DISSERTATION

FINITELY GENERATED MODULES OVER NOETHERIAN RINGS: INTERACTIONS BETWEEN ALGEBRA, GEOMETRY, AND TOPOLOGY

Submitted by

Zachary J. Flores

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2020

Doctoral Committee:

Advisor: Christopher Peterson

Jeanne Duflot
Renzo Cavalieri
Kathryn Ross

ABSTRACT

FINITELY GENERATED MODULES OVER NOETHERIAN RINGS: INTERACTIONS BETWEEN ALGEBRA, GEOMETRY, AND TOPOLOGY

In this dissertation, we aim to study finitely generated modules over several different Noetherian rings and from varying perspectives. This work is divided into four main parts: The first part is a study of algebraic K-theory for a certain class of local Noetherian rings; the second discusses extending well-known results on Lefschetz properties for graded complete intersection algebras to a class of graded finite length modules using geometric techniques; the third discusses the structure of various algebraic and geometric invariants attached to the finite length modules from the previous section; and lastly, we discuss the structure of annihilating ideals of classes of hyperplane arrangements in $\mathbb{P}^n$.

ACKNOWLEDGEMENTS

First and foremost, this dissertation would not have been possible without the amazing graduate student community in the CSU Mathematics Department. Thank you for all of the wonderful time we have spent together.

Having an advisor who supports and encourages you is important, but I am deeply grateful to have had an advisor who let me pursue some of my own interests, as well as helping me to broaden the scope of my research. Thank you, Chris. I could not have asked for a better mentor and friend. This document could not have existed without your unwavering support.

This dissertation would not have been possible if it had not been for the excellent mentoring I received throughout my education. Thank you to Ignacio Uriarte-Tuero, Christel Rotthaus, and Hailong Dao.

Getting through all of the administrative hurdles could not have been done without the fantastic staff in the CSU Mathematics Department. A big thank you to Bryan Elder, Christie Franklin, Hillary Freeman, Sheri Hofeling, and Jane Owen-Maul for always being kind and helpful.

A hearty thank you to my boyfriend Taylor Ball for his support and love throughout the writing of this dissertation during a pandemic.

DEDICATION

To my parents, Mary Williams and Jose Flores, who taught me a love for learning at an early age, spurred my interest in science, and always supported my education.

To my first ever scientific role model, my grandfather and fellow Michigan State Spartan, Dr. Robert B. Foy.

To my grandmother Jayne Foy, for always inspiring me with her wit and charm. I will miss our evening chats over coffee.

To my grandmother Eustolia Flores, who came to the United States to forge a new life filled with opportunities for herself, for her children and grandchildren. Your unconditional love will always be a part of me.

To my grandfather Ramon Flores, for showing me how to persevere through the toughest of times.

TABLE OF CONTENTS

Chapter 1

Preface

This dissertation covers a wide range of topics in commutative algebra, and this preface aims to give a brief introduction to the problems being discussed in each of the respective chapters.

In Chapter 2, we discuss the following problem:

Question 1.0.1. Given a local Cohen-Macaulay ring R, can we give explicit descriptions of the G-groups of R?

The G-groups of R are denoted by $G_i(R) := K_i(\textbf{mod}\, R)$; where the right-hand side is the ith Quillen K-group of the category of finitely generated R-modules. These groups were introduced by Daniel Quillen in [57], and in general, are very difficult to compute.

Our aim in Chapter 2 is to build on the work of Navkal and Holm in [53] and [31], respectively. Specifically, we extend their techniques to provide a structure theorem for the group $G_1(R)$ when R has some additional structure. Moreover, in Chapter 2, we compute $G_1(R)$ explicitly for several classes of hypersurface singularities, building greatly on the work in [53] and [31].

In Chapter 3, we change avenues slightly to study graded commutative algebra with stronger connections to algebraic geometry. Namely, our interest lies in studying the *Weak Lefschetz Property* for a class of finite length modules. Specifically, we ask the following:

Question 1.0.2. Given a vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$, does $H^1_*(\mathbb{P}^2, \mathcal{E})$ have the Weak Lefschetz Property?

We answer Question 1.0.2 in the affirmative in Chapter 3, building on previous work in [34]. The main result of [34] is that codimension three complete intersections have the Weak Lefschetz Property, and we were able to generalize this result to class of finite length *modules* in Chapter 3.

While our aim in Chapter 3 was to generalize the main result of [34], our techniques allow us to encapsulate the proof of the main result into a single paper. In [34], the proof of the main result relies on results from [68], and we can avoid utilizing these.

The work in Chapter 3 grew out of the work in Chapter 4. In fact, our initial aim was to prove the main result of Chapter 3 (see Theorem 3.3.7). However, in solely attempting to utilize the techniques of [34], we were unable to do so. Nonetheless, we still found the techniques useful, providing a more algebraic path to study Lefschetz properties for $H^1_*(\mathbb{P}^2, \mathcal{E})$ than those in Chapter 3. Moreover, we also utilized interesting connections with *Symmetrically Gorenstein modules* coming form [43].

While Question 1.0.2 was also studied in Chapter 4, the following question is also studied in Chapter 4:

Question 1.0.3. What can we say about the non-Lefschetz locus of $H^1_*(\mathbb{P}^2, \mathcal{E})$?

The non-Lefschetz locus is a geometric object associated to the finite length module $H^1_*(\mathbb{P}^2, \mathcal{E})$ originally defined in [7]. Our focus in Chapter 4 was to bring results in [7] to the setting of finite length modules. In this direction, we were successful, but had to incorporate some very different techniques. Namely, we explore and utilize techniques on *Artinian level* modules, and again utilize techniques from [43] on Symmetrically Gorenstein modules.

In Chapter 5, we are interested in the very easy to state question:

Question 1.0.4. What can we say about the structure of the annihilating ideal of (commuting) differential operators of a homogeneous form?

Such annihilating ideals are called *Macaulay duals* or *inverse systems*, and they are always Artinian Gorenstein ideals. In particular, one question we can immediately ask when is the Macaulay dual of a specific class of forms a complete intersection?

In Chapter 5, we ask this question for a class of forms called *generic hyperplane arrangements*. While we answer this question negatively, we succeed in giving a lower bound for the minimal degree of of the Macaulay dual of a generic hyperplane arrangement. Moreover, in the course of this, we also find some interesting connections with star configurations in [25].

While these may seem like very disparate areas of commutative algebra, our collective focus is to gain information from about a ring or a module via the study of a *module action*. For example, an explicit structure theorem for $G_1(R)$ for certain local Cohen-Macaulay rings can help determine when two such rings are not isomorphic; a finite length module with the Weak Lefschetz Property will have a unimodal Hilbert function; and the lower bound for the minimal degree of the Macaulay dual of a generic hyperplane arrangement we give shows that it cannot be a complete intersection if there are too few hyperplanes in the arrangement.

All chapters in this text are independent from one another. However, Chapter 3 and Chapter 4 contain very similar results and could be read together.

Chapter 2

Algebraic K-theory for Cohen-Macaulay Rings

2.1 Introduction

Throughout[1] this section $(R, \mathfrak{m}, k)$ will always denote a local Noetherian ring that is Cohen-Macaulay. Since the introduction of higher algebraic K-theory by Quillen there has been a significant effort to understand the structure of the K-groups $K_i(\mathcal{A})$, for $\mathcal{A}$ an exact category. Our particular interest is when $\mathcal{A} = \textbf{mod}\, R$, the category of finitely generated R-modules. The groups $K_i(\textbf{mod}\, R)$ are denoted by $G_i(R)$. They are, unsurprisingly, called the G-groups of R (they are also called K'-groups in the literature and may be denoted by $K'_i(R)$). In Section 2.2, we will discuss notation and various definitions of K-groups needed in the computation of $G_1(R)$.

Let $\textbf{proj}\, R$ be the subcategory of $\textbf{mod}\, R$ of finitely generated projective R-modules. Now the inclusion $\textbf{proj}\, R \hookrightarrow \textbf{mod}\, R$ induces a map of groups between $K_i(R) := K_i(\textbf{proj}\, R)$ and $G_i(R)$. It is of interest to understand the properties of this induced homomorphism. In particular, when is this map an isomorphism? This is precisely the case when R is regular, following immediately from Quillen's Resolution Theorem ([57], §Theorem 3). However, regular local rings are exceptionally well-behaved, so one cannot expect this behavior in general. Suppose $i = 0$. It is well-known $K_0(R)$ isomorphic to $\mathbb{Z}$ (see ([58], Theorem 1.3.11)), but what of $G_0(R)$? If R is regular, then $G_0(R) = \mathbb{Z}$. However, if R is not regular, but also has *finite Cohen-Macaulay type* (that is, there are, up to isomorphism, finitely many indecomposable maximal Cohen-Macaulay R-modules) then the structure of $G_0(R)$ is elucidated in its entirety by the following.

Theorem 2.1.1. *([69], Theorem 13.7)*

Suppose there are t non-free indecomposable maximal Cohen-Macaulay R-modules and denote by $\mathcal{G}$ the free abelian group on the set of isomorphism classes of indecomposable maximal Cohen-Macaulay R-modules. The map $\mathcal{G} \longrightarrow G_0(R)$ given by $X \longmapsto [X]$ is surjective and its

[1] The main results in this chapter are taken from the paper [23].

kernel is generated by

$$\{X - X' - X'' \mid \exists \text{ an Auslander-Reiten sequence } 0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0\}$$

And $G_0(R) \cong$ *coker*(Υ), *where* $\Upsilon : \mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$ *is the Auslander-Reiten homomorphism.*

The immense usefulness of Theorem 2.1.1 lies in the fact that the computation of $G_0(R)$ has been reduced to linear algebra, as the Auslander-Reiten homomorphism can be readily computed from the Auslander-Reiten quiver. This quickly leads to the explicit computation of $G_0(R)$ for all simple singularities of finite type (see [69], Proposition 13.10). One can quickly see that these groups are often not $\mathbb{Z}$.

Moving up one rung on the K-theory ladder, it is well-known that $K_1(R) := K_1(\textbf{proj}\, R) \cong R^*$ (see ([60], Example 1.6)). However, the structure of $G_1(R)$ was not known for some time until the work of H. Holm in [31] and V. Navkal in [53]. In the former, computing $G_1(R)$ was carried out over an R which has finite Cohen-Macaulay type and it was found that $G_1(R)$ could be computed as an explicit quotient of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$, with M an additive generator for the category maximal Cohen-Macaulay R-modules, $\textbf{mcm}\, R$ (noting such an M exists if and only if R has finite Cohen-Macaulay type). The latter produced the following.

Theorem 2.1.2. *([53], Theorem 1.3)*

Assume that R is Henselian and the category $\textbf{mcm}\, R$ *has an n-cluster tilting object L. Let I be the set of isomorphism classes of indecomposable summands of L and set $I_0 = I \setminus \{R\}$. Then there is a long exact sequence*

$$\cdots \longrightarrow \bigoplus_{L' \in I_0} G_i(\kappa_{L'}) \longrightarrow G_i(\Lambda) \longrightarrow G_i(R) \longrightarrow \bigoplus_{L' \in I_0} G_{i-1}(\kappa_{L'}) \longrightarrow \cdots$$

Where

$$\Lambda = \mathit{End}_R(L)^{op} \quad \text{and} \quad \kappa_{L'} = \mathit{End}_R(L')^{op}/\mathit{rad}(\mathit{End}_R(L')^{op})$$

Moreover, $\kappa_{L'}$ is always a division ring, and when $R/\mathfrak{m} = k$ is algebraically closed, $\kappa_{L'} = k$.

The long exact sequence ends in presentation

$$\bigoplus_{L' \in I_0} G_0(\kappa_{L'}) \longrightarrow G_0(\Lambda) \longrightarrow G_0(R) \longrightarrow 0$$

*of $G_0(R)$. Since $G_0(\Lambda) = \mathbb{Z}^I$ and $\bigoplus_{L' \in I_0} G_0(\kappa_{L'}) = \mathbb{Z}^{I_0}$, the presentation of $G_0(R)$ given above is precisely the one given in Theorem 2.1.1 when L is an additive generator of **mcm** R.*

The definition of an n-cluster tilting object is technical and we refer the reader to Definition 2.2.14 and Section 2.4 for examples. We show in Section 2.3 that utilizing Theorem 2.1.2 and techniques from [31], we can generalize and simplify the results [31] on the structure of $G_1(R)$. Keeping notation as in Theorem 2.1.2), our contribution in this direction is the following.

Theorem 2.1.3. *Let k be an algebraically closed field of characteristic not 2 and R a Henselian k-algebra that admits a dualizing module and is also an isolated singularity. If **mcm** R admits an n-cluster tilting object L such that $\mathit{End}_R(L)^{op}$ has finite global dimension, then there is a subgroup Ξ of $\mathit{Aut}_R(L)_{ab}$, described explicitly in Definition 2.2.21, and a free abelian group $\mathcal{H}$ such that*

$$G_1(R) \cong \mathcal{H} \oplus \mathit{Aut}_R(L)_{ab}/\Xi$$

The utility of Theorem 2.1.3 is that the computation of $G_1(R)$ for some hypersurface singularities becomes tractable, as well as removing the necessity of the injectivity of the Auslander-Reiten homomorphism and the need for R to have finite Cohen-Macaulay type, as required in [31]. In fact, with the long exact sequence of [53] and the machinery of [31], the proof is quite elementary. However, before proving Theorem 2.1.3 in Section 2.1.3, we collect the necessary details on n-cluster tilting objects, noncommutative algebra and functor categories in Section 2.2.

Of course, in order to utilize Theorem 2.1.3, one might want to know when **mcm** R admits an n-cluster tilting object. This is discussed in Section 2.4.

The goal of explicitly computing $G_1(R)$ for specific R would not be possible if we could not compute $\mathrm{Aut}_R(L)_{\mathrm{ab}}$. We expend some energy in Section 2.5 calculating $\mathrm{Aut}_R(L)_{\mathrm{ab}}$ for several concrete examples. This section and the next form the technical heart of our work.

Utilizing the results of Section 2.5, we are able to explicitly compute $G_1(R)$ for several hypersurface rings in Section 2.6. See Examples 2.6.1, 2.6.3, 2.6.4 and Proposition 2.6.6 for details.

In Section 2.7, we discuss the similarities our computations share and make a conjecture.

We now fix notation. We always use A to denote an associative ring with identity that is not necessarily commutative; **mod** A will be the category of finitely generated left A-modules; and **proj** A will be the category of finitely generated projective left A-modules.

We will use the following setup: $(R, \mathfrak{m}, k)$ always denotes a commutative local Cohen-Macaulay ring such that

(a) R is Henselian.

(b) R admits a dualizing module.

(c) **mcm** R admits an n-cluster tilting object.

(d) R is an isolated singularity.

The assumption of (a) give us that *any* maximal Cohen-Macaulay module can be written uniquely as a direct sum of finitely many indecomposable maximal Cohen-Macaulay modules (see ([47], Theorem 1.8 and Exercise 1.19)). In fact, all of the rings for which we compute $G_1(R)$ are complete, so they already satisfy (a) (see ([47], Corollary 1.9)). The assumption of (b) is a standard technical assumption in representation theory of Cohen-Macaulay rings. Currently, the assumption (c) is very much a technical black box, but we will see it is indispensable; see Definition 2.2.14. The assumption in (d) is necessary to make use of the theory of n-cluster tilting objects. When necessary, we will assume that R is a k-algebra and $\mathrm{char}(k) \neq 2$, but we do not use this as a blanket assumption.

2.2 Preliminaries

2.2.1 Some Definitions of K-groups

We begin first by discussing the classical definition lower K-groups.

Definition 2.2.1. The **classical K_0-group** of A, denoted by $K_0^C(A)$, is defined as the Grothendieck group of the category **proj** A. More explicitly, choose an isomorphism class for each $P \in$ **proj** A and let X be the free abelian group on these isomorphism classes. Then $K_0^C(A)$ is the quotient of X by the subgroup of X generated by $\{[P] - [P'] - [P''] : 0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0 \text{ exact}\}$.

The **classical K_1-group** of A, denoted by $K_1^C(A)$, is defined as the abelianization of the infinite general linear group over A. That is, using the obvious embeddings $GL_n(A) \hookrightarrow GL_{n+1}(A)$, we can form the *infinite general linear group* $GL(A) := \bigcup_{n\geq 1} GL_n(A)$. Thus $K_1^C(A)$ is $GL(A)_{\text{ab}}$.

Of principal importance in defining K-groups for our purposes is the following notion.

Definition 2.2.2. An **exact category** $\mathcal{Y}$ is an additive category together with a distinguished class of sequences $Y' \rightarrowtail Y \twoheadrightarrow Y''$ called *coinflations* with a fully faithful additive functor F from $\mathcal{Y}$ into an abelian category $\mathcal{X}$ such that

(a) $Y' \rightarrowtail Y \twoheadrightarrow Y''$ is a conflation in $\mathcal{Y}$ if and only if $0 \longrightarrow F(Y') \longrightarrow F(Y) \longrightarrow F(Y') \longrightarrow 0$ is exact in $\mathcal{X}$.

(b) If $0 \longrightarrow F(Y') \longrightarrow X \longrightarrow F(Y'') \longrightarrow 0$ is exact in $\mathcal{X}$, then $X \cong F(Y)$ for some Y in $\mathcal{Y}$. That is, $\mathcal{Y}$ is closed under extensions in $\mathcal{X}$.

We note any abelian category is an exact category. Moreover, **proj** A is an exact category, where the conflations are taken to be the sequences that are exact in **mod** A. Note that **proj** A is an exact category which is not abelian.

We will need the following notions as they pertain to exact categories.

Definition 2.2.3. $\mathcal{Y}$ denotes an exact category.

(a) We will always work under the assumption that the objects of $\mathcal{Y}$ form a set. In this regard, we say that $\mathcal{Y}$ is **skeletally small**.

(b) We say $\mathcal{Y}$ is a **semisimple exact category** if every conflation splits. The prototypical example of a semisimple exact category is **proj** A.

(c) We write $\mathcal{Y}_0$ to denote $\mathcal{Y}$ viewed as an exact category in which the coinflations $Y' \rightarrowtail Y \twoheadrightarrow Y''$ are such that the corresponding exact sequence in the abelian category $\mathcal{X}$ is split exact. We call this the **trivial exact structure** for $\mathcal{Y}$.

The definition of Bass's K_1 functor rests squarely upon the following notion.

Definition 2.2.4. Let $\mathcal{Y}$ be any category. Its **loop category** $\Omega\mathcal{Y}$ is the category whose objects are pairs (Y, α), Y an object of $\mathcal{Y}$ and $\alpha \in \mathrm{Aut}_{\mathcal{Y}}(Y)$. A morphism in $\Omega\mathcal{Y}$ between two objects (Y, α) and (Y', α') is a commutative diagram in $\mathcal{Y}$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] & Y' \arrow[d, "\alpha'", "\cong"'] \\ Y \arrow[r, "f"'] & Y' \end{tikzcd}$$

Remark 2.2.5. Let $\mathcal{Y}$ be a skeletally small exact category. Its loop category $\Omega\mathcal{Y}$ is also skeletally small and it is not hard to see that $\Omega\mathcal{Y}$ inherits an exact structure such that $(Y', \alpha') \rightarrowtail (Y, \alpha) \twoheadrightarrow (Y'', \alpha'')$ is a coinflation in $\Omega\mathcal{Y}$ if and only if $Y' \rightarrowtail Y \twoheadrightarrow Y''$ is a coinflation in $\mathcal{Y}$.

Definition 2.2.6. Let $\mathcal{Y}$ be a skeletally small exact category and $\Omega\mathcal{Y}$ be its loop category, so that $\Omega\mathcal{Y}$ is also skeletally small and exact. We define **Bass's K_1-group of** $\mathcal{Y}$, denoted by $K_1^B(\mathcal{Y})$, to be the Grothendieck group of $\Omega\mathcal{Y}$ modulo the subgroup generated by the following elements

$$(Y, \alpha) + (Y, \beta) - (Y, \alpha\beta)$$

For (Y, α) in $\Omega\mathcal{Y}$ we denote its image in $K_1^B(\mathcal{Y})$ as $[Y, \alpha]$.

Remark 2.2.7. (a) ([31], 3.4) We note for $Y \in \mathcal{Y}$, we have

$$[Y, 1_Y] + [Y, 1_Y] = [Y, 1_Y 1_Y] = [Y, 1_Y]$$

Hence $[Y, 1_Y]$ is the identity element of $K_1^B(\mathcal{Y})$.

(b) Unexpectedly, K_1^B is a functor from the category of skeletally small exact categories to abelian groups. Indeed, for a morphism F (which is necessarily an exact functor) between $\mathcal{Y}$ and another skeletally small exact category, we have $K_1^B(F)([Y, \alpha]) = [F(Y), F(\alpha)]$.

Remark 2.2.8. ([58], Theorem 3.1.7)

There is an isomorphism

$$\eta_A : K_1^C(A) \xrightarrow{\cong} K_1^B(\textbf{proj}\, A)$$

The isomorphism η_A is such that $\xi \in GL_n(A)$ is mapped to the class $[A^n, \xi] \in K_1^B(\textbf{proj}\, A)$, where elements of A^n are viewed as row vectors and ξ acts by multiplication on the right.

Definition 2.2.9. Let $\mathcal{Y}$ be a skeletally small exact category. The i**th Quillen** K**-group of** $\mathcal{Y}$, denoted by $K_i^Q(\mathcal{Y})$, is defined to be the abelian group $\pi_{i+1}(BQ\mathcal{Y}, 0)$, where $Q\mathcal{Y}$ is *Quillen's Q-construction*; $BQ\mathcal{Y}$ is the classifying space of $Q\mathcal{Y}$; 0 is a fixed zero object; and π_{i+1} denotes the taking of a homotopy group.

By ([57],Section 2, Theorem 1) there is a natural isomorphism of between the Grothendieck group functor and K_0^Q (as functors on the category of skeletally small exact categories). Moreover, $K_1^Q(\textbf{proj}\, A)$ is naturally isomorphic to $K_1^C(A)$ (see ([60], Corollary 2.6 and Theorem 5.1)). Quillen's definition of higher K-theory is stunningly elegant, but does not often lend itself to performing computations with ease. The definition of Bass's functor K_1^B will be more suited for our computational needs and, we will want to exploit this in the sequel. As in ([31], 3.6), we will make strong use of the following theorem.

Theorem 2.2.10. *There exists a natural transformation* $\zeta : K_1^B \longrightarrow K_1^Q$*, which we call the Gersten-Sherman transformation, of functors on the category of skeletally small exact categories* *such that $\zeta_{\mathcal{Y}} : K_1^B(\mathcal{Y}) \longrightarrow K_1^Q(\mathcal{Y})$ is an isomorphism for every semisimple exact category $\mathcal{Y}$. In particular, $\zeta_{\textbf{\textit{proj}}\, A} : K_1^B(\textbf{\textit{proj}}\, A) \longrightarrow K_1^Q(\textbf{\textit{proj}}\, A)$ is an isomorphism for every ring A.*

The name for ζ was introduced in [31] for the following: The existence of ζ was initially sketched by Gersten in ([27], sect. 5) and the details were later filled in by Sherman ([59], sect. 4), whom also proved $\zeta_{\mathcal{Y}}$ is an isomorphism for every semisimple exact category.

2.2.2 n-Auslander-Reiten Theory

We want to discuss generalizations of Auslander-Reiten theory, following [39]. To do so, we will require some precise categorical language. Here $\mathcal{Y}$ denotes any exact category.

Definition 2.2.11. Write $\mathbf{Mod}\,\mathcal{Y}$ for the category of additive contravariant functors $\mathcal{Y} \longrightarrow \mathbf{Ab}$, with $\mathbf{Ab}$ the category of abelian groups. The morphisms in $\mathbf{Mod}\,\mathcal{Y}$ are natural transformations between functors with kernels and cokernels computed pointwise. An easy check shows that $\mathbf{Mod}\,\mathcal{Y}$ is abelian. We write $(\bullet, Y)$ to denote the additive contravariant functor $\mathrm{Hom}_{\mathcal{Y}}(\bullet, Y)$. We say $F \in \mathbf{Mod}\,\mathcal{Y}$ is **finitely presented** if there is an exact sequence

$$(\bullet, Y) \longrightarrow (\bullet, Y') \longrightarrow F \longrightarrow 0$$

in $\mathbf{Mod}\,\mathcal{Y}$. We write $\mathbf{mod}\,\mathcal{Y}$ for the subcategory of finitely presented functors.

For a ring A, let $\mathbf{Mod}\,A$ denote the category of all left A-modules and denote the subcategory of finitely presented left A-modules by $\mathbf{mod}_{\mathrm{fp}}\,A$. Fix a left A-module N and denote by E its endomorphism ring $\mathrm{End}_A(N)$. Then N has a left E-module structure that is compatible with its left A-module structure such that for $e \in E$ and $n \in N$, $e \cdot n = e(n)$. Denote by $\mathbf{add}_A\,N$ the category of A-modules that consists of all direct summands of finite direct sums of N. For $F \in \mathbf{Mod}\,(\mathbf{add}_A\,N)$, the aforementioned left E-module structure on N induces a left-E^{op}-module structure on the abelian group FN such that $e \cdot z = (Fe)(z)$ for $e \in E^{\mathrm{op}}$ and $z \in FN$. We use these facts for the following proposition, which will be essential in the proof of Theorem 2.1.3.

Proposition 2.2.12. *([31], Proposition 6.2)*

There are quasi-inverse equivalences of abelian categories

$$\textbf{\textit{Mod}}\,(\textbf{\textit{add}}_A\,N) \underset{f_N}{\overset{e_N}{\rightleftarrows}} \textbf{\textit{Mod}}\,E^{op} \quad (\simeq)$$

Where the functors e_N and f_N are defined as follows: $e_N(F) = FN$ (evaluation) and $f_N(Z) = Z \otimes_E Hom_A(\bullet, N)|_{\textbf{\textit{add}}_A N}$ (functorification). Also, these quasi-inverse equivalences restrict to equivalences between categories of finitely presented objects

$$\textbf{\textit{mod}}\,(\textbf{\textit{add}}_A\,N) \underset{f_N}{\overset{e_N}{\rightleftarrows}} \textbf{\textit{mod}}_{fp}\,E^{op} \quad (\simeq)$$

Definition 2.2.13. Let $\mathcal{X}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{X}$. We call $\mathcal{C}$ **contravariantly finite**, if for any $X \in \mathcal{X}$ there is a morphism $f : C \longrightarrow X$ with $C \in \mathcal{C}$ such that

$$(\bullet, \mathcal{C}) \xrightarrow{\bullet f} (\bullet, X) \longrightarrow 0$$

is exact (where $\bullet f$ is the map induced by f). Such an f is called a **right-$\mathcal{C}$-approximation** of X. We dually define a **covariantly finite** subcategory and a **left-$\mathcal{C}$-approximation**. A contravariantly and covariantly finite subcategory is called **functorially finite**.

At long last, we are able to define an n-cluster tilting object.

Definition 2.2.14. Let $\mathcal{Y}$ be an exact category with enough projectives. For objects X, Y in $\mathcal{Y}$ we write $X \perp_n Y$ if $\mathrm{Ext}^i_{\mathcal{Y}}(X, Y) = 0$ for $0 < i \leq n$. For an exact subcategory $\mathcal{C} \subset \mathcal{Y}$, we put

$$\mathcal{C}^{\perp_n} = \{X \in \mathcal{Y} : Y \perp_n X \text{ for all } Y \in \mathcal{C}\}$$

$$^{\perp_n}C = \{X \in \mathcal{Y} : X \perp_n Y \text{ for all } Y \in \mathcal{C}\}$$

We call $\mathcal{C}$ an n-**cluster-tilting** subcategory of $\mathcal{Y}$ if it is functorially finite and $\mathcal{C} = \mathcal{C}^{\perp_{n-1}} =^{\perp_{n-1}} C$. An object L of $\mathcal{Y}$ is called n-**cluster-tilting** if $\mathrm{add}_{\mathcal{Y}}(L)$ is an n-cluster tilting subcategory.

From the definition of n-cluster tilting, if $\mathbf{mcm}\, R$ admits an n-cluster tilting object L, then R is necessarily a direct summand of L. While the definition of n-cluster tilting is quite a bit to digest at once, there are concrete examples of n-cluster tilting objects over familiar rings and we refer the reader to Section 2.4 for several examples.

When R has finite Cohen-Macaulay type, we have the classical notion of an *Auslander-Reiten sequence* or *almost-split sequence*. When $\mathbf{mcm}\, R$ has an n-cluster tilting subcategory, we have the following generalization.

Definition 2.2.15. If $\mathcal{C} \subset \mathbf{mcm}\, R$ is an n-cluster tilting subcategory, given $X \in \mathbf{mcm}\, R$ not free and indecomposable, an exact sequence

$$0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $C_0, \ldots, C_n \in \mathcal{C}$ such that

$$0 \longrightarrow (\bullet, C_{n-1}) \xrightarrow{\bullet f_n} \cdots \xrightarrow{\bullet f_1} (\bullet, C_0) \xrightarrow{\bullet f_0} (\bullet, X) \longrightarrow 0$$

is a minimal projective resolution of $(\bullet, X)/\mathbf{rad}_{\mathbf{mcm}\, R}(\bullet, X)$ in $\mathbf{mod}\,\mathcal{C}$ is called an n-**Auslander-Reiten sequence** (or an n-**almost-split sequence**).

Here $\mathbf{rad}_{\mathbf{mcm}\, R}(\bullet, X)$ is such that

$$\mathbf{rad}_{\mathbf{mcm}\, R}(Y, X) = \{f \in \mathrm{Hom}_R(Y, X) : fg \in \mathrm{rad}(\mathrm{End}_R(Y)) \text{ for all } g \in \mathrm{Hom}_R(Y, X)\}$$

If $\mathcal{C} \subset \mathbf{mcm}\, R$ is an n-cluster tilting subcategory, then n-Auslander-Reiten sequences exist by ([40], Theorem 3.31).

2.2.3 Endomorphism Rings and K-groups

By our blanket assumptions on R, there is a unique decomposition of the n-cluster tilting object $L = L_0^{\oplus l_0} \oplus \cdots \oplus L_t^{\oplus l_t}$, such that $L_i \in \mathbf{mcm}\, R$ is indecomposable and $l_i > 0$ and the L_i are pairwise non-isomorphic. In this section, we will assume that $l_i = 1$. For if we write $L_{\text{red}} = L_0 \oplus \cdots \oplus L_t$, then $\mathbf{add}_R L = \mathbf{add}_R L_{\text{red}}$. Thus L is an n-cluster tilting object for $\mathbf{mcm}\, R$ if and only if L_{red} is. Moreover, we will see in Section 2.3, that in the context of Theorem 2.1.3, the choice of L_{red} over L is immaterial. Write $\mathcal{C} = \mathbf{add}_R L$. The following construction is from ([31], Construction 2.6).

If $L' \in \mathcal{C}$, we can write $L' = L_0^{\oplus m_0} \oplus \cdots \oplus L_t^{\oplus m_t}$ for uniquely determined $m_0, \ldots, m_t \geq 0$. Set $q = q(L') = \max\{m_0, \ldots, m_t\}$ and $v_j = v_j(L') = q - m_j$. Notice that q is the smallest integer such that L' is a direct summand of $L^{\oplus q}$. Now form the R-module $L'' = L_0^{\oplus v_0} \oplus \cdots \oplus L_t^{\oplus v_t}$ and let $\psi : L' \oplus L'' \longrightarrow L^{\oplus q}$ be the R-linear isomorphism that takes the element

$$((\underline{x}_0, \ldots, \underline{x}_t), (\underline{y}_0, \ldots, \underline{y}_t) \in L' \oplus L'' = (L_0^{\oplus m_0} \cdots \oplus L_t^{\oplus m_t}) \oplus (L_0^{\oplus v_0} \oplus \cdots \oplus L_t^{\oplus v_t})$$

where $\underline{x}_j \in L_j^{\oplus m_j}$ and $\underline{y}_j \in L_j^{\oplus v_j}$, to the element

$$((z_{01}, \ldots, z_{t1}), \ldots, (z_{0q}, \ldots, z_{tq})) \in L^{\oplus q} = (L_0 \oplus \cdots \oplus L_t)^{\oplus q}$$

with $z_{j1}, \ldots, z_{jq} \in L_j$ given by

$$(z_{j1}, \ldots, z_{jq}) = (\underline{x}_j, \underline{y}_j) \in L_j^{\oplus q} = L_j^{\oplus(m_j + v_j)}$$

Now for $\alpha \in \mathrm{Aut}_R(L')$, we define $\widetilde{\alpha}$ to be the automorphism on $L^{\oplus q}$ given by $\psi(\alpha \oplus 1_{L''})\psi^{-1}$. Note that $\widetilde{\alpha} = (\widetilde{\alpha_{ij}})$, with $\widetilde{\alpha_{ij}}$ uniquely determined endomorphisms of L. In particular, $\widetilde{\alpha} \in \mathbb{M}_q(\mathrm{End}_R L)$. As in [31], we refer to this construction as the **tilde construction**.

Remark 2.2.16. We note a special case of the tilde construction. Keep notation as above. Suppose $\alpha = a1_{L'}$ with $a \in R^*$. If $L' = L_{i_1}^{\oplus q} \oplus \cdots \oplus L_{i_h}^{\oplus q}$ with $0 \leq i_1 < i_2 < \cdots < i_h \leq t$. Then $\widetilde{\alpha} : L^{\oplus q} \longrightarrow L^{\oplus q}$ is the automorphism given by $e1_{L^{\oplus q}}$ with $e \in \mathrm{Aut}_R(L)$ given by

$$\mathrm{diag}(1_{L_0}, \ldots, a1_{L_{i_1}}, \ldots, a1_{L_{i_h}}, \ldots, 1_{L_t})$$

Hence, $(\widetilde{a1_{L'}})^{-1} = \widetilde{a^{-1}1_{L'}}$.

