THESIS

"STRESSES NEAR A CIRCULAR CRACK IN A HALF SPACE"

Submitted by

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ABSTRACT

"STRESSES NEAR A CIRCULAR CRACK IN A HALF SPACE"

Solutions are presented for a circular crack embedded in a semi-infinite solid and for a part circular crack in the surface of a semi-infinite solid. The stresses in the neighborhood of the crack were found by a superposition and iteration technique. Recurrence relationships and series solutions for some integrals involving Bessel functions were derived to facilitate the computations. A computer program was written to compute the stresses and stress intensity factor for two types of normal loading and for several crack geometries.

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INTRODUCTION

The problem of structural failure by catastrophic crack propagation at the average stress well below the yield strength has been known for many years. The crack propagation can be a result of mechanical or thermal loading. When an elastic body which contains a crack is stressed, redistribution of stresses will occur which in turn causes high stresses near the crack tip and as a result a plastic zone at the crack tip. These elevated stresses are the actual cause of crack propagation.

Although the state of stress in the plastic zone cannot be determined by elastic analysis, the elastic stresses on these regions can be used as a measure or an indicator of the stresses or strains within the plastic zone. In fracture mechanics it is customary to identify the elastic stress or strain at the crack border by the stress intensity factor. This factor which can be calculated theoretically and measured experimentally, then can be used as a means to predict the load at which unstable crack propagation will occur.

In recent years a great amount of experimental and theoretical work has been done to evaluate the stress intensity factor and the

criterion at which failure occurs. Most of the theoretical work has so far been focused on two dimensional aspects of crack analysis. Relatively few three-dimensional crack problems have been solved.

In 1945 Sneddon (1)* formulated the distribution of stress in the neighborhood of a circular crack in an elastic solid when the crack surface is subjected to a constant normal pressure. Since 1945 solutions for several problems involving the circular crack in an infinite solid have been presented. Various types of mechanical loading and displacement on the crack surface have been considered by authors including Keer (2), Collins (3), Westman (4), Muki (5) and others.

In 1960 Green and Sneddon (6) solved the problem of a flat elliptical crack in an infinite solid. More recent work on the embedded elliptical crack includes that of Kassir and Sih (7) who solved the problem for prescribed shear on the crack surface and later solved it for a linearly varying normal pressure on crack surface (8). The solution for linearly varying loads and a technique for solving problems with quadratic and higher order loading was presented by Segedin (9). The problems of an embedded hyperbola and an embedded parabola were recently solved by Shah (10).

The work cited above is all restricted to cracks in infinite solids subjected to various types of loading conditions. Only limited work has been done on three-dimensional solution of cracks in finite

* Numbers in parenthesis refer to references at the end of this thesis.

solids. In 1962 Irwin (11) using the result obtained by Green and Sneddon, estimated the stress intensity factor for a part through elliptical crack in a flat plate. In 1965 Smith (12) solved the semi-circular edge crack problem when the crack surface is loaded with any kind of mechanical or thermal loading symmetrical with respect to the plane of the crack. In this solution, Smith first found a solution for a circular crack embedded in an elastic solid subjected to a prescribed load. He then introduced a plane normal to the crack surface through the center of the crack. The resulting normal stresses on this plane were then found. To make the plane free of stresses, the normal stresses were removed by applying the opposite stresses which in turn cause stresses on the crack surface. Using the circular crack solution, the stresses from the crack surfaces were again removed. This procedure was repeated until the crack surface and free plane were stress free.

It is the purpose of this thesis to solve the problem of a circular crack embedded in a semi-infinite solid as well as part through circular edge cracks. The stress intensity factors will then be determined for several geometrics as shown in Figures 1 and 2. These geometries represent the common flaw shapes in service failures, therefore the evaluation of the stress intensity factor for these geometries is of major practical importance.

The procedure that was used to solve the present problem is the same as that used by Smith (12). This problem is more complex however in that shears as well as normal stresses must be removed from the plane during the iteration procedure. Chapter one describes the problem in detail as well as the method of solution. Chapter two describes the circular crack solution as used in this work. Chapter three discusses the normal stress and shear solutions for a half space which are required for the iteration procedure described above. Chapter four presents the results of this work.

CHAPTER I

THE PROBLEM

Consider an embedded or a part through circular crack as shown in Figures 1 and 2. The crack surfaces which are located on the plane $z = 0$, are opened due to application of a system of mechanical or thermal loading. The problem is to determine the distribution of stresses in the nieghborhood of these cracks, and calculate the stress intensity factor.

In treating these problems, the loading is restricted to be symmetrical with respect to the plane $z = 0$, which implies that:

1) The shearing stresses are zero everywhere on the plane $z = 0$.

2) The normal stress u_z is zero outside the crack on the plane $z = 0$.

Due to extreme mathematical difficulty the solution is obtained by a superposition and iteration technique which involves the following steps:

I) It is assumed first that there exists no crack. The normal stresses at the location of crack surfaces due to applied load are found.

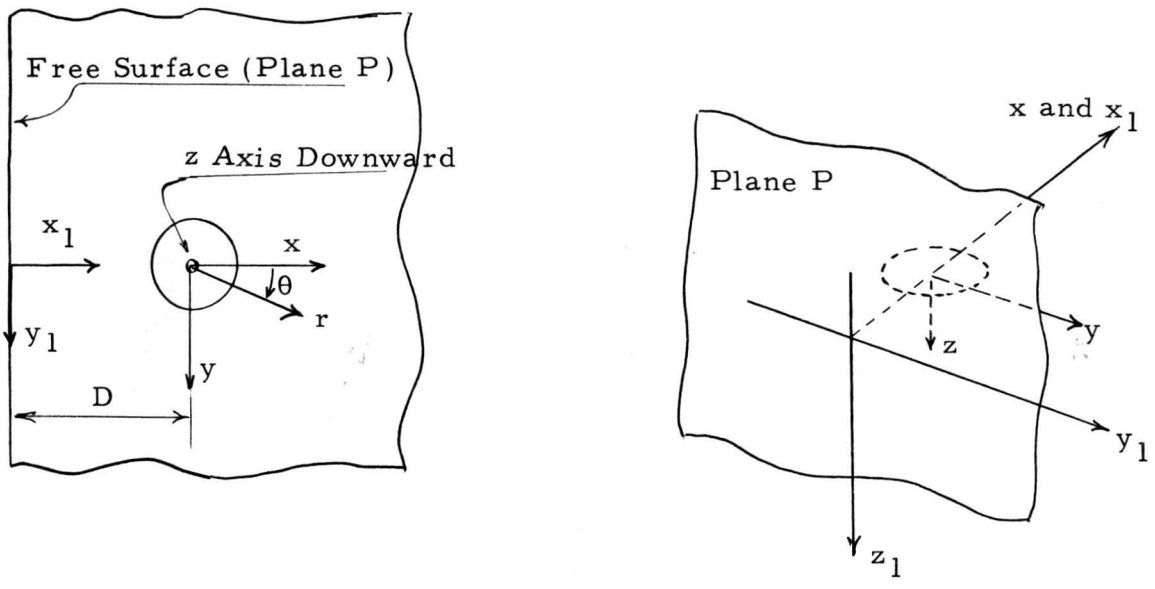
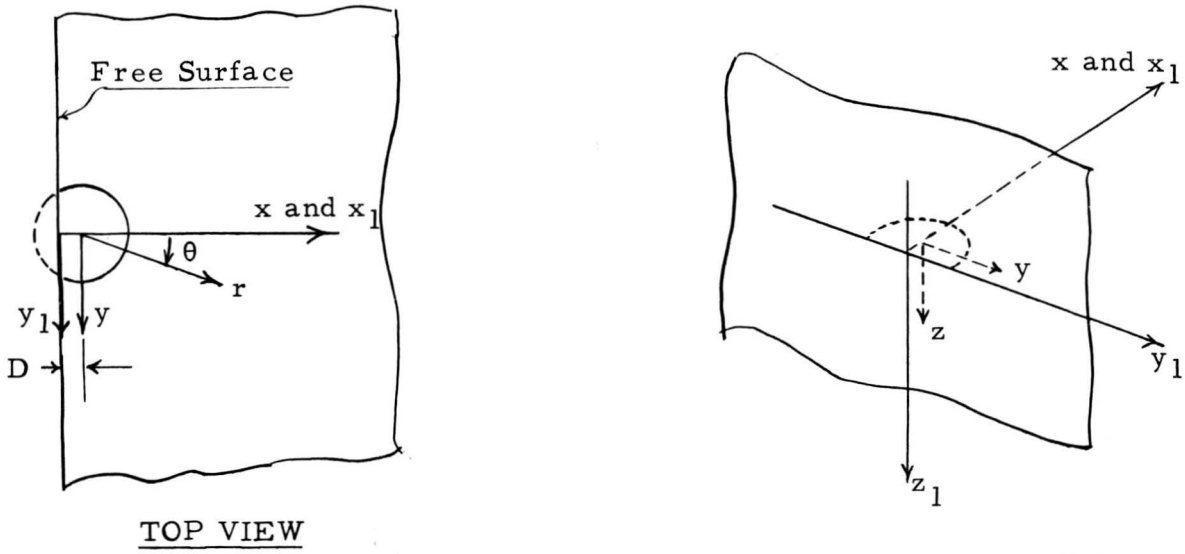
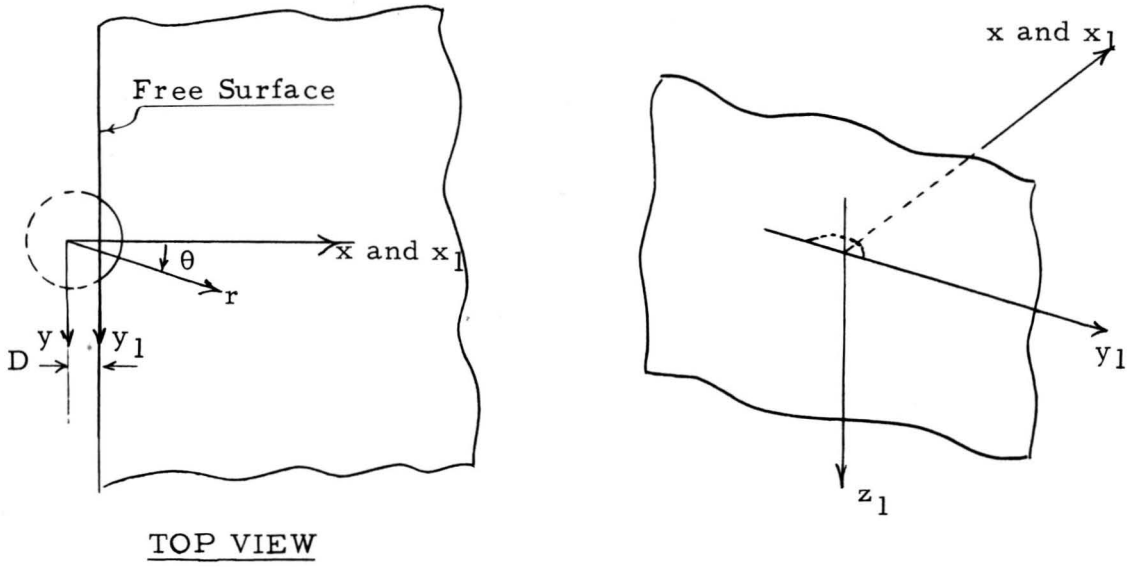


Figure 1. Embedded Circular Crack



CASE I



CASE II

Figure 2. Part Through Circular Edge Crack

II) It is then assumed there exists a crack. Since the crack surfaces cannot support normal tension, the normal tension found in step I must then be removed by applying equal and opposite normal stresses to the crack surfaces. In order to do this, it is necessary to find a solution for a circular crack embedded in an infinite solid subjected to an arbitrary normal loading on the crack surface. This solution was developed for use in an iteration procedure by Smith (12), but the author has developed an improved numerical evaluation technique for this step.

III) A free plane normal to the crack surface located an arbitrary distance from the center of the crack is introduced. On this plane, normal and shearing stresses caused by the removal of the stresses from the crack surface are computed using the solution described in II above.

IV) The normal and shearing stresses on the free plane must be removed. To remove these stresses it was necessary to find a solution for a semi-infinite solid with no crack when its bounding plane is subjected to a system of normal and shearing stresses. The normal stresses on the crack surfaces resulting from the removal of stresses from the free plane are then computed.

V) Steps II through IV are repeated until both the crack surface and the plane are stress free. The final solution for normal stresses on the plane of the crack will be obtained by superposition of steps I through IV.

VI) The stress intensity factor will finally be computed from the circular crack solution for each iteration and superimposed to obtain the final result.

The iteration procedure described above is ideally suited for a high speed digital computer. A program was written and the solution obtained using the Colorado State University CDC 6400.

CHAPTER II

STRESS DISTRIBUTION IN THE NEIGHBORHOOD OF A CIRCULAR CRACK

2.1 The General Problem Statement

Consider an elastic solid containing a circular crack of radius $0 \leq r \leq a$ in the plane $z = 0$ which is opened by a system of normal pressure $p(r,\theta)$ symmetrical with respect to the plane $z = 0$. Because of loading symmetry a cylindrical coordinate system (r, θ, z) will be employed to describe the distribution of the stresses in the neighborhood of the crack (Figure 3). The distribution of stress is the same as that in a semi-infinite solid $z \geq 0$ subjected to a boundary conditions:

$$\left.\tau_{rz}\right|_{z=0} = \left.\tau_{\theta z}\right|_{z=0} = 0$$

$$\left.\tau_{zz}\right|_{z=0} = -P(r,\theta) \qquad 0 \leq r < a \tag{2.1.1}$$

$$\left.U_z\right|_{z=0} = 0 \qquad r > a$$

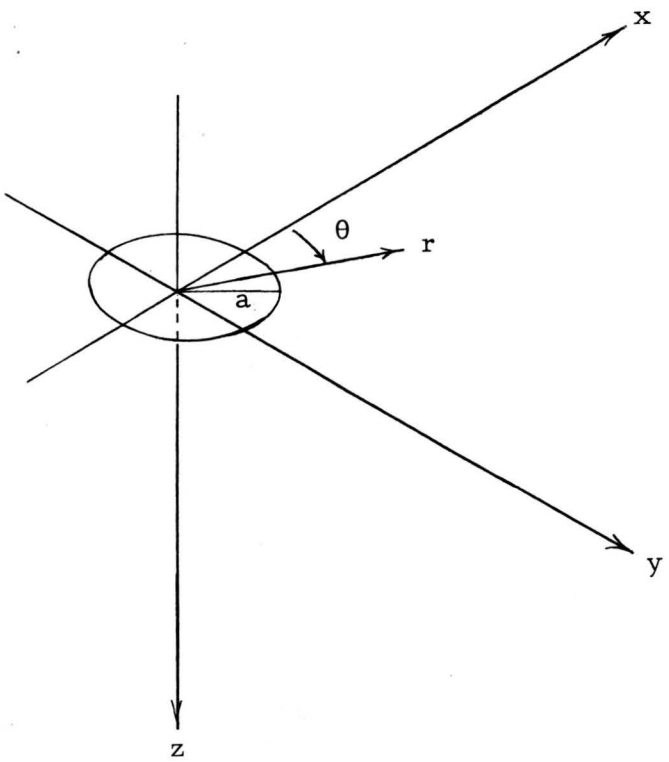


Figure 3. Circular Crack in Infinite Solid

The formulation for this special class of problems, in which the shearing stresses vanish at all points in the plane $z = 0$ is given by Green and Zerna (13) in the form of:

$$U_r = \frac{\partial \phi}{\partial r} + (\beta^2 - 1)\, z \frac{\partial^2 \phi}{\partial r\, \partial z}$$

$$U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + (\beta^2 - 1) \frac{z}{r} \frac{\partial^2 \phi}{\partial \theta\, \partial z} \qquad (2.1.2)$$

$$U_z = -\beta^2 \frac{\partial \phi}{\partial z} + (\beta^2 - 1)\, z \frac{\partial^2 \phi}{\partial z^2}$$

$$\beta^2 = \frac{\lambda + \mu}{\mu} = \frac{2(1-\nu)}{1-2\nu}$$

where the function ϕ is harmonic, that is $\nabla^2 \phi = 0$ and λ and μ are the Lame' constants.

These displacements satisfy the Navier's equation:

$$\mu \nabla^2 U_i + (\lambda + \mu)\, e_{,i} = 0 \qquad (2.1.3)$$

In which as usual the dilatation e is defined as:

$$e = \epsilon_x + \epsilon_y + \epsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

It should be noticed that in the above derivation U_r, U_θ and U_z have been nondimensionalized through division by "a" the crack radius.

Using the generalized Hookes law

$$\tau_{ij} = \lambda \, \delta_{ij} \, e + 2\mu \, \epsilon_{ij} \tag{2.1.4}$$

the stress components can easily be found as

$$\tau_{rr} = -2(\beta^2 - 2) \frac{\partial^2 \phi}{\partial z^2} + 2 \frac{\partial^2 \phi}{\partial r^2} + 2(\beta^2 - 1) z \frac{\partial^3 \phi}{\partial r^2 \partial z}$$

$$\tau_{\theta\theta} = -2(\beta^2 - 2) \frac{\partial^2 \phi}{\partial z^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{2}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$+ 2(\beta^2 - 1) \frac{z}{r^2} \frac{\partial^3 \phi}{\partial \theta^2 \partial z} + 2(\beta^2 - 1) \frac{z}{r} \frac{\partial^2 \phi}{\partial r \partial z}$$

$$\tau_{zz} = -2(\beta^2 - 1) \frac{\partial^2 \phi}{\partial z^2} + 2(\beta^2 - 1) z \frac{\partial^3 \phi}{\partial z^3}$$

$$\tau_{r\theta} = \frac{2}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + 2(\beta^2 - 1) \frac{z}{r} \frac{\partial^3 \phi}{\partial r \partial \theta \partial z}$$

$$- \frac{2}{r^2} \frac{\partial \phi}{\partial \theta} - 2(\beta^2 - 1) \frac{z}{r^2} \frac{\partial^2 \phi}{\partial \theta \partial z} \tag{2.1.5}$$

$$\tau_{rz} = 2(\beta^2 - 1) z \frac{\partial^3 \phi}{\partial r \partial z^2}$$

$$\tau_{\theta z} = 2(\beta^2 - 1) \frac{z}{r} \frac{\partial^3 \phi}{\partial \theta \partial z^2}$$

These stress components have been nondimensionalized through division by μ the shearing modulus of elasticity.

The first of the boundary conditions (2.1.1) is satisfied automatically by the formulation. The remaining boundary conditions will be satisfied if the stress function ϕ is found to be such that on the plane $z = 0$

$$\frac{\partial^2 \phi}{\partial z^2} = - \frac{\tau_{zz}}{2(\beta^2 - 1)} = \frac{-P(r, \theta)}{2(\beta^2 - 1)} \qquad 0 \leq r \leq 1 \qquad (2.1.6)$$

$$\frac{\partial \phi}{\partial z} = - \frac{U_z}{\beta^2} = 0 \qquad r > 1$$

2.2 The Stress Function ϕ

The unknown stress function ϕ must satisfy two conditions:

1) it must be harmonic

2) it should be such that all of the stress components vanish when r and z are large.

Let us assume ϕ be in the form of

$$\phi = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \frac{1}{\xi} J_n(\xi r) e^{-z\xi} f_n(\xi) \, d\xi \qquad (2.2.1)$$

where $J_n(\xi r)$ indicates the Bessel function of the first kind of order n, and $f_n(\xi)$ is an unknown function. The Laplacian of ϕ is equal to:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \frac{1}{\xi} f_n(\xi) e^{-z\xi} \left\{ \frac{d^2}{dr^2} J_n(\xi r) \right. \tag{2.2.2}$$

$$\left. \frac{1}{r}\frac{d}{dr} J_n(\xi r) + \frac{1}{r^2}(\xi^2 r^2 - n^2) J_n(\xi r) \right\} d\xi$$

Since the $J_n(\xi r)$ is a solution to the Bessel differential equation, the right hand side of the above expression is zero, which implies that ϕ is harmonic. Furthermore, since $J_n(\xi r)$ and $e^{-z\xi}$ both tend to zero when z and r go to infinity, the second condition imposed on stress function ϕ is also satisfied. Therefore the boundary conditions (2.1.6) can be rewritten as:

$$\frac{\partial^2\phi}{\partial z^2} = \sum_{n=0}^{\infty} \cos(n\theta) \int^{\infty} \xi J_n(\xi r) f_n(\xi) e^{-z\xi} d\xi \qquad 0 \le r \le 1 \tag{2.2.3}$$

$$\frac{\partial\phi}{\partial z} = \sum_{n=0}^{\infty} \cos(n\theta) \int^{\infty} J_n(\xi r) f_n(\xi) e^{-z\xi} d\xi = 0 \qquad r > 1$$

To find the unknown function $f_n(\xi)$ such that the conditions (2.2.3) are satisfied, let us express the loading $p(r,\theta)/2(\beta^2-1)$ in the Fourier cosine series that is:

$$\frac{p(r,\theta)}{2(\beta^2-1)} = \sum_{n=0}^{\infty} B_n(r) \cos(n\theta) \qquad (2.2.4)$$

The Fourier coefficients $B_n(r)$ for such an expansion are:

$$B_o(r) = \frac{1}{2\pi(\beta^2-1)} \int_o^{\pi} p(r,\theta)\, d\theta$$

$$B_n(r) = \frac{1}{\pi(\beta^2-1)} \int_o^{\pi} p(r,\theta) \cos(n\theta)\, d\theta \qquad (2.2.5)$$

Using the equation (2.2.4) the boundary conditions (2.2.3) can be written as:

$$B_n(r) = \int_o^{\infty} \xi\, J_n(\xi r)\, f_n(\xi)\, e^{-z\xi} d\xi \qquad 0 \le r \le 1$$

$$\int_o^{\infty} J_n(\xi r)\, f_n(\xi)\, e^{-z\xi} d\xi = 0 \qquad r > 1 \qquad (2.2.6)$$

Therefore, the problem reduces to that of solving the dual integral equations (2.2.6) for unknown function $f_n(\xi)$. The solution of dual integrals of this type is due to Busbridge and is given by Sneddon in the Fourier Transforms (14) as:

$$f_n(\xi) = \sqrt{\frac{2}{\pi}}\xi^{1/2} \int_0^1 \eta^{3/2} J_{n+1/2}(\xi\eta)\, d\eta \int_0^1 B_n(\eta\zeta) \frac{\zeta^{n+1}}{\sqrt{1-\zeta^2}}\, d\zeta \qquad (2.2.7)$$

By determining $f_n(\xi)$ from the above relationship the stress function ϕ is completely defined.

2.3 The Stress Components

The five stress components, namely, τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$, τ_{rz} and $\tau_{\theta z}$ which will be needed later can be found using the equations (2.1.5) and by appropriate differentiation of the stress function.

