{7.3}$$

Weerahandi defined the standardized between-group sum of squares $\bar{S}_b$ by

$$\bar{S}_b = \bar{S}_b(\sigma_1^2, \cdots, \sigma_k^2) = \sum_{i=1}^{k} \frac{n_i(\bar{Y}_i)^2}{\sigma_i^2} - \frac{\left(\sum_{i=1}^{k} \frac{n_i \bar{Y}_i}{\sigma_i^2}\right)^2}{\left(\sum_{i=1}^{k} \frac{n_i}{\sigma_i^2}\right)}$$

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Denote the realized value of $\bar{S}_b$ by $\bar{s}_b$. Weerahandi (1995) gave the following formulas for GTV and GPV for testing H_0 in Equation (7.3).

$$T = \frac{\bar{S}_b(\sigma_1^2, \cdots, \sigma_k^2)}{\bar{S}_b\left(\frac{s_1^2}{S_1^2\sigma_1^2}, \cdots, \frac{s_k^2}{S_k^2\sigma_k^2}\right)}$$

$$p = 1 - E\left\{H_{k-1,N-k}\left[\frac{N-k}{k-1}s_b\left(\frac{n_1 s_1^2}{B_1 B_2 \cdots B_{k-1}}, \frac{n_2 s_2^2}{(1-B_1)B_2 \cdots B_k - 1}, \frac{n_3 s_3^2}{(1-B_2)B_3 \cdots B_{k-1}}, \cdots, \frac{n_k s_k^2}{1-B_{k-1}}\right)\right]\right\}$$

where $H_{k-1,N-k}$ is the cdf of the F distribution with $k-1$ and $N-k$ degrees of freedom and the expectation is taken with respect to independent beta random variables $B_1, \ldots, B_{k-1}$ defined by

$$B_j = \frac{\sum_{i=1}^{j} \frac{n_i S_i^2}{\sigma_i^2}}{\sum_{i=1}^{j+1} \frac{n_i S_i^2}{\sigma_i^2}} \sim \quad Beta\left(\left(\sum_{i=1}^{j} \frac{n_i - 1}{2}\right), \frac{n_{j+1} - 1}{2}\right), \qquad j = 1, 2, \cdots, k-1,$$

where

$$\frac{n_i S_i^2}{\sigma_i^2} \sim \chi^2_{n_i - 1}$$

In 2000, Seo and Srivastava considered the two components mixed linear model

$$Y_{ij} = \mu_j + \alpha_i + \epsilon_{ij} \qquad i = 1, \cdots, n_j, \ j = 1, \cdots, p$$

where α_i and ϵ_{ij} are jointly independent, α_i are iid $N(0, \sigma_\alpha^2)$ and ϵ_{ij} are iid $N(0, \sigma_\epsilon^2)$. They developed exact tests as well as large sample tests of

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_p$$

Witkovský (2002) concerned with calculating GPV for multiple comparisons of cell means in unbalanced heterogeneous one-way ANOVA. He considered the following null hypothesis and GPV.

$$H_0 : \theta^{ij} = \mu_i - \mu_j = \theta_0^{ij} \ i = 1, \cdots, k, \ j = i+1, \cdots, k$$

$$p^{ij} = 1 - pr\left\{\chi^2_{k-1}\left(\frac{s_i^2/n_i}{\chi^2_{f_i}/f_i} + \frac{s_j^2/n_i}{\chi^2_{f_j}/f_i}\right) \leq \left((\bar{y}_i - \bar{y}_j) - \theta_0^{ij}\right)^2\right\} \tag{7.4}$$

He considered that the null hypothesis will be rejected if $\min_{i,j} p^{ij} < \alpha$.

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7.2 A General Approach for Constructing Generalized Tests

In this section we outline a general procedure for constructing generalized tests of hypothesis about vector of parameters in the presence of nuisance parameters. We then apply the procedure to the one-way ANOVA model with unequal sample sizes and unequal variances and observe that the resulting test is the one given in Weerahandi (1995).

Proposition 7.2.1

Let $\boldsymbol{Y}$ be a random vector whose distribution depends on a p-dimensional parameter $\boldsymbol{\xi}$ where $\boldsymbol{\xi} = (\boldsymbol{\theta}, \boldsymbol{\phi})$ where $\boldsymbol{\theta}$ is q-dimensional and $\boldsymbol{\phi}$ is $p - q$ dimensional. The component $\boldsymbol{\theta}$ is the parameter of interest and the component $\boldsymbol{\phi}$ is the nuisance parameter. Suppose we wish to test the following null hypotheses.

$$H_0 : h_i(\boldsymbol{\theta}) = 0, \qquad \text{for } \; i = 1, \ldots, r. \tag{7.5}$$

Let $\boldsymbol{y}$ denote the realized value of $\boldsymbol{Y}$. Suppose the following conditions hold.

1. A FGPQ is available for $\boldsymbol{\phi}$. Denote this by $R_{\boldsymbol{\phi}}$.
2. A quantity $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ exists with the property that, when H_0 holds, the distribution of $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ is free of all unknown parameters.
3. The joint distribution of $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ and $R_{\boldsymbol{\phi}}$, under H_0, is free of unknown parameters.

Let the quantity $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ be a generalized test variable for testing H_0 such that.

$$T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}) = \frac{Q(\boldsymbol{Y}, \boldsymbol{\phi})}{Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})}.$$

The generalized P-value for the test is $Pr[T > 1|H_0]$ as long as $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ takes large values when H_0 in not true.

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Proof:

We verify that $T(\boldsymbol{Y};\boldsymbol{y},\boldsymbol{\xi})$ satisfies the requirements for it to be a GTV.

(a) The realized value of $T(\boldsymbol{Y},\boldsymbol{y},\boldsymbol{\xi})$ is $t = T(\boldsymbol{y};\boldsymbol{y},\boldsymbol{\xi})$ which is easily seen to equal 1 because the realized value of $R_{\boldsymbol{\xi}}$ is $\boldsymbol{\xi}$. Hence $T(\boldsymbol{y},\boldsymbol{y},\boldsymbol{\xi})$ is free of all unknown parameters.

(b) Since joint distribution of $Q(\boldsymbol{Y},\boldsymbol{\phi})$ and $R_{\boldsymbol{\phi}}$, under H_0, is free of unknown parameters, it follows that the distribution of $T(\boldsymbol{Y},\boldsymbol{y},\boldsymbol{\xi})$ is free of unknown parameters under H_0. Hence $T(\boldsymbol{Y},\boldsymbol{y},\boldsymbol{\xi})$ satisfies the requirements for it to be a GTV for testing H_0.

Remark

To rule out uninteresting generalized tests, the following condition may be added

$$pr\left\{T(\boldsymbol{Y};\boldsymbol{y},\boldsymbol{\xi}) \geq t\right\} \geq pr\left\{T(\boldsymbol{Y};\boldsymbol{y},\boldsymbol{\xi}_0) \geq t\right\}$$

This condition ensures that the test is unbiased.

7.2.1 Testing Equality of Cell-Means in Unbalanced Heterogeneous One-Way ANOVA

Consider the linear model

$$Y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, \cdots, a; \quad j = 1, \cdots, n_i$$

where Y_{ij} are random variables representing the observations which are taken from a populations with realized values y_{ij}, μ_i is the mean of i^{th} population, and n_i is the sample size from the i^{th} population. The error variables ϵ_{ij} are independently and normally distributed as $N(0, \sigma_i^2)$. The variance parameters $\sigma_i^2, i = 1, \ldots, a$, are unknown. The random vector $\boldsymbol{Y}$ is a vector whose elements are Y_{ij} arranged in lexicographic order such that $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$ where

$$\boldsymbol{\mu} = \boldsymbol{W\beta}, \quad \boldsymbol{\beta}' = \left(\mu_1, \mu_2, \ldots, \mu_a\right),$$

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$$\boldsymbol{W} = \begin{pmatrix} \boldsymbol{J}_{n_1} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{J}_{n_2} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{J}_{n_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \boldsymbol{J}_{n_a} \end{pmatrix}, \text{ and } \boldsymbol{V} = \begin{pmatrix} \sigma_1^2 \boldsymbol{I}_{n_1} & 0 & \dots & 0 \\ 0 & \sigma_2^2 \boldsymbol{I}_{n_2} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_a^2 \boldsymbol{I}_{n_a} \end{pmatrix}.$$

Consider the problem of testing $H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$ versus H_a: $\mu_i \neq \mu_j$ for some $i \neq j$. The null hypothesis H_0 can be expressed as

$$H_0 : \boldsymbol{H\beta} = \boldsymbol{0} \quad \text{versus} \quad H_a : \boldsymbol{H\beta} \neq \boldsymbol{0}$$

where

$$\boldsymbol{H} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}$$

Let $\hat{\boldsymbol{\mu}}$ denote the BLUE of $\boldsymbol{\beta}$ assuming σ_i^2 are known. It is easily verified that

$$\hat{\boldsymbol{\mu}} = \left(\boldsymbol{W}'\boldsymbol{V}^{-1}\boldsymbol{W}\right)^{-1} \boldsymbol{W}'\boldsymbol{V}^{-1}\boldsymbol{Y} = \bar{\boldsymbol{Y}}$$

where $\bar{\boldsymbol{Y}}$ is the sample mean vector of size $a \times 1$. We have

$$\hat{\boldsymbol{\mu}} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{7.6}$$

where

$$\boldsymbol{\Sigma} = (\boldsymbol{W}'\boldsymbol{V}^{-1}\boldsymbol{W})^{-1} = diag\left(\frac{\sigma_1^2}{n_1}, \dots, \frac{\sigma_a^2}{n_a}\right).$$

Let $\boldsymbol{\xi}$ denote the parameter vector consisting of $\mu_1, \dots, \mu_a, \sigma_1^2, \dots, \sigma_a^2$. Let $\boldsymbol{\theta}$ be the component of $\boldsymbol{\xi}$ consisting of $\mu_1, \dots, \mu_a$ and let $\boldsymbol{\phi}$ be the component consisting of $\sigma_1^2, \dots, \sigma_a^2$. Define $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ by

$$Q(\boldsymbol{Y}, \boldsymbol{\phi}) = (\boldsymbol{H}\hat{\boldsymbol{\beta}})'(Var(\boldsymbol{H}\hat{\boldsymbol{\beta}}))^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}}). \tag{7.7}$$

The quantity $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ can be simplified and rewritten as

$$Q(\boldsymbol{Y}, \boldsymbol{\phi}) = \sum_{i=1}^{a} \bar{Y}_{i.}^2 \frac{n_i}{\sigma_i^2} - \frac{1}{\sum_{i=1}^{a} \frac{n_i}{\sigma_i^2}} \left(\sum_{i=1}^{a} \bar{Y}_{i.} \frac{n_i}{\sigma_i^2}\right)^2$$

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From standard linear model theory (Graybill, 1976), Q($\boldsymbol{Y}$,$\boldsymbol{\phi}$) has the distribution

$$Q(\boldsymbol{Y},\boldsymbol{\phi}) \sim \chi^2_{a-1}(\lambda)$$

where $\chi^2_{a-1}(\lambda)$ denotes a chi-squared distribution with $a-1$ degrees of freedom and noncentrality parameter λ given by

$$\lambda = (\boldsymbol{H\beta})'(Var(\boldsymbol{H\hat{\beta}}))^{-1}(\boldsymbol{H\beta})$$

which is zero if and only if H_0 holds. Observe that

$$R_{\sigma_i^2} \;=\; \sigma_i^2 \frac{s_i^2}{S_i^2},$$

is a FGPQ for σ_i^2, where s_i^2 is the realized value of S_i^2. Since $V_i = \dfrac{(n_i-1)S_i^2}{\sigma_i^2}$ has a chi-squared distribution with $n_i - 1$ degrees of freedom, one may write

$$R_{\sigma_i^2} = \frac{(n_i-1)s_i^2}{V_i}.$$

In this form it is clear that the distribution of $R_{\sigma_i^2}$ is free of unknown parameters. Define $R_{\boldsymbol{\phi}} = (R_{\sigma_1^2}, \ldots, R_{\sigma_a^2})$.

