DISSERTATION

COMBINATORIAL STRUCTURES OF HYPERELLIPTIC HODGE INTEGRALS

Submitted by

Adam Afandi

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Spring 2021

Doctoral Committee:

Advisor: Renzo Cavalieri
Co-Advisor: Mark Shoemaker

Henry Adams
Ashok Prasad

ABSTRACT

COMBINATORIAL STRUCTURES OF HYPERELLIPTIC HODGE INTEGRALS

This dissertation explores the combinatorial structures that underlie hyperelliptic Hodge integrals. In order to compute hyperelliptic Hodge integrals, we use Atiyah-Bott (torus) localization on a stack of stable maps to $[\mathbb{P}^1/\mathbb{Z}_2] = \mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2$. The dissertation culminates in two results: a closed-form expression for hyperelliptic Hodge integrals with one λ-class insertion, and a structure theorem (polynomiality) for Hodge integrals with an arbitrary number of λ-class insertions.

ACKNOWLEDGEMENTS

First and foremost, I want to express my gratitude to my thesis advisors, Renzo Cavalieri and Mark Shoemaker. Their guidance, their encouragement, and their friendship, is the reason why I was able to write a thesis that I am proud of. Thanks guys, for everything.

I want to thank all of the wonderful friends that I have made during my time in grad school: Bragdons, Ben(zo) and JG, Pines, Derek, Catalina, Wes, Levi, Andy, Steven and Emily, the darts crew: thank you all for making my grad school experience one that I will look back on fondly.

To all of my teachers from Lowell, Davis, SFSU, and CSU: thank you for imparting your knowledge, and being a part of my education.

To KKN, my second family: thanks for always picking up where we left off, no matter how much time passes by.

To Mathematics: you let a poor Cham kid realize his dream, and for that, I'll always be thankful that you let me into your world.

To my family: despite having so little, you always made me feel like we had everything. Your love and support is the reason I've been able to get this far. Look Ma, I did it!

Last, but never least, I want to thank Bayka, my love. You've put up with my shenanigans, and you pour into my life a love that astounds and humbles me. To that, I say, "Many women have done excellently, but you surpass them all." (Prov. 31:29).

DEDICATION

This one's for The City, and The Bay

Math let me see the world

But nothing ever compares to Home

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

One of the central goals of modern algebraic geometry is to gain a deeper understanding of $\overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford moduli space of stable genus g curves with n marked points. Beginning with the pioneering work of Deligne and Mumford [1], algebraic geometers have sought a better understanding of the intersection theory of $\overline{\mathcal{M}}_{g,n}$ i.e. a better understanding of the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ (see [2]).

This dissertation makes a contribution towards the understanding of $A^*(\overline{\mathcal{M}}_{g,n})$ by computing the degree of a certain class of degree-0 Chow cycles in $A_0(\overline{\mathcal{M}}_{g,2g+2})$, namely, the class of *hyperelliptic Hodge integrals*.

Let us begin by explaining the geometric context of how these cycles arise.

Denote by $\overline{\mathcal{H}}_{g,2g+2} \subset \overline{\mathcal{M}}_{g,2g+2}$ the $(2g-1)$-dimensional moduli space of hyperelliptic curves of genus g with $2g+2$ marked Weierstrass points. We refer to this moduli space as the *hyperelliptic locus*. A point $p = [\varphi : (C_g, w_1, \ldots, w_{2g+2}) \to (C_0, b_1, \ldots, b_{2g+2})] \in \overline{\mathcal{H}}_{g,2g+2}$ is comprised of the following data:

- A curve $(C_g, w_1, \ldots, w_{2g+2}) \in \overline{\mathcal{M}}_{g,2g+2}$
- A curve $(C_0, b_1, \ldots, b_{2g+2}) \in \overline{\mathcal{M}}_{0,2g+2}$
- An *admissible cover* $\varphi : C_g \to C_0$ of degree 2 with branch locus $\{b_i\}$ and ramification locus $\{w_i\}$

There are two natural maps defined on the hyperelliptic locus, the source map $s : \overline{\mathcal{H}}_{g,2g+2} \to \overline{\mathcal{M}}_{g,2g+2}$, and the branch map $\mathrm{br} : \overline{\mathcal{H}}_{g,2g+2} \to \overline{\mathcal{M}}_{0,2g+2}$. These maps are defined as follows:

$$s : [\varphi : (C_g, w_1, \ldots, w_{2g+2}) \to (C_0, b_1, \ldots, b_{2g+2})] \mapsto (C_g, w_1, \ldots, w_{2g+2})$$
$$\text{br} : [\varphi : (C_g, w_1, \ldots, w_{2g+2}) \to (C_0, b_1, \ldots, b_{2g+2})] \mapsto (C_0, b_1, \ldots, b_{2g+2})$$

In other words, the source map and the branch map remember the source and target of the hyperelliptic cover φ, respectively.

There is a variant of the hyperelliptic locus that arises in this dissertation. Denote by $\overline{\mathcal{H}}_{g,2g+2,2}$ the $2g$-dimensional moduli space of marked hyperelliptic curves as before, but with an additional pair of markings. The two extra points that are marked are a *conjugate pair* i.e. a pair of points that are interchanged under the hyperelliptic involution.

In summary, we have the following two diagrams:

$$\begin{array}{ccc} \overline{\mathcal{H}}_{g,2g+2} & \xrightarrow{s} & \overline{\mathcal{M}}_{g,2g+2} \\ \downarrow{\scriptstyle \text{br}} & & \\ \overline{\mathcal{M}}_{0,2g+2} & & \end{array}$$

$$\begin{array}{ccc} \overline{\mathcal{H}}_{g,2g+2,2} & \xrightarrow{s} & \overline{\mathcal{M}}_{g,2g+4} \\ \downarrow{\scriptstyle \text{br}} & & \\ \overline{\mathcal{M}}_{0,2g+3} & & \end{array}$$

There are two types of vector bundles on $\overline{\mathcal{M}}_{g,n}$ that will play central roles in this dissertation. Denote by $\mathbb{E}_g$ the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$. For a point $(C_g, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$, the fiber of the Hodge bundle over this point is defined to be

$$(\mathbb{E}_g)\,|_{(C_g,p_1,\ldots,p_n)} := \Omega^1(C_g)$$

where $\Omega^1(g)$ is the rank-g vector bundle of holomorphic 1-forms on C_g. The remaining vector bundles, denoted $\mathbb{L}_j$, where $1 \leq j \leq n$, are called the j^{th} universal cotangent line bundles over $\overline{\mathcal{M}}_{0,n}$. Given a point $(C_0, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}$, the fiber of the line bundle $\mathbb{L}_j$ over this point is defined to be

$$(\mathbb{L}_j)\,|_{(C_0,p_1,\ldots,p_n)} := T^*_{p_j}(C_0)$$

where $T^*_{p_j}(C_0)$ is the cotangent space to C_0 at the j^{th} marked points p_j. There are two types of elements in the Chow ring of the hyperelliptic locus that comprise the integrands of the intersection numbers we want to compute. They are defined by taking Chern classes of $\mathbb{E}_g$ and $\mathbb{L}_j$. Specifically,

$$(0 \le i \le g) \qquad \lambda_i := c_i(s^*\mathbb{E}_g) \in A^i(\overline{\mathcal{H}}_{g,2g+2}), A^i(\overline{\mathcal{H}}_{g,2g+2,2})$$
$$(1 \le j \le 2g+2) \qquad \psi_j := c_1(\text{br}^*\mathbb{L}_j) \in A^1(\overline{\mathcal{H}}_{g,2g+2})$$
$$(1 \le j \le 2g+3) \qquad \psi_j := c_1(\text{br}^*\mathbb{L}_j) \in A^1(\overline{\mathcal{H}}_{g,2g+2,2})$$

We are now ready to define the class of intersection numbers under investigation:

Definition 1. *Let $\vec{i} := (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\ge 0}$, and define $\lambda_{\vec{i}} := \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}$, and $|\vec{i}| := i_1 + \ldots + i_n$. Hyperelliptic Hodge integrals are intersection numbers on $\overline{\mathcal{H}}_{g,2g+2}$ and $\overline{\mathcal{H}}_{g,2g+2,2}$ of the form*

$$D_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{2g-1-|\vec{i}|}\,\lambda_{\vec{i}}$$
$$d_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_{2g+3})^{2g-|\vec{i}|}\,\lambda_{\vec{i}}$$

It turns out that the intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$ display a remarkable amount of combinatorial structure and symmetry. There are still open questions and conjectures concerning these intersection numbers (see Chapter 7 of this dissertation), the most glaring one being, “can one find closed form expressions for $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$?" The author is willing to assert that the answer is “most likely, yes", but there is still much work that needs to be done.

1.1 Motivations From Orbifold Gromov-Witten Theory

Let $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ be the moduli space of n-pointed genus g stable maps into the classifying space of a finite group G. If one gains a deeper understanding of the intersection theory of $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$, this would have applications to orbifold Gromov-Witten theory. This is due to the fact that when one computes Gromov-Witten invariants of an orbifold $\mathcal{X}$ using torus localization, one needs to compute *Hurwitz-Hodge integrals* i.e. intersection numbers on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ that involve Hodge classes and ψ-classes.

Although Hurwitz-Hodge integrals are the building blocks of calculations in orbifold Gromov-Witten theory, there are very few results that indicate efficient ways to evaluate them. They have a reputation of being a class of intersection numbers that are notoriously difficult to compute. Two common approaches used to tackle them are Atiyah-Bott localization, and the orbifold Grothendieck-Riemann-Roch (GRR) theorem. In both of these approaches, the main obstacle is usually an inability to tame the combinatorial complexity of the calculations.

In addition to localization and GRR, one can use the *mirror theorem* for the toric orbifolds $[\mathbb{C}^n/G]$ to compute Hurwitz-Hodge integrals. Established in [3], the mirror theorem finds a connection between two seemingly unrelated functions, the twisted I-function, and the twisted J-function. The twisted J-function is a cohomology-valued generating function for Hurwitz-Hodge integrals, whereas the twisted I-function is a hypergeometric series. The mirror theorem says that these two functions coincide after a change of variables, along with a complicated procedure involving Birkhoff factorization. In order to extract closed form expressions for Hurwitz-Hodge integrals from the twisted J-function, one must find an expression for the inverse of the *mirror map*. This allows one to invert the formal generating function parameters between the twisted I-function and the twisted J-function. In theory, this method of computing Hurwitz-Hodge integrals is incredibly powerful due to its breadth and scope of application. However, in practice, using the mirror theorem to compute Hurwitz-Hodge has proven to be difficult.

In the language of [3], this dissertation computes the genus 0 orbifold Gromov-Witten theory of the stack quotient $[\mathbb{C}^n/\mathbb{Z}_2]$ for all n, without appealing to the mirror theorem. We carefully analyze the recursions obtained when using torus localization. Our approach is computationally involved, however we are successful in our combinatorial analysis. We discover a rich combinatorial structure (see Theorem 9 and Theorem 10) for Hurwitz-Hodge integrals that does not obviously follow from the mirror theorem. One exciting part of this development is that we now know what kind of structure to look for as we generalize beyond the hyperelliptic case. The most optimistic outlook is that the combinatorial formulas we see in the case when $g = 0$ and $G = \mathbb{Z}_2$ is simply a shadow of a much more general phenomenon as g increases, and as G varies.

1.2 A History of Hyperelliptic Hodge Integrals

The earliest result concerning hyperelliptic Hodge integrals goes back to Faber and Pandharipande ([4], Corollary 2) in which they discovered that

$$D_{(g-1,g),2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} \lambda_{g-1}\lambda_g = \frac{2^{2g}-1}{2g}|B_{2g}| \tag{1.1}$$

where B_{2g} is the $2g^{th}$ Bernoulli number. In [5], Cavalieri generalized the integrand in Equation (1.1), and considered hyperelliptic Hodge integrals of the form

$$D_{(g-i,g),2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{i-1} \lambda_{g-i}\lambda_g \tag{1.2}$$

Cavalieri was able to find recursions that completely determine the hyperelliptic Hodge integrals in Equation (1.2). He packaged them into a single generating function, and discovered that

$$D_i(u) := \sum_{g\geq 0} D_{(g-i,g),2g+2}\frac{u^{2g}}{2g!} = \left(\frac{2^{i-1}}{i!}\right)\left[\ln\left(\frac{2\sin\left(\frac{u}{2}\right)}{\sin(u)}\right)\right]^i$$

In [6], while investigating the Gromov-Witten theory of $\mathrm{Sym}^2(\mathbb{P}^2)$, Wise discovered the following relation among hyperelliptic Hodge integrals with two λ-class insertions:

$$\sum_{i+j+k=2g-1}\left(\frac{1}{2}\right)^i(-1)^{j+k}\int_{\overline{\mathcal{H}}_{g,2g+2}}\psi_\ell^i\lambda_j\lambda_k = \left(\frac{-1}{4}\right)^g$$

In [7], Johnson, Pandharipande, and Tseng found an algorithm to compute the linear hyperelliptic Hodge integrals $D_{i,2g+2}$ in terms of double Hurwitz numbers. Finally, as mentioned in Section 1.1, in [3], Coates, Corti, Iritani, and Tseng used toric mirror symmetry to realize hyperelliptic Hodge integrals as coefficients of the twisted J-function of $\mathcal{B}\mathbb{Z}_2$.

1.3 Methods and Results

The main computational technique used in this dissertation is the Atiyah-Bott localization theorem [8]. The general strategy is as follows. First, we identify hyperelliptic loci with spaces of stable maps into stacky points (see Chapter 2):

$$\overline{\mathcal{H}}_{g,2g+2} \cong \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$$
$$\overline{\mathcal{H}}_{g,2g+2,2} \cong \overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$$

We then use Atiyah-Bott localization on an auxiliary moduli space $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ (see Chapter 3) to compute recursions for $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$. Conducting a meticulous analysis of these recursions culminates in three new findings concerning hyperelliptic Hodge integrals, which are summarized in Theorem 1, Theorem 2 and Theorem 3 below.

Theorem 1. *There exists a set of recursions that completely determine the intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$. The only initial conditions required for these recursions are*

$$D_{\vec{0},2g+2} = d_{\vec{0},2g+2} = \frac{1}{2} \tag{1.3}$$

The integrals in Equation (1.3) were first discovered in [9].

In Chapter 4, we solve the recursions in the special case that the vector $\vec{i}$ is a 1-tuple.

Theorem 2. *The hyperelliptic Hodge integrals $D_{i,2g+2}$ and $d_{i,2g+2}$ have the following closed formula:*

$$D_{i,2g+2} = \left(\frac{1}{2}\right)^{i+1} e_i(1,3,5,\ldots,2g-1)$$
$$d_{i,2g+2} = \left(\frac{1}{2}\right)^{i+1} e_i(2,4,6,\ldots,2g)$$

Finally, we make a very general statement concerning hyperelliptic Hodge integrals:

Theorem 3. *The quantities $2^{|\vec{i}|+1}D_{\vec{i},2g+2}$ and $2^{|\vec{i}|+1}d_{\vec{i},2g+2}$ are integer-valued polynomials in g. The degrees of these polynomials is bounded by $|\vec{i}|^2+1$.*

1.4 Outline of Dissertation

We begin our exposition in Chapter 2 by discussing the geometry of the hyperelliptic locus. The purpose of Chapter 2 is to establish the dictionary between admissible covers and stable maps to the stacky $\mathcal{B}\mathbb{Z}_2$.

Once this dictionary is established, we can begin to make computations of hyperelliptic Hodge integrals by using Atiyah-Bott localization on the auxiliary moduli space $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. These computations are carefully laid out in Chapter 3. Chapter 3 provides the computational/technical heart of this dissertation. It is in this Chapter that we prove there are enough recursions to completely determine the intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$, thereby establishing Theorem 1.

Once we have recursions for $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$, we can begin a careful analysis of these recursions. In Chapter 4, we prove Theorem 2 by showing that the recursions obtained in Theorem 1 are satisfied by the purported closed-form expressions stated in Theorem 2.

We then translate the recursions in Theorem 1 into systems of ordinary differential equations for certain generating functions of $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$. We inductively show that the solutions to these systems take a form that implies the conclusion of Theorem 3.

Packaging the numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$ into generating functions that allow $|\vec{i}|$ to vary, in Chapter 6, we show that these generating functions satisfy a system of non-linear partial differential equations.

Finally, we end our exposition by discussing various conjectures and open problems.

Chapter 2

The Hyperelliptic Locus

In this Chapter, we provide the broad geometric context of this dissertation. The main purpose of this Chapter is to establish a dictionary. This dictionary will interpolate between two perspectives of the moduli space of hyperelliptic curves: admissible covers, and orbifold Gromov-Witten theory.

2.1 The Perspective From Admissible Covers

We say that a smooth algebraic curve C of genus g is *hyperelliptic* if it admits a degree 2 map $\varphi : C \to \mathbb{P}^1$. We call φ the hyperelliptic covering. By the Riemann-Hurwitz formula, such a map must be branched at $2g+2$ points. The points of ramification $\{q_1, \ldots, q_{2g+2}\} \subset C$ are called the Weierstrass points of C.

Let $\mathcal{H}_{g,2g+2} \subseteq \overline{\mathcal{M}}_{g,2g+2}$ be the moduli space of smooth hyperelliptic curves whose Weierstrass points are marked. In order to compactify this space, we use the theory of *admissible covers* (see [10]). The space of admissible covers provides a compactification of all Hurwitz schemes. In this dissertation, we are solely concerned with hyperelliptic curves, so we restrict to the case of degree 2 admissible covers.

Definition 2. *Let $[C_g, q_1, \ldots, q_{2g+2}] \in \overline{\mathcal{M}}_{g,2g+2}$ and $[C_0, p_1, \ldots, p_{2g+2}] \in \overline{\mathcal{M}}_{0,2g+2}$ be two moduli points, and let $(C_g, q_1, \ldots, q_{2g+2})$ and $(C_0, p_1, \ldots, p_{2g+2})$ be representative marked curves in the former and latter isomorphism classes of moduli points, respectively. We say a map $\varphi : (C_g, q_1, \ldots, q_{2g+2}) \to (C_0, p_1, \ldots, p_{2g+2})$ is an admissible cover of degree 2 if φ is a map of degree 2, and*

1. *The map φ is branched at p_i for all i*
2. *$\varphi(q_i) = p_i$ (i.e $\{q_i\}$ is the smooth ramification locus)*
3. *φ maps nodes to nodes*

4. *(Balancing Condition) Let $C_g' \subseteq C_g$ be an irreducible component of C_g. If $n \in \varphi(C_g') \subseteq C_0$ is a node of the curve C_0, and $I := \{p_i\}_{p_i \in \varphi(C_g')}$ is the set of marked points on $\varphi(C_g')$, then n is a branch point if $|I|$ is odd, and is not a branch point otherwise.*

See Example 1 below for a better understanding of the Balancing Condition.

Allowing for admissible covers compactifies the space $\mathcal{H}_{g,2g+2}$. We denote the compactified space by $\overline{\mathcal{H}}_{g,2g+2} \subseteq \overline{\mathcal{M}}_{g,2g+2}$ and call it the *hyperelliptic locus*.

As mentioned in the Introduction, there are two natural maps associated to $\overline{\mathcal{H}}_{g,2g+2}$. Given a point in $\overline{\mathcal{H}}_{g,2g+2}$, we can either keep track of the stable genus g curve of the hyperelliptic covering, along with the ramification locus, or, we can keep track of the stable rational curve, along with the branch locus. We denote by s the former map, and br the latter:

$$\begin{array}{ccc} \overline{\mathcal{H}}_{g,2g+2} & \xrightarrow{s} & \overline{\mathcal{M}}_{g,2g+2} \\ \big\downarrow \text{br} & & \\ \overline{\mathcal{M}}_{0,2g+2} & & \end{array}$$

Given a rational curve $[C, p_1, \ldots, p_{2g+2}] \in \overline{\mathcal{M}}_{0,2g+2}$, one can construct a genus g hyperelliptic covering of C with branch locus $\{p_i\}$. This covering is unique up to automorphisms of the genus g curve. Therefore, $\overline{\mathcal{M}}_{0,2g+2}$ can be interpreted as the *coarse moduli space* of $\overline{\mathcal{H}}_{g,2g+2}$. In particular, this argument tells us that the dimension of the moduli space $\overline{\mathcal{H}}_{g,2g+2}$ is the same as $\overline{\mathcal{M}}_{0,2g+2}$, which is known to be $2g - 1$.

There is a slight variant of the moduli space $\overline{\mathcal{H}}_{g,2g+2}$ that we will consider in this dissertation. Denote by $\overline{\mathcal{H}}_{g,2g+2,2}$ the moduli space of hyperelliptic curves as before, but with an additional 2 marked points. These 2 marked points consists of a conjugate pair, that is, a pair of points that are exchanged under the hyperelliptic involution. The dimension of this space is $2g$ (the extra degree of freedom coming from the marked conjugate pair).

As g becomes large, the various boundary components of $\overline{\mathcal{H}}_{g,2g+2}$ can beome arduous to enumerate. However, for the purposes of this dissertation, we only need to describe the *boundary divisors* of $\overline{\mathcal{H}}_{g,2g+2}$

Example 1. *In this example, we describe all of the boundary divisors of* $\overline{\mathcal{H}}_{2,6}$*. The boundary divisors of* $\overline{\mathcal{H}}_{2,6}$ *are pulled back from the boundary divisors of* $\overline{\mathcal{M}}_{0,6}$*. In general, the boundary divisors of* $\overline{\mathcal{M}}_{0,n}$ *are enumerated by subsets* $S \subset \{1, 2, \ldots n\}$*. Such a divisor parametrizes rational curves* $C = C_1 \cup C_2$*, where* C_i *is isomorphic to* $\mathbb{P}^1$*, and* C_1 *and* C_2 *are attached at a node. The irreducible component* C_1 *contains the marked points indexed by* S*, and* C_2 *contains the marked points indexed by the complement of* S*. By this definition, there is a combinatorial symmetry in this notation, in that the divisor corresponding to* S *is the same as the divisor corresponding to* S^c*. In the case of* $\overline{\mathcal{H}}_{2,6}$*, there are two combinatorial types of divisors, corresponding to either* $S = \{i, j\}$ *or* $S = \{i, j, k\}$ *(notice that the case* $S = \{i\}$ *cannot occur due to stability constraints on* $\overline{\mathcal{M}}_{0,n}$*). There are* $\binom{6}{2}$ *divisors in the former combinatorial type, and* $\binom{6}{3}$ *divisors in the latter, respectively. This count simply comes from choosing which of the two/three marked points reside on* C_1*. Lastly, we note that each of these boundary divisors is a product of hyperelliptic loci. In the case of the combinatorial type* $S = \{i, j\}$*, the divisor corresponds to the product* $\overline{\mathcal{H}}_{0,2,2} \times \overline{\mathcal{H}}_{1,4,2}$*. In the case of the combinatorial type* $S = i, j, k$*, the divisor corresponds to the product* $\overline{\mathcal{H}}_{1,4} \times \overline{\mathcal{H}}_{1,4}$*. See Figure 2.1 and Figure 2.2.*

2.2 Hyperelliptic Hodge Integrals

In this section, we define the main objects of interest in this dissertation: *hyperelliptic Hodge integrals*. We first need the definition of two vector bundles on the hyperelliptic locus, the *Hodge bundle* and the *universal cotangent line bundle*.