Using the following well known relationships between Bessel functions and their derivatives:

$$\frac{d}{dr}\left[J_n(\xi r)\right] = \frac{n}{r} J_n(\xi r) - \xi J_{n+1}(\xi r) \qquad (2.3.1)$$

$$\frac{d^2}{dr^2}\left[J_n(\xi r)\right] = \left(-\frac{n}{r^2} + \frac{n^2}{r^2} - \xi^2\right) J_n(\xi r) + \frac{\xi}{r} J_{n+1}(\xi r)$$

these stress components are found to be:

$$\tau_{rr} = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \left[a_1 J_n(\xi r) + a_2 \frac{1}{\xi} J_n(\xi r)\right.$$

$$\left. + a_3 \xi J_n(\xi r) + a_4 \xi^2 J_n(\xi r) + a_5 J_{n+1}(\xi r) + a_6 \xi J_{n+1}(\xi r)\right] f_n(\xi) e^{-z\xi} d\xi$$

$$\tau_{\theta\theta} = \sum_{n=0}^{\infty} \cos(n\theta) \int_{o}^{\infty} \left[-a_1 J_n(\xi r) - a_2 \frac{1}{\xi} J_n(\xi r) \right.$$

$$\left. + a_9 \xi J_n(\xi r) - a_5 J_{n+1}(\xi r) - a_6 \xi J_{n+1}(\xi r) \right] f_n(\xi) e^{-z\xi} d\xi$$

$$\tau_{r\theta} = \sum_{n=0}^{\infty} \sin(n\theta) \int_{o}^{\infty} \left[-a_1 J_n(\xi r) - a_2 \frac{1}{\xi} J_n(\xi r) \right.$$

$$\left. + a_7 J_n(\xi r) + a_8 \xi J_{n+1}(\xi r) \right] f_n(\xi) e^{-z\xi} d\xi$$

$$\tau_{rz} = \sum_{n=0}^{\infty} \cos(n\theta) \int_{o}^{\infty} \left[-a_8 \xi J_n(\xi r) - a_4 \xi^2 J_{n+1}(\xi r) \right] f_n(\xi) e^{-z\xi} d\xi$$

$$\tau_{\theta z} = \sum_{n=0}^{\infty} \cos(n\theta) \int_{o}^{\infty} a_8 \xi J_n(\xi r) f_n(\xi) e^{-z\xi} d\xi \quad .$$

Where a_1, a_2, ... and a_9 are:

$$a_1 = \frac{-2(\beta^2 - 1) n (n-1) z}{r^2}$$

$$a_2 = \frac{2n (n-1)}{r^2}$$

$$a_3 = -2(\beta^2 - 1)$$

$$a_4 = 2z(\beta^2 - 1)$$

$$a_5 = \frac{2}{r}$$

$$a_6 = -\frac{2(\beta^2-1)\,z}{r}$$

$$a_7 = \frac{2n}{r}$$

$$a_8 = \frac{-2(\beta^2-1)\,n\,z}{r}$$

$$a_9 = -2(\beta^2-2)$$

2.4 Stress Transformation

For this work it will be necessary to determine the stresses on the plane $\underline{P}$ which is located a distance "D" from the center of the crack (Figures 1 and 2). Therefore, the stresses must be transformed from the cylindrical coordinate system (r, θ, r) to the rectangular coordinate system (x_1, y_1, z_1). This can be easily accomplished by using the stress transformation law:

$$\tau_{\alpha\beta} = a_{\alpha i}\, a_{\beta j}\, \tau_{ij} \qquad (2.4.1)$$

where $a_{\alpha i}$ and $a_{\beta j}$ denote the direction cosines of the rotated system (x_1, y_1, z_1) with respect to original system (r, θ, z). These can be found from the following tabular form.

	r	θ	z
x_1	$\cos\theta$	$-\sin\theta$	0
y_1	$\sin\theta$	$\cos\theta$	0
z_1	0	0	1

(2.4.2)

Therefore the stresses on the plane P are:

$$\tau_{x1x1} = \tau_{rr} \cos^2\theta + \tau_{\theta\theta} \sin^2\theta - 2\tau_{r\theta} \cos\theta \sin\theta$$

$$\tau_{x1y1} = (\tau_{rr} - \tau_{\theta\theta}) \cos\theta \sin\theta + \tau_{r\theta} (\cos^2\theta - \sin^2\theta) \qquad (2.4.3)$$

$$\tau_{x1z1} = \tau_{rz} \cos\theta - \tau_{\theta z} \sin\theta$$

2.5 The Function $f_n(\zeta)$

In order to evaluate the stresses from equations (2.3.2) further development of the function $f_n(\zeta)$ is necessary. It is convenient at this point to expand the Fourier coefficients $B_n(r)$ in the power series

$$B_n(r) = \sum_{p=0}^{N} C_n^p \, r^p \qquad (2.5.1)$$

which implies

$$B_n(\zeta\eta) = \sum_{p=0}^{N} C_n^p \, \zeta^p \, \eta^p$$

By substituting $B_n(\zeta\eta)$ into the equation (2.2.7) we obtain

$$f_n(\xi) = \sqrt{\frac{2}{\pi}} \sum_{p=0}^{N} C_n^p \; \xi^{1/2} \int_0^1 \eta^{p+3/2} \; J_{n+1/2}(\xi\eta)\, d\eta \int_0^1 \frac{\zeta^{p+n+1}}{\sqrt{1-\zeta^2}}\, d\zeta$$

The last integral above is given by

$$\int_0^1 \frac{\zeta^{p+n+1}}{\sqrt{1-\zeta^2}} = \frac{\sqrt{\pi}}{2} \; \frac{\Gamma\left(\frac{p+n+2}{2}\right)}{\Gamma\left(\frac{p+n+3}{2}\right)}$$

Therefore $f_n(\xi)$ may be written as

$$f_n(\xi) = \frac{1}{\sqrt{2}} \sum_{p=0}^{N} C_n^p \; H_n^p \; \xi^{1/2} \int_0^1 \eta^{p+3/2} \; J_{n+1/2}(\xi\eta)\, d\eta \qquad (2.5.2)$$

where

$$H_n^p = \frac{\Gamma\left(\frac{p+n+2}{2}\right)}{\Gamma\left(\frac{p+n+3}{2}\right)} \qquad (2.5.3)$$

The series expansion for the Bessel functions $J_{n+1/2}(\xi\eta)$ is

$$J_{n+1/2}(\xi\eta) = \sum_{k=0}^{\infty} \frac{(-1)^k (\xi\eta)^{2k+n+1/2}}{2^{2k+n+1/2} \; k! \; \Gamma(k+n+3/2)} \; . \qquad (2.5.4)$$

Substituting (2.5.4) into (2.5.2) and performing the integration results in

$$f_n(\xi) = \sum_{p=0}^{N} \sum_{k=0}^{\infty} \frac{(-)^k \, C_n^p \, H_n^p \; \xi^{2k+n+1}}{2^{2k+n+1} \; k! \; (2k+n+p+3) \; \Gamma(k+n+3/2)}$$

If A(n, p, k) is defined as

$$A(n,\ p,\ k) = \frac{(-)^{k}\ H_{n}^{p}}{2^{2k+n+1}\ k\ !\ (2k+n+p+3)\,\Gamma(k+n+3\ 2)} \tag{2.5.5}$$

then the function $f_n(\ )$ may be rewritten as

$$f_n(\xi) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p\ \ 2k+n+1 \tag{2.5.6}$$

2.6 Final Stress Formulation

By substituting (2.5.6) into the equation (2.3.2) the final formulation for stresses may be obtained as:

$$\tau_{rr} = \sum_{p=0}^{\infty} \cos(n\theta) \sum_{p=0}^{N} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p$$

$$\cdot \Big[a_1\ S(n,\ 2k+n+1) + a_2\ S(n,\ 2k+n) +$$

$$a_3\ S(n,\ 2k+n+2) + a_4\ S(n,\ 2k+n+3)$$

$$+ a_5\ S(n+1,\ 2k+n+1) + a_6\ S(n+1,\ 2k+n+2) \Big] ,$$

$$\tau_{\theta\theta} = \sum_{n=0}^{\infty} \cos(n\theta) \sum_{p=0}^{N} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p$$

$$\cdot \Big[-a_1\ S(n,\ 2k+n+1) - a_2\ S(n,\ 2k+n)$$

$$+ a_9\, S(n,\ 2k + n + 2) - a_5\, S(n + 1,\ 2k + n + 1)$$

$$\left[- a_6\, S(n + 1,\ sk + n + 2) \right.$$

$$\tau_{r\theta} = \sum_{n=0}^{\infty} \mathrm{Sin}\,(n\theta) \sum_{p=0}^{N} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p$$

$$\left[-a_1\, s(n,\ 2k + n + 1) - a_2\, s(n,\ 2k + n) \right.$$

$$\left. + a_7\, s(n+1,\ 2k+n+1) + a_8\, s(n+1,\ 2k+n+2) \right]$$

$$\tau_{rz} = \sum_{n=0}^{\infty} \cos\,(n\theta) \sum_{p=0}^{N} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p \qquad (2.6.1)$$

$$\left[- a_8\, s(n,\ 2k + n + 2) - a_4\, s(n + 1,\ 2k + n + 3) \right]$$

$$\tau_{\theta z} = \sum_{n=0}^{\infty} \mathrm{Sin}\,(n\theta) \sum_{p=0}^{N} \sum_{k=0}^{\infty} A(n,\ p,\ k)\ C_n^p$$

$$\left[a_8\ s(n,\ 2k + n + 2) \right]$$

where $s(\nu,\ n)$ is defined as

$$s(\nu,\ n) = \int_0^{\infty} \xi^{\eta}\, J_{\nu}(\xi r)\, e^{-z\xi}\, d\xi \qquad (2.6.2)$$

Therefore, all of the stress components can be computed if the integrals $s(\nu, n)$ are evaluated.

2.7 Recurrence Formula for the Integrals $s(\nu, n)$

In the following recurrence formulae are derived from which the integrals $s(\nu, n)$ may be evaluated.

Using the definition of the integral $s(0, n+1)$ and the relationship

$$\xi J_o(\xi r) = \frac{1}{r} \frac{\partial}{\partial \xi} \left[\xi J_1(\xi r) \right]$$

integration by parts yields

$$s(0, n+1) = \int_o^\infty \xi^{n+1} J_o(\xi r) e^{-z\xi} d\xi \qquad (2.7.1)$$

$$= \frac{1}{r} \int_o^\infty \xi^n e^{-z\xi} \frac{\partial}{\partial \xi} \left[\xi J_1(\xi r) \right] d\xi$$

or

$$s(0, n+1) = \frac{1}{r} \left[-n\, s(1, n) + z\, s(1, n+1) \right]$$

On the other hand

$$J_1(\xi r) = \frac{1}{r} \frac{\partial}{\partial \xi} \left[J_o(\xi r) \right]$$

Therefore

$$s(1,\ n+1) = \int_0^\infty \xi^{n+1} J_1(\xi r)\, e^{-z\xi} d\xi$$

$$= -\frac{1}{r} \int_0^\infty \xi^{n+1} e^{-z\xi} \frac{\partial}{\partial \xi} \left[J_o(\xi r) \right] d\xi$$

Integrating by parts results in

$$s(1,\ n+1) = \frac{1}{r} \left[(n+1)\, s(0,n) - z\, s(0,\ n+1) \right] . \qquad (2.7.2)$$

Inserting the recurrence formula

$$J_\eta(\xi r) = \frac{\xi r}{2\nu} \left[J_{\nu+1}(\xi r) + J_{\nu-1}(\xi r) \right]$$

into the definition (2.6.2) results in

$$s(\nu+1,\ n+1) = \frac{2\nu}{r}\ s(\nu,\ n) - s(\nu-1,\ n+1) \qquad (2.7.3)$$

Solving the equations (2.7.1) and (2.7.2) for $s(0,\ n+1)$ and $s(1,\ n+1)$ gives

$$s(o,\ n+1) = \frac{(n+1)\ z}{r^2+z^2}\ s(0,n) - \frac{nr}{r^2+z^2}\ s(1,n) \qquad (2.7.4)$$

$$s(1,\ n+1) = \frac{(n+1)}{r^2+z^2}\ s(0,n) + \frac{nz}{r^2+z^2}\ s(1,n) \qquad (2.7.5)$$

Since

$$s(\nu, 0) = \int_0^\infty J_\nu(\xi r)\, e^{-z\xi} d\xi = \frac{1}{\sqrt{z^2+r^2}} \left\{ \frac{\sqrt{z^2+r^2}-z}{r} \right\}$$

All of the unknown integrals $s(\nu, n)$, then can be found using the recurrence formula (2.7.4), (2.7.5) and (2.7.3), respectively. A computer program has been written which generates all of the integrals $s(\nu, n)$ from the known integrals $s(\nu, o)$ using the above recurrence relationships. To the author's knowledge the solution for these integrals does not appear in the literature.

By knowing the integrals $s(\nu, n)$ all of the stress components can be easily found using the series solution (2.6.1). A computer program has been written to evaluate these stresses. Although restricted to large values of z and r the series converges after a few terms, the series does not converge when z and r are both less than the crack radius. Therefore, it is necessary to find another solution for this case (when z and r are both less than one) and thus complete the circular crack solution.

2.8 Stress Components for Small Values of z and r

To complete the solution for small values z and r it is necessary to find a closed form solution for the stresses as defined in (2.3.2). To do so, consider only the first three terms of the Fourier cosine expansion for the loading, as well as the first three terms of the power series expansion for the Fourier coefficients.

That is

$$\frac{p(r,\theta)}{2(\beta^2-1)} = B_o(r) + B_1(r)\cos(\theta) + B_2(r)\cos(2\theta) \tag{2.8.1}$$

$$B_n(r) = C_n^o r^o + C_n^1 r^1 + C_n^2 r^2 \qquad (C_n^o = 0 \quad n \neq 0)$$

This procedure can be carried out to as many terms as required, but the above approximation for the Fourier expansion of the loading and Fourier coefficients will be used due to the extreme complexity of including more terms in the series.

To evaluate the function $f_n(\xi)$ defined in (2.5.2) the integral

$$\int_o^1 \eta^{p+3/2} J_{n+1/2}(\xi\eta)\, d\eta \tag{2.8.2}$$

must be evaluated for $n = 0, 1, 2$ and $p = 0, 1, 2$.

These integrations were performed and the results are as follows:

$$\xi^{1/2} \int_o^1 \eta^{3/2} J_{1/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{3/2}(\xi) \qquad (n=0,\ p=0)$$

$$\xi^{1/2} \int_o^1 \eta^{5/2} J_{1/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{3/2}(\xi) - 2\xi^{-3}(1-\cos) + \xi^{-3/2} J_{1/2}(\xi)$$

$$(n=0,\ p=1)$$

$$\xi^{1/2} \int_o^1 \eta^{7/2} J_{1/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{3/2}(\xi) - 2\xi^{-3/2} J_{5/2}(\xi)$$

$$(n=0,\ p=2)$$

$$\xi^{1/2} \int_0^1 \eta^{5/2} J_{3/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{5/2}(\xi) \qquad (n=1,\ p=1)$$

$$\xi^{1/2} \int_0^1 \eta^{7/2} J_{3/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{5/2}(\xi) - 8\xi^{-4}(1-\cos\xi)$$

$$+ 4\xi^{-5/2} J_{1/2}(\xi) + \xi^{-3/2} J_{3/2}(\xi) \qquad (n=1,\ p=2)$$

$$\xi^{1/2} \int_0^1 \eta^{5/2} J_{5/2}(\xi\eta)\, d\eta = 8\xi^{-3}(1-\cos\xi) - 4\xi^{-3/2} J_{1/2}(\xi) - \xi^{-1/2} J_{3/2}(\xi)$$

$$(n=2,\ p=1) \qquad (2.8.3)$$

$$\xi^{1/2} \int_0^1 \eta^{7/2} J_{5/2}(\xi\eta)\, d\eta = \xi^{-1/2} J_{7/2}(\xi) \qquad (n=2,\ p=2)$$

The stress components can be found by inserting $f_n(\xi)$ in (2.3.2). Two new sets of integrals are involved, namely SI (n, m, p) and CI (n, m+n+2) which are defined as

$$SI(n,\ m,\ p) = \int_0^\infty J_n(\xi r)\, J_{m+1/2}(\xi) e^{-z\xi}\, \xi^{p-1/2} d\xi \qquad (2.8.4)$$

$$CI(n,\ m+n+2) = \int_0^\infty J_n(\xi r) e^{-z\xi} (1-\cos\xi)^{m+(n+2)} d\xi \qquad m+n+2 \geq 0 \qquad (2.8.5)$$

Therefore, in order to compute stresses for small r and z these integrals must be evaluated.

2.9 Evaluation of integrals SI (n, m, p)

The closed form solution for these integrals is not available, and even the available solution for some special cases involves elliptic integrals.

A series technique was developed for evaluation of these integrals when $r \leq 1$ as follows:

Let us substitute the series expansion for $J_n(\xi r)$ in the definition (2.8.4) then:

$$SI(n, m, p) = \sum_{s=0}^{\infty} \frac{(-)^s (\frac{1}{2} r)^{n+2s}}{S\,!\,(n+s)\,!} \cdot \int_0^{\infty} J_{m+1/2}(\xi) e^{-z\xi} \xi^{n+2s+p-1/2} d\xi \qquad (2.9.1)$$

Using the hypergeometric form solution available given by Watson(15) for the integrals of the type

$$\int_0^{\infty} e^{-at} J_{\nu}(bt)\, t^{\mu-1}\, dt = \frac{(\frac{1}{2}b)^{\nu} \Gamma(\mu+\nu)}{(a^2+b^2)^{\frac{(\mu+\nu)}{2}} \Gamma(\nu+1)} \cdot {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{b^2}{a^2+b^2}\right) \qquad (2.9.2)$$

where

$$R(\mu + \nu) > 0$$

$$R(a \pm ib) > 0$$

the integrals SI (n, m, p) may be written as

$$SI(n, m, p) = \sum_{s=0}^{\infty} \frac{(-)^s \left(\frac{1}{2} r\right)^{n+2s} \left(\frac{1}{2}\right)^{m+1/2}}{s \; ! \; (n+s) \; !} \cdot$$

$$\cdot \frac{\Gamma(m+n+2s+p+1)}{(1+z^2)^{\frac{m+n+2s+p+1}{2}} \Gamma(m+3/2)} \cdot {}_2F_1\left(\frac{m+n+2s+p+1}{2}, \frac{m-n-2s-p+1}{2}; m+3/2; \frac{1}{z^2+1}\right)$$

$$(m+n+2s+p+1) > 0 \qquad z > 0 \qquad (2.9.3)$$

where as usual, ${}_2F_1(\alpha, \beta; \gamma; x)$ is defined by the equation

$${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\gamma \beta}{1 \; ! \; \gamma} x + \frac{\alpha(\alpha+1) \beta (\beta+1)}{2 \; ! \; \gamma(\gamma+1)} x^2 + \ldots\ldots \qquad (2.9.4)$$

The hypergeometric series in equation (2.9.3) always converges, therefore the integrals of SI (n, m, p) can be evaluated from the series solution. A computer program was written which generates these integrals whenever they exist.

2.10 Evaluation of Integrals CI (n, m+n+2)

The closed form solutions to these integrals may be found from some known integrals by integration by parts and using the relationships

$$J_1(\xi r) = -\frac{1}{r} \frac{\partial}{\partial \xi} \left[J_o(\xi r) \right]$$

$$\xi^{\nu} J_{\nu-1}(\xi r) = \frac{1}{r} \frac{\partial}{\partial \xi} \left[\xi^{\nu} J_{\nu}(\xi r) \right] \qquad (2.10.1)$$

$$J_{\nu}(\xi r) = \frac{\xi r}{2\nu} \left[J_{\nu+1}(\xi r) + J_{\nu-1}(\xi r) \right]$$

For example, CI (0, 0) can be found from CI (1, 2) given in reference (16) as follows

$$CI(1,2) = \int_0^{\infty} \frac{(1-\cos\xi)}{\xi} J_1(\xi r) e^{-z\xi} d\xi$$

$$= -\frac{1}{r} \int_0^{\infty} \frac{(1-\cos\xi)}{\xi} e^{-z\xi} \frac{\partial}{\partial \xi} \left[J_o(\xi r) \right] d\xi$$

Integrating by parts results in

$$CI(1,2) = \frac{1}{r} \left\{ \int_0^{\infty} \frac{\sin\xi}{\xi} J_o(\xi r) e^{-z\xi} d\xi \right.$$

$$- \int_0^{\infty} \frac{1-\cos\xi}{2} e^{-z} J_o(r) d\xi$$

$$\left. -z \int_0^{\infty} \frac{1-\cos\xi}{\xi} e^{-z\xi} J_o(\xi r)\, d\xi \right\}$$

or

$$CI(0,0) = -r\, CI(1,2) - z\, CI(0,1) + S01$$

where

$$S01 = \int_0^{\infty} \frac{\sin\xi}{\xi} J_o(\xi r)\, e^{-z\xi}\, d\xi$$

By the same procedure all of these integrals may be found. It is first necessary to derive the integrals

$$Snm = \int_0^\infty \frac{\text{Sin } \xi}{\xi^m} J_n(\xi r) e^{-z\xi} d\xi \qquad (2.10.11)$$

$$Cnm = \int_0^\infty \frac{\text{Cos } \xi}{\xi^m} J_n(\xi r) e^{-z\xi} d\xi$$

These integrals may be evaluated with the results shown below.

$$S01 = \tan^{-1} \frac{1}{X} \qquad z \neq 0$$

$$S01 = \frac{\pi}{2} \qquad z = 0$$

$$S11 = \frac{1}{r} (1 - R^{-1/2} \text{Sin } \frac{\phi}{2})$$

$$C11 = \frac{1}{r} (R^{1/2} \text{Cos } \frac{\phi}{2} - z)$$

$$S21 = \frac{1}{r} (C11 - z\ S11)$$

$$C21 = \frac{1}{2} - \frac{1}{r} \left[z\ C11 + S11 \right]$$

$$S22 = \frac{1}{3} \left[r\ S11 - z\ S21 + C21 \right]$$

$$C22 = \frac{1}{3} \left[r\ C11 - z\ C21 - S21 \right]$$

$$S23 = \frac{1}{4} \left[r\ S12 - z\ S22 + C22 \right]$$

where

$$2X^2 = \left[(r^2 + z^2 - 1)^2 + 4\,z^2\right]^{1/2} + r^2 + z^2 - 1$$

$$2Y^2 = \left[(r^2 + z^2 - 1)^2 + 4\,z^2\right]^{1/2} - (r^2 + z^2 - 1)$$

$$R = (r^2 + z^2 - 1^2) + 4\,z^2$$

$$\tan\phi = \frac{z^2}{r^2+z^2-1}$$

Knowing these results it is now possible to derive the solutions for the integrals CI(n, m + n + 2) given below

$$CI(0, 1) = \frac{1}{2}\log\left[(z + X)^2 + (1 + Y)^2\right] - \log\left[z + (r^2+z^2)^{1/2}\right] \qquad r \neq 0$$

$$CI(0, 1) = \frac{1}{2}\left[1 + z^2\right] - \log z \qquad r = 0$$

$$CI(1, 2) = \frac{1}{r}\left[(r^2 + z^2)^{1/2} - X\right] \qquad r \neq 0$$

$$CI(1, 2) = 0 \qquad r = 0$$

$$CI(1, 1) = \frac{1}{2r}\left\{1 - Y + z\left[X - (r^2 + z^2)^{1/2}\right]\right\} + \frac{1}{4}\,r\log\left[(z + X)^2 + (1 + Y)^2\right] - \frac{1}{r}\log\left[z+(r^2+z^2)^{1/2}\right] \qquad r \neq 0$$

$$CI(1, 1) = 0 \qquad r = 0$$

(2.10.12)

$$CI(0, 0) = -r\,CI(1, 2) - z\,CI(0, 1) + S01$$

$$CI(1, 0) = \frac{1}{3}\left[r\,CI(0, 0) - z\,CI(1, 1) + S12\right]$$

$$CI(2,2) = \frac{2}{r}\, CI(1,0) - CI(0,0)$$

$$CI(2,3) = \frac{2}{r}\, CI(1,1) - CI(0,1)$$

$$CI(2,1) = \frac{1}{4}\left[r\, CI(1,1) - z\, CI(2,2) + S22\right]$$

$$CI(2,0) = \frac{1}{5}\left[r\, CI(1,0) - z\, CI(2,1) + S23\right]$$

$$CI(3,2) = \frac{4}{r}\left[CI(2,0) - CI(1,0)\right]$$

$$CI(3,3) = \frac{4}{r}\left[CI(2,1) - CI(1,1)\right]$$

$$CI(4,5) = \frac{4}{r}\left[CI(2,2) - CI(1,2)\right]$$

Having the integrals of SI(n, m, p) and CI(n, m+n+2) all of the stress components may be found from the equations (2.3.2). For example:

$$\tau_{\theta z} = \frac{1}{\sqrt{2}}\, C_1^1\, H_1^1 \left[a_8\, SI(1,2,1)\right] \operatorname{Sin}\theta$$

$$+ \frac{1}{\sqrt{2}}\, C_1^1\, H_1^2 \left\{ a_8 \left[SI(1,2,1) + 4\, SI(1,0,-1) \right.\right.$$

$$\left.\left. + SI(1,1,0) - 8\sqrt{\frac{2}{\pi}}\, CI(1,0) \right]\right\} \operatorname{Sin}\theta$$

$$+ \frac{1}{\sqrt{2}}\, C_2^1\, H_2^1 \left\{ a_8 \left[-4\, SI(2,0,0) - SI(2,1,1) \right.\right.$$

$$\left.\left. + 8\sqrt{\frac{2}{\pi}}\, CI(2,2) \right]\right\} \operatorname{Sin}(2\theta)$$

$$+ \frac{1}{\sqrt{2}}\, C_2^2\, H_2^2 \left[a_8\, SI(2,3,1)\right] \operatorname{Sin}(2\theta)$$

The other expressions for stresses are very lengthy, and therefore, do not appear here. However, they may be found in Appendix "A" as well as in the computer program.