Proposition 7.2.2

$T(\boldsymbol{Y}, \boldsymbol{y}, \boldsymbol{\xi}) = \dfrac{Q(\boldsymbol{Y},\boldsymbol{\phi})}{Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})}$ is a GTV for testing H_0.

Proof:

We will verify that the three assumptions of Proposition 7.2.1 are satisfied.

(1) $R_{\boldsymbol{\phi}}$ is a FGPQ for $\boldsymbol{\phi}$.

(2) The distribution of $Q(\boldsymbol{Y},\boldsymbol{\phi})$ is free of unknown parameters when H_0 holds. In this case $Q(\boldsymbol{Y},\boldsymbol{\phi})$ has a central chi-squared distribution with $a-1$ degrees of freedom.

(3) Observe that $\{\bar{Y}_1, \ldots, \bar{Y}_a\}$ and $\{S_1^2, \ldots, S_a^2\}$ are independent. Since $R_{\boldsymbol{\phi}}$ is a function of $S_i^2, i = 1, \ldots, a$ but $Q(\boldsymbol{Y},\boldsymbol{\phi})$ is a function of $\bar{Y}_i$, we can conclude $Q(\boldsymbol{Y},\boldsymbol{\phi})$

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and $Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})$ are independent. By construction, the distribution of $Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})$ is free of unknown parameters and the distribution of $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ is free of unknown parameters when H_0 holds. Therefore the distribution of $T(\boldsymbol{Y}, \boldsymbol{y}, \boldsymbol{\xi})$ is free of unknown parameters when H_0 holds.

Since the assumptions of Proposition 7.2.2 are satisfied we conclude that $T(\boldsymbol{Y}, \boldsymbol{y}, \xi)$ is a GTV for testing H_0.

Straightforward algebra shows that the GTV of Proposition 7.2.2 is the same as the GTV for testing H_0 given in Weerahandi (1995).

Remark

The unbiasedness condition is satisfied here because The denominator of $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ is constant, say c. Under H_0 the numerator has χ^2_{a-1} but if the null hypothesis is false, then the numerator has $\chi^2_{a-1,\lambda}$. It is known that $pr\left\{\chi^2_{d,\lambda} \geq c\right\} \geq pr\left\{\chi^2_{d} \geq c\right\}$. Thus, $pr\left\{T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}) \geq t\right\} \geq p\left\{T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}_0) \geq t\right\}$ for all $\boldsymbol{\theta}$ and given any fixed $t, \boldsymbol{y}$ and $\boldsymbol{\phi}$.

7.3 Simulation Study

A simulation study was conducted to assess the performance of the GTV and GPV for testing the equality of the cell means. Without loss of generality, it was assumed that $\boldsymbol{\mu} = \mathbf{0}$. The simulation settings were chosen to exactly match those used by Dunnett (1980).

The simulations were used to estimate the coverage error rates. The sample sizes, variance multipliers and multiplication factors used in the simulations are summarized in Table 4.3.1.

7.3.1 Simulation Details

The simulation was carried out using the following steps.

Step 1. Set $\mu_i = 0,\ i = 1, \cdots, a$. Select one of the settings for sample size, variance multiplier and multiplication factor as summarized in Table 4.3.1.

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Step 2. Generate $\bar{Y}_i$ independent of S_i^2 as follows

$$\begin{aligned}\bar{Y}_i &= \mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_i}}, \qquad i = 1, \cdots, a \\ S_i^2 &= \frac{\sigma_i^2 U_i}{n_i - 1}, \qquad i = 1, \cdots, a\end{aligned}$$

where V_i are independent $N(0,1)$ distributions and U_i are independent $\chi^2_{n_i-1}$ distributions and all variables are jointly independent. The value σ_i^2 is the desired value for the variance of the i^{th} group. The observed values of $\bar{Y}_i$ and S_i^2 are $\bar{y}_i$ and s_i^2 respectively.

Step 3. For $q = 1, \ldots, Q$, generate independent random vectors

$$\left(U_1^{(q)}, \ldots, U_a^{(q)}\right)$$

where the component random variables are jointly independent, $U_i^{(q)} \sim \chi^2_{n_i-1}$, for $i = 1, \ldots, a$. We used $Q = 10000$. Define

$$GTV^{(q)} = \frac{Q(\mathbf{Y}, \phi)}{Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})} \tag{7.8}$$

where the numerator say U in Equation (7.8) has χ^2_{a-1} under H_0 and the denominator has the form

$$Q^{(q)}(\boldsymbol{y}, R_{\boldsymbol{\phi}}) = \sum_{i=1}^{a} \frac{\bar{y}_{i.}^2 n_i U_i^q}{(n_i - 1)\left(s_i^2\right)} - \left(\frac{1}{\sum_{i=1}^{a} \frac{n_i U_i^q}{(n_i-1)\left(s_i^2\right)}}\right)\left(\sum_{i=1}^{a} \frac{\bar{y}_{i.} n_i U_i^q}{(n_i - 1)\left(s_i^2\right)}\right)^2$$

Define the observed value of GTV to be t. The value of t is 1. Thus GPV has the form

$$GPV^{(q)} = pr\left\{GTV^{(q)} > t\right\}, \quad t = 1$$

Thus

$$GPV^{(q)} = 1 - pr\left(U^q \leq \sum_{i=1}^{a} \frac{\bar{y}_{i.}^2 n_i U_i^q}{(n_i - 1)\left(s_i^2\right)} - \left(\frac{1}{\sum_{i=1}^{a} \frac{n_i U_i^q}{(n_i-1)\left(s_i^2\right)}}\right)\left(\sum_{i=1}^{a} \frac{\bar{y}_{i.} n_i U_i^q}{(n_i - 1)\left(s_i^2\right)}\right)^2\right)$$

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$$= 1 - E\left(G^q\left[\sum_{i=1}^{a}\frac{\bar{y}_{i.}^2 n_i U_i^q}{(n_i-1)\left(s_i^2\right)} - \left(\frac{1}{\sum_{i=1}^{a}\frac{n_i U_i^q}{(n_i-1)\left(s_i^2\right)}}\right)\left(\sum_{i=1}^{a}\frac{\bar{y}_{i.} n_i U_i^q}{(n_i-1)\left(s_i^2\right)}\right)^2\right]\right)$$

where G^q is the cdf of χ^2_{a-1} and the E is the expectation with respect to independent $U_i^{(q)} \sim \chi^2_{n_i-1}$. The expectation is approximated by using simulation to obtain G^q and then sample mean is calculated. Calculate the proportion of values which satisfies GPV less than $\alpha = 0.05$ and consider as a false rejection.

Step 4. Repeat the steps 1-3 for M times (we used M=1000), calculate the proportion of false rejection in M repetitions. Then These provides empirical estimates of the type-I error rates. Table 7.3.1 summarizes the results of the simulation runs.

7.3.2 Discussion of the Simulation Results

When the sample sizes are big and equal, the estimated type-I error rates are close to 0.05. When the sample sizes are different, the estimated type-I error rates are close to 0.05 most of the cases.

7.4 Example

An example is discussed to illustrate the computation of GTV and GPV for testing equality of the cell-means for the model given in Equation (7.1). Four populations are being compared. The following summary of statistics were obtained from Weerahandi (1995).

$$\begin{array}{lll} n_1 = 6, & \bar{y}_1 = 13.1, & s_1 = 1.90 \\ n_2 = 8, & \bar{y}_2 = 14.1, & s_2 = 1.70 \\ n_3 = 5, & \bar{y}_3 = 14.6, & s_3 = 0.89 \\ n_4 = 7, & \bar{y}_4 = 12.9, & s_4 = 0.55 \end{array}$$

where $\bar{y}_i$ and s_i^2 are observed values of $\bar{Y}_i$ and S_i^2 which based on MLE of the parameters. The null hypothesis is $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$. Using the GTV of the previous section, a generalized P value was calculated for H_0.

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Table 7.3.1 Estimated Type-I Error Rates for Generalized Test for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$

Variance multiplier	Simulation (1) $n_i = (7,7,7,7)$				Simulation (2) $n_i = (7,9,11,13)$				Simulation (3) $n_i = (7,7,7,7,7,7,7,7)$			
	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞	x_1	x_2	x_8	x_∞
0.5	0.062	0.062	0.050	0.054	0.053	0.062	0.049	0.050	0.097	0.069	0.059	0.048
1.0	0.065	0.058	0.047	0.052	0.052	0.055	0.041	0.052	0.083	0.060	0.059	0.053
2.0	0.072	0.070	0.046	0.054	0.047	0.060	0.042	0.052	0.082	0.066	0.064	0.056
4.0	0.080	0.072	0.051	0.052	0.050	0.056	0.052	0.053	0.088	0.073	0.065	0.058
10.0	0.081	0.073	0.048	0.048	0.054	0.062	0.053	0.048	0.097	0.078	0.062	0.054

We obtained $GPV = 0.046$ which is less than $\alpha = 0.05$. So H_0 may be rejected at the 0.05 level. Weerahandi (1995) calculated a GPV for this example and he obtained $GPV = 0.046$. Using the procedures of GTV and GPV lead to the same test as the one in Weerahandi (1995) for the equality of means in an unbalanced, heterogeneous, one-way ANOVA model. However, these procedures are applicable to more general situations as well.