Definition 3. *The Hodge bundle, denoted* $\mathbb{E}_g \to \overline{\mathcal{H}}_{g,2g+2}$ *is defined to be the rank-*g *vector bundle whose fiber over the point* $[\varphi : (C_g, q_1, \ldots, q_{2g+2}) \to (C_0, p_1, \ldots, p_{2g+2})] \in \overline{\mathcal{H}}_{g,2g+2}$ *is* $\Omega^1(C_g)$*, the* g*-dimensional vector space of holomorphic* 1*-forms on* C_g*. The Hodge bundle* $\mathbb{E}_g \to \overline{\mathcal{H}}_{g,2g+2,2}$ *is defined similarly. We define* $\lambda_i := c_i(\mathbb{E}_g)$*, where* c_i *is the* i^{th} *Chern class.*

Definition 4. *Let* $1 \leq j \leq 2g+2$*. The universal cotangent line bundle, denoted* $\mathbb{L}_j \to \overline{\mathcal{H}}_{g,2g+2}$*, is the line bundle whose fiber over the point* $[\varphi : (C_g, q_1, \ldots, q_{2g+2}) \to (C_0, p_1, \ldots, p_{2g+2})] \in \overline{\mathcal{H}}_{g,2g+2}$

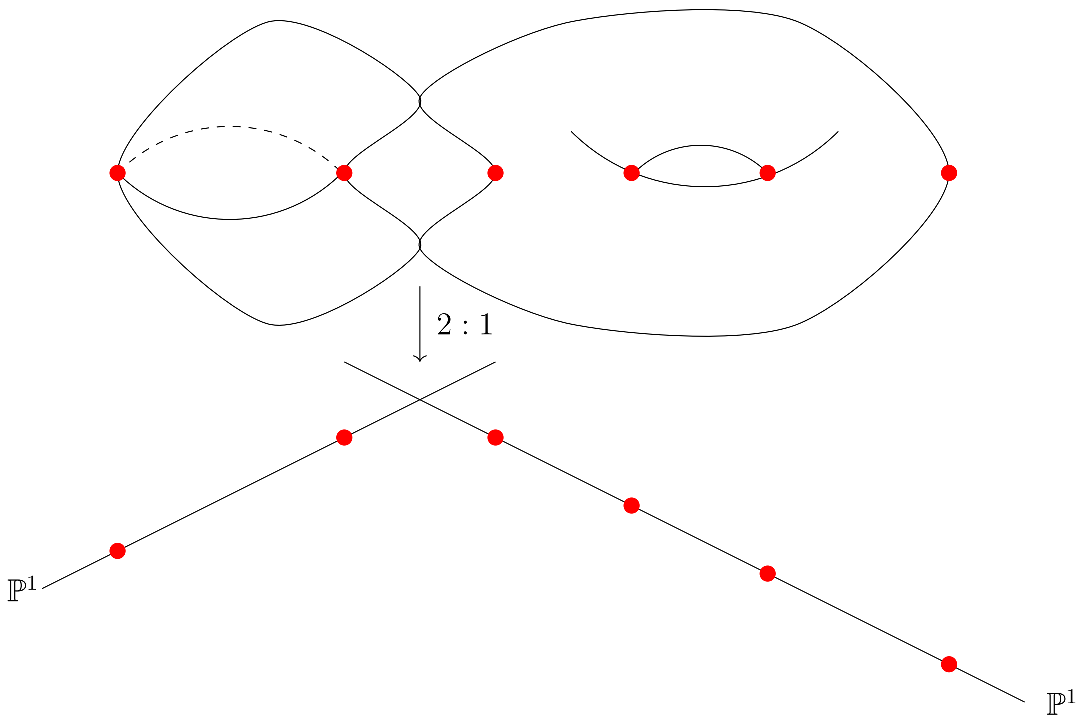


Figure 2.1: This is a depiction of a boundary divisor on $\overline{\mathcal{H}}_{2,6}$ corresponding to the combinatorial type $S = \{i, j\}$. Notice that the node on the target curve is not a ramification point since the number of marked points on each irreducible component is even.

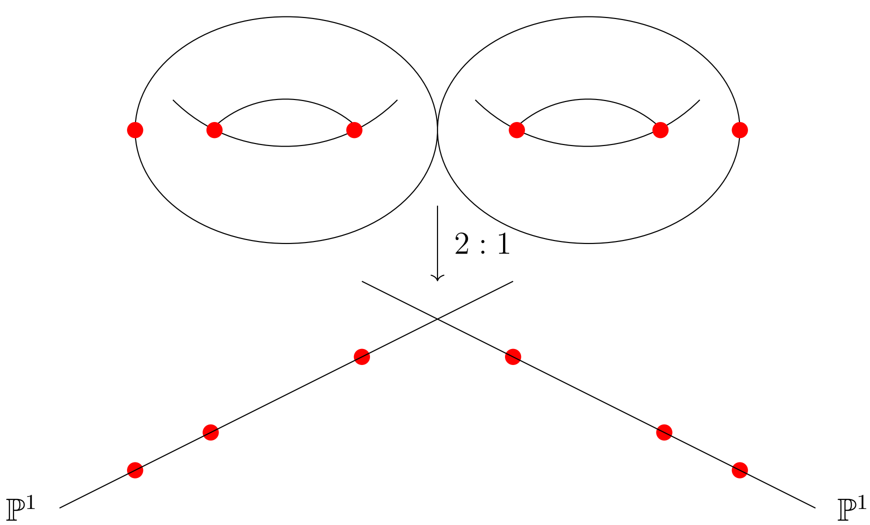


Figure 2.2: This is a depiction of a boundary divisor on $\overline{\mathcal{H}}_{2,6}$ with combinatorial type $S = \{i, j, k\}$. Notice that the node on the source curve is a ramification point since the number of marked points on each irreducible component is odd.

is $T^*_{p_j}C_0$, the cotangent space to C_0 at the j^{th} marked point. For $1 \leq j \leq 2g+3$, $\mathbb{L}_j$ is defined similarly over $\overline{\mathcal{H}}_{g,2g+2,2}$. We define $\psi_j := c_1(\mathbb{L}_j)$, where c_1 is the first Chern class.

We are now ready to define the main objects under investigation:

Definition 5. *Let $\vec{i} := (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ be a tuple of non-negative integers. Define $\lambda_{\vec{i}} := \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}$, and $|\vec{i}| := i_1 + \ldots + i_n$. A hyperelliptic Hodge integral is any zero-dimensional Chow cycle of the following two forms:*

$$D_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{\vec{i}}$$

$$d_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{\vec{i}}$$

2.3 The Perspective From Orbifold Gromov-Witten Theory

In order to gain access to hyperelliptic Hodge integrals, the first step is to reinterpret the hyperelliptic locus as the moduli space of *stable maps to the orbifold $\mathcal{B}\mathbb{Z}_2$*. In this section, we explain how this reinterpretation works.

The study of stable maps to orbifolds properly sits in the field of *orbifold Gromov-Witten theory*. Providing a complete exposition on orbifold Gromov-Witten theory does not contribute to the goals of this dissertation. Instead, we introduce just enough of the technical machinery that is required to prove the main results. For thorough expositions, see [11] and [12], and for an exposition aligned with the requirements of this dissertation, see Section 1 of [13]

Classical Gromov-Witten theory (see [14]) is a 'curve-counting theory'. The goal of Gromov-Witten theory is to understand the intersection theory of

$$\overline{\mathcal{M}}_{g,n}(X,d)$$

the moduli space of genus g, n-marked, degree-d stable maps into a smooth projective varity X. A point in this space is the data of a map $f : (C, p_1, \ldots, p_n) \to X$, where $(C, p_1, \ldots, p_n)$ is an n-marked curve of genus g, X is a smooth projective variety, and $f_*([C]) = d \in H_*(X, \mathbb{C})$. Stability amounts to requiring that every contracted component of the map f must satisfy the following condition: if the contracted component is rational, then this component must have at least three special points, and if the component has genus 1, it must have at least one special point (by 'special point', we mean either a marked point or a node).

Orbifold Gromov-Witten theory generalizes the above situation by allowing the target space to be an *orbifold*, and the source curves are allowed to be *orbifold curves* or *orbicurves*. The key difference between orbifolds and varieties is that, locally, orbifolds are modeled by *quotients* of affine charts. In other words, each point of an orbifold comes equipped with a local chart and a group action on that chart. This group is called the *isotropy group* associated to this point. This local picture can be glued to form a global geometric object. For a general treatment of orbifolds, see [15].

Let $\overline{\mathcal{M}}_{0,n}(\mathcal{X}, d)$ be the moduli stack of genus 0 degree d stable maps with n marked points, into the orbifold $\mathcal{X}$. The *inertia stack* of $\mathcal{X}$ (see [[13], Section 1]), denoted $\mathcal{I}\mathcal{X}$, is the fiber product

$$\begin{array}{ccc} \mathcal{I}\mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

where Δ is the diagonal map. The product is taken in the 2-category of stacks. The points in $\mathcal{I}\mathcal{X}$ can be identified with all pairs (x, g), where $x \in \mathcal{X}$ and $g \in \mathrm{Aut}_{\mathcal{X}}(x)$. In the case that $\mathcal{X} = [V/G]$, where V is a smooth projective variety, and G is a finite abelian group, we have

$$\mathcal{I}\mathcal{X} = \coprod_{g \in G} [V^g/G]$$

where V^g is the g-fixed subset of V.

In this dissertation, we will only need to understand the genus 0 orbifold Gromov-Witten theory for two orbifold targets. Let $\mathbb{Z}_2$ be the finite simple group of order 2. The two orbifold targets we are interested in are:

1. The stack quotient $[\mathbb{P}^1/\mathbb{Z}_2] = \mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts trivially on $\mathbb{P}^1$

2. The stacky point $\mathcal{B}\mathbb{Z}_2 := [pt./\mathbb{Z}_2]$

Let's first consider the orbifold $\mathcal{X} = [\mathbb{P}^1/\mathbb{Z}_2] = \mathbb{P} \times \mathcal{B}\mathbb{Z}_2$. By the description of the inertia stack above,

$$\mathcal{I}\mathcal{X} = \left(\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2\right) \amalg \left(\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2\right) := \mathcal{I}\mathcal{X}_0 \amalg \mathcal{I}\mathcal{X}_1$$

where $\mathcal{I}\mathcal{X}_0$ corresponds to the component with trivial isotropy, and $\mathcal{I}\mathcal{X}_1$ corresponds to the component with non-trivial isotropy. Similarly, the inertia stack of $\mathcal{B}\mathbb{Z}_2$ is

$$\mathcal{I}\mathcal{X} = (\mathcal{B}\mathbb{Z}_2) \amalg (\mathcal{B}\mathbb{Z}_2) := \mathcal{I}\mathcal{X}_0 \amalg \mathcal{I}\mathcal{X}_1$$

In orbifold Gromov-Witten theory, the source curves are allowed to be *orbicurves*, but in a restricted way: only the marked points and the nodes are allowed to have non-trivial orbifold structure. Furthermore, the evaluation maps no longer land in the target space, but instead land in the *rigidified* inertia stack,

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,k}(\mathcal{X}, d) \to \overline{\mathcal{I}}\mathcal{X}$$

The definition of the rigidified inertia stack is technical, and we refer the reader to [[11], Section 3] for details. However, as explained in [[11], Section 6], even though there is not a well defined evaluation map from the stack of stable maps to the inertia stack, because there is an isomorphism between the cohomology groups of $\overline{\mathcal{I}}\mathcal{X}$ and $\mathcal{I}\mathcal{X}$, there is a well defined map

$$\mathrm{ev}_i^* : H^\bullet(\mathcal{I}\mathcal{X}) \to H^\bullet(\overline{\mathcal{M}}_{0,k}(\mathcal{X}, d))$$

If $\mathcal{X} = [\mathbb{P}^1/\mathbb{Z}_2]$, since $\mathcal{I}\mathcal{X}$ only consists of two components, the marked points on the source curve are either 'untwisted' or 'twisted', i.e. maps to $\mathcal{I}\mathcal{X}_0$ or maps to $\mathcal{I}\mathcal{X}_1$. In the former case, the marked point has trivial isotropy, and in the latter, it has non-trivial isotropy. With this, we have the following definitions:

Definition 6. *The substack $\overline{\mathcal{M}}_{0,kt,\ell u}([\mathbb{P}^1/\mathbb{Z}_2], d) \subset \overline{\mathcal{M}}_{0,k+l}([\mathbb{P}^1/\mathbb{Z}_2], d)$ is defined to be the space of degree d stable orbifold maps of genus 0 curves into $[\mathbb{P}^1/\mathbb{Z}_2]$, in which the first k marked points are twisted, and the last ℓ marked points are untwisted. Similarly, the substack $\overline{\mathcal{M}}_{0,kt,\ell u}(\mathcal{B}\mathbb{Z}_2) \subset \overline{\mathcal{M}}_{0,k+\ell}(\mathcal{B}\mathbb{Z}_2)$ is the space of degree 0 stable orbifold maps of genus 0 curves into the stacky point $\mathcal{B}\mathbb{Z}_2 = [pt./\mathbb{Z}_2]$, in which the first k marked points are twisted, and the last ℓ marked points are untwisted.*

Remark 1. *In the case that $\ell = 0$, we suppress ℓ from the notation, and simply indicate the number of twisted points.*

Let $[\mathcal{C} \to \mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2] \in \overline{\mathcal{M}}_{0,(2g+2)t}(\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2, 1)$. If we compose the map $\mathcal{C} \to \mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2$ with projection onto the $\mathcal{B}\mathbb{Z}_2$ factor, we get the map $\mathcal{C} \to \mathcal{B}\mathbb{Z}_2$, which, by definition of classifying space, is equivalent to the data of a principal $\mathbb{Z}_2$-bundle over $\mathcal{C}$ branched over the $2g+2$ marked points. By Riemann-Hurwitz, the total space of this bundle is a curve C of genus g. In particular, notice that this implies that the space $\overline{\mathcal{M}}_{0,kt}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is non-empty if and only if k is even.

If $[\mathcal{C} \to \mathcal{B}\mathbb{Z}_2] \in \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$, then again, the map $\mathcal{C} \to \mathcal{B}\mathbb{Z}_2$ is equivalent to the data of a degree 2 branched covering of $\mathcal{C}$, whose branch locus is the $2g+2$ marked points on the source curve. Similarly, if $[\mathcal{C} \to \mathcal{B}\mathbb{Z}_2] \in \overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$, the map $\mathcal{C} \to \mathcal{B}\mathbb{Z}_2$ is equivalent to a degree 2 branched covering of $\mathcal{C}$, in which the branch locus is the $(2g+2)$ twisted points on the source curve, but the *last* marked point, which is untwisted/has trivial isotropy, is *not* a branch point of the covering. The preimage of the untwisted point is a pair of conjugate points that are interchanged under the hyperelliptic involution. Again, in particular, this implies that $\overline{\mathcal{M}}_{0,kt}(\mathcal{B}\mathbb{Z}_2)$ and $\overline{\mathcal{M}}_{0,kt,\ell u}(\mathcal{B}\mathbb{Z}_2)$ are non-empty spaces if and only if k is even.

With the geometric descriptions of stable maps in the previous two paragraphs, we see that

$$\overline{\mathcal{H}}_{g,2g+2} \cong \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$$

$$\overline{\mathcal{H}}_{g,2g+2,2} \cong \overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$$

In particular, the dimension of $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ is $2g-1$, and the dimension of $\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$ is $2g$. For $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$, we have

$$\dim(\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)) = 2g+2$$

See [16] for a derivation.

What remains left to do in constructing our dictionary between admissible covers and stable maps is to understand how the Hodge bundle and the universal cotangent line bundle are defined on $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2), \overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$, and $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$.

Definition 7. *Let $[\mathcal{C} \to \mathcal{B}\mathbb{Z}_2] \in \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$, and let $C \to \mathcal{C}$ be the corresponding genus g hyperelliptic covering. The Hodge bundle $\mathbb{E}_g \to \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ is defined to be the rank-g vector bundle whose fiber over the point $[\mathcal{C} \to \mathcal{B}\mathbb{Z}_2]$ is $\Omega^1(C)$, the g-dimensional vector space of holomorphic 1-forms on the genus-g curve C. The Hodge bundle is defined similarly for the space $\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$. Now let $[\mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. Composing the map $\mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]$ with the canonical map $[\mathbb{P}^1/\mathbb{Z}_2] \to \mathcal{B}\mathbb{Z}_2$ onto the $\mathcal{B}\mathbb{Z}_2$ factor, we get a point in $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$. Therefore, we the map $\mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]$ induces a genus-g hyperelliptic covering $C \to \mathcal{C}$. We define the Hodge bundle on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$ to be the rank-g vector bundle whose fiber over the point $[\mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]]$ is $\Omega^1(C)$, the g-dimensional vector space of holomorphic 1-forms on the genus-g curve C. As in the admissible covers perspective, we define $\lambda_i := c_i(\mathbb{E}_g)$.*

Definition 8. *The j^{th} universal cotangent line bundle $\mathbb{L}_j \to \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ is defined to be the line bundle whose fiber over the point $[(\mathcal{C}, p_1, \ldots, p_{2g+2}) \to \mathcal{B}\mathbb{Z}_2] \in \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ is $T^*_{p_j}\mathcal{C}$, the cotangent space to the source curve $\mathcal{C}$ at the j^{th} marked point. A similar definition is made for $\mathbb{L}_j \to \overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$ and $\mathbb{L}_j \to \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. As in the admissible covers perspective, we define $\psi_j := c_1(\mathbb{L}_j)$.*

With the Hodge bundle and universal cotangent line bundle defined on the stable maps side of the dictionary, we can now translate the definition of hyperelliptic Hodge integral as an intersection number on the moduli of stable maps:

$$D_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{\vec{i}} = \int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{\vec{i}}$$
$$d_{\vec{i},2g+2} := \int_{\overline{\mathcal{H}}_{g,2g+2,2}} (\psi_j)^{2g-|\vec{i}|}\lambda_{\vec{i}} = \int_{\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{\vec{i}}$$

For the remainder of this dissertation, we will mostly work on the stable maps side of this dictionary.

Chapter 3

Atiyah-Bott Localization

In this Chapter, we describe a method to compute intersection numbers on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. The main technique is called *Atiyah-Bott localization*, or sometimes referred to as *torus localization*. The general idea is as follows. Since $\mathbb{P}^1$ has a $\mathbb{C}^*$-action, one can post-compose this action with a map $f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]$, and therefore, the space $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$ has an induced $\mathbb{C}^*$-action coming from the action on $\mathbb{P}^1$. Once a $\mathbb{C}^*$-action is established, one can use the Atiyah-Bott localization theorem to compute intersection numbers on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. However, in order to use localization, two tasks must be accomplished:

1. Enumerate/describe the irreducible components of the $\mathbb{C}^*$-fixed locus

2. Compute the $\mathbb{C}^*$-equivariant Euler class of the normal bundles to the irreducible components

3.1 Torus Equivariant Cohomology of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$

For the purposes of this dissertation, we state the Atiyah-Bott localization theorem in the following generality:

Theorem 4 (Atiyah-Bott)**.** *Let X be a Deligne-Mumford stack with a $\mathbb{C}^*$-action. If $\alpha \in A^*(X)$, then*

$$\int_X \alpha = \sum_i \int_{\Gamma_i} \frac{\alpha|_{\Gamma_i}}{e(N_{\Gamma_i})} \tag{3.1}$$

where the sum runs over all the irreducible components Γ_i of the $\mathbb{C}^*$-fixed loci, and $e(N_{\Gamma_i})$ is the $\mathbb{C}^*$-equivariant Euler class of the normal bundle to Γ_i

See [8] for the original exposition on this theorem, and see [17] for localization in the context of stable maps.

Recall the stack of stable maps $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$, introduced in Chapter 2. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $u \cdot [x_0 : x_1] = [x_0 : ux_1]$. Notice that the $\mathbb{C}^*$-fixed points are $0 = [1 : 0] \in \mathbb{P}^1$ and $\infty =$

$[0:1] \in \mathbb{P}^1$. Since the $\mathbb{C}^*$-action commutes with the trivial $\mathbb{Z}_2$-action, this $\mathbb{C}^*$-action lifts to the quotient $[\mathbb{P}^1/\mathbb{Z}_2]$. Given a point $[f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$, we can post compose f with the $\mathbb{C}^*$-action on $[\mathbb{P}^1/\mathbb{Z}_2]$, and therefore, we obtain a $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. Since $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ has a $\mathbb{C}^*$-action, we can apply the localization theorem to compute integrals on this space. It should be noted that these integrals only serve an *auxiliary* role, which will soon become apparent.

When we use localization to compute integrals on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$, the calculations take place in the $\mathbb{C}^*$-equivariant cohomology ring of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. We recall some basic notions from equivariant cohomology (see [14], Chapter 4).

Let G be an abelian group. The *classifying space* of G, denoted BG is defined to be EG/G, where EG is a contractible space with a free G action. The choice of EG is unique up to homotopy. The G-equivariant cohomology of a point is defined to be

$$H^*_G(\{pt.\}) := H^*(BG) = H^*(EG/G)$$

Now let us consider the case of $G = \mathbb{C}^*$. One choice of $E\mathbb{C}^*$ is $\mathbb{C}^\infty$ with $\mathbb{C}^*$-action $\lambda \cdot (x_0, x_1, \ldots) = (\lambda x_0, \lambda x_1, \ldots)$. With this choice, $E\mathbb{C}^*/\mathbb{C}^* = \mathbb{P}^\infty$. If we denote by t the hyperplane class of $\mathbb{P}^\infty$ (alternatively, $t := c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$), we see that

$$H^*_{\mathbb{C}^*}(\{pt.\}) = \mathbb{C}[t]$$

We call t the *equivariant parameter*. If $\mathcal{M}$ is any manifold with a $\mathbb{C}^*$-action, the map $\mathcal{M} \to \{pt.\}$ induces an injective ring homomorphism $H^*_{\mathbb{C}^*}(\{pt.\}) = \mathbb{C}[t] \to H^*_{\mathbb{C}}(\mathcal{M})$, and therefore, $H^*_{\mathbb{C}^*}(\mathcal{M})$ is a module over $\mathbb{C}[t]$. If $F \subset \mathcal{M}$ is the $\mathbb{C}^*$-fixed locus, and $i : F \to \mathcal{M}$ the inclusion map, it can be shown (see [8]) that the induced map

$$H^*_{\mathbb{C}^*}(\mathcal{M}) \otimes \mathbb{C}(t) \to H^*_{\mathbb{C}^*}(F) \otimes \mathbb{C}(t) \tag{3.2}$$

is an *isomorphism*. That is to say, computing intersection numbers on $\mathcal{M}$ is equivalent to computing intersection numbers solely on the fixed locus, but with the caveat that we tensor with $\mathbb{C}(t)$ i.e. we *invert* the equivariant parameter. This is an important point, the localization formula requires us to invert the $\mathbb{C}^*$-equivariant Euler class to the normal bundle, which is only possible after tensoring with $\mathbb{C}(t)$.

3.2 Localization Graphs

The first step in applying localization to $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is to describe and enumerate the irreducible components of the $\mathbb{C}^*$-fixed loci. Let $n_0, n_\infty \subseteq \{1, 2, \ldots, 2g+2\}$ be subsets such that $n_0 \amalg n_\infty = \{1, 2, \ldots, 2g+2\}$. If a point $[f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is fixed by the $\mathbb{C}^*$-action, then f must be a map of the following type:

1. The source curve of f consists of (at most) three irreducible components, which we will denote by $C_0, C_{\mathbb{P}^1}$, and C_∞. The components C_0 and C_∞ may be empty

2. The component C_0 is a smooth rational component that is contracted to $0 \in [\mathbb{P}^1/\mathbb{Z}_2]$. The marked points indexed by n_0 lie on the component C_0.

3. The component C_∞ is a smooth rational component that is contracted to $\infty \in [\mathbb{P}^1/\mathbb{Z}_2]$. The marked points indexed by n_∞ lie on the component C_∞

4. In the case that $|n_0|$ and $|n_\infty|$ are both odd, the rational component $C_{\mathbb{P}^1}$ is a rational orbifold curve that maps with degree 1 (and hence isomorphically) to $[\mathbb{P}^1/\mathbb{Z}_2]$, and has non-trivial isotropy over 0 and ∞. In the case that $|n_0|$ and $|n_\infty|$ are even, $C_{\mathbb{P}^1}$ is a copy of $\mathbb{P}^1$ with no non-trivial isotropy.

With this description of the irreducible components of the $\mathbb{C}^*$-fixed locus, it is convenient to index the irreducible components with combinatorial objects called *localization graphs*. Localization graphs were first introduced in [18] and [17]. In the present context, the combinatorial description of these graphs is as follows.

Definition 9. *A localization graph Γ for $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is a decorated graph with the following properties:*

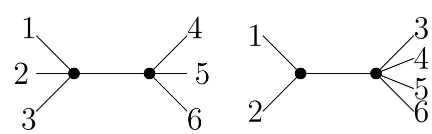
1. *Γ has two vertices, denoted v_0 and v_∞. They correspond to contracted components over 0 and ∞, respectively.*

2. *Γ has one edge connecting v_0 and v_∞. This edge corresponds to the component mapping with degree 1 to the target.*

3. *The vertices v_0 and v_∞ can be incident to* half edges. *We denote the set of half edges incident to v_0 and v_∞ as e_0 and e_∞, respectively. We require that $|e_0| + |e_\infty| = 2g + 2$*

4. *The half edges are* labelled, *i.e. there is a bijective map $\nu : e_0 \cup e_\infty \to \{1, 2, \ldots, 2g + 2\}$.*

Example 2. *Consider the space $\overline{\mathcal{M}}_{0,6t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. As described above, each localization graph of this space is determined by which marked points are mapped to zero, and which marked points are mapped to ∞. Here are two examples of localization graphs for this space:*



Denote the localization graph on the left by Γ_1, and the localization graph on the right by Γ_2. The graph Γ_1 corresponds to an irreducible component of the $\mathbb{C}^*$-fixed locus. This component consists of maps that contract a rational component to zero, and the first three marked points lie on this rational component. These maps also contract a rational component to ∞, and this rational component contains the last three marked points. Similar statements can be made for Γ_2, except this time, the first two marked points map to zero, and the last four marked points map to ∞.