As we will often be working explicitly with highly noncommutative rings, we need to discuss important ideas at the intersection of noncommutative algebra and K-theory. Let $J(A)$ be the Jacobson radical of the not necessarily commutative ring A. Recall that A is said to be *semilocal* if $A/J(A)$ is semisimple. That is, every left $A/J(A)$-module has the property that each of its submodules is a direct summand of $A/J(A)$. In the case that A is commutative, this is equivalent to A having only finitely many maximal ideals ([45], Proposition 20.2). Of great importance to us is the following situation: If A is a commutative semilocal Noetherian ring and N is a nonzero finitely generated A-module, then $\text{End}_A(N)$ is semilocal in the preceding sense ([31], Lemma 5.1). We will see how the following remark utilizes this small but essential fact in the proof of Theorem 2.1.3.

Remark 2.2.17. ([31], Paragraph 5.2)

For arbitrary A, denote the composition of the following group homomorphisms

$$A^* = GL_1(A) \hookrightarrow GL(A) \twoheadrightarrow GL(A)_{\text{ab}} = K_1^C(A)$$

by ϑ_A. Since $K_1^C(A)$ is abelian, there is an induced map $\theta_A : A^*_{\text{ab}} \longrightarrow K_1^C(A)$. If A is semilocal, then ([3], V§9 Theorem 9.1) shows that ϑ_A is surjective, hence so is θ_A. When A contains a field k with $\text{char}(k) \neq 2$, a result of Vaserstein ([66], Theorem 2) shows that θ_A is an isomorphism. In particular, if R is a k-algebra, $\text{char}(k) \neq 2$ and M is a finitely generated R-module with $E = \text{End}_R(M)$, then θ_E and $\theta_{E^{\text{op}}}$ are isomorphisms.

Suppose now A is a commutative semilocal ring, so that the commutator subgroup $[A^*, A^*]$, is trivial, hence $\theta_A : A^* \longrightarrow K_1^C(A)$ is surjective. In ([31], Remark. 5.4), if θ_A is an isomorphism, an explicit inverse to θ_A is constructed: The determinant homomorphisms $\det_n : GL_n(A) \longrightarrow A^*$ induce a homomorphism $\det_A : K_1^C(A) \longrightarrow A^*$ (since each $\det_n$ is trivial on commutators in $GL(A)$) which satisfies $\det_A \theta_A = 1_{A^*}$, so that $\theta_A^{-1} = \det_A$.

Using Remark 2.2.17 as motivation, the following definition is made in [31].

Definition 2.2.18. Let A be a ring for which the map $\theta_A : A^*_{\text{ab}} \longrightarrow K_1^C(A)$ is an isomorphism. The inverse θ_A^{-1} is denoted by $\det_A$ and is is called the **generalized determinant**.

The following proposition makes use of the tilde construction and will be useful in proving Theorem 2.1.3. We note it is essentially proven in [31], where it is a synthesis of ([31], Lemma 6.5) and the proof of ([31], Proposition 8.8). We also note that the assumptions in ([31], Proposition 8.8) are that R has finite Cohen-Macaulay type. However, we note that under our assumptions, the portion of the proof we are referencing ([31], equation (8.8.1)) still holds.

Proposition 2.2.19. *Keeping our general assumptions, suppose in addition that R is an algebra over its residue field k and the characteristic of k is not two. Let $L_0, \ldots, L_t \in$ **mcm** R and L be their direct sum. Set $\Lambda = \mathit{End}_R(L)^{op}$. Let $\mathcal{C}_0 = \textbf{\textit{add}}_R(L)$ be equipped with the trivial exact structure. If Λ has finite global dimension, then there is an isomorphism of groups*

$$\tau : K_1^B(\mathcal{C}_0) \longrightarrow \mathit{Aut}_R(L)_{ab}$$

such that for any $L' \in \mathcal{C}_0$ and any $\alpha \in \mathit{Aut}_R(L')$, $\tau([L', \alpha]) = \det_{\Lambda^{op}}(\widetilde{\alpha})$.

Remark 2.2.20. ([31], Observation 8.9)

Let A be any commutative Noetherian local ring and η_A be the isomorphism from Remark 2.2.8 and $\theta_A : A^* \longrightarrow K_1^C(A)$ be the induced map from Remark 2.2.17. Then θ_A is an isomorphism by ([60], Example 1.6). Thus the composition $\rho_A = \eta_A \theta_A : A^* \longrightarrow K_1^B(\textbf{proj}\, A)$ is an isomorphism such that $a \in A^*$ is mapped to $[A, a1_A]$.

We now combine the the above preliminaries with the tilde construction to define the subgroup Ξ of $\mathrm{Aut}_R(L)_{\text{ab}}$ in Theorem 2.1.3.

Definition 2.2.21. Recall that we are assuming that $\mathbf{mcm}\,R$ has n-cluster-tilting object of the form $L = L_0 \oplus \cdots \oplus L_t$. We assume that $L_0 = R$ and that for $j > 0$, the L_j are non-free pairwise non-isomorphic and indecompsable objects in $\mathbf{mcm}\,R$. Suppose also that R is a k-algebra, char$(k) \neq 2$ and k is algebraically closed. If $\mathbf{mcm}\,R$ has an n-cluster tilting object L such that $\Lambda := \mathrm{End}_R(L)^{\mathrm{op}}$ has finite global dimension, we define a subgroup Ξ of $\mathrm{Aut}_R(L)_{\mathrm{ab}}$ as follows: For $j > 0$, let

$$0 \longrightarrow C_n^j \longrightarrow \cdots \longrightarrow C_0^j \longrightarrow L_j \longrightarrow 0$$

be the n-Auslander-Reiten sequence ending in L_j (see Definition 2.2.15). By Remark 2.2.17, $\theta_{\Lambda^{\mathrm{op}}} : \mathrm{Aut}_R(L)_{\mathrm{ab}} \longrightarrow K_1^C(\Lambda^{\mathrm{op}})$ is an isomorphism with inverse given by $\det_{\Lambda^{\mathrm{op}}}$. Then Ξ is the subgroup generated by the elements given by

$$\widetilde{a1_{L_j}} \prod_{i=1}^{n+1} \det{}_{\Lambda^{\mathrm{op}}}(\widetilde{a1_{C^j_{i-1}}})^{(-1)^i}$$

where a runs over all elements of k^* and $j = 1, \ldots, t$.

2.3 The Structure of $G_1(R)$

In this section, unadorned K-groups are the Quillen K-groups. Our goal of this section is to prove Theorem 2.1.3. We always assume that $\mathbf{mcm}\,R$ has an n-cluster tilting object $L = L_0^{l_0} \oplus \cdots \oplus L_t^{\oplus l_t}$, with $L_0 = R$, and $L_1, \ldots, L_t$ non-free, non-isomorphic indecomposable maximal Cohen-Macaulay R-modules such that $\Lambda := \mathrm{End}_R(L)^{\mathrm{op}}$ has finite global dimension. In addition to our blanket assumptions, we assume that k is algebraically closed of characteristic not two and R is a k-algebra. We begin with an easy reduction.

Lemma 2.3.1. *Set $L_{red} = L_0 \oplus \cdots \oplus L_t$. If $\Lambda_{red} = End_R(L_{red})^{op}$, then Λ and Λ_{red} are Morita-equivalent. In particular, $G_i(\Lambda) \cong G_i(\Lambda_{red})$ for all $i \geq 0$.*

Proof. The desired Morita equivalence is from ([17], Lemma 2.2). Thus the categories of left Λ and Λ_{red} modules are equivalent, hence there is an equivalence of exact categories between $\mathbf{mod}\,\Lambda$

and $\textbf{mod}\,\Lambda_{\text{red}}$. It is well-known this yields an isomorphism in G-theory, hence $G_i(\Lambda) \cong G_i(\Lambda_{\text{red}})$ for all $i \geq 0$.

$\square$

It is easy to see $\textbf{add}_R\,L = \textbf{add}_R\,L_{\text{red}}$. Moreover, since Λ has finite global dimension, the Morita equivalence of Lemma 2.3.1 gives that Λ_{red} also has has finite global dimension. Since Λ has finite global dimension and is a semilocal algebra over a field of characteristic not two, by Quillen's Resolution Theorem ([57], §Theorem 3), ([60], Corollary 2.6 and Theorem 5.1), and ([66], Theorem 2) we have isomorphisms

$$G_1(\Lambda) \cong K_1(\Lambda) \cong K_1^C(\Lambda) = \Lambda^*_{\text{ab}} = \text{Aut}_R(L)_{\text{ab}}$$

As noted above, Λ_{red} has finite global dimension, hence the same arguments apply, so that the above remarks and Lemma 2.3.1 give

$$\text{Aut}_R(L_{\text{red}})_{\text{ab}} = (\Lambda_{\text{red}})^*_{\text{ab}} \cong G_1(\Lambda_{\text{red}}) \cong G_1(\Lambda) \cong \Lambda^*_{\text{ab}} = \text{Aut}_R(L)_{\text{ab}}$$

Thus may safely assume that the n-cluster tilting object L for $\textbf{mcm}\,R$ has the form $L_0 \oplus \cdots \oplus L_t$, where the L_i are non-isomorphic indecomposable maximal Cohen-Macaulay. Henceforth, we always use Λ to denote $\text{End}_R(L)^{\text{op}}$ with $L = L_0 \oplus \cdots \oplus L_t$, $L_0 = R$ and for $j > 0$, the L_j are non-free, non-isomorphic indecomposable objects in $\textbf{mcm}\,R$.

Since k is algebraically closed, $\kappa_{L_j} = \text{End}_R(L_j)^{\text{op}}/\text{rad}(\text{End}_R(L_j)^{\text{op}}) = k$ for all j (this is essentially Nakayamma's lemma). By Theorem 2.1.2, there is an exact sequence of abelian groups

$$G_1(k)^{\oplus t} \xrightarrow{\gamma} G_1(\Lambda) \longrightarrow G_1(R) \longrightarrow G_0(k)^{\oplus t} \longrightarrow G_0(\Lambda) \longrightarrow G_0(R) \longrightarrow 0$$

By Theorem 2.1.2, $G_0(\Lambda) = \mathbb{Z}^{\oplus(t+1)}$. Moreover, is well-known that $G_0(k) = \mathbb{Z}$. In particular, the above exact sequence becomes

$$G_1(k)^{\oplus t} \xrightarrow{\gamma} G_1(\Lambda) \longrightarrow G_1(R) \longrightarrow \mathcal{H} \longrightarrow 0 \tag{$\star$}$$

where $\mathcal{H}$ is the kernel of a map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$. Now $\mathcal{H}$ is free, being the subgroup of a free group, hence the exactness of $(\star)$ gives an isomorphism

$$G_1(R) \cong \text{coker}(\gamma) \oplus \mathcal{H}$$

Thus to prove Theorem 2.1.3, that is, in order to calculate Ξ, we need to explicitly describe the map γ. In this direction, we first define $\mathcal{C}_0$ to be the category $\mathcal{C} := \textbf{add}_R L = \textbf{add}_R (L_0, \ldots, L_t)$ equipped with trivial exact structure. As we are assuming Λ has finite global dimension, ([31], Lemma 6.5) gives an isomorphism $K_1(\mathcal{C}_0) \cong K_1(\textbf{mod}\,\mathcal{C})$ that is induced by the exact Yoneda functor $y_L : \mathcal{C}_0 \longrightarrow \textbf{mod}\,\mathcal{C}$, where $y_L(X) = \text{Hom}_R(\bullet, X)|_{\mathcal{C}}$. Since Λ is left Noetherian, Proposition 2.2.12 gives that the evaluation functor $e_L : \textbf{mod}\,\mathcal{C} \longrightarrow \textbf{mod}\,\Lambda$ is an equivalence, hence induces an isomorphism $K_1(\textbf{mod}\,\mathcal{C}) \cong K_1(\Lambda)$. Moreover, Λ has finite global dimension, so that Quillen's Resolution Theorem ([57], §Theorem 3) yields that the inclusion functor $\textbf{proj}\,\Lambda \longrightarrow \textbf{mod}\,\Lambda$ induces an isomorphism $K_1(\Lambda) \cong G_1(\Lambda)$. Hence there is a map $\alpha : G_1(k)^{\oplus t} \longrightarrow K_1(\mathcal{C}_0)$ such that the diagram

$$\begin{tikzcd}
 & K_1(\mathcal{C}_0) \arrow[d, "\cong"] \\
G_1(k)^{\oplus t} \arrow[ur, "\alpha"] \arrow[r, "\gamma"'] & G_1(\Lambda)
\end{tikzcd}$$

commutes. This gives $\text{coker}(\gamma) \cong \text{coker}(\alpha)$. Thus to prove Theorem 2.1.3, it suffices to compute $\text{coker}(\alpha)$. In fact, α is computed in the discussion of ([53], Section 7.2). The details will be useful and we recall them. Now $L = L_0 \oplus \cdots \oplus L_t$, with $L_0 = R$ and $L_1, \ldots, L_t$ are the non-free indecomposable and non-isomorphic summands of L. We set $I = \{L_0, \ldots, L_t\}$ and $I_0 = I \setminus \{R\}$. For $j > 0$ let

$$0 \longrightarrow C_n^j \longrightarrow \cdots \longrightarrow C_0^j \longrightarrow L_j \longrightarrow 0$$

be the n-Auslander-Reiten sequence ending in L_j (see Definition 2.2.15). Denote by k_j the object of $\oplus_{I_0}\mathbf{mod}\, k$ which is k in the L_j-coordinate and 0 in the others. We remark that to define a k-linear functor out of $\oplus_{I_0}\mathbf{mod}\, k$, one needs only to specify the image of each object k_j. We define k-linear functors

$$a_i : \bigoplus_{I_0} \mathbf{mod}\, k \longrightarrow \mathcal{C}_0 \qquad (0 \leq i \leq n+1)$$

by

$$\begin{cases} a_i(k_j) = C^j_{i-1} & (1 \leq i \leq n+1) \\ a_0(k_j) = L_j \end{cases}$$

It is shown in ([53], Section 7.2) that $\alpha = \sum_{i=0}^{n+1}(-1)^i K_1(a_i)$. We have the following.

Proposition 2.3.2. *If Ξ is the subgroup of Λ^*_{ab} from Definition 2.2.21, there is an isomorphism coker$(\alpha) \cong \Lambda^*_{ab}/\Xi$.*

Now Proposition 2.3.2 implies Theorem 2.1.3, so the proof of Proposition 2.3.2 will conclude this section.

Proof. Since the morphisms $a_i : \bigoplus_{I_0} \mathbf{mod}\, k \longrightarrow \mathcal{C}_0$ are functors on exact categories, they also define maps $K_1^B(a_i) : K_1^B(\bigoplus_{I_0} \mathbf{mod}\, k) \longrightarrow K_1^B(\mathcal{C}_0)$ on the Bass K_1-groups. Now $|I_0| = t$, so that $K_1^B(\bigoplus_{I_0} \mathbf{mod}\, k) = \bigoplus_{I_0} K_1^B(\mathbf{mod}\, k) = K_1^B(\mathbf{mod}\, k)^{\oplus t}$. Let $\beta : K_1^B(\mathbf{mod}\, k)^{\oplus t} \longrightarrow K_1^B(\mathcal{C}_0)$ be the map given by $\sum_{i=0}^{n+1}(-1)^i K_1^B(a_i)$. Our first task is to show that coker$(\alpha) \cong$ coker(β). The Gersten-Sherman transformation (see Theorem 2.2.10) $\zeta : K_1^B \longrightarrow K_1$ provides the following commutative diagram for $i = 0, 1, \ldots, n+1$

$$\begin{tikzcd} K_1(\mathbf{mod}\, k)^{\oplus t} \arrow[r, "K_1(a_i)"] \arrow[d, "\zeta^{\oplus t}_{\mathbf{mod}\, k}"', "\cong"] & K_1(\mathcal{C}_0) \arrow[d, "\zeta_{\mathcal{C}_0}", "\cong"'] \\ K_1^B(\mathbf{mod}\, k)^{\oplus t} \arrow[r, "K_1^B(a_i)"'] & K_1^B(\mathcal{C}_0) \end{tikzcd}$$

Where the vertical isomorphisms come courtesy of Theorem 2.2.10, as $\mathcal{C}_0$ and $\mathbf{mod}\, k$ are semisimple exact categories. Hence there is a commutative diagram

$$\begin{tikzcd}
K_1(\mathbf{mod}\,k)^{\oplus t} \arrow[r, "\alpha"] \arrow[d, "\zeta^{\oplus t}_{\mathbf{mod}\,k}"', "\cong"] & K_1(\mathcal{C}_0) \arrow[d, "\cong"', "\zeta_{\mathcal{C}_0}"] \\
K_1^B(\mathbf{mod}\,k)^{\oplus t} \arrow[r, "\beta"'] & K_1^B(\mathcal{C}_0)
\end{tikzcd}$$

This gives that $\mathrm{coker}(\alpha) \cong \mathrm{coker}(\beta)$. To finish the proof, first note that Remark 2.2.20 furnishes an isomorphism $\rho_k : k^* \longrightarrow K_1^B(\mathbf{mod}\,k)$ such that $a \mapsto [k, a1_k]$, hence there is an isomorphism $\rho_k^{\oplus t} : (k^*)^{\oplus t} \longrightarrow K_1^B(\mathbf{mod}\,k)^{\oplus t}$. Now recall the isomorphism $\tau : K_1^B(\mathcal{C}_0) \longrightarrow \Lambda^*_{\text{ab}}$ (noting $\Lambda^*_{\text{ab}} = \mathrm{Aut}_R(L)_{\text{ab}}$) of Proposition 2.2.19. The map τ is such that for $L' \in \mathcal{C}_0$ and any $f \in \mathrm{Aut}_R(L')$, $\tau([L', f]) = \det_{\Lambda^{\text{op}}}(\widetilde{f})$, where $\det_{\Lambda^{\text{op}}}$ is the generalized determinant of Definition 2.2.18 and $\widetilde{f} \in \mathrm{Aut}_R(L)$ is the map obtained from the tilde construction of Subsection 2.2.3. In particular, $\mathrm{coker}(\beta) \cong \mathrm{coker}(\tau\beta\rho_k^{\oplus t})$, hence we calculate the latter. Restricting to the jth coordinate of $(k^*)^{\oplus t}$, by slight abuse of notation, we have for $a \in k^*$

$$\beta\rho_k(a) = \beta([k, a1_k]) = [L_j, a1_{L_j}] + \sum_{i=1}^{n+1} (-1)^i [C^j_{i-1}, a1_{C^j_{i-1}}]$$

By definition, $\det_{\Lambda^{\text{op}}}(\widetilde{a1_{L_j}}) = \widetilde{a1_{L_j}}$, so that

$$\tau\beta\rho_k(a) = \tau\left([L_j, a1_{L_j}] + \sum_{i=1}^{n+1} (-1)^i [C^j_{i-1}, a1_{C^j_{i-1}}]\right) = \widetilde{a1_{L_j}} \prod_{i=1}^{n+1} \mathrm{det}_{\Lambda^{\text{op}}}(\widetilde{a1_{C^j_{i-1}}})^{(-1)^i}$$

This is precisely the subgroup Ξ of Definition 2.2.21, whence the result.

$\square$

2.4 Existence of n-Cluster Tilting Objects in $\mathbf{mcm}\,R$

Naturally, the usefulness of Theorem 2.1.3 would be limited if the situations in which $\mathbf{mcm}\,R$ contained an n-cluster tilting object were sparse. Fortunately for us, they are not. Moreover, if $\mathbf{mcm}\,R$ admits an n-cluster tilting object L, we require that $\Lambda := \mathrm{End}_R(L)^{\text{op}}$ has finite global dimension. At first glance, this condition might also seem limiting, but is in fact quite common, as seen in the following theorem.

Theorem 2.4.1. *([39], Theorem 3.12(a))*

Suppose $\dim R = d$ *and that* ***mcm*** R *contains an* n*-cluster tilting object* L *with* $d \leq n$*. Then* Λ *has global dimension at most* $n + 1$.

The most well-studied situation in which **mcm** R admits an n-cluster tilting object is the following.

2.4.1 Finite Cohen-Macaulay Type

Recall that we say that R has finite Cohen-Macaulay type (or *finite type* for short) when R has only finitely many indecomposable maximal Cohen-Macaulay modules. Now the only 1-cluster tilting subcategory of **mcm** R is **mcm** R itself. Thus the existence of a 1-cluster tilting object for **mcm** R is equivalent to R having finite type. In particular, when R has finite type, **mcm** R has an additive generator M. For practical and computational purposes, when R has finite type, we will often work with the R-module $M = M_0 \oplus M_1 \oplus \cdots \oplus M_t$, with $M_0 = R$ and $M_1, \ldots, M_t$ the pairwise non-isomorphic and non-free indecomposable maximal Cohen-Macaulay R-modules. Moreover, ([46], Theorem 6) shows that $\mathrm{End}_R(M)^{\mathrm{op}}$ has finite global dimension, hence Theorem 2.1.3 is applicable in this situation. In fact, in this case, if the Auslander-Reiten homomorphism $\Upsilon : \mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$ is injective, Theorem 2.1.3 is just ([31], Theorem 2.12), the result which inspired Theorem 2.1.3.

ADE Singularities

The most important examples of rings that have finite type are the simple surface singularities. These are called the ADE singularities. Let $S = k[[x, y, z_2, z_3, \ldots, z_d]]$ and assume k is algebraically closed with characteristic different from $2, 3$ and 5. Set $R = S/fS$ with f nonzero and $f \notin (x, y, z_2, \ldots, z_d)^2$. The f for which R has finite type are exactly the following ([47], Theorem 9.8)

$$
\begin{array}{lll}
(A_n) & x^2 + y^{n+1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 1) \\
(D_n) & x^2 y + y^{n-1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 4) \\
(E_6) & x^3 + y^4 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\
(E_7) & x^3 + xy^3 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\
(E_8) & x^3 + y^5 + z_2^2 + z_3^2 + \cdots + z_d^2 &
\end{array}
$$

2.4.2 Invariant Subrings

Let k be a field and S the ring $k[[x_1, \ldots, x_n]]$. Suppose G is a finite subgroup of $GL_n(k)$ that does not contain any nontrivial pseudo-reflections and with $|G|$ invertible in k. Let R be the invariant subring $k[[x_1, \ldots, x_n]]^G$ of S, where G acts by a linear change of variables on S. If R is an isolated singularity, then the R-module S is an $(n-1)$-cluster tilting object (see ([40], 2.5)).

The *skew group ring* of S, denoted by $S\#G$, is given by $S\#G = \bigoplus_{\sigma \in G} S \cdot \sigma$, with multiplication defined by $(s \cdot \sigma)(t \cdot \tau) = s\sigma(t) \cdot \sigma\tau$. In this situation, $S\#G$ has global dimension equal to n ([47], Corollary 5.8) and there is an isomorphism $\mathrm{End}_R(S) \cong S\#G$ ([47], Theorem 5.15). In particular, Theorem 2.1.3 is applicable in this situation.

2.4.3 Reduced Hypersurface Singularities

Dimension One

Let k be an algebraically closed field of characteristic not two and $S = k[[x, y]]$. For $f \in (x, y)$, let $R = S/fS$ be a reduced hypersurface singularity. Suppose f has prime factorization and $f = f_1 \cdots f_n$, $S_i = S/(f_1 \cdots f_i)S$ and L is the R-module $S_1 \oplus \cdots \oplus S_n$. If $f_i \notin (x, y)^2$ for all i, then [13] shows that L is a 2-cluster tilting object for $\mathbf{mcm}\, R$. Moreover, Theorem 2.4.1 shows that $\mathrm{End}_R(L)^{\mathrm{op}}$ has global dimension at most three. Hence we can apply Theorem 2.1.3 in this situation. Note, in particular, if $\lambda_1, \ldots, \lambda_n$ are distinct elements of k, then Theorem 2.1.3 is applicable to the ring S/fS with $f = (x - \lambda_1 y) \cdots (x - \lambda_n y)$.

Dimension Three

Keep notation as above, but set $S' = k[[x,y,u,v]]$ and $R' = S'/(f+uv)S'$. Then **mcm** R' has a 2-cluster tilting object if $f_i \notin (x,y)^2$ for all i and it is given by $L := U_1 \oplus \cdots \oplus U_n$ with $U_i = (u, f_1 \cdots f_i) \subset R'$ ([39], Theorem 4.17). Moreover, ([39], Theorem 4.17) also says $\mathrm{End}_{R'}(L)^{\mathrm{op}}$ has finite global dimension, so Theorem 2.1.3 is applicable in this situation.

2.5 Abelianization of Automorphism Groups

Of course, the usefulness of Theorem 2.1.3 would be limited if one were unable to compute $\mathrm{Aut}_R(L)_{\mathrm{ab}}$. We make several computations, though each computation is tailored specifically to each ring and it seems difficult to find results that hold generally. Our computations rely significantly upon the general framework laid out by [31] and this work serves strongly as inspiration for our results. The purpose of this section is to prove the following.

Proposition 2.5.1. *Let k be an algebraically closed field of characteristic not equal to two. Then*

(a) if $R = k[x]/x^n k[x]$ and $M = R \oplus xR \oplus \cdots \oplus x^{n-1}R$, then $Aut_R(M)_{ab} \cong (k^*)^{\oplus n}$.

(b) if k also has characteristic not equal to 3 or 5, $R = k[[t^2, t^{2n+1}]]$, $n \geq 0$ and $M = R \oplus R_1 \oplus \cdots \oplus R_n$, with $R_i = k[[t^2, t^{2(n-i)+1}]]$, then $Aut_R(M)_{ab} \cong (k^*)^{\oplus n} \oplus k[[t]]^*$.

(c) if $R = k[[s^2, st, t^2]]$, then $Aut_R(R \oplus (s^2, st)R)_{ab} \cong k^* \oplus R^*$.

(d) if $S = k[[x,y]]$, $f_1, \ldots, f_n \in (x,y)$ are irreducible such that

(i) $f = f_1 \cdots f_n$, $R := S/fS$, is an isolated singularity (i.e. $(f_i) \neq (f_j)$),

(ii) $f_i \notin (x,y)^2$ for all i,

(iii) $(f_i, f_{i+1}) = (x,y)$,

$S_i = S/(f_1 \cdots f_i)S$, and $L = S_1 \oplus \cdots \oplus S_n$, then

$$Aut_R(L)_{ab} \cong (S/f_1S)^* \oplus \cdots \oplus (S/f_nS)^* = \overline{R}^*$$

Where $\overline{R} = S/f_1S \oplus \cdots \oplus S/f_nS$ is the integral closure of R in its total quotient ring.

(e) if k has characteristic zero, $S' = k[[x, y, u, v]]$, $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n$ with $f_i \in k[[x, y]]$ satisfying the conditions in (d), $U_i = (u, f_1 \cdots f_i)$, and $L = U_1 \oplus \cdots \oplus U_n$, then

$$Aut_{R'}(L)_{ab} \cong R'^{*} \oplus k[[w, z]]^{*\oplus(n-1)}$$

where w, z are variables over k.

Of course, the purpose of Proposition 2.5.1 is to combine it with Theorem 2.1.3 calculate explicit examples of $G_1(R)$ for several hypersurface singularities. This will be done in Section 2.6.

We set up some useful notation. Let $N_1, \ldots, N_s$ be A-modules and consider the A-module $N := N_1 \oplus \cdots \oplus N_s$. We view the elements of N as column vectors and the endomorphism ring of N has a matrix-like structure: For $f \in \mathrm{End}_A(N)$, we can write $f = (f_{ij})$ with $f_{ij} \in \mathrm{Hom}_A(N_j, N_i)$ and composition with another endomorphism $g = (g_{ij})$ can be accomplished in the same manner one would multiply matrices with entries in A. We write a $\mathrm{diag}(\alpha_1, \ldots, \alpha_s)$ for the diagonal endomorphism of N with $\alpha_i \in \mathrm{End}_A(N_i)$. For $\alpha \in \mathrm{Aut}_A(N_j)$, we denote by $d_j(\alpha)$ the automorphism of N given by $\mathrm{diag}(1_{N_1}, \ldots, 1_{N_{j-1}}, \alpha, 1_{N_{j+1}}, \ldots, 1_{N_s})$. For $i \neq j$ and $\beta \in \mathrm{Hom}_A(N_j, N_i)$, we denote by $e_{ij}(\beta)$ the automorphism of N with diagonal entries $1_{N_1}, \ldots, 1_{N_s}$ and (i, j)th entry given by β and zeros elsewhere. Before we begin, we discuss calculations that will be used often in the sequel.

Lemma 2.5.2. *([31], Lemma 9.2)*

Let A be a ring in which 2 is invertible, $N_1, \ldots, N_s$ be A-modules and $N := N_1 \oplus \cdots \oplus N_s$. If $i \neq j$ and $\alpha \in Hom_A(N_j, N_i)$, then $e_{ij}(\alpha)$ is a commutator in $Aut_A(N)$.

Proof. Given β, γ in $\mathrm{Aut}_A(N)$, the commutator of β and γ is $[\beta, \gamma] = \beta\gamma\beta^{-1}\gamma^{-1}$. It is not hard to see that $e_{ij}(\alpha) = [e_{ij}(\frac{\alpha}{2}), d_j(-1_{N_j})]$. □

Lemma 2.5.3. *Let $(A, \mathfrak{n})$ be commutative and local such that 2 is invertible in A. Let $N_1, \ldots, N_s$ be A-modules and set $N = N_1 \oplus \cdots \oplus N_s$. Let $a \in 1 + \mathfrak{n}$, and consider the automorphism $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$ of N. Suppose either*

(a) $N_i \supseteq N_{i+1}$ and $\mathfrak{n}N_i \subseteq N_{i+1}$ or

(b) $N_i \subseteq N_{i+1}$ and $(1-a)N_{i+1} \subseteq N_i$

then $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$ is in the commutator subgroup of Aut$_A(N)$.

Proof. In the case of (a), Let $\iota_i : N_{i+1} \longrightarrow N_i$ be inclusion. Now note that $a^{-1} \in 1 + \mathfrak{n}$, so that we have the following decomposition of $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$:

$$e_{i+1,i}((a^{-1}-1)1_{N_i})e_{i,i+1}(\iota_i)e_{i+1,i}((a-1)1_{N_i})e_{i,i+1}(-a^{-1}\iota_i)$$

We apply Lemma 2.5.2 to see that $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$ is in the commutator subgroup of $\mathrm{Aut}_A(N)$.

In the case of (b), notice that our hypothesis implies $(a^{-1}-1)N_{i+1} \subseteq N_i$. Let $\iota_i : N_i \longrightarrow N_{i+1}$ be the inclusion map. We have the following decomposition of $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$:

$$e_{i,i+1}((a^{-1}-1)1_{N_{i+1}})e_{i+1,i}(\iota_i)e_{i,i+1}((a-1)1_{N_{i+1}})e_{i+1,i}(-a^{-1}\iota_i)$$

and once again, we apply Lemma 2.5.2 to see that $d_i(a1_{N_i})d_{i+1}(a^{-1}1_{N_{i+1}})$ is in the commutator subgroup of $\mathrm{Aut}_A(N)$. □

2.5.1 Truncated Polynomial Rings in One Variable

Our aim here is to prove (a) of Proposition 2.5.1. That is k is algebraically closed and has characteristic not two, $R = k[x]/x^n k[x]$, $\mathfrak{m}$ is its maximal ideal xR, then with $M = R \oplus \mathfrak{m} \oplus \cdots \oplus \mathfrak{m}^{n-1}$, we have $\mathrm{Aut}_R(M)_{\mathrm{ab}} \cong (k^*)^{\oplus n}$.

Proof. Denote by R_j the ring $k[x]/x^j k[x]$ for $1 \leq j \leq n$. Note that $R_{j-1} \subset R_j$ and $R = R_n$. Let $\mathfrak{m}$ denote the maximal ideal xR of R. Then $\text{End}_R(\mathfrak{m}^i)$ is isomorphic to the local ring R_{n-i}. Let M be the R-module $R \oplus \mathfrak{m} \oplus \cdots \oplus \mathfrak{m}^{n-1}$. We set $E = \text{End}_R(M)$ and seek to show $E^*_{\text{ab}} \cong (k^*)^{\oplus n}$.

For $n = 1$, this is clear. For $n = 2$, we have $\text{End}_R(\mathfrak{m}) = k$, so that $E^*_{\text{ab}} \cong (k^*)^{\oplus 2}$ by ([31], Proposition 9.6).

Suppose now $n \geq 3$. We first show that there is a surjection $E^*_{\text{ab}} \longrightarrow (k^*)^{\oplus n}$ such that the kernel consists of diagonal matrices $\alpha = (\alpha_{ii})$ with $\alpha_{ii} \in \text{Aut}_R(\mathfrak{m}^{i-1}) = R^*_{n-i+1}$. By ([31], Proposition 9.4), $(\alpha_{ij}) \in E$ is invertible if and only if α_{ii} is invertible for all i. In particular, this gives that every two-sided maximal ideal of E is of the form $\mathfrak{n}_i := \{(\alpha_{ij}) : \alpha_{ii} \in J(\text{End}_R(\mathfrak{m}^{i-1}))\}$. Hence the Chinese Remainder Theorem gives $E/J(E) \cong E/\mathfrak{n}_1 \times \cdots \times E/\mathfrak{n}_n = k \times \cdots \times k$. In particular, there is an induced surjection $\varphi : E^*_{\text{ab}} \twoheadrightarrow (k^*)^{\oplus n}$. We appeal to ([31], Corollary 9.5) to see every element of E^*_{ab} can be represented by a diagonal automorphism. Moreover, it is clear elements in the kernel φ are given by (α_{ii}) such that α_{ii} is multiplication by an element in $1 + J(\text{End}_R(\mathfrak{m}^{i-1})) = 1 + xR_{n-i+1}$ for all i.