7.5 Test of Equality of Cell-Means in the Balanced Three-factor Crossed Mixed Linear Model with Interactions

We consider the balanced three-factor mixed linear model given by

$$\begin{aligned} Y_{ijkr} &= \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + e_{ijkr}, \\ & i = 1, \cdots a; \quad j = 1, \cdots b; \quad k = 1, \cdots c, \quad r = 1, \cdots n, \end{aligned} \tag{7.9}$$

where Y_{ijkr} denotes the r^{th} response corresponding to i^{th} level of factor A (fixed), j^{th} level of factor B (random), and k^{th} level of factor C (random), α_i is the effect of i^{th} level of the main effect A, $\beta_j \sim N(0, \sigma_B^2)$ is the effect of j^{th} level of the random main effect B, $\gamma_k \sim N(0, \sigma_C^2)$ is the effect of k^{th} level of the random main effect C, $(\alpha\beta)_{ij} \sim N(0, \sigma_{AB}^2)$ is the effect of ij^{th} level of the interaction effect between factors A and B, $(\alpha\gamma)_{ik} \sim N(0, \sigma_{AC}^2)$ is the effect of ik^{th} level of the interaction between factors A and C, $e_{ijkr} \sim N(0, \sigma_e^2)$ are the error terms. It is assumed that all random variables are mutually independent. Thus $\boldsymbol{Y} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma}^*)$ where

$$\begin{aligned} \boldsymbol{\Sigma}^* &= \sigma_B^2 \boldsymbol{V}_2 + \sigma_C^2 \boldsymbol{V}_3 + \sigma_{AB}^2 \boldsymbol{V}_{12} + \sigma_{AC}^2 \boldsymbol{V}_{13} + \sigma_e^2 \boldsymbol{I}_{abcn}, \\ \boldsymbol{V}_2 &= \boldsymbol{U}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{U}_c \otimes \boldsymbol{U}_n, \\ \boldsymbol{V}_3 &= \boldsymbol{U}_a \otimes \boldsymbol{U}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{U}_n, \\ \boldsymbol{V}_{12} &= \boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{U}_c \otimes \boldsymbol{U}_n, \\ \boldsymbol{V}_{13} &= \boldsymbol{I}_a \otimes \boldsymbol{U}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{U}_n. \end{aligned}$$

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where $\boldsymbol{I}_a$ is the $a \times a$ identity matrix and $\boldsymbol{U}_a$ is the matrix of size $a \times a$ with all elements equal to one. Define

$$\bar{Y}_{i..} = \frac{1}{bcn} \sum_{j,k,r} Y_{ijkr}$$

so that $\bar{Y}_{i..}$ is the marginal sample cell-mean corresponding to level i of factor A. Let the $a \times 1$ vector $\bar{\boldsymbol{Y}}$ be the vector of sample marginal cell means for factor A. Then the distribution of $\bar{\boldsymbol{Y}}$ is given by

$$\bar{\boldsymbol{Y}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\Sigma}$ is the $a \times a$ matrix given by

$$\boldsymbol{\Sigma} \;=\; \frac{1}{bcn}\left[nc\sigma_{AB}^2 + nb\sigma_{AC}^2 + \sigma_e^2\right]\boldsymbol{I}_a + \frac{1}{bc}\left[c\sigma_B^2 + b\sigma_C^2\right]\boldsymbol{U}_a$$

The ANOVA table for the model is in Table 7.4.1 as follows.

Table 7.4.1: ANOVA Table for Balanced Three-factor Crossed Mixed Linear Model with Interactions

Source	df	SS	EMS
A	$a-1$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nc\sigma_{AB}^2 + nb\sigma_{AC}^2$ $+\sigma_e^2 + nbcQ(\boldsymbol{\mu})$
B	$b-1$	$\boldsymbol{Y}'\left(\frac{\boldsymbol{U}_a}{a} \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nac\sigma_B^2 + nc\sigma_{AB}^2 + \sigma_e^2$
C	$c-1$	$\boldsymbol{Y}'\left(\frac{\boldsymbol{U}_a}{a} \otimes \frac{\boldsymbol{U}_b}{b} \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nab\sigma_C^2 + nb\sigma_{AC}^2 + \sigma_e^2$
AB	$(a-1)(b-1)$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \boldsymbol{S}_b \otimes \frac{\boldsymbol{U}_c}{c} \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nc\sigma_{AB}^2 + \sigma_e^2$
AC	$(a-1)(c-1)$	$\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \frac{\boldsymbol{U}_b}{b} \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	$nb\sigma_{AC}^2 + \sigma_e^2$
Residual	$a(bcn-b-c+1)$	$\boldsymbol{Y}'\left(\boldsymbol{I}_a \otimes \boldsymbol{I}_b \otimes \boldsymbol{I}_c \otimes \boldsymbol{S}_n\right)\boldsymbol{Y}+$ $\boldsymbol{Y}'\left(\frac{\boldsymbol{U}_a}{a} \otimes \boldsymbol{S}_b \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}+$ $\boldsymbol{Y}'\left(\boldsymbol{S}_a \otimes \boldsymbol{S}_b \otimes \boldsymbol{S}_c \otimes \frac{\boldsymbol{U}_n}{n}\right)\boldsymbol{Y}$	σ_e^2

where $Q(\boldsymbol{\mu})$ is a function of expected responses; $\boldsymbol{\mu}$. Let $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_a)'$ where $\mu_i = \mu + \alpha_i$ denote the expected response corresponding to level i of factor A.

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We aimed to test the null hypothesis $H_0 : \mu_1 = \cdots = \mu_a$ against the alternative hypothesis H_1 that at least two cell means are different. The null hypothesis of equal means can also be written as $H_0 : \alpha_1 = \cdots = \alpha_a = 0$. Under the null hypothesis of equal cell-means, the expected mean square corresponding to factor A is

$$\begin{aligned} EMS(A) &= nc\sigma_{AB}^2 + nb\sigma_{AC}^2 + \sigma_e^2 \\ &= EMS(AB) + EMS(AC) - EMS(Residual) \end{aligned}$$

So the mean square of the factor A, under H_0, does not have a scaled central chi-squared distribution but has a distribution that is the same as a linear combination of central chi-squared distributions. Thus there is no exact test for H_0. In this situation a generalized test variable can be constructed . This is facilitated by first obtaining FGPQs for σ_{AB}^2, σ_{AC}^2 and σ_e^2. Define

$$\begin{aligned} U_1 &= \frac{SS(AB)}{\psi_{AB}}, \quad \psi_{AB} = nc\sigma_{AB}^2 + \sigma_e^2 \\ U_2 &= \frac{SS(AC)}{\psi_{AC}}, \quad \psi_{AC} = nb\sigma_{AC}^2 + \sigma_e^2 \\ U_3 &= \frac{SS(e)}{\psi_e}, \quad \psi_e = \sigma_e^2, \end{aligned}$$

where $U_1 \sim \chi^2_{(a-1)(b-1)}$, $U_2 \sim \chi^2_{(a-1)(c-1)}$, $U_3 \sim \chi^2_{a(bcn-b-c+1)}$ and U_1, U_2, U_3 are jointly independent. The FGPQs for σ_{AB}^2, σ_{AC}^2, and σ_e^2 are given by

$$\begin{aligned} R_{\sigma_{AB}^2} &= \frac{ss(AB)}{ncU_1} - \frac{ss(e)}{ncU_3} \\ &= \frac{ss(AB)}{nc}\frac{nc\sigma_{AB}^2 + \sigma_e^2}{SS(AB)} - \frac{ss(e)}{nc}\frac{\sigma_e^2}{SS(e)} \\ R_{\sigma_{AC}^2} &= \frac{ss(AC)}{nbU_2} - \frac{ss(e)}{nbU_3} \\ &= \frac{ss(AC)}{nb}\frac{nb\sigma_{AC}^2 + \sigma_e^2}{SS(AC)} - \frac{ss(e)}{nb}\frac{\sigma_e^2}{SS(e)} \\ R_{\sigma_e^2} &= \frac{ss(e)}{U_3} \\ &= \frac{ss(e)}{SS(e)}\sigma_e^2 \end{aligned}$$

where $ss(AB)$, $ss(AC)$, $ss(e)$ are the realized values of $SS(AB)$, $SS(AC)$, $SS(e)$, respectively.

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Let ξ be the parameter vector whose components are $\mu_1, \ldots, \mu_a, \sigma_B^2, \sigma_C^2, \sigma_{AB}^2, \sigma_{AC}^2, \sigma_e^2$. Let $\boldsymbol{\theta}$ be the component of ξ consisting of $\mu_1, \ldots, \mu_a$ and $\boldsymbol{\phi}$ be the component consisting of $\sigma_B^2, \sigma_C^2, \sigma_{AB}^2, \sigma_{AC}^2$ and σ_e^2. Define $R_{\boldsymbol{\phi}} = \left\{R_{\sigma_{AB}^2}, R_{\sigma_{AC}^2}, R_{\sigma_e^2}\right\}$.

Proposition 7.5.1

Define

$$Q(\boldsymbol{Y}, \boldsymbol{\phi}) = \frac{SS(A)}{\psi_A}$$

so that

$$Q\left(\boldsymbol{y}, R_{\boldsymbol{\phi}}\right) = \frac{ss(A)}{R_{\psi_A}} = \frac{ss(A)}{\frac{ss(AB)}{U_1} + \frac{ss(AC)}{U_2} - \frac{ss(e)}{U_3}}.$$

Then $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}) = \frac{Q(\boldsymbol{Y}, \boldsymbol{\phi})}{Q\left(\boldsymbol{y}, R_{\boldsymbol{\phi}}\right)}$ is a GTV for testing $H_0 : \mu_1 = \ldots = \mu_a$.

Proof:

$R_{\boldsymbol{\phi}}$ is FGPQ for $\boldsymbol{\phi}$. Note that, under H_0, $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ has a central χ^2 distribution with $a-1$ degrees of freedom. Furthermore $Q(\boldsymbol{Y}, \boldsymbol{\phi})$, $R_{\sigma_{AB}^2}$, $R_{\sigma_{AC}^2}$, $R_{\sigma_e^2}$, are jointly independent. The distribution of $Q(\boldsymbol{Y}, \boldsymbol{\phi})$ is free of unknown parameters under H_0. The distribution of $Q\left(\boldsymbol{y}, R_{\boldsymbol{\phi}}\right)$ is free of unknown parameters by construction. Thus the distribution of $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ is free of unknown parameters under H_0. It follows that the conditions of Proposition 7.2.1 are satisfied. Hence $T(\boldsymbol{Y}, \boldsymbol{y}, \xi)$ defined by

$$T(\boldsymbol{Y}, \boldsymbol{y}, \boldsymbol{\xi}) = \frac{Q(\boldsymbol{Y}, \boldsymbol{\phi})}{Q(\boldsymbol{y}, R_{\boldsymbol{\phi}})} = \frac{SS(A)\, R_{\psi(A)}}{ss(A)\, \psi_A} \tag{7.10}$$

is a GTV for testing H_0. Also the GPV for the test is

$$GPV = Pr[T(\boldsymbol{Y}, \boldsymbol{y}, \xi) > 1 \,|\, H_0].$$

The null hypothesis is rejected if GPV is smaller than the specified α.