To each localization graph Γ, we associate a moduli space $\overline{\mathcal{M}}_\Gamma$. There exists a finite map from $\overline{\mathcal{M}}_\Gamma$ to the corresponding irreducible component of the $\mathbb{C}^*$-fixed locus. The degree of this map is called the *gluing factor* of $\overline{\mathcal{M}}_\Gamma$. Consequently, integrals over an irreducible component of the fixed locus may be computed as an integral over the corresponding space $\overline{\mathcal{M}}_\Gamma$, after correcting by the gluing factor [19]. The space $\overline{\mathcal{M}}_\Gamma$ is determined by the *vertices* of the localization graph,

$$\overline{\mathcal{M}}_\Gamma := \overline{\mathcal{M}}_{v_0} \times \overline{\mathcal{M}}_{v_\infty}$$

The spaces $\overline{\mathcal{M}}_{v_0}$ and $\overline{\mathcal{M}}_{v_\infty}$ are described as follows. If $|e_0|$ is odd, then $\overline{\mathcal{M}}_{v_0} := \overline{\mathcal{M}}_{0,(|e_0|+1)t}(\mathcal{B}\mathbb{Z}_2)$, and if $|e_0|$ is even, $\overline{\mathcal{M}}_{v_0} := \overline{\mathcal{M}}_{0,|e_0|t,1u}(\mathcal{B}\mathbb{Z}_2)$. Similarly, if $|e_\infty|$ is odd, $\overline{\mathcal{M}}_{v_\infty} := \overline{\mathcal{M}}_{0,(|e_\infty|+1)t}(\mathcal{B}\mathbb{Z}_2)$, and if $|e_\infty|$ is even, $\overline{\mathcal{M}}_{v_\infty} := \overline{\mathcal{M}}_{0,|e_\infty|t,1u}(\mathcal{B}\mathbb{Z}_2)$.

In general, localization graphs will correspond to products of hyperelliptic loci, which is precisely the reason why we obtain recursions of intersection numbers over these spaces.

Example 3. *Recall the localization graphs Γ_1 and Γ_2 in Example 2. We have*

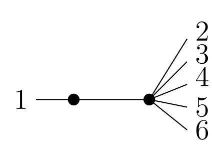
$$\overline{\mathcal{M}}_{\Gamma_1} = \overline{\mathcal{M}}_{0,4t}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,4t}(\mathcal{B}\mathbb{Z}_2) \qquad \overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{0,2t,1u}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,4t,1u}(\mathcal{B}\mathbb{Z}_2)$$

3.3 The $\mathbb{C}^*$-Equivariant Euler Class

Our next task is to compute the $\mathbb{C}^*$-equivariant Euler class of the normal bundle of $\overline{\mathcal{M}}_\Gamma$.

Definition 10. *Let Γ be a localization graph of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$, and let $\overline{\mathcal{M}}_\Gamma$ be the corresponding moduli space. The gluing factor for $\overline{\mathcal{M}}_\Gamma$ is defined to be 2^{n-1}, where n is the number of nodes on any source curve C, where $[C \to [\mathbb{P}^1/\mathbb{Z}_2]]$ is a generic point in $\overline{\mathcal{M}}_\Gamma$.*

Example 4. *Recall the localization graphs Γ_1 and Γ_2 from Example 2. Every source curve of every stable map inside the loci $\overline{\mathcal{M}}_{\Gamma_1}$ and $\overline{\mathcal{M}}_{\Gamma_2}$ has two non-empty contracted components over zero and ∞, so these source curves have two nodes. Therefore, $\overline{\mathcal{M}}_{\Gamma_1}$ and $\overline{\mathcal{M}}_{\Gamma_2}$ both have gluing factors of 2. However, consider the localization graph*



Lets call this graph Γ_3. Notice that every stable map $[\mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{\Gamma_3}$ does not have a contracted component over 0. Indeed, if there were a contracted component over zero, this would cause an unstable component since the contracted component only has two special points. Therefore, we see that the gluing factor for $\overline{\mathcal{M}}_{\Gamma_3}$ is 1.

Let $N_{\overline{\mathcal{M}}_\Gamma}$ denote the normal bundle to the fixed locus $\overline{\mathcal{M}}_\Gamma$. In order to rigorously compute $\frac{1}{e(N_{\overline{\mathcal{M}}_\Gamma})}$, as required for localization, one needs to carefully understand the deformation theory of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. For the purposes of this dissertation, we will not proceed with a completely rigorous derivation for the expression of $\frac{1}{e(N_{\overline{\mathcal{M}}_\Gamma})}$. Instead, we opt for an informal outline of how this derivation goes. The reader who is interested in a meticulous derivation should consult ([14], Chapter 27), and all references therein.

Let $[f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ be a point in the space of stable maps. We want to gain access to the space of normal directions, or the normal bundle, to this point. In deformation theory, one begins this endeavor by looking at the *tangent-obstruction sequence*:

$$0 \longrightarrow H^0(\mathcal{C}, T\mathcal{C}) \longrightarrow H^0(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])) \longrightarrow \mathcal{T}^1 \longrightarrow$$

$$\longrightarrow H^1(\mathcal{C}, T\mathcal{C}) \longrightarrow H^1(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])) \longrightarrow \mathcal{T}^2 \longrightarrow 0$$

The Euler class to the normal bundle at $[f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is defined to be

$$e(N) := \frac{e(\mathcal{T}^1)}{e(\mathcal{T}^2)}$$

Using the multiplicative properties of the Euler class applied to the tangent-obstruction sequence, we obtain

$$\frac{e(H^0(\mathcal{C}, T\mathcal{C}))e(\mathcal{T}^1)e(H^1(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])))}{e(H^0(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])))e(H^1(\mathcal{C}, T\mathcal{C}))e(\mathcal{T}^2)} = 1$$

and therefore,

$$\frac{1}{e(N)} := \frac{e(\mathcal{T}^2)}{e(\mathcal{T}^1)} = \frac{e(H^0(\mathcal{C}, T\mathcal{C}))e(H^1(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])))}{e(H^0(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2])))e(H^1(\mathcal{C}, T\mathcal{C}))} \tag{3.3}$$

Now let Γ be a localization graph for $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. What remains is to evaluate all of the terms on the right hand side of Equation (3.3) when restricted to $\overline{\mathcal{M}}_\Gamma$, thereby computing $\frac{1}{e(N_{\overline{\mathcal{M}}_\Gamma})}$. When this restriction is made, each of the terms on the right hand side of Equation (3.3) depends only on the graph Γ. Again, we refer the reader to ([14], Chapter 27) for a complete derivation. We will expedite the process by stating the results explicitly. First, we make the following definition:

Definition 11. *Let Γ be a localization graph of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. We define the following subsets of vertices of Γ:*

- *$Val_0(1) := \{$vertices of valence 1 over 0$\}$*
- *$Val_\infty(1) := \{$vertices of valence 1 over $\infty\}$*
- *$Val_0(3) := \{$vertices of valence 3 over 0$\}$*
- *$Val_\infty(3) := \{$vertices of valence 3 over $\infty\}$*
- *$Val_0(\geq 3) := \{$vertices of valence at least 3 over 0$\}$*
- *$Val_\infty(\geq 3) := \{$vertices of valence at least 3 over $\infty\}$*

Before we state the next proposition, let us explain a product notation we use for elements in $H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_\Gamma)$. Since $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{v_0} \times \overline{\mathcal{M}}_{v_\infty}$, we have two canonical projection maps $\pi_1 : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{v_0}$ and $\pi_2 : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{v_\infty}$. Whenever we denote an element $(a) \times (b) \in H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_\Gamma)$, what we mean is $\pi^*(a) \cdot \pi^*(b)$.

Proposition 1. *Let Γ be a localization graph of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$, and let $[f : \mathcal{C} \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_\Gamma$. All of the terms appearing in Equation* (3.3) *restricted to $\overline{\mathcal{M}}_\Gamma$ are as follows:*

1. $e(H^0(\mathcal{C}, T\mathcal{C})) = t^{|Val_0(1))|}(-t)^{|Val_\infty(1)|}$

2. $e(H^1(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2]))) = 1$

3. $e(H^0(\mathcal{C}, f^*(T[\mathbb{P}^1/\mathbb{Z}_2]))) = -t^2$

4. $e(H^1(\mathcal{C}, T\mathcal{C})) = \left((t - \psi_0)^{|Val_0(\geq 3)|}\right) \times \left((-t - \psi_\infty)^{|Val_\infty(\geq 3)|}\right)$

where ψ_0 is the ψ-class at the node over 0, and ψ_∞ is the ψ-class at the node over ∞. Incorporating the gluing factor of $\overline{\mathcal{M}}_\Gamma$, which is

$$2^{|Val_0(3)|+|Val_\infty(3)|-1}$$

we obtain the following closed formula for the inverse of the $\mathbb{C}^*$-equivariant Euler class to the normal bundle of $\overline{\mathcal{M}}_\Gamma$:

$$\frac{1}{e(N_{\overline{\mathcal{M}}_\Gamma})} = 2^{|Val_0(3)|+|Val_\infty(3)|-1} t^{|Val_0(1))|} (-t)^{|Val_\infty(1)|} \left(\frac{-1}{t^2}\right) \left(\frac{1}{t - \psi_0}\right) \times \left(\frac{1}{-t - \psi_\infty}\right)$$

In the context of our computations, the Atiyah-Bott localization theorem can be stated as follows:

Corollary 1. *Let $\alpha \in A^*(\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1))$. Then*

$$\int_{\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)} \alpha = \sum_\Gamma \int_{\overline{\mathcal{M}}_\Gamma} \frac{\alpha|_{\overline{\mathcal{M}}_\Gamma}}{e(N_{\overline{\mathcal{M}}_\Gamma})}$$

where the sum is over all localization graphs Γ of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$.

3.4 Restrictions of Classes

In order to compute hyperelliptic Hodge integrals, we will make auxiliary computations on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. In order to use localization, we need to understand how various Chow classes in $A^*(\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1))$ restrict the fixed loci $\overline{\mathcal{M}}_\Gamma$.

Let us first explain how λ-classes and the classes $\mathrm{ev}_i^*(0)$ and $\mathrm{ev}_i^*(\infty)$ restrict to the fixed loci $\overline{\mathcal{M}}_\Gamma$. These are standard facts in the literature:

$$\lambda_i|_{\overline{\mathcal{M}}_\Gamma} = \sum_{j+k=i} \lambda_j|_{\overline{\mathcal{M}}_{v_0}} \times \lambda_k|_{\overline{\mathcal{M}}_{v_\infty}} \tag{3.4}$$

$$\mathrm{ev}_i^*(0)|_{\overline{\mathcal{M}}_\Gamma} = \begin{cases} t & \overline{\mathcal{M}}_\Gamma \text{ consists of curves that map the } i^{th} \text{ marked point to } 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

$$\mathrm{ev}_i^*(\infty)|_{\overline{\mathcal{M}}_\Gamma} = \begin{cases} -t & \overline{\mathcal{M}}_\Gamma \text{ consists of curves that map the } i^{th} \text{ marked point to } \infty \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

The remaining Chow classes are Chern classes of a certain vector bundle. Before we can describe this vector bundle, we need a bit of technical machinery from the theory of lines bundles on orbifold curves.

First, recall the *orbifold Riemann-Roch Theorem* (see [11]), applied to orbifold curves and line bundles. Let L be a line bundle over an orbifold curve $\mathcal{X}$ of genus g, and suppose $\mathcal{X}$ has only finitely many points $p_1, \ldots, p_n \in \mathcal{X}$ that have non-trivial isotropy. If G_{p_i} is the local group at p_i, then G_{p_i} acts on L_{p_i}. This action takes the form $z \mapsto e^{\frac{2\pi i}{r}} z$, where $r \in \{1, 2, \ldots, |G_{p_i}|\}$. We define $\mathrm{age}_{p_i}(L) := \frac{1}{r}$, and so the Euler characteristic of L is

$$\chi(L) = (1-g) + \deg(L) - \sum_{i=1}^{n} \mathrm{age}_{p_i}(L)$$

Consider the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ over $[\mathbb{P}^1/\mathbb{Z}_2]$. By this we mean, take $\mathcal{O}(-1)$ on $\mathbb{P}^1$, and pullback along the map $[\mathbb{P}^1/\mathbb{Z}_2] \to \mathbb{P}^1$ that forgets the orbifold structure, and let $\mathbb{Z}_2$ act non-trivially on the fibers. Let $[f : (C, p_1, \ldots, p_{2g+2}) \to [\mathbb{P}^1/\mathbb{Z}_2]] \in \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$. Then by orbifold Riemann-Roch, we have

$$\chi(f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = -\sum_{i=1}^{2g+2}\frac{1}{2} = -g-1$$

Since $h^0(f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, we have $h^1(f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = g+1$. This observation justifies the following:

Proposition 2. *Let π be the universal family over $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$,*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad f \quad} & [\mathbb{P}^1/\mathbb{Z}_2] \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1) & & \end{array}$$

Then $R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)$ is a vector bundle over $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. The fiber over the point $[f:(C,p_1,\ldots,p_{2g+2}\to[\mathbb{P}^1/\mathbb{Z}_2]]$ is $H^1(C,f^*\mathcal{O}_{\mathbb{P}^1}(-1))$.

Now, lets see how the vector bundle bundle $R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)$ restricts to the $\mathbb{C}^*$-fixed loci of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. In order to localize, we make a choice of *torus weights* on $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see [14] for more details). This amounts to determining how $\mathbb{C}^*$ acts on $\mathcal{O}_{\mathbb{P}^1}(-1)|_0$ and $\mathcal{O}_{\mathbb{P}^1}(-1)|_\infty$. Throughout, we choose the torus weights to be a on $\mathcal{O}_{\mathbb{P}^1}(-1)|_0$ and $a+1$ on $\mathcal{O}_{\mathbb{P}^1}(-1)|_\infty$. In other words, for $\gamma\in\mathbb{C}^*$, the torus action on the fiber over 0 is $\gamma\cdot z=\gamma^a z$, and over ∞ it is $\gamma\cdot z=\gamma^{a+1}z$.

Let Γ be a localization graph of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$. Let $[f:(C,p_1,\ldots,p_{2g+2})\to[\mathbb{P}^1/\mathbb{Z}_2]]$ be a point in $\overline{\mathcal{M}}_\Gamma$. We denote C_{v_0} and C_{v_∞} as the contracted components over 0 and ∞, respectively. We assume C_{v_0} and C_{v_∞} are of positive dimension. Consider the *normalization exact sequence* for C,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{v_0}\oplus\mathcal{O}_{v_\infty}\oplus\mathcal{O}_e \longrightarrow \mathcal{O}_{n_0}\oplus\mathcal{O}_{n_\infty} \longrightarrow 0$$

where

- $\mathcal{O}_{v_0}$ is the structure sheaf on C_0, the contracted component over 0
- $\mathcal{O}_{v_\infty}$ is the structure sheaf on C_∞, the contracted component over ∞
- $\mathcal{O}_e$ is the structure sheaf on C_e, the rational curve nodal to the contracted components

- $\mathcal{O}_{n_0}$ is the skyscraper sheaf supported on n_0, the node above 0
- $\mathcal{O}_{n_\infty}$ is the skyscraper sheaf supported on n_∞, the node above ∞

Tensoring this sequence with $f^*\mathcal{O}_{\mathbb{P}^1}(-1)$, and applying cohomology, we get the long exact sequence

$$0 \longrightarrow H^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow H^0(C_0, \mathbb{C} \times C_0) \oplus H^0(C_\infty, \mathbb{C} \times C_\infty) \oplus H^0(C_e, \mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\longrightarrow H^0(n_0, \mathbb{C} \times n_0) \oplus H^0(n_\infty, \mathbb{C} \times n_\infty) \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\longrightarrow H^1(C_0, \mathbb{C} \times C_0) \oplus H^1(C_\infty, \mathbb{C} \times C_\infty) \oplus H^1(C_e, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow 0$$

It is clear that $H^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(C_e, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. However, we also claim that $H^0(C_0, \mathbb{C} \times C_0) = H^0(C_\infty, \mathbb{C} \times C_\infty) = 0$. To see this let $s : C_0 \to \mathbb{C} \times C_0$. be a constant section. The section s must be $\mathbb{Z}_2$-equivariant, and in particular, if $p_i \in C_0$ is a point with non-trivial isotropy, then $s(p_i) = -s(p_i)$, and therefore, s is the zero section. The same argument works if we choose a section $s : C_\infty \to \mathbb{C} \times C_\infty$. In summary, the above sequence reduces to the short exact sequence

$$0 \longrightarrow H^0(n_0, \mathbb{C} \times n_0) \oplus H^0(n_\infty, \mathbb{C} \times n_\infty) \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\longrightarrow H^1(C_0, \mathbb{C} \times C_0) \oplus H^1(C_\infty, \mathbb{C} \times C_\infty) \oplus H^1(C_e, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow 0$$

Lets first consider the case when the corresponding moduli space for Γ is of the form

$$\overline{\mathcal{M}}_{0,k_1t,1u}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,k_2t,1u}(\mathcal{B}\mathbb{Z}_2)$$

Define $g_1 := \frac{k_1-2}{2}$ and $g_2 := \frac{k_2-2}{2}$. If $s : n_0 \to \mathbb{C} \times n_0$ is an element of $H^0(n_0, \mathbb{C} \times n_0$, then since n_0 has trivial isotropy, imposing $\mathbb{Z}_2$ equivariance on s does nothing. Therefore, s can be any constant section. A similar argument holds for n_∞. Therefore, normalization exact sequence becomes

$$0 \longrightarrow L_a \oplus L_{a+1} \to H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \mathbb{E}_{g_1}^\vee \oplus \mathbb{E}_{g_2}^\vee \longrightarrow 0 \tag{3.7}$$

where L_a is defined to be the trivial equivariant line bundle with torus weight a (see Definition 16). Now consider the case when Γ corresponds to a moduli space of the form $\overline{\mathcal{M}}_{0,k_1t}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,k_2t}(\mathcal{B}\mathbb{Z}_2)$. We have $H^0(n_0, \mathbb{C} \times n_0) = H^0(n_\infty, \mathbb{C} \times n_\infty) = 0$. Indeed, if $s \in H^0(n_0, \mathbb{C} \times n_0)$, then since n_0 has non-trivial isotropy, $\mathbb{Z}_2$ equivariance forces s to be the zero section. The same holds for n_∞. In which case, we get the very simple two term sequence

$$0 \to H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{E}_{g_1}^\vee \oplus \mathbb{E}_{g_2}^\vee \oplus L_{a+\frac{1}{2}} \longrightarrow 0 \tag{3.8}$$

The term $L_{a+\frac{1}{2}}$ shows up due to the following argument. If C is a $\mathbb{P}^1$ with non-trivial $\mathbb{Z}_2$-isotropy at 0 and ∞, and $f : C \to [\mathbb{P}^1/\mathbb{Z}_2]$ is a map of degree 1, then the orbifold Riemann-Roch theorem says that $h^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 1$. Therefore, it only remains to compute the torus weights of this line bundle. One way to do this is to use the perspective of admissible covers: the data of f is equivalent to the data of a hyperelliptic covering $\widetilde{f} : \mathbb{P}^1 \to C$, branched at 0 and ∞. It turns out that $H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \widetilde{f}^*(f^*\mathcal{O}_{\mathbb{P}^1}(-1)))$, and from this, one can derive that the torus weight is $(a + \frac{1}{2})$, see ([10], pg. 22) for a complete derivation.

The sequences (3.7) and (3.8) show up prominently when we compute hyperelliptic Hodge integrals that have no ψ-classes i.e. pure Hodge integrals.

We are now ready to begin making localization computations.

3.5 Recursions for Hyperelliptic Hodge Integrals

In this section, we provide one of the key results in this dissertation. Recall the definition of the intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$ from Chapter 2

Theorem 5. *There exists a set of recursions that completely determine the intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{i},2g+2}$. The only initial conditions required of these recursions are the intersection numbers $D_{\vec{0},2g+2} = d_{\vec{0},2g+2} = \frac{1}{2}$, established in [9].*

We begin by distinguishing between the notion of a *pure Hodge integral* and a *non-pure Hodge integral*.

Definition 12. *Pure Hodge integrals are intersection numbers $D_{\vec{i},2g+2}$ and $d_{\vec{j},2g+2}$ such that $|\vec{i}| = 2g-1$ and $|\vec{j}| = 2g$. Otherwise, we say that such an integral is a non-pure Hodge integral*

Concretely, pure Hodge integrals have no ψ-classes in their integrand, whereas non-pure Hodge integrals have at least ψ-class in their integrand.

3.5.1 Non-Pure Hodge Integrals

In this section, we compute two families of auxiliary integrals that provide recursions for non-pure Hodge integrals. The calculations in this section will include sums over multi-indices. We make the following definition to allow for readability:

Definition 13. *For $\vec{\ell} = (\ell_1, \dots, \ell_n), \vec{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$, we say*

$$\vec{\ell} \leq \vec{j} \iff \ell_i \leq j_i \,\forall\, i$$

For a fixed $\vec{i}$ such that $|\vec{i}| < 2g-1$, and genus $g > 0$, let k be an integer such that $0 \leq k \leq 2g-2-|\vec{i}|$. We define

$$I_k := \left(\int_{\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)} \lambda_{\vec{i}} \left(\prod_{i=1}^{2+k} \mathrm{ev}_i^*(0) \right) \mathrm{ev}_{2g+2}^*(\infty) \right) |_{t=1} = 0 \tag{3.9}$$

Remark 2. *There is a slight abuse of notation. We use 't' to refer to the equivariant parameter of the Chow ring of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$, but we we also use '$t$' as a decoration on marked points to indicate non-trivial isotropy. However, the contexts in which these two uses occur are distinct and discernible, so no confusion should arise.*

Let us explain the intuition of what's happening in the integrand of Equation (3.9). First of all, notice that the integrand in Equation (3.9) is a Chow class of codimension

$$\begin{aligned} |\vec{i}| + (2+k) + 1 &= |\vec{i}| + (k+3) \\ &\leq |\vec{i}| + (2g - 2 - |\vec{i}|) \\ &= 2g - 2 \\ &< 2g + 2 \end{aligned}$$

and since the dimension of $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2], 1)$ is $2g+2$, this is the reason why the integral in Equation (3.9) is identically zero. Furthermore, the class

$$\left(\prod_{i=1}^{2+k} \mathrm{ev}_i^*(0)\right) \mathrm{ev}_{2g+2}^*$$

tames the combinatorial complexity when we localize the integral in Equation (3.9). Indeed, the only localization graphs that contribute to the computation of Equation (3.9) are those which correspond to curves that map the first $2+k$ marked points to 0, and the last marked point to ∞. Therefore, it is convenient to establish the following notation/definition to capture which localization graphs contribute to I_k.

Definition 14. *Define Γ_j as the set of all localization graphs with the following properties:*

1. *The first $2+k$ marked points map to 0*

2. *The last marked point maps to ∞*

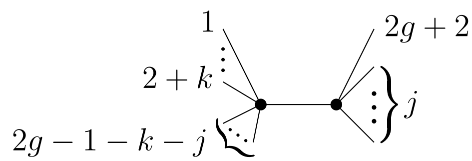


Figure 3.1: These are the localization graphs that appear in the set Γ_j (defined in Definition 14). Each graph in Γ_j has the first $2+k$ marked points mapping to zero, the last marked point mapping to ∞, and j arbitrary marked points also mapping to ∞. The remaining marked points map to zero.

3. *There are j marked points, distinct from the first $2+k$ marked points and the last marked point, that map to ∞*

4. *The remaining $2g-1-k-j$ marked points map to 0*

See Figure 3.1 for a depiction of the graphs in Γ_j.