We now demonstrate the injectivity of φ. Let $\alpha \in E^*_{\text{ab}}$ such that $\varphi(\alpha)$ is trivial. By the above, we can write $\alpha = (\alpha_{ii})$ such that α_{ii} is multiplication by an element of $1 + xR^*_{n-i+1}$. Now every endomorphism on $\mathfrak{m}^{n-1}$ is given by an element of $1 + xR_1 = \{1\}$, so that we can write $\alpha = d_1(\alpha_{11}) \cdots d_{n-1}(\alpha_{n-1,n-1})$. It suffices to show each $d_i(\alpha_{ii})$ is in the commutator subgroup of E^*. We do this below.

We show by decreasing induction on i that $d_i(\beta)$ can be written as a product of commutators, where β is given by multiplication by an element of $1+xR_{n-i+1}$. For $i = n-1$, write $\beta = r1_{\mathfrak{m}^{n-2}}$, where $r \in 1 + xR_2$. Notice that $r^{-1} \in 1 + xR_2$ as well, hence multiplication by r^{-1} restricts to the identity on $\mathfrak{m}^{n-1}$. This gives

$$d_{n-1}(\beta) = d_{n-1}(r1_{\mathfrak{m}^{n-2}}) = d_{n-1}(r1_{\mathfrak{m}^{n-2}})d_n(r^{-1}1_{\mathfrak{m}^{n-1}})$$

By Lemma 2.5.3, $d_{n-1}(r1_{\mathfrak{m}^{n-2}})d_n(r^{-1}1_{\mathfrak{m}^{n-1}})$ is in the commutator subgroup of E^*, hence so is $d_{n-1}(\beta)$. Suppose now $i < n-1$ and $\beta \in \mathrm{Aut}_R(\mathfrak{m}^{i-1})$ is given by multiplication on $\mathfrak{m}^{i-1}$ by an element of $1 + xR_{n-i+1}$. We have

$$d_i(\beta) = d_i(\beta)d_{i+1}(\beta^{-1}|_{\mathfrak{m}^i})d_{i+1}(\beta|_{\mathfrak{m}^i})$$

By the induction hypothesis, $d_{i+1}(\beta|_{\mathfrak{m}^i})$ is in the commutator subgroup of E^*. By Lemma 2.5.3, $d_i(\beta)d_{i+1}(\beta^{-1}|_{\mathfrak{m}^i})$ is in the commutator subgroup of E^*, hence so is $d_i(\beta)$. This completes the induction step and gives that $E^*_{\mathrm{ab}} \cong (k^*)^{\oplus n}$. □

2.5.2 Singularty of Type A_{2n} in Dimension One

Our aim here is to prove (b) of Proposition 2.5.1. Thus, k is an algebraically closed field of characteristic not equal to $2, 3$ or 5 and R the ring $k[[t^2, t^{2n+1}]]$. Set $R = R_0$ and let M be the R-module $R_0 \oplus R_1 \oplus \cdots \oplus R_n$, where $R_i = k[[t^2, t^{2(n-i)+1}]]$ for $i = 0, \ldots, n$. Then we want to show $\mathrm{Aut}_R(M)_{\mathrm{ab}} \cong (k^*)^{\oplus n} \oplus k[[t]]^*$. Before we begin, we prove the following.

Lemma 2.5.4. *Let $0 \leq i, j \leq n$. If*

(a) $i \leq j$, then $\mathrm{Hom}_R(R_i, R_j) = R_j$.

(b) $i > j$, then $\mathrm{Hom}_R(R_i, R_j)$ can be viewed as a subset of R. In particular, it is contained in $R_n = k[[t]]$.

As a consequence of the above, we can view $E := \mathrm{End}_R(M)$ as a subring of $\mathbb{M}_{n+1}(R_n) = \mathbb{M}_{n+1}(k[[t]])$.

Proof. (a) We claim

$$\mathrm{Hom}_R(R_i, R_j) = \mathrm{Hom}_{R_i}(R_i, R_j)$$

Indeed, since $R \subseteq R_i$, there is a natural inclusion $\mathrm{Hom}_{R_i}(R_i, R_j) \subseteq \mathrm{Hom}_R(R_i, R_j)$. We demonstrate the reverse inclusion. Let $s \in R_i$, $f \in R_i$ and $\varphi \in \mathrm{Hom}_R(R_i, R_j)$. It is not hard to see that there is a nonzero $r \in R$ such that $rs \in R$ (for example, by noting that $t^{2n+1} \in R$, $t^{2n+1}k[[t]] \subseteq R$, and $R_i \subseteq k[[t]]$). We have

$$r\varphi(sf) = \varphi(rsf) = rs\varphi(f)$$

and r is nonzero, so that $\varphi(sf) = s\varphi(f)$. This proves the claim. Thus, we have

$$\mathrm{Hom}_R(R_i, R_j) = \mathrm{Hom}_{R_i}(R_i, R_j)$$

and the latter is naturally isomorphic to R_j.

(b) By ([63], Lemma 2.4.3), there is an isomorphism of R-modules:

$$\mathrm{Hom}_R(R_i, R_j) \cong (R_j :_R R_i)$$

Where $(R_j :_R R_i)$ is the ideal of R consisting of $f \in R$ such that $fR_i \subseteq R_j$.

Utilizing (a) and (b), we see that E can be viewed as the subring of $\mathbb{M}_{n+1}(R_n) = \mathbb{M}_{n+1}(k[[t]])$ given by

$$\begin{pmatrix} R_0 & R_1 & R_2 & R_3 & \cdots & R_n \\ (R_1 :_R R_0) & R_1 & R_2 & R_3 & \cdots & R_n \\ (R_2 :_R R_0) & (R_2 :_R R_1) & R_2 & R_3 & \cdots & R_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (R_n :_R R_0) & (R_n :_R R_1) & (R_n :_R R_2) & (R_n :_R R_3) & \cdots & R_n \end{pmatrix}$$

□

We now proceed with the proof of (b) of Proposition 2.5.1.

Proof. Note the R_i are finitely generated R-modules; each R_i is local with maximal ideal $\mathfrak{m}_i = (t^2, t^{2(n-i)+1})R_i$; we have inclusions $R_i \subseteq R_{i+1}$ and $\mathfrak{m}_i \subseteq \mathfrak{m}_{i+1}$; and each R_i has k as a residue field.

This is clear for $n = 0$. For $n = 1$, this is just ([31], Proposition 9.6), since $k[[t]] \cong (t^2, t^3)k[[t^2, t^3]]$ as $k[[t^2, t^3]]$-modules.

Suppose now $n \geq 2$. Our goal is to construct a map from E^* to the abelian group $(k^*)^{\oplus n} \oplus k[[t]]^*$, so that we obtain an induced map $E^*_{\text{ab}} \longrightarrow (k^*)^{\oplus n} \oplus k[[t]]^*$ that we will later show is an isomorphism.

First we construct a map from E^* to $(k^*)^{\oplus n}$. The proof that there is group homomorphism from $E^* \longrightarrow (k^*)^{\oplus n}$ works in exactly the same manner as as it did in the proof of (a) of Proposition 2.5.1. Noting of course that with $\mathfrak{n}_i = \{(\alpha_{ij}) : \alpha_{ii} \in J(\text{End}_R(R_{i-1}))\}$, (a) of Lemma 2.5.4 gives that $E/\mathfrak{n}_i = \text{End}_R(R_{i-1})/J(\text{End}_R(R_{i-1})) = R_{i-1}/\mathfrak{m}_{i-1} = k$. Thus we obtain an induced map $E^*_{\text{ab}} \longrightarrow (k^*)^{\oplus n}$.

As Lemma 2.5.4 allows us to view E as a subring of $\mathbb{M}_{n+1}(R_n) = \mathbb{M}_{n+1}(k[[t]])$, E^* is naturally a subset of $GL_{n+1}(k[[t]])$, the group of invertible $(n+1) \times (n+1)$ matrices over $k[[t]]$. By taking the determinant, we obtain a map from $E^* \longrightarrow k[[t]]^*$. Now $k[[t]]^*$ is abelian, hence this induces a group homomorphism $E^*_{\text{ab}} \longrightarrow k[[t]]^*$.

Regarding E as a matrix subring of $\mathbb{M}_{n+1}(k[[t]])$, we combine our preceding work to see there is a group homomorphism $\Phi : E^*_{\text{ab}} \longrightarrow (k^*)^{\oplus n} \oplus R_n^*$ such that the image of $\alpha = (\alpha_{ij}) \in E^*_{\text{ab}}$ under Φ is

$$(\alpha_{11} + \mathfrak{m}_0, \ldots, \alpha_{nn} + \mathfrak{m}_{n-1}, \det(\alpha))$$

We note Φ is surjective: For $(a_1, \ldots a_n, f) \in (k^*)^{\oplus n} \oplus R_n^*$, $(a_1, \ldots a_n, f)$ is the image under Φ of

$$\text{diag}(a_1, a_2, \ldots, a_n, (a_1 a_2 \cdots a_n)^{-1} f)$$

To see that Φ is injective, let $\alpha \in E^*_{\text{ab}}$ such that $\Phi(\alpha)$ is trivial. By ([31], Corollary 9.5), we may assume that $\alpha \in E^*_{\text{ab}}$ is diagonal. Write $\alpha = \text{diag}(f_0, f_1, \ldots, f_{n-1}, f_n)$, with $f_{i-1} \in (R_{i-1})^*$ by Lemma 2.5.4. Since $\Phi(\alpha)$ is trivial, $f_{i-1} \in 1 + \mathfrak{m}_{i-1}$ for $i = 1, \ldots, n$ and $f_0 f_1 \cdots f_n = 1$ in $R_n^* = k[[t]]^*$. Hence for $i = 1, \ldots, n$, α is the product of the diagonal automorphisms $\beta_i = d_i(f_{i-1}) d_{n+1}(f_{i-1}^{-1})$. Consider the automorphisms $\gamma_i = d_i(f_{i-1}) d_{i+1}(f_{i-1}^{-1})$ and note that $\beta_i = \gamma_i \cdots \gamma_n$. To see that γ_i is in the commutator subgroup, note that f_{i-1}^{-1} is in $1 + \mathfrak{m}_{i-1}$, hence multiplication by $f_{i-1} - 1$ maps R_i into R_{i-1}. Indeed, multiplication by $\mathfrak{m}_{i-1}$ on $\mathfrak{m}_i$ takes $\mathfrak{m}_i$ into

$\mathfrak{m}_{i-1}$. Moreover, any unit in R_i is a power series with nonzero constant term, hence multiplication on R_i by an element in $\mathfrak{m}_{i-1}$ takes R_i into R_{i-1}. Thus the hypotheses of Lemma 2.5.3 are satisfied, so that each γ_i is in the commutator subgroup of E^*, hence so is each β_i, and ultimately so is α. Thus Φ is injective, hence an isomorphism. □

2.5.3 Generalities for Invariant Subrings

Let k be a field. Recall from Section 2.4 that S is the ring $k[[x_1, \ldots, x_n]]$, G is a finite subgroup of $GL_n(k)$ that does not contain any nontrivial pseudo-reflections with $|G|$ invertible in k and R is the invariant subring S^G of S (where G acts by a linear change of variables on S). Then if R is an isolated singularity, the R-module S is an $(n-1)$-cluster tilting object in $\mathbf{mcm}\, R$.

We need the following lemmas for the proof of (c) of Proposition 2.5.1.

Lemma 2.5.5. *Let A be a local Cohen-Macaulay integral domain of dimension $d > 1$ such that A is an isolated singularity. Then A is normal and Hom$_A(I, I) \cong A$ for any ideal I of height one.*

Proof. Clearly A satisfies Serre's criterion for normality. For the second part, choose $x \in I$ to be nonzero. Then ([63], Lemma 2.4.3) shows that $\mathrm{Hom}_A(I, I)$ can be identified with the A-submodule $\frac{1}{x}(xI :_A I)$ of the quotient field of A. Now $(Ix :_A I)$ is nonzero and contained in I, so must have height one. If I is principal, it is clear that $(xI :_A I) = xA$. However, as $(xI :_A I)$ has height one and A is an isolated singularity, $A_\mathfrak{p}$ is a discrete valuation ring for every associated prime $\mathfrak{p}$ of $(xI :_A I)$, hence $(xIA_\mathfrak{p} :_{A_\mathfrak{p}} IA_\mathfrak{p}) = xA_\mathfrak{p}$. Thus $(xI :_A I) = xA$ and $\mathrm{Hom}_A(I, I) \cong A$.

□

Lemma 2.5.6. *([14], Lemma 5.4)*

Let A be a commutative Noetherian ring. Then for any ideal I and module M such that grade$(I, M) \geq 2$, we have Hom$_A(I, M) \cong$ Hom$_A(A, M) \cong M$.

2.5.4 Singularity of Type A_1 in Dimension Two

Our aim here is to prove (c) of Proposition 2.5.1. Thus R is the A_1 singularity $k[[s^2, st, t^2]]$ in dimension two with $\text{char}(k) \neq 2$. If $I = (s^2, st)R$, then $\text{Aut}_R(R \oplus I)_{\text{ab}} \cong k^* \oplus R^*$.

Proof. By ([47], Example 5.25), the indecomposable maximal Cohen-Macaulay modules of M are R and I. That is, R has finite type. Thus by ([69], Theorem 4.22), R is an isolated singularity. Moreover, since R is of finite type, $R \oplus I$ is an additive generator for **mcm** R, so that $\text{End}_R(R \oplus I)^{\text{op}}$ has finite global dimension by ([46], Theorem 6). Now I has height one, so that $\text{Hom}_R(I, I) \cong R$ by Lemma 2.5.5. Moreover, as I is maximal Cohen-Macaulay, we have $\text{Hom}_R(I, R) \cong R$ by Lemma 2.5.6. Thus $\text{End}_R(R \oplus I)$ is isomorphic to the subring of $\mathbb{M}_2(R)$ given by

$$\begin{pmatrix} R & R \\ I & R \end{pmatrix}$$

By ([56], Corollary 2.8), there is an isomorphism

$$\text{Aut}_R(R \oplus I)_{\text{ab}} \cong K_1^C(R) \oplus K_1^C(R/I) = R^* \oplus k[[t^2]]^*$$

Thus if $\mathfrak{m}$ denotes the maximal ideal of R, we have

$$\begin{aligned} R^* \oplus k[[t^2]]^* &\cong k^* \oplus 1 + \mathfrak{m} \oplus k[[t^2]]^* \\ &\cong k^* \oplus k[[t^2]][[s^2, st, t^2]]^* \\ &= k^* \oplus R^* \end{aligned}$$

□

2.5.5 Generalities for Reduced Hypersurface Singularities

Before we prove parts (d) and (e) of Proposition 2.5.1, we discuss another route for computing the group $\text{Aut}_R(L)_{\text{ab}}$ that we plan to utilize for the proof. We begin with another aside on noncommuatative algebra. A ring A with Jacobson radical $J(A)$ is said to be *semiperfect* if A is semilocal and idempotents of $A/J(A)$ lift to idempotents of A. We assume that **mcm** R contains an n-cluster tilting object L of the form $L_0 \oplus L_1 \oplus \cdots \oplus L_t$ and $L_0, L_1, \ldots, L_t$ are pairwise non-isomorphic and indecomposable. As $\mathrm{End}_R(L_i)$ is local for all i, it is the case that $\Lambda = \mathrm{End}_R(L)^{\mathrm{op}}$ is semiperfect by ([45], Theorem 23.8) (noting that Λ is semiperfect if and only if Λ^{op} is semiperfect). In particular, if R is a k-algebra, the characteristic of k is not two, then by ([66], Theorem 2), there is an isomorphism

$$K_1^C(\Lambda) \cong \Lambda^*_{\mathrm{ab}} = \mathrm{Aut}_R(L)_{\mathrm{ab}}$$

Since Λ is semiperfect, with the above isomorphism, we can utilize ([56], Theorem 2.2) to obtain an isomorphism

$$\mathrm{Aut}_R(L)_{\mathrm{ab}} \cong K_1^C(\Lambda) \cong \left(\bigoplus_{i=0}^{t} \mathrm{Aut}_R(L_i) \right) \Big/ HC$$

Where C is the subgroup of $\bigoplus_{i=0}^{t} \mathrm{Aut}_R(L_i)$ generated by all elements of the form

$$(1 + \alpha_i \beta_i)(1 + \beta_i \alpha_i)^{-1}$$

with $\alpha_i, \beta_i \in \mathrm{End}_R(L_{i-1})$ such that $1 + \alpha_i \beta_i \in \mathrm{Aut}_R(L_{i-1})$, and H is the subgroup generated by all elements of the form

$$(1 + \alpha_{ij} \alpha_{ji})(1 + \alpha_{ji} \alpha_{ij})^{-1}$$

with $\alpha_{ij} \in \mathrm{Hom}_R(L_{i-1}, L_{j-1})$, $i \neq j$, and $1 + \alpha_{ij}\alpha_{ji} \in \mathrm{Aut}_R(L_{i-1})$.

However each $\alpha_{ij}\alpha_{ji}$ is never an automorphism when $i \neq j$, since otherwise L_{i-1} would be a direct summand of L_{j-1} (see ([31], Lemma 9.3). Since each of the rings $\mathrm{End}_R(L_{i-1})$ are local, this implies that $1 + \alpha_{ij}\alpha_{ji} \in \mathrm{Aut}_R(L_{i-1})$ *for all* $i \neq j$.

We now continue with the proof of (d) of Proposition 2.5.1.

2.5.6 Reduced Hypersurface Singularities in Dimension One

Our aim here is to prove (d) of Proposition 2.5.1. Here, k is an algebraically closed field of characteristic not two and $S = k[[x, y]]$, R is the ring S/fS with $f \in (x, y)$ is such that in its prime factorization, $f = f_1 \cdots f_n$ we have $(f_i) \neq (f_j)$ for $i \neq j$, $f_i \notin (x, y)^2$, $(f_i, f_{i+1}) = (x, y)$. Then if $S_i = S/(f_1 \cdots f_i)S$ and $L := S_1 \oplus \cdots \oplus S_n$, we have $\mathrm{Aut}_R(L)_{\mathrm{ab}} \cong \overline{R}^*$, where $\overline{R}$ is the integral closure of R in its total quotient ring. We first prove a useful lemma.

Lemma 2.5.7. *With notation as above, we have*

$$\mathit{Hom}_R(S_j, S_i) \cong \begin{cases} (f_{j+1} \cdots f_i)/(f_1 \cdots f_i) & j < i \\ S_i & i \leq j \end{cases}$$

Proof. The isomorphisms

$$\mathrm{Hom}_R(S_j, S_i) \cong \mathrm{Hom}_R(R/(f_1 \cdots f_j), R/(f_1 \cdots f_i)) \cong (0 :_{R/(f_1 \cdots f_i)} (f_1 \cdots f_j))$$

make the statement clear. □

We now proceed with the proof of (d) of Proposition 2.5.1.

Proof. Now by ([13], 4.7), L is a 2-cluster-tilting object for **mcm** R. As $\Lambda := \mathrm{End}_R(L)^{\mathrm{op}}$ has finite global dimension by Theorem 2.4.1, the remarks of Subsection 2.5.5 give

$$\mathrm{Aut}_R(L)_{\mathrm{ab}} \cong \left(\bigoplus_{i=1}^{n} \mathrm{Aut}_R(S_i) \right) /HC$$

Where C is the subgroup of $\bigoplus_{i=1}^n \mathrm{Aut}_R(S_i)$ generated by all elements of the form $(1 + \alpha_i\beta_i)(1 + \beta_i\alpha_i)^{-1}$ such that $\alpha_i, \beta_i \in \mathrm{End}_R(S_i)$ and $1 + \alpha_i\beta_i \in \mathrm{Aut}_R(S_i)$. By Lemma 2.5.7, $\mathrm{End}_R(S_i) = S_i$, so that C is trivial and $\mathrm{Aut}_R(L)_{\mathrm{ab}} \cong (S_1^* \oplus \cdots \oplus S_n^*)/H$. We now describe the subgroup H. Again by the remarks in subsection 2.5.5, H is the subgroup generated by all elements of the form

$$(1+\alpha_{ij}\alpha_{ji})(1+\alpha_{ji}\alpha_{ij})^{-1}$$

where $\alpha_{ij} \in \mathrm{Hom}_R(S_j, S_i)$, $\alpha_{ji} \in \mathrm{Hom}_R(S_i, S_j)$), and $i \neq j$. In fact, we can consider the subgroup generated by such elements where $i < j$. We note $\alpha_{ij}\alpha_{ji} \in \mathrm{End}_R(S_i) = S_i$ and $\alpha_{ji}\alpha_{ij} \in \mathrm{End}_R(S_j) = S_j$. Utilizing Lemma 2.5.7, we can give a more concise description of H as follows (note $i < j$). The subgroup H is generated by the elements $h_{ij}(s)$, which we now describe:

(i) the ith entry of $h_{ij}(s)$ is the image of an element $s \in 1 + (f_{i+1} \cdots f_j) \subset S$ in the unit group S_i^*;

(ii) the jth entry of $h_{ij}(s)$ is the image of s^{-1}, with s from (i) in the unit group S_j^*;

(iii) $h_{ij}(s)$ is trivial elsewhere.

Let $H_{i,j}$ be the subgroup of H generated by the $h_{ij}(s)$, with s defined above. We have $H = \oplus_{i<j} H_{i,j}$. By projecting onto the jth coordinate, it is easy to see $H_{i,j}$ is isomorphic to the subgroup $1 + (f_{i+1} \cdots f_j)$ of S_j^*. For $1 \leq i < n$, we call the subgroup $H_{i,i+1} \oplus \cdots \oplus H_{i,n}$ of H the *ith layer* of H. It is easy to see that $S_1^* \oplus \cdots \oplus S_n^*$ modulo the direct sum of the first m layers of H is

$$\bigoplus_{u=1}^{m+1} (S/f_u S)^* \oplus \bigoplus_{v=m+2}^{n} (S/(f_{m+1} \cdots f_v)S)^*$$

As H is the direct sum of its $n-1$ layers of H, we see that $S_1^* \oplus \cdots \oplus S_n^*$ modulo H is just

$$(S/f_1 S)^* \oplus \cdots \oplus (S/f_n S)^*$$

And this is just $\overline{R}^*$.

$\square$

2.5.7 Reduced Hypersurface Singularities in Dimension Three

Our aim here is to prove (e) of Proposition 2.5.1. Keep notation as in Subsection 2.5.6 with the exception that we now require k be an algebraically closed field of characteristic zero. Set

$S' = k[[x,y,u,v]]$ and $R' = S'/(f+uv)S'$. Then a 2-cluster tilting object for $\mathbf{mcm}\, R$ is given by $L := U_1 \oplus \cdots \oplus U_n$, with $U_i = (u, f_1 \cdots f_i) \subset R'$ (see Section 2.4). Then we aim to show $\mathrm{Aut}_R(L)_{\mathrm{ab}} \cong R'^* \oplus k[[w,z]]^{*\oplus(n-1)}$. In order to understand $\mathrm{Aut}_R(L)_{\mathrm{ab}}$, we first need to understand the structure of the modules $\mathrm{Hom}_{R'}(U_i, U_j)$, so that we are able to use the remarks of Subsection 2.5.5 to compute $\mathrm{Aut}_R(L)_{\mathrm{ab}}$. This is the first step we make below.

Proposition 2.5.8. *Let R' and U_i be as above. Then*

$$\mathit{Hom}_{R'}(U_i, U_j) \cong \begin{cases} U_j & j < i \\ R' & i \leq j \end{cases}$$

Proof. Now R' is Gorenstein of dimension three and an isolated singularity. Since U_i is an ideal of R' of height one, we may apply Lemma 2.5.5 to see that $\mathrm{Hom}_{R'}(U_i, U_i) \cong R'$ for all i. If $i \neq j$, ([63], Lemma 2.4.3) says we may identify $\mathrm{Hom}_{R'}(U_i, U_j)$ with the the R'-submodule $\frac{1}{u}(uU_j :_{R'} U_i)$ of the quotient field of R'. Now $(uU_j :_{R'} U_i)$ is nonzero and $(uU_j :_{R'} U_i) \subset U_j$, hence $(uU_j :_{R'} U_i)$ has height one. Let $\mathfrak{p}$ be a minimal prime of $(uU_i :_{R'} U_j)$. As R' is an isolated singularity, $R'_{\mathfrak{p}}$ is a discrete valuation ring. Write $R'_{\mathfrak{p}} = A$ and let μ be a generator for the maximal ideal of A. Suppose u maps to $c\mu^a$, with $a > 0$ and $c \in A^*$. Write $(U_i)_{\mathfrak{p}} = \mu^{n_i} A$ and $(U_j)_{\mathfrak{p}} = \mu^{n_j} A$, with n_j, n_i nonnegative integers. Then

$$(uU_j :_{R'} U_i)_{\mathfrak{p}} = (\mu^{a+n_j} :_A \mu^{n_i})$$

If $i \leq j$, then $U_j \subseteq U_i$, hence $n_j \geq n_i$. We have

$$(\mu^{a+n_j} :_A \mu^{n_i}) = \mu^{a+n_j-n_i} A \subset \mu^a A$$

Thus $(uU_j :_{R'} U_i)_{\mathfrak{p}} = (uR')_{\mathfrak{p}}$. In this case, $(uU_i :_{R'} U_j) = uR'$, so that $\mathrm{Hom}_{R'}(U_i, U_j) \cong R'$.

Now if $j < i$, then $U_i \subset U_j$ and $n_i \geq n_j$. Notice $u \in U_i$, so that $a \geq n_i$. We have

$$(\mu^{a+n_j} :_A \mu^{n_i}) = \mu^{a+n_j-n_i} = \mu^{a-n_i}(U_j)_{\mathfrak{p}}$$

And $\mu^{a-n_i}A = (\mu^a :_A \mu^{n_i}) = (u :_{R'} U_i)_{\mathfrak{p}}$. We have $(u :_{R'} U_i) = (u :_{R'} f_1 \cdots f_i)$. Thus $(uU_j :_{R'} U_i) = (u :_{R'} f_1 \cdots f_i)U_j$. When $i = n$, $(f_1 \cdots f_n)R' = (uv)R'$, hence $(u :_{R'} f_1 \cdots f_n) = R'$. This gives $U_j = (uU_j :_{R'} U_n)$, hence there is an isomorphism of R'-modules $\mathrm{Hom}_{R'}(U_n, U_j) \cong \frac{1}{u}U_j \cong U_j$.

To analyze the ideal $(u :_{R'} f_1 \cdots f_i)$ for $i < n$, note that $f_{i+1} \cdots f_n \in (u :_{R'} f_1 \cdots f_i)$ and that the ideals $(u, f_{i+1})R', \ldots, (u, f_n)R'$ are prime. In particular, the minimal primes of $(u :_{R'} f_1 \cdots f_i)$ are $(u, f_{i+1})R', \ldots, (u, f_n)R'$. Let $\mathfrak{q}$ denote the prime ideal $(u, f_s)R'$, with $i + 1 \le s \le n$. Then $(f_1 \cdots f_i)R'_{\mathfrak{q}} = R'_{\mathfrak{q}}$, as $f_1, \ldots, f_i \notin \mathfrak{q}$ and hence $(u :_{R'} f_1 \cdots f_i)_{\mathfrak{q}} = (uR')_{\mathfrak{q}}$. Thus $(u :_{R'} f_1 \cdots f_i) = uR'$, so that $(u :_{R'} f_1 \cdots f_i)U_j = uU_j$, and hence $\mathrm{Hom}_{R'}(U_i, U_j) \cong U_j$. This gives the result. $\square$

We now proceed with the proof of (e) of Proposition 2.5.1.

Proof. By ([39], Theorem 4.17), $\mathrm{End}_{R'}(L)^{\mathrm{op}}$ has finite global dimension, so the remarks of Subsection 2.5.5 are applicable. Thus there is an isomorphism:

$$\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong \left(\bigoplus_{i=1}^{n} \mathrm{Aut}_{R'}(U_i) \right) / HC$$

Where C is the subgroup of $\bigoplus_{i=1}^{n} \mathrm{Aut}_{R'}(U_i)$ generated by $(1+\alpha_i\beta_i)(1+\beta_i\alpha_i)^{-1}$ such that $\alpha_i, \beta_i \in \mathrm{End}_{R'}(U_i)$ and $1 + \alpha_i\beta_i \in \mathrm{Aut}_{R'}(U_i)$. By Proposition 2.5.8, $\mathrm{End}_{R'}(U_i) = R'$, so that C is trivial, hence $\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong (R'^*)^{\oplus n}/H$. We now describe the subgroup H. Again by the remarks in subsection 2.5.5, H is the subgroup generated by all elements of the form

$$(1 + \alpha_{ij}\alpha_{ji})(1 + \alpha_{ji}\alpha_{ij})^{-1}$$

where $\alpha_{ij} \in \mathrm{Hom}_{R'}(U_j, U_i)$, $\alpha_{ji} \in \mathrm{Hom}_{R'}(U_i, U_j)$), and $i \neq j$. In fact, we can consider the subgroup generated by such elements where $i < j$. We now give a more concise description of H.

Utilizing Proposition 2.5.8, H is the subgroup of $(R'^*)^{\oplus n}$ generated by the elements $h_{ij}(g)$ with $i < j$ and $g \in U_i$ such that that:

(i) the ith entry of $h_{ij}(g)$ is $1 + g$;

(ii) the jth entry of $h_{ij}(g)$ is $(1 + g)^{-1}$;

(iii) $h_{ij}(g)$ is trivial elsewhere.

For fixed i and j, let $H_{i,j}$ be the subgroup generated by the elements $h_{ij}(g)$. Thus $H = \oplus_{i<j} H_{i,j}$ and $H_{i,j} \cong 1 + U_i \subset R'^*$. For $i < n$, we call the subgroup $H_{i,i+1} \oplus H_{i,i+2} \oplus \cdots \oplus H_{i,n}$ the *ith layer* of H. As $U_n \subset U_n \subset \cdots \subset U_1$, it is easy to see that $(R'^*)^{\oplus n}$ modulo the direct sum of layers $n-1, n-2, \ldots, n-i$ is isomorphic to

$$(R'^*)^{\oplus(n-i)} \oplus (R'/U_{n-i})^{*\oplus i}$$

Now the direct sum of layers $n-1, n-2, \ldots, 1$ is just H, so that we see

$$\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong R'^* \oplus (R'/U_1)^{*\oplus(n-1)}$$

Moreover, since $U_1 = (u, f_1)$ and $f_1 \in (x, y)\backslash(x, y)^2 \subseteq k[[x, y]]$, we see $R'/U_1 \cong k[[w, z]]$, for variables w, z over k. Thus

$$\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong R'^* \oplus k[[w, z]]^{*\oplus(n-1)}$$

□

2.6 Computing $G_1(R)$

The aim of this section is to utilize Theorem 2.1.3 to explicitly calculate $G_1(R)$ for several hypersurface singularities. Our results are the following:

Example 2.6.1. Let k be an algebraically closed field of characteristic not two. If $n \geq 1$ and $R = k[x]/x^n k[x]$, then $G_1(R) \cong k^*$.

Remark 2.6.2. We note that Example 2.6.1 follows immediately from Quillen's Dévissage Theorem ([57], §5 Theorem 4), but we find the calculation illustrative of our methods as well as allowing us to generalize ([31], Example 10.2).

Example 2.6.3. Let k be an algebraically closed field of characteristic not two, three or five. If R is the finite-type singularity $k[[t^2, t^{2n+1}]]$ for $n \geq 0$, then $G_1(R) \cong \overline{R}^* = k[[t]]^*$;

Example 2.6.4. Let k be an algebraically closed field of characteristic not two. If $S = k[[x, y]]$ let $f_1, \ldots, f_n \in (x, y)$ be irreducible and $f = f_1 \cdots f_n$ be such that

(i) $R := S/fS$ is an isolated singularity (ie. $(f_i) \neq (f_j)$)

(ii) $f_i \notin (x, y)^2$ for all i.

(iii) $(f_i, f_{i+1}) = (x, y)$.

Then $G_1(R) \cong \mathbb{Z}^{\oplus(n-1)} \oplus \overline{R}^*$ (where $\overline{R}$ is the integral closure of R);

Remark 2.6.5. We note here that Examples 2.6.3 and 2.6.4 follow from the use of more classical technology. Let $A \subseteq B$ be an inclusion of commutative Noetherian such that B is a module-finite extension of A. Let $I \subseteq A$ and $J \subseteq B$ be ideals such that $IB \subseteq J$. Set $X = \operatorname{Spec}(A) \setminus \operatorname{Spec}(A/I)$, $Y = \operatorname{Spec}(B) \setminus \operatorname{Spec}(B/J)$, and suppose that the induced morphism of schemes $X \longrightarrow Y$ is an isomorphism. Then Quillen's Localization Theorem ([57], Theorem 5) yields long exact sequences

$$G_i(A/I) \longrightarrow G_i(A) \longrightarrow G_i(X) \longrightarrow G_{i-1}(A/I) \longrightarrow \cdots$$

and

$$G_i(B/J) \longrightarrow G_i(B) \longrightarrow G_i(Y) \longrightarrow G_{i-1}(B/J) \longrightarrow \cdots$$

Where we note that for a Noetherian scheme $\mathcal{S}$, $G_i(\mathcal{S})$ is the ith Quillen K-group of the category of coherent $\mathcal{O}_{\mathcal{S}}$-modules. Now restriction of scalars induces the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & G_i(B/J) \arrow[r, "\delta_{B/J}"] \arrow[d, "\varepsilon_{B/J}"] & G_i(B) \arrow[r, "\delta_B"] \arrow[d, "\varepsilon_B"] & G_i(Y) \arrow[r, "\delta_Y"] \arrow[d, "\varepsilon_Y"] & G_{i-1}(B/J) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & G_i(A/I) \arrow[r, "\delta_{A/I}"'] & G_i(A) \arrow[r, "\delta_A"'] & G_i(X) \arrow[r, "\delta_X"'] & G_{i-1}(A/I) \arrow[r] & \cdots
\end{tikzcd}$$

Where we note that ε_Y is an isomorphism. Some rather involved but straightforward diagram chasing gives a Mayer-Vietoris-like sequence of G-groups that we denote by $(\star)$:

$$\cdots \longrightarrow G_i(B/J) \xrightarrow{\alpha} G_i(A/I) \oplus G_i(B) \xrightarrow{\beta} G_i(A) \xrightarrow{\gamma} G_{i-1}(B/J) \longrightarrow \cdots$$

Where $\alpha = \begin{pmatrix} \varepsilon_{B/J} \\ \delta_{B/J} \end{pmatrix}$, $\beta = (\delta_{A/I}, -\varepsilon_B)$, and $\gamma = \delta_Y \varepsilon_Y^{-1} \delta_A$.