Remark

The unbiasedness condition is satisfied here because the denominator of $T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi})$ is constant, say c. Under H_0 the numerator has χ_{a-1}^2 but if the null hypothesis is false, then the numerator has $\chi_{a-1,\lambda}^2$. It is known that $pr\left\{\chi_{d,\lambda}^2 \geq c\right\} \geq pr\left\{\chi_d^2 \geq c\right\}$ which leads to $pr\left\{T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}) \geq t\right\} \geq pr\left\{T(\boldsymbol{Y}; \boldsymbol{y}, \boldsymbol{\xi}_0) \geq t\right\}$.

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7.6 Simulation Study

A simulation study was conducted to assess the performance of the generalized test of the null hypothesis $H_0 : \mu_1 = \ldots = \mu_a$ in the model given in Equation (7.9) and to compare it with the performance of the commonly used approximate F-test using Satterthwaite's degrees of freedom. Several different combinations of number of levels of factors and values of σ^2_{AB}, σ^2_{AC} and σ^2_e were considered. These different combinations (except the last two combinations of levels of factors) were obtained from Calvin (1985) who conducted a simulation study for a related problem. Different combinations of variance parameters were chosen as follows. Each of σ^2_{AB}, σ^2_{AC} and σ^2_e was allowed to take values $0, 0.2, 0.4, 0.6, 0.8, 1$ subject to the requirement that $\sigma^2_{AB} + \sigma^2_{AC} \leq 1$ and $\sigma^2_e = 1 - \sigma^2_{AB} - \sigma^2_{AC}$. If this assignment resulted in σ^2_e taking the value 0, then the value of σ^2_e was set at 0.01 and the value of the larger remaining variance parameters was decreased by 0.01. The familywise type-I error rates were estimated via simulation. The sample size settings that were used in the simulation study are given in Table 7.6.1. The value of n was taken to be 1 in every case. Each combination of values for (a, b, c) and $(\sigma^2_{AB}, \sigma^2_{AC}, \sigma^2_e)$ was considered.

Table 7.6.1: Factor Level Settings for the Simulation Study

Quantity	Values					
(a,b,c)	(2,3,3),	(2,3,4),	(2,3,10),	(3,3,3),	(3,3,5),	(3,3,10),
	(3,5,10),	(4,4,4),	(5,5,10),	(10,10,10),	(10,10,100),	(10,100,100)

7.6.1 Simulation Details

The simulation study is conducted using the following steps.

Step 1. Select one of the settings for factor levels (a, b, c) and one of the allowed combinations for $(\sigma^2_{AB}, \sigma^2_{AC}, \sigma^2_e)$.

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Step 2. Generate the objects (with independent components) ($SS(A)$, $SS(AB)$, $SS(AC)$, $SS(Residual)$) and ($MS(A)$, $MS(AB)$, $MS(AC)$, $MS(Residual)$) as follows.

$$\begin{aligned}
SS(A) &= \left(nc\sigma_{AB}^2 + nb\sigma_{AC}^2 + \sigma_e^2\right)\chi^2_{(a-1)}, \quad MS(A) = \frac{SS(A)}{(a-1)} \\
SS(AB) &= \left(nc\sigma_{AB}^2 + \sigma_e^2\right)\chi^2_{(a-1)(b-1)}, \quad MS(AB) = \frac{SS(AB)}{(a-1)(b-1)} \\
SS(AC) &= \left(nb\sigma_{AC}^2 + \sigma_e^2\right)\chi^2_{(a-1)(c-1)}, \quad MS(AC) = \frac{SS(AC)}{(a-1)(c-1)} \\
SS(e) &= \left(\sigma_e^2\right)\chi^2_{a(bcn-b-c+1)}, \quad MS(e) = \frac{SS(e)}{a(bcn-b-c+1)},
\end{aligned}$$

where χ_f^2 denotes a chi-squared random variable with f degrees of freedom. Denote the corresponding realized values by $(ss(A), ss(AB), ss(AC), ss(e))$ and $(ms(A), ms(AB), ms(AC), ms(e))$, respectively.

Step 3. (*Satterthwaite*) Calculate

$$F - ratio = \frac{ms(A)}{ms(AB) + ms(AC) - ms(e)}$$

and $F - tabulated = 1 - \alpha$ percentile from a central F distribution with numerator degrees of freedom equal to $a-1$ and denominator degrees of freedom equal to $\hat{\nu}^*$ where

$$\hat{\nu}^* = \frac{(ms(AB) + ms(AC) - ms(e))^2}{\frac{(ms(AB))^2}{(a-1)(b-1)} + \frac{(ms(AC))^2}{(a-1)(c-1)} + \frac{(ms(e))^2}{a(bcn-b-c+1)}}$$

If $ms(AB) + ms(AC) - ms(e)$ in the numerator of $\hat{\nu}^*$ and the denominator of the $F-$ ratio is zero or negative, then there are two possible courses of action one could take. One possibility is to replace $ms(AB) + ms(AC) - ms(e)$ by 0.0001 in both the numerator of $\hat{\nu}^*$ and the denominator of $F-$ ratio. This leads to the following calculations.

$$\begin{aligned}
F - ratio &= \frac{ms(A)}{\max\{ms(AB) + ms(AC) - ms(e), 0.0001\}}, \quad \text{and} \\
\hat{\nu}^* &= \frac{(\max\{ms(AB) + ms(AC) - ms(e), 0.0001\})^2}{\frac{(ms(AB))^2}{(a-1)(b-1)} + \frac{(ms(AC))^2}{(a-1)(c-1)} + \frac{(ms(e))^2}{a(bcn-b-c+1)}}. \qquad (7.11)
\end{aligned}$$

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The other possibility is to use 0.0001 in the denominator of the $F-$ ratio and keep the negative value in the numerator of $\hat{\nu}^*$. Thus the following alternate calculations.

$$F-ratio = \frac{ms(A)}{\max\{ms(AB)+ms(AC)-ms(e), 0.0001\}}, \quad \text{and}$$
$$\hat{\nu}^* = \frac{(ms(AB)+ms(AC)-ms(e))^2}{\frac{(ms(AB))^2}{(a-1)(b-1)}+\frac{(ms(AC))^2}{(a-1)(c-1)}+\frac{(ms(e))^2}{a(bcn-b-c+1)}}. \tag{7.12}$$

We considered both possibilities. If F-ratio is greater than or equal to $F-tabulated$, then H_0 is rejected. This is counted as a false rejection since the statistic $SS(A)$ is generated from a central chi-squared distribution assuming that H_0 holds.

Step 4. (*GPV*) For $r = 1, \cdots, P$, generate independent realizations of the following chi-squared random variables.

$$U^{(r)} \sim \chi^2_{a-1}, \quad U_1^{(r)} \sim \chi^2_{(a-1)(b-1)}, \quad U_2^{(r)} \sim \chi^2_{(a-1)(c-1)}, \quad \text{and} \quad U_3^{(r)} \sim \chi^2_{a(bcn-b-c+1)}.$$

We used $P = 10000$. Then calculate $\frac{ss(AB)}{U_1^{(r)}}$, $\frac{ss(AC)}{U_2^{(r)}}$ and $\frac{ss(e)}{U_3^{(r)}}$. Calculate $Q\left(\boldsymbol{y}, R_{\boldsymbol{\phi}}\right) = \frac{ss(A)}{R_{\psi_A}} = \frac{ss(A)}{\frac{ss(AB)}{U_1^{(r)}}+\frac{ss(AC)}{U_2^{(r)}}-\frac{ss(Residual)}{U_3^{(r)}}}$. Using these, calculate the generalized test variable $T^{(r)}(\boldsymbol{Y}, \boldsymbol{y}, \xi)$. Since R_{ψ_A} can be negative with nonzero probability, one might consider two alternative modified definitions of $T^{(r)}(\boldsymbol{Y}, \boldsymbol{y}, \xi)$. One choice is to define

$$T^{(r)}(\boldsymbol{Y}, \boldsymbol{y}, \xi) = \frac{SS(A)\max\{R_{\psi(A)}, 0\}}{ss(A)\,\psi_A} \tag{7.13}$$

and the other choice is to use

$$T^{(r)}(\boldsymbol{Y}, \boldsymbol{y}, \xi) = \frac{SS(A)\,|R_{\psi(A)}|}{ss(A)\,\psi_A}. \tag{7.14}$$

Although it appears to be no logical reasoning leading to the Equation 7.14, we consider it here because other authors (e.g., Weerahandi, 1993) have tried this approach.

Next, calculate the proportion of the values $T^{(r)}(\boldsymbol{Y}, \boldsymbol{y}, \xi)$, $r = 1, \ldots, P$, that are greater than 1. Denote this proportion by p. This is an estimate of the GPV for testing H_0. If p is less than α then record this as a false rejection.

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Step 5. Repeat the steps 1-4 M times (we used $M = 1000$). Calculate the proportion of false rejections for the Satterthwaite's approach and for the generalized test. These provide empirical estimates of the type-I error rates for the two procedures. Table 7.6.2 summarize the results of the simulation study.

7.6.2 Discussion of the Simulation Results

Using GPV: when the number of levels of the factors in the model is small, the test is quite conservative since the type-I error rates are smaller than the nominal value of $\alpha = 0.05$. When Equation (7.14) is used to calculate the GTV, the estimated type-I error rate is closer to the nominal value when compared to the results obtained by using Equation (7.13). (We do not, at this time, have a theoretical explanation for this). As the number of levels of at least one of random effects increases, the difference between the two approaches becomes negligible for most of the cases. As the number of levels of random factors is large in comparison with the number of levels of the fixed factor, the estimated type-I error rates approach the nominal value of 0.05 for most of the cases.

Using Satterthwaite's procedure: when the number of levels of the factors in the model is small, the empirical error rates are smaller than the nominal value of $\alpha = 0.05$. When using Equation (7.12), the empirical error rates get closer to 0.05. When the number of levels of the fixed factor is small but the number of levels of random factors is moderate or large, then the empirical type-I error rates using either Equation (7.11) or Equation (7.12) are nearly the same for most of the cases considered. Increasing the number of levels for factors such that the number of levels of random factors is larger than that of the fixed factor leads to empirical error rates that are close to 0.05 in most of the cases.