In order to localize I_k, we need to compute the contributions coming from all of the localization graphs in Γ_j for $0 \le j \le 2g-1-k$. If we let α be the integrand in I_k, and if we define $\text{Cont}(\Gamma_j) := \sum_{\Gamma\in\Gamma_j} \int_{\overline{\mathcal{M}}_\Gamma} \frac{\alpha|_\Gamma}{e(N_\Gamma)}$, we see that

$$I_k = \sum_{j=0}^{2g-1-k} \text{Cont}(\Gamma_j)|_{t=1}$$

Since $|\Gamma_j| = \binom{2g-1-k}{j}$, and since each contribution is the same for each $\Gamma \in \widetilde{\Gamma}_j$, when we compute $\text{Cont}(\Gamma_j)|_{t=1}$, we pick up a factor of $\binom{2g-k}{j}$

First, consider Γ_0. This set contains only one localization graph Γ, and its corresponding moduli space is

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2) \times \{pt.\} = \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$$

Since $\frac{1}{e\left(N_{\overline{\mathcal{M}}_\Gamma}\right)} = \frac{-1}{t^2(t-\psi_{v_0})}$, the contribution coming from Γ, evaluated at $t=1$, is

$$(-1)\int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} \frac{-\lambda_{\vec{i}}}{1-\psi_j}$$

and therefore,

$$\mathrm{Cont}(\Gamma_0)|_{t=1} = D_{(\vec{i}),2g+2} \tag{3.10}$$

Now consider Γ_j where j is odd and $1 \leq j \leq 2\left\lfloor \frac{2g-2-k}{2} \right\rfloor + 1$. Let $g_1 := \frac{2g-1-j}{2}$ and $g_2 := \frac{j-1}{2}$. For all $\Gamma \in \Gamma_j$, the corresponding moduli space is

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g_1+2)t,1u}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,(2g_2+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$$

Since $\frac{1}{e\left(N_{\overline{\mathcal{M}}_\Gamma}\right)} = 2\left(\frac{-1}{t^2(t-\psi_{v_0})}\right) \times \left(\frac{1}{-t-\psi_{v_\infty}}\right)$, we have

$$\begin{aligned}
\mathrm{Cont}(\Gamma)|_{t=1} &= 2\binom{2g-1-k}{j}(-1)\sum_{\vec{\ell}_1+\vec{\ell}_2=\vec{i}} \int_{\overline{\mathcal{M}}_{0,(2g_1+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{\lambda_{\vec{\ell}_1}}{1-\psi_{2g_1+3}} \int_{\overline{\mathcal{M}}_{0,(2g_2+2)t,1u}} \frac{\lambda_{\vec{\ell}_2}}{1+\psi_{2g_2+3}} \\
&= 2\binom{2g-1-k}{2g_2+1}(-1)\sum_{\vec{\ell}_1+\vec{\ell}_2=\vec{i}} (-1)^{2g_2-|\vec{\ell}_2|} d_{\vec{\ell}_1,2g_1+2} d_{\vec{\ell}_2,2g_2+2} \\
&= -2\binom{2g-1-k}{2g_2+1}\sum_{\vec{\ell}_2\leq\vec{i}} (-1)^{|\vec{\ell}_2|} d_{(\vec{i}-\vec{\ell}_2),2g_1+2} d_{\vec{\ell}_2,2g_2+2}
\end{aligned}$$

and therefore,

$$\sum_{\substack{j \text{ odd} \\ 1\leq j\leq 2\left\lfloor \frac{2g-2-k}{2}\right\rfloor+1}} \mathrm{Cont}(\Gamma_j)|_{t=1} = -2 \sum_{\substack{g_1+g_2=g-1 \\ 0\leq g_2\leq \left\lfloor \frac{2g-2-k}{2}\right\rfloor \\ \vec{\ell}_2\leq\vec{i}}} (-1)^{|\vec{\ell}_2|}\binom{2g-1-k}{2g_2+1} d_{(\vec{i}-\vec{\ell}_2),2g_1+2} d_{\vec{\ell}_2,2g_2+2} \tag{3.11}$$

Next, we consider Γ_j where j is even, and $2 \leq j \leq 2\left\lfloor \frac{2g-1-k}{2} \right\rfloor$. Let $g_1 := \frac{2g-j}{2}$ and $g_2 := \frac{j}{2}$. For each $\Gamma \in \Gamma_j$, the corresponding moduli space is

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g_1+2)t}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,(2g_2+2)t}(\mathcal{B}\mathbb{Z}_2)$$

Since $\frac{1}{e(N_\Gamma)} = 2\left(\frac{-1}{t^2(t-\psi_{v_0}}\right) \times \left(\frac{1}{-t-\psi_\infty}\right)$, we have

$$\mathrm{Cont}(\Gamma)|_{t=1} = (2)\binom{2g-1-k}{j}(-1)\sum_{\vec{\ell}_1+\vec{\ell}_2=\vec{i}} \int_{\overline{\mathcal{M}}_{0,(2g_1+2)t}(\mathcal{B}\mathbb{Z}_2)} \frac{\lambda_{\vec{\ell}_1}}{1-\psi_j} \int_{\overline{\mathcal{M}}_{0,2g_2+2}(\mathcal{B}\mathbb{Z}_2)} \frac{\lambda_{\vec{\ell}_2}}{1+\psi_j}$$
$$= (2)\binom{2g-1-k}{2g_2}(-1)\sum_{\vec{\ell}_1+\vec{\ell}_2=\vec{i}} (-1)^{2g_2-1-|\vec{\ell}_2|} D_{\vec{\ell}_1,2g_1+2} D_{\vec{\ell}_2,2g_2+2}$$

and therefore,

$$\sum_{\substack{j \text{ even} \\ 2\leq j\leq 2\lfloor \frac{2g-1-k}{2} \rfloor}} \mathrm{Cont}(\Gamma_j)|_{t=1} = 2 \sum_{\substack{g_1+g_2=g \\ 1\leq g_2\leq \lfloor \frac{2g-1-k}{2} \rfloor \\ \vec{\ell}\leq \vec{i}}} (-1)^{|\vec{\ell}|}\binom{2g-1-k}{2g_2} D_{(\vec{i}-\vec{\ell}),2g_1+2} D_{\vec{\ell},2g_2+2} \tag{3.12}$$

Now we consider a different auxiliary integral. For a fixed $\vec{i}$ and genus $g > 0$, let $0 \leq k \leq 2g-1-|\vec{i}|$ and define

$$\widetilde{I}_k := \left(\int_{\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)} \lambda_{\vec{i}} \left(\prod_{i=1}^{2+k} \mathrm{ev}_i^*(0)\right)\right)|_{t=1} = 0$$

As we saw in the case of I_k, we first need to enumerate the localization graphs that contribute to $\widetilde{I}_k$.

Definition 15. *Define* $\widetilde{\Gamma}_j$ *as the set of all localization graphs of* $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$ *with the following properties:*

1. *The first* $2+k$ *marked points map to 0*
2. *There are* j *marked points, distinct from the first* $2+k$ *marked points, that map to* ∞.
3. *The remaining* $2g-k-j$ *marked points map to* 0.

See Figure 3.2 for a depiction of the graphs appearing in $\widetilde{\Gamma}_j$.

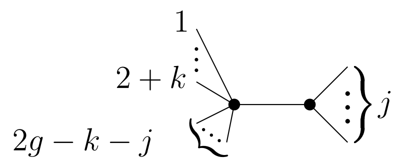


Figure 3.2: These are the localization graphs appearing in the set $\widetilde{\Gamma}_j$ (as defined in Definition 3.2)

Let α be the integrand appearing in $\widetilde{I}_k$, and define $\mathrm{Cont}(\widetilde{\Gamma}_j) := \sum_{\Gamma \in \Gamma_j^B} \int_{\overline{\mathcal{M}}_\Gamma} \frac{\alpha|_\Gamma}{e(N_\Gamma)}$. When we localize $\widetilde{I}_k$, we need to compute the contributions coming from the sets $\widetilde{\Gamma}_j$,

$$\widetilde{I}_k = \sum_{j=0}^{|\vec{i}|+1} \mathrm{Cont}(\widetilde{\Gamma}_j)|_{t=1}$$

Since $|\widetilde{\Gamma}_j| = \binom{2g-k}{j}$, and since each contribution is the same for each $\Gamma \in \widetilde{\Gamma}_j$, when we compute $\mathrm{Cont}(\widetilde{\Gamma}_j)|_{t=1}$, we pick up a factor of $\binom{2g-k}{j}$.

The computations for $\mathrm{Cont}(\widetilde{\Gamma}_j)|_{t=1}$ are analogous to the computations made in the case of I_k i.e. to the results in Equation (3.10), Equation (3.11), and Equation (3.12). We expedite the process by simply stating the results:

$$\mathrm{Cont}(\widetilde{\Gamma}_0)|_{t=1} = d_{\vec{i},2g+2} \tag{3.13}$$

$$\sum_{\substack{j \text{ odd} \\ 1 \le j \le 2\left\lfloor \frac{2g-1-k}{2} \right\rfloor + 1}} \mathrm{Cont}(\widetilde{\Gamma}_j)|_{t=1} = -2 \sum_{\substack{g_1+g_2=g \\ 0 \le g_2 \le \left\lfloor \frac{2g-1-k}{2} \right\rfloor \\ \vec{\ell} \le \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-k}{2g_2+1} D_{(\vec{i}-\vec{\ell}),2g_1+2} D_{\vec{\ell},2g_2+2} \tag{3.14}$$

$$\sum_{\substack{j \text{ even} \\ 2 \le j \le 2\left\lfloor \frac{2g-k}{2} \right\rfloor}} \mathrm{Cont}(\widetilde{\Gamma}_j)|_{t=1} = 2 \sum_{\substack{g_1+g_2=g-1 \\ 0 \le g_2 \le \left\lfloor \frac{2g-k}{2} \right\rfloor - 1 \\ \vec{\ell} \le \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-k}{2g_2+2} d_{(\vec{i}-\vec{\ell}),2g_1+2} d_{\vec{\ell},2g_2+2} \tag{3.15}$$

Combining all of the localization computations in Equations (3.10), (3.11), (3.12), (3.13), (3.14), and (3.15), we obtain the following theorem:

Theorem 6. *Fix $\vec{i}$ and $g > 0$ such that $|\vec{i}| < 2g-1$. For $0 \leq k \leq 2g-2-|\vec{i}|$,*

$$D_{\vec{i},2g+2} = 2 \sum_{\substack{g_1+g_2=g-1 \\ 0 \leq g_2 \leq \left\lfloor \frac{2g-2-k}{2} \right\rfloor \\ \vec{\ell} \leq \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-1-k}{2g_2+1} d_{(\vec{i}-\vec{\ell}),2g_1+2} d_{\vec{\ell},2g_2+2} - 2 \sum_{\substack{g_1+g_2=g \\ 1 \leq g_2 \leq \left\lfloor \frac{2g-1-k}{2} \right\rfloor \\ \vec{\ell} \leq \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-1-k}{2g_2} D_{(\vec{i}-\vec{\ell}),2g_1+2} D_{\vec{\ell},2g_2+2} \tag{3.16}$$

and for $0 \leq k \leq 2g-1-|\vec{i}|$,

$$d_{\vec{i},2g+2} = 2 \sum_{\substack{g_1+g_2=g \\ 0 \leq g_2 \leq \left\lfloor \frac{2g-1-k}{2} \right\rfloor \\ \vec{\ell} \leq \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-k}{2g_2+1} D_{(\vec{i}-\vec{\ell}),2g_1+2} D_{\vec{\ell},2g_2+2} - 2 \sum_{\substack{g_1+g_2=g-1 \\ 0 \leq g_2 \leq \left\lfloor \frac{2g-k}{2} \right\rfloor - 1 \\ \vec{\ell} \leq \vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g-k}{2g_2+2} d_{(\vec{i}-\vec{\ell}),2g_1+2} d_{\vec{\ell},2g_2+2} \tag{3.17}$$

3.5.2 Pure Hodge Integrals

In this section, we compute recursions for pure Hodge integrals, that is, we compute Hodge integrals of the form $D_{\vec{i},2g+2}$, where $|\vec{i}| = 2g-1$. Notice that if $|\vec{i}| = 2g$, then the pure Hodge integral $d_{\vec{i},2g+2} = 0$. This is due to the fact that Hodge monomials on $\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$ are pulled back from $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ along the map that forgets the untwisted point, and since such a monomial is zero in Chow on $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ for dimensions reason, it's also zero in Chow on $\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$.

Before we begin the localization computation, we establish the following notation:

Definition 16. *Define L_a to be the $\mathbb{C}^*$-equivariant trivial line bundle on $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$ with torus weight a. In other words, $\mathbb{C}^*$ acts on each fiber by $\lambda \cdot z = \lambda^a z$.*

Notice that $c_{\mathbb{C}^*}(L_a) = 1 + at$. In our localization computations below, we make repeated use of the following fact concerning Chern classes (see [20]):

Proposition 3. *Let $\mathbb{E}$ be a rank r vector bundle, and let L be a line bundle. Then*

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) c_1(L)^j$$

Let $\vec{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ such that $|\vec{i}| = 2g-1$. Without loss of generality, assume $i_n > 0$. For notational convenience, we define $\vec{m} := (i_1, \ldots, i_{n-1})$, the tuple $\vec{i}$ that is missing the last entry. Consider the auxiliary integral

$$I := \left(\int_{\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)} \lambda_{\vec{m}} \cdot c_{i_n+1}(R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cdot \mathrm{ev}_1^*(\infty) \right) |_{t=1} = 0$$

where we choose the torus weights of $\mathcal{O}_{\mathbb{P}^1}(-1)$ to be -1 over 0 and 0 over ∞ (we refer the reader to Section 3.4 for all of the definitions and conventions regarding the vector bundle $R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)$).

Using localization, we express I as a sum of integrals over fixed loci, which, as in the case of non-pure Hodge integrals, results in recursions for the integrals

$$D_{(\vec{i}),2g+2} := \int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} \lambda_{\vec{i}}$$

In order to enumerate the localization graphs that contribute to I, we make the following definition:

Definition 17. *Let Γ_i denote the set of all localization graphs for $\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1)$ with the following properties:*

1. *The first marked point lies over ∞*
2. *There are i marked points, distinct from the first marked point, that lie over ∞*
3. *The remaining $2g+1-i$ points lie over 0*

In order to compute I, we compute the contributions coming from the sets Γ_i, as i ranges from 0 to $2g+1$. Let $\alpha \in A^*(\overline{\mathcal{M}}_{0,(2g+2)t}([\mathbb{P}^1/\mathbb{Z}_2],1))$ represent the integrand in I. If we define Cont$(\Gamma_i) := \sum_{\Gamma\in\Gamma_i}\int_{\overline{\mathcal{M}}_\Gamma}\frac{\alpha|_\Gamma}{e(N_\Gamma)}$ then

$$I = \sum_{i=0}^{2g+1} \text{Cont}(\Gamma_i)|_{t=1}$$

Since $|\Gamma_i| = \binom{2g+1}{i}$, and since each contribution is the same for each $\Gamma \in \Gamma_i$, when we compute Cont$(\Gamma_i)|_{t=1}$, we pick up a factor of $\binom{2g+1}{i}$.

First, we show that

$$\text{Cont}(\Gamma_{2g-1})|_{t=1} = \text{Cont}(\Gamma_{2g+1})|_{t=1} = 0$$

Consider the set Γ_{2g-1}. Each localization graph $\Gamma \in \Gamma_{2g-1}$ corresponds to the moduli space

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,2t,1u}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,(2(g-1)+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$$

Applying cohomology to the normalization exact sequence, we get

$$0 \longrightarrow L_{-1}\oplus L_0 \longrightarrow H^1(C, f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) \longrightarrow H^1(C_{v_\infty}, \mathbb{C}\times C_{v_\infty}) \longrightarrow 0$$

so for any graph $\Gamma \in \Gamma_{2g-1}$, we have

$$
\begin{aligned}
c_{i_n+1}(R^1(\pi_* f^*\mathcal{O}(-1)))|_{\overline{\mathcal{M}}_\Gamma, t=1} &= c^{\mathbb{C}^*}_{i_n+1}\left(L_{-1} \oplus L_0 \oplus \left(\mathbb{E}^\vee_{g-1} \otimes L_0\right)\right)|_{t=1} \\
&= c^{\mathbb{C}^*}_{i_n+1}(L_{-1} \oplus (\mathbb{E}^\vee_{g-1} \otimes L_0))|_{t=1} \\
&= c^{\mathbb{C}^*}_{i_n+1}\left(\mathbb{E}^\vee_{g-1} \otimes L_0\right) - c^{\mathbb{C}^*}_{i_n}\left(\mathbb{E}^\vee_{g-1} \otimes L_0\right)|_{t=1} \\
&= \sum_{r=0}^{i_n+1} \binom{g-1-i_n-1+r}{r} c_{i_n+1-r}\left(\mathbb{E}^\vee_{g-1}\right)(0)^r \\
&- \sum_{r=0}^{i_n} \binom{g-1-i_n+r}{r} c_{i_n-r}\left(\mathbb{E}^\vee_{g-1}\right)(0)^r \\
&= (-1)^{i_n+1}\lambda_{i_n+1} - (-1)^{i_n}\lambda_{i_n} \\
&= (-1)^{i_n+1}(\lambda_{i_n+1} + \lambda_{i_n})
\end{aligned}
$$

For each $\Gamma \in \Gamma_{2g-1}$, $\frac{1}{e(N_\Gamma)} = \left(\frac{-1}{t^2(t-\psi)}\right) \times \left(\frac{1}{-t-\psi}\right)$. Therefore, we have

$$
\begin{aligned}
\text{Cont}(\Gamma_{2g-1})|_{t=1} &= \binom{2g+1}{2g-1}(-1)\int_{\overline{\mathcal{M}}_{0.2t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{-1}{1-\psi_3} \\
&\times \int_{\overline{\mathcal{M}}_{0,(2(g-1)+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{(-1)^{i_n+1}\lambda_{i_n+1}\lambda_{\vec{m}} + (-1)^{i_n+1}\lambda_{i_n}\lambda_{\vec{m}}}{-1-\psi_{2g+1}} \\
&= \binom{2g+1}{2g-1}(-1)^{i_n}\int_{\overline{\mathcal{M}}_{0,(2(g-1)+2)t}(\mathcal{B}\mathbb{Z}_2)} \frac{\lambda_{\vec{m}}\lambda_{i_n+1} + \lambda_{\vec{m}}\lambda_{i_n}}{1+\psi_{2g+1}} \\
&= 0
\end{aligned}
$$

where the last equality holds for dimension reasons, since dim $\left(\overline{\mathcal{M}}_{0,(2(g-1)+2)t}(\mathcal{B}\mathbb{Z}_2)\right) = 2g-2$.

Now consider the set Γ_{2g+1}. Each localization graph $\Gamma \in \Gamma_{2g+1}$ corresponds to the moduli space

$$
\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g+2)t,1u}
$$

Applying cohomology to the normalization long exact sequence, we get

$$0 \longrightarrow L_0 \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow H^1(C_{v_\infty}, \mathbb{C} \times C_{v_\infty}) \longrightarrow 0$$

Therefore, we have

$$\begin{aligned} c_{i_n+1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma, t=1} &= c^{\mathbb{C}^*}_{i_n+1}(L_0 \oplus (\mathbb{E}_g^\vee \otimes L_0))|_{t=1} \\ &= c^{\mathbb{C}^*}_{i_n+1}(\mathbb{E}_g^\vee \otimes L_0)|_{t=1} \\ &= \sum_{r=0}^{i_n+1} \binom{g - i_n - 1 + r}{r} c_{i_n+1-r}(\mathbb{E}_g^\vee)(0)^r \\ &= (-1)^{i+1}\lambda_{i+1} \end{aligned}$$

Since $\frac{1}{e(N_\Gamma)} = \frac{-1}{t^2(-t-\psi_{v_\infty}}$,

$$\mathrm{Cont}(\Gamma_{2g+1})|_{t=1} = (-1)\int_{\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{(-1)^{i_n+1}\lambda_{i_n+1}\lambda_{\vec{m}}}{-1-\psi_{2g+3}} = 0$$

The reason for the above vanishing is the following: in the integrand above we have a monomial of λ-classes whose total degree is $2g$. Every monomial of λ classes on the space $\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)$ is pulled back from $\overline{\mathcal{M}}_{0,(2g+2)t}$ (along the map that forgets the untwisted point). But since the dimension of $\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$ is $2g-1$, this monomial is zero in Chow.

For the remainder of this localization computation, we organize the workflow in the following way. First, we compute $\mathrm{Cont}(\Gamma_0)|_{t=1}, \mathrm{Cont}(\Gamma_1)|_{t=1}, \mathrm{Cont}(\Gamma_{2g})|_{t=1}$; these contributions are special in that the corresponding moduli spaces are not products. We split the remaining cases into two situations: $\{\mathrm{Cont}(\Gamma_j)|_{t=1}\}_{2,\leq j \leq 2g-2, j \text{ even}}$, and $\{\mathrm{Cont}(\Gamma_j)|_{t=1}\}_{3 \leq j \leq 2g-3, j \text{ odd}}$. In the former case, the corresponding moduli spaces are products of spaces with no untwisted points, and the latter case corresponds to spaces with one untwisted point.

Lets begin with $\mathrm{Cont}(\Gamma_0)$. There is only graph $\Gamma \in \Gamma_0$, and the corresponding moduli space is

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$$

Applying cohomology to the normalization long exact sequence, we get

$$0 \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow H^1(C_{v_0}, \mathbb{C} \times C_{v_0}) \oplus H^1(C_e, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow 0$$

and therefore,

$$\begin{aligned}
c_{i_n+1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma, t=1} &= c^{\mathbb{C}^*}_{i_n+1}((\mathbb{E}_g^\vee \otimes L_{-1}) \oplus L_{-\frac{1}{2}})|_{t=1} \\
&= c^{\mathbb{C}^*}_{i_n+1}(\mathbb{E}_g^\vee \otimes L_{-1}) - \frac{1}{2} t c^{\mathbb{C}^*}_{i_n}(\mathbb{E}_g^\vee \otimes L_{-1})|_{t=1} \\
&= \sum_{r=0}^{i_n+1} \binom{g-i_n-1+r}{r} c_{i_n+1-r}(\mathbb{E}_g^\vee)(-1)^r \\
&- \frac{1}{2} \sum_{r=0}^{i_n} \binom{g-i_n+r}{r} c_{i_n-r}(\mathbb{E}_g^\vee)(-1)^r \\
&= \sum_{r=0}^{i_n+1} \binom{g-i_n-1+r}{r} (-1)^{i_n+1} \lambda_{i_n+1-r} \\
&- \frac{1}{2} \sum_{r=0}^{i_n} \binom{g-i_n+r}{r} (-1)^{i_n} \lambda_{i_n-r} \\
&= (-1)^{i_n+1} \\
&\times \left[\sum_{r=0}^{i_n+1} \binom{g-i_n-1+r}{r} \lambda_{i_n+1-r} + \frac{1}{2} \sum_{r=0}^{i_n} \binom{g-i_n+r}{r} \lambda_{i_n-r} \right]
\end{aligned}$$

Since $\frac{1}{e(N_\Gamma)} = \frac{-1}{t^2(t-\psi_{v_0})}$, we have

$$\text{Cont}(\Gamma_0)|_{t=1} =$$

$$(-1) \int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} \frac{(-1)(-1)^{i_n+1} \left[\sum_{r=0}^{i_n+1} \binom{g-i_n-1+r}{r} \lambda_{\vec{m}} \lambda_{i_n+1-r} + \frac{1}{2} \sum_{r=0}^{i_n} \binom{g-i_n+r}{r} \lambda_{\vec{m}} \lambda_{i_n-r} \right]}{1-\psi_1}$$

After some simplification, we obtain:

$$\mathrm{Cont}(\Gamma_0)|_{t=1} = (-1)^{i_n+1}\left(g - i_n + \frac{1}{2}\right) D_{(\vec{i}),2g+2}$$
$$+ (-1)^{i_n+1}\sum_{r=1}^{i_n}\left[\binom{g-i_n+r}{r+1} + \frac{1}{2}\binom{g-i_n+r}{r}\right] D_{(\vec{m},i_n-r),2g+2}$$

Now consider Γ_1. For all $\Gamma \in \Gamma_1$, we have

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2(g-1)+2)t,1u}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,2t,1u}(\mathcal{B}\mathbb{Z}_2)$$

Applying cohomology to the normalization long exact sequence, we get

$$0 \longrightarrow L_{-1} \oplus L_0 \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow H^1(C_{v_0}, \mathbb{C} \times C_{v_0}) \longrightarrow 0$$

Therefore, we have

$$\begin{aligned}
c_{i_n+1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma, t=1} &= c^{\mathbb{C}^*}_{i_n+1}(L_{-1} \oplus L_0 \oplus (\mathbb{E}^\vee_{g-1} \otimes L_{-1}))|_{t=1} \\
&= c^{\mathbb{C}^*}_{i_n+1}(L_{-1} \oplus (\mathbb{E}^\vee_{g-1} \otimes L_{-1}))|_{t=1} \\
&= c^{\mathbb{C}^*}_{i_n+1}(\mathbb{E}_{g-1} \otimes L_{-1})|_{t=1} - t c^{\mathbb{C}^*}_{i_n}(\mathbb{E}^\vee_{g-1} \otimes L_{-1})|_{t=1} \\
&= \sum_{r=0}^{i_n+1}\binom{g-1-i_n-1+r}{r} c_{i_n+1-r}(\mathbb{E}^\vee_{g-1})(-1)^r \\
&- \sum_{r=0}^{i_n}\binom{g-1-i_n+r}{r} c_{i_n-r}(\mathbb{E}^\vee_{g-1})(-1)^r \\
&= (-1)^{i_n+1} \\
&\times \left[\sum_{r=0}^{i_n+1}\binom{g-2-i_n+r}{r}\lambda_{i_n+1-r} + \sum_{r=0}^{i}\binom{g-1-i_n+r}{r}\lambda_{i_n-r}\right]
\end{aligned}$$