2.11 Special Case for $z = 0$

On the plane $z = 0$ many terms in the stress formulation vanish ($a_1 = a_4 = a_6 = a_8 = 0$). The stress formulations for this case are:

$$\tau_{rr} = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \left[a_2 \frac{1}{\xi} J_n(\xi r) + a_3 \xi J_n(\xi r) + a_5 J_{n+1}(\xi r) \right] f_n(\xi) d\xi$$

$$\tau_{\theta\theta} = \sum_{n=0}^{\infty} \cos(n\theta) \int_0^{\infty} \left[-a_2 \frac{1}{\xi} J_n(\xi r) + a_9 \xi J_n(\xi r) - a_5 J_{n+1}(\xi r) \right] f_n(\xi) d\xi$$

$$\tau_{r\theta} = \sum_{n=0}^{\infty} \mathrm{Sin}(n\theta) \int_0^{\infty} \left[-a_1 \frac{1}{\xi} J_n(\xi r) + a_7 J_{n+1}(\xi r) \right] f_n(\xi) d\xi \qquad (2.11.1)$$

$$\tau_{rz} = 0$$

$$\tau_{\theta z} = 0$$

Inserting the values of $f_n(\xi)$ from the equations (2.8.3) in the above equations, two new sets of integrals, namely SI´(n, m, p) and CI´(n, m+n+2) similar to SI(n, m, p) and CI(n, m+n+2) will be found where

$$S'I(n, m, p) = \int_0^{\infty} J_n(\xi r) J_{m+1/2}(\xi) \xi^{p-1/2} d\xi \qquad (2.11.2)$$

$$C\,\mathfrak{I}(n,\, m+n+2) \;=\; \int_0^{\infty} J_n(\xi r)\,(1 - \cos\xi)^{m+n+2}\, d\xi$$

The integral S I(n, m, p) is a special case of the Weber-Schafheitlin type integral (17) for which the solution can be written in the form of:

$$S\,\mathfrak{I}(n,\, m,\, p) = \frac{r^n\ \Gamma\left(\frac{m+n+p+1}{2}\right)}{2^{-p+1/2}\,\Gamma(n+1)\,\Gamma\left(\frac{m-n-p+2}{2}\right)} \cdot$$

$$\cdot\ {}_2F_1\left(\frac{m+n+p+1}{2}\,,\ \frac{-m-n+p}{2}\,;\ n+1,\ \frac{1}{r^2}\right)$$

$$0 < r < 1\ ,\quad (m+n+p+1) > 0 \quad \text{and} \quad \left(-p+\tfrac{1}{2}\right) > -1$$

(2.11.3)

A computer program was written which generates these integrals whenever they exist. The integrals C I(n, n+m+2) may be found in closed form from some known integrals by exactly the same procedure as used for the integrals CI(n, m+n+2). These integrals are found to be

$$C'I(1,\, 2) \;=\; 1 \qquad\qquad r < 1$$

$$C'I(1,\, 2) \;=\; 1 - \frac{1}{r}\,(1 - r^2)^{1/2} \qquad\qquad r > 1$$

$$C'I(1,\, 1) \;=\; \frac{1}{2r} \qquad\qquad r > 1$$

$$C'I(1,\, 1) \;=\; \frac{1}{2r}\left[1 - (1-r^2)^{1/2}\right] + \frac{r}{2}\left\{\log\left[1+(1-r^2)^{1/2}\right] - \log r\right\}$$

$$r < 1$$

$$S'01 = \text{Sin}^{-1}\left(\frac{1}{r}\right) \qquad r > 1$$

$$S'01 = \frac{\pi}{2} \qquad r < 1$$

$$S'11 = \frac{1}{r} \qquad r > 1$$

$$S'11 = \frac{r}{1+(1-r^2)^{1/2}} \qquad r < 1$$

$$C'11 = \frac{1}{r}(r^2-1)^{1/2} \qquad r > 1$$

$$C'11 = 0 \qquad r < 1 \qquad (2.11.4)$$

$$S'21 = \frac{1}{2}\ \text{Sin}\left[2\ \text{Arc Sin}\left(\frac{1}{r}\right)\right] \qquad r > 1$$

$$S'21 = 0 \qquad r < 1$$

$$C'21 = \frac{1}{2}\ \text{Cos}\left[2\ \text{Arc Sin}\left(\frac{1}{r}\right)\right] \qquad r > 1$$

$$C'21 = \frac{-r^2/2}{1+(1-r^2)^{1/2}} \qquad r < 1$$

where

$$S'nm = \int_0^\infty \frac{\text{Sin}\ \xi}{\xi^m}\ J_n(\xi r)\ d\xi \qquad (2.11.5)$$

$$S'nm = \int_0^\infty \frac{\text{Cos}\ \xi}{\xi^m}\ J_n(\xi r)\ d\xi$$

Having these integrals, all components of stresses on the plane $z = 0$ may be found from the equations (2.11.1). The expressions for these stresses are listed in Appendix "A."

2.12 The Fourier Coefficients $B_n(r)$

The Fourier coefficients $B_n(r)$ as defined in Section (1.2) were found by numerical integration, using the Weddle's Rule with 18 equally spaced intervals ($h = \frac{\pi}{18}$):

$$B_o(r) = \frac{1}{2\pi(\beta^2-1)} \int_o^{\pi} p(r,\theta)\, d\theta =$$

$$\frac{.3\,h}{2\pi(\beta^2-1)} \left[y_o + 5y_1 + y_2 + \ldots\ldots\ldots y_{18} \right]$$

(2.12.2)

$$B_n(r) = \frac{1}{\pi(\beta^2-1)} \int_o^{\pi} p(r,\theta)\cos(n\theta)\, d\theta =$$

$$\frac{.3\,h}{\pi(\beta^2-1)} \left[y_o + 5y_1 + y_2 + \ldots\ldots\ldots y_{18} \right]$$

where the function y identifies the function $p(r,\theta)$ or $p(r,\theta)\cos(n\theta)$.

The equations (2.12.1) are evaluated at six equally spaced values of r for $0 < r < 1$.

2.13 The Constants C_n^p

In Section (2.5) the Fourier coefficients of the loading were expanded in the power series

$$B_n(r) = \sum_{p=0}^{N} C_n^p\, r^p \qquad (2.13.1)$$

The constants C_n^p in equation (2.13.1) are determined by least squares fitting a second order polynominal to six values of the function $B_n(r)$ which are calculated as described in the preceding paragraph.

2.14 The Stress Intensity Factor

The stress intensity factor k_I is defined as

$$k_I = \lim_{\delta \to 0} \sqrt{2\pi\delta}\ \tau_{zz} \Big|_{z=0} \qquad (2.14.1)$$

where δ is a small distance measured from the crack tip, and τ_{zz} is the normal stress on the plane of the crack.

If k_I reaches a certain value k_{IC} called the critical stress intensity factor, the crack starts to propagate. The critical stress intensity factor is a basic material property and can be measured in the laboratory. Substituting for τ_{zz} in equation (2.14.1) and performing the limiting procedure yields:

$$k_I = 2\mu\,(\beta^2-1) \sum_{n=0}^{\infty} \cos(n\theta) \sum_{p=0}^{N} C_n^p\ H_n^p \qquad (2.14.2)$$

Where H_n^p and C_n^p are defined by equations (2.5.3) and (2.13.1). This derivation was first presented by Smith (12).

CHAPTER III

HALF SPACE SOLUTION

3.1 The General Problem

In the step III of the solution it is necessary to remove the stresses from the plane boundary of the semi-infinite solid (plane P, Figures 1 and 2) with no crack. This will be accomplished by the following approximate technique. Divide the plane boundary into small rectangular areas, and assume that the surface shears and normal stress are uniform inside of each rectangle and have the values corresponding to that of the center of the rectangles. Therefore, the problem reduces to that of finding a solution for a semi-infinite solid when a small rectangular area of its bounding plane is subjected to uniform shears and normal stress. This solution for constant normal stress was presented by Love (18). The solution for constant shears on the rectangle did not seem to be available in the literature. Therefore, this solution was derived and is described in the following paragraphs.

3.2 The distribution of stresses in a semi-infinite solid when a small rectangular area of its boundary plane is subjected to given surface forces.

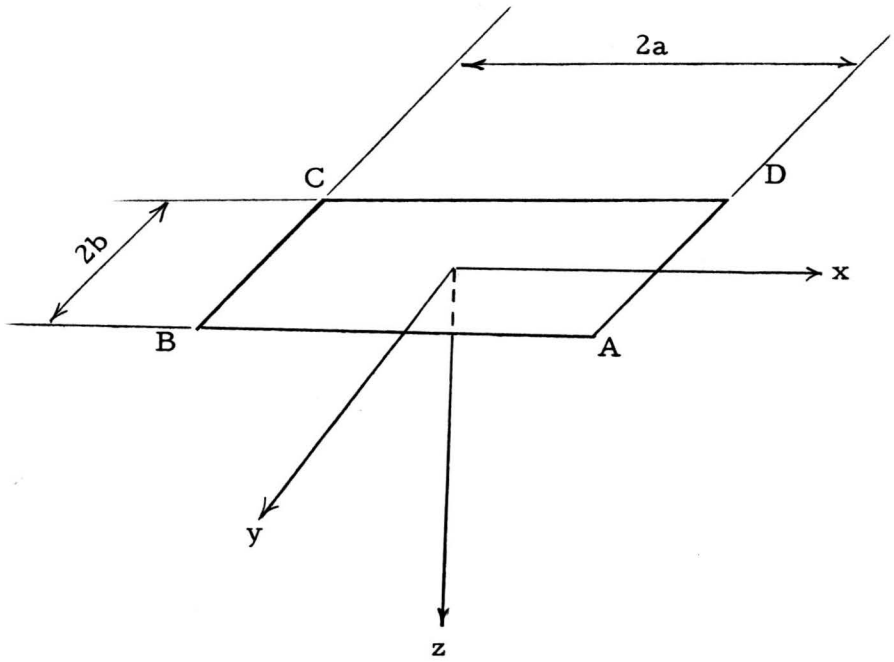


Figure 4. Rectangular Coordinate System for Half Space Solution

Consider the plane boundary of the semi-infinite solid as horizontal, and the solid below it as shown in Figure 4. Let (x, y, z) be coordinates of a point within the solid and $(x', y', 0)$ be coordinates of a point inside of the rectangular area. We denote the distance between these points by R so that:

$$R^2 = (x - x')^2 + (y - y')^2 + z^2$$

The displacements (u, v, w) produced at any point (x, y, z) by the surface forces applied over the rectangular area ABCD are given by:

$$u = \frac{1}{2\pi\mu}\frac{\partial F}{\partial z} - \frac{1}{4\pi\mu}\frac{\partial H}{\partial x} + \frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial \Psi_1}{\partial x} - \frac{1}{4\pi\mu} z \frac{\partial \Psi}{\partial x}$$

(3.2.1)

$$v = \frac{1}{2\pi\mu}\frac{\partial G}{\partial z} - \frac{1}{4\pi\mu}\frac{\partial H}{\partial y} + \frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial \Psi_1}{\partial y} - \frac{1}{4\pi\mu} z \frac{\partial \Psi}{\partial y}$$

$$w = \frac{1}{4\pi\mu}\frac{\partial H}{\partial z} + \frac{1}{4\pi(\lambda+\mu)}\Psi - \frac{1}{4\pi\mu} z \frac{\partial \Psi}{\partial z}$$

where λ and μ are the Lame' constants. This formulation satisfies the equations of elasticity and boundary conditions if the potential function χ and the harmonic functions F, G, H, Ψ, $\curlywedge$, F_1, G_1, H_1 and Ψ_1 are defined as:

$$\chi = \log (z + R)$$

$$F = \iint_S X_\nu \chi' \, dx' dy' = X_\nu \int_{-b}^{b} \int_{-a}^{a} \chi \, dx' dy'$$

$$G = \iint_S Y_\nu \chi \, dx' dy' = Y_\nu \int_{-b}^{b} \int_{-a}^{a} \chi \, dx' dy'$$

$$H = \iint_S Z_\nu \chi \, dx' dy' = z_\nu \int_{-b}^{b} \int_{-a}^{a} \chi \, dx' dy' \qquad (3.2.2)$$

$$\Psi = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}$$

$$\curvearrowright = z \log (z + R) - R$$

$$F_1 = \iint X_\nu \curvearrowright \, dx' dy' = X_\nu \int_{-b}^{b} \int_{-a}^{a} \curvearrowright \, dx' dy'$$

$$G_1 = \iint Y_\nu \curvearrowright \, dx' dy' = Y_\nu \int_{-b}^{b} \int_{-a}^{a} \curvearrowright \, dx' dy'$$

$$H_1 = \iint Z_\nu \curvearrowright \, dx' dy' = z_\nu \int_{-b}^{b} \int_{-a}^{a} \curvearrowright \, dx' dy'$$

$$\Psi_1 = \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z}$$

Where X_ν, Y_ν, and Z_ν are the components of the surface force per unit area at the center of the rectangle (19).

All of these integrations are taken over the rectangular area ABCD given by the equations

$$z = 0 \qquad x = \pm a \qquad y = \pm b$$

Using the generalized Hookes law (2.1.4) the stress components can be found from the equations (3.2.1).

The σ_{yy} component is the only stress which is needed for iteration technique. This stress is composed of three parts, namely, σ_{yy_1}, σ_{yy_2}, and σ_{yy_3} which correspond to the effect of surface forces applied over rectangle in the x, y, and z directions, respectively when each is considered separately. Considering the effect of a surface force in x direction only results in

$$\sigma_{yy_1} = \frac{1}{2\pi} \left\{ \frac{\lambda}{\lambda+\mu} \frac{\partial \psi}{\partial z} + \frac{\lambda}{\lambda+\mu} \frac{\partial^2 \psi_1}{\partial y^2} - z \frac{\partial^2 \psi}{\partial y^2} \right\} \qquad (3.2.3)$$

The functions $\frac{\partial \psi}{\partial z}$, $\frac{\partial^2 \psi}{\partial y^2}$ and $\frac{\partial^2 \psi_1}{\partial y^2}$ may be found from the equations (3.2.2) as:

$$\frac{\partial \psi}{\partial z} = X_\nu \log \left\{ \frac{\left[d_4 - (b+y) \quad b_2 + (b-y) \right]}{\left[a_1 + (b+y) \quad c_3 + (b+y) \right]} \right\} \qquad (3.2.4)$$

$$\frac{\partial^2 \psi}{\partial y_2} = X_\nu \left\{ - \frac{(b-y)}{a_1(z+a_1)} + \frac{(b-y)}{b_2(z+b_2)} \right.$$

$$\left. + \frac{(b+y)}{c_3(z+c_3)} - \frac{(b+y)}{d_4(z+d_4)} \right\} \qquad (3.2.5)$$

$$\frac{\partial^2 \psi_1}{\partial y^2} = z \frac{\partial^2 \psi}{\partial y^2} + X_\nu \left\{ \frac{(b-y)}{a_1} - \frac{(b-y)}{b_2} \right. \tag{3.2.6}$$

$$\left. - \frac{(b+y)}{c_3} + \frac{(b+y)}{d_4} \right\}$$

Considering the effect of surface force in the y direction we find:

$$\sigma yy_2 = \frac{1}{2\pi} \left\{ \frac{3\lambda + 2\mu}{\lambda + \mu} \frac{\partial \psi}{\partial z} + \frac{\lambda}{\lambda + \mu} \frac{\partial^2 \psi_1}{\partial y^2} - z \frac{\partial^2 \psi}{\partial y^2} \right\} \tag{3.2.7}$$

where:

$$\frac{\partial \psi}{\partial z} = Y_\nu \log \left\{ \frac{\left[d_4 + (a-x) \quad b_2 - (a+x) \right]}{\left[a_1 + (a-x) \quad c_3 - (a+x) \right]} \right\} \tag{3.2.8}$$

$$\frac{\partial^2 \psi}{\partial x^2} = Y_\nu \left\{ - \frac{(a-x)}{a_1(z+a_1)} - \frac{(a+x)}{b_2(z+b_2)} \right.$$

$$\left. + \frac{(a+x)}{c_3(z+c_3)} + \frac{(a-x)}{d_4(z+d_4)} \right\} \tag{3.2.9}$$

$$\frac{\partial^2 \psi}{\partial z^2} = Y_\nu \left\{ \frac{z}{(b-y)^2 + z^2} \left(\frac{a-x}{a_1} + \frac{a+x}{b_2} \right) \right.$$

$$\left. - \frac{z}{(b+y)^2 + z^2} \left(\frac{a-x}{d_4} + \frac{a+x}{c_3} \right) \right\} \tag{3.2.10}$$

$$\frac{\partial^2 \Psi}{\partial y^2} = -\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) \tag{3.2.11}$$

$$\frac{\partial^2 \Psi_1}{\partial z^2} = \frac{\partial \Psi}{\partial z} \tag{3.2.12}$$

$$\frac{\partial^2 \Psi_1}{\partial x^2} = z \frac{\partial^2 \Psi}{\partial x^2} + Y_\nu \left[\frac{(a-x)}{a_1} + \frac{(a-x)}{b_2} - \frac{(a-x)}{c_3} - \frac{(a-x)}{d_4} \right] \tag{3.2.13}$$

$$\frac{\partial^2 \Psi_1}{\partial y^2} = -\left(\frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial z^2} \right) \tag{3.2.14}$$

Considering the effect of a surface force in the z direction results in

$$\sigma_{yy_3} = \frac{1}{2\pi} \left\{ \frac{\lambda}{\lambda+\mu} \frac{\partial \psi}{\partial z} - \frac{\mu}{\lambda+\mu} \frac{\partial^2 \chi}{\partial y^2} - z \frac{\partial^2 \psi}{\partial y^2} \right\} \tag{3.2.15}$$

where:

$$\frac{\partial^2 \chi}{\partial x^2} = z_\nu \left[\tan^{-1} \frac{(b-y)}{(a-x)} + \tan^{-1} \frac{b+y}{(a-x)} - \tan^{-1} \frac{z(b-y)}{(a-x)\, a_1} \right.$$

$$- \tan^{-1} \frac{z(b+y)}{(a-x) d_4} + \tan^{-1} \frac{(b-y)}{(a+x)} + \tan^{-1} \frac{(b+y)}{(a+x)}$$

$$\left. - \tan^{-1} \frac{z(b-y)}{(a+x) b_2} - \tan^{-1} \frac{z(b+y)}{(a+x) c_3} \right] \qquad (3.2.16)$$

$$\frac{\partial^2 \chi}{\partial y^2} = Z_\nu \left[\tan^{-1} \frac{(a-x)}{(b-y)} + \tan^{-1} \frac{(a+x)}{(b-y)} \right.$$

$$-\tan^{-1} \frac{z(a-x)}{(b-y) a_1} - \tan^{-1} \frac{z(a+x)}{(b-y)\, b_2}$$

$$\left. + \tan^{-1} \frac{(a-x)}{b+y} + \tan^{-1} \frac{(a+x)}{(b+y)} - \tan^{-1} \frac{z(a-x)}{(b+y) d_4} \right]$$

$$- \tan^{-1} \frac{z(a+x)}{(b+y) c_3} \qquad (3.2.17)$$

$$\frac{\partial \psi}{\partial z} = - \left(\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} \right) \qquad (3.2.18)$$

$$\frac{\partial^2 \psi}{\partial y^2} = - Z_\nu \left\{ \frac{(b-y)}{(b-y)^2 + z^2} \left(\frac{(a-x)}{a_1} + \frac{(a+x)}{b_2} \right) + \right. \qquad (3.2.19)$$

$$\left. \frac{(b+y)}{(b+y)^2 + z^2} \left(\frac{a-x}{d_4} + \frac{a+x}{c_3} \right) \right\}$$

In these derivations a_1, b_2, c_3 and d_4 represent the distance from any point (x, y, z) to the corners A, B, C, and D of the rectangular area ABCD that is

$$a_1 = \left[(a-x)^2 + (b-y)^2 + z^2\right]^{1/2}$$

$$b_2 = \left[(a+x)^2 + (b-y)^2 + z^2\right]^{1/2} \qquad (3.2.20)$$

$$c_3 = \left[(a+x)^2 + (b+y)^2 + z^2\right]^{1/2}$$

$$d^4 = \left[(a-x)^2 + (b+y)^2 + z^2\right]^{1/2}$$

Finally, the stress component σyy produced by application of surface shears and normal stress on the rectangular area ABCD can be found by superposition:

$$\sigma yy = \sigma yy_1 + \sigma yy_2 + \sigma yy_3 \qquad (3.2.21)$$

or

$$\sigma yy = X_\nu f_1(x, y, z) + Y_\nu f_2(x, y, z) + Z_\nu f_3(x, y, z) \qquad (3.2.22)$$

where f_1, f_2 and f_3 characterize the equations (3.2.3), (3.2.7) and (3.2.15) and have the following symmetry properties:

f_1 is odd x even y

f_2 is even x odd y $\qquad (3.2.23)$

f_3 is even x even y

3.3 Generalization of the Solution

The solution thus found applies to a small rectangular area. However, it can be generalized to many rectangles or a plane boundary. Let ξ, η and ζ be a local coordinate system located at the center of each rectangle as shown in Figure 5. Let $\bar{y}_{1i}$ and $\bar{z}_{1j}$ designate the coordinate of the center of these rectangles. It is apparent that:

$$\begin{aligned} \xi &= y_1 = \bar{y}_{1i} \\ \eta &= z_1 - \bar{z}_{1i} \\ \zeta &= x_1 \end{aligned} \tag{3.3.1}$$

The x_ν, y_ν and z_ν for each rectangle correspond to those stresses found from the circular crack solution on the plane boundary as found in Section (2.4), that is:

$$\begin{aligned} X_\nu &= -\tau_{x1y1} \\ Y_\nu &= -\tau_{x1z1} \\ Z_\nu &= -\tau_{x1x1} \end{aligned} \tag{3.3.2}$$