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Table 7.6.2: Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
2	3	3	0	0	1	0	0	0	0.133
2	3	3	0	0.2	0.2	0	0.008	0.001	0.043
2	3	3	0	0.2	0.4	0	0.005	0	0.069
2	3	3	0	0.2	0.6	0	0.004	0	0.085
2	3	3	0	0.2	0.8	0	0.004	0	0.094
2	3	3	0	0.4	0.2	0	0.013	0.004	0.034
2	3	3	0	0.4	0.4	0	0.008	0.001	0.045
2	3	3	0	0.4	0.6	0	0.005	0	0.053
2	3	3	0	0.6	0.2	0	0.016	0.006	0.029
2	3	3	0	0.6	0.4	0	0.012	0.002	0.039
2	3	3	0	0.8	0.2	0.001	0.021	0.009	0.029
2	3	3	0	0.99	0.01	0.031	0.043	0.035	0.035
2	3	3	0.2	0	0.2	0	0.012	0.001	0.04
2	3	3	0.2	0	0.4	0	0.005	0	0.072
2	3	3	0.2	0	0.6	0	0.003	0	0.084
2	3	3	0.2	0	0.8	0	0.002	0	0.097
2	3	3	0.2	0.2	0.2	0	0.009	0.002	0.019
2	3	3	0.2	0.2	0.4	0	0.006	0.001	0.037
2	3	3	0.2	0.2	0.6	0	0.005	0.001	0.049
2	3	3	0.2	0.4	0.2	0	0.010	0.002	0.012
2	3	3	0.2	0.4	0.4	0	0.009	0.001	0.029
2	3	3	0.2	0.6	0.2	0	0.009	0.007	0.016
2	3	3	0.2	0.79	0.01	0.012	0.013	0.047	0.047
2	3	3	0.4	0	0.2	0	0.020	0.003	0.025
2	3	3	0.4	0	0.4	0	0.012	0.001	0.041
2	3	3	0.4	0	0.6	0	0.008	0	0.060
2	3	3	0.4	0.2	0.2	0	0.011	0.005	0.014
2	3	3	0.4	0.2	0.4	0	0.008	0.002	0.025
2	3	3	0.4	0.4	0.2	0	0.010	0.005	0.009
2	3	3	0.4	0.59	0.01	.009	0.009	0.038	0.038
2	3	3	0.6	0	0.2	0	0.022	0.006	0.020
2	3	3	0.6	0	0.4	0	0.016	0.002	0.031
2	3	3	0.6	0.2	0.2	0	0.012	0.007	0.012
2	3	3	0.59	0.4	0.01	0.009	0.010	0.041	0.041
2	3	3	0.8	0	0.2	0	0.027	0.008	0.019
2	3	3	0.79	0.2	0.01	0.015	0.015	0.052	0.052
2	3	3	0.99	0	0.01	0.044	0.049	0.051	0.051

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
2	3	4	0	0	1	0	0	0	0.106
2	3	4	0	0.2	0.2	0.001	0.018	0.004	0.025
2	3	4	0	0.2	0.4	0	0.009	0.002	0.043
2	3	4	0	0.2	0.6	0	0.006	0	0.053
2	3	4	0	0.2	0.8	0	0.004	0	0.063
2	3	4	0	0.4	0.2	0.005	0.027	0.010	0.019
2	3	4	0	0.4	0.4	0.001	0.018	0.004	0.025
2	3	4	0	0.4	0.6	0	0.012	0.004	0.035
2	3	4	0	0.6	0.2	0.007	0.030	0.012	0.018
2	3	4	0	0.6	0.4	0.002	0.023	0.008	0.021
2	3	4	0	0.8	0.2	0.008	0.032	0.015	0.020
2	3	4	0	0.99	0.01	0.049	0.052	0.051	0.051
2	3	4	0.2	0	0.2	0	0.016	0.001	0.022
2	3	4	0.2	0	0.4	0	0.010	0	0.044
2	3	4	0.2	0	0.6	0	0.007	0	0.059
2	3	4	0.2	0	0.8	0	0.004	0	0.069
2	3	4	0.2	0.2	0.2	0.001	0.012	0.005	0.008
2	3	4	0.2	0.2	0.4	0	0.011	0.002	0.017
2	3	4	0.2	0.2	0.6	0	0.011	0.001	0.028
2	3	4	0.2	0.4	0.2	0.002	0.014	0.011	0.012
2	3	4	0.2	0.4	0.4	0.001	0.012	0.004	0.011
2	3	4	0.2	0.6	0.2	0.002	0.015	0.021	0.021
2	3	4	0.2	0.79	0.01	0.018	0.018	0.045	0.045
2	3	4	0.4	0	0.2	0.001	0.025	0.004	.017
2	3	4	0.4	0	0.4	0	0.016	0.001	.023
2	3	4	0.4	0	0.6	0	0.012	0	0.038
2	3	4	0.4	0.2	0.2	0.003	0.012	0.012	0.014
2	3	4	0.4	0.2	0.4	0.001	0.012	0.004	0.013
2	3	4	0.4	0.4	0.2	0.004	0.013	0.013	0.014
2	3	4	0.4	0.59	0.01	0.013	0.013	0.040	0.040
2	3	4	0.6	0	0.2	0.002	0.025	0.011	0.019
2	3	4	0.6	0	0.4	0.001	0.020	0.004	0.021
2	3	4	0.6	0.2	0.2	0.004	0.014	0.021	0.023
2	3	4	0.59	0.4	0.01	0.014	0.014	0.039	0.039
2	3	4	0.8	0	0.2	0.004	0.027	0.014	0.020
2	3	4	0.79	0.2	0.01	0.016	0.016	0.050	0.050
2	3	4	0.99	0	0.01	0.036	0.044	0.038	0.038

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
2	3	10	0	0	1	0	0.013	0	0.074
2	3	10	0	0.2	0.2	0.017	0.022	0.028	0.028
2	3	10	0	0.2	0.4	0.003	0.020	0.016	0.020
2	3	10	0	0.2	0.6	0.001	0.019	0.004	0.011
2	3	10	0	0.2	0.8	0	0.016	0.004	0.013
2	3	10	0	0.4	0.2	0.023	0.024	0.037	0.037
2	3	10	0	0.4	0.4	0.017	0.022	0.028	0.028
2	3	10	0	0.4	0.6	0.009	0.020	0.019	0.021
2	3	10	0	0.6	0.2	0.028	0.028	0.041	0.041
2	3	10	0	0.6	0.4	0.021	0.023	0.034	0.034
2	3	10	0	0.8	0.2	0.029	0.029	0.043	0.043
2	3	10	0	0.99	0.01	0.047	0.047	0.048	0.048
2	3	10	0.2	0	0.2	0.011	0.054	0.022	0.031
2	3	10	0.2	0	0.4	0.004	0.045	0.010	0.023
2	3	10	0.2	0	0.6	0.001	0.041	0.005	0.023
2	3	10	0.2	0	0.8	0.001	0.033	0.001	0.028
2	3	10	0.2	0.2	0.2	0.026	0.028	0.059	0.059
2	3	10	0.2	0.2	0.4	0.017	0.027	0.038	0.038
2	3	10	0.2	0.2	0.6	0.010	0.027	0.023	0.025
2	3	10	0.2	0.4	0.2	0.023	0.023	0.046	0.046
2	3	10	0.2	0.4	0.4	0.021	0.024	0.042	0.042
2	3	10	0.2	0.6	0.2	0.019	0.019	0.042	0.042
2	3	10	0.2	0.79	0.01	0.019	0.019	0.043	0.043
2	3	10	0.4	0	0.2	0.029	0.061	0.037	0.043
2	3	10	0.4	0	0.4	0.011	0.054	0.022	0.032
2	3	10	0.4	0	0.6	0.005	0.047	0.014	0.027
2	3	10	0.4	0.2	0.2	0.036	0.040	0.066	0.066
2	3	10	0.4	0.2	0.4	0.027	0.037	0.051	0.051
2	3	10	0.4	0.4	0.2	0.027	0.028	0.068	0.068
2	3	10	0.4	0.59	0.01	0.024	0.024	0.058	0.058
2	3	10	0.6	0	0.2	0.036	0.061	0.040	0.045
2	3	10	0.6	0	0.4	0.019	0.058	0.030	0.040
2	3	10	0.6	0.2	0.2	0.041	0.042	0.069	0.069
2	3	10	0.59	0.4	0.01	0.033	0.033	0.068	0.068
2	3	10	0.8	0	0.2	0.041	0.061	0.041	0.043
2	3	10	0.79	0.2	0.01	0.048	0.048	0.081	0.081
2	3	10	0.99	0	0.01	0.063	0.063	0.062	0.062

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
3	3	3	0	0	1	0	0.003	0	0.061
3	3	3	0	0.2	0.2	0	0.018	0.005	0.015
3	3	3	0	0.2	0.4	0	0.010	0.001	0.02
3	3	3	0	0.2	0.6	0	0.003	0	0.014
3	3	3	0	0.2	0.8	0	0.006	0	0.026
3	3	3	0	0.4	0.2	0.004	0.025	0.011	0.016
3	3	3	0	0.4	0.4	0	0.018	0.005	0.015
3	3	3	0	0.4	0.6	0	0.013	0.001	0.015
3	3	3	0	0.6	0.2	0.010	0.034	0.022	0.024
3	3	3	0	0.6	0.4	0.002	0.020	0.008	0.013
3	3	3	0	0.8	0.2	0.020	0.037	0.026	0.027
3	3	3	0	0.99	0.01	0.050	0.050	0.048	0.048
3	3	3	0.2	0	0.2	0.002	0.022	0.006	0.010
3	3	3	0.2	0	0.4	0	0.010	0.002	0.016
3	3	3	0.2	0	0.6	0	0.008	0	0.022
3	3	3	0.2	0	0.8	0	0.008	0	0.027
3	3	3	0.2	0.2	0.2	0.005	0.017	0.013	0.013
3	3	3	0.2	0.2	0.4	0.001	0.016	0.004	0.009
3	3	3	0.2	0.2	0.6	0	0.014	0.001	0.011
3	3	3	0.2	0.4	0.2	0.006	0.020	0.018	0.018
3	3	3	0.2	0.4	0.4	0.002	0.019	0.007	0.008
3	3	3	0.2	0.6	0.2	0.011	0.020	0.024	0.024
3	3	3	0.2	0.79	0.01	0.018	0.018	0.040	0.040
3	3	3	0.4	0	0.2	0.009	0.034	0.021	0.022
3	3	3	0.4	0	0.4	0.002	0.022	0.006	0.010
3	3	3	0.4	0	0.6	0.001	0.012	0.004	0.015
3	3	3	0.4	0.2	0.2	0.007	0.015	0.022	0.022
3	3	3	0.4	0.2	0.4	0.005	0.016	0.010	0.012
3	3	3	0.4	0.4	0.2	0.009	0.019	0.023	0.023
3	3	3	0.4	0.59	0.01	0.017	0.017	0.038	0.038
3	3	3	0.6	0	0.2	0.017	0.040	0.027	0.027
3	3	3	0.6	0	0.4	0.006	0.031	0.013	0.015
3	3	3	0.6	0.2	0.2	0.011	0.017	0.027	0.027
3	3	3	0.59	0.4	0.01	0.013	0.013	0.041	0.041
3	3	3	0.8	0	0.2	0.025	0.045	0.037	0.037
3	3	3	0.79	0.2	0.01	0.020	0.020	0.057	0.057
3	3	3	0.99	0	0.01	0.058	0.058	0.056	0.056