Since $\frac{1}{e(N_\Gamma)} = \left(\frac{-1}{t^2(t-\psi_{v_0})}\right) \times \left(\frac{1}{-t-\psi_{v_\infty}}\right)$, the contribution becomes

$$\binom{2g+1}{1}(-1)$$
$$\times \int_{\overline{\mathcal{M}}_{0,(2(g-1)+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{(-1)(-1)^{i_n+1}\left[\sum_{r=0}^{i_n+1}\binom{g-2-i_n+r}{r}\lambda_{\vec{m}}\lambda_{i_n+1-r}+\sum_{r=0}^{i_n}\binom{g-1-i_n+r}{r}\lambda_{\vec{m}}\lambda_{i_n-r}\right]}{1-\psi_{2g+1}}$$
$$\times \int_{\overline{\mathcal{M}}_{0,2t,1u}(\mathcal{B}\mathbb{Z}_2)} \frac{1}{-1-\psi_3}$$

After some simplification, we get:

$$\text{Cont}(\Gamma_1)|_{t=1} = (2g+1)(-1)^{i_n}\left[\sum_{r=1}^{i_n}\binom{g-i_n+r}{r+1}d_{(\vec{m},i_n-r),2(g-1)+2}\right]$$

Now consider Γ_{2g}. For all $\Gamma \in \Gamma_{2g}$, we have

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)$$

Applying cohomology to the normalization long exact sequence, we get

$$0 \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow H^1(C_e, f^*\mathcal{O}_{\mathbb{P}^1}) \oplus H^1(C_{v_\infty}, \mathbb{C}\times C_{v_\infty}) \longrightarrow 0$$

and therefore,

$$\begin{aligned}
c_{i_n+1}(R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma,t=1} &= c_{i_n+1}^{\mathbb{C}^*}(L_{\frac{-1}{2}} \oplus (\mathbb{E}_g^\vee \otimes L_0))|_{t=1} \\
&= c_{i_n+1}^{\mathbb{C}^*}(\mathbb{E}_g^\vee \otimes L_0)|_{t=1} - \frac{1}{2}tc_{i_n}^{\mathbb{C}^*}(\mathbb{E}_g^\vee \otimes L_0)|_{t=1} \\
&= \sum_{r=0}^{i_n+1}\binom{g-i_n-1+r}{r}c_{i_n+1-r}(\mathbb{E}_g^\vee)(0)^r \\
&- \frac{1}{2}\sum_{r=0}^{i_n}\binom{g-i_n+r}{r}c_{i_n-r}(\mathbb{E}_g^\vee)(0)^r \\
&= (-1)^{i_n+1}\left[\lambda_{i_n+1}+\frac{1}{2}\lambda_{i_n}\right]
\end{aligned}$$

Since $\frac{1}{e(N_\Gamma)} = \frac{-1}{t^2(t-\psi_{v_\infty}}$, it follows that

$$\mathrm{Cont}(\Gamma_{2g})|_{t=1} = \binom{2g+1}{2g}(-1)\int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} \frac{(-1)(-1)^{i_n+1}\left[\lambda_{\vec{m}}\lambda_{i_n+1} + \frac{1}{2}\lambda_{\vec{m}}\lambda_{i_n}\right]}{-1-\psi_1}$$

After some simplification, we have

$$\mathrm{Cont}(\Gamma_{2g})|_{t=1} = (2g+1)(-1)^{i_n}\left(\frac{1}{2}\right) D_{(\vec{i}),2g+2}$$

Now consider Γ_j, where $2 \leq j \leq 2g-2$, and j is even. First, we make the variable substitutions $g_1 := \frac{2g-j}{2}$ and $g_2 := \frac{j}{2}$, so that for $\Gamma \in \Gamma_j$, we have

$$\begin{aligned}\overline{\mathcal{M}}_\Gamma &= \overline{\mathcal{M}}_{0,(2g+2-j)t}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,(j+2)t}(\mathcal{B}\mathbb{Z}_2)\\ &= \overline{\mathcal{M}}_{0,(2g_1+2)t}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0,(2g_2+2)t}(\mathcal{B}\mathbb{Z}_2)\end{aligned}$$

Applying cohomology to the normalization long exact sequence, we get

$$0 \longrightarrow H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\longrightarrow H^1(C_e, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus H^1(C_{v_0}, \mathbb{C}\times C_{v_0}) \oplus H^1(C_{v_\infty}, \mathbb{C}\times C_{v_\infty}) \longrightarrow 0$$

Therefore, we have

$$c_{i_n+1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma, t=1} = c^{\mathbb{C}^*}_{i_n+1}(L_{\frac{-1}{2}} \oplus (\mathbb{E}^\vee_{g_1} \otimes L_{-1}) \oplus (\mathbb{E}^\vee_{g_2} \otimes L_0))|_{t=1}$$

After a bit of unraveling, the above becomes

$$(-1)^{i_n+1}$$
$$\times\left[\sum_{p+q=i_n+1}\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{p-r}\right)\times(\lambda_q)+\frac{1}{2}\sum_{p+q=i_n}\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{p-r}\right)\times(\lambda_q)\right]$$

Since $\frac{1}{e(N_\Gamma)}=(2)\left(\frac{-1}{t^2(t-\psi_{v_0})}\right)\times\left(\frac{1}{-t-\psi_{v_\infty}}\right)$,

$$\text{Cont}(\Gamma_j)|_{t=1}=$$
$$\binom{2g+1}{j}(-1)(2)\left[(-1)^{i_n+1}\sum_{\substack{\vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n+1}}\int_{\overline{\mathcal{M}}_{0,(2g_1+2)t}(\mathcal{B}\mathbb{Z}_2)}\frac{(-1)\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{\vec{\ell}_1}\lambda_{p-r}\right)}{1-\psi_1}\right.$$
$$\times\int_{\overline{\mathcal{M}}_{0,(2g_2+2)t}(\mathcal{B}\mathbb{Z}_2)}\frac{\lambda_{\vec{\ell}_2}\lambda_q}{-1-\psi_1}$$
$$\left.+\frac{1}{2}(-1)^{i_n+1}\sum_{\substack{\vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n}}\int_{\overline{\mathcal{M}}_{0,(2g_1+2)t}(\mathcal{B}\mathbb{Z}_2)}\frac{(-1)\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{\vec{\ell}_1}\lambda_{p-r}\right)}{1-\psi_1}\int_{\overline{\mathcal{M}}_{0,(2g_2+2)t}(\mathcal{B}\mathbb{Z}_2)}\frac{\lambda_{\vec{\ell}_2}\lambda_q}{-1-\psi_1}\right]$$

Since summing across all even j such that $2\leq j\leq 2g-2$ is equivalent to summing across all $g_1,g_2,g_i>0$ such that $g_1+g_2=g$, we have

$$\sum_{\substack{2\leq j\leq 2g-2\\ j\text{ even}}}\text{Cont}(\Gamma_j)|_{t=1}=$$
$$(2)(-1)^{i_n}\sum_{\substack{g_1+g_2=g,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n+1\\ 0\leq r\leq p}}(-1)^{2g_2-1-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2}\binom{g_1-p+r}{r}D_{(\vec{\ell}_1,p-r),2g_1+2}D_{(\vec{\ell}_2,q),2g_2+2}$$
$$+(-1)^{i_n}\sum_{\substack{g_1+g_2=g,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n\\ 0\leq r\leq p}}(-1)^{2g_2-1-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2}\binom{g_1-p+r}{r}D_{(\vec{\ell}_1,p-r),2g_1+2}D_{(\vec{\ell}_2,q),2g_2+2}$$

Lastly, consider Γ_j where $3 \leq j \leq 2g-3$, and j is odd. At this point, the reader should understand how these contributions are calculated, and therefore, we expedite the exposition by simply stating the results of the computations. For all $\Gamma \in \Gamma_j$, we have

$$c_{i_n+1}(R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^1}(-1))|_{\overline{\mathcal{M}}_\Gamma,t=1} = (-1)^{i_n+1}\left[\sum_{p+q=i_n+1}\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{p-r}\right)\times(\lambda_q)\right.$$
$$\left. + \sum_{p+q=i_n}\left(\sum_{r=0}^{p}\binom{g_1-p+r}{r}\lambda_{p-r}\right)\times(\lambda_q)\right]$$

and therefore,

$$\sum_{\substack{3\leq j\leq 2g-3\\ j\text{ odd}}}\text{Cont}(\Gamma_j)|_{t=1} =$$
$$(-1)^{i_n}(2)\sum_{\substack{g_1+g_2=g-1,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n+1\\ 0\leq r\leq p}}(-1)^{2g_2-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2+1}\binom{g_1-p+r}{r}d_{(\vec{\ell}_1,p-r),2g_1+2}d_{(\vec{\ell}_2,q),2g_2+2}$$
$$+(-1)^{i_n}(2)\sum_{\substack{g_1+g_2=g-1,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n\\ 0\leq r\leq p}}(-1)^{2g_2-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2+1}\binom{g_1-p+r}{r}d_{(\vec{\ell}_1,p-r),2g_1+2}d_{(\vec{\ell}_2,q),2g_2+2}$$

3.6 Proof of Theorem 5

Proof of Theorem 5. We have recursions for Hodge integrals that have at lease one ψ-class insertion (see Theorem 6). In order to obtain a complete set of recursions that determine all of the intersection numbers $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$, we need a recursion for Hodge integrals $D_{(\vec{i}),2g+2}$, where $|\vec{i}| = 2g-1$. In Section 3.5.2, we computed a vanishing localization computation that involved integrals $D_{(\vec{i}),2g+2}$, where $\vec{i} = (i_1, \ldots, i_n)$, $|\vec{i}| = 2g-1$, and without loss of generality, we assumed $i_n > 0$. It only remains to isolate the term $D_{(\vec{i}),2g+2}$, and check that the remaining terms involve either Hodge monomials of lower degree, or Hodge integrals of lower genus.

Since $I = 0$,

$$\left[\text{Cont}(\Gamma_0) + \text{Cont}(\Gamma_1) + \text{Cont}(\Gamma_{2g}) + \sum_{\substack{2 \le j \le 2g-2 \\ j \text{ even}}} \text{Cont}(\Gamma_j) + \sum_{\substack{3 \le j \le 2g-3 \\ j \text{ odd}}} \text{Cont}(\Gamma_j) \right] |_{t=1} = 0$$

The only terms that contain $D_{(\vec{i}),2g+2}$ are $\text{Cont}(\Gamma_0)$ and $\text{Cont}(\Gamma_{2g})$. and therefore,

$$\begin{aligned} &(-1)^{i_n+1} \left(g - i_n + \frac{1}{2} \right) D_{(\vec{i}),2g+2} + (2g+1)(-1)^{i_n} \left(\frac{1}{2} \right) D_{(\vec{i}),2g+2} \\ &+ (-1)^{i_n+1} \sum_{r=1}^{i_n} \left[\binom{g - i_n + r}{r+1} + \frac{1}{2} \binom{g - i_n + r}{r} \right] D_{(\vec{m}, i_n - r),2g+2} \\ &+ \left[\text{Cont}(\Gamma_1) + \sum_{\substack{2 \le j \le 2g-2 \\ j \text{ even}}} \text{Cont}(\Gamma_j) + \sum_{\substack{3 \le j \le 2g-3 \\ j \text{ odd}}} \text{Cont}(\Gamma_j) \right] |_{t=1} = 0 \end{aligned}$$

Since

$$(-1)^{i_n+1} \left(g - i_n + \frac{1}{2} \right) + (2g+1)(-1)^{i_n} \left(\frac{1}{2} \right) = i_n (-1)^{i_n} \neq 0$$

then we can isolate the term $D_{(\vec{i}),2g+2}$, to get

$$D_{(\vec{i}),2g+2} = \frac{1}{i_n(-1)^{i_n+1}} \left[(-1)^{i_n+1} \sum_{r=1}^{i_n} \left[\binom{g-i_n+r}{r+1} + \frac{1}{2}\binom{g-i_n+r}{r} \right] D_{(\vec{m},i_n-r),2g+2} \right.$$

$$\left. + \operatorname{Cont}(\Gamma_1)|_{t=1} + \sum_{\substack{2\leq j\leq 2g-2 \\ j \text{ even}}} \operatorname{Cont}(\Gamma_j)|_{t=1} + \sum_{\substack{3\leq j\leq 2g-3 \\ j \text{ odd}}} \operatorname{Cont}(\Gamma_j)|_{t=1} \right] \tag{3.18}$$

The Theorem now follows from the following Lemma:

Lemma 1. *All Hodge integrals on the right hand side of Equation* (3.18) *involve Hodge monomials* $\lambda_{\vec{v}}$ *such that* $|\vec{v}| < 2g - 1$.

Proof. This is simply a consequence of meticulously analyzing all of the terms on the right hand side of Equation (3.18), which we follow through with below.

The first summation of Hodge integrals on the right hand side of Equation (3.18) involves integrals of the form $D_{(\vec{m},i_n-r),2g+2}$, where $1 \leq r \leq i_n$. Therefore, $|(\vec{m}, i_n - r)| = 2g - 1 - r < 2g - 1$. The same argument holds for all Hodge integrals appearing in $\operatorname{Cont}(\Gamma_1)|_{t=1}$.

Now consider the Hodge integrals appearing in

$$\sum_{\substack{2\leq j\leq 2g-2 \\ j \text{ even}}} \operatorname{Cont}(\Gamma_j)|_{t=1}$$

Every term appearing in this sum is a product of Hodge integrals $D_{(\vec{v}_1),2g_1+2}D_{(\vec{v}_2),2g_2+2}$ where $g_1+g_2 = g$ and $g_i > 0$. In particular, this means $g_i \leq g-1$ Therefore, the Hodge integrals that occur are over moduli spaces of the form $\overline{\mathcal{M}}_{0,(2g_i+2)t}(\mathcal{B}\mathbb{Z}_2)$, but since $\dim(\overline{\mathcal{M}}_{0,(2g_i+2)t}(\mathcal{B}\mathbb{Z}_2) = 2g_i - 1 \leq 2(g-1) - 1 = 2g - 3$. this implies that the products of Hodge integrals appearing in this sum are non-zero if and only if the vectors $\vec{v}_i$ have the property that $|\vec{v}_i| \leq 2g - 3 < 2g - 1$, as desired. A completely analogous argument works for the Hodge integrals appearing in $\sum_{\substack{3\leq j\leq 2g-3 \\ j \text{ odd}}} \operatorname{Cont}(\Gamma_j)|_{t=1}$.

□

With Lemma 1 established, this concludes the proof.

□

Chapter 4

Closed form Expressions Involving Elementary Symmetric Functions

In this Chapter, we find a closed form expression for *linear* hyperelliptic Hodge integrals i.e. hyperelliptic Hodge integrals with one λ-class insertion,

$$D_{i,2g+2} := \int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} \psi_i^{2g-1-i}\lambda_i$$
$$d_{i,2g+2} := \int_{\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} \psi_{2g+3}^{2g-i}\lambda_i$$

Before we state the main result in this Chapter, we need the following definition:

Definition 18. *The i^{th} elementary symmetric function in n indeterminates $x_1, \dots, x_n$, denoted $e_i(x_1, \dots, x_n)$ is defined to be*

$$e_i(x_1, \dots, x_n) = \sum_{1 \le a_1 < a_2 < \dots < a_i \le n} x_{a_1} x_{a_2} \dots x_{a_i}$$

Example 5. *Here are the elementary symmetric functions on 4 indeterminates:*

$$e_1(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$$
$$e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$
$$e_3(x_1, x_2, x_3, x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$
$$e_4(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4$$

Here is the main result obtained in this Chapter:

Theorem 7. *The hyperelliptic Hodge integrals $D_{i,2g+2}$ and $d_{i,2g+2}$ have a closed form expression involving elementary symmetric functions evaluated on arithmetic progressions of integers. Specifically,*

$$D_{i,2g+2} = \left(\frac{1}{2}\right)^{i+1} e_i(1, 3, \ldots, 2g-1)$$
$$d_{i,2g+2} = \left(\frac{1}{2}\right)^{i+1} e_i(2, 4, \ldots, 2g)$$

In order to prove Theorem 7, we simply need to check that the purported expressions for $D_{i,2g+2}$ and $d_{i,2g+2}$ in Theorem 7 satisfy the recursions obtained in the previous Chapter. First, recall the recursions from Theorem 6. Setting the parameter k equal to 0 in Theorem 6, and setting the vector $\vec{i}$ equal to a vector of a single entry, which we denote by i, we obtain

$$\begin{aligned} D_{i,2g+2} = {} & 2 \sum_{\substack{g_1+g_2=g-1 \\ 0 \le g_2 \le g-1 \\ \ell \le i}} (-1)^\ell \binom{2g-1}{2g_2+1} d_{i-\ell,2g_1+2} d_{\ell,2g_2+2} \\ & - 2 \sum_{\substack{g_1+g_2=g \\ 1 \le g_2 \le g-1 \\ \ell \le i}} (-1)^\ell \binom{2g-1}{2g_2} D_{i-\ell,2g_1+2} D_{\ell,2g_2+2} \end{aligned} \tag{4.1}$$

$$\begin{aligned} d_{i,2g+2} = {} & 2 \sum_{\substack{g_1+g_2=g \\ 0 \le g_2 \le g-1 \\ \ell \le i}} (-1)^\ell \binom{2g}{2g_2+1} D_{i-\ell,2g_1+2} D_{\ell,2g_2+2} \\ & - 2 \sum_{\substack{g_1+g_2=g-1 \\ 0 \le g_2 \le g-1 \\ \ell \le i}} (-1)^\ell \binom{2g}{2g_2+2} d_{i-\ell,2g_1+2} d_{\ell,2g_2+2} \end{aligned} \tag{4.2}$$

In Equations (4.1) and (4.2), we make the following linear variable substitution:

$$k - 3 := 2g - 1$$

With this variable substitution, Equations (4.1) and (4.2) can be rewritten as

$$D_{i,k} = 2\sum_{j \text{ odd}} \binom{k-3}{j} \left(\sum_{\ell=0}^{i} (-1)^{\ell} d_{i-\ell,k-1-j} d_{\ell,j+1} \right) - 2\sum_{j \text{ even}} \binom{k-3}{j} \left(\sum_{\ell=0}^{i} (-1)^{\ell} D_{i-\ell,k-j} D_{\ell,j+2} \right) \tag{4.3}$$

$$d_{i,k} = 2\sum_{j \text{ odd}} \binom{k-2}{j} \left(\sum_{\ell=0}^{i} (-1)^{\ell} D_{i-\ell,k-j+1} D_{\ell,j+1} \right) - 2\sum_{j \text{ even}} \binom{k-2}{j} \left(\sum_{\ell=0}^{i} (-1)^{\ell} d_{i-\ell,k-j} d_{\ell,j} \right) \tag{4.4}$$

With the recursions in Equations (4.3) and (4.4), we are ready to prove our main theorem. First, we need some preliminary results.

Lemma 2. *For $p \leq n$, we have the following identity:*

$$\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k} k^p = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} k^p = 0$$

Proof. The proof of this identity can be found in [21] on pg. 30. □

Furthermore, we have a simple corollary to the above lemma

Corollary 2. *If $(m_1, \ldots, m_n) \in \mathbb{R}^n$, we have*

$$\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k} \prod_{i=1}^{n} (m_i - k) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \prod_{i=1}^{n} (m_i - k) = 0$$

Proof. We have

$$\begin{aligned}\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}\prod_{i=1}^{n}(m_i-k) &= \sum_{k=1}^{2n-1}(-1)^k\binom{2n-1}{k}\left(\sum_{i=0}^{n}e_i(m_1,\dots,m_n)k^{n-i}\right)\\ &= \sum_{i=0}^{n}e_i(m_1,\dots,m_n)\left(\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}k^{n-i}\right)\\ &= 0\end{aligned}$$

A similar argument is used to show the vanishing of the second expression.

□

We are now ready to state and prove our main theorem. We state the theorem in with respect the variable change $k-3 := 2g-1$.

Theorem 8. *The following equalities hold for $D_{i,k}$ and $d_{i,k}$:*

$$\begin{aligned}D_{i,k} &= \left(\frac{1}{2}\right)^{i+1} e_i(1,3,\dots,k-3)\\ d_{i,k} &= \left(\frac{1}{2}\right)^{i+1} e_i(2,4,\dots,k-2)\end{aligned}$$

Proof. In order to prove the theorem, we simply need to check that the purported expressions of $D_{i,k}$ and $d_{i,k}$ satisfy the recursions obtained in the previous chapter. When we plug in the purported expression for $D_{i,k}$ into the recursion obtained in Equation (4.3), we obtain

$$\begin{aligned}&\left(\frac{1}{2}\right)^{i+1} e_i(1,3,\dots,k-3)\\ &= 2\sum_{j\text{ odd}}\binom{k-3}{j}\left(\sum_{\ell=0}^{i}(-1)^\ell\left(\frac{1}{2}\right)^{i+2} e_{i-\ell}(2,4,\dots,k-3-j)e_\ell(2,4,\dots,j-1)\right)\\ &- 2\sum_{j\text{ even}}\binom{k-3}{j}\left(\sum_{\ell=0}^{i}(-1)^\ell\left(\frac{1}{2}\right)^{i+2} e_{i-\ell}(1,3,\dots,k-3-j)e_\ell(1,3,\dots,j-1)\right)\end{aligned}$$

Before we can proceed, we need a few standard combinatorial facts (see [22]). The elementary symmetric functions have a very nice description via their generating functions,

$$e_i(x_1, x_2, \ldots, x_n) = [t^i]\prod_{j=1}^{n}(1 + x_j t)$$

where by $[t^i]p(t)$, we mean the degree i coefficient of $p(t)$. Furthermore, recall that, if $f(t) = \sum_{i\geq 0} a_i t^i$ and $g(t) = \sum_{i\geq 0} b_i t^i$ are generating functions for sequences a_i and b_i, then $f(t)g(-t)$ is the generating function of the sequence c_i, where

$$c_i = \sum_{j=0}^{i}(-1)^j a_{i-j} b_j$$

Using the above facts, we see that

$$e_i(1, 3, \ldots, k-3) = [t^i]\cdot\prod_{n=1}^{\frac{k-2}{2}}(1 + (2n-1)t)$$

$$\sum_{\ell=0}^{i}(-1)^\ell e_{i-\ell}(2, 4, \ldots, k-3-j)e_\ell(2, 4, \ldots, j-1) = [t^i]\prod_{n=1}^{\frac{k-3-j}{2}}(1 + 2nt)\prod_{n=1}^{\frac{j-1}{2}}(1 - 2nt)$$

$$\sum_{\ell=0}^{i}(-1)^\ell e_{i-\ell}(1, 3, \ldots, k-3-j)e_\ell(1, 3, \ldots, j-1) = [t^i]\prod_{n=1}^{\frac{k-2-j}{2}}(1 + (2n-1)t)\prod_{n=1}^{\frac{j}{2}}(1 - (2n-1)t)$$

and therefore, after a bit of simplification, we see that the recursion is satisfied if and only if the following equality between polynomials holds:

$$\prod_{n=1}^{\frac{k-2}{2}}(1+(2n-1)t) \tag{4.5}$$

$$= \sum_{j \text{ odd}} \binom{k-3}{j} \prod_{n=1}^{\frac{k-3-j}{2}} (1+2nt) \prod_{n=1}^{\frac{j-1}{2}}(1-2nt)$$

$$- \sum_{j \text{ even}} \binom{k-3}{j} \prod_{n=1}^{\frac{k-2-j}{2}} (1+(2n-1)t) \prod_{n=1}^{\frac{j}{2}}(1-(2n-1)t)$$

where we define $\prod_{n=1}^{0}(1-2nt) := 1$. Having the foresight of eventually using the previous lemmas (as they are stated), we make the variable substitution $g := \frac{k-2}{2}$, so that Equation (4.5) becomes

$$\prod_{n=1}^{g}(1+(2n-1)t) = \sum_{j \text{ odd}} \binom{2g-1}{j} \prod_{n=1}^{\frac{2g-1-j}{2}} (1+2nt) \prod_{n=1}^{\frac{j-1}{2}}(1-2nt)$$

$$- \sum_{j \text{ even}} \binom{2g-1}{j} \prod_{n=1}^{\frac{2g-j}{2}} (1+(2n-1)t) \prod_{n=1}^{\frac{j}{2}}(1-(2n-1)t)$$