Inspection of equations (2.4.3) reveals that these stresses have certain symmetry properties, that is:

τ_{x1y1} is odd y_1 even z_1

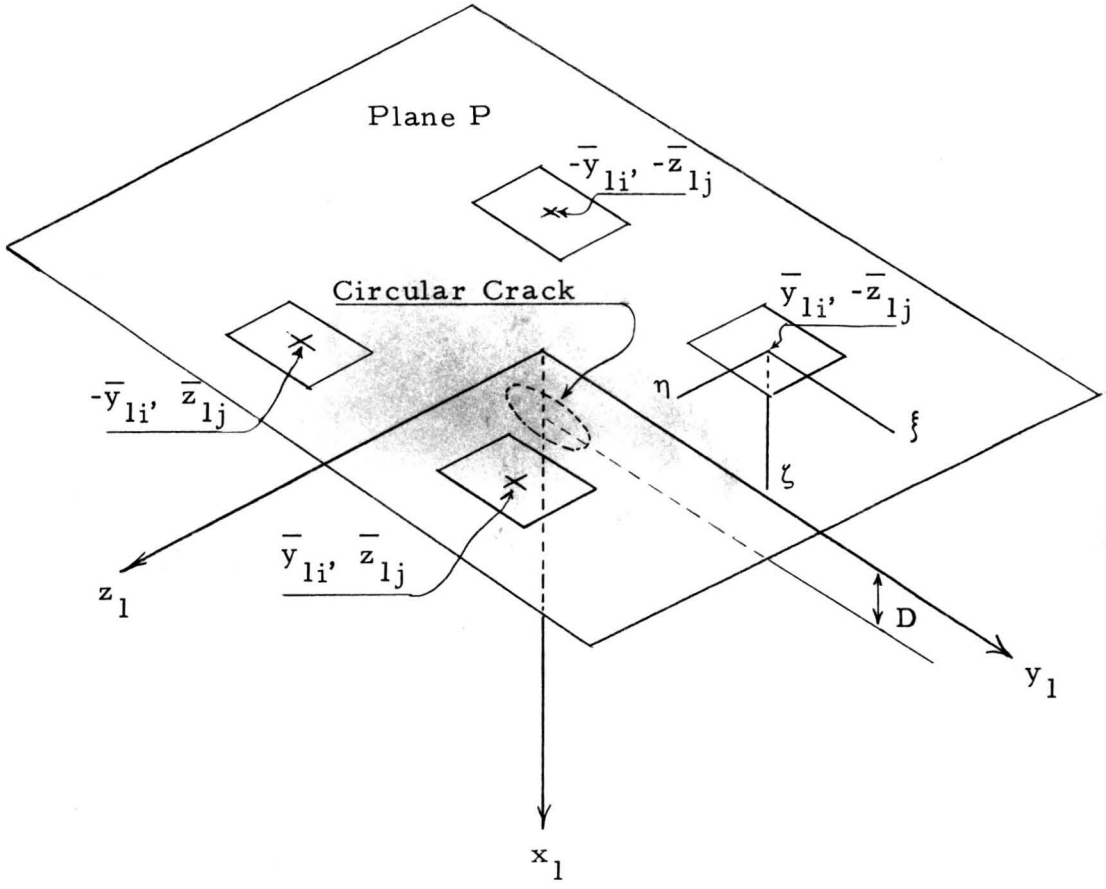


Figure 5. Symmetrical Distribution of the Areas On Plane P Located a Distance D from the Center of the Crack

τ_{xlzl} is even y_1 odd z_1 (3.3.3)

τ_{xlxl} is even y_1 even z_1

Now consider the effect of surface forces on four rectangular areas, symmetrically located in four quadrants (Figure 5). The stress σ_{zlzl}, at any point due to these forces is:

$$\sigma_{zlzl} = \sigma_{zz} = X_\nu(\bar{y}_{1i}, \bar{z}_{1j}) \left[f_1(y_1-\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) - f_1(y_1+\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) + f_1(y_1-\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) - f_1(y_1+\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) \right]$$

$$+ Y_\nu(\bar{y}_{1i}, \bar{z}_{1j}) \left[f_2(y_1-\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) + f_2(y_1+\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) - f_2(y_1-\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) - f_2(y_1+\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) \right]$$

$$+ Z_\nu(y_{1i}, z_{1j}) \left[f_3(y_1-\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) \; f_3(y_1+\bar{y}_{1i}, z_1-\bar{z}_{1j}, x_1) \; f_3(y_1-\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) \; f_3(y_1+\bar{y}_{1i}, z_1+\bar{z}_{1j}, x_1) \right] \quad (3.3.4)$$

Considering the symmetry properties described in (3.2.23) and considering all of the rectangular areas on the plane P, the total stress at any point $(x_1, y_1, 0)$ located on the plane $z_1 = 0)$ is found to be

$$\sigma_{z1z1} = \sum_{i=1}^{M}\sum_{j=1}^{N} 2X_{\nu}(\bar{y}_{1i}, \bar{z}_{1j}) \left[f_1(y_1 - \bar{y}_{1i}, \bar{z}_{1j}, x) - f_1(y_1 + \bar{y}_{1i}, \bar{z}_{1j}, x) \right]$$

$$- \sum_{i=1}^{M}\sum_{j=1}^{N} 2Y_{\nu}(\bar{y}_{1i}, \bar{z}_{1j}) \left[f_2(y_1 - \bar{y}_{1i}, \bar{z}_{1j}, x) + f_2(y_1 + \bar{y}_{1i}, \bar{z}_{1j}, x) \right]$$

$$+ \sum_{i=1}^{M}\sum_{j=1}^{N} 2Z_{\nu}(\bar{y}_{1i}, \bar{z}_{1j}) \left[f_3(y_1 - \bar{y}_{1i}, \bar{z}_{1j}, x) + f_3(y_1 + \bar{y}_{1i}, \bar{z}_{1j}, x) \right]$$

(3.3.5)

where the summation is taken over all of the rectangles on the first quadrant.

If we indicate the coordinates of a general point on the surface of the circular crack by $(r, \theta, 0)$ the stress produced at this point due to removal of stress from the plane $\underline{P}$, may be found from the equations (3.3.5). From Figure 6 it is obvious that:

$$f(y_1 - \bar{y}_{1i}, \bar{z}_{1j}, x_1) = f(r \sin\theta - \bar{y}_{1i}, \bar{z}_{1j}, r \cos\theta + d)$$

$$f(y_1 + y_{1i}, z_{1j}, x_1) = f(r \sin\theta + \bar{y}_{1i}, \bar{z}_{1j}, r \cos\theta + d)$$

(3.3.6)

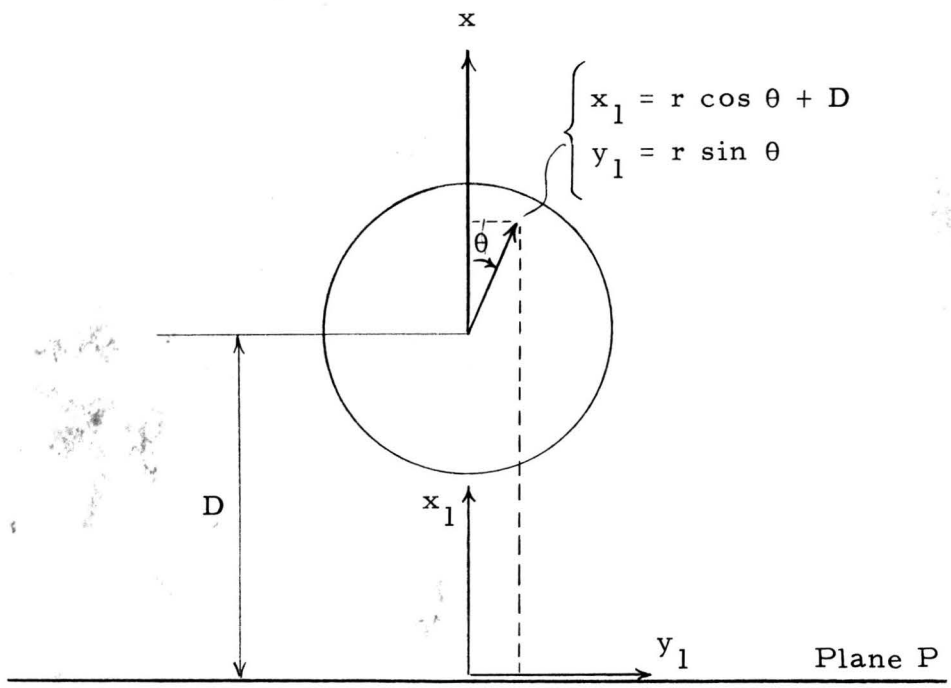


Figure 6. Relationship Between Cylindrical and Cartesian Coordinate System

CHAPTER IV

RESULTS

4.1 General Remarks

A computer program has been written using the solution and technique described in the preceeding chapters to evaluate the stresses and stress intensity factors for several classes of problems indicated in Figures 1 and 2. The stresses in the solid and on the free plane were found from the circular crack solution given by equations (2.6.2) and 2.10.12). The results obtained for a uniform tension from this solution agrees completely with the solution obtained by Sneddon (1) and Smith (12). To remove the stresses from the plane p a grid having the boundaries r = 3, and z = 4 was first attempted. These boundaries were adequate since all of the stresses vanish at these distances as was observed from the case of simple tension. Figure (7) indicates the final grid which was used in the analysis. The number of rectangles in the final analysis were 4 times larger than the first grid tried, but the maximum change in the stress intensity factor was not greater than 10%. The stresses on the plane p and on the crack surface vanish after a few cycles of iteration. For example, when the free surface is located at a distance D = 1.1, complete convergence is obtained after three

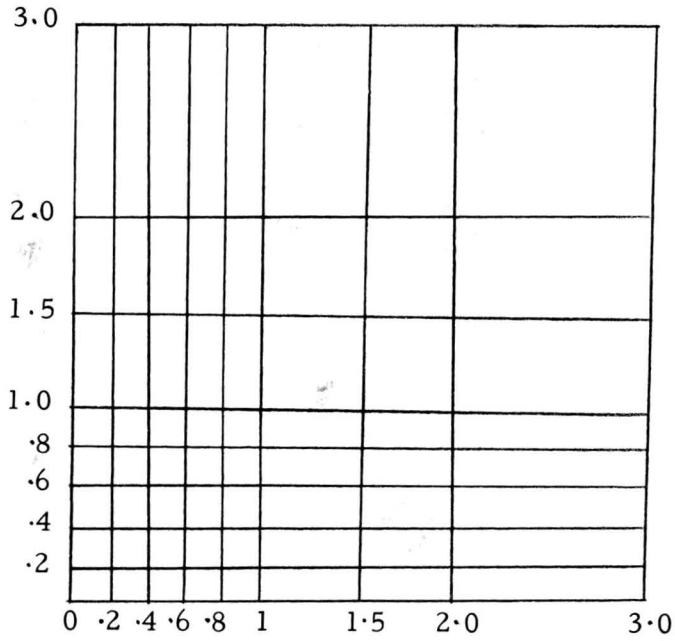


Figure 7. Grid Spacing on the Bounding Plane

iterations as indicated in Table 1. Two types of loading were considered: constant load $p(r, \theta) = \sigma$ and linearly varying load $p(r, \theta) = \sigma(1 + r \cos\theta)$. The graphs in Figures 8, 9, 10 and 11 present the stress intensity factors for several types of loading and crack geometries. The stress intensity factor for some other types of loading can be found from superposition of these results. It can be seen from the results of Figure (7) that the maximum stress intensity factor does not change if the free surface is located at a distance $D > 2a$.

4.2 Comparison with Available Solutions

Some comparison with two dimensional analysis, and other available solutions can be made to check the validity of the results.

1) The magnification factor in a strip subjected to uniform tension based on two dimensional analysis is approximately 1.35 for $D = 1.5$ (see Figure 12). This factor for a circular crack for $D = 1.5$ was found to be 1.03. This reduction in stress intensity factor was expected, since the state of stress for two dimensional analyses is more severe than three dimensional.

2) When the free plane is close to the center of the crack ($D = 0.1$), the stress intensity factor obtained is within 2% of the results obtained by Smith (12) for a semi-circular edge crack (Figure 10). Part of the error is due to the fact that $D = .1$ is not exactly a semi-circular crack.

3) When the free plane is located a distance D from the center of the crack as shown in Figure 2, Case 2, the stress intensity factor obtained can be compared to the stress intensity factor for an ellipse embedded in an infinite solid as shown in Figure 13. The stress intensity factor for an ellipse with major and minor axes of $2c$ and $2a_e$ is

$$K_I = \frac{\sigma \sqrt{\pi a}}{\Phi} \left[\frac{a^2}{c^2} \sin^2 \phi + \cos^2 \phi \right]^{1/4}$$

$$m = 1 - \frac{a^2}{c^2}$$

where $\Phi = E(m)$ is a complete elliptic integral of second kind that is

$$\Phi = \int_0^{\pi} \sqrt{1 - m^2 \sin^2 \Phi} \; d\Phi$$

for $\phi = 0$, the stress intensity factor will have its maximum value

$$K_{Imax} = \frac{\sigma \sqrt{\pi a}}{\Phi}$$

For the case when $D = .5$, a_e for the ellipse is .5 and is half of the radius of the circle so that

$$m = .6666$$

and

$$\Phi = E(.6666) = 1.26$$

Therefore

$$K_{Imax} = \frac{1}{1.26} \sqrt{\pi a}_e = .794 \sqrt{\pi a}_e$$

for the embedded ellipse. The stress intensity obtained for the present edge crack is

$$K_{Imax} = 0.807 \sigma \sqrt{\pi a}_e$$

The magnification factor is therefore 1.02 which agrees quite well with the value estimated by Smith (23).

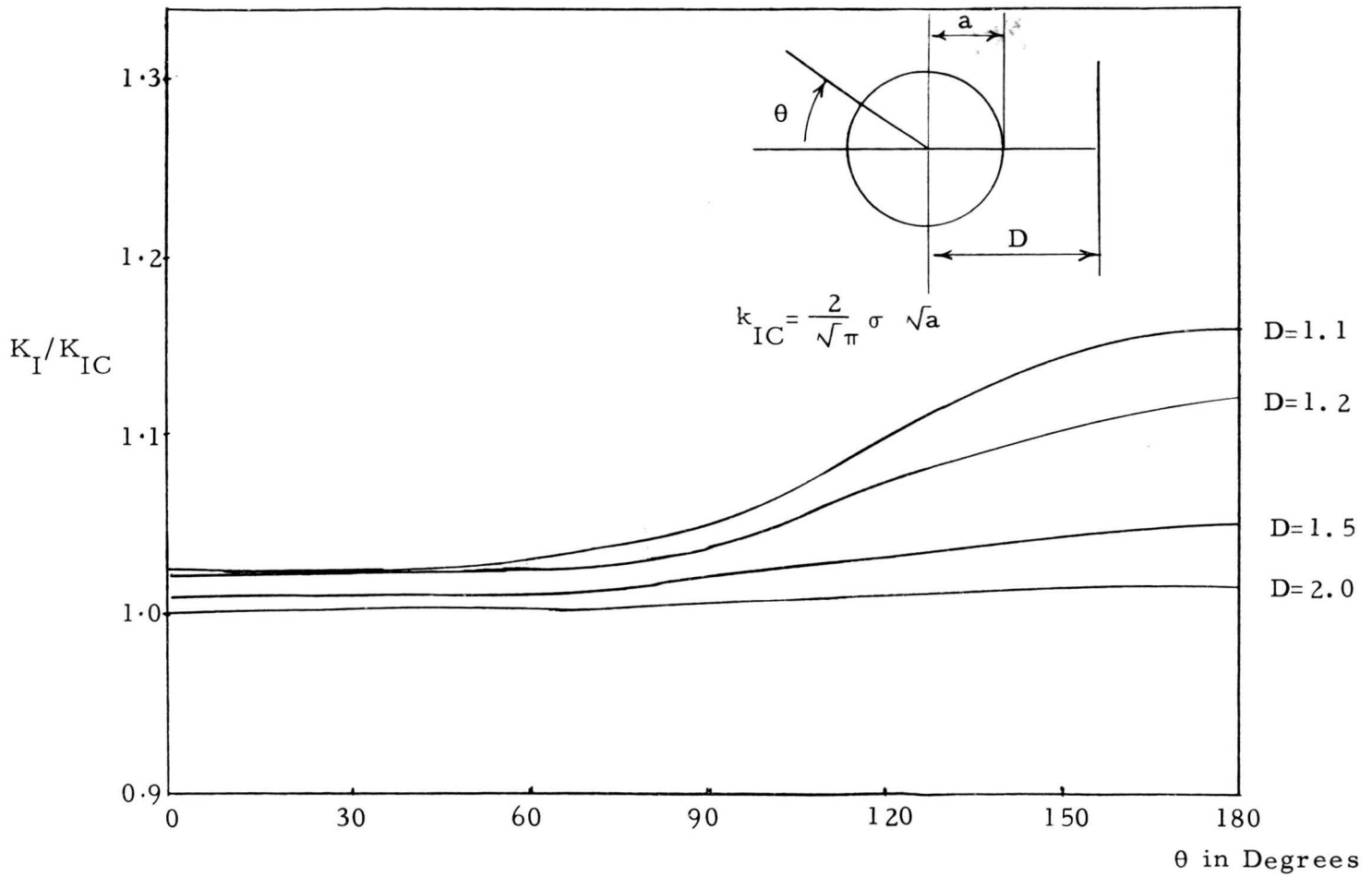


Figure 8. Stress Intensity Factor for an Embedded Circular Crack in a Semi-Infinite Solid in Simple Tension

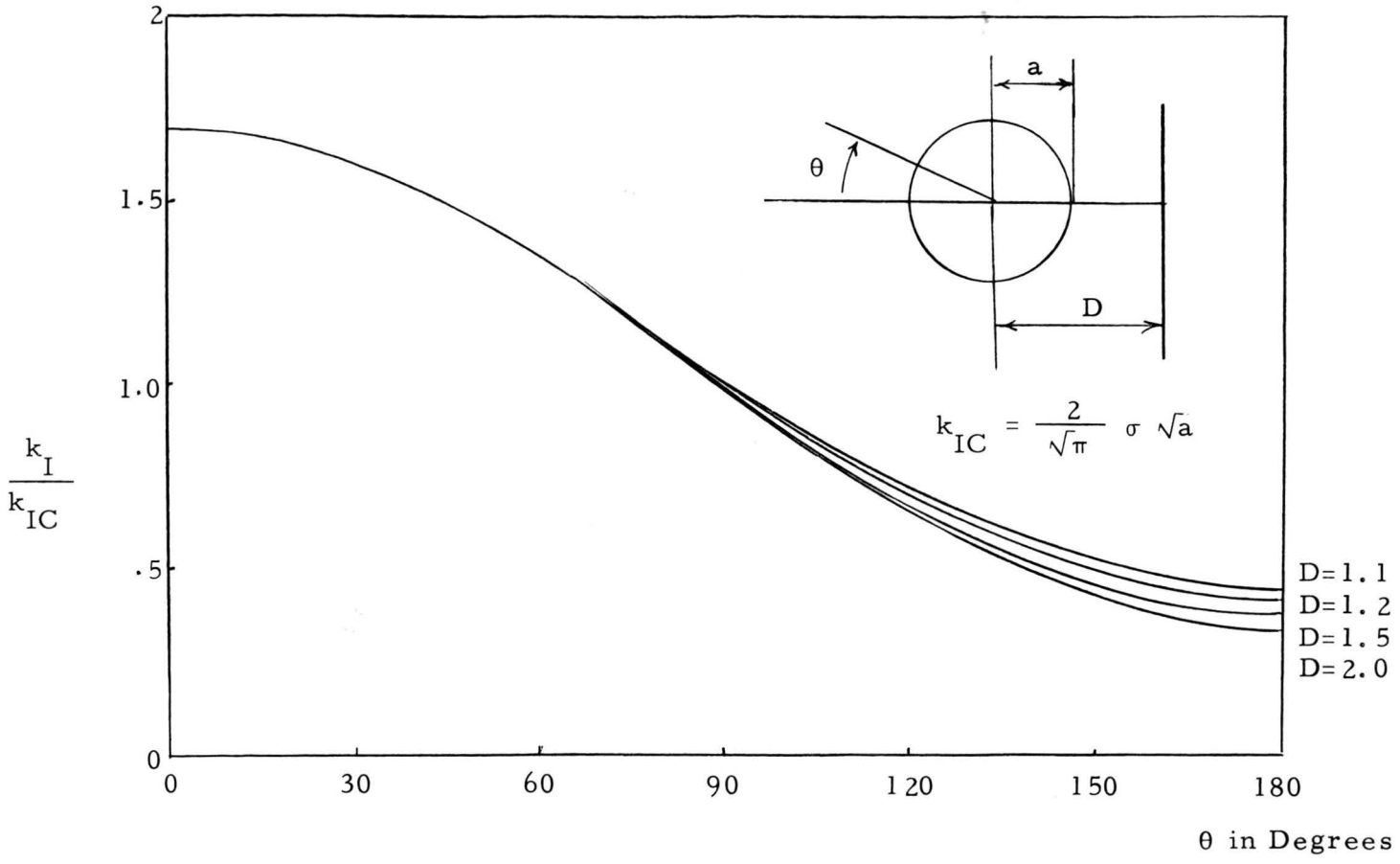


Figure 9. Stress Intensity Factor for an Embedded Circular Crack in a Semi-Infinite Solid for a Linear Varying Load of $p(r, \theta) = \sigma (1 + r \cos \theta)$

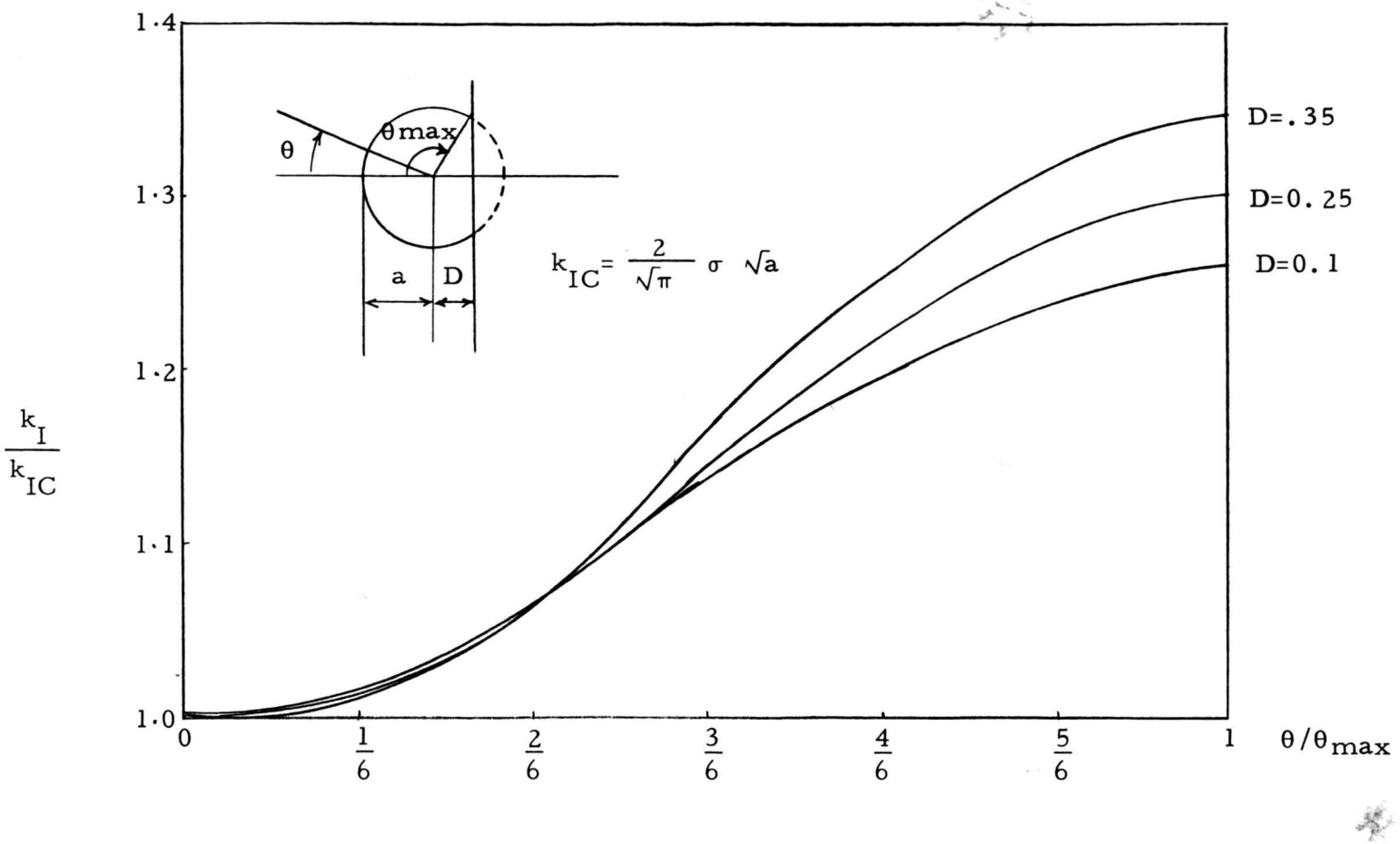


Figure 10. Stress Intensity Factor for a Part Through Circular Crack in a Semi-Infinite Solid in Simple Tension