To see how we can recover the claims in Example 2.6.3 using $(\star)$, let $I = (t^2, t^{2n+1}) \subseteq k[[t^2, t^{2n+1}]] = A$ and $J = (t) \subseteq B = k[[t]]$. We note that B is a module-finite extension of A and Spec$(A) \setminus$ Spec$(A/I) = \{(0)\} \cong$ Spec$(B) \setminus$ Spec$(B/J) = \{(0)\}$, so the above requirements are met. Using $(\star)$, we obtain a long exact sequence

$$G_i(k) \xrightarrow{\alpha} G_i(k) \oplus G_i(B) \xrightarrow{\beta} G_i(A) \xrightarrow{\gamma} G_{i-1}(k) \xrightarrow{\alpha'} G_{i-1}(k) \oplus G_{i-1}(B)$$

Where $\alpha' = \begin{pmatrix} \varepsilon'_{B/J} \\ \delta'_{B/J} \end{pmatrix}$. As $I = J \cap A$, the induced map $A/I \longrightarrow B/J$ is an isomorphism, so that $\varepsilon_{B/J}$ is an isomorphism. In particular, we obtain the exact sequence

$$0 \longrightarrow G_i(B)/\mathrm{im}(\delta_{B/J}) \longrightarrow G_i(A) \xrightarrow{\gamma} G_{i-1}(k) \xrightarrow{\alpha'} G_{i-1}(k) \oplus G_{i-1}(B)$$

Now $\delta_{B/J} = \delta'_{B/J} = 0$, so that we easily obtain from the above exact sequence $G_i(B) \cong G_i(A)$. In particular, $G_1(A) = G_1(k[[t^2, t^{2n+1}]] \cong G_1(B) = G_1(k[[t]]) = k[[t]]^*$. We note, unlike the restriction on the characteristic we encounter using Theorem 2.1.3 below, this holds regardless of the characteristic.

To see how we can recover the claims in Example 2.6.4 using $(\star)$, we let $A = S/(f_1 \cdots f_n)$ with I the maximal ideal of A and $B = S/(f_1) \oplus \cdots \oplus S/(f_n)$ with $J = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$, where $\mathfrak{m}_i$ is the maximal ideal of the local ring $S/(f_i)$. It is easy to see that B is a module-finite extension of A. Moreover, B is also the integral closure of A in its total quotient ring. We also have

$$X = \text{Spec}(A) \setminus \text{Spec}(A/I) = \{(f_i)/(f_1 \cdots f_n) : 1 \leq i \leq n\}$$

$$Y = \text{Spec}(B) \setminus \text{Spec}(B/J) = \left\{ S/(f_1) \oplus \cdots \oplus \underbrace{0}_{i} \oplus \cdots \oplus S/(f_n) : 1 \leq i \leq n \right\}$$

From the above, it is clear the induced map $X \longrightarrow Y$ is given by $(f_i)/(f_1 \cdots f_n) \mapsto S/(f_1) \oplus \cdots \oplus \underbrace{0}_{i} \oplus \cdots \oplus S/(f_n)$, hence is clearly an isomorphism. From $(\star)$, we obtain a long exact sequence

$$G_i(k)^{\oplus n} \xrightarrow{\alpha} G_i(k) \oplus G_i(B) \xrightarrow{\beta} G_i(A) \xrightarrow{\gamma} G_{i-1}(k)^{\oplus n} \xrightarrow{\alpha'} G_{i-1}(k) \oplus G_{i-1}(B)$$

Now the first component of α and α' is the summing map. Moreover, $\delta^{B/J} = \delta'_{B/J} = 0$ (where $\delta'_{B/J}$ is the second component of α') so that we obtain an exact sequence

$$0 \longrightarrow G_i(B) \longrightarrow G_i(A) \longrightarrow G_{i-1}(k)^{\oplus(n-1)} \longrightarrow 0$$

As $f_j \notin (x,y)^2$, $S/(f_j)$ is regular, hence $G_i(B) = K_i(B)$. Specializing to $i = 1$, we obtain the exact sequence

$$0 \longrightarrow K_1(B) \longrightarrow G_1(A) \longrightarrow \mathbb{Z}^{\oplus(n-1)} \longrightarrow 0$$

Since the above sequence splits, we obtain $G_1(A) \cong K_1(B) \oplus \mathbb{Z}^{n-1}$. As $K_1(S/(f_j)) = (S/(f_j))^*$, it is easy to see that $K_1(B) = B^*$. This gives Example 2.6.4. We note that (iii) in the hypothesis of Example 2.6.4 is not needed, so this is more general.

While we can recover Examples 2.6.3 and 2.6.4 from these methods, we find that our work expands on calculations given in ([53], Section 7.3 and Proposition 7.26) and ([31], Example 10.5).

Lastly, our results that do not follow from our arguments in Remark 2.6.5 are the following:

Proposition 2.6.6. *Let k be an algebraically closed field of characteristic not two.*

(a) If $R = k[[s^2, st, t^2]]$, then $G_1(R) \cong R^*$.

(b) Suppose k has characteristic zero, $S = k[[x, y]]$, $S' = k[[x, y, u, v]]$, and $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n \in S = k[[x, y]]$ is such that

(i) S/fS is an isolated singularity (ie. $(f_i) \neq (f_j)$)

(ii) $f_i \notin (x, y)^2$ for all i.

(iii) $(f_i, f_{i+1}) = (x, y)$.

Then $G_1(R') \cong \mathbb{Z}^{\oplus(n-1)} \oplus \left(R'^* \oplus k[[w, z]]^{*\oplus(n-1)}\right)/\Xi$, with Ξ the subgroup of Definition 2.2.21 and w, z variables over k.

2.6.1 The n-Auslander-Reiten Matrix

Before we can use Theorem 2.1.3 to perform the calculation of Examples 2.6.1, 2.6.3, 2.6.4, and prove Proposition 2.6.6, we need to explicitly define the free group $\mathcal{H}$ occurring in the decomposition of $G_1(R)$ in Theorem 2.1.3. Our assumptions are as usual and we also require that R is a k-algebra and k is algebraically closed of characteristic not two. We use $L = L_0 \oplus L_1 \oplus \cdots \oplus L_t$ to denote an n-cluster tilting object of $\mathbf{mcm}\, R$ such that $\Lambda = \mathrm{End}_R(L)^{\mathrm{op}}$ has finite global dimension. We assume that $L_0, L_1, \ldots, L_t$ are the pairwise non-isomorphic summands of L (each occurs with multiplicity one in the decomposition of L) and that $L_0 = R$. Let $I = \{L_0, L_1, \ldots, L_t\}$ and $I_0 = I \setminus \{L_0\}$. We set $\mathcal{C} = \mathbf{add}_R L$. Recall, for $j > 0$, there is an exact sequence, called the n-Auslander-Reiten ending in L_j (see Definition 2.2.15):

$$0 \longrightarrow C_n^j \longrightarrow \cdots \longrightarrow C_0^j \longrightarrow L_j \longrightarrow 0$$

with $C_0^j, C_1^j, \ldots, C_n^j \in \mathcal{C}$. Given $N \in \mathcal{C}$, let $\#(i, N)$ be the number of L_i-summands ($0 \leq i \leq t$) appearing in a decomposition of N into the indecompsables R-modules $L_0, L_1, \ldots, L_t$.

Following [53], we define a $(t+1) \times t$ integer matrix T whose ij-th entry is $\#(i, L_j) + \sum_{u=0}^{n}(-1)^{u+1}\#(i, C_u^j) = \delta_{ij} + \sum_{u=0}^{n}(-1)^{u+1}\#(i, C_u^j)$ (note that T has a 0th row but no 0th column). As $G_0(k) = \mathbb{Z}$ and $G_0(\Lambda) = \mathbb{Z}^{\oplus t}$, Theorem 2.1.2 gives us a map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$. It is shown in ([53], Section 7.2) that T defines the map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$ afforded to us by Theorem 2.1.2. We call T the *n-Auslander-Reiten matrix* or the *n-Auslander-Reiten homomorphism*. Moreover, this is the same map given in Theorem 2.1.1 when **mcm** R has a 1-cluster tilting object. For our needs, recall Theorem 2.1.3 says $G_1(R) = \mathcal{H} \oplus \mathrm{Aut}_R(L)_{\mathrm{ab}}/\Xi$, so that now $\mathcal{H} = \ker(T)$.

We make a useful observation before our computations.

Lemma 2.6.7. *Let $1 \leq i_1 < \cdots < i_h \leq t$ and $L' = L_{i_1}^{\oplus q} \oplus \cdots \oplus L_{i_h}^{\oplus q}$ with $q > 0$. Then for $a \in R^*$, we have $\det_{\Lambda^{op}}(\widetilde{a1_{L'}}) = \alpha$, where $\alpha \in (\Lambda^{op})^*$ is given by diag$(1_{L_0}, \ldots, a^q 1_{L_{i_1}}, \ldots, a^q 1_{L_{i_h}}, \ldots, 1_{L_t})$*

Proof. From Remark 2.2.16, we see that $\widetilde{a1_{L'}} : L^{\oplus q} \longrightarrow L^{\oplus q}$ is the map $e1_{L^{\oplus q}}$, where $e \in (\Lambda^{\mathrm{op}})^*$ is given by $\mathrm{diag}(1_{L_0}, \ldots, a1_{L_{i_1}}, \ldots, a1_{L_{i_h}}, \ldots, 1_{L_t})$. Now recall the injection $GL_1(\Lambda^{\mathrm{op}}) = (\Lambda^{\mathrm{op}})^* \hookrightarrow GL_q(\Lambda^{\mathrm{op}})$ that takes $\gamma \in (\Lambda^{\mathrm{op}})^*$ to the automorphism $d_1(\gamma) = \mathrm{diag}(\gamma, 1_L, \ldots, 1_L) \in GL_q(\Lambda^{\mathrm{op}})$. Now

$$(e1_{L^{\oplus q}})^{-1} \cdot d_1(\alpha) = e^{-1}1_{L^{\oplus q}} \cdot d_1(\alpha) = \beta_1 \cdots \beta_{q-1}$$

where $\beta_u := d_1(e)d_{u+1}(e^{-1}) \in GL_q(\Lambda^{\mathrm{op}})$. Consider the element $\gamma_u := \mathrm{diag}(e, 1, \ldots, 1)$ in $GL_u(\Lambda^{\mathrm{op}})$. Then, by slight abuse of notation, the matrix $\delta_u := \mathrm{diag}(\gamma_u, \gamma_u^{-1})$ in GL_{2u} is in the commutator subgroup of $GL_{2u}(\Lambda^{\mathrm{op}})$ by ([58], Corollary 2.1.3). Thus by ([58], Proposition 2.1.4), each δ_u is in the commutator subgroup of $GL(\Lambda^{\mathrm{op}})$. In $GL_(\Lambda^{\mathrm{op}})$, either β_u is the image of δ_u under the injection $GL_{2u}(\Lambda^{\mathrm{op}}) \hookrightarrow GL_q(\Lambda^{\mathrm{op}})$, or δ_u is the image of β_u under the injection $GL_q(\Lambda^{\mathrm{op}}) \hookrightarrow GL_{2u}(\Lambda^{\mathrm{op}})$. In either case we see that β_u is in the commutator subgroup of $GL(\Lambda^{\mathrm{op}})$. Hence $e1_{L^{\oplus q}} \equiv d_1(\alpha)$ in $GL(\Lambda^{\mathrm{op}})_{\mathrm{ab}}$.

Since $\det_{\Lambda^{\mathrm{op}}} : GL(\Lambda^{\mathrm{op}})_{\mathrm{ab}} \longrightarrow (\Lambda^{\mathrm{op}})^*_{\mathrm{ab}}$ is the inverse of the isomorphism induced by the map

$$(\Lambda^{\mathrm{op}})^* = GL_1(\Lambda^{\mathrm{op}}) \hookrightarrow GL(\Lambda^{\mathrm{op}}) \twoheadrightarrow GL(\Lambda^{\mathrm{op}})_{\mathrm{ab}}$$

We see that $\det_{\Lambda^{\mathrm{op}}}(e1_{L^{\oplus q}}) = \alpha$. $\square$

We note R has finite type if and only if R has a 1-cluster tilting object M. In this case, $\mathbf{mcm}\, R = \mathbf{add}_R\, M$ and $M = M_0 \oplus M_1 \oplus \cdots \oplus M_t$, with $M_0 = R$ and $M_1, \ldots, M_t$ the non-free indecomposable maximal Cohen-Macaulay R-modules. For $j > 0$, we call the 1-Auslander-Reiten sequence ending in M_j the Auslander-Reiten sequence ending in M_j and the 1-Auslander-Reiten matrix is referred to as the Auslander-Reiten matrix. The Auslander-Reiten matrix is a classical invariant and we denote it by Υ.

We now make use of Theorem 2.1.3 perform the calculations of Examples 2.6.1, 2.6.3, 2.6.4, and prove Proposition 2.6.6. That is, in the context of Theorem 2.1.3, we must compute the kernel of the n-Auslander-Reiten homomorphism and the subgroup Ξ of $\mathrm{Aut}_R(L)_{\mathrm{ab}}$. In each computation, it will also be clear that R is a k-algebra.

2.6.2 Truncated Polynomial Rings in One Variable

Our aim here is to utilize Theorem 2.1.3 to perform the calculation in Example 2.6.1. That is, if $R = k[x]/x^n k[x]$, then $G_1(R) \cong k^*$.

Proof. For $n = 1$, $R = k$, so $G_1(R) = K_1(R) \cong k^*$.

We now suppose $n \geq 2$. Let $\mathfrak{m}$ denote the maximal ideal xR. By the proof of ([47], Theorem 3.3), R has finite type and the indecomposable R-modules are given by $R, \mathfrak{m}, \ldots, \mathfrak{m}^{n-1}$. Let M be the R-module given by $R \oplus \mathfrak{m} \oplus \cdots \oplus \mathfrak{m}^{n-1}$ and denote its endomorphism ring by E. Using ([69], Lemma 2.9) it is not hard to see the Auslander-Reiten sequences ending in $\mathfrak{m}^j$ are given by

$$0 \longrightarrow \mathfrak{m}^j \longrightarrow \mathfrak{m}^{j-1} \oplus \mathfrak{m}^{j+1} \longrightarrow \mathfrak{m}^j \longrightarrow 0 \qquad (1 \leq j \leq n-1) \tag{$\star$}$$

Thus for $1 \leq j \leq n-2$, Υ has its jth column given by $(0, \ldots, -1, 2, -1, \ldots, 0)^T$, where $-1, 2$ and -1 occur in rows $j-1, j$ and $j+1$, respectively. And the $(n-1)$st column is given by $(0, \ldots, 0, -1, 2)^T$. It is easy to see that Υ is injective.

We compute the subgroup Ξ of E^*_{ab} from Definition 2.2.21. By $(\star)$ and Lemma 2.6.7, the subgroup Ξ is generated by elements

$$\xi_{a,j} = (\widetilde{a^2 1_{\mathfrak{m}^j}}) \cdot (\widetilde{a^{-1}1_{\mathfrak{m}^{j-1}} \oplus a^{-1}1_{\mathfrak{m}^{j+1}}}) \qquad (1 \leq j \leq n-2)$$

$$\xi_{a,n-1} = (\widetilde{a^2 1_{\mathfrak{m}^{n-1}}}) \cdot (\widetilde{a^{-1}1_{\mathfrak{m}^{n-2}}})$$

where a runs over k^*. Again by Lemma 2.6.7, we have

$$\xi_{a,j} = \text{diag}(1_R, \ldots, a^{-1}1_{\mathfrak{m}^{j-1}}, a^2 1_{\mathfrak{m}^j}, a^{-1}1_{\mathfrak{m}^{j+1}}, \ldots, 1_{\mathfrak{m}^{n-1}})$$

$$\xi_{a,n-1} = \text{diag}(1_R, \ldots, \ldots, a^{-1}1_{\mathfrak{m}^{n-2}}, a^2 1_{\mathfrak{m}^{n-1}})$$

By (a) of Proposition 2.5.1, there is an isomorphism $E^*_{\text{ab}} \cong (k^*)^{\oplus n}$. We regard Ξ as a subgroup of $(k^*)^{\oplus n}$ and abuse notation to write

$$\xi_{a,j} = (1, \ldots, a^{-1}, a^2, a^{-1}, \ldots 1)$$

$$\xi_{a,n-1} = (1, \ldots, a^{-1}, a^2)$$

Where a^{-1}, a^2 and a^{-1} occur in $\xi_{a,j}$ at positions j, $j+1$ and $j+2$, respectively. Let $\Psi : (k^*)^{\oplus n} \longrightarrow k^*$ be the map such that $\Psi(a_1, \ldots, a_n) = a_1^n a_2^{n-1} \cdots a_n$. Then Ψ is a surjective group homomorphism such that $\Xi \subseteq \ker(\Psi)$. Let $(a_1, \ldots, a_n) \in \ker(\Psi)$, so that $a_1^n a_2^{n-1} \cdots a_n = 1$. Then $(a_1, \ldots, a_n) = \zeta_1 \cdots \zeta_{n-1}$, where

$$\zeta_j = \prod_{v=j}^{n-1} \xi_{a_j^{j-v-1}, v}$$

Thus Ψ induces an isomorphism $\overline{\Psi} : (k^*)^{\oplus n}/\Xi \longrightarrow k^*$, hence $G_1(R) \cong k^*$ by Theorem 2.1.3.

$\square$

2.6.3 Singularity of Type A_{2n} in Dimension One

The ADE singularity of type A_{2n} is given by the ring $R = k[[t^2, t^{2n+1}]]$. Here we utilize Theorem 2.1.3 to perform the calculation in Example 2.6.3. That is, if the characteristic of k is not $2, 3$ or 5, then $G_1(R) \cong \overline{R}^* = k[[t]]^*$.

Proof. For $n = 0$, $R = k[[t]]$, a regular local ring, so that $G_1(R) \cong K_1(R) \cong R^* = k[[t]]^*$ by Quillen's Resolution Theorem ([57], §Theorem 3).

We now suppose $n \geq 1$. Now R has finite type and the indecomposable maximal Cohen-Macaulay R-modules are $R_j = k[[t^2, t^{2(n-j)+1}]]$, with $j = 0, \ldots, n$ by ([69], Proposition 5.11). Thus M is the R-module $R_0 \oplus R_1 \oplus \cdots \oplus R_n$ ($R_0 = R$). Let E be the endomorphism ring of M. By the proof of ([69], Proposition 5.11), the Auslander-Reiten sequence ending in R_j is

$$0 \longrightarrow R_j \longrightarrow R_{j-1} \oplus R_{j+1} \longrightarrow R_j \longrightarrow 0 \quad (1 \leq j < n)$$

$$0 \longrightarrow R_n \longrightarrow R_{n-1} \oplus R_n \longrightarrow R_n \longrightarrow 0$$

Thus the Auslander-Reiten matrix Υ, for $1 \leq j \leq n-1$, has jth column given by $(0, \ldots, -1, 2, -1, \ldots, 0)^T$, with $-1, 2$ and -1 occur in rows $j-1, j$ and $j+1$, respectively. The nth column is given by $(0, \ldots, 0, -1, 1)^T$. Now Υ is clearly injective, hence $G_1(R) \cong E^*_{\text{ab}}/\Xi$ by Theorem 2.1.3. We calculate the subgroup Ξ occurring of Definition 2.2.21. By Lemma 2.6.7, the subgroup Ξ is generated by the elements

$$\xi_{a,j} = \widetilde{a^2 1_{R_j}} \cdot \widetilde{a^{-1} 1_{R_{j-1}} \oplus a^{-1} 1_{R_{j+1}}} \qquad (1 \leq j < n)$$

$$\xi_{a,n} = \widetilde{a^2 1_{R_n}} \cdot \widetilde{a^{-1} 1_{R_{n-1}} \oplus a^{-1} 1_{R_n}}$$

Where a runs over k^*. We abuse notation and regard Ξ as a subgroup of $(k^*)^{\oplus(n+1)}$. We compute compute $(k^*)^{\oplus(n+1)}/\Xi$, viewing the elements of Ξ as a row vectors in $(k^*)^{\oplus(n+1)}$. Hence the elements that generate Ξ are given by

$$\xi_{a,j} = (1, \ldots, a^{-1}, a^2, a^{-1}, \ldots, 1) \qquad (1 \leq j < n)$$

$$\xi_{a,n} = (1, \ldots, a^{-1}, a)$$

Where a^{-1}, a^2 and a^{-1} occur in positions $j, j+1$ and $j+2$ for $1 \leq j < n$. Let $\chi : (k^*)^{\oplus(n+1)} \longrightarrow k^*$ be given by $\chi(a_1, \ldots, a_{n+1}) = a_1 \cdots a_{n+1}$. Then $\ker(\chi)$ is generated by elements of the form $(a_1, \ldots, a_{n+1})$ such that

$$(a_1, \ldots, a_{n+1}) = (a_2^{-1}, a_2, 1, \ldots, 1)(a_3^{-1}, 1, a_3, 1, \ldots, 1) \cdots (a_{n+1}^{-1}, 1, 1, \ldots, a_{n+1})$$

We show $\Xi = \ker(\chi)$. Obviously, $\Xi \subseteq \ker(\chi)$. For the converse, it suffices to show the elements $\zeta_{a,j} = (a^{-1}, 1 \ldots, a, \ldots, 1)$, where a is in the jth position and $2 \leq j \leq n+1$, are in Ξ. Indeed, note that $\zeta_{2,a} = \xi_{a,1}\xi_{a,2} \cdots \xi_{a,n}$ and for $j > 2$, we have $\zeta_{a,j} = \zeta_{a,j-1}\xi_{a,j-1}\xi_{a,j} \cdots \xi_{a,n}$. Thus $\ker(\chi) = \Xi$ as needed. Combining the above and using (b) of Proposition 2.5.1, we have

$$G_1(R) \cong (k^*)^{\oplus(n+1)}/\Xi \oplus (1 + tk[[t]]) \cong k^* \oplus (1 + tk[[t]]) \cong k[[t]]^*$$

□

2.6.4 Reduced Hypersurface Singularities in Dimension One

Our aim here is use Theorem 2.1.3 to perform the calculation in Example 2.6.4. We recall Example 2.6.4. We let $S = k[[x, y]]$, $f_1, \ldots, f_n \in (x, y)\backslash(x, y)^2$, with f_i irreducible, $f = f_1 \cdots f_n$, $R = S/fS$ is an isolated singularity (i.e. $f_iS \neq f_jS$), and $(f_i, f_{i+1}) = (x, y)$. Then $G_1(R) \cong \mathbb{Z}^{\oplus(n-1)} \oplus \overline{R}^*$.

Proof. Now $L = S_1 \oplus \cdots \oplus S_n$, with $S_i = S/(f_1 \cdots f_i)$, is a 2-cluster tilting object in **mcm** R. In order to compute $G_1(R)$, we need to understand the structure of the 2-Auslander-Reiten sequences in $\mathcal{C} = \mathbf{add}_R\, L$. By ([39], Proof of Theorem 4.11) the 2-Auslander-Reiten sequences ending in S_j are

$$0 \longrightarrow S_j \longrightarrow S_{j+1} \oplus S_{j-1} \longrightarrow S_{j+1} \oplus S_{j-1} \longrightarrow S_j \longrightarrow 0 \qquad (1 \leq j < n)$$

From this and Lemma 2.6.7 it is clear that the subgroup Ξ of $\mathrm{Aut}_R(L)_{\mathrm{ab}}$ is trivial. Moreover, from this, it is easy to see that the 2-Auslander-Reiten matrix $T : \mathbb{Z}^{\oplus(n-1)} \longrightarrow \mathbb{Z}^{\oplus n}$ is zero. Thus by Theorem 2.1.3 and (d) of Proposition 2.5.1

$$G_1(R) \cong \ker(T) \oplus \mathrm{Aut}_R(L)_{\mathrm{ab}} \cong \mathbb{Z}^{\oplus(n-1)} \oplus \overline{R}^*$$

$\square$

2.6.5 Singularity of Type A_1 in Dimension Two

Our aim here is to prove (a) of Proposition 2.6.6. That is, if R is the ring $k[[s^2, st, t^2]]$ then $G_1(R) \cong R^*$.

Proof. By ([47], Example 5.25 and 13.21) R has finite type and the indecomposable maximal Cohen-Macaulay R-modules are R and $I = (s^2, st)$. Moreover, the Auslander-Reiten sequence ending in I is given by

$$0 \longrightarrow I \longrightarrow R^2 \longrightarrow I \longrightarrow 0$$

Set $M = R \oplus I$ and let E be its endomorphism ring.

An easy calculation shows that the Auslander-Reiten homomorphism $\Upsilon : \mathbb{Z} \longrightarrow \mathbb{Z}^{\oplus 2}$ is injective. Now Ξ is the subgroup of E^*_{ab} generated by the elements

$$\widetilde{a1_I} \cdot \mathrm{det}_E(\widetilde{a1_{R^2}})^{-1} \cdot \widetilde{a1_I} = \widetilde{a^2 1_I} \cdot \mathrm{det}_E(\widetilde{a1_{R^2}})^{-1} \ \ (a \in k^*)$$

The automorphism of M, $\widetilde{a^2 1_I}$, is given by $\mathrm{diag}(1_R, a^2 1_I)$. Using Lemma 2.6.7, $\mathrm{det}_E(\widetilde{a1_{R^2}})$ is the image of the automorphism $\mathrm{diag}(a^2 1_R, 1_I)$ in E^*_{ab}. Thus Ξ is the subgroup of E^*_{ab} generated by the elements

$$\mathrm{diag}(1_R, a^2 1_I) \cdot \mathrm{diag}(a^{-2} 1_R, 1_I) = \mathrm{diag}(a^{-2} 1_R, a^2 1_I)$$

As groups, $\Xi \cong k^{*2} = \{a^2 : a \in k^*\}$. Since k is algebraically closed, $k^{*2} = k^*$. Using (c) of Proposition 2.5.1, we have $E^*_{\text{ab}} \cong k^* \oplus R^*$, hence $E^*_{\text{ab}}/\Xi \cong R^*$. Thus $G_1(R) \cong R^*$ by Theorem 2.1.3, since Υ is injective. $\square$

2.6.6 Reduced Hypersurface Singularities in Dimension Three

Our aim here is to prove (b) of Proposition 2.6.6. We recall (b). If $S' = k[[x, y, u, v]]$, $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n$ with $f_i \in (x, y)\backslash(x, y)^2 \subseteq S = k[[x, y]]$ are such that then $G_1(R') \cong \mathbb{Z}^{\oplus(n-1)} \oplus \left(R'^* \oplus k[[w, z]]^{*\oplus(n-1)}\right)/\Xi$, with Ξ the subgroup from Theorem 2.1.3 and w, z variables over k.

Proof. If $U_i = (u, f_1 \cdots f_i)$, then $L = U_1 \oplus \cdots \oplus U_n$ is a 2-cluster tilting object. Then by ([53], Proposition 7.28), the 2-Auslander-Reiten matrix T is zero. By (e) of Proposition 2.5.1, $\text{Aut}_{R'}(L)_{\text{ab}} \cong R'^* \oplus (k[[w, z]])^{*\oplus(n-1)}$ (w and z variables over k), thus Theorem 2.1.3 yields

$$G_1(R') \cong \mathbb{Z}^{\oplus(n-1)} \oplus \left(R'^* \oplus k[[w, z]]^{*\oplus(n-1)}\right)/\Xi$$

Where Ξ is the subgroup of $R'^* \oplus k[[w, z]]^{*\oplus(n-1)}$ of Definition 2.2.21.

$\square$

2.7 Discussion

It is of interest to note that in the calculations of $G_1(R)$ for R of positive dimension, either $G_1(R) \cong \overline{R}^*$ ($\overline{R}$ is the integral closure of R in its total quotient ring), or $G_1(R)$ contains $\overline{R}^*$ a direct summand. Our methods were ad hoc and tailored specifically to each singularity via the calculation of the group $\text{Aut}_R(L)_{\text{ab}}$, so a deeper look into the relationship between $\Lambda = \text{End}_R(L)^{\text{op}}$ and $\overline{R}$ could shed some light on the structure of $G_1(R)$ for hypersurface singularities.

In fact, the key to the relationship seems to be understanding the relationship between the derived categories of **mod** $\text{End}_R(L)^{\text{op}}$ and **mod** $\overline{R}$. Indeed, in [18], it is shown that if A and B are Noetherian (not necessarily commutative) rings whose derived categories are equivalent as triangulated categories, then there is an isomorphism $G_i(A) \cong G_i(B)$ for $i \geq 0$. Of course, one should not expect an equivalence of the derived categories of $\mathbf{mod}\,\mathrm{End}_R(L)^{\mathrm{op}}$ and $\mathbf{mod}\,\overline{R}$ since our examples (see Proposition 2.5.1) indicate for positive-dimensional rings that $G_1(\mathrm{End}_R(L)^{\mathrm{op}}) \cong \mathrm{Aut}_R(L)_{\mathrm{ab}}$ only contains $\overline{R}^*$ as a direct summand. Moreover, it may also be too much to ask that $G_1(\overline{R})$ is a direct summand of $G_1(\mathrm{End}_R(L)^{\mathrm{op}})$, as $G_1(\overline{R})$ is not always isomorphic to $\overline{R}^*$. However, if R is a reduced one-dimensional local Noetherian ring, then $\overline{R} = \overline{R/\mathfrak{p}_1} \times \cdots \times \overline{R/\mathfrak{p}_s}$, where the $\mathfrak{p}_j$ are the minimal primes of R and each ring $\overline{R/\mathfrak{p}_j}$ is a semilocal principal ideal domain. In this situation

$$G_1(\overline{R}) \cong G_1(\overline{R/\mathfrak{p}_1}) \times \cdots \times G_1(\overline{R/\mathfrak{p}_s})$$

Now $\overline{R/\mathfrak{p}_j}$ is semilocal and has finite global dimension, hence if R is an algebra over a field k with $\mathrm{char}(k) \neq 2$, then Quillen's Resolution Theorem ([57], §Theorem 3), ([60], Corollary 2.6 and Theorem 5.1), and ([66], Theorem 2) show there are isomorphisms

$$G_1(\overline{R/\mathfrak{p}_j}) \cong K_1(\overline{R/\mathfrak{p}_j}) \cong K_1^C(\overline{R/\mathfrak{p}_j}) = (\overline{R/\mathfrak{p}_j})^*$$

Thus $G_1(\overline{R}) \cong \overline{R}^*$ in this case. Nevertheless, we conjecture that if R satisfies the hypotheses of Theorem 2.1.3 and has positive dimension, then $\mathrm{Aut}_R(L)_{\mathrm{ab}}/\Xi \cong \overline{R}^*$ and hence $G_1(R)$ is isomorphic to the direct sum of the kernel of the n-Auslander-Reiten homomorphism and $\overline{R}^*$.

Chapter 3

The Weak Lefschetz for a Class of Finite Length Modules Over $\mathbb{K}[x, y, z]$

3.1 Introduction

Let[2] $\mathbb{K}$ be an algebraically closed field and S the polynomial ring $\mathbb{K}[x_0, \ldots, x_r]$ with standard grading and irrelevant maximal ideal $\mathfrak{m} = (x_0, \ldots, x_r)$. All S-modules considered are finitely generated. In particular, all Artinian S-modules have finite length. Thus if N is a graded finite length S-module, then $N = \oplus_{j \in \mathbb{Z}} N_j$, where all but finitely many of the N_j are nonzero and $\dim_{\mathbb{K}}(N_j) < \infty$. We begin with the following.

Definition 3.1.1. If N is a graded Artinian S-module, then we say that N has the *Weak Lefschetz Property* if there is a *general* linear form $\ell \in S_1$ such that the $\mathbb{K}$-linear map $\times \ell : N_j \longrightarrow N_{j+1}$ has maximal rank for all j.

Richard Stanley and others showed how the Weak Lefschetz Property, a property that is geometric and algebraic in nature, ties in with several interesting problems of a combinatorial nature [12,48,61,62]. In particular, Stanley utilized the property to complete the proof of McMullen's conjecture on the characterization of f-vectors of simplicial polytopes. In honor of the influential works of Stanley, the Weak Lefschetz Property is also referred to as the *Weak Stanley Property* in the literature. There has been a rich body of research establishing the existence or non-existence of the Weak Lefschetz Property for various types of Artinian algebras, in particular for Artinian Gorenstein algebras [5,32,34,37,68] and other Artinian algebras with special structure [49,51,70]. Within this rapidly growing body of research involving the Weak Lefschetz Property, we found the following survey type works to be very helpful [33,50].

[2]The main results in this chapter can be found in the paper [21], which is joint with Gioia Failla and Chris Peterson.

Throughout the remainder of this chapter, we focus on the case when $r = 2$, and write $R = \mathbb{K}[x, y, z]$.

There were two papers that played a major role in inspiring us to utilize an approaching using vector bundle techniques that ultimately led to a proof of our main result. The first was [34], which made use of the Grauert-Mülich theorem to gain further insight into the Weak Lefschetz Property of a height three complete intersection. The Grauert-Mülich theorem enabled them to pinpoint the generic splitting type of a stable, normalized, rank two vector bundle on $\mathbb{P}^2$ which enabled precise homological conclusions to be made. The second influential work for us was the paper by Brenner and Kaid [8] which made further use of the Grauert-Mülich theorem for higher rank bundles on $\mathbb{P}^2$ and solidified the connection between the generic splitting type of a bundle and the Weak Lefschetz Property.