Notice that

$$j \text{ odd} \implies \prod_{n=1}^{\frac{2g-1-j}{2}} (1+2nt) \prod_{n=1}^{\frac{j-1}{2}}(1-2nt) = \prod_{n=1}^{g}(1+(2g-1-j-2(n-1))t)$$

$$j \text{ even} \implies \prod_{n=1}^{\frac{2g-j}{2}} (1+(2n-1)t) \prod_{n=1}^{\frac{j}{2}}(1-(2n-1)t) = \prod_{n=1}^{g}(1+(2g-1-j-2(n-1))t)$$

We see that the desired result follows if the following polynomial vanishes

$$P(t) := \sum_{j=0}^{2g-1}(-1)^j \binom{2g-1}{j} \prod_{n=1}^{g}(1+(2g-1-j-2(n-1))t)$$

Consider the following variable transformation for $P(t)$

$$\widehat{P}(t) := t^g P\left(\frac{1}{t}\right) = \sum_{j=0}^{2g-1}(-1)^j \binom{2g-1}{j} \prod_{n=1}^{g}((t+2g-1-2(n-1))-j)$$

For $1 \leq n \leq g$, define $m_n(t) := t + 2g - 1 - 2(n-1)$, so that $\widehat{P}(t)$ becomes

$$\widehat{P}(t) = \sum_{j=1}^{2g-1}(-1)^j \binom{2g-1}{j} \prod_{n=1}^{g}(m_n(t) - j)$$

By direct application of Corollary 2,

$$\widehat{P}(1) = \widehat{P}(2) = \ldots = \widehat{P}(g+1) = 0$$

Therefore, $\widehat{P}(t)$ has $g+1$ distinct roots. But since the degree of $\widehat{P}(t)$ is g, it follows that $\widehat{P}(t) = 0$, and therefore, $P(t) = 0$. This shows that

$$D_{i,k} = \left(\frac{1}{2}\right)^{i+1} e_i(1, 3, \ldots, k-3)$$

In order to prove that $d_{i,k} = \left(\frac{1}{2}\right)^{i+1} e_i(2, 4, \ldots, k-2)$, we plug in this expression into the recursion in Equation (4.4), and run through the same calculations as in the case of $D_{i,k}$. The desired result then follows after showing the vanishing of the polynomial

$$Q(t) := \sum_{j=0}^{2g}(-1)^j \binom{2g}{j} \prod_{n=1}^{g}(1 + (2g+1-j-2(n-1))t)$$

We will follow the same strategy used to show the vanishing of $P(t)$. Consider the following variable transformation for $Q(t)$,

$$\widehat{Q}(t) := t^g Q\left(\frac{1}{t}\right) = \sum_{j=0}^{2g}(-1)^j \binom{2g}{j} \prod_{j=1}^{g}(t+2g-1-2(n-1)-j)$$

For $1 \leq n \leq g$, define $r_n(t) := t + 2g - 1 - 2(n-1)$, so that $\widehat{Q}(t)$ becomes

$$\widehat{Q}(t) = \sum_{j=0}^{2g}(-1)^j \binom{2g}{j} \prod_{n=1}^{g}(r_n(t) - j)$$

By direct application of Corollary 2,

$$\widehat{Q}(1) = \widehat{Q}(2) = \ldots = \widehat{Q}(g+1) = 0$$

Therefore, $\widehat{Q}(t)$ has $g+1$ distinct roots. But since the degree of $\widehat{Q}(t)$ is g, it follows that $\widehat{Q}(t) = 0$, and therefore, $Q(t) = 0$. This shows that

$$d_{i,k} = \left(\frac{1}{2}\right)^{i+1} e_i(2, 4, \ldots, k-2)$$

□

Chapter 5

Integrality and Polynomiality of Hyperelliptic Hodge Integrals

This chapter is devoted proving the following two theorems:

Theorem 9. *The hyperelliptic Hodge integrals* $2^{|\vec{i}|+1}D_{\vec{i},2g+2}$ *and* $2^{|\vec{i}|+1}d_{\vec{i},2g+2}$ *are integral.*

Theorem 10. *The hyperelliptic Hodge integrals* $D_{\vec{i},2g+2}$ *and* $d_{\vec{i},2g+2}$ *are polynomials in* g*. These polynomials have degree less than or equal to* $|\vec{i}|^2 + 1$.

5.1 Integrality

Proof of Theorem 9. This proof requires simultaneous induction on $|\vec{i}|$ and g. However, since the recursions in Theorem 6 are a two-step recursion, and Equation (3.18) involves many terms, our argument has many moving parts. To remedy this, let us summarize the argument:

1. We induct on $|\vec{i}|$
2. There are two cases to consider, either $|\vec{i}|$ is odd, or $|\vec{i}|$ is even.
3. In the case that $|\vec{i}|$ is odd, this means $D_{(\vec{i}),2g_0+2}$ is a pure Hodge integrals for some g_0. Using Equation (3.18), we confirm the result holds in this base case, and then induct on g. Then we use the recursion obtained in Theorem 6 to confirm that the result holds for the base case $d_{(\vec{i}),2g_0+2}$, and then induct on g
4. In the case that $|\vec{i}|$ is even, this means $D_{(\vec{i}),2g+2}$ is not a pure Hodge integral for any g, so we can solely refer to the recursions in Theorem 6. As in Step 3, we verify the result for base cases i.e. the smallest genus g_0 for which $D_{(\vec{i}),2g_0+2}$ and $d_{(\vec{i}),2g_0+2}$ are non-zero, and then induct on g

Fix $\vec{i}$, and suppose that the integrality result holds for all vectors $\vec{v}$ where $|\vec{v}| < |\vec{i}|$.

Case 1: $|\vec{i}|$ is odd.

Since $|\vec{i}| = i_1 + i_2 + \dots i_n$ is odd, this means there exists a g_0 such that $D_{(\vec{i}),2g_0+2}$ is a pure Hodge integral. By definition, g_0 is the smallest genus g for which $D_{(\vec{i}),2g+2}$ is non-zero. Similarly, g_0 is the smallest genus g for which $d_{(\vec{i}),2g+2}$ is non-zero. Using Equation (3.18), and multiplying through by $2^{|\vec{i}|+1} i_n (-1)^{i_n+1}$, we have

$$
\begin{aligned}
2^{|\vec{i}|+1} i_n (-1)^{i_n+1} D_{(\vec{i}),2g_0+2} = (-1)^{i_n+1} 2^{|\vec{i}|+1} \sum_{r=1}^{i_n} \left[\binom{g_0 - i_n + r}{r+1} + \frac{1}{2} \binom{g_0 - i_n + r}{r} \right] D_{(\vec{m}, i_n - r), 2g_0+2} \\
+ 2^{|\vec{i}|+1} \mathrm{Cont}(\Gamma_1)|_{t=1} + 2^{|\vec{i}|+1} \sum_{\substack{2 \le j \le 2g_0 - 2 \\ j \text{ even}}} \mathrm{Cont}(\Gamma_j)|_{t=1} \\
+ 2^{|\vec{i}|+1} \sum_{\substack{3 \le j \le 2g_0 - 3 \\ j \text{ odd}}} \mathrm{Cont}(\Gamma_j)|_{t=1}
\end{aligned}
\tag{5.1}
$$

Consider the first term on the right hand side of Equation (5.1). It simplifies to

$$
\begin{aligned}
&(-1)^{i_n+1} \left[2^{|\vec{i}|+1} \sum_{r=1}^{i_n} \binom{g_0 - i_n + r}{r+1} D_{(\vec{m}, i_n - r), 2g_0+2} + 2^{|\vec{i}|} \sum_{r=1}^{i_n} \binom{g_0 - i_n + r}{r} D_{(\vec{m}, i_n - r), 2g_0+2} \right] \\
&= (-1)^{i_n+1} \left[\sum_{r=1}^{i_n} \binom{g_0 - i_n + r}{r+1} 2^r 2^{|\vec{m}| + i_n - r + 1} D_{(\vec{m}, i_n - r), 2g_0+2} \right. \\
&\left. + \sum_{r=1}^{i_n} \binom{g_0 - i_n + r}{r} 2^{r-1} 2^{|\vec{m}| + i_n - r + 1} D_{(\vec{m}, i_n - r), 2g_0+2} \right]
\end{aligned}
$$

which is a sum of integers by Lemma 1 and the induction hypothesis, and therefore, an integer.

Now consider the term $2^{|\vec{i}|+1} \mathrm{Cont}(\Gamma_1)|_{t=1}$. We have

$$2^{|\vec{i}|+1}\mathrm{Cont}(\Gamma_1)|_{t=1} = (2g_0+1)(-1)^{i_n}\left[\sum_{r=1}^{i_n}\binom{g_0-i_n+r}{r+1}2^r 2^{|\vec{m}|+i_n-r+1}d_{(\vec{m},i_n-r),2(g_0-1)+2}\right]$$

which is a sum of integers by Lemma 1 and the induction hypothesis, and therefore, an integer.

The term $2^{|\vec{i}|+1}\sum_{\substack{2\le j\le 2g_0-2\\ j \text{ even}}}\mathrm{Cont}(\Gamma_j)|_{t=1}$ is equal to

$$\begin{aligned}
&(-1)^{i_n}\sum_{\substack{g_1+g_2=g,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n+1\\ 0\le r\le p}}(-1)^{2g_2-1-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2}\binom{g_1-p+r}{r}\\
&\times 2^{|\vec{i}|-|\vec{\ell}_1|-p+r-|\vec{\ell}_2|-q}2^{|\vec{\ell}_1|+p-r+1}D_{(\vec{\ell}_1,p-r),2g_1+2}2^{|\vec{\ell}_2|+q+1}D_{(\vec{\ell}_2,q),2g_2+2}\\
&+(-1)^{i_n}\sum_{\substack{g_1+g_2=g,\ g_i>0\\ \vec{\ell}_1+\vec{\ell}_2=\vec{m}\\ p+q=i_n\\ 0\le r\le p}}(-1)^{2g_2-1-q-|\vec{\ell}_2|}\binom{2g+1}{2g_2}\binom{g_1-p+r}{r}\\
&\times 2^{|\vec{i}|-|\vec{\ell}_1|-p+r-|\vec{\ell}_2|-q-1}2^{|\vec{\ell}_1|+p-r+1}D_{(\vec{\ell}_1,p-r),2g_1+2}2^{|\vec{\ell}_2|+q+1}D_{(\vec{\ell}_2,q),2g_2+2}
\end{aligned} \tag{5.2}$$

The integrality of the first summation in Equation (5.2) follows if $|\vec{i}|-|\vec{\ell}_1|-p+r-|\vec{\ell}_2|-q\ge 0$. Indeed,

$$\begin{aligned}
|\vec{i}|-|\vec{\ell}_1|-p+r-|\vec{\ell}_2|-q &= |\vec{m}|+i_n-(|\vec{\ell}_1|+|\vec{\ell}_2|)-(p+q)+r\\
&= |\vec{m}|-|\vec{m}|+i_n-(i_n+1)+r\\
&= r-1
\end{aligned}$$

By Lemma 1, we have

$$
\begin{aligned}
&(|\vec{\ell}_1| + p - r) + (|\vec{\ell}_2| + q) < 2|\vec{i}| \\
\implies &|\vec{m}| + (p+q) - r < 2|\vec{i}| \\
\implies &|\vec{m}| + i_n + 1 - r < 2(|\vec{m}| + i_n) = 2|\vec{m}| + 2i_n \\
\implies &1 - r < |\vec{m}| + i_n \\
\implies &r - 1 > |\vec{i}| \geq 0
\end{aligned}
$$

as desired. The integrality of the second summation in Equation (5.2) follows if $|\vec{i}| - |\vec{\ell}_1| - p + r - |\vec{\ell}_2| - q - 1 \geq 0$. Indeed,

$$
\begin{aligned}
|\vec{i}| - |\vec{\ell}_1| - p + r - |\vec{\ell}_2| - q - 1 &= |\vec{m}| + i_n - (|\vec{\ell}_1| + |\vec{\ell}_2|) - (p+q) + (r-1) \\
&= |\vec{m}| - |\vec{m}| + i_n - i_n + r - 1 \\
&= r - 1 \\
&\geq 0
\end{aligned}
$$

as desired.

When confirming the integrality of the term

$$
2^{|\vec{i}|+1} \sum_{\substack{3 \leq j \leq 2g_0 - 3 \\ j \text{ odd}}} \text{Cont}(\Gamma_j)|_{t=1}
$$

one uses similar/analagous calculations as the ones used in the analysis of Equation (5.2). Therefore, we conclude that $2^{|\vec{i}|+1} D_{(\vec{i}),2g_0+2}$ is an integer. The remaining base case is the integrality of $d_{(\vec{i}),2g_0+2}$. Using Theorem 6, specializing to the parameter $k = 0$, and multiplying through by $2^{|\vec{i}|+1}$, we obtain

$$2^{|\vec{i}|+1}d_{(\vec{i}),2g_0+2} = \sum_{\substack{g_1+g_2=g_0 \\ 0\le g_2\le g-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2g_0}{2g_2+1} 2^{|\vec{i}|-|\vec{\ell}|+1} D_{(\vec{i}-\vec{\ell}),2g_1+2}(2)^{|\vec{\ell}|+1} D_{(\vec{\ell}),2g_2+2} \qquad (5.3)$$

$$+ \sum_{\substack{g_1+g_2=g_0-1 \\ 0\le g_2\le g-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2g_0}{2g_2+2}(2)^{|\vec{i}|-|\vec{\ell}|+1}(2)^{|\vec{\ell}|+1} d_{(\vec{i}-\vec{\ell}),2g_1+2} d_{(\vec{\ell}),2g_2+2}$$

The summations on the right hand side of Equation (5.3) are all integers by the induction hypothesis, except for the cases when $\vec{\ell} = \vec{0}$ and $\vec{\ell} = \vec{i}$. However, the integrals corresponding to these terms are zero for dimension reasons. Therefore, we conclude that $d_{(\vec{i}),2g_0+2}$ is an integer.

With the base cases established, we now induct on g. Suppose that the integrality results holds for $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ for all $g < \widetilde{g}$, where $\widetilde{g} > g_0$. Notice that $D_{(\vec{i}),2\widetilde{g}+2}$ is not a pure-Hodge integral, which means we can use Theorem 6 to calculate it. Specializing to the parameter $k = 0$ in Theorem 6, we have

$$2^{|\vec{i}|+1}D_{\vec{i},2\widetilde{g}+2} = \sum_{\substack{g_1+g_2=\widetilde{g}-1 \\ 0\le g_2\le \widetilde{g}-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2\widetilde{g}-1}{2g_2+1}(2)^{|\vec{i}|-|\vec{\ell}|+1} d_{(\vec{i}-\vec{\ell}),2g_1+2}(2)^{|\vec{\ell}|+1} d_{(\vec{\ell}),2g_2+2} \qquad (5.4)$$

$$- \sum_{\substack{g_1+g_2=\widetilde{g} \\ 1\le g_2\le \widetilde{g}-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2\widetilde{g}-1}{2g_2}(2)^{|\vec{i}|-|\vec{\ell}|+1} D_{(\vec{i}-\vec{\ell}),2g_1+2}(2)^{|\vec{\ell}|+1} D_{\vec{\ell},2g_2+2}$$

$$2^{|\vec{i}|+1}d_{\vec{i},2\widetilde{g}+2} = \sum_{\substack{g_1+g_2=\widetilde{g} \\ 0\le g_2\le \widetilde{g}-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2\widetilde{g}}{2g_2+1}(2)^{|\vec{i}|-|\vec{\ell}|+1} D_{(\vec{i}-\vec{\ell}),2g_1+2}(2)^{|\vec{\ell}|+1} D_{\vec{\ell},2g_2+2} \qquad (5.5)$$

$$\sum_{\substack{g_1+g_2=\widetilde{g}-1 \\ 0\le g_2\le \widetilde{g}-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2\widetilde{g}}{2g_2+2}(2)^{|\vec{i}|-|\vec{\ell}|+1} d_{(\vec{i}-\vec{\ell}),2g_1+2}(2)^{|\vec{\ell}|+1} d_{\vec{\ell},2g_2+2}$$

The right hand sides of both equations above are all integers, either by the induction hypothesis on $|\vec{i}|$, or the induction hypothesis on $g < \widetilde{g}$.

This concludes the case when $|\vec{i}|$ is odd.

Case 2: $|\vec{i}|$ is even.

As before, define g_0 as the smallest genus g for which $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ are non-zero. Since $|\vec{i}|$ is even, this means $D_{(\vec{i}),2g+2}$ is not a pure Hodge integral for all g. Therefore, we can solely refer to Theorem 6 for its computation.

We again specialize to the parameter $k = 0$ in Theorem 6. To compute $(2)^{|\vec{i}|+1} D_{(\vec{i}),2g_0+2}$, we have

$$
\begin{aligned}
2^{|\vec{i}|+1} D_{\vec{i},2g_0+2} = & \sum_{\substack{g_1+g_2=g_0-1 \\ 0\le g_2\le g_0-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g_0-1}{2g_2+1} (2)^{|\vec{i}|-|\vec{\ell}|+1} d_{(\vec{i}-\vec{\ell}),2g_1+2} (2)^{|\vec{\ell}|+1} d_{(\vec{\ell}),2g_2+2} \\
& - \sum_{\substack{g_1+g_2=g_0 \\ 1\le g_2\le g_0-1 \\ \vec{\ell}\le\vec{i}}} (-1)^{|\vec{\ell}|} \binom{2g_0-1}{2g_2} (2)^{|\vec{i}|-|\vec{\ell}|+1} D_{(\vec{i}-\vec{\ell}),2g_1+2} (2)^{|\vec{\ell}|+1} D_{\vec{\ell},2g_2+2}
\end{aligned}
$$

The sums above involve summation over vectors $\vec{\ell}$ such that $\vec{\ell} \le \vec{i}$. However, we can refine the inequality, and turn the above into sums over $\vec{\ell}$ such that $\vec{0} < \vec{\ell} < \vec{i}$. This follows for dimension reasons: when $\vec{\ell} = \vec{0}$ or $\vec{\ell} = \vec{i}$, the resulting terms vanish for dimension reasons. Since the summation is refined to $\vec{0} < \vec{\ell} < \vec{i}$, the integrality result holds for $D_{\vec{i},2g_0+2}$ by the induction hypothesis on $|\vec{i}|$. An analogous argument goes through when we compute $2^{|\vec{i}|+1} d_{(\vec{i}),2g_0+2}$. This establishes the base cases. When we induct on g, one can use the same argument/computations used in the analysis of Equation (5.4) and Equation (5.5).

This concludes the proof of Theorem 9

□

5.2 Polynomiality

Before we begin the proof of Theorem 10, we establish a few useful lemmas along the way. Using the theory of integer valued polynomials ([23]), we have the following lemma:

Lemma 3. *Suppose a_g is an integer valued polynomial in g of degree at most i. Then there exists integers $c_0, \dots c_i$ such that*

$$a_g = \sum_{k=0}^{i} c_k \binom{g}{k}$$

□

Lemma 4. *The exponential generating function of the binomial coefficient $\binom{g}{k}$ is*

$$f(t) := \sum_{g \geq k} \binom{g}{k} \frac{t^g}{g!} = \frac{t^k}{k!} e^t$$

Proof. This follows from direct computation:

$$\begin{aligned}
\frac{t^k}{k!} e^t &= \frac{t^k}{k!} \sum_{g \geq 0} \frac{t^g}{g!} \\
&= \sum_{g \geq 0} \frac{t^{g+k}}{k! g!} \\
&= \sum_{g \geq k} \frac{t^g}{k!(g-k)!} \\
&= \sum_{g \geq k} \frac{g!}{g!} \frac{t^g}{k!(g-k)!} \\
&= \sum_{g \geq k} \binom{g}{k} \frac{t^g}{g!}
\end{aligned}$$

□

Lemma 5. *Let $0 \le k \le n$. We have the following combinatorial identity,*

$$\sum_{\ell+m=2(n-k)} (-1)^{\ell} \binom{n}{\ell} \binom{n}{m} = (-1)^k \binom{n}{k}$$

Proof. We use the notation $[x^i]p(x)$ to mean the degree i coefficient of the polynomial $p(x)$. The Lemma follows from direct computation, as shown below,

$$\begin{aligned}
\sum_{\ell+m=2(n-k)} (-1)^{\ell} \binom{n}{\ell} \binom{n}{m} &= [x^{2(n-k)}](x-1)^n(x+1)^n \\
&= [x^{2(n-k)}](x^2-1)^n \\
&= [x^{2(n-k)}] \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} (x^2)^{\ell} \\
&= [x^{2(n-k)}] \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} (x^{2\ell}) \\
&= (-1)^{n-(n-k)} \binom{n}{n-k} \\
&= (-1)^k \binom{n}{k}
\end{aligned}$$

□

Lemma 6. *Let a_g be an integer valued sequence, and let $f(t) = \sum_{g \ge 0} a_g \frac{t^g}{g!}$ be its exponential generating function. Then a_g is a polynomial in g of degree at most i if and only if there exists a polynomial $p(t)$ of degree at most i such that $f(t) = p(t)e^t$.*

Proof. We first prove the forward direction. Suppose that a_g is a polynomial in g of degree at most i. By Lemma 3, there exists constant $c_0, \dots c_i$ such that

$$a_g = \sum_{k=0}^{i} c_k \binom{g}{k}$$

and therefore, the exponential generating function of a_g is

$$
\begin{aligned}
f(t) &:= \sum_{g\geq 0} a_g \frac{t^g}{g!} \\
&= \sum_{g\geq 0}\left(\sum_{k=0}^{i} c_k \binom{g}{k}\right)\frac{t^g}{g!} \\
&= \sum_{k=0}^{i} c_k \left(\sum_{g\geq 0}\binom{g}{k}\frac{t^g}{g!}\right) \\
\text{(by Lemma 4)} \qquad &= \sum_{k=0}^{i} c_k \frac{t^k}{k!} e^t \\
&= \left(\sum_{k=0}^{i} c_k \frac{t^k}{k!}\right) e^t
\end{aligned}
$$

Since $\sum_{k=0}^{i} c_k \frac{t^k}{k!}$ is a polynomial of degree at most i, this proves the forward direction.

Now we prove the converse. Suppose there exists a polynomial $p(t)$ of degree at most i such that $f(t) := \sum_{g\geq 0} a_g \frac{t^g}{g!} = p(t)e^t$. If we express $p(t)$ as $p(t) = \sum_{n=0}^{i} b_n t^n$, then

$$
\begin{aligned}
f(t) = p(t)e^t = \left(\sum_{n=0}^{i} b_n t^n\right)\left(\sum_{g\geq 0}\frac{t^g}{g!}\right) &= \sum_{n=0}^{i}\left(\sum_{g\geq 0} b_n \frac{t^{g+n}}{g!}\right) \\
&= \sum_{n=0}^{i}\left(\sum_{g\geq 0} b_n (g+n)(g+(n-1))\dots(g+1)\frac{t^{g+n}}{(g+n)!}\right) \\
&= \sum_{n=0}^{i}\left(\sum_{g\geq n} b_n (g)(g-1)\dots(g-(n-1))\frac{t^{g}}{g!}\right) \\
&= \sum_{n=0}^{i}\left(\sum_{g\geq n} b_n n!\binom{g}{n}\frac{t^{g}}{g!}\right) \\
\left(\text{since } \binom{g}{n} = 0 \text{ for } g<n\right) \quad &= \sum_{n=0}^{i}\left(\sum_{g\geq 0} b_n n!\binom{g}{n}\frac{t^{g}}{g!}\right) = \sum_{g\geq 0}\left(\sum_{n=0}^{i} b_n n!\binom{g}{n}\right)\frac{t^g}{g!}
\end{aligned}
$$

Since $\sum_{n=0}^{i} b_n n! \binom{g}{n}$ is a polynomial in g of degree at most i, the Lemma follows. $\square$

Proof of Theorem 10. Throughout, we define

$$F_{\vec{i}}(t) := \sum_{g\geq 0} D_{(\vec{i}),2g+2} \frac{t^g}{g!}$$
$$G_{\vec{i}}(t) := \sum_{g\geq 0} d_{(\vec{i}),2g+2} \frac{t^g}{g!}$$

The proof proceeds as follows:

1. We consider the recursions obtained in Theorem 6. For the recursion in which $D_{(\vec{i}),2g+2}$ is the principal part, we specialize to the parameter $k = 2g - 2 - |\vec{i}|$, and for the recursion in which $d_{(\vec{i}),2g+2}$ is the principal part, we specialize to the parameter $k = 2g - 1 - |\vec{i}|$.

2. We translate the recursions into systems of ordinary differential equations for $F_{\vec{i}}$ and $G_{\vec{i}}$

3. We use induction on $|\vec{i}|$, and in the Laplace space, we show that $\mathcal{L}\{F_{\vec{i}}\}$ and $\mathcal{L}\{G_{\vec{i}}\}$ satisfy the desired result.