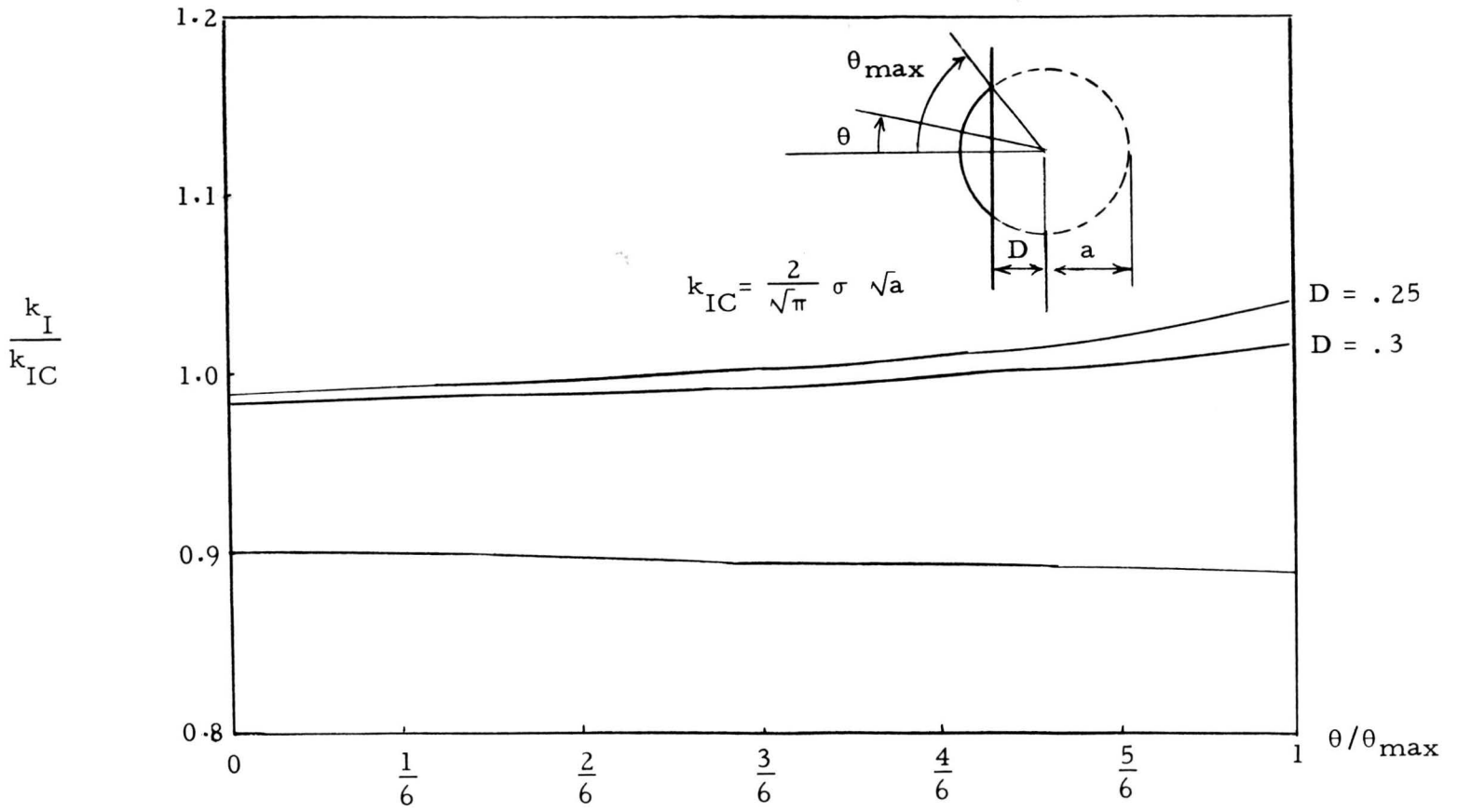
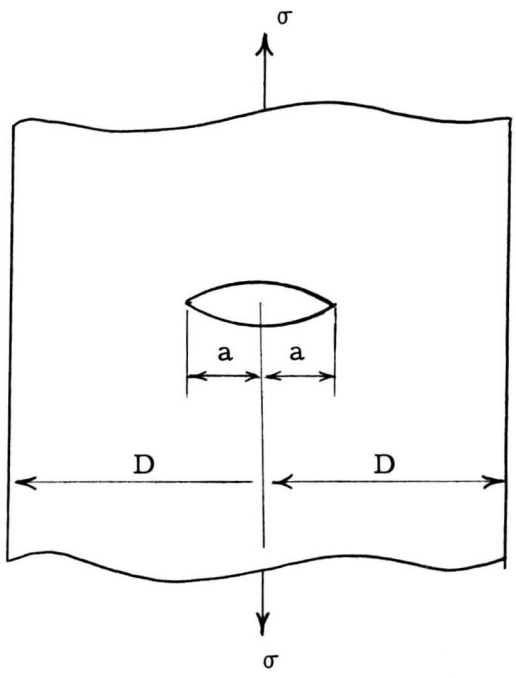
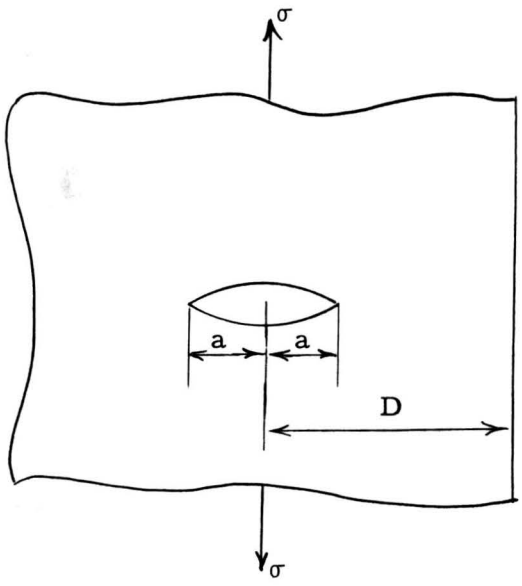


Figure 11. Stress Intensity Factor for a Part Through Circular Crack in a Semi-Infinite Solid in Simple Tension

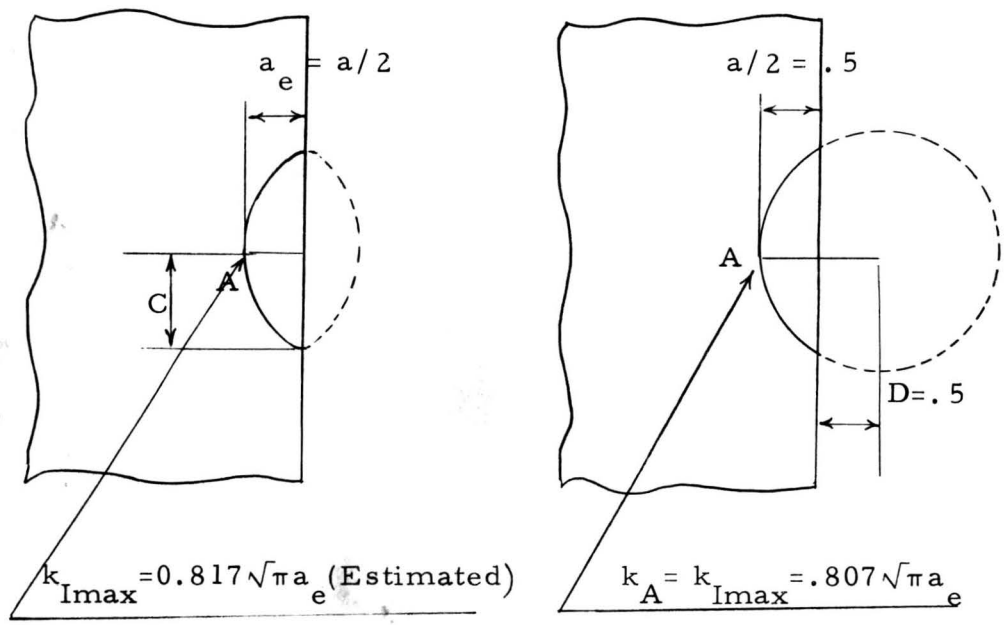


A central crack in a strip subjected to tension



A circular crack in a semi-infinite solid

Figure 12. Comparison of a Circular Crack in a Semi-Infinite Solid with a Central Crack in a Strip



Semi Elliptical Edge Crack

Part Through Circular Edge Crack

Figure 13. Comparison of a Semi-Elliptical Edge Crack With a Part Through Circular Edge Crack

Table 1. Stress on crack surface load = Unity

Applied Load

θ/r	0·	·2	·4	·6	·8	1·0
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
π/4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
π/2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Iteration No. 1

θ/r	0·	·2	·4	·6	·8	1·0
0	.0559	.0404	.0297	.0222	.0168	.0129
	.0559	.0405	.0299	.0224	.0170	.0130
	.0559	.0411	.0306	.0230	.0176	.0135
	.0559	.0419	.0317	.0242	.0186	.0144
	.0559	.0432	.0334	.0258	.0201	.0157
	.0559	.0448	.0356	.0281	.0223	.0176
	.0559	.0467	.0384	.0312	.0252	.0202
	.0559	.0491	.0420	.0352	.0291	.0238
	.0559	.0518	.0463	.0403	.0343	.0287
π/4	.0559	.0548	.0516	.0469	.0413	.0354
	.0559	.0581	.0579	.0553	.0506	.0447
	.0559	.0616	.0652	.0659	.0632	.0576
	.0559	.0652	.0735	.0790	.0801	.0755
	.0559	.0688	.0825	.0950	.1026	.1002
	.0559	.0721	.0918	.1134	.1316	.1325
	.0559	.0750	.1007	.1332	.1664	.1663
	.0559	.0773	.1081	.1515	.2018	.1757
	.0559	.0787	.1131	.1649	.2285	.1352
π/2	.0559	.0792	.1149	.1698	.2384	.1014

Table 1. (Continued)

Iteration No. 2

θ/r	0·	·2	·4	·6	1·0	
0	.0046	.0032	.0023	.0017	.0012	.0009
	.0046	.0032	.0023	.0017	.0012	.0009
	.0046	.0033	.0023	.0017	.0013	.0010
	.0046	.0033	.0024	.0018	.0014	.0010
	.0046	.0034	.0026	.0019	.0015	.0011
	.0046	.0036	.0028	.0021	.0016	.0013
	.0046	.0038	.0030	.0024	.0019	.0015
	.0046	.0040	.0033	.0027	.0022	.0017
	.0046	.0042	.0037	.0032	.0026	.0021
π/4	.0046	.0045	.0042	.0037	.0032	.0027
	.0046	.0048	.0048	.0045	.0040	.0035
	.0046	.0051	.0055	.0055	.0052	.0046
	.0046	.0055	.0063	.0068	.0068	.0064
	.0046	.0058	.0072	.0084	.0092	.0090
	.0046	.0061	.0081	.0104	.0124	.0129
	.0046	.0064	.0090	.0126	.0165	.0177
	.0046	.0067	.0098	.0147	.0211	.0204
	.0046	.0068	.0104	.0163	.0246	.0169
π/2	.0046	.0068	.0106	.0168	.0260	.0132

Iteration No. 3

θ/r	0·	·2	·4	·6	·8	1·0
0	.0004	.0003	.0002	.0002	.0001	.0001
	.0004	.0003	.0002	.0002	.0001	.0001
	.0004	.0003	.0002	.0002	.0001	.0001
	.0004	.0003	.0002	.0002	.0001	.0001
	.0004	.0003	.0002	.0002	.0001	.0001
	.0004	.0003	.0003	.0002	.0002	.0001
	.0004	.0003	.0003	.0002	.0002	.0001
	.0004	.0004	.0003	.0002	.0002	.0002
	.0004	.0004	.0003	.0003	.0002	.0002
π/4	.0004	.0004	.0004	.0003	.0003	.0002
	.0004	.0004	.0004	.0004	.0004	.0003
	.0004	.0005	.0005	.0005	.0005	.0004
	.0004	.0005	.0006	.0006	.0006	.0006
	.0004	.0005	.0007	.0008	.0009	.0008
	.0004	.0006	.0008	.0010	.0012	.0012
	.0004	.0006	.0009	.0012	.0016	.0017
	.0004	.0006	.0009	.0014	.0020	.0020
	.0004	.0006	.0010	.0016	.0024	.0017
π/2	.0004	.0006	.0010	.0016	.0025	.0013

CONCLUSION

The work reported here makes significant contributions in the area of fracture mechanics by providing solutions for a class of problems that have never been solved or for which only estimates are available. This is only the second application known to the author for the Schwartz alternating technique to the solution of three-dimensional elasto-static problems. In the course of the theoretical development a recurrence technique for evaluating integrals of the type

$$s(\nu,\ n) = \int_{\xi}^{\infty} \xi^{n} J_{\nu}(\xi r)\, e^{-z\xi} d\xi$$

was developed which is applied for all values of ν and n. At the present only solutions for special cases of this integral are available in the literature. Further solutions were developed for the integrals of the type

$$SI(n,\ m,\ p) = \int_{\xi}^{\infty} J_{n}(\xi r)\, J_{m+1/2}(\xi)\, e^{-z\xi}\, \xi^{p-1/2}\, d\xi$$

$$CI(n,\ m+n+2) = \int_{\xi}^{\infty} J_{n}(\xi r)\, e^{-z\xi} (1-\cos\xi)^{m+n+2}\, d\xi$$

for all values n, m and p for which the integral exist. Again only solutions for special cases of these integrals appear in the literature. Solutions are given for several special cases of the integrals

$$S_{nm} = \int_0^\infty \frac{\text{Sin } \xi}{\xi^m} J_n(\xi\nu) e^{-z\xi} d\xi$$

$$C_{nm} = \int_0^\infty \frac{\text{Cos}}{\xi^m} J_n(\xi\nu) e^{-z\xi} d\xi$$

which do not appear in the literature.

A solution for applied shear on a rectangular area on the surface of a half space was developed. The basic formulation for this solution was given by Cerruti (19) but the closed form solution does not appear in the literature.

APPENDIX A

THE STRESS COMPONENTS FOR SMALL VALUES OF z AND r

$$\tau_{rr} = (T_0^{\,0} + T_0^{\,1} + T_0^{\,2}) + (T_1^{\,1} + T_1^{\,2}) \cos\theta + (T_2^{\,1} + T_2^{\,2}) \cos(2\theta)$$

where:

$$T_0^{\,0} = \frac{1}{\sqrt{2}} C_0^{\,0} H_0^{\,0} \left[a_3 \, SI(0, 1, 1) + a_4 \, SI(0, 1, 2) + a_5 \, SI(1, 1, 0) + a_6 \, SI(1, 1, 1) \right]$$

$$T_0^{\,1} = \frac{1}{\sqrt{2}} C_0^{\,1} H_0^{\,1} \left\{ a_3 \left[SI(0, 1, 1) + SI(0, 0, 0) - \sqrt{\frac{8}{\pi}} \, CI(0, 0) \right] + a_4 \left[SI(0, 1, 2) + SI(0, 0, 1) - \sqrt{\frac{8}{\pi}} \, CI(0, 1) \right] + a_5 \left[SI(1, 1, 0) + SI(1, 0, -1) - \sqrt{\frac{8}{\pi}} \, CI(1, 0) \right] + a_6 \left[SI(1, 1, 1) + SI(1, 0, 0) - \sqrt{\frac{8}{\pi}} \, CI(1, 1) \right] \right\}$$

$$T_0^{\,2} = \frac{1}{\sqrt{2}} C_0^{\,2} H_0^{\,2} \left\{ a_3 \left[SI(0, 1, 1) - 2SI(0, 2, 0) \right] + a_4 \left[SI(0, 1, 2) - 2SI(0, 2, 1) \right] + a_5 \left[SI(1, 1, 0) - 2SI(1, 2, -1) \right] + a_6 \left[SI(1, 1, 1) - 2SI(1, 2, 0) \right] \right\}$$

$$T_1^1 = \frac{1}{\sqrt{2}} C_1^1 H_1^1 \left[a_3 \ SI(1, 2, 1) + a_4 \ SI(1, 2, 2) + a_5 \ SI(2, 2, 0) + a_6 \ SI(2, 2, 1) \right]$$

$$T_1^2 = \frac{1}{\sqrt{2}} C_1^2 H_1^2 \left\{ a_3 \left[SI(1, 2, 1) + 4SI(1, 0, -1) + SI(1, 1, 0) - 8\sqrt{\frac{2}{\pi}} \ CI(1, 0) \right] + a_4 \left[SI(1, 2, 2) + 4SI(1, 0, 0) + SI(1, 1, 1) - 8\sqrt{\frac{2}{\pi}} \ CI(1, 1) \right] + a_5 \left[SI(2, 2, 0) + 4SI(2, 0, -2) + SI(2, 1, -1) - 8\sqrt{\frac{2}{\pi}} \ CI(2, 0) \right] + a_6 \left[SI(2, 2, 1) + 4SI(2, 0, -1) + SI(2, 1, 0) - 8\sqrt{\frac{2}{\pi}} \ CI(2, 1) \right] \right\}$$

$$T_2^1 = \frac{1}{\sqrt{2}} C_2^1 H_2^1 \left\{ a_1 \left[-4SI(2, 0, -1) - SI(2, 1, 0) + 8\sqrt{\frac{2}{\pi}} \ CI(2, 1) \right] + a_2 \left[-4SI(2, 0, -2) - SI(2, 1, -1) + 8\sqrt{\frac{2}{\pi}} \ CI(2, 0) \right] + a_3 \left[-4SI(2, 0, 0) - SI(2, 1, 1) + 8\sqrt{\frac{2}{a}} \ CI(2, 2) \right] + a_4 \left[-4SI(2, 0, 1) - SI(2, 1, 2) + 8\sqrt{\frac{2}{\pi}} \ CI(2, 3) \right] + a_5 \left[-4SI(3, 0, -1) - SI(3, 1, 0) + 8\sqrt{\frac{2}{\pi}} \ CI(3, 2) \right] + a_6 \left[-4SI(3, 0, 0) - SI(3, 1, 1) + 8\sqrt{\frac{2}{\pi}} \ CI(3, 3) \right] \right\}$$

$$T_2^2 = \frac{1}{\sqrt{2}} C_2^2 H_2^2 \left[a_1 \ SI(2, 3, 0) + a_2 \ SI(2, 3, -1) + a_3 \ SI(2, 3, 1) + a_4 \ SI(2, 3, 2) + a_5 \ SI(3, 3, 0) + a_6 \ SI(3, 3, 1) \right]$$

$$T_{\theta\theta} = (P_0^{\,0} + P_0^{\,1} + P_0^{\,2}) + (P_1^{\,1} + P_1^{\,2}) \cos\theta + (P_2^{\,1} + P_2^{\,2}) \cos(2\theta)$$

where:

$$P_0^{\,0} = \frac{1}{\sqrt{2}} C_0^{\,0} H_0^{\,0} \left[a_9 \, SI(0, 1, 1) - a_5 \, SI(1, 1, 0) - a_6 \, SI(1, 1, 1) \right]$$

$$P_0^{\,1} = \frac{1}{\sqrt{2}} C_0^{\,1} H_0^{\,1} \left\{ a_9 \left[SI(0, 1, 1) + SI(0, 0, 0) - \sqrt{\frac{8}{\pi}} \, CI(0, 0) \right] \right.$$
$$- a_5 \left[SI(1, 1, 0) \; SI(1, 0, -1) - \sqrt{\frac{8}{\pi}} \, CI(1, 0) \right]$$
$$\left. - a_6 \left[SI(1, 1, 1) + SI(1, 0, 0) - \sqrt{\frac{8}{\pi}} \, CI(1, 1) \right] \right\}$$

$$P_0^{\,2} = \frac{1}{\sqrt{2}} C_0^{\,2} H_0^{\,2} \left\{ a_9 \left[SI(0, 1, 1) - 2SI(0, 2, 0) \right] \right.$$
$$- a_5 \left[SI(1, 1, 0) - 2SI(1, 2, -1) \right]$$
$$\left. - a_6 \left[SI(1, 1, 1) - 2SI(1, 2, 0) \right] \right\}$$

$$P_1^{\,1} = \frac{1}{\sqrt{2}} C_1^{\,1} H_1^{\,1} \left[a_9 \, SI(1, 2, 1) - a_5 \, SI(2, 2, 0) - a_6 \, SI(2, 2, 1) \right]$$

$$P_1^{\,2} = \frac{1}{\sqrt{2}} C_1^{\,2} H_1^{\,2} \left\{ a_9 \left[SI(1, 2, 1) + 4SI(1, 0, -1) + SI(1, 1, 0) - 8\sqrt{\frac{2}{\pi}} \, CI(1, 0) \right] \right.$$
$$- a_5 \left[SI(2, 2, 0) + 4SI(2, 0, -2) + SI(2, 1, -1) - 8\sqrt{\frac{2}{\pi}} \, CI(2, 0) \right]$$
$$\left. - a_6 \left[SI(2, 2, 1) + 4SI(2, 0, -1) + SI(2, 1, 0) - 8\sqrt{\frac{2}{\pi}} \, CI(2, 1) \right] \right\}$$

$$P_2^1 = \frac{1}{\sqrt{2}} C_2^1 H_2^1 \left\{ - a_1 \left[-4SI(2, 0, -1) - SI(2, 1, 0) + 8\sqrt{\frac{2}{\pi}} CI(2, 1) \right] \right.$$

$$- a_2 \left[-4SI(2, 0, -2) - SI(2, 1, -1) + 8\sqrt{\frac{2}{\pi}} CI(2, 0) \right]$$

$$+ a_9 \left[-4SI(2, 0, 0) - SI(2, 1, 1) + 8\sqrt{\frac{2}{\pi}} CI(2, 2) \right]$$

$$- a_5 \left[-SI(3, 0, -1) - SI(3, 1, 0) + 8\sqrt{\frac{2}{\pi}} CI(3, 2) \right]$$