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
3	3	5	0	0	1	0	0.011	0.001	0.061
3	3	5	0	0.2	0.2	0.008	0.027	0.015	0.061
3	3	5	0	0.2	0.4	0.003	0.028	0.005	0.008
3	3	5	0	0.2	0.6	0.002	0.025	0.003	0.013
3	3	5	0	0.2	0.8	0.002	0.023	0.002	0.018
3	3	5	0	0.4	0.2	0.022	0.032	0.027	0.027
3	3	5	0	0.4	0.4	0.008	0.027	0.015	0.016
3	3	5	0	0.4	0.6	0.005	0.028	0.007	0.011
3	3	5	0	0.6	0.2	0.03	0.033	0.034	0.034
3	3	5	0	0.6	0.4	0.016	0.030	0.017	0.017
3	3	5	0	0.8	0.2	0.034	0.035	0.037	0.037
3	3	5	0	0.99	0.01	0.041	0.041	0.041	0.041
3	3	5	0.2	0	0.2	0.019	0.043	0.024	0.026
3	3	5	0.2	0	0.4	0.008	0.042	0.013	0.018
3	3	5	0.2	0	0.6	0.006	0.038	0.008	0.019
3	3	5	0.2	0	0.8	0.004	0.035	0.007	0.022
3	3	5	0.2	0.2	0.2	0.023	0.03	0.040	0.040
3	3	5	0.2	0.2	0.4	0.013	0.032	0.017	0.018
3	3	5	0.2	0.2	0.6	0.012	0.031	0.014	0.016
3	3	5	0.2	0.4	0.2	0.025	0.027	0.044	0.044
3	3	5	0.2	0.4	0.4	0.017	0.030	0.027	0.027
3	3	5	0.2	0.6	0.2	0.027	0.027	0.046	0.046
3	3	5	0.2	0.79	0.01	0.027	0.027	0.052	0.052
3	3	5	0.4	0	0.2	0.027	0.046	0.029	0.029
3	3	5	0.4	0	0.4	0.019	0.043	0.024	0.026
3	3	5	0.4	0	0.6	0.011	0.042	0.020	0.024
3	3	5	0.4	0.2	0.2	0.032	0.036	0.051	0.051
3	3	5	0.4	0.2	0.4	0.019	0.037	0.033	0.033
3	3	5	0.4	0.4	0.2	0.031	0.032	0.051	0.051
3	3	5	0.4	0.59	0.01	0.025	0.025	0.052	0.052
3	3	5	0.6	0	0.2	0.034	0.046	0.034	0.034
3	3	5	0.6	0	0.4	0.024	0.045	0.027	0.028
3	3	5	0.6	0.2	0.2	0.039	0.041	0.054	0.054
3	3	5	0.59	0.4	0.01	0.030	0.030	0.055	0.055
3	3	5	0.8	0	0.2	0.039	0.046	0.039	0.039
3	3	5	0.79	0.2	0.01	0.038	0.038	0.061	0.061
3	3	5	0.99	0	0.01	0.047	0.047	0.049	0.049

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
3	3	10	0	0	1	0	0.022	0	0.037
3	3	10	0	0.2	0.2	0.022	0.025	0.031	0.031
3	3	10	0	0.2	0.4	0.012	0.026	0.021	0.022
3	3	10	0	0.2	0.6	0.010	0.028	0.013	0.014
3	3	10	0	0.2	0.8	0.004	0.026	0.010	0.014
3	3	10	0	0.4	0.2	0.026	0.026	0.038	0.038
3	3	10	0	0.4	0.4	0.022	0.025	0.031	0.031
3	3	10	0	0.4	0.6	0.017	0.027	0.025	0.025
3	3	10	0	0.6	0.2	0.028	0.028	0.039	0.039
3	3	10	0	0.6	0.4	0.027	0.027	0.037	0.037
3	3	10	0	0.8	0.2	0.032	0.032	0.041	0.041
3	3	10	0	0.99	0.01	0.038	0.038	0.039	0.039
3	3	10	0.2	0	0.2	0.032	0.042	0.032	0.033
3	3	10	0.2	0	0.4	0.017	0.038	0.021	0.023
3	3	10	0.2	0	0.6	0.013	0.038	0.014	0.018
3	3	10	0.2	0	0.8	0.008	0.037	0.010	0.014
3	3	10	0.2	0.2	0.2	0.030	0.030	0.047	0.047
3	3	10	0.2	0.2	0.4	0.028	0.029	0.041	0.041
3	3	10	0.2	0.2	0.6	0.025	0.032	0.034	0.035
3	3	10	0.2	0.4	0.2	0.028	0.028	0.042	0.042
3	3	10	0.2	0.4	0.4	0.028	0.028	0.042	0.042
3	3	10	0.2	0.6	0.2	0.024	0.024	0.043	0.043
3	3	10	0.2	0.79	0.01	0.024	0.024	0.045	0.045
3	3	10	0.4	0	0.2	0.040	0.042	0.036	0.036
3	3	10	0.4	0	0.4	0.032	0.042	0.032	0.033
3	3	10	0.4	0	0.6	0.023	0.041	0.024	0.026
3	3	10	0.4	0.2	0.2	0.035	0.035	0.052	0.052
3	3	10	0.4	0.2	0.4	0.033	0.034	0.047	0.047
3	3	10	0.4	0.4	0.2	0.032	0.032	0.047	0.047
3	3	10	0.4	0.59	0.01	0.027	0.027	0.046	0.046
3	3	10	0.6	0	0.2	0.042	0.042	0.039	0.039
3	3	10	0.6	0	0.4	0.037	0.041	0.035	0.035
3	3	10	0.6	0.2	0.2	0.035	0.035	0.052	0.052
3	3	10	0.59	0.4	0.01	0.033	0.033	0.053	0.053
3	3	10	0.8	0	0.2	0.042	0.042	0.041	0.041
3	3	10	0.79	0.2	0.01	0.038	0.038	0.053	0.053
3	3	10	0.99	0	0.01	0.042	0.042	0.042	0.042

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
3	5	10	0	0	1	0.006	0.043	0.004	0.014
3	5	10	0	0.2	0.2	0.043	0.043	0.051	0.051
3	5	10	0	0.2	0.4	0.047	0.047	0.045	0.045
3	5	10	0	0.2	0.6	0.042	0.045	0.039	0.039
3	5	10	0	0.2	0.8	0.034	0.044	0.035	0.035
3	5	10	0	0.4	0.2	0.051	0.051	0.055	0.055
3	5	10	0	0.4	0.4	0.043	0.043	0.051	0.051
3	5	10	0	0.4	0.6	0.045	0.045	0.047	0.047
3	5	10	0	0.6	0.2	0.052	0.052	0.055	0.055
3	5	10	0	0.6	0.4	0.046	0.046	0.054	0.054
3	5	10	0	0.8	0.2	0.053	0.053	0.056	0.056
3	5	10	0	0.99	0.01	0.055	0.055	0.055	0.055
3	5	10	0.2	0	0.2	0.057	0.058	0.060	0.060
3	5	10	0.2	0	0.4	0.053	0.056	0.054	0.054
3	5	10	0.2	0	0.6	0.049	0.054	0.046	0.046
3	5	10	0.2	0	0.8	0.045	0.053	0.040	0.040
3	5	10	0.2	0.2	0.2	0.049	0.049	0.059	0.059
3	5	10	0.2	0.2	0.4	0.048	0.048	0.056	0.056
3	5	10	0.2	0.2	0.6	0.047	0.047	0.053	0.053
3	5	10	0.2	0.4	0.2	0.045	0.045	0.060	0.060
3	5	10	0.2	0.4	0.4	0.042	0.042	0.057	0.057
3	5	10	0.2	0.6	0.2	0.044	0.044	0.056	0.056
3	5	10	0.2	0.79	0.01	0.045	0.045	0.060	0.060
3	5	10	0.4	0	0.2	0.058	0.058	0.060	0.060
3	5	10	0.4	0	0.4	0.057	0.058	0.060	0.060
3	5	10	0.4	0	0.6	0.055	0.057	0.056	0.056
3	5	10	0.4	0.2	0.2	0.054	0.054	0.066	0.066
3	5	10	0.4	0.2	0.4	0.051	0.051	0.064	0.064
3	5	10	0.4	0.4	0.2	0.050	0.050	0.059	0.059
3	5	10	0.4	0.59	0.01	0.044	0.044	0.060	0.060
3	5	10	0.6	0	0.2	0.060	0.060	0.060	0.060
3	5	10	0.6	0	0.4	0.057	0.057	0.060	0.060
3	5	10	0.6	0.2	0.2	0.054	0.054	0.064	0.064
3	5	10	0.59	0.4	0.01	0.050	0.050	0.066	0.066
3	5	10	0.8	0	0.2	0.060	0.060	0.060	0.060
3	5	10	0.79	0.2	0.01	0.055	0.055	0.065	0.065
3	5	10	0.99	0	0.01	0.061	0.061	0.060	0.060