It is very natural to study codimension three complete intersections via the Koszul complex. First of all, the Koszul complex is exact for complete intersections. Second, by sheafifying the Koszul complex, one can identify the first cohomology module of an associated rank two locally free sheaf as the Artinian module $R/(f_1, f_2, f_3)$, where f_1, f_2, f_3 is a regular sequence of homogeneous polynomials in R defining the complete intersection. A natural generalization can be obtained via the Buchsbaum-Rim complex associated to a graded R-linear map $\varphi : \mathbb{F} \to \mathbb{G}$ where $\mathbb{F}$ is a free R-module of rank $n+2$ and $\mathbb{G}$ is a free R-module of rank n. In particular, if the cokernel of φ is of codimension three, which over R corresponds to the cokernel being a module of finite length, then the Buchsbaum-Rim complex is exact. By sheafifying the Buchsbaum-Rim complex we can again identify the first cohomology module of an associated rank two locally free sheaf, $\mathcal{E}$, as the cokernel of φ. As in the papers [8, 34], it is crucial to understand the generic splitting type of $\mathcal{E}$ and its relationship to the multiplication between consecutive graded components of the cokernel of φ induced by a general linear form.

This chapter is broken into four sections. In section two of this paper we provide background meant to clarify the connection between the Buchsbaum-Rim complex for a certain class of finite length R-modules and rank 2 vector bundles on $\mathbb{P}^2$. The third section contains the statement and proofs of the main results of the paper. In particular, we show that the first cohomology module of any rank 2 vector bundle on $\mathbb{P}^2$ satisfies the Weak Lefschetz Property. The final section consists of examples, some potential paths for future research, and concluding remarks.

3.2 The Buchsbaum-Rim complex

Let $\mathbb{F} = \oplus_{j=1}^{n+2} R(-b_j)$, let $\mathbb{G} = \oplus_{i=1}^{n} R(-a_i)$, let $a = a_1 + \cdots + a_n$, and let $b = b_1 + \cdots + b_{n+2}$. Given a graded map of degree zero $\varphi : \mathbb{F} \to \mathbb{G}$ we have a kernel E, cokernel M and an exact sequence

$$0 \to E \to \mathbb{F} \to \mathbb{G} \to M \to 0. \tag{3.1}$$

In addition, we have a Buchsbaum-Rim complex associated to $\phi : \mathbb{F} \to \mathbb{G}$ [9, 10]. If the cokernel of ϕ has the "expected codimension", which in this case corresponds to requiring that M has finite length, then the Buchsbaum-Rim complex is exact and has the form (see Section 4.2.3 for details):

$$0 \to \mathbb{G}^\vee(a-b) \to \mathbb{F}^\vee(a-b) \to \mathbb{F} \to \mathbb{G} \to M \to 0 \tag{3.2}$$

This complex is one of a much larger family of complexes associated to sufficiently general maps between R-modules. These complexes are exact if a certain genericity condition is met and they can be derived by considering "strands" of a particular Koszul complex (see ([19], Appendix A2.6) for details).

If we sheafify (3.2) then we get an exact sequence of locally free sheaves

$$0 \to \mathcal{G}^\vee(a-b) \to \mathcal{F}^\vee(a-b) \to \mathcal{F} \to \mathcal{G} \to 0 \tag{3.3}$$

which can be decomposed into the two short exact sequences

$$0 \to \mathcal{G}^\vee(b-a) \to \mathcal{F}^\vee(a-b) \to \mathcal{E} \to 0 \tag{3.4}$$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0. \tag{3.5}$$

Note that (3.5) is also the sheafification of (3.1). The locally free sheaf $\mathcal{E}$ has rank two and is an example of a (first) Buchsbaum-Rim sheaf. The apparent symmetry of the Buchsbaum-Rim complex is closely related to the fact that a rank 2 locally free sheaf is self-dual (up to a twist by a line bundle). In general, the structure found in the Buchsbaum-Rim complex is reflected in properties of $\mathcal{E}$, in properties of its sections, and in properties of its cohomology modules [42, 52]. In particular, the rigidity of the Buchsbaum-Rim complex, when it is exact, suggests that properties of the objects involved reduce to combinatorial considerations of the a_i and b_j involved in the definitions of $\mathbb{F}$ and $\mathbb{G}$. In the next section, we will see that this is indeed the case. Let $H^0_*(\mathbb{P}^2, \mathcal{E})$ denote the module $\oplus_{i\in\mathbb{Z}} H^0(\mathbb{P}^2, \mathcal{E}(i))$. If we apply the global section functor to the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

we obtain the long exact sequence

$$0 \to H^0_*(\mathbb{P}^2, \mathcal{E}) \to H^0_*(\mathbb{P}^2, \mathcal{F}) \to H^0_*(\mathbb{P}^2, \mathcal{G}) \to H^1_*(\mathbb{P}^2, \mathcal{E}) \to H^1_*(\mathbb{P}^2, \mathcal{F}) \to \dots. \tag{3.6}$$

Note that $H^1_*(\mathbb{P}^2, \mathcal{F}) = 0$ since $\mathcal{F} = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^2}(-a_i)$ and that (3.6) is actually a recovery of (3.1). In particular, we have

$$H^1_*(\mathbb{P}^2, \mathcal{E}) = M.$$

In general, finite length R-modules that can be expressed as cokernels of maps of the form $\varphi : \oplus_{j=1}^{n+2} R(-b_j) \to \oplus_{i=1}^n R(-a_i)$ correspond to finite length modules of the form $H^1_*(\mathbb{P}^2, \mathcal{E})$, where $\mathcal{E}$ is a rank 2 locally free sheaf on $\mathbb{P}^2$.

3.3 Main Results

In this section, we collect the key definitions and theorems that form the heart of the paper. Many of the needed tools can be found in the books [36, 54].

Definition 3.3.1. Let $\mathcal{E}$ be a torsion free sheaf on $\mathbb{P}^n$. Let $c_1(\mathcal{E})$ denote the first Chern class of $\mathcal{E}$ and let $rank(\mathcal{E})$ denote its rank.

1) The *slope* of $\mathcal{E}$ is the rational number $\mu(\mathcal{E}) = c_1(\mathcal{E})/rank(\mathcal{E})$

2) $\mathcal{E}$ is said to be *stable* if for any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ the slopes satisfy $\mu(\mathcal{F}) < \mu(\mathcal{E})$

3) $\mathcal{E}$ is said to be *semistable* if for any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ the slopes satisfy $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$

4) $\mathcal{E}$ is *unstable* if it is not semistable.

In various contexts, the definition of stability given above is sometimes referred to by other names including slope stability, μ-stability, Mumford stability, or Mumford-Takemoto stability.

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ and let e denote the rank of $\mathcal{E}$. We say that $\mathcal{E}$ is a *normalized* bundle if $c_1(\mathcal{E}) \in \{-e+1, \ldots, 0\}$. In general, there exists a unique $a \in \mathbb{Z}$ such that $\mathcal{E}(a)$ is a normalized bundle. In particular, if $\mathcal{E}$ is a normalized rank 2 vector bundle, then $c_1(\mathcal{E}) \in \{-1, 0\}$. The following lemma is a quick application of the definition of stability (see Chapter II of [54] for a more detailed discussion of stability and Lemma 1.2.5 on pg. 166-167 for the statement and proof of the lemma).

Lemma 3.3.2. *Let $\mathcal{E}$ be an normalized rank 2 vector bundle on $\mathbb{P}^n$.*

1) $\mathcal{E}$ is stable if and only if $H^0(\mathbb{P}^n, \mathcal{E}) = 0$.

2) If $c_1(\mathcal{E}) = -1$ then $\mathcal{E}$ is semistable if and only if $\mathcal{E}$ is stable

3) If $c_1(\mathcal{E}) = 0$ then $\mathcal{E}$ is semistable if and only if $H^0(\mathbb{P}^n, \mathcal{E}(-1)) = 0$.

The following is the Grauert-Mülich Theorem for rank 2 bundles on $\mathbb{P}^n$. For a more detailed discussion of the Grauert-Mülich theorem and its role in the classification of vector bundles, see [29] for the original result or see ([54], Ch. 2, Sec. 2) for a general discussion of the splitting behavior of vector bundle and ([54], Corollary 2, pg. 206) for the specifics of the Grauert-Mülich theorem.

Proposition 3.3.3. *Let $\mathcal{E}$ be a semistable, normalized, rank 2 vector bundle on $\mathbb{P}^n$. Let L be a general line.*

1) *If $c_1(\mathcal{E}) = 0$ then the restriction to L splits as $\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.*

2) *If $c_1(\mathcal{E}) = -1$ then the restriction to L splits as $\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.*

Definition 3.3.4. If $\mathcal{E}$ is an unstable, normalized, rank 2 vector bundle on $\mathbb{P}^n$ then the largest a such that $H^0(\mathbb{P}^n, \mathcal{E}(-a)) \neq 0$ is called the *index of instability* of $\mathcal{E}$.

From the above lemma, if $\mathcal{E}$ is an unstable, normalized, rank 2 vector bundle on $\mathbb{P}^n$ and $c_1(\mathcal{E}) = 0$ then its index of instability is greater than zero. Similarly, if $c_1(\mathcal{E}) = -1$ then its index of instability is at least zero. If $\mathcal{E}$ is a vector bundle on $\mathbb{P}^2$, we can make a stronger statement:

Proposition 3.3.5. *Let $\mathcal{E}$ be an unstable, normalized, rank 2 vector bundle on $\mathbb{P}^2$. Let k be the index of instability of $\mathcal{E}$. Let L be a general line in $\mathbb{P}^2$. Then*

1) *Every nonzero section $s \in H^0(\mathbb{P}^2, \mathcal{E}(-k))$ is regular.*

2) *If $c_1(\mathcal{E}) = 0$ then $k > 0$ and $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$.*

3) *If $c_1(\mathcal{E}) = -1$ then $k \geq 0$ and $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$.*

Proof. Let s be a nonzero section in $H^0(\mathbb{P}^2, \mathcal{E}(-k))$. Using s we can build a short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E}(-k) \to \mathcal{Q}(-k) \to 0$$

which we can twist by $\mathcal{O}_{\mathbb{P}^2}(k)$ to get the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{E} \to \mathcal{Q} \to 0$$

If s is not regular (i.e. its vanishing locus is not of codimension 2 or greater), then the vanishing locus of s contains a curve component. This curve is of codimension 1 in $\mathbb{P}^2$ thus can be identified with a form $F \in R$. If we factor out F from s we obtain a nonzero section $s' \in H^0(\mathbb{P}^2, \mathcal{E}(-k-d))$ where d is the degree of F (see [2], Lem. 2, pg. 128). Since k is the largest integer such that $H^0(\mathbb{P}^2, \mathcal{E}(-k)) \neq 0$, we get a contradiction. Therefore s is regular.

Suppose first that $c_1(\mathcal{E}) = 0$. If $L = \mathbb{P}^1$ is a general line in $\mathbb{P}^2$ then L does not meet the zero locus of s. As a consequence, the restriction of the short exact sequence to L is still a short exact sequence and by Chern class considerations, the restriction of $\mathcal{Q}$ to L is $\mathcal{O}_{\mathbb{P}^1}(-k)$. Thus, restricting the exact sequence to L leads to

$$0 \to \mathcal{O}_{\mathbb{P}^1}(k) \to \mathcal{E}|_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-k) \to 0.$$

Since $\mathcal{E}$ has rank 2, is unstable, and has $c_1 = 0$, we know that $H^0(\mathbb{P}^2, \mathcal{E}(-1)) \neq 0$ thus we can conclude that $k > 0$. Using this fact, we can conclude that $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-k), \mathcal{O}_{\mathbb{P}^1}(k)) = 0$. As a consequence, $\mathcal{E}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$.

Now suppose that $c_1(\mathcal{E}) = -1$. Like before, the restriction of the short exact sequence to L is still a short exact sequence except now, by Chern class considerations, the restriction of $\mathcal{Q}$ to L is $\mathcal{O}_{\mathbb{P}^1}(-k-1)$. Thus we get the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(k) \to \mathcal{E}|_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-k-1) \to 0.$$

Since $\mathcal{E}$ has rank 2, is unstable, and has $c_1 = -1$, we know that $H^0(\mathbb{P}^2, \mathcal{E}) \neq 0$ thus we can conclude that $k \geq 0$. Using this fact, we can conclude that $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-k-1), \mathcal{O}_{\mathbb{P}^1}(k)) = 0$. As a consequence, $\mathcal{E}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$. $\square$

Proposition 3.3.6. *If $\mathcal{E}$ is an unstable, normalized, rank 2 vector bundle on $\mathbb{P}^2$ with index of instability k then*

- *If* $c_1(\mathcal{E}) = 0$ *then* $h^0(\mathbb{P}^2, \mathcal{E}(t)) = \binom{k+t+2}{2}$ *for* $t < k$.

- *If* $c_1(\mathcal{E}) = -1$ *then* $h^0(\mathbb{P}^2, \mathcal{E}(t)) = \binom{k+t+2}{2}$ *for* $t \leq k$.

Proof. Consider the exact sequence

$$0 \to \mathcal{E}(t-1) \to \mathcal{E}(t) \to \mathcal{E}(t)|_L \to 0.$$

If we apply the global section functor we get the exact sequence

$$0 \to H^0(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^0(\mathbb{P}^2, \mathcal{E}(t)) \to H^0(\mathbb{P}^2, \mathcal{E}(t)|_L) \to \dots$$

From this exact sequence, we have

$$h^0(\mathbb{P}^2, \mathcal{E}(t)) \leq h^0(\mathbb{P}^2, \mathcal{E}(t-1)) + h^0(\mathbb{P}^2, \mathcal{E}(t)|_L).$$

If L is a general line then from Proposition 3.3.5 we have that

$$\text{if } c_1(\mathcal{E}) = 0 \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k+t) \oplus \mathcal{O}_{\mathbb{P}^1}(k+t))$$
$$\text{if } c_1(\mathcal{E}) = -1 \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k-1+t) \oplus \mathcal{O}_{\mathbb{P}^1}(k+t))$$

As a consequence

$$\text{if } c_1(\mathcal{E}) = 0 \text{ and if } t < 2k \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(-k+t)|_L) = max\{0, t+1\}$$
$$\text{if } c_1(\mathcal{E}) = -1 \text{ and if } t \leq 2k \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(-k+t)|_L) = max\{0, t+1\}$$

Since there exists a nonzero section $s \in H^0(\mathbb{P}^2, \mathcal{E}(-k))$, we can tensor this section by forms of degree t and produce sections in $H^0(\mathbb{P}^2, \mathcal{E}(-k+t))$. As a consequence, we have

$$h^0(\mathbb{P}^2, \mathcal{E}(t)) \geq \binom{k+t+2}{2} \text{ or equivalently } h^0(\mathbb{P}^2, \mathcal{E}(-k+t)) \geq \binom{t+2}{2}$$

We can now establish the claim of the proposition by an inductive approach. In the interest of space, we let $h^0(\mathcal{E})$ denote $h^0(\mathbb{P}^2, \mathcal{E})$. Recalling that $h^0(\mathcal{E}(-k-1)) = 0$ and that $h^0(\mathbb{P}^2, \mathcal{E}(-k+t)|_L) = t+1$ (provided t is in the proper range) we have the following inequalities:

$$1 \leq h^0(\mathcal{E}(-k+0)) \leq h^0(\mathcal{E}(-k-1)) + h^0(\mathcal{E}(-k+0)|_L) = 0+1 = 1$$

$$3 \leq h^0(\mathcal{E}(-k+1)) \leq h^0(\mathcal{E}(-k+0)) + h^0(\mathcal{E}(-k+1)|_L) = 1+2 = 3$$

$$6 \leq h^0(\mathcal{E}(-k+2)) \leq h^0(\mathcal{E}(-k+1)) + h^0(\mathcal{E}(-k+2)|_L) = 3+3 = 6$$

$$\cdots$$

$$\tbinom{t+2}{2} \leq h^0(\mathcal{E}(-k+t)) \leq h^0(\mathcal{E}(-k+t-1)) + h^0(\mathcal{E}(-k+t)|_L) = \tbinom{t+1}{2} + t + 1 = \tbinom{t+2}{2}$$

$$\cdots$$

For $c_1(\mathcal{E}) = 0$, following the inequalities through one at a time leads to the constraint

$$\binom{t+2}{2} \leq h^0(\mathcal{E}(-k+t)) \leq \binom{t+2}{2} \text{ for } t < 2k$$

or equivalently

$$\binom{k+t+2}{2} \leq h^0(\mathcal{E}(t)) \leq \binom{k+t+2}{2} \text{ for } t < k.$$

Thus we conclude that

$$\text{if } c_1(\mathcal{E}) = 0 \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(t)) = \binom{k+t+2}{2} \text{ for } t < k.$$

In a similar manner, we can also conclude that

$$\text{if } c_1(\mathcal{E}) = -1 \text{ then } h^0(\mathbb{P}^2, \mathcal{E}(t)) = \binom{k+t+2}{2} \text{ for } t \leq k.$$

$\square$

Theorem 3.3.7. *Let $\mathcal{E}$ be a normalized, rank 2, locally free sheaf on $\mathbb{P}^2$. Let $\ell \in R$ be a general linear form. Let $H^1_*(\mathbb{P}^2, \mathcal{E}) = \oplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^2, \mathcal{E}(t))$. Let $\varphi_\ell(t) : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ be the linear map induced by ℓ.*

1) $H^1_*(\mathbb{P}^2, \mathcal{E})$ has the Weak Lefschetz Property

2) Let $\mathcal{E}$ be stable.

- *If $c_1(\mathcal{E}) = 0$ then $\varphi_\ell(t)$ is injective for $t \le -1$ and surjective for $t \ge -1$*
- *If $c_1(\mathcal{E}) = -1$ then $\varphi_\ell(t)$ is injective for $t \le -1$ and surjective for $t \ge 0$.*

3) Let $\mathcal{E}$ be unstable with index of instability k.

- *If $c_1(\mathcal{E}) = 0$ then $\varphi_\ell(t)$ is injective for $t \le k-1$ and surjective for $t \ge -k-1$*
- *If $c_1(\mathcal{E}) = -1$ then $\varphi_\ell(t)$ is injective for $t \le k$ and surjective for $t \ge -k-1$*

Proof. In order to prove the theorem, we will first prove 2) and 3) which immediately imply 1).

We denote by L the general plane defined by ℓ. Consider the short exact sequence of sheaves

$$0 \to \mathcal{E}(t-1) \to \mathcal{E}(t) \to \mathcal{E}(t)|_L \to 0. \tag{3.7}$$

If we apply the global section functor we get the long exact sequence

$$\begin{aligned}
0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{E}(t-1)) &\longrightarrow H^0(\mathbb{P}^2, \mathcal{E}(t)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{E}(t)|_L) \\
\longrightarrow H^1(\mathbb{P}^2, \mathcal{E}(t-1)) &\longrightarrow H^1(\mathbb{P}^2, \mathcal{E}(t)) \longrightarrow H^1(\mathbb{P}^2, \mathcal{E}(t)|_L) \\
\longrightarrow H^2(\mathbb{P}^2, \mathcal{E}(t-1)) &\longrightarrow H^2(\mathbb{P}^2, \mathcal{E}(t)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0
\end{aligned}$$

To show that $H^1_*(\mathbb{P}^2, \mathcal{E})$ has the Weak Lefschetz Property, we need to show that for each $t \in \mathbb{Z}$, the map $H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ is either injective or surjective. From the long exact sequence above, we have the following observations:

- The map is injective if and only if $h^0(\mathbb{P}^2, \mathcal{E}(t-1)) - h^0(\mathbb{P}^2, \mathcal{E}(t)) + h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$.
- The map is injective if $h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$.
- The map is surjective if and only if $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) - h^2(\mathbb{P}^2, \mathcal{E}(t-1)) + h^2(\mathbb{P}^2, \mathcal{E}(t)) = 0$.
- The map is surjective if $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$.

If the generic splitting type of $\mathcal{E}$ is $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ then, by Serre Duality, $h^1(\mathbb{P}^2, \mathcal{E}|_L) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-b-2))$ and $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a-t-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-b-t-2))$. As a consequence, we can easily compute the value of $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L)$. In particular, if $-a-t-2 \leq -1$ and $-b-t-2 \leq -1$ then $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$. We collect the following facts:

A) Since $\mathcal{E}$ is locally free on $\mathbb{P}^2$, by duality we have $h^2(\mathbb{P}^2, \mathcal{E}(t)) = h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-3))$.

B) If we restrict $\mathcal{E}$ to a general line L we have $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-2)|_L)$.

C) Since $\mathcal{E}$ has rank two, if $c_1(\mathcal{E}) = 0$ then $\mathcal{E}^\vee \cong \mathcal{E}$ and if $c_1(\mathcal{E}) = -1$ then $\mathcal{E}^\vee \cong \mathcal{E}(1)$.

We now assume that $\mathcal{E}$ is stable and use the above considerations to establish a range of values of t where the map, $\varphi_\ell(t) : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$, is injective and a range of values where the map is surjective. It is important to note that the following shows that for *every* value of t, the map is either injective or surjective.

Suppose $\mathcal{E}$ is stable and that $c_1(\mathcal{E}) = 0$. By Proposition 3.3.3, $\mathcal{E}$ splits on L as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. In this case, $h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$ for $t \leq -1$ and $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$ for $t \geq -1$. Thus $\varphi_\ell(t)$ is injective for $t \leq -1$ and surjective for $t \geq -1$.

Suppose $\mathcal{E}$ is stable and that $c_1(\mathcal{E}) = -1$. By Proposition 3.3.3, $\mathcal{E}$ splits on L as $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. In this case, $h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$ for $t \leq -1$ and $h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0$ for $t \geq 0$.

Suppose $\mathcal{E}$ is unstable and that $c_1(\mathcal{E}) = 0$. If the index of instability is k then by Proposition 3.3.5, $k > 0$ and $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$. In this case, Proposition 3.3.6 allows us to conclude that

$$h^0(\mathbb{P}^2, \mathcal{E}(t-1)) - h^0(\mathbb{P}^2, \mathcal{E}(t)) + h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0 \text{ for } t \leq k-1.$$

This implies that $\varphi_\ell(t)$ is injective for $t \leq k-1$. Using A) and B) above, we note that

$$h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) - h^2(\mathbb{P}^2, \mathcal{E}(t-1)) + h^2(\mathbb{P}^2, \mathcal{E}(t))$$

can be expressed as

$$h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-2)|_L) - h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-2)) + h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-3)).$$

Using C) and rearranging, we can then express this as

$$h^0(\mathbb{P}^2, \mathcal{E}(-t-3) - h^0(\mathbb{P}^2, \mathcal{E}(-t-2)) + h^0(\mathbb{P}^2, \mathcal{E}(-t-2)|_L).$$

By Proposition 3.3.6 this quantity is equal to 0 for $-t-2 \leq k-1$. In other words, $\varphi_\ell(t)$ is surjective for $-k-1 \leq t$.

Suppose $\mathcal{E}$ is unstable and that $c_1(\mathcal{E}) = -1$. If the index of instability is k then by Proposition 3.3.5, $k \geq 0$ and $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$. In this case, Proposition 3.3.6 allows us to conclude that

$$h^0(\mathbb{P}^2, \mathcal{E}(t-1)) - h^0(\mathbb{P}^2, \mathcal{E}(t)) + h^0(\mathbb{P}^2, \mathcal{E}(t)|_L) = 0 \text{ for } t \leq k.$$

This implies that $\varphi_\ell(t)$ is injective for $t \leq k$. Using A) and B) above, we note that

$$h^1(\mathbb{P}^2, \mathcal{E}(t)|_L) - h^2(\mathbb{P}^2, \mathcal{E}(t-1)) + h^2(\mathbb{P}^2, \mathcal{E}(t))$$

can be expressed as

$$h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-2)|_L) - h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-2)) + h^0(\mathbb{P}^2, \mathcal{E}^\vee(-t-3)).$$

Using C) and rearranging, we can then express this as

$$h^0(\mathbb{P}^2, \mathcal{E}(-t-2) - h^0(\mathbb{P}^2, \mathcal{E}(-t-1)) + h^0(\mathbb{P}^2, \mathcal{E}(-t-1)|_L).$$

By Proposition 3.3.6 this quantity is equal to 0 for $-t-1 \leq k$. In other words, $\varphi_\ell(t)$ is surjective for $-k-1 \leq t$.

In each of these cases, we see that for each $t \in \mathbb{Z}$, the map $H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ is either injective or surjective. Thus $H^1_*(\mathbb{P}^2, \mathcal{E})$ has the Weak Lefschetz Property for any rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$. □

Corollary 3.3.8. *If f_1, f_2, f_3 is a regular sequence of homogeneous polynomials in R, then $R/(f_1, f_2, f_3)$ has the Weak Lefschetz Property.*

Corollary 3.3.9. *If $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$ then $H^1_*(\mathbb{P}^2, \mathcal{E})$ is unimodal.*

Proof. In the proof of Theorem 3.3.7, we saw that for any rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$, there exists an r such that for $t < r$ the map $\times\ell : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ is injective and for $t \geq r$ the map $\times\ell : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ was surjective. This fact establishes unimodality. □

3.4 An Example and Further Remarks

In this section, we first give an example to illustrate the theorems of the paper and the structure of the Buchsbaum-Rim complexes. In each of the following two examples, the associated locally free sheaf is unstable. After giving the two examples, we conclude the paper with a few remarks and considerations for possible further research.

Example 3.4.1. Consider a map $\varphi : R(-7) \oplus R(-2)^3 \to R(-1) \oplus R$ whose cokernel is a finite length module M. An elementary computation show that $M = M_0 \oplus \cdots \oplus M_9$ has Hilbert function $(1, 4, 6, 7, 7, 7, 7, 6, 4, 1)$. The Buchsbaum-Rim complex associated to φ is:

$$0 \to R(-12) \oplus R(-11) \to R(-10)^3 \oplus R(-5) \to R(-7) \oplus R(-2)^3 \to R(-1) \oplus R \to M \to 0 \tag{3.8}$$

If we sheafify (3.8) and tensor by $\mathcal{O}_{\mathbb{P}^2}(6)$ we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-4)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)^3 \to \mathcal{O}_{\mathbb{P}^2}(5) \oplus \mathcal{O}_{\mathbb{P}^2}(6) \to 0 \tag{3.9}$$

We can break (3.9) into two short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-4)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{E} \to 0 \tag{3.10}$$

and

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(4)^3 \to \mathcal{O}_{\mathbb{P}^2}(5) \oplus \mathcal{O}_{\mathbb{P}^2}(6) \to 0 \tag{3.11}$$

where $\mathcal{E}$ is a normalized rank 2 locally free sheaf with $c_1(\mathcal{E}) = 0$. From the exact sequence (3.10), we see that $H^0(\mathbb{P}^2, \mathcal{E}(-1)) > 0$ and $H^0(\mathbb{P}^2, \mathcal{E}(-2)) = 0$. Therefore $\mathcal{E}$ is unstable with index of instability $k = 1$. By Theorem 3.3.7, $\varphi_\ell(t)$ is injective for $t \leq 0$ and surjective for $t \geq -2$ This corresponds to saying that the map $\times \ell : M_{d-1} \to M_d$ is injective for $d \leq 6$ and surjective for

$d \geq 4$. Note that this implies bijectivity for $4 \leq d \leq 6$ thus M_3, M_4, M_5 and M_6 all have the same dimension. Further note that for every value of d, the map $\times \ell : M_{d-1} \to M_d$ is either injective or surjective, thus M has the Weak Lefschetz Property.

Example 3.4.2. Consider a map $\varphi : R(-8) \oplus R(-2)^4 \to R(-1) \oplus R^2$ whose cokernel is a finite length module M. An elementary computation show that $M = M_0 \oplus \cdots \oplus M_{12}$ has Hilbert function $(2, 7, 11, 14, 16, 17, 17, 17, 16, 14, 11, 7, 2)$. The Buchsbaum-Rim complex associated to φ is:

$$0 \to R(-15)^2 \oplus R(-14) \to R(-13)^4 \oplus R(-7) \to R(-8) \oplus R(-2)^4 \to R(-1) \oplus R^2 \to M \to 0 \tag{3.12}$$

If we sheafify (3.12) and tensor by $\mathcal{O}_{\mathbb{P}^2}(7)$ we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-8)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-7) \to \mathcal{O}_{\mathbb{P}^2}(-6)^4 \oplus \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(5)^4 \to \mathcal{O}_{\mathbb{P}^2}(6) \oplus \mathcal{O}_{\mathbb{P}^2}(7)^2 \to 0 \tag{3.13}$$

We can break (3.13) into two short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-8)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-7) \to \mathcal{O}_{\mathbb{P}^2}(-6)^4 \oplus \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to 0 \tag{3.14}$$

and

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(5)^4 \to \mathcal{O}_{\mathbb{P}^2}(6) \oplus \mathcal{O}_{\mathbb{P}^2}(7)^2 \to 0 \tag{3.15}$$

where $\mathcal{E}$ is a normalized rank 2 locally free sheaf with $c_1(\mathcal{E}) = -1$. From exact sequence (3.14), we see that $H^0(\mathbb{P}^2, \mathcal{E}) > 0$ and $H^0(\mathbb{P}^2, \mathcal{E}(-1)) = 0$. Therefore $\mathcal{E}$ is unstable with index of instability $k = 0$. By Theorem 3.3.7, $\varphi_\ell(t)$ is injective for $t \leq 0$ and surjective for $t \geq -1$ This corresponds to saying that the map $\times \ell : M_{d-1} \to M_d$ is injective for $d \leq 7$ and surjective for $d \geq 6$. Note that this implies bijectivity for $6 \leq d \leq 7$ thus M_5, M_6, M_7 all have the same dimension. Further note that for every value of d, the map $\times \ell : M_{d-1} \to M_d$ is either injective or surjective thus M has the Weak Lefschetz Property.

In this paper, we have shown that the first cohomology module of a rank two locally free sheaf on $\mathbb{P}^2$ must have the Weak Lefschetz Property. This is equivalent to showing that if M is a finite length module arising as the cokernel of a map of the form $\varphi : \mathbb{F} \to \mathbb{G}$ with $\mathbb{F} = \oplus_{j=1}^{n+2} R(-b_j)$ and $\mathbb{G} = \oplus_{i=1}^{n} R(-a_i)$, then M has the Weak Lefschetz Property. As a special case, every codimension three complete intersection has the Weak Lefschetz Property (proved first in [34] and proved again in [8]).

The key piece needed in the proofs of the main theorems is that $\mathcal{E}$ is a rank two locally free sheaf on a surface. Many of the key conclusions ultimately follow from this fact. This suggests that there may be generalizations of Theorem 3.3.7 to the case of rank two locally free sheaves on weighted projective planes and on $\mathbb{P}^1 \times \mathbb{P}^1$. We note the interesting paper by Harima and Watanabe where they considered the strong Lefschetz property for Artinian algebras with non-standard grading [35]. It is hoped that additional progress may be made in the understanding of Lefschetz Properties by considering the more general problem for modules over rings with a non-standard grading.

Chapter 4

Algebraic and Geometric Properties Associated to the Weak Lefschetz for Finite Length Modules

4.1 Introduction

Let[3] $\mathbb{K}$ be an algebraically closed field and S the polynomial ring $\mathbb{K}[x_0, \ldots, x_r]$ with standard grading and irrelevant maximal ideal $\mathfrak{m} = (x_0, \ldots, x_r)$. All S-modules considered are finitely generated. In particular, all Artinian S-modules have finite length.

We also set $R = \mathbb{K}[x, y, z]$ and let $\varphi : \mathbb{F} \to \mathbb{G}$ with $\mathbb{F} = \oplus_{j=1}^{n+2} R(-b_j)$ and $\mathbb{G} = \oplus_{i=1}^{n} R(-a_i)$ be an R-linear map with Artinian cokernel M.

The original motivation for the work in this chapter grew out of wanting to generalize the main result of [34] that complete intersections in R have the Weak Lefschetz Property, by showing that M has the Weak Lefschetz Property. We were partially successful in this direction (see Theorem 4.4.3), as there were restrictions on the a_i and b_j; these restrictions were removed in [21] (see Theoremch 3.3.7).

However, in attempting to prove Theorem 4.4.3, we spent a significant time discussing when M has symmetric and unimodal Hilbert function. As is well-known, complete intersections are Gorenstein, hence have symmetric Hilbert functions. There is not a widely-used analogue for the Gorenstein condition for modules of finite length, however, there is a proposed analogue defined in [43] (see Definition 4.3.4) that suits our needs perfectly. Using [43], we are able to determine when M has symmetric Hilbert function (see Proposition 4.3.9). Moreover, using this, we are able to determine when M has unimodal Hilbert function (see Proposition 4.5.3). While the use of such results was to determine when M has the Weak Lefschetz, we find they are of independent interest.

[3] The main results in this paper are taken from the paper [24].

Inspired by [7], we define and discuss the *non-Lefschetz locus* for an Artinian graded S-module N. To wit, given an Artinian S-module $N = \bigoplus_{j\in\mathbb{Z}} N_j$, the S-module structure of N is determined by a sequence of $\mathbb{K}$-linear maps $\phi_j : S_1 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(N_j, N_{j+1})$. In particular, given a linear form $\ell = t_0x_1 + \cdots + t_rx_r$, $\phi_j(\ell)$ is a matrix X_j of linear forms in $t_0, \ldots, t_r$. Regarding $t_0, \ldots, t_r$ as variables, we look at the scheme defined by the vanishing of the maximal minors of the matrix X_j, and this is our object of study. In particular, we discuss some issues that are raised when attempting to generalize results of [7], but make use of some connections with results on *Artin level modules* from [6], that we also find are of independent interest.

This chapter is organized as follows: In Section 4.2, we compute the minimal free resolution of a M. This is essential for Section 4.3, where we discuss symmetry and unimodality properties of M, most notably using an analogue of the Gorenstein condition for Artinian modules defined in [43]. In Section 2.4, we discuss when the R-module M has the Weak Lefschetz, recover ([34], Theorem 2.3), and give an example a family of non-cyclic R-modules that have the Weak Lefschetz Property. In Section 4.5, we discuss the *non-Lefschetz locus* for a graded S-module N and give some generalizations from work in [7]. Most importantly, we discuss what conditions we can place on N so that is the non-Lefschetz locus is given by at most two degrees, and, in some cases, a single degree.