$$\left. - a_6 \left[-4SI(3, 0, 0) - SI(3, 1, 1) + 8\sqrt{\frac{2}{\pi}} CI(3, 3) \right] \right\}$$

$$P_2^2 = \frac{1}{\sqrt{2}} C_2^2 H_2^2 \left[-a_1 SI(2, 3, 0) - a_2 SI(2, 3, -1) \right.$$

$$\left. + a_9 SI(2, 3, 1) - a_5 SI(3, 3, 0) - a_6 SI(3, 3, 1) \right]$$

$$\tau_{r\theta} = (Q_1^1 + Q_1^2) \text{ Sin } \theta + (Q_2^1 + Q_2^2) \text{ Sin } (2\theta)$$

where:

$$Q_1^1 = \frac{1}{\sqrt{2}} C_1^1 H_1^1 \left[a_7 SI(2, 2, 0) + a_8 SI(2, 2, 1) \right]$$

$$Q_1^2 = \frac{1}{\sqrt{2}} C_1^2 H_1^2 \left\{ a_7 \left[SI(2, 2, 0) + 4SI(2, 0, -2) + SI(2, 1, -1) - 8\sqrt{\frac{2}{\pi}} CI(2, 0) \right] \right.$$

$$\left. + a_8 \left[SI(2, 2, 1) + 4SI(2, 0, -1) + SI(2, 1, 0) - 8\sqrt{\frac{2}{\pi}} CI(2, 1) \right] \right\}$$

$$Q_2^1 = \frac{1}{\sqrt{2}} C_2^1 H_2^1 \left\{ - a_1 \left[-4SI(2, 0, -1) - SI(2, 1, 0) + 8\sqrt{\frac{2}{\pi}} CI(2, 1) \right] \right.$$
$$- a_2 \left[-4SI(2, 0, -2) - SI(2, 1, -1) + 8\sqrt{\frac{2}{\pi}} CI(2, 0) \right]$$
$$+ a_7 \left[-4SI(3, 0, -1) - SI(3, 1, 0) + 8\sqrt{\frac{2}{\pi}} CI(3, 2) \right]$$
$$\left. + a_8 \left[-4SI(3, 0, 0) - SI(3, 1, 1) + 8\sqrt{\frac{2}{\pi}} CI(3, 3) \right] \right\}$$

$$Q_2^2 = \frac{1}{\sqrt{2}} C_2^2 H_2^2 \left[- a_1 SI(2, 3, 0) - a_2 SI(2, 3, -1) + a_7 SI(3, 3, 0) + a_8 SI(3, 3, 1) \right]$$

$$\tau_{rz} = (R_0^0 + R_0^1 + R_0^2) + (R_1^1 + R_1^2) \cos\theta + (R_2^1 + R_2^2) \cos(2\theta)$$

where:

$$R_0^0 = \frac{1}{\sqrt{2}} C_0^0 H_0^0 \left[- a_4 SI(1, 1, 2) \right]$$

$$R_0^1 = \frac{1}{\sqrt{2}} C_0^1 H_0^1 \left\{ - a_4 \left[SI(1, 1, 2) + SI(1, 0, 1) - \sqrt{\frac{8}{\pi}} CI(1, 2) \right] \right\}$$

$$R_0^2 = \frac{1}{\sqrt{2}} C_0^2 H_0^2 \left\{ - a_4 \left[SI(1, 1, 2) - 2SI(1, 2, 1) \right] \right\}$$

$$R_1^1 = \frac{1}{\sqrt{2}} C_1^1 H_1^1 \left[- a_8 SI(1, 2, 1) - a_4 SI(2, 2, 2) \right]$$

$$R_1^2 = \frac{1}{\sqrt{2}} C_1^1 H_1^2 \left\{ - a_8 \left[SI(1, 2, 1) + 4SI(1, 0, -1) + SI(1, 1, 0) - 8\sqrt{\frac{2}{\pi}} CI(1, 0) \right] \right.$$
$$\left. - a_4 SI(2, 2, 2) + 4SI(2, 0, 0) + SI(2, 1, 1) - 8\sqrt{\frac{2}{\pi}} CI(2, 2) \right] \right\}$$

$$R_2^1 = \frac{1}{\sqrt{2}} C_2^1 H_2^1 \left\{ - a_8 \left[-4SI(2, 0, 0) - SI(2, 1, 1) + 8 \sqrt{\frac{2}{\pi}} CI(2, 2) \right.\right.$$

$$\left.\left. -4SI(3, 0, 1) - SI(3, 1, 2) + 8 \sqrt{\frac{2}{\pi}} CI(3, 4) \right] \right\}$$

$$R_2^2 = \frac{1}{\sqrt{2}} C_2^2 H_2^2 \left[- a_8 SI(2, 3, 1) - a_4 SI(3, 3, 2) \right]$$

STRESS COMPONENTS ON THE PLANE z = 0

$$\tau_{rr} = (T_0^0 + T_0^1 + T_0^2) + (T_1^1 + T_1^2) \cos\theta + (T_2^1 + T_2^2) \cos(2\theta)$$

where:

$$T_0^0 = \frac{1}{\sqrt{2}} C_0^0 H_0^0 \left[a_3 \; SI'(0, 1, 1) + a_5 SI'(1, 1, 0) \right]$$

$$T_0^1 = \frac{1}{\sqrt{2}} C_0^1 H_0^1 \left\{ a_3 \left[SI'(0, 1, 1) + SI'(0, 0, 0) - \sqrt{\frac{8}{\pi}} \; CI'(0, 0) \right] \right.$$

$$\left. + a_5 \left[SI'(1, 1, 0) + SI'(1, 0, -1) - \sqrt{\frac{8}{\pi}} \; CI'(1, 0) \right] \right\}$$

$$T_0^2 = \frac{1}{\sqrt{2}} C_0^2 H_0^2 \left\{ a_3 \left[SI'(0, 1, 1) - 2 SI'(0, 2, 0) \right] \right.$$

$$\left. + a_5 \left[SI'(1, 1, 0) - 2 SI'(1, 2, -1) \right] \right\}$$

$$T_1^1 = \frac{1}{\sqrt{2}} C_1^1 H_1^1 \left[a_3 \; SI'(1, 2, 1) + a_5 SI'(2, 2, 0) \right]$$

$$T_1^{\,2} = \frac{1}{\sqrt{2}} C_1^{\,2} H_1^{\,2} \left\{ a_3 \left[SI'(1, 2, 1) + 4SI'(1, 0, -1) + SI'(1, 1, 0) - 8\sqrt{\frac{2}{\pi}} CI'(1, 0) \right] \right.$$

$$\left. + a_5 \left[SI'(2, 2, 0) + 4SI'(2, 0, -2) + SI'(2, 1, -1) - 8\sqrt{\frac{2}{\pi}} CI'(2, 0) \right] \right\}$$

$$T_2^{\,1} = \frac{1}{\sqrt{2}} C_2^{\,1} H_2^{\,1} \left\{ a_2 \left[-4SI'(2, 0, -2) - SI'(2, 1, -1) + 8\sqrt{\frac{2}{\pi}} CI'(2, 0) \right] \right.$$

$$+ a_3 \left[-4SI'(2, 0, 0) - SI'(2, 1, 1) + 8\sqrt{\frac{2}{\pi}} CI'(2, 2) \right]$$

$$\left. + a_5 \left[-4SI'(3, 0, -1) - SI'(3, 1, 0) + 8\sqrt{\frac{2}{\pi}} CI'(3, 2) \right] \right\}$$

$$T_2^{\,2} = \frac{1}{\sqrt{2}} C_2^{\,2} H_2^{\,2} \left[a_2 SI'(2, 3, -1) + a_3 SI'(2, 3, 1) + a_5 SI'(3, 3, 0) \right]$$

$$\tau_{\theta\theta} = (P_0^{\,0} + P_0^{\,1} + P_0^{\,2}) + (P_1^{\,1} + P_1^{\,2}) \cos\theta + (P_2^{\,1} + P_2^{\,2}) \cos(2\theta)$$

where:

$$P_0^{\,0} = \frac{1}{\sqrt{2}} C_0^{\,0} H_0^{\,0} \left[a_9 SI'(0, 1, 1) - a_5 SI'(1, 1, 0) \right]$$

$$P_0^{\,1} = \frac{1}{\sqrt{2}} C_0^{\,1} H_0^{\,1} \left\{ a_9 \left[SI'(0, 1, 1) + SI'(0, 0, 0) - \sqrt{\frac{8}{\pi}} CI'(0, 0) \right] \right.$$

$$\left. - a_5 \left[SI'(1, 1, 0) + SI'(1, 0, -1) - \sqrt{\frac{8}{\pi}} CI'(1, 0) \right] \right\}$$

$$P_0^{\,2} = \frac{1}{\sqrt{2}} C_0^{\,2} H_0^{\,2} \left\{ a_9 \left[SI'(0, 1, 1) - 2\, SI'(0, 2, 0) \right] - a_5 \left[SI'(1, 1, 0) - 2\, SI'(1, 2, -1) \right] \right\}$$

$$P_1^{\,1} = \frac{1}{\sqrt{2}} C_1^{\,1} H_1^{\,1} \left[a_9\, SI'(1, 2, 1) - a_5\, SI'(2, 2, 0) \right]$$

$$P_1^{\,2} = \frac{1}{\sqrt{2}} C_1^{\,1} H_1^{\,1} \left\{ a_9 \left[SI'(1, 2, 1) + 4SI'(1, 0, -1) + SI'(1, 1, 0) - 8\sqrt{\frac{2}{\pi}}\, CI'(1, 0) \right] - a_5 \left[SI'(2, 2, 0) + 4SI'(2, 0, -2) + SI(2, 1, -1) - 8\sqrt{\frac{2}{\pi}}\, CI(2, 0) \right] \right\}$$

$$P_2^{\,1} = \frac{1}{\sqrt{2}} C_2^{\,1} H_2^{\,1} \left\{ - a_2 \left[-4SI'(2, 0, -2) - SI'(2, 1, -1) + 8\sqrt{\frac{2}{\pi}}\, CI'(2, 0) \right] + a_9 \left[-4SI'(2, 0, 0) - SI'(2, 1, 1) + 8\sqrt{\frac{2}{\pi}}\, CI'(2, 2) \right] - a_5 \left[-4SI'(3, 0, -1) - SI'(3, 1, 0) + 8\sqrt{\frac{2}{\pi}}\, CI'(3, 2) \right] \right\}$$

$$P_2^{\,2} = \frac{1}{\sqrt{2}} C_2^{\,2} H_2^{\,2} \left[-a_2\, SI'(2, 3, -1) + a_9\, SI'(2, 3, 1) - a_5 SI'(3, 3, 0) \right]$$

$$\tau_{r\theta} = (Q_1^{\,1} + Q_1^{\,2}) \sin\theta + (Q_2^{\,1} + Q_2^{\,2}) \sin(2\theta)$$

where

$$Q_1^{\,1} = \frac{1}{\sqrt{2}} C_1^{\,1} H_1^{\,1} \left[a_7\, SI'(2, 2, 0) \right]$$

$$Q_1^{\,2} = \frac{1}{\sqrt{2}} C_1^{\,2} H_1^{\,2} \left\{ a_7 \left[SI'(2, 2, 0) + 4SI'(2, 0, -2) + SI'(2, 1, -1) - 8\sqrt{\frac{2}{\pi}}\, CI'(2, 0) \right] \right\}$$

$$Q_2^{\,1} = \frac{1}{\sqrt{2}} C_2^{\,1} H_2^{\,1} \left\{ - a_2 \left[-4SI'(2, 0, -2) - SI'(2, 1, -1) + 8\sqrt{\frac{2}{\pi}}\, CI'(2, 0) \right] \right.$$

$$\left. + a_7 \left[-4SI'(3, 0, -1) - SI'(3, 1, 0) + 8\sqrt{\frac{2}{\pi}}\, CI'(3, 2) \right] \right\}$$

$$Q_2^{\,2} = \frac{1}{\sqrt{2}} C_2^{\,2} H_2^{\,2} \left[- a_2\, SI'(2, 3, -1) + a_7\, SI'(3, 3, 0) \right]$$

$$\tau_{rz} = 0$$

$$T_{\theta z} = 0$$

NOMENCLATURE

τ_{ij}	Stress Components
U_i	Displacement Components
λ, μ	Lame' Constants
ν	Poisson's Ratio
β^2	$2(1-\nu) / (1-2\nu)$
ϕ	Stress Function
e	Dilatation
r, θ, z	Circular Cylindrical Coordinates
x, y, z	Rectangular Cartesian Coordinates
x_1, y_1, z_1	Rectangular Cartesian Coordinates
k_I	Stress Intensity Factor