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
4	4	4	0	0	1	0	0.019	0	0.017
4	4	4	0	0.2	0.2	0.037	0.046	0.040	0.040
4	4	4	0	0.2	0.4	0.020	0.041	0.020	0.022
4	4	4	0	0.2	0.6	0.012	0.038	0.012	0.015
4	4	4	0	0.2	0.8	0.004	0.033	0.005	0.008
4	4	4	0	0.4	0.2	0.049	0.050	0.048	0.048
4	4	4	0	0.4	0.4	0.037	0.046	0.040	0.04
4	4	4	0	0.4	0.6	0.027	0.043	0.029	0.031
4	4	4	0	0.6	0.2	0.052	0.052	0.053	0.053
4	4	4	0	0.6	0.4	0.045	0.049	0.045	0.045
4	4	4	0	0.8	0.2	0.051	0.051	0.054	0.054
4	4	4	0	0.99	0.01	0.055	0.055	0.056	0.056
4	4	4	0.2	0	0.2	0.028	0.035	0.024	0.024
4	4	4	0.2	0	0.4	0.015	0.033	0.014	0.014
4	4	4	0.2	0	0.6	0.007	0.031	0.009	0.009
4	4	4	0.2	0	0.8	0.003	0.032	0.002	0.003
4	4	4	0.2	0.2	0.2	0.029	0.029	0.040	0.040
4	4	4	0.2	0.2	0.4	0.027	0.031	0.033	0.033
4	4	4	0.2	0.2	0.6	0.023	0.030	0.028	0.028
4	4	4	0.2	0.4	0.2	0.032	0.032	0.047	0.047
4	4	4	0.2	0.4	0.4	0.032	0.034	0.044	0.044
4	4	4	0.2	0.6	0.2	0.040	0.040	0.056	0.056
4	4	4	0.2	0.79	0.01	0.041	0.041	0.060	0.060
4	4	4	0.4	0	0.2	0.038	0.040	0.034	0.034
4	4	4	0.4	0	0.4	0.028	0.035	0.024	0.024
4	4	4	0.4	0	0.6	0.022	0.033	0.019	0.019
4	4	4	0.4	0.2	0.2	0.028	0.028	0.040	0.040
4	4	4	0.4	0.2	0.4	0.027	0.029	0.037	0.037
4	4	4	0.4	0.4	0.2	0.029	0.029	0.044	0.044
4	4	4	0.4	0.59	0.01	0.033	0.033	0.048	0.048
4	4	4	0.6	0	0.2	0.037	0.038	0.035	0.035
4	4	4	0.6	0	0.4	0.032	0.036	0.033	0.033
4	4	4	0.6	0.2	0.2	0.029	0.029	0.039	0.039
4	4	4	0.59	0.4	0.01	0.029	0.029	0.044	0.044
4	4	4	0.8	0	0.2	0.038	0.038	0.036	0.036
4	4	4	0.79	0.2	0.01	0.032	0.032	0.037	0.037
4	4	4	0.99	0	0.01	0.036	0.036	0.038	0.038

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
5	5	10	0	0	1	0.019	0.048	0.012	0.012
5	5	10	0	0.2	0.2	0.051	0.051	0.055	0.055
5	5	10	0	0.2	0.4	0.053	0.053	0.055	0.055
5	5	10	0	0.2	0.6	0.052	0.052	0.055	0.055
5	5	10	0	0.2	0.8	0.051	0.051	0.049	0.049
5	5	10	0	0.4	0.2	0.051	0.051	0.052	0.052
5	5	10	0	0.4	0.4	0.051	0.051	0.055	0.055
5	5	10	0	0.4	0.6	0.054	0.054	0.054	0.054
5	5	10	0	0.6	0.2	0.051	0.051	0.051	0.051
5	5	10	0	0.6	0.4	0.052	0.052	0.053	0.053
5	5	10	0	0.8	0.2	0.049	0.049	0.049	0.049
5	5	10	0	0.99	0.01	0.048	0.048	0.047	0.047
5	5	10	0.2	0	0.2	0.051	0.051	0.054	0.054
5	5	10	0.2	0	0.4	0.053	0.053	0.050	0.050
5	5	10	0.2	0	0.6	0.049	0.049	0.050	0.050
5	5	10	0.2	0	0.8	0.049	0.049	0.047	0.047
5	5	10	0.2	0.2	0.2	0.045	0.045	0.049	0.049
5	5	10	0.2	0.2	0.4	0.046	0.046	0.049	0.049
5	5	10	0.2	0.2	0.6	0.046	0.046	0.050	0.050
5	5	10	0.2	0.4	0.2	0.040	0.040	0.048	0.048
5	5	10	0.2	0.4	0.4	0.039	0.039	0.048	0.048
5	5	10	0.2	0.6	0.2	0.042	0.042	0.046	0.046
5	5	10	0.2	0.79	0.01	0.041	0.041	0.048	0.048
5	5	10	0.4	0	0.2	0.053	0.053	0.054	0.054
5	5	10	0.4	0	0.4	0.051	0.051	0.054	0.054
5	5	10	0.4	0	0.6	0.052	0.052	0.051	0.051
5	5	10	0.4	0.2	0.2	0.048	0.048	0.053	0.053
5	5	10	0.4	0.2	0.4	0.048	0.048	0.053	0.053
5	5	10	0.4	0.4	0.2	0.044	0.044	0.049	0.049
5	5	10	0.4	0.59	0.01	0.043	0.043	0.046	0.046
5	5	10	0.6	0	0.2	0.053	0.053	0.054	0.054
5	5	10	0.6	0	0.4	0.054	0.054	0.054	0.054
5	5	10	0.6	0.2	0.2	0.048	0.048	0.056	0.056
5	5	10	0.59	0.4	0.01	0.047	0.047	0.055	0.055
5	5	10	0.8	0	0.2	0.053	0.053	0.054	0.054
5	5	10	0.79	0.2	0.01	0.048	0.048	0.057	0.057
5	5	10	0.99	0	0.01	0.054	0.054	0.054	0.054

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test Empirical Type-I Error		Satterthwaite Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
10	10	10	0	0	1	0.053	0.053	0.048	0.048
10	10	10	0	0.2	0.2	0.056	0.056	0.057	0.057
10	10	10	0	0.2	0.4	0.055	0.055	0.055	0.055
10	10	10	0	0.2	0.6	0.055	0.055	0.056	0.056
10	10	10	0	0.2	0.8	0.055	0.055	0.056	0.056
10	10	10	0	0.4	0.2	0.057	0.057	0.057	0.057
10	10	10	0	0.4	0.4	0.056	0.056	0.057	0.057
10	10	10	0	0.4	0.6	0.056	0.056	0.056	0.056
10	10	10	0	0.6	0.2	0.058	0.058	0.058	0.058
10	10	10	0	0.6	0.4	0.057	0.057	0.057	0.057
10	10	10	0	0.8	0.2	0.058	0.058	0.059	0.059
10	10	10	0	0.99	0.01	0.059	0.059	0.059	0.059
10	10	10	0.2	0	0.2	0.054	0.054	0.054	0.054
10	10	10	0.2	0	0.4	0.053	0.053	0.053	0.053
10	10	10	0.2	0	0.6	0.051	0.051	0.052	0.052
10	10	10	0.2	0	0.8	0.051	0.051	0.051	0.051
10	10	10	0.2	0.2	0.2	0.049	0.049	0.052	0.052
10	10	10	0.2	0.2	0.4	0.050	0.050	0.053	0.053
10	10	10	0.2	0.2	0.6	0.050	0.050	0.052	0.052
10	10	10	0.2	0.4	0.2	0.052	0.052	0.053	0.053
10	10	10	0.2	0.4	0.4	0.051	0.051	0.054	0.054
10	10	10	0.2	0.6	0.2	0.052	0.052	0.055	0.055
10	10	10	0.2	0.79	0.01	0.053	0.053	0.054	0.054
10	10	10	0.4	0	0.2	0.055	0.055	0.056	0.056
10	10	10	0.4	0	0.4	0.054	0.054	0.054	0.054
10	10	10	0.4	0	0.6	0.054	0.054	0.054	0.054
10	10	10	0.4	0.2	0.2	0.047	0.047	0.049	0.049
10	10	10	0.4	0.2	0.4	0.048	0.048	0.049	0.049
10	10	10	0.4	0.4	0.2	0.049	0.049	0.052	0.052
10	10	10	0.4	0.59	0.01	0.050	0.050	0.052	0.052
10	10	10	0.6	0	0.2	0.055	0.055	0.056	0.056
10	10	10	0.6	0	0.4	0.055	0.055	0.056	0.056
10	10	10	0.6	0.2	0.2	0.046	0.046	0.050	0.050
10	10	10	0.59	0.4	0.01	0.048	0.048	0.050	0.050
10	10	10	0.8	0	0.2	0.055	0.055	0.055	0.055
10	10	10	0.79	0.2	0.01	0.050	0.050	0.053	0.053
10	10	10	0.99	0	0.01	0.054	0.054	0.054	0.054

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
10	10	100	0	0	1	0.047	0.047	0.047	0.047
10	10	100	0	0.2	0.2	0.054	0.054	0.054	0.054
10	10	100	0	0.2	0.4	0.055	0.055	0.056	0.056
10	10	100	0	0.2	0.6	0.056	0.056	0.058	0.058
10	10	100	0	0.2	0.8	0.054	0.054	0.058	0.058
10	10	100	0	0.4	0.2	0.054	0.054	0.054	0.054
10	10	100	0	0.4	0.4	0.054	0.054	0.054	0.054
10	10	100	0	0.4	0.6	0.055	0.055	0.055	0.055
10	10	100	0	0.6	0.2	0.054	0.054	0.054	0.054
10	10	100	0	0.6	0.4	0.054	0.054	0.054	0.054
10	10	100	0	0.8	0.2	0.053	0.053	0.053	0.053
10	10	100	0	0.99	0.01	0.051	0.051	0.051	0.051
10	10	100	0.2	0	0.2	0.043	0.043	0.044	0.044
10	10	100	0.2	0	0.4	0.043	0.043	0.044	0.044
10	10	100	0.2	0	0.6	0.043	0.043	0.044	0.044
10	10	100	0.2	0	0.8	0.043	0.043	0.044	0.044
10	10	100	0.2	0.2	0.2	0.045	0.045	0.046	0.046
10	10	100	0.2	0.2	0.4	0.045	0.045	0.046	0.046
10	10	100	0.2	0.2	0.6	0.045	0.045	0.046	0.046
10	10	100	0.2	0.4	0.2	0.047	0.047	0.050	0.050
10	10	100	0.2	0.4	0.4	0.047	0.047	0.050	0.050
10	10	100	0.2	0.6	0.2	0.052	0.052	0.053	0.053
10	10	100	0.2	0.79	0.01	0.052	0.052	0.054	0.054
10	10	100	0.4	0	0.2	0.043	0.043	0.044	0.044
10	10	100	0.4	0	0.4	0.043	0.043	0.044	0.044
10	10	100	0.4	0	0.6	0.043	0.043	0.044	0.044
10	10	100	0.4	0.2	0.2	0.045	0.045	0.046	0.046
10	10	100	0.4	0.2	0.4	0.045	0.045	0.046	0.046
10	10	100	0.4	0.4	0.2	0.045	0.045	0.047	0.047
10	10	100	0.4	0.59	0.01	0.046	0.046	0.047	0.047
10	10	100	0.6	0	0.2	0.043	0.043	0.044	0.044
10	10	100	0.6	0	0.4	0.043	0.043	0.044	0.044
10	10	100	0.6	0.2	0.2	0.044	0.044	0.045	0.045
10	10	100	0.59	0.4	0.01	0.045	0.045	0.046	0.046
10	10	100	0.8	0	0.2	0.043	0.043	0.044	0.044
10	10	100	0.79	0.2	0.01	0.044	0.044	0.045	0.045
10	10	100	0.99	0	0.01	0.043	0.043	0.049	0.049

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Table 7.6.2: (continued) Estimated Type-I Error Rates for Satterthwaite's Procedure and GTV Procedure for Testing $H_0 : \mu_1 = \cdots = \mu_a$ at Level $\alpha = 0.05$.