When specializing to the parameter $k = 2g - 2 - |\vec{i}|$ in Equation (3.16), and to the parameter $k = 2g - 1 - |\vec{i}|$ in Equation (3.17), we have

$$\begin{aligned}
D_{(\vec{i}),2g+2} &= \sum_{g_2=0}^{\left\lfloor \frac{|\vec{i}|}{2} \right\rfloor} \binom{|\vec{i}|+1}{2g_2+1} d_{(\vec{i}),2(g-(g_2+1))+2} - \sum_{g_2=1}^{\left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor} \binom{|\vec{i}|+1}{2g_2} D_{(\vec{i}),2(g-g_2)+2} \\
&+ 2 \sum_{g_2=0}^{\left\lfloor \frac{|\vec{i}|}{2} \right\rfloor} \sum_{0<\vec{\ell}\leq\vec{i}} (-1)^{|\vec{\ell}|} \binom{|\vec{i}|+1}{2g_2+1} d_{(\vec{i}-\vec{\ell}),2(g-(g_2+1))+2} d_{(\vec{\ell}),2g_2+2} \\
&- 2 \sum_{g_2=1}^{\left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor} \sum_{0<\vec{\ell}\leq\vec{i}} (-1)^{|\vec{\ell}|} \binom{|\vec{i}|+1}{2g_2} D_{(\vec{i}-\vec{\ell}),2(g-g_2)+2} D_{(\vec{\ell}),2g_2+2}
\end{aligned}$$

$$d_{(\vec{i}),2g+2} = \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor} \binom{|\vec{i}|+1}{2g_2+1} D_{(\vec{i}),2(g-g_2)+2} - \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-1} \binom{|\vec{i}|+1}{2g_2+2} d_{(\vec{i}),2(g-(g_2+1))}$$
$$+ 2\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor} \sum_{0<\vec{\ell}\leq i} (-1)^{|\vec{\ell}|} \binom{|\vec{i}|+1}{2g_2+1} D_{(\vec{i}-\vec{\ell}),2(g-g_2)+2} D_{(\vec{\ell}),2g_2+2}$$
$$- 2\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-1} \sum_{0<\vec{\ell}\leq\vec{i}} (-1)^{|\vec{\ell}|} \binom{|\vec{i}|+1}{2g_2+2} d_{(\vec{i}-\vec{\ell}),2(g-(g_2+1))} d_{(\vec{\ell}),2g_2+2}$$

Suppose that the polynomiality result holds for all vectors $\vec{v}$ such that $|\vec{v}| < |\vec{i}|$, for some vector $\vec{i}$. Translating the above recursions into a system of ordinary differential equations for $F_{\vec{i}}$ and $G_{\vec{i}}$, and applying Lemma 6, there exist polynomials $P_{\vec{i}}(t)$ and $Q_{\vec{i}}(t)$ of degree at most $(|\vec{i}|-1)^2+1$ such that

$$\partial_t^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor+1} F_{\vec{i}} = \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor} \binom{|\vec{i}|+1}{2g_2+1} \partial_t^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor-g_2} G_{\vec{i}} - \sum_{g_2=1}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor} \binom{|\vec{i}|+1}{2g_2} \partial_t^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor+1-g_2} F_{\vec{i}} + P_{\vec{i}}(t)e^t$$

$$\partial_t^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor} G_{\vec{i}} = \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor} \binom{|\vec{i}|+1}{2g_2+1} \partial_t^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-g_2} F_{\vec{i}} - \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-1} \binom{|\vec{i}|+1}{2g_2+2} \partial_t^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-(g_2+1)} G_{\vec{i}} + Q_{\vec{i}}(t)e^t$$

Denote by $\mathcal{L}$ the Laplace transform, and define $\widetilde{F}_{\vec{i}}(s) := \mathcal{L}\{F_{\vec{i}}(t)\}(s)$, and $\widetilde{G}_{\vec{i}}(s) := \mathcal{L}\{G_{\vec{i}}(t)\}(s)$. Recall that $\mathcal{L}\{t^n e^t\}(s) = \frac{n!}{(s-1)^{n+1}}$. Using this fact, when we take the Laplace transform of the above and combine like terms, we see that there exist constants a_k, b_k, for $0 \leq k \leq (|\vec{i}|-1)^2+1$, such that

$$\left(\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2}s^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor+1-g_2}\right)\widetilde{F}_{\vec{i}}(s)=\left(\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2+1}s^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor-g_2}\right)\widetilde{G}_{\vec{i}}(s)+\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{a_k}{(s-1)^{k+1}}$$

$$\left(\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2}s^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-g_2}\right)\widetilde{G}_{\vec{i}}(s)=\left(\sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2+1}s^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-g_2}\right)\widetilde{F}_{\vec{i}}(s)+\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{b_k}{(s-1)^{k+1}}$$

To ease notation, we make the following definitions:

$$\begin{aligned}
A_{\vec{i},1} &:= \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2}s^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor+1-g_2}\\
A_{\vec{i},2} &:= \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2+1}s^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor-g_2}\\
A_{\vec{i},3} &:= \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2}s^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-g_2}\\
A_{\vec{i},4} &:= \sum_{g_2=0}^{\left\lfloor\frac{|\vec{i}|}{2}\right\rfloor}\binom{|\vec{i}|+1}{2g_2+1}s^{\left\lfloor\frac{|\vec{i}|+1}{2}\right\rfloor-g_2}
\end{aligned}$$

After some simplification, we have

$$\begin{aligned}
(A_{\vec{i},1}A_{\vec{i},3}-A_{\vec{i},2}A_{\vec{i},4})\widetilde{F}_{\vec{i}}(s) &= A_{\vec{i},2}\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{b_k}{(s-1)^{k+1}}+A_{\vec{i},3}\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{a_k}{(s-1)^{k+1}}\\
(A_{\vec{i},1}A_{\vec{i},3}-A_{\vec{i},2}A_{\vec{i},4})\widetilde{G}_{\vec{i}}(s) &= A_{\vec{i},4}\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{a_k}{(s-1)^{k+1}}+A_{\vec{i},1}\sum_{k=0}^{(|\vec{i}|-1)^2+1}\frac{b_k}{(s-1)^{k+1}}
\end{aligned}$$

Case 1: $|\vec{i}|$ is odd

In this case, $\left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor = \frac{|\vec{i}|+1}{2}$ and $\left\lfloor \frac{|\vec{i}|}{2} \right\rfloor = \frac{|\vec{i}|-1}{2}$, so

$$\begin{aligned}
A_{\vec{i},1}A_{\vec{i},3} - A_{\vec{i},2}A_{\vec{i},4} &= \left(\sum_{g_2=0}^{\frac{|\vec{i}|+1}{2}} \binom{|\vec{i}|+1}{2g_2} s^{\frac{|\vec{i}|+1}{2}-g_2} \right)^2 - \left(\sum_{g_2=0}^{\frac{|\vec{i}|-1}{2}} \binom{|\vec{i}|+1}{2g_2+1} s^{\frac{|\vec{i}|-1}{2}-g_2} \right) \\
&\times \left(\sum_{g_2=0}^{\frac{|\vec{i}|-1}{2}} \binom{|\vec{i}|+1}{2g_2+1} s^{\frac{|\vec{i}|+1}{2}-g_2} \right) \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=|\vec{i}|+1-k} \binom{|\vec{i}|+1}{2\ell} \binom{|\vec{i}|+1}{2m} \right) s^k \\
&- \sum_{k=0}^{|\vec{i}|} \left(\sum_{\ell+m=|\vec{i}|-k} \binom{|\vec{i}|+1}{2\ell+1} \binom{|\vec{i}|+1}{2m+1} \right) s^k \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=|\vec{i}|+1-k} \binom{|\vec{i}|+1}{2\ell} \binom{|\vec{i}|+1}{2m} \right. \\
&\left. - \sum_{\ell+m=|\vec{i}|-k} \binom{|\vec{i}|+1}{2\ell+1} \binom{|\vec{i}|+1}{2m+1} \right) s^k \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=2(|\vec{i}|+1-k)} (-1)^\ell \binom{|\vec{i}|+1}{\ell} \binom{|\vec{i}|+1}{m} \right) s^k \\
\text{(by Lemma 5)} &= \sum_{k=0}^{|\vec{i}|+1} (-1)^k \binom{|\vec{i}|+1}{k} s^k \\
&= (s-1)^{|\vec{i}|+1}
\end{aligned}$$

Case 2: $|\vec{i}|$ is even

In this case, $\left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor = \left\lfloor \frac{|\vec{i}|}{2} \right\rfloor = \frac{|\vec{i}|}{2}$, so

$$\begin{aligned}
A_{\vec{i},1}A_{\vec{i},3} - A_{\vec{i},2}A_{\vec{i},4} &= \left(\sum_{g_2=0}^{\frac{|\vec{i}|}{2}} \binom{|\vec{i}+1}{2g_2} s^{\frac{|\vec{i}|}{2}+1-g_2}\right)\left(\sum_{g_2=0}^{\frac{|\vec{i}|}{2}} \binom{|\vec{i}|+1}{2g_2} s^{\frac{|\vec{i}|}{2}-g_2}\right) \\
&- \left(\sum_{g_2=0}^{\frac{|\vec{i}|}{2}} \binom{|\vec{i}|+1}{2g_2+1} s^{\frac{|\vec{i}|}{2}-g_2}\right)^2 \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=|\vec{i}|+1-k} \binom{|\vec{i}|+1}{2\ell}\binom{|\vec{i}|+1}{2m}\right) s^k \\
&- \sum_{k=0}^{|\vec{i}|} \left(\sum_{\ell+m=|\vec{i}|-k} \binom{|\vec{i}|+1}{2\ell+1}\binom{|\vec{i}|+1}{2m+1}\right) s^k \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=|\vec{i}|+1-k} \binom{|\vec{i}|+1}{2\ell}\binom{|\vec{i}|+1}{2m} - \right. \\
&\left. \sum_{\ell+m=|\vec{i}|-k} \binom{|\vec{i}|+1}{2\ell+1}\binom{|\vec{i}|+1}{2m+1}\right) s^k \\
&= \sum_{k=0}^{|\vec{i}|+1} \left(\sum_{\ell+m=2(|\vec{i}|+1-k)} (-1)^\ell \binom{|\vec{i}|+1}{\ell}\binom{|\vec{i}|+1}{m}\right) s^k \\
\text{(by Lemma 5)} &= \sum_{k=0}^{|\vec{i}|+1} (-1)^k \binom{|\vec{i}|+1}{k} s^k \\
&= (s-1)^{|\vec{i}|+1}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(s-1)^{|\vec{i}|+1}\widetilde{F}_{\vec{i}}(s) &= A_{\vec{i},2} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{b_k}{(s-1)^{k+1}} + A_{\vec{i},3} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{a_k}{(s-1)^{k+1}} \\
(s-1)^{|\vec{i}|+1}\widetilde{G}_{\vec{i}}(s) &= A_{\vec{i},4} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{a_k}{(s-1)^{k+1}} + A_{\vec{i},1} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{b_k}{(s-1)^{k+1}}
\end{aligned}$$

Dividing through by $(s-1)^{|\vec{i}|+1}$, we get

$$\widetilde{F}_{\vec{i}}(s) = A_{\vec{i},2} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},3} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}} \tag{5.6}$$

$$\widetilde{G}_{\vec{i}}(s) = A_{\vec{i},4} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},1} \sum_{k=0}^{(|\vec{i}|-1)^2+1} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}} \tag{5.7}$$

The highest power of $(s-1)$ on the right hand sides of the above equations is $|\vec{i}|+2+(|\vec{i}-1)^2+1 = |\vec{i}|^2 - |\vec{i}| + 4$. Collecting the terms under this common denominator, we get

$$\widetilde{F}_{\vec{i}}(s) = \frac{\sum_{k=0}^{(|\vec{i}|-1)^2+1} (A_{\vec{i},2} b_k + A_{\vec{i},3} a_k)(s-1)^{(|\vec{i}|-1)^2+1-k}}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4}}$$

$$\widetilde{G}_{\vec{i}}(s) = \frac{\sum_{k=0}^{(|\vec{i}|-1)^2+1} (A_{\vec{i},4} a_k + A_{\vec{i},1} b_k)(s-1)^{(|\vec{i}|-1)^2+1-k}}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4}}$$

The next step is to make sure that the degrees of the polynomials occurring in the numerator of $\widetilde{F}_{\vec{i}}$ and $\widetilde{G}_{\vec{i}}$ are strictly less than $|\vec{i}|^2 - |\vec{i}| + 4$.

Lets first consider the numerator of $\widetilde{F}_{\vec{i}}$. The degree of this polynomial, which we denote by d_1, is

$$d_1 := \max\left\{ \left\lfloor \frac{|\vec{i}|}{2} \right\rfloor, \left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor \right\} + (|\vec{i}|-1)^2 + 1 = \left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor + (|\vec{i}|-1)^2 + 1$$

If $|\vec{i}|$ is odd, the degree is $\frac{|\vec{i}|+1}{2} + (|\vec{i}|-1)^2 + 1 = |\vec{i}|^2 - \frac{3}{2}|\vec{i}| + \frac{5}{2}$, and if $\vec{i}$ is even, the degree is $\frac{|\vec{i}|}{2} + (|\vec{i}|-1)^2 + 1 = |\vec{i}|^2 - \frac{3}{2}|\vec{i}| + 2$. In either case, $d_1 < |\vec{i}|^2 - |\vec{i}| + 4$.

Now consider the numerator of $\widetilde{G}_{\vec{i}}$, The degree of this polynomial, which we denote by d_2, is

$$d_2 := \max\left\{ \left\lfloor \frac{|\vec{i}|}{2} \right\rfloor + 1, \left\lfloor \frac{|\vec{i}|+1}{2} \right\rfloor \right\} + (|\vec{i}|-1)^2 + 1$$

When $|\vec{i}|$ is odd, $\max\left\{\left\lfloor \frac{|\vec{i}|}{2}\right\rfloor + 1, \left\lfloor \frac{|\vec{i}|+1}{2}\right\rfloor\right\} = \frac{|\vec{i}|+1}{2}$, and $d_2 = |\vec{i}|^2 - \frac{3}{2}|\vec{i}| + \frac{5}{2}$. When $|\vec{i}|$ is even, $d_2 = |\vec{i}|^2 - \frac{3}{2}|\vec{i} + 3$. In either case, $d_2 < |\vec{i}|^2 - |\vec{i}| + 4$.

Expanding the numerator polynomials of $\widetilde{F}_{\vec{i}}$ and $\widetilde{G}_{\vec{i}}$ at $s = 1$, we see that there exist constants f_k, g_k such that

$$\widetilde{F}_{\vec{i}}(s) = \frac{\sum_{k=0}^{d_1} f_k(s-1)^k}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4}} = \sum_{k=0}^{d_1} \frac{f_k}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4-k}}$$
$$\widetilde{G}_{\vec{i}}(s) = \frac{\sum_{k=0}^{d_2} g_k(s-1)^k}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4}} = \sum_{k=0}^{d_2} \frac{g_k}{(s-1)^{|\vec{i}|^2-|\vec{i}|+4-k}}$$

Taking the inverse Laplace transform, we see that

$$F_{\vec{i}}(t) = \left(\sum_{k=0}^{d_1} \frac{f_k}{(|\vec{i}|^2 - |\vec{i}| + 3 - k)!} t^{|\vec{i}|^2-|\vec{i}|+4-k}\right) e^t$$
$$G_{\vec{i}}(t) = \left(\sum_{k=0}^{d_2} \frac{g_k}{(\vec{i}|^2 - |\vec{i}| + 3 - k)!} t^{|\vec{i}|^2-|\vec{i}|+4-k}\right) e^t$$

Since $\sum_{k=0}^{d_1} \frac{f_k}{(|\vec{i}|^2 - |\vec{i}| + 3 - k)!} t^{|\vec{i}|^2-|\vec{i}|+4-k}$ and $\sum_{k=0}^{d_2} \frac{g_k}{(\vec{i}|^2 - |\vec{i}| + 3 - k)!} t^{|\vec{i}|^2-|\vec{i}|+4-k}$ are polynomials of degree at most $|\vec{i}|^2 - |\vec{i}| + 4$, by Lemma 6, we see that $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ are polynomial in g, with degree at most $|\vec{i}|^2 - 3|\vec{i}| + 4 \leq |\vec{i}|^2 + 1$, as desired.

□

5.3 Example

To demonstrate the practical use of the theorems in this paper, we go through an example of computing $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ as polynomials in g. Since we know that $2^{|\vec{i}|+1}D_{(\vec{i}),2g+2}$ and

$2^{|\vec{i}|+1}d_{(\vec{i}),2g+2}$ are integer valued polynomials in g, these polynomials can be written uniquely as linear combinations of

$$\binom{g}{0}, \binom{g}{1}, \ldots, \binom{g}{|\vec{i}|^2+1}$$

where the coefficients are integers. This is summarized in the following Corollary:

Corollary 3. *Let* $\vec{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$. *Define the integers* $m_{\vec{i}}$ *and* $\widetilde{m}_{\vec{i}}$ *by*

$$m_{\vec{i}} := \textit{min}\left\{ g : \int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{i_1}\ldots\lambda_{i_n} \neq 0 \right\}$$
$$\widetilde{m}_{\vec{i}} := \textit{min}\left\{ g : \int_{\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{i_1}\ldots\lambda_{i_n} \neq 0 \right\}$$

Then there exists integers $c_{\vec{i},k}$ *and* $\widetilde{c}_{\vec{i},k}$, *where* $0 \leq k \leq |\vec{i}|^2+1$, *such that, for all* g,

$$2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,(2g+2)t}(\mathcal{B}\mathbb{Z}_2)} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{i_1}\ldots\lambda_{i_n} = \sum_{k=m_{\vec{i}}}^{|\vec{i}|^2+1} c_{\vec{i},k}\binom{g}{k}$$
$$2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,(2g+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{i_1}\ldots\lambda_{i_n} = \sum_{k=\widetilde{m}_{\vec{i}}}^{|\vec{i}|^2+1} \widetilde{c}_{\vec{i},k}\binom{g}{k}$$

With the aid of Theorem 6 and Equation (3.18), *the integers* $c_{\vec{i},k}$ *and* $\widetilde{c}_{\vec{i},k}$ *are computed recursively by*

$$
\begin{aligned}
& c_{\vec{i},0} = 2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,2t}(\mathcal{B}\mathbb{Z}_2)} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{i_1}\dots\lambda_{i_n} \\
& \widetilde{c}_{\vec{i},0} = 2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,2t,1u}(\mathcal{B}\mathbb{Z}_2)} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{i_1}\dots\lambda_{i_n} \\
(k>0) \qquad & c_{\vec{i},k} = 2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,(2k+2)t}(\mathcal{B}\mathbb{Z}_2)} (\psi_j)^{2g-1-|\vec{i}|}\lambda_{i_1}\dots\lambda_{i_n} - \sum_{j=0}^{k-1} c_{\vec{i},j}\binom{k}{j} \\
(k>0) \qquad & \widetilde{c}_{\vec{i},k} = 2^{|\vec{i}|+1}\int_{\overline{\mathcal{M}}_{0,(2k+2)t,1u}(\mathcal{B}\mathbb{Z}_2)} (\psi_{2g+3})^{2g-|\vec{i}|}\lambda_{i_1}\dots\lambda_{i_n} - \sum_{j=0}^{k-1} \widetilde{c}_{\vec{i},j}\binom{k}{j}
\end{aligned}
$$

It is difficult to implement the recursions by hand. The author was aided by the use of the computer algebra software Maple.

Example 6. $\vec{i} = (1,2)$

Since $|\vec{i}| = 3$ *and* $m_{(1,2)} = \widetilde{m}_{(1,2)} = 2$*, we have*

$$
\begin{aligned}
2^4 D_{(1,2),2g+2} &= \sum_{k=2}^{10} c_{(1,2),k}\binom{g}{k} \\
2^4 d_{(1,2),2g+2} &= \sum_{k=2}^{10} \widetilde{c}_{(1,2),k}\binom{g}{k}
\end{aligned}
$$

Using the recursions for $c_{\vec{i},k}$ *and* $\widetilde{c}_{\vec{i},k}$ *in Corollary 3, we have*

$$
\begin{aligned}
c_{(1,2),2} &= 2^4 D_{(1,2),6} = 2 \\
c_{(1,2),3} &= 2^4 D_{(1,2),8} - 2\binom{3}{2} = 61 \\
c_{(1,2),4} &= 2^4 D_{(1,2),10} - 62\binom{4}{3} - 2\binom{4}{2} = 364 \\
c_{(1,2),5} &= 2^4 D_{(1,2),12} - 364\binom{5}{4} - 61\binom{5}{3} - 2\binom{5}{2} = 660 \\
c_{(1,2),6} &= 2^4 D_{(1,2),14} - 660\binom{6}{5} - 364\binom{6}{4} - 61\binom{6}{3} - 2\binom{6}{2} = 360 \\
c_{(1,2),7} &= c_{(1,2),8} = c_{(1,2),9} = c_{(1,2),10} = 0
\end{aligned}
$$

A similar calculation results in

$$\{\widetilde{c}_{(1,2),k}\}_{k\geq 2} = \{8, 168, 640, 840, 360, 0, 0, \ldots\}$$

Therefore, we have

$$
\begin{aligned}
2^4 D_{(1,2),2g+2} &= 2\binom{g}{2} + 61\binom{g}{3} + 364\binom{g}{4} + 660\binom{g}{5} + 360\binom{g}{6} \\
2^4 d_{(1,2),2g+2} &= 8\binom{g}{2} + 168\binom{g}{3} + 640\binom{g}{4} + 840\binom{g}{5} + 360\binom{g}{6}
\end{aligned}
$$

Chapter 6

Generating Functions of Hyperelliptic Hodge Integrals

In this chapter, we translate the recursions found previously into a system of partial differential equations for the generating functions for hyperelliptic Hodge integrals. First, we define how we package hyperelliptic Hodge integrals into generating functions.

Definition 19. *Define* $s := (s_1, s_2, \ldots)$, *and for* $\vec{i} = (i_1, \ldots, i_n)$, *define* $s^{\vec{i}} := s_1^{i_1} \ldots s_n^{i_n}$. *We define the generating functions* $F_{\vec{i}}$ *and* $G_{\vec{i}}$ *by*

$$F_{\vec{i}}(s,t) := \sum_{\substack{g\geq 0 \\ \vec{i}\geq\vec{0}}} (D_{\vec{i},2g+2}) s^{\vec{i}} \frac{t^{2g+2}}{(2g+2)!}$$

$$G_{\vec{i}}(s,t) := \sum_{\substack{g\geq 0 \\ \vec{i}\geq\vec{0}}} (d_{\vec{i},2g+2}) s^{\vec{i}} \frac{t^{2g+2}}{(2g+2)!}$$

Theorem 11. *The generating functions* $F_{\vec{i}}$ *and* $G_{\vec{i}}$ *satisfy the following partial differential equations:*

$$2\partial_t^3 F(\vec{s},t)\partial_t^2 F(-\vec{s},t) = 2\partial_t^2 G(\vec{s},t)\partial_t G(-\vec{s},t) \tag{6.1}$$

$$2\partial_t^2 G(\vec{s},t) G(-\vec{s},t) = 2\partial_t^3 F(\vec{s},t)\partial_t F(-\vec{s},t) - \partial_t^2 G(\vec{s},t) \tag{6.2}$$

Proof of Theorem 11. The Theorem follows from direct computation and Theorem 6.