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COMPUTER PROGRAM

```
*CHARGE,1123700
*NAME,ALAVI, M. J.
*LIMIT,T=12,PR=50
*FORTRAN
      PROGRAM MIR
      REAL NU
      COMMON    S(10,140),A(10,10,62 ),C(10,10)=SRR(65),SP(65),SQQ(65),
     1 SRQ(65),SRP(65),SRZ(65),SQZ(65),SK(10), FF(4 ),SC(6,6,19),YP(65),
     2GG(4) , H(3,3) , CI(4,5) , SI(5,5,6),ZP(65),AA(65),BB(65),FK(65)
     3,SXX(65), SXY(65), SXZ(65) ,FCK(6,3)
      READ(5,2423)( YP(I), I=1, 64 )
      READ(5,2423)( ZP(I), I=1, 64 )
      READ(5,2423)( AA(I), I=1, 64 )
      READ(5,2423)( BB(I), I=1, 64 )
2423  FORMAT(8 F10.3)
      READ (5,8) N,M,K,NU,D ,ITER ,ID
8     FORMAT  (3I10 ,2F10.5,2I10)
      WRITE(6,1009) D
1009  FORMAT(/////50X,3HD =,F4.2////)
      NN = N+1
      NM = M+1
      NK = K+1
      A3= 2.*(1. - NU) / (1. - 2.*NU)
      A1 = 2.*(A3- 1.)
      A2 = 2.*(A3- 2.)
C
      DO211I =3, 4
      FF(1 ) =1   .
      FF(2 ) =1   .
      XQ = I - 1
      FF(I) = XQ * FF(I- 1 )
211   CONTINUE
C
      DO 22I=2, 4
```

```
      W = I - 1
      GG(1)=SQRT(3.141592653589793)
      GG(I)= (2.*W - 1. )*GG(I - 1 )/2.
22    CONTINUE
C
      DO 33I=1, 3
      DO 33J=1, 3
      II = I-1
      JJ = J-1
      IF(  (2*((II+JJ)/2)) .EQ. (II+JJ)  )  H(I-J) =
     1 (FF( (II+JJ+2)/2  )  /  GG( (II+JJ+2)/2 +1  ) )*(SQRT(2. ) )/2.
      IF(  (2*((II+JJ)/2)) .NE. (II+JJ)  )  H(ITJ) =
     1 (GG( (II+JJ+1)/2 +1  )  /  FF( (II+JJ+3)/2  ) ) *(SQRT (2. ) )/2.
33    CONTINUE
C
      DO 707I =1, 64
      OP(I) = ATAN( YP(I) / D  )
      RP(I) = SQRT(  D*D + (YP(I))*(YP(I))  )
707   CONTINUE
C       TO EVALUATE THE INITIAL STRESSES ON THE CRACK SURFCE
      HA = .087266463 *2.
      DO 9909 IIK =1,1
      DO 1178 I =1, 6
      DO 1178 J =1, 19
       R = I - 1
      R = R* .2
      XY = J - 1
      TET = XY *HA
      IF(  IIK .EQ. 1) GO TO 1176
      IF(  IIK .EQ. 2) GO TO 1179
      IF(  IIK .EQ. 3) GO TO 1180
      IF(  IIK .EQ. 4) GO TO 1181
      IF(  IIK .EQ. 5) GO TO 1182
      IF(  IIK .EQ. 6) GO TO 1183
1176  SC(1, I, J ) = 1.
      GO TO 1178
1179  SC(1, I, J ) = R* R
```

```
      GO TO 1178
1180  SC(1, I, J ) = R* COS( TET) + 1.
      GO TO 1178
1181  SC(1, I, J ) = R* R * COS( TET )
      GO TO 1178
1182  SC(1, I, J ) = R* COS( 2.* TET )
      GO TO 1178
1183  SC(1, I, J ) = R* R * COS( 2. * TET )
1178  CONTINUE
C
      CALL        COEFF(N,M,K,A   )
C
      DO 8793 I =1, 7
8793  FK(I) = 0.
C
C
      DO 1118 INDIC=1, ITER
C
C
      WRITE(6,199)INDIC
199   FORMAT(1H1,30X,*ITERATION NO.*,I2)
C
C
      GO TO 717
7071  INDIC = JNDIC
C
717   CALL CONST(A1, INDIC, SC, C )
      IF( ID  .EQ.  2 ) GO TO 4443
      IF ( D  .GE.  1. ) GO TO 137
      IF( D  .LT. 1. ) Q1 = ASIN(D)
      Q2 = ( Q1 + 3.14159/2.  ) / 6.
      GO TO 137
4443  Q2 =(ACOS(D) ) /6.
C
C
C
C
```

```
C
C
C
137   DO 8877 IK =1, 7
      XY = IK - 1
      O = XY * ( 3.14159 )/ 6.
      IF( D .LT. 1. ) O = XY * O2
C
C
      SUM1 = 0.
      DO 4477 I =1,3
      SUM2 = 0.
      DO 3377 J =1,3
      I = I - 1
      SUM2 = SUM2 + C(I,J)*H(I,J) *A1
      FCK( INDIC,I) = SUM2 * SQRT( 3.141592/2.)
3377  CONTINUE
      SUM1 = SUM1 + (COS(I*O) )*SUM2
4477  CONTINUE
      SK(IK) = SUM1*SQRT(    3.141596/2.)
      FK( IK ) = FK( IK ) + SK( IK )
C
C
8877  CONTINUE
      WRITE(6,303)( SK(I), I=1, 7)
303   FORMAT(1X,*SINLSS INTENSITY FACTOR K =*,7F10.4/)
C
      IF( INDIC  .EQ. ( ITER + 1 )  ) GO TO 8803
C
      DO 100  I2 =1,64
C
C
      Z = ZP(I2)
      R = RP(I2)
      O = 3.14159  - OP(I2)
      IF( ID  .EQ.  2) O = OP(I2)
      IF ( R  .EQ.  1.   .AND.   Z  .EQ. 0.  ) R = 1.1
```

```
      IF( R  .EQ.  0. ) R = .0001
      IF( R  .LT.  1.   .AND.  Z  .LE.  1. ) GO TO 888
      CALL        BESSEL(N,K,R,Z, S)
      SRR3= 0.
      SOO3= 0.
      SRO3= 0.
      SRZ3= 0.
      SOZ3= 0.
C
C
      DO 10 I = 1,  NN
C
C
      W = I-1
      C1 = -A1*W*(W-1.)*Z / (R**2.)
      C2 = 2.*W*(W-1.) / (R**2.)
      C3 = -A1
      C4 = A1*Z
      C5 = 2./R
      C6 = -A1*Z /R
      C7 = 2.*W/R
      C8 = -A1*W*Z/R
      C9 = -A2
      SRR1 = 0.
      SOO1 = 0.
      SRO1 = 0.
      SRZ1 = 0.
      SOZ1 = 0.
C
C
      DO 20 J = 1,  NM
C
C
      SRR2 = 0.
      SOO2 = 0.
      SRO2 = 0.
      SRZ2 = 0.
```

```
      SOZ2 = 0.
C
C
      DO 30 L = 1,  NK
C
C
      T = C(I,J)* A(I,J,L)
      LL = 2*(L-1) + I
      P = S(I,LL   )
      P1= S(I,LL+1)
      P2= S(I,LL+2)
      P3= S(I,LL+3)
      P4= S(I+1,LL+1)
      P5= S(I+1,LL+2)
      P6= S(I+1,LL+3)
      SRR2= SRR2+ ( C1*P1+C2*P+C3*P2+C4*P3+C5*P4+C6*P5 )* T
      SOO2= SOO2+ (-C1*P1-C2*P+C9*P2-C5*P4-C6*P5 )* T
      SRO2= SRO2+ (-C1*P1-C2*P+C7*P4+C8*P5 ) * T
      SRZ2= SRZ2+ (-C8*P2-C4*P6 ) * T
      SOZ2= SOZ2+ (+C8*P2 ) * T
30    CONTINUE
      SRR1 = SRR1 + SRR2
      SOO1 = SOO1 + SOO2
      SRO1 = SRO1 + SRO2
      SRZ1 = SRZ1 + SRZ2
      SOZ1 = SOZ1 + SOZ2
20    CONTINUE
      SRR3 = SRR3 +SRR1*COS(W*O)
      SOO3 = SOO3 +SOO1*COS(W*O)
      SRO3 = SRO3 +SRO1*SIN(W*O)
      SRZ3 = SRZ3 +SRZ1*COS(W*O)
      SOZ3 = SOZ3 +SOZ1*SIN(W*O)
10    CONTINUE
      SRR(I2)     = SRR3
      SOO(I2)     = SOO3
      SRO(I2)     = SRO3
      SRZ(I2    ) = SRZ3
```

```
      SOZ(I2)     = SOZ3
      GO TO 109
888   CALL        HYPER (R, Z, SI, CI )
C
      SRRR= 0.
      SOOO= 0.
      SROO= 0.
      SRZZ= 0.
      SOZZ= 0.
C
      W = 0.
      C1 = -A1*W*(W-1.)*Z / (R**2.)
      C2 = 2.*W*(W-1.) / (R**2.)
      C3 = -A1
      C4 = A1*Z
      C5 = 2./R
      C6 = -A1*Z /R
      C7 = 2.*W/R
      C8 = -A1*W*Z/R
      C9 = -A2
C
      IF( Z  .NE.  0.  ) GO TO 660
C
C
      T00 =(C3*SI(1,2,5) + C5*SI(2,2,4) )* C(1,1)*H(1,1)
      T01 = (C3* ( SI(1,2,5) + SI(1,1,4) - ( SQRT(8. /3.14159))*CI(1,1))
     1       +C5*( SI(2,2,4) + SI(2,1,3) - ( SQRT(8. /3.14159))*CI(2,1))
     2  )* C(1,2)* H(1,2)
      T02 = (C3*(SI(1,2,5)-2.*SI(1,3,4))+ C5*(SI(2,2,4)-2.* SI(2,3,3)))
     1*C(1,3)*H(1,3)
      P00 = (C9*SI(1,2,5) - C5*SI(2,2,4))*C(1,1D*H(1,1)
      P01 = ( C9*( SI(1,2,5) + SI(1,1,4) - ( SQRT(8. /3.14159))*CI(1,1)
     1       -C5*( SI(2,2,4) + SI(2,1,3) - ( SQRT(8. /3.14159))*CI(2,1))
     2  )* C(1,2)* H(1,2)
      P02 = ( C3* ( SI(1,2,5) - 2.* SI(1,3,4) ) - C5* ( SI(2,2,4)
     1 -2.* SI(2,3,3)  )  )* C(1,3) * H(1,3)
      R00 = 0.
```

```
      R01 = 0.
      R02 = 0.
      GO TO 550
C
C
660   T00 = (C3 *SI(1,2,5) + C4 * SI(1,2,6) + C5 * SI(2,2,4) + C6 *
     1SI(2,2,5)  ) * C(1,1) * H(1,1)
      T01 = (C3* ( SI(1,2,5) + SI(1,1,4) - ( SQRT(8. /3.14159))*CI(1,1))
     1       +C4*( SI(1,2,6) + SI(1,1,5) - ( SQRT(8. /3.14159))*CI(1,2))
     2       +C5*( SI(2,2,4) + SI(2,1,3) - ( SQRT(8. /3.14159))*CI(2,1))
     3       +C6*( SI(2,2,5) + SI(2,1,4) - ( SQRT(8. /3.14159))*CI(2,2))
     4  )* C(1,2)* H(1,2)
      T02 = ( C3*(SI(1,2,5) - 2.* SI(1,3,4) ) +
     1        C4*(SI(1,2,6) - 2.* SI(1,3,5) ) +
     2        C5*(SI(2,2,4) - 2.* SI(2,3,3) ) +
     3        C6*(SI(2,2,5) - 2.* SI(2,3,4) ) ) * C(1,3) * H(1,3)
C
      P00 = (C9*SI(1,2,5) - C5*SI(2,2,4) - C6 *SI(2,2,5) )*C(1,1)*H(1,1)
      P01 = ( C9*( SI(1,2,5) + SI(1,1,4) - ( SQRT(8. /3.14159))*CI(1,1))
     1       -C5*( SI(2,2,4) + SI(2,1,3) - ( SQRT(8. /3.14159))*CI(2,1))
     2       -C6*( SI(2,2,5) + SI(2,1,4) - ( SQRT(8. /3.14159))*CI(2,2))
     3  )* C(1,2)* H(1,2)
      P02 = ( C3*(SI(1,2,5) - 2.* SI(1,3,4) )-
     1        C5*(SI(2,2,4) - 2.* SI(2,3,3) )-
     2        C6*(SI(2,2,5) - 2.* SI(2,3,4) ) ) * C(1,3) * H(1,3)
C
      R00 = ( - C4 * SI(2,2,6) ) * C(1,1) *H(1,1)
      R01 = (-C4*( SI(2,2,6) + SI(2,1,5) - ( SQRT(8. /3.14159))*CI(2,3))
     1  )* C(1,2)* H(1,2)
      R02 =  -C4*(SI(2,2,6) - 2.* SI(2,3,5) ) * C(1,3) * H(1,3)
C
550   SRRR= SRRR+ T00 + T01 + T02
      S000= S000+ P00 + P01 + P02
      SRZZ= SRZZ+ R00 + R01 + R02
C
C
      W = 1.
```

```
      C1 = -A1*W*(W-1.)*Z / (R**2.)
      C2 = 2.*W*(W-1.) / (R**2.)
      C7 = 2.*W/R
      C8 = -A1*W*Z/R
C
      IF( Z  .NE.  0.  ) GO TO 661
C
      T11 = ( C3* SI(2,3,5) + C5* SI(3,3,4) )* C(2,2) * H(2,2)
      T12 = (C3*(SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1) )  +
     2        C5*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     3CI(3,1) ) ) *  C(2,3) * H(2,3)
      P11 = ( C9* SI(2,3,5) - C5* SI(3,3,4) )* C(2,2) * H(2,2)
      P12 = (C9*(SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1)) -
     2        C5*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     3CI(3,1)) )  * C(2,3) * H(2,3)
      Q11 = ( C7* SI(3,3,4) )* C(2,2) * H(2,2)
      Q12 = (C7*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     1CI(3,1)) )  * C(2,3) * H( 2,3)
      R11 = 0.
      R12 = 0.
      Z11 = 0.
      Z12 = 0.
      GO TO 551
C
C
661   T11 = (C3*SI(2,3,5)+C4*SI(2,3,6)+C5*SI(3,3,4)+C6*SI(3,3,5))*
     1 C(2,2) * H(2,2)
      T12 = (C3*(SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1) )  +
     2        C4*(SI(2,3,6)+4.*SI(2,1,4)+SI(2,2,5)-(SQRT(128./3.14159))*
     3CI(2,2) ) +
     4        C5*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     5CI(3,1) ) +
     6        C6*(SI(3,3,5)+4.*SI(3,1,3)+SI(3,2,4)-(SQRT(128./3.14159))*
     7CI(3,2) )) * C(2,3) * H(2,3)
```

```
C
      P11 = (C9*SI(2,3,5)-C5*SI(3,3,4)-C6*SI(3,*,5))* C(2,2) *H(2,2)
      P12 = (C9*(SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1)) -
     2        C5*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     3CI(3,1)) -
     4        C6*(SI(3,3,5)+4.*SI(3,1,3)+SI(3,2,4)-(SQRT(128./3.14159))*
     5CI(3,2)) ) * C(2,3) * H(2,3)
C
      Q11 = ( C7* SI(3,3,4) + C8 * SI(3,3,5) ) MC(2,2) *H(2,2)
      Q12 = (C7*(SI(3,3,4)+4.*SI(3,1,2)+SI(3,2,3)-(SQRT(128./3.14159))*
     1CI(3,1)) +
     2        C8*(SI(3,3,5)+4.*SI(3,1,3)+SI(3,2,4)-(SQRT(128./3.14159))*
     3CI(3,2)) ) * C(2,3) * H(2,3)
C
      R11 = (-C8* SI(2,3,5) - C4 *SI(3,3,6) )*CU2,2) * H(2,2)
      R12 =(-C8*(SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1 ))-
     2        C4*(SI(3,3,6)+4.*SI(3,1,4)+SI(3,2,5)-(SQRT(128./3.14159))*
     3CI(3,3 ))) * C(2,3) * H(2,3)
C
      Z11 = (C8 * SI(2,3,5) )* C(2,2) * H(2,2)
      Z12 = ( C8*SI(2,3,5)+4.*SI(2,1,3)+SI(2,2,4)-(SQRT(128./3.14159))*
     1CI(2,1 ) ) * C(2,3) *H(2,3)
C
551   SRRR= SRRR+ ( T11 + T12 ) * COS( W * O )
      SOOO= SOOO+ ( P11 + P12 ) * COS( W * O )
      SROO= SROO+ ( Q11 + Q12 ) * SIN( W * O ),
      SRZZ= SRZZ+ ( R11 + R12 ) * COS( W * O )
      SOZZ= SOZZ+ ( Z11 + Z12 ) * SIN( W * O )
C
      W = 2.
      C1 = -A1*W*(W-1.)*Z / (R**2.)
      C2 = 2.*W*(W-1.) / (R**2.)
      C7 = 2.*W/R
      C8 = -A1*W*Z/R
C
```

```
      IF( Z  .NE.  0.  ) GO TO 662
      T21=(C3*(-4.*SI(3,1,4)-SI(3,2,5)+( SQRT(128./3.14159))*CI(3,3))
     1    +C5*(-4.*SI(4,1,3)-SI(4,2,4)+( SQRT(128./3.14159))*CI(4,3))
     2    +C2*(-4.*SI(3,1,2)-SI(3,2,3)+( SQRT(128./3.14159))*CI(3,1))
     3)*C(3,2) * H(3,2)
      T22 = ( C3*SI(3,4,5)+ C5* SI(4,4,4)+C2*SI(3,4,3))*C(3,3)*H(3,3)
      P21 =   (C9*(-4.*SI(3,1,4)-SI(3,2,5)+(SQRO(128./3.14159))*CI(3,3))
     1        -C5*(-4.*SI(4,1,3)-SI(4,2,4)+(SQRT(128./3.14159))*CI(4,3))
     2        -C2*(-4.*SI(3,1,2)-SI(3,2,3)+(SQR*(128./3.14159))*CI(3,1))
     3 )*C(3,2) * H(3,2)
      P22 = ( C9* SI(3,4,5) - C5* SI(4,4,4)-C2*SI(3,4,3))*C(3,3)*H(3,3)
      Q21 =   (C7*(-4.*SI(4,1,3)-SI(4,2,4)+(SQRT(128./3.14159))*CI(4,3))
     1        -C2*(-4.*SI(3,1,2)-SI(3,2,3)+(SQR*(128./3.14159))*CI(3,1))
     2 )*C(3,2) * H(3,2)
      Q22 = ( C7* SI(4,4,4) -C2*SI(3,4,3))*C(3,*)*H(3,3)

C
      R21 = 0.
      R22 = 0.
      Z21 = 0.
      Z22 = 0.
      GO TO 552
C
C
C
662   T21=(C1*(-4.*SI(3,1,3)-SI(3,2,4)+( SQRT(128./3.14159))*CI(3,2))
     1    +C2*(-4.*SI(3,1,2)-SI(3,2,3)+( SQRT(128./3.14159))*CI(3,1))
     2    +C3*(-4.*SI(3,1,4)-SI(3,2,5)+( SQRT(128./3.14159))*CI(3,3))
     3    +C4*(-4.*SI(3,1,5)-SI(3,2,6)+( SQRT(128./3.14159))*CI(3,4))
     4    +C5*(-4.*SI(4,1,3)-SI(4,2,4)+( SQRT(128./3.14159))*CI(4,3))
     5    +C6*(-4.*SI(4,1,4)-SI(4,2,5)+( SQRT(128./3.14159))*CI(4,4)) )
     6 *C(3,2) * H(3,2)
      T22 = ( C1* SI(3,4,4) + C2*SI(3,4,3) + C3M SI(3,4,5) + C4*
     1SI(3,4,6) + C5*SI(4,4,4) + C6*SI(4,4,5) )*C(3,3)* H(3,3)
C
      P21 = ( -C1*(-4.*SI(3,1,3)-SI(3,2,4)+(SQRT(128./3.14159))*CI(3,2))
     1        -C2*(-4.*SI(3,1,2)-SI(3,2,3)+(SQRO(128./3.14159))*CI(3,1))
```

```
     2         +C9*(-4.*SI(3,1,4)-SI(3,2,5)+(SQRT(128./3.14159))*CI(3,3))
     3         -C5*(-4.*SI(4,1,3)-SI(4,2,4)+(SQRO(128./3.14159))*CI(4,3))
     4         -C6*(-4.*SI(4,1,4)-SI(4,2,5)+(SQRT(128./3.14159))*CI(4,4))
     5 )*C(3,2) * H(3,2)
      P22 = ( -C1*SI(3,4,4) -C2*SI(3,4,3) +C9* SI(3,4,5) -C5* SI(4,4,4)
     1- C6* SI(4,4,5) ) * C(3,3) * H(3,3)
C
      Q21 = ( -C1*(-4.*SI(3,1,3)-SI(3,2,4)+(SQR (128./3.14159))*CI(3,2))
     1         -C2*(-4.*SI(3,1,2)-SI(3,2,3)+(SQRT(128./3.14159))*CI(3,1))
     2         +C7*(-4.*SI(4,1,3)-SI(4,2,4)+(SQRO(128./3.14159))*CI(4,3))
     3         +C8*(-4.*SI(4,1,4)-SI(4,2,5)+(SQR3(128./3.14159))*CI(4,4))
     4 )*C(3,2) * H(3,2)
      Q22 = ( -C1*SI (    3,4,4) -C2*SI(3,4,3) + C7*SI(4,4,4) +
     1C8* SI(4,4,5) ) * C(3,3 ) *H(3,3)
C
      R21 = ( -C8*(-4.*SI(3,1,4)-SI(3,2,5)+(SQR (128./3.14159))*CI(3,3))
     1         -C4*(-4.*SI(4,1,5)-SI(4,2,6)+(SQRT(128./3.14159))*CI(4,5))
     2 )*C(3,2) * H(3,2)
      R22 = (-C8 *SI(3,4,5) - C4 * SI(4,4,6) )*C(3,3)*H(3,3)
C
      Z21 = (  C8*(-4.*SI(3,1,4)-SI(3,2,5)+(SQRT(128./3.14159))*CI(3,3))
     1 )*C(3,2) * H(3,2)
      Z22 = ( C8 *SI(3,4,5) ) *C(3,3) * H(3,3)
552   SRRR= SRRR+ ( T21 + T22 ) * COS( W * O )
      SOOO= SOOO+ ( P21 + P22 ) * COS( W * O )
      SROO= SROO+ ( Q21 + Q22 ) * SIN( W * O )
      SRZZ= SRZZ+ ( R21 + R22 ) * COS( W * O )
      SOZZ= SOZZ+ ( Z21 + Z22 ) * SIN( W * O )
      SRR(I2)     = SRRR
      SOO(I2)     = SOOO
      SRO(I2)     = SROO
      SRZ(I2)     = SRZZ
      SOZ(I2)     = SOZZ
C
C
109   SXX(I2)=(SRR(I2))*( COS(O) ) *( COS(O) ) +(SOO(I2))* (  SIN(O) ) *
     1( SIN(O) ) - 2. * ( COS(O) )* ( SIN(O) )*U SRO(I2) )
```

```
      SXY(I2)=(SRR(I2))*( COS(O) ) *( SIN(O) ) -(SOO(I2))* (  COS(O) ) *
     1( SIN(O) )+(SRO(I2))*( (COS(O) )* ( COS(OD)-( SIN(O) )*( SIN(O) ))
      SXZ(I2) =(SRZ(I2))* ( COS(O) ) -(SOZ(I2))M ( SIN(O) )
C
C
100   CONTINUE
C
      WRITE(6,285) ( SRR(I ), I=1, 32 )
      WRITE(6,286) ( SOO(I ), I=1, 32 )
      WRITE(6,287) ( SRO(I ), I=1, 32 )
      WRITE(6,288) ( SRZ(I ), I=1, 32 )
      WRITE(6,289) ( SOZ(I ), I=1, 32 )
      WRITE(6,290) ( SXX(I ), I=1, 32 )
      WRITE(6,291) ( SXY(I ), I=1, 32 )
      WRITE(6,292) ( SXZ(I ), I=1, 32 )
285   FORMAT(1X,5HSRR =,32F4.2)
286   FORMAT(1X,5HSOO =,32F4.2)
287   FORMAT(1X,5HSRO =,32F4.2)
288   FORMAT(1X,5HSRZ =,32F4.2)
289   FORMAT(1X,5HSOZ =,32F4.2)
290   FORMAT(1X,5HSXX =,32F4.2)
291   FORMAT(1X,5HSXY =,32F4.2)
292   FORMAT(1X,5HSXZ =,32F4.2)
C
C
C
C     TO EVALUATE STESSES ON THE CRACK SURFACE DUE TO REMOVABLE  OF
C     STRESSES FROM THE PLANE D
C
C
       JNDIC = INDIC + 1
      HA= .087266463 * 2.
      DO 9999  I = 1, 6
      DO 9999  J = 1, 19
C
      R = I - 1
      R = R * .2
```

```
C
      XY = J - 1
      TETA = XY * HA
C
C
      SUM1 = 0.
      SUM2 = 0.
      SUM3 = 0.
C
C
      DO2500 II = 1, 64
      X1 = R* SIN( TETA ) - YP( II )
      X2 = R* SIN( TETA ) + YP( II )
      Y1 = ZP( II )
      Z1 = R* COS( TETA ) + D
      IF( ID  .EQ.  2) Z1 = R* COS( TETA ) - D
      A8 = AA(II)
      B8 = BB(II)
      IF(Z1.LT. 0. ) Z1= ABS( Z1)
C
C
      CALL  LOVE(A8, B8, NU, X1, Y1, Z1, SYY1, 2YY2, SYY3 )
      IF( II .LE.  8) GO TO 937
      SUM1 = SUM1 - 2. *          SXY(II)* SYY1
      SUM2 = SUM2 + 2. *          SXZ(II)* SYY2
      SUM3 = SUM3 + 2. *          SXX(II)* SYY3
      GO TO 938
937   SUM1 = SUM1 -               SXY(II)* SYY1
      SUM2 = SUM2 +               SXZ(II)* SYY2
      SUM3 = SUM3 +               SXX(II)* SYY3
938   CONTINUE
C
      CALL  LOVE(A8, B8, NU, X2, Y1, Z1, SYY1, SYY2, SYY3 )
      IF( II .LE.  8) GO TO 939
      SUM1 = SUM1 + 2. *          SXY(II)* SYY1
      SUM2 = SUM2 + 2. *          SXZ(II)* SYY2
      SUM3 = SUM3 + 2. *          SXX(II)* SYY3
```

```
      GO TO 940
939   SUM1 = SUM1 +                SXY(II)* SYY1
      SUM2 = SUM2 +                SXZ(II)* SYY2
      SUM3 = SUM3 +                SXX(II)* SYY3
940   CONTINUE
C
C
2500  CONTINUE
C
      SC(JNDIC,I, J)= -SUM1 -SUM2 +SUM3
9999  CONTINUE
C
C
C
      IF( JNDIC  .EQ. ( ITER + 1 )  ) GO TO 7071
1118  CONTINUE
8803  WRITE(6,303)( FK(I), I=1, 7)
      WRITE(6,2299)    ( (SC(     1,I,J),I=1,6), J=1,19 )
      WRITE(6,2299)    ( (SC(     2,I,J),I=1,6), J=1,19 )
      WRITE(6,2299)    ( (SC(     3,I,J),I=1,6), J=1,19 )
      WRITE(6,2299)    ( (SC(     4,I,J),I=1,6), J=1,19 )
      WRITE(6,2299)    ( (SC(     5,I,J),I=1,6), J=1,19 )
2299  FORMAT(//(1X,6F15.4)//)
      CALL SIF(FCK, ITER)
      WRITE(6,3308)(  (FCK(I,J),J=1,3)    ,I=1,I*ER)
3308  FORMAT(1X,3F10.5)
9909  CONTINUE
      STOP
      END
      SUBROUTINE COEFF(N,M,K,A  )
      DIMENSION F(150 ) , G(75  ) , H( 10,10 ) , A( 10,10,62)
      NN = N+1
      NM = M+1
      NK = K+1
C
C
C     TO EVALUATE  K FACTORIAL OR , (P+N+1)/2 FACTORIAL WHICHEVER IS
```

```
C       GREATER
C
        M2 = 2*( N + K + 3 )
        DO 1 I =3, M2
        F( 1 ) =1   .
        F( 2 ) =1   .
        XX = I - 1
        F( I) = XX * F(I - 1 )
1       CONTINUE
C
C
C       TO EVALUATE GAMA FUNCTION OF (P+N+2)/2 +1/2 , OR GAMA FUNTION OF
C        (K+N+1)/2 +1/2 WHICHEVER IS GREATER
C
        IF( 2*(K+N+1) .GE. (M+N+2) ) N1=(K+N+1)
        IF( 2*(K+N+1) .LT. (M+N+2) ) N1=( (M+N+2) /2 +1 )
        N2 = N1 +1
        DO 2 I=2, N2
        W = I - 1
        G(1) =SQRT(3.1415926535897 93)
        G(I) = (2.*W - 1. )*G(I-1)/2.
2       CONTINUE
C
C       TO EVALUATE H(N,P)  WHICH IS EQUAL TO    GAMA FUNCTION OF (P+N+2)2
C       DIVIDED BY GAMA FUNCTION OF (P+N+3)/2
C
C
        DO 3 I=1, NN
        DO 3 J=1, NM