4.2 The Minimal Free Resolution of M

Our setup for this section is as follows: R is the polynomial ring $\mathbb{K}[x, y, z]$, where $\mathbb{K}$ is algebraically closed (we will restrict the characteristic when necessary); $n > 0$ is a positive integer; φ is a degree zero graded homomorphism of free R-modules from $\bigoplus_{j=1}^{n+2} R(-b_j)$ to $\bigoplus_{i=1}^{n} R(-a_i)$ with finite length cokernel M such that $b_1 \le b_2 \le \cdots \le b_{n+2}$ and $a_1 \le a_2 \le \cdots \le a_n$; the map $\varphi = (\varphi_{ij})$ is such that either $\varphi_{ij} = 0$ or $\varphi_{ij} \in R_{e_{ij}}$ with $e_{ij} > 0$; and if I denotes the ideal generated by the $n \times n$ minors of φ, then I has codimension 3, as M has finite length .

Since I has codimension 3, by ([19], Theorem A.210), the Buchsbaum-Rim complex provides the minimal free resolution of M. That is, there is an exact sequence

$$\mathbb{F}_\bullet : 0 \longrightarrow \bigoplus_{i=1}^{n} R(-d_i) \xrightarrow{\delta} \bigoplus_{j=1}^{n+2} R(-c_j) \xrightarrow{\varepsilon} \bigoplus_{j=1}^{n+2} R(-b_j) \xrightarrow{\varphi} \bigoplus_{i=1}^{n} R(-a_i) \longrightarrow M \longrightarrow 0$$

where the entries of all maps are in $\mathfrak{m}$. In this section, we determine the values of the c_j and d_i. To do so, we first need information about the maps ε and δ. Before we proceed, we note the following lemma that will be used frequently in the sequel.

Lemma 4.2.1. *If $\varphi : \bigoplus_{j=1}^{n+2} R(-b_j) \longrightarrow \bigoplus_{i=1}^{n} R(-a_i)$ is as above, then $b_i > a_i$ for $i = 1, \dots, n$.*

Proof. Suppose not. Then there is an i such that $b_i \leq a_i$. We recall that $b_1 \leq \cdots \leq b_{n+2}$ and $a_1 \leq \cdots \leq a_n$, hence this implies that if $u \leq i$ and $v \geq i$, then $b_u \leq a_v$. In particular, φ contains a zero submatrix of size $(n - i + 1) \times i$. Let $t(\varphi)$ denote the half-perimeter of this zero submatrix, so that $t(\varphi) = n + 1$. Then ([28], Théorème 1.6.2) says that the codimension of I is at most $n + 3 - t(\varphi)$. In particular, I has codimension at most 2, contrary to our assumption.

□

4.2.1 The map ε

For ease of notation, set $\mathbb{F}_1 = \bigoplus_{j=1}^{n+2} R(-b_j)$ and $\mathbb{F}_2 = \bigoplus_{j=1}^{n+2} R(-c_j)$. Let $f_{11}, \dots, f_{1,n+2}$ be a basis for $\mathbb{F}_1$ and $f_{21}, \dots, f_{2,n+2}$ be a basis for $\mathbb{F}_2$. Then by ([19], Section A2.6.1) ε is the map such that

$$\varepsilon(f_{2j}) = \sum_{K_{pj} \subset H_j} \text{sgn}(K_{pj} \subset H_j) \det(\varphi_{K_{pj}}) f_{1p}$$

Where for $j = 1, \dots, n+2$, $H_j := \{1, \dots, n+2\} \setminus \{j\}$; for $p \in H_j$, $K_{pj} = H_j \setminus \{p\}$; $\varphi_{K_{pj}}$ is the the $n \times n$ submatrix of φ indexed by the elements of K_{pj}; and $\text{sgn}(K_{pj} \subset H_j)$ is the sign of the permutation of H_j that puts the elements of K_{pj} into the first n positions of H_j. Thus the jth column of a matrix ε is given by

$$\begin{bmatrix} \operatorname{sgn}(K_{1j} \subset H_j) \det(\varphi_{K_{1j}}) \\ \vdots \\ \operatorname{sgn}(K_{j-1,j} \subset H_j) \det(\varphi_{K_{j-1,j}}) \\ 0 \\ \operatorname{sgn}(K_{j+1,j} \subset H_j) \det(\varphi_{K_{j+1,j}}) \\ \vdots \\ \operatorname{sgn}(K_{n+2,j} \subset H_j) \det(\varphi_{K_{n+2,j}}) \end{bmatrix}$$

Noting the 0 occurs in the jth row. When $1 \leq p < j$, it is not hard to see that $\operatorname{sgn}(K_{pj} \subset H_j) = (-1)^{n-p+1}$. Now for $j < p \leq n+2$, it is also easy to see we have $\operatorname{sgn}(K_{pj} \subset I_j) = (-1)^{n-p+2} = (-1)^{n-p}$. If $\Phi_{pj} = \det(\varphi_{K_{pj}})$, then the jth column of the matrix of ε is given by

$$\begin{bmatrix} (-1)^n \Phi_{1j} \\ \vdots \\ (-1)^{n+2-j} \Phi_{j-1,j} \\ 0 \\ (-1)^{n+1-j} \Phi_{j+1,j} \\ \vdots \\ \Phi_{n+2,j} \end{bmatrix}$$

4.2.2 The map δ

For ease of notation, set $\mathbb{F}_3 = \bigoplus_{i=1}^{n} R(-d_i)$ and let $f_{31}, \ldots, f_{3n}$ be a basis for $\mathbb{F}_3$. By ([19], Section A.2.6.1) the map $\delta : \mathbb{F}_3 \longrightarrow \mathbb{F}_2$ is such that

$$f_{3i} \mapsto \sum_{j=1}^{n+2} (-1)^{j+1} \varphi_{ij} f_{2j}$$

In particular, the ith column of the matrix for δ is given by

$$\begin{bmatrix} \varphi_{i1} \\ -\varphi_{i2} \\ \vdots \\ (-1)^{j+1}\varphi_{ij} \\ \vdots \\ (-1)^{n+2}\varphi_{i,n+1} \\ (-1)^{n+3}\varphi_{i,n+2} \end{bmatrix}$$

so a matrix for δ is given by

$$\begin{bmatrix} \varphi_{11} & \varphi_{21} & \cdots & \varphi_{n1} \\ -\varphi_{12} & -\varphi_{22} & \cdots & -\varphi_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{n+2}\varphi_{1,n+1} & (-1)^{n+2}\varphi_{2,n+1} & \cdots & (-1)^{n+2}\varphi_{n,n+1} \\ (-1)^{n+3}\varphi_{1,n+2} & (-1)^{n+3}\varphi_{2,n+2} & \cdots & (-1)^{n+3}\varphi_{n,n+2} \end{bmatrix}$$

4.2.3 Computing the c_j and d_i

We first calculate the degrees of the Φ_{pj}. This follows from the following general lemma, which is probably well-known, but we could not find an exact source.

Lemma 4.2.2. *Let $S = \mathbb{K}[x_0, \ldots, x_r]$ and $\alpha : \bigoplus_{i=1}^{t} S(-v_i) \longrightarrow \bigoplus_{i=1}^{t} S(-u_i)$ be a homogeneous S-linear map such that $v_i > u_i$ for all i. If $\alpha = (\alpha_{ij})$, where either $\alpha_{ij} = 0$ or $\alpha_{ij} \in S_{t_{ij}}$ with $t_{ij} > 0$, we denote the determinant of α by Φ and assume Φ is nonzero. Then Φ is homogeneous of degree $\sum_{i=1}^{t}[v_i - u_i]$.*

Proof. Before we begin, notice that if α_{ij} is nonzero, then $\deg(\alpha_{ij}) = t_{ij} = v_j - u_i > 0$.

We proceed by induction on t. For $t = 1$, this is just the statement that a graded map $S(-v_1) \longrightarrow S(-u_1)$ is given by multiplication of a homogeneous element of S of degree $v_1 - u_1$. This is easy to see. Suppose that $t > 1$ and write

$$\Phi = \alpha_{11}\Phi_1 - \alpha_{12}\Phi_2 \cdots + (-1)^{t+1}\alpha_{1t}\Phi_t$$

Where Φ_i is the determinant of the $(t-1) \times (t-1)$ submatrix of α obtained by deleting the first row and the ith column. By hypothesis, Φ is nonzero, so that there is an h such that both α_{1h} and Φ_h are nonzero. In this case, note that Φ_h is the determinant of a homogeneous linear map from $\bigoplus_{j \neq h} S(-v_j)$ to $\bigoplus_{i \neq 1} S(-u_i)$. The induction hypothesis gives that Φ_h is homogeneous of degree $\sum_{j \neq h} v_j - \sum_{i \neq 1} u_i$, hence $\alpha_{1h}\Phi_h$ is homogeneous of degree $\sum_{i=1}^{t}[v_i - u_i]$, as needed. This gives that Φ is homogeneous of the required degree. $\square$

Set $d = \sum_{j=1}^{n+2} b_j - \sum_{i=1}^{n} a_i$, so that we have the following:

Corollary 4.2.3. *Let Φ_{pj} be the maximal minor of φ corresponding to the set $K_{pj} = H_j \setminus \{p\} = \{1, \ldots, n+2\} \setminus \{p, j\}$ (so that Φ_{pj} is the minor of φ obtained by deleting columns p and j of φ). If Φ_{pj} is nonzero, then the degree of Φ_{pj} is $d - b_p - b_j$.*

Suppose for $1 \leq j \leq n+2$ that there is an $p \in H_j$ such that $\Phi_{pj} \neq 0$. Then we have $c_j = b_p + \deg(\Phi_{pj}) = d - b_j$. Thus we need to know if for all p, there is an $p \in H_j$ such that Φ_{pj} is nonzero. We do this below.

Lemma 4.2.4. *Given $1 \leq j \leq n+2$ there is an $p \in H_j$ such that Φ_{pj} is nonzero. In particular, $c_j = d - b_j$.*

Proof. The sequence $\mathbb{F}_\bullet$ is exact, so that if no Φ_{pj} is nonzero, then the jth column of ε is zero. This implies that $\mathbf{u} = [0, \ldots, 1, \ldots, 0]^T \in \mathbb{F}_2$ is in $\ker(\varepsilon)$, where the lone 1 occurs in row j. By the exactness of $\mathbb{F}_\bullet$, we can write $\mathbf{u} = \delta(\beta)$, where $\beta = [\beta_1, \ldots, \beta_n]^T \in \mathbb{F}_3$. This gives the equation

$$\sum_{i=1}^{n} \varphi_{ij}\beta_i = (-1)^{j+1}$$

This gives a contradiction, as the sum on the left is either homogeneous of positive degree or zero. $\square$

Corollary 4.2.5. $d_i = d - a_i$

Proof. Up to sign of entries, the ith column of the matrix for δ is the ith row of the matrix φ. In particular, by Lemma 4.2.1, φ_{ii} is nonzero, so that $e_{ii} - d_i = -c_i$. By Lemma 4.2.4, $c_i = d - b_i$. This gives $d_i = e_{ii} + d - b_i = d - a_i$.

□

4.3 The Unimodality and Symmetry of the Hilbert Function of the R-module M

As previously mentioned, our motivation for wanting to study to the unimodality and symmetry of the R-module M was to understand when M has the Weak Lefschetz Property. However, the question of whether or not a graded Artinian module N over $S = \mathbb{K}[x_0, \ldots, x_r]$ has the Weak Lefschetz Property is more subtle if N is not generated in a single degree. For example, let N be an Artinian S-module with Hilbert function h_N such that N_{j+1} contains a minimal generator of N and $h_N(j) \geq h_N(j+1)$. Then $\times \ell : N_j \longrightarrow N_{j+1}$ cannot be surjective. Naturally, we would like to avoid situations such as this and seek to understand when our specific R-module M has a strictly unimodal Hilbert function over R (that is, where it is increasing or decreasing, it does so strictly). In particular, we look for numerical conditions on the a_i and b_j that make it so that the Hilbert function of M is strictly unimodal and symmetric.

The following lemma will be used frequently. Its proof is essentially that of ([44], Lemma 1.3), but we provide details.

Lemma 4.3.1. *Let $S = \mathbb{K}[x_0, \ldots, x_r]$ and N be a graded Artinian S-module with minimal free resolution $\mathbb{G}_\bullet$. If $\mathbb{G}_{r+1} = \bigoplus_{j=1}^{v} S(-u_j)$ is the last module occurring in $\mathbb{G}_\bullet$, then there is a graded degree zero isomorphism*

$$\mathit{Soc}(N) \cong \bigoplus_{j=1}^{v} \mathbb{K}(-(u_j - r - 1))$$

Proof. We have $\mathrm{Tor}^S_{r+1}(N,\mathbb{K}) = H_{r+1}(\mathbb{F}_\bullet \otimes \mathbb{K}) = \bigoplus_{j=1}^{v} \mathbb{K}(-u_j)$. If $\mathbb{C}_\bullet$ is the Koszul complex on $x_0,\ldots,x_r$, then we also have $\mathrm{Tor}^S_{r+1}(N,\mathbb{K}) = H_{r+1}(\mathbb{C}_\bullet \otimes N) = \mathrm{Soc}(N)(-r-1)$.

□

With Corollary 4.2.5 in hand, the following is immediate from Lemma 4.3.1.

Corollary 4.3.2. *M has maximal socle degree $d - a_1 - 3$.*

We turn our discussion to graded duals of Artinian modules over $S = \mathbb{K}[x_0,\ldots,x_r]$.

Definition 4.3.3. Let N be a graded Artinian S-module. Denote by $N^\vee$ the S-module $\mathrm{Hom}_{\mathbb{K}}(N,\mathbb{K})$. Then $N^\vee$ is a graded S-module with $N^\vee_j = \mathrm{Hom}_{\mathbb{K}}(N_{-j},\mathbb{K})$. The S-module action on $N^\vee$ is such that for $a \in S_i$ and $f \in N^\vee_j$, then $af \in N^\vee_{i+j}$ is the $\mathbb{K}$-linear map from $N_{-i-j} \longrightarrow \mathbb{K}$ with $(af)(\lambda) = f(a\lambda)$.

Following [43], we now define an analogue of the Gorenstein condition for Artinian S-modules.

Definition 4.3.4. A graded Artinian S-module N is *Symmetrically Gorenstein* if there is an isomorphism $\tau : N \longrightarrow \mathrm{Hom}_{\mathbb{K}}(N,\mathbb{K})(-s)$ such that $\tau = \mathrm{Hom}_{\mathbb{K}}(\tau,\mathbb{K})(-s)$.

With the above definition in hand, consider the following.

Lemma 4.3.5. *Let N be a non-negatively graded Artinian S-module, say $N = N_0 \oplus \cdots \oplus N_c$. We suppose that N_0 and N_c are nonzero. Suppose there is a graded isomorphism $\tau : N \xrightarrow{\cong} N^\vee(-s)$ for some $s \in \mathbb{Z}$. That is, $\tau(N_j) \subseteq N^\vee(-s)_{j+d}$ for some $d \in \mathbb{Z}$. Then N has symmetric Hilbert function.*

Proof. We have $\tau(N_0) \subseteq N^\vee_{d-s}$, which gives $-c \leq d-s \leq 0$, as $N^\vee$ is concentrated in degrees $-c$ to 0. Also, $\tau(N_c) \subseteq N^\vee_{c+d-s}$ and $\tau(N_c)$ is nonzero, so we have $-c \leq c+d-s \leq 0$. Thus $s - c = d$, which gives $\tau(N_j) \subseteq N^\vee(-s)_{j+s-c} = N^\vee_{j-c} = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, N_{c-j})$. Hence we obtain an isomorphism of vector spaces over $\mathbb{K}$:

$$\tau|_{N_j} : N_j \longrightarrow \mathrm{Hom}_k(N_{c-j}, \mathbb{K})$$

Thus for $j = 0, 1, \ldots, \lfloor \frac{c}{2} \rfloor$, we obtain $\dim_{\mathbb{K}} N_j = \dim_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(N_{c-j}, \mathbb{K}) = \dim_{\mathbb{K}} N_{c-j}$. That is, the Hilbert function of N is symmetric. $\square$

In particular, Lemma 4.3.5 gives that Hilbert function of a non-negatively graded Symmetrically Gorenstein S-module in which the component in degree zero is nonzero is symmetric. As one might guess, we want our module M over R to be Symmetrically Gorenstein. Since we have spent a significant amount of time analyzing the minimal free resolution of M over R in the previous section, one might hope there is a characterization of a Symmetrically Gorenstein module in terms of its minimal free resolution. This is indeed the case.

Theorem 4.3.6. *([43], Theorem 1.3)*

Suppose $\mathbb{K}$ has characteristic not two. Let $S = \mathbb{K}[x_0, \ldots, x_r]$ and N be a graded Artinian S-module with maximal socle degree c. Set $d = c+r+1$ and $(\bullet)^{\vee d} = Hom_S(\bullet, S(-d))$. Let $a \geq 3$ be an odd integer and $b = \frac{a-1}{2}$. Then N is Symmetrically Gorenstein if and only if its minimal graded free resolution has the following form

$$0 \longrightarrow (\mathbb{G}_0)^{\vee d} \xrightarrow{\psi_1^{\vee d}} (\mathbb{G}_1)^{\vee d} \longrightarrow \cdots \longrightarrow (\mathbb{G})_b^{\vee d} \xrightarrow{\psi_b^{\vee d}} \mathbb{G}_b \longrightarrow \cdots \longrightarrow \mathbb{G}_1 \xrightarrow{\psi_1} \mathbb{G}_0 \longrightarrow N \longrightarrow 0$$

To this end, we utilize Theorem 4.3.6 to show that under mild restrictions, M is a Symmetrically Gorenstein R-module, hence by Lemma 4.3.5, M will have a symmetric Hilbert function.

Remark 4.3.7. Write $\varepsilon = [\Phi_1, \ldots, \Phi_{n+2}]$, with Φ_j the jth column of ε. Consider the matrix $\varepsilon' : \bigoplus_{j=1}^{n+2} R(-c_j) \longrightarrow \bigoplus_{j=1}^{n+2} R(-b_j)$ with $\varepsilon' = [-\Phi_1, \ldots, (-1)^j \Phi_j, \ldots, (-1)^{n+2} \Phi_{n+2}]$. For $1 \leq j < r$, we have

$$\varepsilon'_{rj} = (-1)^{n+2-r+j} \Phi_{rj} = (-1)^{n+2+r-j} \Phi_{rj}$$

$$\varepsilon'_{jr} = (-1)^{n+2-r+j-1}\Phi_{jr} = (-1)^{n+2+r-j-1}\Phi_{rj}$$

Thus $\varepsilon'_{jr} = -\varepsilon'_{rj}$, so ε' is antisymmetric. We utilize ε' for the following.

Lemma 4.3.8. *The sequence*

$$\mathbb{F}'_\bullet : 0 \longrightarrow \mathbb{F}_3 \xrightarrow{g'\delta} \mathbb{F}_2 \xrightarrow{\varepsilon'} \mathbb{F}_1 \xrightarrow{\varphi} \mathbb{F}_0 \longrightarrow M \longrightarrow 0$$

is exact. Where $g' : \bigoplus_{j=1}^{n+2} R(-c_j) \longrightarrow \bigoplus_{j=1}^{n+2} R(-c_j)$ *is the map such that*

$$g' \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_j \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} -\beta_1 \\ \vdots \\ (-1)^j \beta_j \\ \vdots \\ (-1)^m \beta_m \end{bmatrix}$$

In particular, there is an isomorphism of minimal free resolutions of M

$$\mathbb{F}_\bullet \cong \mathbb{F}'_\bullet$$

Proof. We know the sequence

$$\mathbb{F}_\bullet : 0 \longrightarrow \bigoplus_{i=1}^{n} R(-d_i) \xrightarrow{\delta} \bigoplus_{j=1}^{n+2} R(-c_j) \xrightarrow{\varepsilon} \bigoplus_{j=1}^{n+2} R(-b_j) \xrightarrow{\varphi} \bigoplus_{i=1}^{n} R(-a_i) \longrightarrow M \longrightarrow 0$$

is exact. Clearly $g'\delta$ is injective, since g' is an isomorphism. Obviously, $\varepsilon' g' = \varepsilon$. This gives $\text{im}(\varepsilon') = \text{im}(\varepsilon) = \ker(\varphi)$. We have

$$\varepsilon' g' \delta = \varepsilon\delta = 0$$

Hence $\mathrm{im}(g'\delta) \subseteq \ker(\varepsilon')$. If $\varepsilon'(\alpha') = 0$, then $\alpha' = g'(\alpha)$, for some α necessarily in $\ker(\varepsilon)$ (as g' is its own inverse). Thus $\alpha = \delta(\beta)$, for some $\beta \in \bigoplus_{i=1}^{n} R(-d_i)$. That is, $\alpha' = g'\delta(\beta)$. Thus $\mathbb{F}'_\bullet$ is exact, which gives that $\mathbb{F}'_\bullet$ is a graded minimal free resolution of M, whence the isomorphism of complexes.

□

Proposition 4.3.9. *The R-module M is Symmetrically Gorenstein and its Hilbert function of M is symmetric if $a_1 = 0$ and $\mathbb{K}$ has characteristic not two.*

Proof. By Corollary 4.3.2, the maximal socle degree of M is $d - 3$, where $d = \sum b_j - \sum a_i$. As in the statement of Theorem 4.3.6, we let $(\bullet)^{\vee d}$ be the functor $\mathrm{Hom}_R(\bullet, R(-d))$. By Lemma 4.3.8,

$$\mathbb{F}'_\bullet : 0 \longrightarrow \bigoplus_{i=1}^{n} R(-d_i) \longrightarrow \bigoplus_{j=1}^{n+2} R(-c_j) \xrightarrow{\varepsilon'} \bigoplus_{j=1}^{n+2} R(-b_j) \longrightarrow \bigoplus_{i=1}^{n} R(-a_i) \longrightarrow M \longrightarrow 0$$

is the graded minimal free resolution of M. By Corollary 4.2.3, $c_j = d - b_j$ and by Corollary 4.2.5, $d_i = d - a_i$. Hence

$$\left(\bigoplus_{j=1}^{n+2} R(-b_j)\right)^{\vee d} = \bigoplus_{j=1}^{n+2} \mathrm{Hom}_R(R(-b_j), R(-d)) = \bigoplus_{j=1}^{n+2} R(b_j - d) = \bigoplus_{j=1}^{n+2} R(-c_j)$$

$$\left(\bigoplus_{i=1}^{n} R(-a_i)\right)^{\vee d} = \bigoplus_{i=1}^{n} \mathrm{Hom}_R(R(-a_i), R(-d)) = \bigoplus_{i=1}^{n} R(a_i - d) = \bigoplus_{i=1}^{n} R(-d_i)$$

Thus the minimal graded free resolution of M is given by

$$0 \longrightarrow \mathbb{F}_0^{\vee d} \longrightarrow \mathbb{F}_1^{\vee d} \xrightarrow{\varepsilon'} \mathbb{F}_1 \longrightarrow \mathbb{F}_0 \longrightarrow M$$

The map ε' is antisymmetric by Remark 4.3.7, hence by Theorem 4.3.6, M is Symmetrically Gorenstein. By our assumption that $a_1 = 0$, M is non-negatively graded and $M_0 \neq 0$. By Lemma 4.3.5, we obtain that the Hilbert function of M is symmetric. $\square$

Proposition 4.3.9 answers the question of when the Hilbert function is symmetric. This was a subtle but crucial point in showing that complete intersections in R have the Weak Lefschetz in [34]. However, as mentioned at the beginning of this section, a decreasing Hilbert function and having generators in degree greater than zero may cause M to lack the Weak Lefschetz Property. However, the following proposition shows that the Hilbert function of M is indeed strictly unimodal.

Proposition 4.3.10. *Suppose $\mathbb{K}$ has characteristic not two. The Hilbert function of M is strictly unimodal if $a_1 = 0$ and*

(a) d is even and $d' + b_{n+1} + 2 > b_{n+2}$.

(b) d is odd and $d' + b_{n+1} + 1 > b_{n+2}$.

where $d = \sum b_j - \sum a_i$ and $d' = \sum_{i=1}^{n}(b_i - a_i)$.

Proof. By ([20], Corollary 1.2), Lemma 4.2.4, and Corollary 4.2.5, the Hilbert function $h_M(t)$ of M is given by

$$\sum_{i=1}^{n}\left[\binom{t+2-a_i}{2} - \binom{t+2+a_i-d}{2}\right] + \sum_{j=1}^{n+2}\left[\binom{t+2+b_j-d}{2} - \binom{t+2-b_j}{2}\right] \qquad (\star)$$

As $a_1 = 0$, the maximal socle degree of M is $c := d-3$ by Corollary 4.3.2. We first claim that for $t \leq \lfloor \frac{c}{2} \rfloor$, we have $\binom{t+2+a_i-d}{2} = 0$ for all i and $\binom{t+2+b_j-d}{2} = 0$ for all j. It suffices to show $\lfloor \frac{c}{2} \rfloor + 2 + b_{n+2} - d \leq 1$, as $a_i < b_{n+2}$ by Lemma 4.2.1 and $b_j \leq b_{n+2}$ by hypothesis. Note this equivalent to showing that $b_{n+2} \leq \lfloor \frac{d}{2} \rfloor + 1$. Hence if d is even, this is equivalent to showing $2b_{n+2} \leq d+2$, and if d is odd, this equivalent to showing $2b_{n+2} \leq d+1$. These inequalities both follow immediately from the assumptions in (a) and (b), respectively.

Thus by ($\star$) and the above remarks, for $t \leq \lfloor \frac{c}{2} \rfloor$, $h_M(t)$ is given by

$$\sum_{i=1}^{n} \binom{t+2-a_i}{2} - \sum_{j=1}^{n+2} \binom{t+2-b_j}{2} \tag{$\star\star$}$$

Recalling that by Lemma 4.2.1, $a_i \leq a_n < b_n$, ($\star\star$) gives the following for $t \leq \lfloor \frac{c}{2} \rfloor$:

(1) if $t \geq b_{n+2}$, then $h_M(t) = -t^2 + ct + \alpha$ for $\alpha \in \mathbb{Z}$.

(2) if $t \in [b_{n+1}, b_{n+2})$, then $h_M(t) = -\frac{1}{2}t^2 + \left(c + \frac{3-2b_{n+2}}{2}\right) t + \beta$, where $\beta \in \mathbb{Z}$.

(3) if $t \in [b_n, b_{n+1})$, then $h_M(t) = d't + \gamma$, where $d' = \sum_{i=1}^{n}(b_i - a_i)$ and $\gamma \in \mathbb{Z}$.

(3) if $t \in [a_n, b_n) \cap [b_u, b_{u+1})$ for $u < n$ or $t \in [a_v, a_{v+1}) \cap [b_u, b_{u+1})$ for $u, v < n$, then we first note by Lemma 4.2.1, we must have $u \leq v$. Then either

   (i) $h_M(t) = \frac{1}{2}(n-u)t^2 + d_{u,n}t + \delta_{u,n}$, where $d_{u,n} = \frac{3}{2}(n-u) + \left(\sum_{j=1}^{u} b_j - \sum_{i=1}^{v} a_i\right)$ and $\delta_{u,n} \in \mathbb{Z}$.

   (ii) $h_M(t) = \frac{1}{2}(v-u)t^2 + d_{u,v}t + \delta_{u,v}$, where $d_{u,v} = \frac{3}{2}(v-u) + \left(\sum_{j=1}^{u} b_j - \sum_{i=1}^{v} a_i\right)$, $\delta_{u,v} \in \mathbb{Z}$, and $u \leq v < n$.

(4) if $t < b_1$ and $t \in [a_v, a_{v+1})$ for $v < n$, then $h_M(t) = \frac{1}{2}vt^2 + \left(\frac{3}{2}v - \sum_{i=1}^{v} a_i\right) t + \varepsilon_v$, with $\varepsilon_v \in \mathbb{Z}$.

Now we want to show in all of the intervals given above that $h_M(t)$ is increasing. In particular, for $b_{n+2} \leq t < \lfloor \frac{c}{2} \rfloor$, we immediately obtain by differentiation:

($1'$) $h_M(t)$ is strictly increasing if $t \geq b_{n+2}$.

Now for $t \in [b_{n+1}, b_{n+2})$, if d is even, then our assumption in (a) shows that $2b_{n+2} < d+2$, hence $2b_{n+2} \leq d+1$. As d is even, we have $2b_{n+2} \leq d$. If d is odd, then our assumption in (b) gives $2b_{n+2} < d+1$, hence $2b_{n+2} \leq d$. We have

$$c + \frac{3-2b_{n+2}}{2} \geq \frac{d-3}{2} \geq \left\lfloor \frac{c}{2} \right\rfloor \geq b_{n+2} > t$$

Thus differentiation of $h_M(t)$ on this interval yields:

$(2')$ $h_M(t)$ is strictly increasing for $t \in [b_{n+1}, b_{n+2})$.

Lemma 4.2.1 gives that $d' > 0$, hence we obtain after differentiation of $h_M(t)$:

$(3')$ $h_M(t)$ is strictly increasing for $t \in [b_n, b_{n+1})$.

Now we want to show that $h_M(t)$ is strictly increasing on $[a_n, b_n) \cap [b_u, b_{u+1})$ for $u < n$ and on $[a_v, a_{v+1}) \cap [b_u, b_{u+1})$ for $v < n$ and $u < n$ and $v \geq u$. For the first, we must show that for $t \in [a_n, b_n) \cap [b_u, b_{u+1})$ and $u < n$, that $(n-u)t + d_{u,n} > 0$. As $t \geq a_n$, we have $(n-u)t + d_{u,n} \geq (n-u)a_n + d_{u,n}$. By Lemma 4.2.1, we have

$$d_{u,n} = \frac{3}{2}(n-u) + \sum_{j=1}^{u} b_j - \sum_{i=1}^{n} a_i \geq u - \sum_{i=u+1}^{n} a_i \geq u - (n-u)a_n$$

This gives

$$(n-u)t + d_{u,n} \geq (n-u)a_n + \frac{3}{2}(n-u) + u - (n-u)a_n = \frac{3}{2}(n-u) + u > 0$$

For the second statement, note that Lemma 4.2.1 implies $h_M(t)$ is increasing if $v = u$. For $u < v$, we have we have $(n-u)t + d_{u,n} \geq (n-u)a_v + d_{u,v}$. By Lemma 4.2.1, we have

$$d_{u,v} = \frac{3}{2}(v-u) + \sum_{j=1}^{u} b_j - \sum_{i=1}^{v} a_i \geq u - \sum_{i=u+1}^{v} a_i \geq u - (v-u)a_v$$

This gives

$$(v-u)t + d_{u,v} \geq (v-u)a_v + \frac{3}{2}(v-u) + u - (v-u)a_v = \frac{3}{2}(v-u) + u > 0$$

Hence, differentiation yields:

$(4')$ (i) $h_M(t)$ is strictly increasing on $[a_n, b_n) \cap [b_u, b_{u+1})$ for $u < n$

(ii) $h_M(t)$ is increasing $[a_v, a_{v+1}) \cap [b_u, b_{u+1})$ for $v < n$ and $u < n$ and $v \geq u$.

$(5')$ To show that $h_M(t)$ is increasing for $t < b_1$ and $t \in [a_v, a_{v+1}]$ for $v < n$, we must show $tv + \frac{3}{2}v - \sum_{i=1}^{v} a_i > 0$. To wit, we have

$$tv + \frac{3}{2}v - \sum_{i=1}^{v} a_i \geq a_v v + \frac{3}{2}v - \sum_{i=1}^{v} a_i \geq \frac{3}{2}v > 0$$

By Proposition 4.3.9, $h_M(t)$ is symmetric, hence (1′)-(5′) give that $h_M(t)$ is strictly unimodal with maximum occurring at $t = \lfloor \frac{c}{2} \rfloor$.

□

4.4 The Weak Lefschetz for M

We utilize the same setup in this section as in Section 4.2, except we suppose $\mathbb{K}$ has characteristic zero. Set $E = \ker(\varphi)$ and let $\mathcal{E}$ be the sheafification of E, so that $\mathcal{E}$ is a vector bundle of rank two on $\mathbb{P}^2$. In [34], when $M = R/I$ with I a complete intersection, conditions were sought to force the semistability of the vector bundle $\mathcal{E}$. In fact, if $\ell \in R$ is general linear form and $\overline{R} = R/\ell R$, it was shown, using a theorem of Grauert-Mülich ([54], pg. 206) that the first syzygy of $\overline{I}$ was given by $\overline{R}(e_1) \oplus \overline{R}(e_2)$ with $|e_1 - e_2| = 0$ or 1. This allowed for a nearly immediate conclusion that R/I has the Weak Lefschetz. We show that the same tools that allowed this conclusion generalize to our setting.