First, we compute the formal expressions of the functions appearing in Equation (6.1) and Equation (6.2) :

$$\partial_t F(-\vec{s},t) = \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} (-1)^{|\vec{i}|} D_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g+1}}{(2g+1)!}$$

$$\partial_t^2 F(-\vec{s},t) = \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} (-1)^{|\vec{i}|} D_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g}}{(2g)!}$$

$$\partial_t^3 F(\vec{s},t) = \sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}} D_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g-1}}{(2g-1)!}$$

$$G(-\vec{s},t) = \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} (-1)^{|\vec{i}|} d_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g+2}}{(2g+2)!}$$

$$\partial_t G(-\vec{s},t) = \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} (-1)^{|\vec{i}|} d_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g+1}}{(2g+1)!}$$

$$\partial_t^2 G(\vec{s},t) = \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} d_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g}}{(2g)!}$$

The right hand side of Equation (6.1) is

$$\begin{aligned}
2\partial_t^2 G(\vec{s},t)\partial_t G(-\vec{s},t) &= 2\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} d_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g}}{(2g)!}\right)\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} (-1)^{|\vec{i}|} d_{(\vec{i}),2g+2} s^{\vec{i}} \frac{t^{2g+1}}{(2g+1)!}\right)\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{\vec{\ell}_1+\vec{\ell}_2=\vec{i}\\ 2g_1+2g_2+1=2g-1}} (-1)^{|\vec{\ell}_2|}\frac{1}{(2g_1)!}\frac{1}{(2g_2+1)!} d_{(\vec{\ell}_1),2g_1+2} d_{(\vec{\ell}_2),2g_2+1} s^{\vec{\ell}_1} s^{\vec{\ell}_2}\right) t^{2g-1}\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq 0}}\left(\sum_{\substack{g_1+g_2=g-1\\ 0\leq g_2\leq g\\ \vec{\ell}\leq\vec{i}}} (-1)^{|\vec{\ell}|}\binom{2g-1}{2g_2+1} d_{(\vec{i}-\vec{\ell}),2g_1+2} d_{(\vec{\ell}),2g_2+2}\right) s^{\vec{i}}\frac{t^{2g-1}}{(2g-1)!}
\end{aligned}$$

The left hand side of Equation (6.1) is

$$
\begin{aligned}
2\partial_t^3 F(\vec{s},t)\partial_t^2 F(-\vec{s},t) &= 2\left(\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}} D_{(\vec{i}),2g+2}s^{\vec{i}}\frac{t^{2g-1}}{(2g-1)!}\right)\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}}(-1)^{|\vec{i}|}D_{(\vec{i}),2g+2}s^{\vec{i}}\frac{t^{2g}}{(2g)!}\right)\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{2g_1-1+2g_2=2g-1\\ \vec{\ell}_2+\vec{\ell}_2=\vec{i}}}(-1)^{|\vec{\ell}_2|}\frac{1}{(2g_1-1)!}\frac{1}{(2g_2)!}D_{(\vec{\ell}_1),2g_1+2}D_{(\vec{\ell}_2),2g_2+2}s^{\vec{\ell}_1}s^{\vec{\ell}_2}\right)t^{2g-1}\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{g_1+g_2=g\\ 1\leq g_1\leq g\\ \vec{\ell}\leq\vec{i}}}(-1)^{|\vec{\ell}|}\binom{2g-1}{2g_2}D_{(\vec{i}-\vec{\ell}),2g_1+2}D_{(\vec{\ell}),2g_2+2}\right)s^{\vec{i}}\frac{t^{2g-1}}{(2g-1)!}\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{g_1+g_2=g\\ 1\leq g_2\leq g-1\\ \vec{\ell}\leq\vec{i}}}(-1)^{|\vec{\ell}|}\binom{2g-1}{2g_2}D_{(\vec{i}-\vec{\ell}),2g_1+2}D_{(\vec{\ell}),2g_2+2}+\frac{1}{2}D_{(\vec{i}),2g+2}\right)s^{\vec{i}}\frac{t^{2g-1}}{(2g-1)!}
\end{aligned}
$$

And therefore, Equation (6.1) follows by Equation (3.16) in Theorem 6 with $k=0$.

Now we proceed with Equation (6.2). The left hand side of Equation (6.2) is

$$
\begin{aligned}
2\partial_t^2 G(\vec{s},t)G(-\vec{s},t) &= 2\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} d_{(\vec{i}),2g+2}s^{\vec{i}}\frac{t^{2g}}{(2g)!}\right)\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}}(-1)^{|\vec{i}|}d_{(\vec{i}),2g+2}s^{\vec{i}}\frac{t^{2g+2}}{(2g+2)!}\right)\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{2g_1+2g_2+2=2g\\ \vec{\ell}_1+\vec{\ell}_2=\vec{i}}}(-1)^{|\vec{\ell}_2|}\frac{1}{(2g_1)!}\frac{1}{(2g_2+2)!}d_{(\vec{\ell}_1),2g_1+2}d_{(\vec{\ell}_2),2g_2+2}s^{\vec{\ell}_1}s^{\vec{\ell}_2}\right)t^{2g}\\
&= 2\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\sum_{\substack{g_1+g_2=g-1\\ 0\leq g_2\leq g-1\\ \vec{\ell}\leq\vec{i}}}(-1)^{|\vec{\ell}|}\binom{2g}{2g_2+2}d_{(\vec{i}-\vec{\ell}),2g_1+2}d_{(\vec{\ell}),2g_2+2}\right)s^{\vec{i}}\frac{t^{2g}}{(2g)!}
\end{aligned}
$$

The right hand side of Equation (6.2) is

$$
\begin{aligned}
2\partial_t^3 F(\vec{s},t)\partial_t F(-\vec{s},t) - \partial_t^2 G(\vec{s},t) &= 2\left(\sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}} D_{(\vec{i}),2g+2}\, s^{\vec{i}}\frac{t^{2g-1}}{(2g-1)!}\right) \\
&\times\left(\sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}}(-1)^{|\vec{i}|}D_{(\vec{i}),2g+2}\, s^{\vec{i}}\frac{t^{2g+1}}{(2g+1)!}\right) - \sum_{\substack{g\geq 0\\ \vec{i}\geq\vec{0}}} d_{(\vec{i}),2g+2}\, s^{\vec{i}}\frac{t^{2g}}{(2g)!} \\
&= \sum_{\substack{g\geq 1\\ \vec{i}\geq\vec{0}}}\left(\left(\sum_{\substack{g_1+g_2=g\\ 1\leq g_2\leq g-1\\ \vec{\ell}\leq\vec{i}}}(-1)^{|\vec{\ell}|}\binom{2g}{2g_2+1}D_{(\vec{i}-\vec{\ell}),2g_1+2}D_{(\vec{\ell}),2g_2+2}\right) - d_{(\vec{i}),2g+2}\right)s^{\vec{i}}\frac{t^{2g}}{(2g)!}
\end{aligned}
$$

Therefore, Equation (6.2) follows from Equation (3.17) in Theorem 6 with $k = 0$. □

Chapter 7

Conjectures and Open Problems

The combinatorial structure that governs the $\mathbb{Z}_2$ Hurwitz-Hodge integrals $D_{(\vec{i},2g+2}$ and $d_{(\vec{i}),2g+2}$ is far from being completely understood. Here, we mention some out standing problems concerning these intersection numbers, and outline some of the avenues of investigation for future work.

By Theorem 10, we know that $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ are polynomials in g of degree at most $|\vec{i}|^2 + 1$. However, the data in the Appendix suggests the following much sharper bound:

Conjecture 1. *The integrals $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ are polynomials in g, and their degrees are precisely* $2|\vec{i}|$

Let us take a moment to explain what happens when one tries to use the proof techniques in the proof of Theorem 10 to tackle Conjecture 1. When we try to sharpen the bound on the degrees of the polynomials $D_{(\vec{i}),2g+2}$ and $d_{(\vec{i}),2g+2}$ to $2|\vec{i}|$, the only difference in the proof would be the induction step, and Equations (5.6) and (5.7) become

$$\widetilde{F}_{\vec{i}}(s) = A_{\vec{i},2} \sum_{k=0}^{2(|\vec{i}|-1)} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},3} \sum_{k=0}^{2(|\vec{i}|-1)} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}}$$
$$\widetilde{G}_{\vec{i}}(s) = A_{\vec{i},4} \sum_{k=0}^{2(|\vec{i}|-1)} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},1} \sum_{k=0}^{2(|\vec{i}|-1)} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}}$$

and therefore,

$$\widetilde{F}_{\vec{i}}(s) = \frac{\sum_{k=0}^{2(|\vec{i}|-1)} (A_{\vec{i},2}b_k + A_{\vec{i},3}a_k)(s-1)^{2(|\vec{i}|-1)-k}}{(s-1)^{3|\vec{i}|}} = \frac{\sum_{k=0}^{2(|\vec{i}|-1)} (A_{\vec{i},2}b_k + A_{\vec{i},3}a_k)(s-1)^{|\vec{i}|-1-k}}{(s-1)^{2|\vec{i}|+1}}$$

$$\widetilde{G}_{\vec{i}}(s) = \frac{\sum_{k=0}^{2(|\vec{i}|-1)} (A_{\vec{i},4}a_k + A_{\vec{i},1}b_k)(s-1)^{2(|\vec{i}|-1)-k}}{(s-1)^{3|\vec{i}|}} = \frac{\sum_{k=0}^{2(|\vec{i}|-1)} (A_{\vec{i},4}a_k + A_{\vec{i},1}b_k)(s-1)^{|\vec{i}|-1-k}}{(s-1)^{2|\vec{i}|+1}}$$

At this point, the proof would go through as before only if the numerators are guaranteed to be *polynomials*, but we can't guarantee this since the expressions in the numerators have poles at $s = 1$ for $k > |\vec{i}| - 1$.

In fact, the techniques used in the proof of Theorem 10 do not accommodate any linear bound in $|\vec{i}|$. Indeed, if we wanted to prove that the polynomial degrees were bounded by $n|\vec{i}|$, then Equations (5.6) and (5.7) become

$$\widetilde{F}_{\vec{i}}(s) = A_{\vec{i},2} \sum_{k=0}^{n(|\vec{i}|-1)} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},3} \sum_{k=0}^{n(|\vec{i}|-1)} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}}$$

$$\widetilde{G}_{\vec{i}}(s) = A_{\vec{i},4} \sum_{k=0}^{n(|\vec{i}|-1)} \frac{a_k}{(s-1)^{|\vec{i}|+k+2}} + A_{\vec{i},1} \sum_{k=0}^{n(|\vec{i}|-1)} \frac{b_k}{(s-1)^{|\vec{i}|+k+2}}$$

and therefore,

$$\widetilde{F}_{\vec{i}}(s) = \frac{\sum_{k=0}^{n(|\vec{i}|-1)} (A_{\vec{i},2}b_k + A_{\vec{i},3}a_k)(s-1)^{n(|\vec{i}|-1)-k}}{(s-1)^{(n+1)|\vec{i}|+(2-n)}} = \frac{\sum_{k=0}^{n(|\vec{i}|-1)} (A_{\vec{i},2}b_k + A_{\vec{i},3}a_k)(s-1)^{(n-1)|\vec{i}|-1-k}}{(s-1)^{n|\vec{i}|+1}}$$

$$\widetilde{G}_{\vec{i}}(s) = \frac{\sum_{k=0}^{n(|\vec{i}|-1)} (A_{\vec{i},4}a_k + A_{\vec{i},1}b_k)(s-1)^{n(|\vec{i}|-1)-k}}{(s-1)^{(n+1)|\vec{i}|+(2-n)}} = \frac{\sum_{k=0}^{n(|\vec{i}|-1)} (A_{\vec{i},4}a_k + A_{\vec{i},1}b_k)(s-1)^{(n-1)|\vec{i}|-1-k}}{(s-1)^{n|\vec{i}|+1}}$$

in which case we still cannot guarantee that the expression in the numerators are polynomials due to poles at $s = 1$ when $k > (n-1)|\vec{i}| - 1$. Alas, proving Conjecture 1 will require a different approach.

By Theorem 9 we know that $|\vec{i}| + 1$ is a *sufficient* power of 2 that will normalize the Hodge integrals so that they are integral. However, the exponent $|\vec{i}| + 1$ is far from being *necessary*. For example, using the recursions, one can compute

$$\int_{\overline{\mathcal{H}}_{4,10}} \lambda_1\lambda_2\lambda_4 =: D_{(1,2,4),10} = \frac{27}{8}$$

Certainly, $2^{(1+2+4)+1}D_{(1,2,4),10} \in \mathbb{Z}$, but 3 is the *smallest* exponent for which the integrality holds. Thus, we have the following open problem:

Open Problem 1. *Compute the integers $m_1(\vec{i}, g)$ and $m_2(\vec{i}, g)$, where*

$$m_1(\vec{i}, g) := \textit{min}\left\{\alpha : 2^\alpha D_{(\vec{i}),2g+2} \in \mathbb{Z}\right\}$$
$$m_2(\vec{i}, g) := \textit{min}\left\{\alpha : 2^\alpha d_{(\vec{i}),2g+2} \in \mathbb{Z}\right\}$$

By Corollary 3, we know that there exists integers $c_{\vec{i},k}$ and $\widetilde{c}_{\vec{i},k}$ such that

$$2^{|\vec{i}|+1}D_{(\vec{i}),2g+2} = \sum_{k=0}^{|\vec{i}|^2+1} c_{\vec{i},k}\binom{g}{k} \tag{7.1}$$
$$2^{|\vec{i}|+1}d_{(\vec{i}),2g+2} = \sum_{k=0}^{|\vec{i}|^2+1} \widetilde{c}_{\vec{i},k}\binom{g}{k} \tag{7.2}$$

Many examples of the sequences $\{c_{\vec{i},k}\}_k$ and $\{\widetilde{c}_{\vec{i},k}\}_k$ are shown in the Appendix, and from this data, we make the following conjecture:

Conjecture 2. *Let $c_{\vec{i},k}$ and $\widetilde{c}_{\vec{i},k}$ be the integers defined in Equations* (7.1) *and* (7.2). *Then*

1. *The integers $c_{\vec{i},k}$ and $\widetilde{c}_{\vec{i},k}$ are ≥ 0*

2. *For fixed $\vec{i}$, the sequences $\{c_{\vec{i},k}\}_{k\geq 0}$ and $\{\widetilde{c}_{\vec{i},k}\}_{k\geq 0}$ are log-concave*

In ([7],Theorem 2), Johnson et al discover the following vanishing result:

$$\sum_{i=0}^{g}(-2)^i D_{i,2g+2} = 0 \tag{7.3}$$

Equation (7.3) directly follows from the results in Chapter 4. However, using the data in the Appendix, it is likely that the above vanishing result can be generalized:

Conjecture 3. *Let $g > 0$ and let $\ell \in \mathbb{Z}^n_{\geq 0}$ such that $|\vec{\ell}| \leq g-1$. Then*

1. $\sum_{i=0}^{g}(-2)^i D_{(\vec{\ell},i),2g+2} = 0$
2. *The sequences $\left\{2^{|\vec{\ell}|+i+1}D_{(\vec{\ell},i),2g+2}\right\}_{i\geq 0}$ and $\left\{2^{|\vec{\ell}|+i+1}d_{(\vec{\ell},i),2g+2}\right\}_{i\geq 0}$ are log-concave.*

Finally, recall the result of Faber and Pandharipande [4]:

$$D_{(g-1,g),2g+2} = \frac{2^{2g}-1}{2g}|B_{2g}|$$

where B_{2g} is the $2g^{th}$ Bernoulli number. By Theorem 9, we know that $2^{2g}D_{(g-1,g),2g+2} \in \mathbb{Z}$. It turns out that the integer $2^{2g}D_{(g-1,g),2g+2}$ has an enumerative meaning:

Corollary 4. *Let Z_g be the number of alternating permutations on $[1, 2, \ldots, 2g-1]$. Then*

$$2^{2g}D_{(g-1,g),2g+2} = Z_g$$

This observation prompts the following line of questioning:

Question 1. *Does there exist an enumerative interpretation of the integers $2^{|\vec{i}|+1}D_{(\vec{i}),2g+2}$ and $2^{|\vec{i}|+1}d_{(\vec{i}),2g+2}$ as counting subsets of permutations of $[1, 2, \ldots, 2g-1]$, as in Corollary 4? As a starting point, in light of the results obtained in Chapter 4, is there a way to interpret the integers $e_i(1, 3, \ldots, 2g-1)$ and $e_i(2, 4, \ldots, 2g)$ in this way?*

An affirmative answer to this question would provide an elegant governing principle for $\mathbb{Z}_2$ Hurwitz-Hodge integrals. Some new ideas, either coming from combinatorics or geometry, will be required to approach these questions.

Bibliography

[1] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathematiques de lIHES, 36*, pages 75–109, 1969.

[2] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry*, pages 271–328. Springer, 1983.

[3] Allesio Corti, Ton Coates, Hiroshi Iritani, and Hsian-Hua Tseng. Computing twisted genus-zero gromov-witten invariants. *Duke Mathematical Journal*, 147:377–438, 2009.

[4] Carel Faber and Rahul Pandharipande. Logarithmic series and hodge integrals in the tautological ring (with an appendix by d. zagier). *Michigan Math Journal*, 48(1):215–252, 2000.

[5] Renzo Cavalieri. Generating functions for hurwitz–hodge integrals. *Advances in Mathematics*, 218(5):1419–1429, 2008.

[6] Jonathan Wise. A hyperelliptic hodge integral. *Portugaliae Mathematica*, 73(3):207–219, 2016.

[7] Paul Johnson, Rahul Pandharipande, and Hsian-Hua Tseng. Abelian hurwitz-hodge integrals. *The Michigan Mathematical Journal*, 60(1):171–198, 2011.

[8] Michael Atiyah and Raoul Bott. The moment map and equivariant cohqmology. *Collected Papers*, page 241, 1994.

[9] T Jarvis and Takashi Kimura. Orbifold quantum cohomology of the classifying space of a finite group, orbifolds in mathematics and physics (madison, wi, 2001), 123–134. *Contemp. Math*, 310:123âĂŤ134, 2001.

[10] Renzo Cavalieri. *A topological quantum field theory of intersection numbers on moduli spaces of admissible covers*. PhD thesis, Department of Mathematics, University of Utah, 2005.

[11] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov-witten theory of deligne-mumford stacks. *American Journal of Mathematics*, 130(5):1337–1398, 2008.

[12] Weimin Chen and Yongbin Ruan. A new cohomology theory of orbifold. *Communications in Mathematical Physics*, 248(1):1–31, 2004.

[13] Yuan-Pin Lee and Mark Shoemaker. A mirror theorem for the mirror quintic. *Geometry & Topology*, 18(3):1437–1483, 2014.

[14] Kentaro Hori, Richard Thomas, Sheldon Katz, Cumrun Vafa, Rahul Pandharipande, Albrecht Klemm, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1. American Mathematical Soc., 2003.

[15] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology*, volume 171. Cambridge University Press, 2007.

[16] Paul Johnson. Equivariant gw theory of stacky curves. *Communications in Mathematical Physics*, 327(2):333–386, 2014.

[17] Tom Graber and Rahul Pandharipande. Localization of virtual classes. *Inventiones Mathematicae*, 135(2):487–518, 1999.

[18] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves*, pages 335–368. Springer, 1995.

[19] Charles Cadman and Renzo Cavalieri. Gerby localization, $\mathbb{Z}_3$ hodge integrals and the gw theory of $[\mathbb{C}^3/\mathbb{Z}_3]$. *American journal of mathematics*, 131(4):1009–1046, 2009.

[20] William Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[21] C. Faber and R. Pandharipande. Relative maps and tautological classes. *Journal of the European Mathematical Society*, 7:13–49, 2003.

[22] Richard P Stanley. Enumerative combinatorics volume 1 second edition. *Cambridge studies in advanced mathematics*, 2011.

[23] Paul-Jean Cahen and Jean-Luc Chabert. *Integer-valued polynomials*, volume 48. American Mathematical Soc., 1997.

Appendix A

Tables

We collect the values of various hyperelliptic Hodge integrals. One should use the data in this section as supporting evidence for the various conjectures asserted in Chapter 7.

Table A.1: Genus 1, all points twisted

Hodge Monomial $\lambda_{\vec{i}}$	$D_{(\vec{i}),2(1)+2}$
λ_1	$\frac{1}{4}$

Table A.2: Genus 1, one untwisted point

Hodge Monomial $\lambda_{\vec{i}}$	$d_{(\vec{i}),2(1)+2}$
λ_1	$\frac{1}{2}$

Table A.3: Genus 2, all points twisted

Hodge Monomial $\lambda_{\vec{i}}$	$D_{(\vec{i}),2(2)+2}$
λ_1	1
λ_2	$\frac{3}{8}$
$\lambda_1\lambda_2$	$\frac{1}{8}$

Table A.4: Genus 2, one untwisted point

Hodge Monomial $\lambda_{\vec{i}}$	$d_{(\vec{i}),2(2)+2}$
λ_1	$\frac{3}{2}$
λ_2	1
$\lambda_1\lambda_2$	$\frac{1}{2}$

Table A.5: Genus 3, all points twisted

Hodge Monomial $\lambda_{\vec{i}}$	$D_{(\vec{i}),2(3)+2}$
λ_1	$\frac{9}{4}$
λ_2	$\frac{23}{8}$
λ_3	$\frac{15}{16}$
$\lambda_1\lambda_2$	$\frac{67}{16}$
$\lambda_1\lambda_3$	$\frac{15}{16}$
$\lambda_2\lambda_3$	$\frac{1}{4}$

Table A.6: Genus 3, one untwisted point

Hodge Monomial $\lambda_{\vec{i}}$	$d_{(\vec{i}),2(3)+2}$
λ_1	3
λ_2	$\frac{11}{2}$
λ_3	3
$\lambda_1\lambda_2$	12
$\lambda_1\lambda_3$	4
$\lambda_2\lambda_3$	$\frac{3}{2}$

Table A.7: Genus 4, all points twisted

Hodge Monomial $\lambda_{\vec{i}}$	$D_{(\vec{i}),2(4)+2}$
λ_1	4
λ_2	$\frac{43}{4}$
λ_3	11
λ_4	$\frac{105}{32}$
$\lambda_1\lambda_2$	$\frac{155}{4}$
$\lambda_1\lambda_3$	$\frac{221}{8}$
$\lambda_1\lambda_4$	$\frac{105}{16}$
$\lambda_2\lambda_3$	$\frac{403}{16}$
$\lambda_2\lambda_4$	$\frac{147}{32}$
$\lambda_3\lambda_4$	$\frac{17}{16}$
$\lambda_1\lambda_2\lambda_4$	$\frac{27}{8}$

Table A.8: Genus 4, one untwisted point

Hodge Monomial $\lambda_{\vec{i}}$	$d_{(\vec{i}),2(4)+2}$
λ_1	5
λ_2	$\frac{35}{2}$
λ_3	25
λ_4	12
$\lambda_1\lambda_2$	85
$\lambda_1\lambda_3$	85
$\lambda_1\lambda_4$	30
$\lambda_2\lambda_3$	$\frac{211}{2}$
$\lambda_2\lambda_4$	27
$\lambda_3\lambda_4$	$\frac{17}{2}$
$\lambda_1\lambda_2\lambda_4$	27

Table A.9: Genus 5, all points twisted

Hodge Monomial $\lambda_{\vec{i}}$	$D_{(\vec{i}),2(5)+2}$
λ_1	$\frac{25}{4}$
λ_2	$\frac{115}{4}$
λ_3	$\frac{475}{8}$
λ_4	$\frac{1689}{32}$
λ_5	$\frac{945}{64}$
$\lambda_1\lambda_2$	$\frac{1555}{8}$
$\lambda_1\lambda_3$	$\frac{1195}{4}$
$\lambda_1\lambda_4$	$\frac{13185}{64}$
$\lambda_1\lambda_5$	$\frac{1575}{32}$
$\lambda_2\lambda_3$	$\frac{18599}{32}$
$\lambda_2\lambda_4$	$\frac{10179}{32}$
$\lambda_2\lambda_5$	$\frac{4095}{64}$
$\lambda_3\lambda_4$	$\frac{14801}{64}$
$\lambda_3\lambda_5$	$\frac{1185}{32}$
$\lambda_4\lambda_5$	$\frac{31}{4}$
$\lambda_1\lambda_2\lambda_3$	$\frac{56119}{32}$
$\lambda_1\lambda_2\lambda_4$	$\frac{47367}{64}$
$\lambda_1\lambda_2\lambda_5$	$\frac{1845}{16}$
$\lambda_1\lambda_3\lambda_4$	$\frac{11835}{32}$
$\lambda_1\lambda_3\lambda_5$	$\frac{139}{4}$
$\lambda_2\lambda_3\lambda_4$	$\frac{1381}{8}$

Table A.10: Genus 5, one untwisted point

Hodge Monomial $\lambda_{\vec{i}}$	$d_{(\vec{i}),2(5)+2}$
λ_1	$\frac{15}{2}$
λ_2	$\frac{85}{2}$
λ_3	$\frac{225}{2}$
λ_4	137
λ_5	60
$\lambda_1\lambda_2$	$\frac{725}{2}$
$\lambda_1\lambda_3$	725
$\lambda_1\lambda_4$	680
$\lambda_1\lambda_5$	240
$\lambda_2\lambda_3$	$\frac{3637}{2}$
$\lambda_2\lambda_4$	$\frac{2687}{2}$
$\lambda_2\lambda_5$	381
$\lambda_3\lambda_4$	1279
$\lambda_3\lambda_5$	278
$\lambda_4\lambda_5$	$\frac{155}{2}$
$\lambda_1\lambda_2\lambda_3$	$\frac{14295}{2}$
$\lambda_1\lambda_2\lambda_4$	4087
$\lambda_1\lambda_2\lambda_5$	864
$\lambda_1\lambda_3\lambda_4$	2762
$\lambda_1\lambda_3\lambda_5$	$\frac{695}{2}$
$\lambda_2\lambda_3\lambda_4$	$\frac{6905}{4}$