        II = I-1
        JJ = J-1
        IF(  (2*((II+JJ)/2)) .EQ. (II+JJ)  )  H(ITJ) =
      1  F(  (II+JJ+2)/2  )  /  G(  (II+JJ+2)/2 +1  )
        IF(  (2*((II+JJ)/2)) .NE. (II+JJ)  )  H(I,J) =
      1  G(  (II+JJ+1)/2 +1  )  /  F(  (II+JJ+3)/2  )
3       CONTINUE
C
```

```
C
C      TO EVALUATE  A(N,P,K) ;  WHICH IS EQUAL TO        (-1) TO THE POWER
C      K TIMES H(N,P)  )   DIVIDED BY    (2 TO THE POWER OF (2K+N+1/2)
C      TIMES (2K+N+P+3) TIMES GAMA FUNCTION OF (K+N+1+1/2 )
C
C
       DO 5 I =1, NN
       DO 5 J =1, NM
       DO 5 L =1, NK
       IF( 2*(  (L-1)/2  )  .EQ.  (L-1)  )  W =1.
       IF( 2*(  (L-1)/2  )  .NE.  (L-1)  )  W =-1.
       X = I-1
       Y = J-1
       Z = L-1
       A( I, J, L ) =     W* H(I,J) /( ( 2.**(2.*Z+X  +1.))*(2.*Z+X+Y+3.)
      1 *G( I + L) * F( L))
5      CONTINUE
       RETURN
       END
       SUBROUTINE BESSEL(N,K,R,Z,S  )
       DIMENSION S(10,140)
       T = Z*Z + R*R
       P = SQRT ( T )
       NJ = N+2
       DO 2 I=1, NJ
       W = I-1
       S(I,1) = (((P-Z)/R)**W)/P
2      CONTINUE
       NN = 2*K + N + 4
       DO 3 J=1,NN
       W = J-1
       S(1,J+1) = ((W+1.)*Z/T)*S(1,J) - (W*R/T)*S(2,J)
       S(2,J+1) = ((W+1.)*R/T)*S(1,J) + (W*Z/T)*S(2,J)
3      CONTINUE
       NK = N+1
       IF(  (NK-1).EQ.   0  ) GO TO 9
       DO 4 I =2,NK
```

```
      DO 4 J =1,NN
      W = I-1
      S(I+1,J+1) = ((2.*W)/R)*S(I,J) - S(I-1,J+1)
4     CONTINUE
9     RETURN
      END
      SUBROUTINE HYPER ( R, Z, SI, CI )
      DIMENSION AL( 120 ), BE( 120 ), GA( 120 )T X( 120 ), SI(5,5,6),
     1 F( 150), SS(120) , G(120) ,CI(4,5)
      NS = 7
      KK = 80
C
      DO 111 I=1, 4
      DO 111 J=1, 5
111   CI(I,J) = 0.
C
C
C
      DO 303  II1 = 1, 4
      DO 303  JJ1 = 1, 4
      DO 303  KK1 = 1, 6

303   SI( II1, JJ1, KK1 ) = 0.
C
C
      NP = NS
      IF(R.GE.  .5   .AND.  R  .LT.  .7 ) NP = NS + 5
      IF(R.GE.  .7   .AND.  R  .LT.   1.) NP = NS + 10
      NK = KK
      IF(Z  .LT.  R) NK = KK + 5
      NSS = 2*NP + 11
      IF( NSS  .LT.  NK) NSS= NK
      DO 6000 I=3, NSS
      F( 1 ) = 1.
      F( 2 ) = 1.
      XZ = I - 1
      F( I ) = XZ * F( I - 1 )
```

```
6000  CONTINUE
C
C
      DO 2 I=2, 8
      W = I - 1
      G(1) =SQRT(3.141592653589793)
      G(I) = (2.*W - 1. )*G(I-1)/2.
2     CONTINUE
C
C
      IF( Z  .EQ.  0.  .AND.  R  .LT.  1.  )GO 00 2222
C
C
      DO 8000 IN =1, 4
      WN = IN - 1
C
      DO 8000 IM =1, 4
      WM = IM - 1
C
      DO 8000 IP =1, 6
      WP=IP - 4
C
C
        SUM1 = 0.
      DO 9000 IS =1, NP
      IF(2*((IS- 1) /2) .EQ.  (IS- 1) ) W=1.
      IF(2*((IS- 1) /2) .NE.  (IS- 1) ) W=-1.
      WS = IS - 1
      LL = IN-1 + 2*(IS - 1) +(IP- 4) +(IM -1)
      IF(  LL   .LT.  0 )GO TO 8000


C
      TQ = WM+.5
      TZ = WN + 2.*WS +WP - .5
      COEF = (((  .5)**TQ ) * F(  LL +1 )  ) / (((   1. +Z**2.)**
     1 ( (TQ+TZ+1.)/2.))*G(IM+1))
```

```
      SUM = 1.
      DO 7000KS=2, NK
      TK = KS- 2
      AA = (WM+ .5) / 2.   +   (WN + 2.*WS + WP - .5 ) /2. + TK +.5
      BB = (WM+ .5) / 2.   -   (WN + 2.*WS + WP - .5 ) /2. + TK
      GG =       WM + 1.5   + TK
      XX = 1./ (1. +  Z**2. )
      AL( 1 ) = 1.
      BE( 1 ) = 1.
      GA( 1 ) = 1.
      X ( 1 ) = 1.
      AL(KS) = AA * AL(KS - 1 )
      BE(KS) = BB * BE(KS - 1 )
      GA(KS) = GG * GA(KS - 1 )
      X (KS) = XX * X (KS - 1 )
      SS(KS) =              (  AL(KS)*BE(KS)* X(KS) ) / ( F(KS)*GA(KS))
      SUM = SUM + SS(KS)
      SUM2     = SUM      *COEF
      CHECK2 = SS(KS) *COEF
      IF(  ABS(  CHECK2       )  .LT.    .0001) GO TO 888
7000  CONTINUE
C
C
888   SUM1 =        (  W*( (R/2.)**(WN + 2.*WS) D / ( F(IS) * F(IN+IS-1))
     1)*SUM2 + SUM1
C
C
      CHECK1=       (  W*( (R/2.)**(WN + 2.*WS) D / ( F(IS) * F(IN+IS-1))
     1)*SUM2
      IF(  ABS(     CHECK1     )  .LT.    .0001) GO TO 999
9000  CONTINUE
C
999   SI(IN,IM,IP)=  SUM1
8000  CONTINUE
```

```
C
      XXX = R**2. + Z**2. - 1.
      XX = SQRT((SQRT((ABS(    XXX          ))  **2. + 4.*(Z**2.)   ) +
     1 R**2. + Z**2. - 1. )  /2. )
      YY = SQRT((SQRT((ABS(     XXX         ))  **2. + 4.*(Z**2.)   ) -
     1(R**2. + Z**2. - 1.))  /2. )
      RR = SQRT((ABS(      XXX        ))  **2. + 4. * (Z**2.)   )
      TETA = ATAN ( 1. / Z )
      IF(  XXX  .EQ.  0.  )  GO TO 663
      FI= ATAN2( (2.*Z) , (XXX)  )
      GO TO 773
663   FI = 3.14159  /2.
773   IF(  XX   .EQ.  0.  )  GO TO 664
      S01 = ATAN( 1. /XX)
      GO TO 772
664   S01= 3.14159  /2.
772   IF(  R    .EQ.  0.  )  R = .0001
      S11 = ( 1. - (SQRT(RR)) * ( SIN       (FI/2.)     )  ) / R
      C11 = (  (SQRT(RR) )*( COS(FI/2.) ) - Z )  / R
      S12 = ( R* S01 - Z* S11 + C11 ) / 2.
      S21 = ( C11 - Z* S11 ) / R
      C21 =-( S11 + Z* C11 ) / R + .5
      S22 = ( R* S11 - Z* S21 + C21 ) / 3.
      C22 = ( R* C11 - Z* C21 - S21 ) / 3.
      S23 = ( R* S12 - Z* S22 + C22 ) / 4.
C
C
      CI(1,2) = ( ALOG( (Z + XX)**2. + (1. + YYD**2. ) ) /2. - ALOG( Z +
     1 SQRT( R**2. + Z**2. )  )
      IF( R .EQ.  .0001  )CI(1,2) = (  ALOG(1.   Z**2.)  )/2. -ALOG(Z)
      CI(2,3) = ( SQRT( R**2. + Z**2. ) - XX  ) /R
      IF( R .EQ.  .0001  )CI(2,3) = 0.
      CI(2,2) = ( 1. - YY + Z* ( XX - SQRT( R**2. + Z** 2. ) ) )/( 2.*R)
     1 + (R/ 4.) * ALOG( (Z+XX) **2. + (1. + YYD**2. ) - (R/2.) * ALOG(
     2 Z + SQRT( R**2. + Z**2. ) )
      IF( R .EQ.  .0001  )CI(2,2) = 0.
      CI(1,1) = - R* CI(2,3) - Z* CI(1,2) + S01
```

```
      CI(2,1) = ( R* CI(1,1)- Z* CI(2,2)+ S12 ) / 3.
      CI(3, 3) = (2./R) * CI( 2,1) - CI(1,1)
      CI(3, 4) = (2./R) * CI(2,2) - CI(1,2)
      CI(3,2) = ( R* CI(2,2) - Z* CI(3,3) + S22 ) / 4.
      CI(3,1) = ( R* CI(2,1) - Z* CI(3,2) + S23 ) / 5.
      CI(4,3) = ( 4./R )    * CI(3,1) - CI(2,1)
      CI(4,4) = ( 4./R )    * CI(3,2) - CI(2,2)
      CI(4,5) = ( 4./R )    * CI(3,3) - CI(2,3)
C
      GO TO 105
2222  DO 8800 IN =1, 4
      WN = IN - 1
      JN = IN - 1
C
      DO 8800 IM =1, 4
      WM = IM - 1
      JM = IM - 1
C
      DO 8800 IP =1, 5
      WP=IP - 4
      JP=IP - 4
C.
C
C
      ALPHA = (  WM + .5 +WN -( - WP + .5 )  + 1.  )/ 2.
      BETA  = (-(WM + .5)+WN -( - WP + .5 )  + 1.  )/ 2.
      GAMMA = WN + 1.
C
C
      IAL = ALPHA
      IGA = GAMMA
      IBE = 1. - BETA
      IGAA = GAMMA - ALPHA
C
C
      XAL = IAL
      XBE = IBE
```

```
      XGAA = IGAA
C
C
      IF(  (WM + WP + WN + 1. )  .LE.   0.  .OR+  (-WP + .5 )  .LT.  -1.
     1  ) GO TO 8800
C
      IF(  (1. - BETA )  .LE.  0.  ) GO TO 8800
C
      IF( XAL  .EQ.  ALPHA ) CAL = F( IAL   )
      IF( XAL  .NE.  ALPHA ) CAL = G( IAL + 1)
C
C
      IF( XBE  .EQ.  ( 1. - BETA )  ) CBE = F( IBE   )
      IF( XBE  .NE.  ( 1. - BETA )  ) CBE = G( IBE+1 )
C
      TQ = GAMMA - ALPHA - BETA
      COEF                      = (  (R** ( GAMMA - 1. )  )* CAL    )/
     1(    (2.** TQ  )*  F( IGA )  *  CBE    )
C
C
      SUM = 1.
      NK = 30
C
C
      DO 7700KS=2, NK
      TK = KS- 2
C
      AA = ALPHA + TK
C
      BB = BETA + TK
      IF( R  .GT.  1. )  BB = ALPHA - GAMMA + 1. + TK
C
      GG = GAMMA + TK
      IF(  R  .GT.  1.  ) GG = ALPHA - BETA + 1C + TK
C
      XX = R ** 2.
      IF( R  .GT.    1. )  XX = 1. / ( R  **2.)
```

```
C
      AL( 1 ) = 1.
      BE( 1 ) = 1.
      GA( 1 ) = 1.
      X ( 1 ) = 1.
      AL(KS) = AA * AL(KS - 1 )
      BE(KS) = BB * BE(KS - 1 )
      GA(KS) = GG * GA(KS - 1 )
      X (KS) = XX * X (KS - 1 )
      SS(KS) =              (  AL(KS)*BE(KS)* X(KS) ) / ( F(KS)*GA(KS))
      SUM = SUM + SS(KS)
C
C
      SUM2     = SUM      *COEF
7700  CONTINUE
8881  SI(IN, IM, IP ) = SUM2
8800  CONTINUE
C
C
C
C
C
C
      IF( R  .GT.  1. ) SS01 = ASIN( 1. /R)
      IF( R  .LT.  1. ) SS01 = 3.14159  /2.
      IF( R  .GT.  1. ) SS11 = 1. /R
      IF( R  .LT.  1. ) SS11 = R/ ( 1. + SQRT( 1. - R**2.)  )
      IF( R  .GT.  1. ) CC11 = ( SQRT( R**2. - 1. )  )/R
      IF( R  .LT.  1. ) CC11 = 0.
      IF( R  .GT.  1. ) SS21 = ( SIN ( 2.* ASINU1./R) ) ) /2.
      IF( R  .LT.  1. ) SS21 = 0.
      IF( R  .GT.  1. ) CC21 = ( COS( 2.* ASIN(1./R) ) )/2.
      IF( R  .LT.  1. ) CC21 = -(R**2./2.)  /  U  (1. + SQRT(1. - R**2.)
     1 )**2.  )
      SS12=( R* SS01 + CC11 ) / 2.
      SS22 = ( R* SS11 + CC21 ) / 3.
      CC22 = ( R* CC11 - SS21 ) /3.
```

```
      SS23=( R* SS12 + CC22 ) / 4.
C
C
      IF( R  .LT.  1. ) CI(2,3) = 1.
      IF( R  .GT.  1. ) CI(2,3) = 1.- ( SQRT(R*M2. - 1.) )/ R
      IF( R  .GT. 1. ) CI(2,2) = 1./(2.*R)
      IF( R  .LT. 1. ) CI(2,2) =  ( 1. -SQRT(1. - R**2.))/(2.*R) +
     1 (  ALOG( 1. + SQRT(1. - R**2.)  ) - ALOGUR)   )* (R/2. )
      CI(1,1) = - R* CI(2,3) + SS01
      CI(2,1) = ( R* CI(1,1) + SS12 ) / 3.
      CI(3,3) = ( 2. / R) * CI(2,1) - CI(1,1)
      CI(3,2 ) = (R* CI(2,2) + SS22 )/ 4.
      CI(3,1) = ( R* CI(2,1) + SS23 ) /5.
      CI(4,3) = ( 4./R) * CI(3,1) - CI(2,1)
C
C
105   RETURN
      END
      SUBROUTINE CONST( A1, INDIC, SC, C )
      DIMENSION Q(6,19),SC(6,6,19), C(10,10) ,FNR(6)
C
      DO 225 I=1, 3
      DO 225 J=1, 3
225   C(I,J) = 0.
C
      H = .087266463* 2.
C
      DO 200  JJ = 1, 3
      MM = JJ - 1
C
C
      SUM1 = 0.
      SUM2 = 0.
      SUM3 = 0.
      SUM4 = 0.
      SUM5 = 0.
      SUM6 = 0.
```

```
C
C
      DO 100  II = 1, 6
      R = II - 1
      R = R * .2
C
C
C
      IF(R.EQ.0.) I = 1
      IF((R.GE..19).AND.(R.LE..21)) I = 2
      IF((R.GE..39).AND.(R.LE..41)) I = 3
      IF((R.GE..59).AND.(R.LE..61)) I = 4
      IF((R.GE..79).AND.(R.LE..81)) I = 5
      IF((R.GE..99).AND.(R.LE.1.1)) I = 6
      DO 22 J = 1,19
      XM = MM
      TH = J-1
      TH = H*TH
   22 Q(I,J) =SC(INDIC,I,J)*COS(XM*TH)
      N = 18
      I1 = N-6
      A = Q(I,1)+5.*Q(I,2)+Q(I,3)+6.*Q(I,4)+Q(I,5)+5.*Q(I,6)+Q(I,19)
      DO 25 J = 6,I1,6
   25 A = A+2.*Q(I,J+1)+5.*Q(I,J+2)+Q(I,J+3)+6.MQ(I,J+4)+Q(I,J+5)
     1 +5.*Q(I,J+6)
      AA = .6*H*A/(3.1415927 *A1 )
      IF(MM) 34,33,34
   33 AA = .5*AA
      IF( R  .EQ.  0. ) C(1,1) = AA
   34 IF(ABS(AA).LE.1.E-5) AA = 0.
C
      FNR( II ) = AA
C
C
      SUM1 = SUM1 + R
      SUM2 = SUM2 + R**2.
      SUM3 = SUM3 + R**3.
```

```
      SUM4 = SUM4 + R**4.
      SUM5 = SUM5 + R* AA
      SUM6 = SUM6 + (R**2. )*  AA
C
100   CONTINUE
      WRITE(6,879)( FNR( I), I=1,6 )
879   FORMAT(1X,3HKN=,6F10.5)
C
C
      ALPHA = SUM3 * SUM3 - SUM4 * SUM2
      IF(JJ - 1 )122,122,123
C
122   C(1,2) = ( ((-C(1,1)*SUM2 + SUM6)*SUM3) - ( (-C(1,1)*SUM1 + SUM5)*
     1SUM4 ) ) / ALPHA
C
      C(1,3) = ( ((-C(1,1)*SUM1 + SUM5)*SUM3) - ( (-C(1,1)*SUM2 + SUM6)*
     1SUM2 ) ) / ALPHA
C
      GO TO 200
123   C(JJ, 2) = ( SUM6 * SUM3 - SUM5 * SUM4 )/ ALPHA
C
      C(JJ, 3) = ( SUM5 * SUM3 - SUM6 * SUM2 )/ ALPHA
C
200   CONTINUE
      WRITE(6,150)  ( (  C(I,J), J=1,3) ,  I=1,3 )
150   FORMAT(//(1X,3F15.5)//)
      RETURN
      END
      SUBROUTINE LOVE( A, B, NU, X, Y, Z, SYY1, SYY2, SYY3 )
      REAL  NU
C
      C1 = 1. / (2.*3.14159 )
      C2 = 2.* (1. + NU )
      C5 = 2.* NU
      C4 = 1. - 2.*NU
C
C
```

```
C
56      G1 = A+X
        G2 = A-X
        G3 = B+Y
        G4 = B-Y
        IF( G1  .EQ.  0. ) G1 = .000001
        IF( G2  .EQ.  0. ) G2 = .000001
        IF( G3  .EQ.  0. ) G3 = .000001
        IF( G4  .EQ.  0. ) G4 = .000001
C
        A1 = SQRT( G2*G2 + G4*G4 +Z*Z )
        B2 = SQRT( G1*G1 + G4*G4 +Z*Z )
        C3 = SQRT( G1*G1 + G3*G3 +Z*Z )
        D4 = SQRT( G2*G2 + G3*G3 +Z*Z )
C
C
C
        PZ1   = ALOG(((D4-G3)*(B2+G4))/((A1+G4)*(+3-G3)))
        PYY1=-G4/((Z+A1)*A1)+G4/((Z+B2)*B2)+G3/((Z+C3)*C3)-G3/((Z+D4)*D4)
        PXY1=-G2/((Z+A1)*A1)+G2/((Z+D4)*D4)+G1/((Z+C3)*C3)-G1/((Z+B2)*B2)
        PYZ1 = 1./A1 - 1./B2 + 1./C3 - 1./D4
        P1XY1 = Z*PXY1 + G2/A1 + G1/B2 - G1/C3 - G2/D4
        P1YY1 = Z*PYY1 + G4/A1 + G3/D4 - G3/C3 - G4/B2
        FXX=ATAN(G4/G2)+ATAN(G3/G2)-ATAN((Z*G4)/(G2*A1))-ATAN((Z*G3)/(G2*
       1D4))+ATAN(G4/G1)+ATAN(G3/G1)-ATAN((Z*G4)/(G1*B2))-ATAN((Z*G3)/(G1*
       2C3))
        FYY=ATAN(G2/G4)+ATAN(G1/G4)-ATAN((Z*G2)/(G4*A1))-ATAN((Z*G1)/(G4*
       1B2))+ATAN(G2/G3)+ATAN(G1/G3)-ATAN((Z*G2)/UG3*D4))-ATAN((Z*G1)/(G3*
       2C3))
C
        FZZ = -(FXX + FYY)
        FYZ =  ALOG(((D4 + G2)*(B2 - G1))/((A1 + G2)*(C3 - G1)))
C
        SYY1 = C1*(C5*PZ1 + C5*P1YY1 - Z*PYY1)
C
C
```

```
      PZ2    = FYZ
      PXX2 = PXY1
      PZZ2=(Z*(G2/A1+G1/B2))/(G4*G4+Z*Z)-(Z*(G2/D4+G1/C3))/(G3*G3+Z*Z)
      PYY2 = -(PXX2 + PZZ2)
      PYZ2 = -G4*(G2/A1+G1/B2)/(G4*G4+Z*Z)-G3*(G2/D4+G1/C3)/(G3*G3+Z*Z)
      P1ZZ2 = PZ2
      P1XX2 = P1XY1
      P1YY2 = -(P1XX2 + P1ZZ2)
C
C
      SYY2 = C1*(C2*PZ2 + C5*P1YY2 - Z*PYY2)
C
C
      VZ = FZZ
      VYY = PYZ2
C
C
      SYY3 = C1*(C5*VZ - C4*FYY - Z*VYY)
C
C
      RETURN
      END
      SUBROUTINE LINEQ(A,N,M)
      DIMENSION A(30,40),IN1(30),IN2(30)
      COMMON KTH,DET
   34 KTH =0
      L=0
      DET=1.
      DO 20 I=1,N
C     SEARCH FOR LARGEST ELEMENT IN COLUMN
      D=0.
      DO 2 K=I,N
      D1=ABS(A(K,I))
      IF(D-D1) 1,2,2
    1 D=D1
      J=K
    2 CONTINUE
```

```
C       SEARCH FOR SINGULAR  MATRIX
        IF (D-.1E-11)30,30,3
C       ERROR MESSAGE IN CASE OF ALL ZEROS IN A COLUMN ON AND BELOW DIAGONAL
   30   PUNCH 200
        RETURN
  200   FORMAT(1H1,*SINGULAR MATRIX*)
C       INTERCHANGE ROWS--PIVOT-- TO PUT LARGEST ELEMENT IN DIAGONAL
    3   IF(J-I)4,6,4
    4   L=L+1
        IN1(L)=I
        IN2(L)=J
        DO 5K=1,M
        SAV=A(I,K)
        A(I,K)=A(J,K)
    5   A(J,K)=SAV
C       REDUCE ROW ELEMENTS FOR EACH ROW BELOW MAIN DIAGONAL
    6   DO 16 J=1,N
        IF(J-I)7,16,7
    7   IF (ABS (A(J,I))-.1E-9)  9,16,9
    9   RJ= A(J,I)/A(I,I)
        DO 15 K=1,M
        IF(K-I)14,15,14
   14   A(J,K)=A(J,K)-RJ*A(I,K)
   15   CONTINUE
C       REPLACE A(J,I) BY -A(J,I)/A(I,I)
        A(J,I)=-RJ
   16   CONTINUE
        DET=DET*A(I,I)
C       DIVIDE WORKING ROW ELEMENTS BY THE DIAGONAL
        A(I,I)=1./A(I,I)
        DO 20J=1,M
        IF(J-I)17,20,17
   17   A(I,J)=A(I,J)*A(I,I)
   20   CONTINUE
C       INTERCHANGE COLUMNS- INVERSE OF ROW INTERCHANGE
        IF(L-2*(L/2))23,24,23
   23   DET =-DET
```

```
   24 IF(L)27,26,27
   27 I=IN1(L)
      J=IN2(L)
      DO 25 K=1,N
      SAV=A(K,I)
      A(K,I)=A(K,J)
   25 A(K,J)=SAV
      L=L-1
      GO TO 24
   26 RETURN
      END
      SUBROUTINE SIF(FCK, ITER)
      DIMENSION Z(101), T(101),A1(30,40),B1(3,1D,C1(3,1),F (3),
     1SIF(21), DEG(21) ,W(3) ,FCK(6,3)
C
      DO 5 I=1, 3
      W(I) = 0.
      DO 5 J=1, ITER
5     W(I) = W(I) + FCK(J,I)
C
      DO 10 I=1, 101
      TETA = I - 1
10    T(I)=TETA * .031415927
C
      DO 15 I=1, 101
      Z(I) = 0.
      DO 15 J=1, 3
      FI = J - 1
15    Z(I)   = Z(I) + W(J) * COS( FI*T(I) )
C
      B1(1,1) = 0.
      B1(2,1) = 0.
      B1(3,1) = 0.
      A1(1,1) = 101.
      A1(1,2) = 0.
      A1(1,3) = 0.
      A1(2,3)=0.
```

```
      A1(3,3) = 0.
C
      DO 20 I=1, 101
      B1(1,1) = B1(1,1) + Z(I)
      B1(2,1) =  B1(2,1) + Z(I)* T(I)
      B1(3,1) = B1(3,1) + Z(I)* T(I)*T(I)
      A1(1,2)=A1(1,2) + T(I)
      A1(1,3)=A1(1,3) + T(I) **2.
      A1(2,3)=A1(2,3) + T(I)**3.
20    A1(3,3) = A1(3,3) + T(I) **4.
C
      A1(2,1)=A1(1,2)
      A1(2,2) = A1(1,3)
      A1(3,1)=A1(1,3)
      A1(3,2) = A1(2,3)
C
C
      CALL LINEQ (A1,3,3)
C
      DO 8889 I=1, 3
      DO 8889 J = 1,1
      SUM = 0.
      DO 8887 K =1, 3
8887  SUM = SUM + A1(I,K)*B1(K,J)
8889  C1(I,J) =SUM
C
C
      F(1) = C1(1,1) + 3.1415927*C1(2,1)/2. +.333333*C1(3,1)*(3.1415927*
     1*2.)
      DO30 I=2, 3
      G = I-1
      IF( 2*((I-1)/2)  .EQ.  (I-1)  ) WW = 1.
      IF( 2*((I-1)/2)  .NE.  (I-1)  ) WW = -1.
30    F(I) = ((   WW   -1.)*C1(2,1)/(G*G)+ 2.*C1(3,1)* 3.1415927*(
     1WW ) / (G*G)) * (2./3.1415927)
C
      DO35 I=1, 101, 5
```

```
      N =(I-1)/ 5 + 1
      EPS = 0.
      DO 36 J=1, 3
      F1 = J- 1
36    EPS = EPS + ( W(J) - F(J) )* COS(F1* T(I) )
      SIF(N) = C1(1,1) + C1(2,1)* T(I) + C1(3,1D* T(I)*T(I) + EPS
35    DEG(N) = T(I)* 180./3.1415927
C
      WRITE(6,100) ( DEG(N), N=1, 21)
100   FORMAT(1X,*TETA =*,21F6.1)
      WRITE(6,101)( SIF(N), N=1, 21)
101   FORMAT(1X,*S.I.F=*,21F6.3)
      RETURN
      END
*RUN
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5
 .1        .3        .5        .7        .9        1.25      1.75      2.5

 .3        .3        .3        .3        .3        .3        .3        .3
 .5        .5        .5        .5        .5        .5        .5        .5
 .7        .7        .7        .7        .7        .7        .7        .7
 .9        .9        .9        .9        .9        .9        .9        .9
 1.25      1.25       1.25      1.25      1.25      1.25      1.25      1.25
  1.75     1.75      1.75      1.75      1.75      1.75      1.75      1.75
 2.5       2.5       2.5       2.5       2.5       2.5       2.5       2.5
 .1        .1        .1        .1        .1        .25       .25       .5
 .1        .1        .1        .1        .1        .25       .25       .5
 .1        .1        .1        .1        .1        .25       .25       .5
 .1        .1        .1        .1        .1        .25       .25       .5
 .1        .1        .1        .1        .1        .25       .25       .5
 .1        .1        .1        .1        .1        .25       .25       .5
```

```
 .1      .1      .1      .1      .1      .25     .25     .5
 .1      .1      .1      .1      .1      .25     .25     .5
 .2      .2      .2      .2      .2      .2       .2     .2
 .1      .1      .1      .1      .1      .1       .1     .1
 .1      .1      .1      .1      .1      .1       .1     .1
 .1      .1      .1      .1      .1      .1       .1     .1
 .1      .1      .1      .1      .1      .1       .1     .1
 .25     .25     .25     .25     .25     .25      .25    .25
 .25     .25     .25     .25     .25     .25      .25    .25
 1.      1.      1.      1.      1.      1.       1.     1.
   2       2      60     .25     .25      4        2
*END
```