Recall the graded minimal free resolution $\mathbb{F}_\bullet$ of the graded R-module M has the form:

$$0 \longrightarrow \mathbb{F}_3 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_1 \xrightarrow{\varphi} \mathbb{F}_0 \longrightarrow M \longrightarrow 0$$

Set $E = \ker(\varphi)$, so that upon sheafification, we obtain an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{E} \longrightarrow 0 \qquad (*)$$

Here $\mathcal{F}_2 = \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^2}(-c_j)$ and $\mathcal{F}_3 = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^2}(-d_i)$. Now $\mathcal{E}$ is a vector bundle of rank two. Moreover, the additivity of the first Chern class gives

$$\begin{aligned} \mathfrak{c}_1(\mathcal{E}) &= \sum_{i=1}^{n} d_i - \sum_{j=1}^{n+2} c_j \\ &= \sum_{i=1}^{n}(d - a_i) - \sum_{j=1}^{n+2}(d - b_j) \\ &= -d \end{aligned}$$

We would like conditions that force the semistability of $\mathcal{E}$. We first consider the case in which d is even. Write $d = 2e$, so that $\mathfrak{c}_1(\mathcal{E}) = -2e$, so that the normalized bundle of $\mathcal{E}_{\text{norm}}$ is given by $\mathcal{E}(e)$. Twist (*) by $e - 1$ to obtain

$$0 \longrightarrow \mathcal{F}_3(e-1) \longrightarrow \mathcal{F}_2(e-1) \longrightarrow \mathcal{E}_{\text{norm}}(-1) \longrightarrow 0 \qquad (**)$$

Assume now that d is odd and choose e such that $d = 2e + 1$. Then in this case, $\mathcal{E}_{\text{norm}} = \mathcal{E}(e)$ as well. Then twist (*) by e to obtain

$$0 \longrightarrow \mathcal{F}_3(e-1) \longrightarrow \mathcal{F}_2(e-1) \longrightarrow \mathcal{E}_{\text{norm}} \longrightarrow 0 \qquad (***)$$

We utilize the above exact sequences to give a proof of following lemma. We note Lemma 4.4.1 is a generalization of ([34], Lemma 2.1). In fact, it is ([34], Lemma 2.1) when $n = 1$ and $a_1 = 0$. The proof is similar to ([34], Lemma 2.1), but we provide details. The aim of Lemma 4.4.1 to determine when $\mathcal{E}$ is *semistable* (see Definition 3.3.1).

Lemma 4.4.1. *The rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ given above is semistable when*

(a) d is even and $d' + b_{n+1} + 2 > b_{n+2}$.

(b) d is odd and $d' + b_{n+1} + 1 > b_{n+2}$.

where $d = \sum b_j - \sum a_i$ and $d' = \sum_{i=1}^{n}(b_i - a_i)$

Proof. Assume $\mathfrak{c}_1(\mathcal{E})$ is even. Now $\mathcal{E}$ has rank two, so that from ([54], Lemma 1.2.5) we have that $\mathcal{E}$ is semistable if and only if $H^0(\mathbb{P}^2, \mathcal{E}_{\text{norm}}(-1)) = 0$. When $\mathfrak{c}_1(\mathcal{E})$ is odd and $\mathcal{E}$ has rank two, stability and semistability coincide by ([54], pg. 166) and the condition for semistability is $H^0(\mathbb{P}^2, \mathcal{E}_{\text{norm}}) = 0$. Now (**) is given explicitly by

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^2}(-d_i + e - 1) \longrightarrow \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^2}(-c_j + e - 1) \longrightarrow \mathcal{E}_{\text{norm}}(-1) \longrightarrow 0 \qquad (\star\star)$$

And (***) is given by

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^2}(-d_i + e) \longrightarrow \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^2}(-c_j + e) \longrightarrow \mathcal{E}_{\text{norm}} \longrightarrow 0 \qquad (\star\star\star)$$

We first remark that $2a_n < d$. Indeed, from Lemma 4.2.1, we have $a_n < b_n \leq b_{n+1} \leq b_{n+2}$, so that

$$d = d' + b_{n+1} + b_{n+2} > d' + 2a_n > 2a_n$$

Where we note that $d' > 0$ by Lemma 4.2.1.

Now $(\star\star)$ is exact on global sections, so in order for semistability in (a) to hold, we need the following inequalities to hold (noting $e = \frac{d}{2}$):

(i) $e < c_{n+2} + 1$

(ii) $e < d_n + 1$

We show (i) holds. Since $c_{n+2} = d - b_{n+2}$, (i) is equivalent to showing $2b_{n+2} < d + 2$. We have

$$d + 2 - 2b_{n+2} = d' + 2 + b_{n+1} - b_{n+2} > 0$$

Where the inequality above holds by hypothesis. As $d_n = d - a_n$, (ii) is equivalent to showing $2a_n < d + 2$, but we know this holds from the preceding remark.

For (b), as $(\star\star\star)$ is exact on global sections, for the semistability of $\mathcal{E}$, we need the following in inequalities to hold (noting $e = \frac{d-1}{2}$):

(iii) $e < c_{n+2}$

(iv) $e < d_n$

We show (iii) holds. Since $c_{n+2} = d - b_{n+2}$, (iii) is equivalent to showing $2b_{n+2} < d + 1$. Now

$$d + 1 - 2b_{n+2} = d' + 1 + b_{n+1} - b_{n+2} > 0$$

Thus (iii) holds. Now (iv) is equivalent to showing $2a_n < d + 1$, hence this follows from the preceding remark.

□

Using Lemma 4.4.1, we can say the following about the splitting type of $\mathcal{E}$.

Corollary 4.4.2. *Let $\mathcal{E}$ be the rank two vector bundle obtained above and assume that any of the conditions of Lemma 4.4.1 hold. Then the splitting type of $\mathcal{E}$ is*

$$(\lambda_1, \lambda_2) = \begin{cases} (-e, -e) & d = 2e \\ (-e, -e-1) & d = 2e+1 \end{cases}$$

Proof. By Lemma 4.4.1, $\mathcal{E}$ is semistable. The theorem of Grauert and Mülich ([54], pg. 206) says that in characteristic zero the splitting type of the semistable normalized 2-bundle $\mathcal{E}_{\text{norm}} = \mathcal{E}(e)$ over $\mathbb{P}^2$ is

$$(\lambda_1, \lambda_2) = \begin{cases} (0, 0) & \text{if } \mathfrak{c}_1(\mathcal{E}(e)) = 0 \\ (0, -1) & \text{if } \mathfrak{c}_1(\mathcal{E}(e)) = -1 \end{cases}$$

Recall $\mathfrak{c}_1(\mathcal{E}) = -d$. As $\mathcal{E}$ has rank two, the additivity of the first Chern class gives, $\mathfrak{c}_1(\mathcal{E}(e)) = \mathfrak{c}_1(\mathcal{E}) + 2e \in \{-1, 0\}$, as needed. $\square$

Corollary 4.4.2 was crucial in [34] to showing that complete intersections have the Weak Lefschetz in R. In fact, our generalizations of the essential lemmas of [34] show that we can generalize the main result of [34]. The proof of Theorem 4.4.3 works entirely in the same way as the proof ([34], Theorem 2.3), changing only what is necessary, however, we find reviewing the details in this chapter to be helpful.

First, we do note a couple points of caution. Firstly, we must understand the unimodality of the Hilbert function of M before employing the mechanics of the proof of ([34], Theorem 2.3). This is precisely the purpose of Proposition 4.3.10 in this context. Moreover, it is well-known complete intersections have symmetric Hilbert functions and this is a subtle detail in the proof of ([34], Theorem 2.3). However, Proposition 4.3.9 shows this the Hilbert function of M is also symmetric, allowing the proof of ([34], Theorem 2.3) to generalize to our setting.

Theorem 4.4.3. *If* $a_1 = 0$ *and*

(a) d *is even and* $d' + 2 + b_{n+1} > b_{n+2}$.

(b) d *is odd and* $d' + 1 + b_{n+1} > b_{n+2}$.

where $d = \sum b_j - \sum a_i$ *and* $d' = \sum_{i=1}^{n}(b_i - a_i)$*, then* M *has the Weak Lefschetz Property in the sense of Definition 3.1.1.*

Proof. Let ℓ be a general linear form and $\overline{R} = R/\ell R$. We denote by $\overline{f}$ the image of $f \in R$ in $\overline{R}$; by $\overline{\mathbb{F}}_1$ the free $\overline{R}$-module $\bigoplus_{j=1}^{n+2} \overline{R}(-b_j)$; and and $\overline{\mathbb{F}}_0$ for $\bigoplus_{i=1}^{n} \overline{R}(-a_i)$. From the exact sequence

$$0 \longrightarrow E \longrightarrow \mathbb{F}_1 \xrightarrow{\varphi} \mathbb{F}_0 \longrightarrow M \longrightarrow 0$$

we obtain a commutative diagram with exact rows

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}_1(-1) \arrow[r, "\varphi"] \arrow[d, "L"] & \mathbb{F}_0(-1) \arrow[r] \arrow[d, "L'"] & M(-1) \arrow[r] \arrow[d, "(\times \ell)"] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F}_1 \arrow[r, "\varphi"] \arrow[d] & \mathbb{F}_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & & \overline{\mathbb{F}}_1 \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{\mathbb{F}}_0 \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

Where L is the $(n+2) \times (n+2)$ matrix

$$
\begin{pmatrix}
\ell & 0 & \cdots & 0 \\
0 & \ell & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \ell
\end{pmatrix}
$$

And L' is the $n \times n$ matrix given by

$$
\begin{pmatrix}
\ell & 0 & \cdots & 0 \\
0 & \ell & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \ell
\end{pmatrix}
$$

Note the first vertical exact sequence is a direct sum of $n+2$ copies of the exact sequence

$$
0 \longrightarrow R(-1) \xrightarrow{\times \ell} R \longrightarrow \overline{R} \longrightarrow 0
$$

twisted by $-b_1, -b_2, \ldots, -b_{n+2}$, respectively. Notice the induced map on $E(-1) \longrightarrow E$ is just L. Let M' be the finite length cokernel of $\overline{\varphi}$ and E' the kernel of $\overline{\varphi}$. Now using the two vertical exact sequences, the Snake Lemma yields a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] \arrow[d, "L"] & \mathbb{F}_1(-1) \arrow[r, "\varphi"] \arrow[d, "L"] & \mathbb{F}_0(-1) \arrow[r] \arrow[d, "L'"] & M(-1) \arrow[r] \arrow[d, "(\times\ell)"] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & \mathbb{F}_1 \arrow[r, "\varphi"] \arrow[d] & \mathbb{F}_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & \overline{\mathbb{F}}_1 \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{\mathbb{F}}_0 \arrow[r] \arrow[d] & M' \arrow[r] & 0 \\
 & M(-1) & 0 & 0 & &
\end{tikzcd}$$

Let λ be the line in $\mathbb{P}^2$ defined by ℓ and sheafify the above diagram. Noting that the sheafifications of the finite length modules $M(-1)$, M and M' are zero, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}(-1) \arrow[r] \arrow[d, "L"] & \mathcal{F}_1(-1) \arrow[r, "\varphi"] \arrow[d, "L"] & \mathcal{F}_0(-1) \arrow[r] \arrow[d, "L'"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r, "\varphi"] \arrow[d] & \mathcal{F}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}|_\lambda \arrow[r] \arrow[d] & \mathcal{F}_1|_\lambda \arrow[r, "\overline{\varphi}"] \arrow[d] & \mathcal{F}_0|_\lambda \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Where $\mathcal{F}_1 = \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^2}(-b_j)$ so that $\mathcal{F}_1|_\lambda = \bigoplus_{j=1}^{n} \mathcal{O}_\lambda(-b_j)$. If either (a) or (b) are satisfied, then $\mathcal{E}$ is semistable by Lemma 4.4.1, so that by Corollary 4.4.2, we have

$$\mathcal{E}|_\lambda = \begin{cases} \mathcal{O}_\lambda(-e)^2 & d = 2e \\ \mathcal{O}_\lambda(-e) \bigoplus \mathcal{O}_\lambda(-e-1) & d = 2e+1 \end{cases}$$

If $N = \mathrm{im}(\varphi) \subseteq \mathbb{F}_0$, taking global sections in the last row of the above diagram yields the exact sequence

$$0 \longrightarrow \bigoplus_{u=1}^{2} \overline{R}(-e_u) \longrightarrow \overline{\mathbb{F}}_1 \longrightarrow \overline{N} \longrightarrow 0 \tag{$\star$}$$

where $|e_1 - e_2| = 0$ or 1, depending on the parity of d. We show that this implies the theorem. There are two cases to consider: (i) d is even and (ii) d is odd. We prove this first for (i). In this situation, ($\star$) is given by

$$0 \longrightarrow \overline{R}(-e)^2 \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j) \xrightarrow{\overline{\varphi}} \overline{N} \longrightarrow 0$$

By Proposition 4.3.9 and Proposition 4.3.10, the Hilbert function is symmetric and strictly unimodal, so it suffices to show that multiplication by ℓ is injective on the "first half" of M. That is, for $v \leq \lfloor \frac{d-3}{2} \rfloor = e-2$, we need to show that

$$M_v \xrightarrow{\times \ell} M_{v+1}$$

is injective. Now $M = \mathbb{F}_0/N$, so that $M_v = (\mathbb{F}_0)_v/(\mathbb{F}_0)_v \cap N$. If the kernel of $\times\ell$ is nontrivial, there is an $F \in (\mathbb{F}_0)_v \backslash (\mathbb{F}_0)_v \cap N$ such that $\ell F \in (\mathbb{F}_0)_{v+1} \cap N$. Recall that $N = \mathrm{im}(\varphi)$. Write F as a column vector $[F_1, \ldots, F_n]^T$ with $F_i \in R(-a_i)_v$. Then there are forms A_j such that $A = [A_1, \ldots, A_{n+2}]^T$ and

$$\ell F = \varphi A$$

This gives

$$\ell F_i = \varphi_{i1} A_1 + \cdots + \varphi_{i,n+2} A_{n+2}$$

for $i = 1, \ldots, n$. Since F is nonzero, there is at least one F_i that is nonzero. For such an i, there is at least one j for which $\varphi_{ij} A_j$ is nonzero. For such i and j, we have

$$v - a_i + 1 = e_{ij} + \deg(A_j)$$

Which gives $\deg(A_j) = v + 1 - a_i - e_{ij}$. Reducing the equation $\ell F = \varphi A$ modulo ℓ, we obtain that $\overline{\varphi}\overline{A} = 0$ in the exact sequence

$$0 \longrightarrow \overline{R}(-e)^2 \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j) \xrightarrow{\overline{\varphi}} \overline{N} \longrightarrow 0$$

Denote the map $\overline{R}(-e)^2 \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j)$ by $\overline{\psi}$, so that it is given by a matrix

$$\begin{pmatrix} \overline{\psi}_{11} & \overline{\psi}_{12} \\ \vdots & \vdots \\ \overline{\psi}_{n+2,1} & \overline{\psi}_{n+2,2} \end{pmatrix}$$

Then for the $\overline{\psi}_{jr}$ which are nonzero, we have $\deg(\overline{\psi}_{jr}) = e - b_j$. Since $\overline{\varphi}\overline{A} = 0$ and $\overline{A}$ is nonzero, there is a nonzero $\overline{B} = [\overline{B}_1, \overline{B}_2]^T \in \overline{R}(-e)^2$ such that $\overline{\psi}(\overline{B}) = \overline{A}$. Thus there is an r and j such that $\overline{\psi}_{jr}$ and $\overline{B}_r$ are nonzero, this gives

$$\deg(\overline{\psi}_{jr}) + \deg(\overline{B}_r) = \deg(\overline{A}_j) = v + 1 - a_i - e_{ij}$$

Which tells us $\deg \overline{B}_r = v + 1 - a_i - e_{ij} + b_j - e = v + 1 - e$. Now $\deg \overline{B}_r > 0$, so that $v + 1 > e$. However, our assumption was $v + 1 \leq e - 1 < e$, a contradiction.

Now assume that d is odd and write $d = 2e + 1$, so that $(\star)$ becomes

$$0 \longrightarrow \overline{R}(-e) \oplus \overline{R}(-e-1) \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j) \xrightarrow{\overline{\varphi}} \overline{N} \longrightarrow 0$$

By Proposition 4.3.9 and Proposition 4.3.10, the Hilbert function of M is symmetric and strictly unimodal, so it suffices to show that multiplication by ℓ is injective on the “first half" of M. Since $d-3$ is even, we need to show that for $v \leq \lfloor \frac{d-3}{2} \rfloor = e-1$

$$M_v \xrightarrow{\times \ell} M_{v+1}$$

is injective. Write $\overline{\psi}$, for the map $\overline{R}(-e) \oplus \overline{R}(-e-1) \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j)$. We can write $\overline{\psi}$ as

$$\begin{pmatrix} \overline{\psi}_{11} & \overline{\psi}_{12} \\ \vdots & \vdots \\ \overline{\psi}_{n+2,1} & \overline{\psi}_{n+2,2} \end{pmatrix}$$

Where, if $\overline{\psi}_{j1}$ or $\overline{\psi}_{j2}$ are nonzero, then $\deg(\overline{\psi}_{j1}) = e - b_j$ and $\deg(\overline{\psi}_{j2}) = e+1-b_j$. As in the case of d even, if $M_v \xrightarrow{\times \ell} M_{v+1}$ is not injective, we can assume that there is an $A = [A_1, \ldots, A_{n+2}]^T$ such that at least one of the A_j is nonzero with $\ell F = \varphi A$, where $F = [F_1, \ldots, F_n]^T \in (\mathbb{F}_0)_v \backslash (\mathbb{F}_0)_v \cap N$ and $\deg(F_i) = v - a_i$ if F_i is nonzero. As was the case for d even, there are i and j such that F_i and A_j are nonzero, so that

$$v - a_i + 1 = e_{ij} + \deg(A_j)$$

Using the exact sequence

$$0 \longrightarrow \overline{R}(-e) \oplus \overline{R}(-e-1) \longrightarrow \bigoplus_{j=1}^{n+2} \overline{R}(-b_j) \xrightarrow{\overline{\varphi}} \overline{N} \longrightarrow 0$$

and the fact that $\overline{\varphi}\overline{A} = 0$, there is a nonzero $\overline{B} = [\overline{B}_1, \overline{B}_2]^T \in \overline{R}(-e) \oplus \overline{R}(-e-1)$ such that $\overline{\psi}(\overline{B}) = \overline{A}$. Moreover, since F is not in N, $\overline{A}$ is nonzero, so $\overline{B}$ is nonzero. Thus there is a j such that

$$\deg(\overline{\psi}_{j1}) + \deg(\overline{B}_1) = \deg(\overline{A}_j) = v - a_i + 1 - e_{ij}$$

or

$$\deg(\overline{\psi}_{j2}) + \deg(\overline{B}_2) = \deg(\overline{A}_j) = v - a_i + 1 - e_{ij}$$

The first equation gives

$$\deg(\overline{B}_1) = v - a_i + 1 - e_{ij} - e + b_j = v + 1 - e$$

Similarly, the second equation gives

$$\deg(\overline{B}_2) = v - e$$

Recall that we have assumed that $v \leq e - 1$, hence we obtain a contradiction in either case as the degree of one of $\overline{B}_1$ or $\overline{B}_2$ is positive.

□

We we note we obtain ([34], Theorem 2.3) as a corollary of Theorem 4.4.3.

Corollary 4.4.4. *Complete intersections in R have the Weak Lefschetz Property.*

Proof. Suppose f_1, f_2, f_3 is a regular sequence with $\deg(f_j) = d_j$ and $2 \leq d_1 \leq d_2 \leq d_3$ in R. Set $I = (f_1, f_2, f_3)$. Then it is well-known R/I has a unimodal symmetric Hilbert function. Moreover, with notation as in Theorem 4.4.3, we have $a_1 = 0$ and $b_j = d_j$. If $d_3 < d_1 + d_2 + 1$, the associated vector bundle $\mathcal{E}$ will be semistable by Lemma 4.4.1, so that we can apply Theorem 4.4.3. Now ([68], Corollary 3) shows that $d_3 \geq d_1 + d_2 - 3$, then R/I has the Weak Lefschetz Property.

□

Example 4.4.5. Let f_1, f_2, f_3 be a regular sequence of homogeneous elements in R with $\deg f_i = q$ and $q \geq 3$. For $n > 1$, define $\varphi : R(-q)^{n+2} \longrightarrow R^n$ as follows: Let $\mathbf{v}$ be the row vector $[f_1, f_2, f_3, \mathbf{0}] \in R(-q)^{n+2}$ with $\mathbf{0}$ the zero vector of length $n - 1$. Let $\sigma \in S_{n+2}$ be the permutation that acts on $R(-q)^{n+2}$ (thought of as row vectors) as $\sigma(r_1,\ldots,r_{n+2}) = (r_{n+2}, r_1, \ldots, r_{n+1})$. Then we let φ be the linear map with matrix

$$\begin{bmatrix} \mathbf{v} \\ \sigma\mathbf{v} \\ \sigma^2\mathbf{v} \\ \vdots \\ \sigma^{n-1}\mathbf{v} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_1 & f_2 & f_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_1 & f_2 & f_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & f_1 & f_2 & f_3 \end{bmatrix}$$

Let I denote the ideal of $n \times n$ minors of φ. Notice that the minor corresponding to deleting the first two columns of φ is f_3^n, the minor corresponding to deleting the last two columns of φ is f_1^n and the minor corresponding to deleting the first and the last column of φ has the form $f_2^n + f$, with $f \in f_3R$. Thus I has codimension 3, hence $M = \text{coker}(\varphi)$ is a graded Artinian R-module.

Note $d = (n+2)q$ and the conditions of Lemma 4.4.1 are satisfied regardless of the parity of d since $q \geq 3$ and $n > 1$. Thus M has the Weak Lefschetz Property by Theorem 4.4.3. Since $\text{im}(\varphi) \subseteq \mathfrak{m}$, the minimal number of generators of M as an R-module is n, hence M is not cyclic as $n > 1$.

4.5 The non-Lefschetz Locus for Graded Modules

We now turn our attention to the more general setting of working over $S = \mathbb{K}[x_0, \ldots, x_r]$, with $\mathbb{K}$ an algebraically closed field of characteristic zero. All modules considered will be finitely generated. Let $N = \bigoplus_{j\in\mathbb{Z}} N_j$ be a graded Artinian module. In particular, N has finite length.

In [7], the authors defined what they called the *non-Lefschetz locus* for a cyclic S-module S/I. We recall this notion and discussion for graded S-modules of finite length. The S-module structure of N is determined by a sequence of $\mathbb{K}$-linear maps

$$\phi_j : S_1 \longrightarrow \text{Hom}_{\mathbb{K}}(N_j, N_{j+1})$$

where j ranges from the initial degree of N to the penultimate degree where N is not zero. Since the $\mathbb{K}$-dimension of N_j and N_{j+1} is finite, we have that $\phi_j(x_i)$ is a matrix of size $(\dim_{\mathbb{K}} N_{j+1} \times \dim_{\mathbb{K}} N_j)$. Say $\phi_j(x_i) = X_{i,j}$. In particular, given any linear form $\ell = t_0x_1 + \cdots + t_rx_r$, we have

$$\phi_j(\ell) = t_0X_{0,j} + \cdots + t_rX_{r,j} := X_j$$

If we regard $t_0, \ldots, t_r$ as the *dual variables*, then X_j is a matrix of size $(\dim_{\mathbb{K}} N_{j+1} \times \dim_{\mathbb{K}} N_j)$ in $\mathbb{K}[t_0, \ldots, t_r]$ whose entries are linear forms in the dual variables. In particular, the scheme defined by the vanishing of the maximal minors of the matrix X_j can viewed as lying in *dual projective space* $(\mathbb{P}^r)^*$. Denote this scheme by Y_j.

When $\ell \in S_1$, we call ℓ a *Lefschetz element of* N if it satisfies Definition 3.1.1. We view the collection of Lefschetz elements as a, possibly empty, subset of $(\mathbb{P}^r)^*$. We want to know want to know what the relationship between the scheme Y_j and the failure of ℓ to be a Lefschetz element for N is.

Remark 4.5.1. Recall that if A is an $n \times m$ matrix over an integral domain, then the *rank* of A is the maximum t such that there is a non-vanishing $t \times t$ minor. With notation as above, it is easy to see the following are equivalent:

(a) ℓ is not a Lefschetz element for N.

(b) There is a j such that X_j does not have maximal rank as a matrix over $\mathbb{K}[t_0, \ldots, a_r]$.

(c) There is a j such that $Y_j = (\mathbb{P}^r)^*$.

In particular, we see that N has the Weak Lefschetz property in the sense of Definition 3.1.1 if and only if there is an ℓ such that for all j, we have $Y_j \neq (\mathbb{P}^r)^*$. This brings us to the titular notion of this section, where we follow [7].

Definition 4.5.2. Given an Artinian graded S-module N, we define

$$\mathcal{L}_N := \{[\ell] \in \mathbb{P}(S_1) |\ \ell \text{ is not a Lefschetz element of } N\} \subset (\mathbb{P}^r)^*$$

and we call it the *non-Lefschetz locus of* N. For any integer j, we define

$$\mathcal{L}_{N,j} := \{[\ell] \in \mathbb{P}(S_1) |\times \ell : N_j \longrightarrow N_{j+1} \text{ does not have maximal rank}\} \subset (\mathbb{P}^r)^*$$

Of course, we would like to study $\mathcal{L}_{N,j}$ not just as a collection, but as a scheme. Let $A = \mathbb{K}[t_0, \dots, t_r]$ denote the coordinate ring of dual projective space $(\mathbb{P}^r)^*$. We can view $\mathcal{L}_{N,j}$ as the scheme defined by the maximal minors of the matrix representing the map

$$\times \ell : A \otimes_{\mathbb{K}} N_j \longrightarrow A \otimes_{\mathbb{K}} N_{j+1}$$

of free A-modules. In fact, this the matrix representing this map is just $X_{j,\ell}$. Denote the ideal of maximal minors in A defining the scheme $\mathcal{L}_{N,j}$ by $I(\mathcal{L}_{N,j})$. In this way, we have $\mathcal{L}_N = \bigcup_j \mathcal{L}_{N,j}$ and $\mathcal{L}_N$ is defined by the homogeneous ideal $I(\mathcal{L}_N) = \bigcap_j I(\mathcal{L}_{N,j})$.

When studying Artinian Gorenstein *algebras*, it is well-known that an algebra fails to have the Weak Lefschetz Property if injectivity fails in a single degree. In particular, as a set, the non-Lefschetz locus is determined by a single degree (see [51], Proposition 2.1). Moreover, it is also true that the non-Lefschetz locus is defined by a single degree *scheme-theoretically*, as is shown in ([7], Corollary 2.6). While having a suitable analogue of Gorenstein for Artinian modules, (see Definition 4.3.4), we cannot guarantee that certain properties of Artinian algebras with the Weak Lefschetz Property hold for all Artinian *modules*. For example, as seen in the previous section, we have to be careful when discussing unimodality and symmetry of the Hilbert function for Symmetrically Gorenstein modules.

We first begin by recovering a well-known result for Artinian algebras. The proof is roughly the same as (Proposition 3.2, [33]), but we include the details for the reader's convenience.

Proposition 4.5.3. *Suppose $N = S^v/L$, with L a homogeneous S-submodule of the free module S^v generated by elements of positive degree (with respect to the standard grading on S^v). Then N is a nonnegatively graded S-module that is generated as as S-module in degree zero. Furthermore, suppose N is Artinian. If N has the Weak Lefschetz Property then the Hilbert function of N is unimodal.*

Proof. Let $\mathfrak{m}$ be the irrelevant ideal of S and write $N = N_0 \oplus \cdots \oplus N_c$, so that N_c is nonzero and N is generated by N_0. Then $\mathfrak{m}^i N_0$ generates N_i as a vector space over $\mathbb{K}$. Let $j \geq 0$ be the smallest integer such that $\dim_{\mathbb{K}} N_j > \dim_{\mathbb{K}} N_{j+1}$. Since N has the Weak Lefschetz Property, there is an $\ell \in S_1$ such that $\times\ell : N_j \longrightarrow N_{j+1}$ is surjective. Thus $\ell N_j = N_{j+1}$. That is, $\mathfrak{m}^{j+1} N_0 = \ell \mathfrak{m}^j N_0$. Hence for $i \geq j$, we have $\ell N_i = N_{i+1}$, so that $\times\ell : N_i \longrightarrow N_{i+1}$ is surjective. This gives

$$v \leq \dim_{\mathbb{K}} N_1 \leq \dim_{\mathbb{K}} N_2 \leq \cdots \leq \dim_{\mathbb{K}} N_j > \dim_{\mathbb{K}} N_{j+1} \geq \cdots \geq \dim_{\mathbb{K}} N_c$$

□

It is not hard to see that the Buchsbaum-Rim complex in more than three variables will, in general, not provide a minimal free resolution of a cokernel that is Symmetrically Gorenstein. However, under mild restrictions, they fit naturally into a certain class of Artinian modules. We follow [6] in the next definition.

Definition 4.5.4. If $\mathrm{Soc}(N) = (0 :_N \mathfrak{m})$, we say that an Artinian S-module N is *level* if it is generated by N_0 as an S-module and $\mathrm{Soc}(N) = N_c$ for some c.

Recall from Definition 4.3.3 that if N is an S-module, the $\mathbb{K}$-dual of N is the graded S-module $N^\vee := \mathrm{Hom}_{\mathbb{K}}(N, \mathbb{K})$ with grading such that $N_j^\vee = \mathrm{Hom}_{\mathbb{K}}(N_{-j}, \mathbb{K})$. In particular, if N is nonnegatively graded Artinian S-module, say $N = N_0 \oplus \cdots \oplus N_c$ with N_c nonzero, then $N^\vee(-c)$ is Artinian and nonnegatively graded with maximal socle degree c. Even more is true.

Proposition 4.5.5. *([6], Proposition 2.3)*

Assume that N is a graded Artinian S-module that is level in the sense of Definition 4.5.4. If Soc$(N) = N_c$, then $N^\vee(-c)$ is an Artinian graded level S-module.

We utilize Proposition 4.5.5 to recover a well-known result for level algebras (see ([51], Proposition 2.1)).

Proposition 4.5.6. *Suppose $N = S^v/L$ with L a homogeneous S-submodule generated by elements of positive degree with respect to the standard grading on S^v. Suppose N is Artinian, say $N = N_0 \oplus \cdots \oplus N_c$. Let ℓ be a linear form in S. Denote by $\Psi_t : N_t \longrightarrow N_{t+1}$ for $t \geq 0$ multiplication by ℓ on N_t.*

(a) If Ψ_{t_0} is surjective for some t_0, then Ψ_t is surjective for all $t \geq t_0$.

(b) Suppose N is level in the sense of Definition 4.5.4. If Ψ_{t_0} is injective for some $t_0 \geq 0$ then Ψ_t is injective for all $t \leq t_0$.

(c) In particular, if N is level and there is a t_0 such that $\dim_{\mathbb{K}} N_{t_0} = \dim_{\mathbb{K}} N_{t_0+1}$, then N has the Weak Lefschetz Property if and only if Ψ_{t_0} is injective.

Proof. (a) This was shown in the proof of Proposition 4.5.3.

(b) Write $N = N_0 \oplus \cdots \oplus N_c$, so that by hypothesis, Soc$(N) = (0 :_N \mathfrak{m}) = N_c$. Then $N^\vee(-c)$ is level by Proposition 4.5.5, so is generated in degree 0. Now we can consider multiplication by ℓ on $N^\vee(-c)$. Write $t_0 = c - s_0$, for some s_0 between 0 and c. Then the injectivity of Ψ_{t_0} gives that $\times\ell : N^\vee(-c)_{s_0-1} \longrightarrow N^\vee(-c)_{s_0}$ is surjective. Thus, as in the argument for (a), we obtain that $\times\ell : N^\vee(-c)_s \longrightarrow N^\vee(-c)_{s+1}$ is surjective for $s \geq s_0 - 1$. Dualizing, we obtain that $\times\ell$: Hom$_{\mathbb{K}}(N^\vee(-c)_{s+1}, \mathbb{K}) \longrightarrow$ Hom$_{\mathbb{K}}(N^\vee(-c)_s, \mathbb{K})$ is injective. Hence Ψ_{c-s-1} is injective. Since every $t \leq t_0$ has the form $c - s - 1$ for some $s \geq s_0 - 1$, we obtain the statement.

(c) This follows immediately from (a) and (b).

□

Now the next proposition is crucial to our endeavors and it is an analogue of ([7], Proposition 2.5). The proof of ([7], Proposition 2.5) works by changing what is necessary, but we find providing the details instructive and useful to the reader.

Proposition 4.5.7. *Suppose that N is an Artinian nonnegatively graded S-module with Hilbert function h_N. If $h_N(i) \leq h_N(i+1) \leq h_N(i+2)$ and Soc$(N)_i = 0$, then $I(\mathcal{L}_{N,i+1}) \subseteq I(\mathcal{L}_{N,i})$.*

Proof. The ideal $I(\mathcal{L}_{N,i+1})$ is generated by the maximal minors of a matrix, say Ψ, for the map $\times\ell : S \otimes_{\mathbb{K}} N_{i+1} \longrightarrow S \otimes_{\mathbb{K}} N_{i+2}$, where $\ell \in S_1$. In particular, Ψ has size $h_N(i+2) \times h_N(i+1)$. Suppose $h_N(i+1) = u$ and $h_N(i+2) = v$ (so that $u \leq v$), hence we may choose a $\mathbb{K}$-basis $n_1, \ldots, n_v$ for N_{i+2}. For a maximal minor of Ψ, say ψ, let $n_{i_1}, \ldots, n_{i_{v-u}}$ be basis elements of N_{i+2} that correspond to rows $i_1, \ldots, i_{v-u}$ of Ψ that were deleted to compute ψ. Let N' be the R-submodule of N generated by $n_{i_1}, \ldots, n_{i_{v-u}}$ and set $L = N/N'$. As N' is generated by homogeneous elements of degree $i+2$, $L_i = N_i$, $L_{i+1} = N_{i+1}$, and clearly $\dim_{\mathbb{K}} L_{i+2} = \dim_{\mathbb{K}} N_{i+1}$. Now ψ is the determinant of a matrix for the map

$$\times\ell : S \otimes_{\mathbb{K}} L_{i+1} \longrightarrow S \otimes_{\mathbb{K}} L_{i+2}$$

so that we can prove the inclusion $I(\mathcal{L}_{N,i+1}) \subseteq I(\mathcal{L}_{N,i})$ by proving the inclusion for all such quotients $L = N/N'$. Therefore, we assume that $h_N(i+1) = h_N(i+2)$.