						Generalized Test		Satterthwaite	
						Empirical Type-I Error		Empirical Type-I Error	
a	b	c	σ^2_{AB}	σ^2_{AC}	σ^2_e	using (7.13)	using (7.14)	using (7.11)	using (7.12)
10	100	100	0	0	1	0.056	0.056	0.056	0.056
10	100	100	0	0.2	0.2	0.059	0.059	0.059	0.059
10	100	100	0	0.2	0.4	0.059	0.059	0.059	0.059
10	100	100	0	0.2	0.6	0.059	0.059	0.059	0.059
10	100	100	0	0.2	0.8	0.059	0.059	0.059	0.059
10	100	100	0	0.4	0.2	0.060	0.060	0.059	0.059
10	100	100	0	0.4	0.4	0.059	0.059	0.059	0.059
10	100	100	0	0.4	0.6	0.059	0.059	0.059	0.059
10	100	100	0	0.6	0.2	0.060	0.060	0.059	0.059
10	100	100	0	0.6	0.4	0.060	0.060	0.059	0.059
10	100	100	0	0.8	0.2	0.060	0.060	0.059	0.059
10	100	100	0	0.99	0.01	0.060	0.060	0.060	0.060
10	100	100	0.2	0	0.2	0.056	0.056	0.056	0.056
10	100	100	0.2	0	0.4	0.056	0.056	0.056	0.056
10	100	100	0.2	0	0.6	0.056	0.056	0.056	0.056
10	100	100	0.2	0	0.8	0.056	0.056	0.056	0.056
10	100	100	0.2	0.2	0.2	0.056	0.056	0.056	0.056
10	100	100	0.2	0.2	0.4	0.056	0.056	0.056	0.056
10	100	100	0.2	0.2	0.6	0.056	0.056	0.056	0.056
10	100	100	0.2	0.4	0.2	0.054	0.054	0.055	0.055
10	100	100	0.2	0.4	0.4	0.054	0.054	0.054	0.054
10	100	100	0.2	0.6	0.2	0.056	0.056	0.056	0.056
10	100	100	0.2	0.79	0.01	0.056	0.056	0.056	0.056
10	100	100	0.4	0	0.2	0.056	0.056	0.057	0.057
10	100	100	0.4	0	0.4	0.056	0.056	0.056	0.056
10	100	100	0.4	0	0.6	0.056	0.056	0.056	0.056
10	100	100	0.4	0.2	0.2	0.056	0.056	0.056	0.056
10	100	100	0.4	0.2	0.4	0.056	0.056	0.056	0.056
10	100	100	0.4	0.4	0.2	0.056	0.056	0.056	0.056
10	100	100	0.4	0.59	0.01	0.055	0.055	0.056	0.055
10	100	100	0.6	0	0.2	0.056	0.056	0.057	0.057
10	100	100	0.6	0	0.4	0.056	0.056	0.056	0.056
10	100	100	0.6	0.2	0.2	0.056	0.056	0.056	0.056
10	100	100	0.59	0.4	0.01	0.056	0.056	0.056	0.056
10	100	100	0.8	0	0.2	0.056	0.056	0.057	0.057
10	100	100	0.79	0.2	0.	0.056	0.056	0.056	0.056
10	100	100	0.99	0	0.01	0.057	0.057	0.057	0.057

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7.6.3 Comparison of the Simulation Results for Testing $\mu_1 = \cdots = \mu_a$ Using Satterthwaite's Approximation Procedure and the GPV Procedure

When number of levels of the factors in the model is small, $\hat{\alpha}$ from using Satterthwaite approximation procedure is better than GPV for most of cases. By increasing the number of levels of random effects, $\hat{\alpha}$ from using Satterthwaite approximation procedure is sometimes better than GPV and sometimes $\hat{\alpha}$ from using GTV with absolute value in the denominator is better than Satterthwaite approximation procedure. Increasing the number of levels for factors such that the number of levels of random effects is larger than the number of levels of the fixed effect, GPV gives better results for $\hat{\alpha}$ most of the cases compared to Satterthwaite approximation procedure. When the number of levels of the factors is large or very large, $\hat{\alpha}$ from both procedures are very close to each other.

7.7 Example

The data for this example were obtained from Hocking (1996, Exercise 15.8). Littel (1987) examined the data shown in Table 7.7.1 from a semiconductor plant interested in comparing four "modes" (factor A) of a process condition. Other factors in the experiment were "wafer" with 3 levels (factor B) and "position" with 4 levels (factor C). For this example, "mode" was regarded to be a fixed factor and "wafer" and "position" were regarded to be random factors. The response variable is denoted by Y.

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Table 7.7.1 A semiconductor plant data from Hocking (1996)

A	B	C	Y	A	B	C	Y	A	B	C	Y
1	1	1	5.22	2	2	1	5.77	3	3	1	6.22
1	1	2	5.61	2	2	2	6.23	3	3	2	6.29
1	1	3	6.11	2	2	3	5.57	3	3	3	5.63
1	1	4	6.33	2	2	4	5.96	3	3	4	6.36
1	2	1	6.13	2	3	1	6.43	4	1	1	6.75
1	2	2	6.14	2	3	2	5.81	4	1	2	6.97
1	2	3	5.60	2	3	3	5.83	4	1	3	6.02
1	2	4	5.91	2	3	4	6.12	4	1	4	6.88
1	3	1	5.49	3	1	1	5.66	4	2	1	6.22
1	3	2	4.60	3	1	2	6.25	4	2	2	6.54
1	3	3	4.95	3	1	3	5.46	4	2	3	6.12
1	3	4	5.42	3	1	4	5.08	4	2	4	6.61
2	1	1	5.78	3	2	1	6.53	4	3	1	6.05
2	1	2	6.52	3	2	2	6.50	4	3	2	6.15
2	1	3	5.90	3	2	3	6.23	4	3	3	5.55
2	1	4	5.67	3	2	4	6.84	4	3	4	6.13

The sums of square and mean squares were computed from the above data were used for testing simultaneously equality of cell-means corresponding to the levels of the fixed factor. These summary statistics are as follows.

$$ss(A) = 3.112, \quad ss(AB) = 3.195, \quad ss(AC) = 0.809, \quad ss(e) = 2.667,$$

$$ms(A) = 1.037, \quad ms(AB) = 0.532, \quad ms(AC) = 0.090, \quad ms(e) = 0.111.$$

The GPV computed using Equation (7.13) is 0.222. The GPV computed using Equation (7.14) is also 0.222 in this case. The decision is not to reject $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ at level $\alpha = 0.05$. Both versions of the Satterthwaite approximation also yield a P-value equal to 0.222. In particular, the F-ratio is equal to 2.029 and F-tabulated is equal to 5.125 for both versions of the Satterthwaite approach.

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Chapter 8

SUMMARY

Generalized inference is based on the concepts of generalized pivotal quantity (GPQ), generalized confidence interval (GCI), generalized test variable (GTV) and generalized P-value (GPV). These concepts were first introduced by Tsui and Weerahandi (1989) and Weerahandi (1993). Since then, several authors have applied these concepts to solve various inference problems. Most of the previous studies were concerned with applications of generalized inference for a single parameter.

In this dissertation, we explored a systematic way for constructing GTV, GPV and simultaneous GCIs for more than one parameter. Simultaneous GCIs were developed in unbalanced heterogeneous one-way ANOVA and in balanced mixed linear models. Comparing two non-nested linear models was also considered. The GTV and GPV were then calculated in unbalanced heterogeneous one-way ANOVA and in balanced mixed linear models.

In unbalanced heterogeneous one-way ANOVA we dealt with simultaneous GCIs for all pairwise mean differences (extended Tukey) and simultaneous GCIs for pairwise mean comparisons to a control (extended Dunnett). Simulation studies were conducted to assess the performance of extended Tukey and extended Dunnett and

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to compare the extended Tukey with Dunnett (1980) at different number of groups with different sample sizes at α level of 0.05. When sample sizes are large, extended Tukey and extended Dunnett methods work satisfactory.

Systematic procedures were introduced to construct simultaneous GCIs for all cell-means in balanced two-factor crossed mixed linear model with interaction and simultaneous GCIs for all pairwise cell-mean differences in balanced three-factor nested factorial mixed linear model. We carried out simulation studies to examine the performance of simultaneous GCIs. The estimated type-I errors resulted from simultaneous GCIs in balanced three-factor nested factorial model were compared to estimated type-I errors from SAS analysis at α level of 0.05. The proposed procedures work well when number of levels of the factors in the model increased such that number of levels of the random factor is larger than number of levels of the fixed factor(s).

The GCIs were used to compare two non-nested linear models where the response and the predictors have multivariate normal distribution. A simulation study was carried out to assess the performance of the GCIs using the estimated type-I errors and the results were compared with Ahlbrandt (1988) at different sample sizes with α level of 0.01, 0.05 and 0.10. The GCIs performed satisfactory for most of the cases.

Two problems were solved using GTV and GPV. First problem was testing equality of all cell-means in unbalanced heterogeneous one-way ANOVA. Simulation study was conducted to assess the performance of GTV and GPV for different

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number of groups with different sample sizes at α level of 0.05. The proposed procedure performed satisfactory when sample sizes are large. The second problem was testing the equality of all cell-means in balanced three-factor crossed mixed linear model with interactions. A simulation study was carried out to examine the performance of the proposed procedure and to compare it with Satterthwaite's approximate procedure at α level of 0.05. The proposed procedure works satisfactory when the number of levels of random factors is larger than number of levels of fixed factors or when the number of levels is large for all factors. We presented an example for each problem to illustrate the computations.

The methods discussed in this dissertation can be extended to a number of other problems where exact frequentist solutions are as yet unavailable. This can be a subject for future works.

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