Suppose $\mathcal{L}_{N,i+1} = (\mathbb{P}^r)^*$. Then $I(\mathcal{L}_{N,i+1}) = 0$, so the inclusion of ideals is trivial. Suppose that $\mathcal{L}_{N,i} = (\mathbb{P}^r)^*$. This means that none of the linear forms $x_0, \ldots, x_r$ induce a map of maximal rank from $N_i \longrightarrow N_{i+1}$. Suppose, in addition, that $\mathcal{L}_{N,i+1} \neq (\mathbb{P}^r)^*$. As $h_N(i+1) = h_N(i+2)$, there is a linear form ℓ such that $\times\ell : N_{i+1} \longrightarrow N_{i+2}$ is injective. However, $\times\ell : N_i \longrightarrow N_{i+1}$ is not injective, so there is a a nonzero $y \in N_i$ such that $\ell y = 0$. Since Soc$(N)_i = 0$, there is a j such that $x_j y$ is nonzero. We have

$$\ell(x_j y) = x_j(\ell y) = 0$$

However, $\times\ell : N_{i+1} \longrightarrow N_{i+2}$ is injective, so that we have $x_j y = 0$, a contradiction. Thus $\mathcal{L}_{N,i+1} = (\mathbb{P}^r)^*$, and the inclusion of ideals is again trivial.

Therefore, by our preceding work, we may assume that $h_N(i) \leq h_N(i+1) = h_N(i+2)$ and $\mathcal{L}_{i,N} \neq (\mathbb{P}^r)^*$ and $\mathcal{L}_{i+1,N} \neq (\mathbb{P}^r)^*$. Thus there is a linear form ℓ such that $\times\ell : N_{i+1} \longrightarrow N_{i+2}$ is injective. As the preceding argument shows, for such a linear form ℓ, it must be the case that $\times\ell : N_i \longrightarrow N_{i+1}$ is injective. Now $(\mathbb{P}^r)^* = D_1 \cup \cdots \cup D_r$, where

$$D_j = \{[p_0, \ldots, p_j, \ldots, p_r] \in (\mathbb{P}^r)^* : p_j \neq 0\}$$

Since $\mathcal{L}_{i+1,N} \neq (\mathbb{P}^r)^*$, there is a j such that D_j is not contained in in $\mathcal{L}_{N,i+1}$. That is, there is an $\ell = b_0x_0 + \cdots + b_rx_r$ with $b_j \neq 0$ such that $\times\ell : N_{i+1} \longrightarrow N_{i+2}$. Moreover, we can assume that $b_j = 1$. Relabeling if necessary, we may assume that $j = r$. This gives $\ell = x_r + \ell'$, where ℓ' is a linear form in variables $x_0, \ldots, x_{r-1}$. Hence we can perform a linear change of variables and assume that $\times x_r : N_{i+1} \longrightarrow N_{i+2}$ is injective. Again, as $\mathrm{Soc}(N)_i = 0$, our preceding remark gives $\times x_r : N_i \longrightarrow N_{i+1}$ is injective. Consider the commutative diagram

$$\begin{tikzcd}
S \otimes_{\mathbb{K}} N_i \arrow[r, "\times\ell"] \arrow[d, "\times x_r"'] & S \otimes_{\mathbb{K}} N_{i+1} \arrow[d, "\times x_r"] \\
S \otimes_{\mathbb{K}} N_{i+1} \arrow[r, "\times\ell"'] & S \otimes_{\mathbb{K}} N_{i+2}
\end{tikzcd}$$

Note both vertical arrows are injective (as $\mathbb{K}$ is a field). We want to show that we can choose bases for N_i, N_{i+1} and N_{i+2} in such a away that the matrix for the map $\times\ell : S \otimes_{\mathbb{K}} N_i \longrightarrow S \otimes_k N_{i+1}$ under these bases is a submatrix of the matrix for the map $\times\ell : S \otimes_{\mathbb{K}} N_{i+1} \longrightarrow S \otimes_{\mathbb{K}} N_{i+2}$.

To this end, let $\mathcal{M}$ denote the set of monomials in S and let $\mathcal{M}_i$ be the set of degree i monomials in S. If we write $N = S^n/H$, for some graded S-submodule H of S^n and if $e_1, \ldots, e_n$ is the standard basis for S^n, we consider the set $\{me_j : m \in \mathcal{M}_i, 1 \leq j \leq n\}$. A $\mathbb{K}$-basis B_i of N_i is given by the elements in this set which are nonzero modulo H. Since multiplication by x_r is nonzero on N_i, the set x_rB_i can be extended to basis of N_{i+1}, say B_{i+1}. The injectivity of x_r on

N_{i+1} gives that x_rB_{i+1} is a basis for B_{i+2}, as $\dim_{\mathbb{K}} N_{i+1} = \dim_{\mathbb{K}} N_{i+2}$ Write $B_{i+2} = x_rB_{i+1}$. In particular, $S \otimes_{\mathbb{K}} B_i$, $S \otimes_{\mathbb{K}} B_{i+1}$ and $S \otimes_{\mathbb{K}} B_{i+2}$ are bases for $S \otimes_{\mathbb{K}} N_i, S \otimes_{\mathbb{K}} N_{i+1}$ and $S \otimes_{\mathbb{K}} N_{i+2}$ over S, respectively. We note that under these bases, the $v \times v$ identity matrix represents the map $\times x_r : S \otimes_{\mathbb{K}} N_{i+1} \longrightarrow S \otimes_{\mathbb{K}} N_{i+2}$. If W, X and Y are the matrices for the other maps under these bases, we have the commutative diagram

$$\begin{tikzcd}
S \otimes_{\mathbb{K}} N_i \arrow[r, "W"] \arrow[d, "X"'] & S \otimes_{\mathbb{K}} N_{i+1} \arrow[d, "I_v"] \\
S \otimes_{\mathbb{K}} N_{i+1} \arrow[r, "Y"'] & S \otimes_{\mathbb{K}} N_{i+2}
\end{tikzcd}$$

Now the matrix X is given by

$$\begin{bmatrix} I_u & \\ & O \end{bmatrix}$$

Where is O is a zero matrix of an appropriate size. In particular, we find that W can be regarded as submatrix of Y. Now the ideal $I(\mathcal{L}_{N,i+1})$ is principal and is generated by the determinant of the matrix Y. Since Y and W have the same number of rows, the determinant of Y is contained in the ideal generated by the maximal minors of W.

□

With Proposition 4.5.7 in hand, we have the following.

Corollary 4.5.8. *Suppose N is a nonnegatively graded Artinian level S-module of maximal socle degree c. There is a j such that*

$$\mathcal{L}_N = \mathcal{L}_{j-1,N} \cup \mathcal{L}_{j,N}$$

Proof. Suppose N does not have the Weak Lefschetz Property. Let $\ell \in S_1$ be a linear form such that there is a j so that $\times \ell : N_j \longrightarrow N_{j+1}$ does not have maximal rank. In this situation, we have $\mathcal{L}_N = \mathcal{L}_{N,j} = (\mathbb{P}^r)^*$.

Suppose N has the Weak Lefschetz Property. Then its Hilbert function is unimodal by Proposition 4.5.3, so that there is a j such that $h_N(i) \leq h_N(i+1)$ for $i < j$ and $h_N(i) \geq h_N(i+1)$ for $j \leq i$. Now for $i < j$, we may apply Proposition 4.5.7 to see that

$$I(\mathcal{L}_{N,j-1}) \subseteq I(\mathcal{L}_{N,j-2}) \subseteq \cdots \subseteq I(\mathcal{L}_{N,1}) \subseteq I(\mathcal{L}_{N,0})$$

for $i = 0, \ldots, j-1$, hence we obtain

$$\mathcal{L}_{N,i} \subseteq \mathcal{L}_{N,j-1}$$

for $i = 0, \ldots, j-1$.

Now $N^\vee(-c)$ is also an Artinian level module of maximal socle degree c by Proposition 4.5.5. Moreover, we have $N^\vee(-c)_i = \mathrm{Hom}_{\mathbb{K}}(N_{c-i}, \mathbb{K})$, so that $h_{N^\vee(-c)}(i) \leq h_{N^\vee(-c)}(i+1)$ for $i = 0, \ldots, c-j-1$. Now $I(\mathcal{L}_{N,i})$ is defined the vanishing of minors of a map $\phi_i : S_1 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(N_i, N_{i+1})$. The corresponding maps for $N^\vee(-c)$ are given by ϕ^T_{c-i-1}, where T denotes the transpose of a matrix, in particular, we have

$$I(\mathcal{L}_{N^\vee(-c),i}) = I(\mathcal{L}_{N,c-i-1}) \qquad (\star)$$

Then for $i = 0, \ldots, c-j-1$, using Proposition 4.5.7, we obtain

$$I(\mathcal{L}_{N^\vee(-c),c-j-1}) \subseteq I(\mathcal{L}_{N^\vee(-c),c-j-2,}) \subseteq \cdots \subseteq I(\mathcal{L}_{N^\vee(-c),1}) \subseteq I(\mathcal{L}_{N^\vee(-c)}, 0)$$

so that

$$\mathcal{L}_{N^\vee(-c),i} \subseteq \mathcal{L}_{N^\vee(-c),c-j-1}$$

That is, using $(\star)$, we have, for $i = 0, \ldots, c-j-1$,

$$\mathcal{L}_{c-i-1,N} \subseteq \mathcal{L}_{j,N}$$

This gives the statement when N has the Weak Lefschetz Property.

$\square$

Now Corollary 4.5.8 provides us with a nice decomposition of $\mathcal{L}_N$ in the case that N is Artinian and level, however, pinpointing the j for which this occurs can often be difficult in practice. We have another Corollary of Proposition 4.5.7 that does this when N is Symmetrically Gorenstein. It is well-known a Gorenstein algebra is always level. Naturally, we would like it so that Symmetrically Gorenstein modules are level. We answer this in the affirmative below.

Lemma 4.5.9. *Suppose $N = S^v/L$, where L is a homogeneous submodule of S^v generated by elements of positive degree with respect to the standard grading on S^v. If N is Symmetrically Gorenstein, then N is level.*

Proof. If $\mathbb{G}_\bullet$ is the minimal free resolution of N, we have $\mathbb{G}_0 = S^v$. As N is Symmetrically Gorenstein by Theorem 4.3.6, the last free module in $\mathbb{G}_\bullet$ is $(\mathbb{G}_0)^{\vee d} = S(-d)^v$, where $d = c+r+1$ and c is the maximal socle degree of N. By Lemma 4.3.1, N is level.

$\square$

The next lemma is not difficult to prove, but it is quite useful.

Lemma 4.5.10. *If the Hilbert function h_N of the Artinian module $N = N_0 \oplus \cdots \oplus N_c$ is symmetric and unimodal, then it is not hard to see h_N achieves its maximum value at $\lfloor \frac{c}{2} \rfloor$. In particular, if c is even, then h_N takes on its maximum value at the middle term and if c is odd, h_N takes on its maximum value at the middle two terms.*

Proof. That h_N is symmetric means that for $i = 0, 1, \ldots, \lfloor \frac{c}{2} \rfloor$, one has $\dim_{\mathbb{K}} N_i = \dim_{\mathbb{K}} N_{c-i}$. If the Hilbert function h_N is unimodal, then there is a j such that $h_N(i) \leq h_N(i+1)$ for $i < j$ and $h_N(i) \geq h_N(i+1)$ for $i \geq j$. We aim to show $h_N(j) = h_N(\lfloor \frac{c}{2} \rfloor)$. To wit, if $j \leq \lfloor \frac{c}{2} \rfloor$, then

$$h_N(j) = \dim_{\mathbb{K}} N_j = \dim_{\mathbb{K}} N_{c-j} = h_N(c-j)$$

By hypothesis, $h_N(j) \geq h_N(\lfloor \frac{c}{2} \rfloor)$. Moreover, since $c - j \geq c - \lfloor \frac{c}{2} \rfloor \geq \lfloor \frac{c}{2} \rfloor \geq j$, we have $h_N(j) = h_N(c-j) \leq h_N(c - \lfloor \frac{c}{2} \rfloor) \leq h_N(\lfloor \frac{c}{2} \rfloor)$ Thus $h_N(j) = h_N(\lfloor \frac{c}{2} \rfloor)$, as needed.

Suppose now $j > \lfloor \frac{c}{2} \rfloor$. We have $h_N(j) \geq h_N(\lfloor \frac{c}{2} \rfloor)$. Moreover, $c - j \leq \lfloor \frac{c}{2} \rfloor \leq j$, so that $h_N(j) = h_N(c-j) \leq h_N(\lfloor \frac{c}{2} \rfloor)$. Thus we obtain $h_N(\lfloor \frac{c}{2} \rfloor) = h_N(j)$, as needed.

□

We can now generalize ([7], Corollary 2.7).

Proposition 4.5.11. *Suppose $N = N_0 \oplus \cdots \oplus N_c$ is Symmetrically Gorenstein S-module with $N_0 \neq 0$ and $N_c \neq 0$. Then $\mathcal{L}_N = \mathcal{L}_{N,j}$, where $j = \lfloor \frac{c-1}{2} \rfloor$.*

Proof. The Hilbert function of N is symmetric by Lemma 4.3.5. Suppose N does not have the Weak Lefschetz Property. Then the symmetry of the Hilbert function and Proposition 4.5.6 say that $\times\ell$ cannot induce a map of maximal rank from $N_j \longrightarrow N_{j+1}$. In this case, we have $I(\mathcal{L}_{N,j}) = 0$, giving $\mathcal{L}_{N,j} = \mathcal{L}_N = (\mathbb{P}^r)^*$.

Suppose N has the Weak Lefschetz Property. Then the Hilbert function of N is unimodal by Proposition 4.5.3. As the Hilbert function of N is symmetric, by Lemma 4.5.10 the Hilbert function of N assumes its maximum value at $\lfloor \frac{c}{2} \rfloor$. By Lemma 4.5.9, N is level, so that by Corollary 4.5.8, we have

$$\mathcal{L}_N = \mathcal{L}_{\lfloor \frac{c}{2} \rfloor - 1, N} \cup \mathcal{L}_{\lfloor \frac{c}{2} \rfloor, N}$$

If c is odd, then write $c = 2b + 1$, so that $j = \lfloor \frac{c}{2} \rfloor = b$. Then the symmetry of the Hilbert function gives $h_N(b+1) = h_N(c - \lfloor \frac{c}{2} \rfloor) = h_N(\lfloor \frac{c}{2} \rfloor) = h_N(b)$. Thus by Proposition 4.5.7, $I(\mathcal{L}_{b,N}) \subseteq I(\mathcal{L}_{b-1,N})$, hence $\mathcal{L}_N = \mathcal{L}_{j,N}$.

If c is even, write $c = 2b$, so that $j = \lfloor \frac{c}{2} \rfloor - 1 = b - 1$. Now the symmetry of the Hilbert function gives that $h_N(b-1) = h_N(b+1)$, so that $I(\mathcal{L}_{b-1,N}) = I(\mathcal{L}_{b,N})$, which gives $\mathcal{L}_N = \mathcal{L}_{N,j}$. □

Corollary 4.5.12. *Suppose $R = \mathbb{K}[x, y, z]$. We let φ be a degree zero graded homomorphism from $\bigoplus_{j=1}^{n+2} R(-b_j)$ to R^n $(n > 0)$, where $\varphi = (\varphi_{ij})$ and φ_{ij} is either zero or of positive degree and $b_1 \leq \cdots \leq b_{n+2}$. Suppose the ideal of maximal minors of φ has codimension three, so that the cokernel of φ, denoted by M, is Artinian. Then $\mathcal{L}_M = \mathcal{L}_{M,\lfloor \frac{d-4}{2} \rfloor}$, where $d = \sum b_j$.*

Proof. By Corollary 4.3.2, M has maximal socle degree $d - 3$. By Proposition 4.3.9, M is non-negatively graded and Symmetrically Gorenstein, hence we may apply Proposition 4.5.11 to obtain the result.

□

We remark that we do not necessarily need Proposition 4.5.11 for Corollary 4.5.12. Indeed, the proof of Proposition 4.5.3 shows that h_M achieves its maximum value at $\lfloor \frac{c}{2} \rfloor$, hence we may apply Lemma 4.5.9 and Lemma 4.5.8 to give Corollary 4.5.12.

Chapter 5

Macaulay Duals of Hyperplane Arrangements

5.1 Introduction

Given[4] a homogeneous polynomial f of degree d, the *apolar algebra* R_f (while this notation is commonly used in localization, we will not be localizing in this chapter) is the ring of polynomial differential operators modulo those which annihilate f. This algebra has been studied for a variety of reasons; in particular the apolar algebra of a form of degree d is always an Artinian Gorenstein algebra with socle degree d and every Artinian Gorenstein algebra with socle degree d can be represented as the apolar algebra of a form of degree d. This explicit correspondence, via the apolar algebra, between forms of degree d and Artinian Gorenstein algebras with socle degree d is detailed in [38]. The apolar algebra of a homogeneous polynomial f of degree d is also key to studying the *Waring rank* of f, which is the smallest integer r for which there exist linear forms $\ell_1, \ldots, \ell_r$ so that $f = \ell_1^d + \cdots + \ell_k^d$ (we call such a representation a *Waring decomposition*). The Waring rank often depends on the field chosen, and to avoid such complications, we will always assume our ground field to be algebraically closed and have characteristic zero.

In this chapter we study the apolar algebra of a form f of degree d which can be written as a product of d, not necessarily distinct, linear forms. Such forms correspond geometrically to hyperplane arrangements (in the case of distinct linear forms) and hyperplane multi-arrangements (in the case of non-distinct linear forms). To simplify exposition, we conflate a multi-arrangement with its defining equation. For instance, if we refer to the Waring rank of a multi-arrangement, we mean the Waring rank of its defining equation. Our inspiration for studying this problem stems largely from [67], where several questions are posed about apolar algebras of multi-arrangements. In particular, we study when the apolar algebra of a multi-arrangement is a complete intersection.

[4]The main results in this chapter are taken from the paper [16], which is joint with Michael DiPasquale and Chris Peterson.

If the apolar algebra of a form is a complete intersection, it is often easier to compute its Waring rank. Two important classes of examples (all multi-arrangements) serve to illustrate this point. The first is the case of a monomial, whose apolar algebra is generated by powers of variables. The Waring rank of monomials over the field of complex numbers is completely determined in [11]. The second class is when f is the fundamental skew invariant of a complex reflection group W, which is the product of the linear forms defining the pseudo-reflections of W. In this case the apolar algebra R_f is isomorphic to the ring of covariants of W [41, Chapter 26], which is the quotient of the polynomial ring by the ideal generated by invariants of W. This is a complete intersection since the ring of invariants is itself a polynomial ring by the celebrated Chevalley-Shephard-Todd theorem. In [65], Teitler and Woo determine the Waring rank of (and a Waring decomposition of) the fundamental skew invariant of a complex reflection arrangement under some mild conditions.

Following a section providing preliminary background material, we briefly discuss *reducible* arrangements, which are arrangements that can be written as a product of lower dimensional arrangements. In Section 4 we make use of the defining equations of *star configurations* determined by Geramita, Harbourne, and Migliore [25] to give a lower bound on the initial degree of the apolar algebra of a generic arrangement (Proposition 5.4.10). We give two corollaries to Proposition 5.4.10 – the first is a lower bound on the size of a generic arrangement whose apolar ideal is a complete intersection and the second is a lower bound on the Waring rank of a generic arrangement. The final section of the paper provides closing comments and gives suggestions for further research.

5.2 Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and put $R = \mathbb{K}[X_0, \ldots, X_r]$.

Let $S = \mathbb{K}[x_0, \ldots, x_r]$ be the R-module defined by R acting on S via partial differentiation. That is, if $f \in S$ and $\varphi \in R$,

$$\varphi \circ f = \varphi\left(\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_r}\right) f.$$

This is known as the *apolar* action of R on S. The expository article [26] is an excellent introduction to applications of apolarity, and the book [38] can be used to go into more detail.

Given a form $f \in S$, the *apolar ideal* of f is

$$\mathrm{Ann}_R(f) = \{\varphi \in R : \varphi \circ f = 0\}.$$

We write $R_f = R/\mathrm{Ann}_R(f)$; this is the *apolar algebra* of f. The apolar algebra R_f is a graded Artinian Gorenstein algebra, and every graded Artinian Gorenstein algebra arises in this way [38, Lemma 2.12].

Now suppose $f \in S_d$ (where S_d denotes the degree d forms in S). A *Waring decomposition* of f is a decomposition $f = c_1\ell_1^d + \cdots + c_k\ell_k^d$, where $\ell_1, \ldots, \ell_k$ are linear forms and $c_1, \ldots, c_k \in \mathbb{K}$ (we do not strictly need $c_1, \ldots, c_k$ since $\mathbb{K}$ is algebraically closed, but it will be useful for us to consider them). The smallest number of linear forms needed in a Waring decomposition of f is the *Waring rank* of f. The following lemma relates the apolarity action and Waring decompositions (see [38, Lemma 1.15] for a proof). In what follows, we say a linear form $\ell = \sum_{i=0}^n a_i x_i \in \mathbb{K}[x_0, \ldots, x_r]$ is *dual* to the point $P = [a_0 : \cdots : a_r] \in \mathbb{P}^r_{\mathbb{K}}$. Any non-zero constant multiple of ℓ is of course dual to the same point P.

Lemma 5.2.1 (Apolarity Lemma)**.** *Let $f \in S = \mathbb{K}[x_0, \ldots, x_r]$ be a form of degree d, $X = \{P_1, \ldots, P_k\} \subset \mathbb{P}^r_{\mathbb{K}}$ a set of points, and $I_X \subset R$ its corresponding ideal. Write $\ell_1, \ldots, \ell_k$ for linear forms in S dual to the points $P_1, \ldots, P_k$. Then $f = c_1\ell_1^d + \ldots + c_k\ell_k^d$ for some constants $c_1, \ldots, c_k$ if and only if $I_X \subset \mathrm{Ann}_R(f)$.*

From the apolarity lemma we see that the Waring rank of a form is the same as the minimum degree of a zero-dimensional radical ideal contained in its apolar ideal.

We will focus on forms $f \in S = \mathbb{K}[x_0, \ldots, x_r]$ which decompose as a product of (not necessarily distinct) linear forms as $f = \ell_1^{m_1} \cdots \ell_k^{m_k}$. If $g \in S$, write $V(g)$ for the set of points in $\mathbb{K}^{r+1}$ at which g vanishes. A natural geometric object to attach to the product $f = \ell_1^{m_1} \cdots \ell_k^{m_k}$ is the *multi-arrangement* $(\mathcal{A}, \mathbf{m})$ where $\mathcal{A} = \cup_{i=1}^k V(\ell_i)$ is the union of the hyperplanes $V(\ell_i) \subset \mathbb{K}^{n+1}$

and $\mathbf{m}$ is a function which assigns to each hyperplane $H \in \mathcal{A}$ the integer $\mathbf{m}(H)$, where $\mathbf{m}(H)$ is the power to which the corresponding linear form appears in f. We put $|\mathbf{m}| = \sum_H \mathbf{m}(H)$, which is the degree of the polynomial f. If $\mathbf{m}(H) = 1$ for all $H \in \mathcal{A}$ we will say $(\mathcal{A}, \mathbf{m})$ is a *simple* arrangement and write $\mathcal{A}$ instead of $(\mathcal{A}, \mathbf{m})$. Given a multi-arrangement $(\mathcal{A}, \mathbf{m})$ we define $\mathcal{Q}(A, \mathbf{m}) := \prod_{H \in \mathcal{A}} \alpha_H^{\mathbf{m}(H)}$, where α_H is a choice of linear form vanishing on H. If $\mathcal{A}$ is simple then we write $\mathcal{Q}(\mathcal{A})$ for the product $\prod_{H \in \mathcal{A}} \alpha_H$. We call $\mathcal{Q}(\mathcal{A}, \mathbf{m})$ and $\mathcal{Q}(\mathcal{A})$ the *defining polynomial* of the multi-arrangement and arrangement, respectively. Moreover we write $|\mathcal{A}|$ for the number of hyperplanes in $\mathcal{A}$, so that if $f = \mathcal{Q}(\mathcal{A}, \mathbf{m})$, then $|\mathcal{A}|$ is the number of distinct linear factors of f. For simplicity, throughout this note we will conflate a multi-arrangement or arrangement with its defining polynomial. For instance, by “the apolar algebra of an arrangement" we will mean the apolar algebra of its defining equation.

If $\mathcal{A}_1 = \cup_{i=1}^{p} G_i \subset V \cong \mathbb{K}^s$ and $\mathcal{A}_2 = \cup_{j=1}^{q} H_j \subset W \cong \mathbb{K}^t$ are two simple arrangements, then the *product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined by

$$\mathcal{A}_1 \times \mathcal{A}_2 = (\cup_{i=1}^{s} G_i \times W) \cup \left(V \times \cup_{j=1}^{t} H_j\right) \subset V \times W$$

If $(\mathcal{A}_1, \mathbf{m}_1)$ and $(\mathcal{A}_2, \mathbf{m}_2)$ are multi-arrangements, the product multi-arrangement $(\mathcal{A}_1 \times \mathcal{A}_2, \mathbf{m})$ satisfies $\mathbf{m}(H \times W) = \mathbf{m}(H)$ if $H \in \mathcal{A}_1$, and $\mathbf{m}(V \times G) = \mathbf{m}(G)$ if $G \in \mathcal{A}_2$. Following [55], we will say that a simple arrangement $\mathcal{A}$ is *reducible* if, after a change of coordinates, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ for some simple arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. Otherwise we say that $\mathcal{A}$ is *irreducible*.

Suppose $\mathcal{A} \subset \mathbb{K}^{r+1}$ is a reducible arrangement and $\mathcal{Q}(\mathcal{A})$ is its defining polynomial. Then there is a change of variables so that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 \subset \mathbb{K}^s$ and $\mathcal{A}_2 \subset \mathbb{K}^t$ for some positive integers s, t satisfying $s + t = r + 1$. Put $S_1 = \mathbb{K}[x_1, \ldots, x_s]$ and $S_2 = \mathbb{K}[y_1, \ldots, y_t]$. Then, under this change of variables, $\mathcal{Q}(\mathcal{A}) = \mathcal{Q}(\mathcal{A}_1)\mathcal{Q}(\mathcal{A}_2)$. Algebraically, the defining polynomials of reducible arrangements are those which, after an appropriate change of variables, split as a product of two defining polynomials in disjoint sets of variables.

In this note we only consider hyperplane arrangements all of whose hyperplanes pass through the origin (these are called *central* arrangements). Hence we will freely pass between a central arrangement in $\mathbb{K}^{r+1}$ and its natural quotient in $\mathbb{P}^r$, which does not affect the algebra.

5.3 Products of one and two dimensional arrangements

In this section we observe that if $(\mathcal{A}, \mathbf{m})$ is reducible, so $(\mathcal{A}, \mathbf{m}) = (\mathcal{A}_1, \mathbf{m}_1) \times (\mathcal{A}_2, \mathbf{m}_2)$ after a change of variables, then $R_f \cong R_{f_1} \otimes_{\mathbb{K}} R_{f_2}$, where $f = \mathcal{Q}(\mathcal{A}, \mathbf{m}), f_1 = \mathcal{Q}(\mathcal{A}_1, \mathbf{m}_1)$, and $f_2 = \mathcal{Q}(\mathcal{A}_2, \mathbf{m}_2)$. Our observation hinges on the following proposition. We suspect this is well-known but we include a proof since we were not able to find one in the literature.

Proposition 5.3.1. *Suppose s and t are positive integers, $f \in S_1 = \mathbb{K}[x_1, \ldots, x_s]$ and $g \in S_2 = \mathbb{K}[y_1, \ldots, y_t]$. Put $S = S_1 \otimes_{\mathbb{K}} S_2$. Viewing S as the polynomial ring $\mathbb{K}[x_1, \ldots, x_s, y_1, \ldots, y_t]$, we abuse notation by writing fg for the simple tensor $f \otimes g \in S$. We write R_1, R_2, and R for the polynomial rings dual to S_1, S_2, and S. Then*

1. $R_{fg} \cong (R_1)_f \otimes_{\mathbb{K}} (R_2)_g$ *and*

2. $\mathit{Ann}_R(fg) = \mathit{Ann}_{R_1}(f)R_2 + \mathit{Ann}_{R_2}(g)R_1$

Proof. Since $\mathrm{Ann}_{R_1}(f)R_2 + \mathrm{Ann}_{R_2}(g)R_1$ is the kernel of the natural map from R to $R_f \otimes R_g$, it is clear that (1) and (2) are equivalent. We prove (2).

Suppose that $\varphi = \sum_{\alpha,\beta} c_{\alpha,\beta} X^\alpha Y^\beta \in R$, where $\alpha = (\alpha_1, \cdots, \alpha_s) \in \mathbb{Z}^s_{\geq 0}, \beta = (\beta_1, \ldots, \beta_t) \in \mathbb{Z}^t_{\geq 0}$, $X^\alpha = X_0^{\alpha_0} \cdots X_s^{\alpha_s}$, $Y^\beta = Y_0^{\beta_0} \cdots Y_t^{\beta_t}$, and $c_{\alpha,\beta} \in \mathbb{K}$. Then

$$\varphi \circ (fg) = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial y^\beta}.$$

Similarly, if $\varphi_1 \in R_1$ and $\varphi_2 \in R_2$, then $\varphi_1\varphi_2 \circ fg = (\varphi_1 \circ f)(\varphi_2 \circ g)$. From this observation it is clear that $\mathrm{Ann}_{R_1}(f)R_2 + \mathrm{Ann}_{R_2}(g)R_1 \subseteq \mathrm{Ann}_R(fg)$.

We prove that $\mathrm{Ann}_R(fg) \subseteq \mathrm{Ann}_{R_1}(f)R_2 + \mathrm{Ann}_{R_2}(g)R_1$. For this we consider several maps: $\alpha_f : R_1 \to S_1$ given by $\varphi \to \varphi \circ f$, $\alpha_g : R_2 \to S_2$ by $\varphi \to \varphi \circ g$, the tensor product maps

$\alpha'_f := \alpha_f \otimes_{\mathbb{K}} \mathrm{id}_{R_2} : R_1 \otimes_{\mathbb{K}} R_2 \to S_1 \otimes_{\mathbb{K}} R_2$ and $\alpha'_g := \mathrm{id}_{S_1} \otimes_{\mathbb{K}} \alpha_g : S_1 \otimes_{\mathbb{K}} R_2 \to S_1 \otimes_{\mathbb{K}} S_2$. By the above observations, $\mathrm{Ann}_R(fg) = \ker(\alpha'_g \circ \alpha'_f)$.

Suppose $\varphi = \sum_{\alpha,\beta} c_{\alpha,\beta} X^\alpha Y^\beta \in \mathrm{Ann}_R(fg)$. Then

$$\varphi \circ fg = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial y^\beta} = 0. \tag{5.1}$$

Suppose the monomial x^γ appears in $\frac{\partial f}{\partial x^\alpha}$ with coefficient $d_{\gamma,\alpha} \in \mathbb{K}$. Equating coefficients of x^γ in Equation (5.1) yields

$$x^\gamma \sum_{\alpha,\beta} d_{\gamma,\alpha} c_{\alpha,\beta} \frac{\partial g}{\partial y^\beta} = 0.$$

It follows that $\sum_{\alpha,\beta} d_{\gamma,\alpha} c_{\alpha,\beta} Y^\beta \in \ker(\alpha'_g) = \mathrm{Ann}_{R_2}(g)$. Thus

$$\alpha'_f(\varphi) = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\partial f}{\partial x^\alpha} Y^\beta \in \mathrm{Ann}_{R_2}(g)\alpha_f(R_1).$$

Notice that

$$\alpha'_f(\mathrm{Ann}_{R_1}(f)R_2 + \mathrm{Ann}_{R_2}(g)R_1) = \mathrm{Ann}_{R_2}(g)\alpha_f(R_1).$$

Since $\alpha'_f(\mathrm{Ann}_R(fg)) \subseteq \mathrm{Ann}_{R_2}(g)\alpha_f(R_1)$ and $\ker(\alpha'_f) = \mathrm{Ann}_{R_1}(f)R_2$, we have $\mathrm{Ann}_R(fg) \subseteq \mathrm{Ann}_{R_1}(f)R_2 + \mathrm{Ann}_{R_2}(g)R_1$, as desired. □

Corollary 5.3.2. *Suppose $S \cong S_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} S_k$, where S_i is a polynomial ring in one or two variables for $i = 1, \ldots, k$. If a form $f \in S$ factors as $f = f_1 \cdots f_k$ where $f_i \in S_i$ for $i = 1, \ldots, k$, then Ann$_R(f)$ is a complete intersection.*

Proof. It is well known that the apolar algebra of a homogeneous polynomial in one or two variables is a complete intersection (since Gorenstein coincides with complete intersection in one and two variables). The corollary follows directly from this fact and Proposition 5.3.1. □

Remark 5.3.3. Over an algebraically closed field it is clear that the factors $f_1, \ldots, f_k$ in Corollary 5.3.2 are in fact products of linear forms.

Remark 5.3.4. Corollary 5.3.2 shows that the apolar algebra of a multi-arrangement which is a product of one and two dimensional arrangements is a complete intersection. One may ask the reverse question: if the apolar algebra of $\mathcal{Q}(\mathcal{A}, \mathbf{m})$ is a complete intersection for every choice of multiplicity $\mathbf{m}$, is $\mathcal{A}$ necessarily a product of one and two dimensional arrangements? A similar question has an affirmative answer: in [1] it is proved that if the module of multi-derivations $D(\mathcal{A}, \mathbf{m})$ is free for every multiplicity $\mathbf{m}$, then $\mathcal{A}$ is indeed a product of one and two dimensional arrangements.

5.4 Generic arrangements

In this section we derive a lower bound on the initial degree of the apolar ideal of a *generic* arrangement $\mathcal{A} \subset \mathbb{P}^r$ with at least $r+1$ hyperplanes (Proposition 5.4.10). All arrangements in this section are simple arrangements.

Definition 5.4.1. An arrangement in $\mathbb{P}^r$ is generic if the intersection of any k of its hyperplanes has codimension $\min\{k, r+1\}$.

In preparation we give several lemmas and definitions. Given a form $G \in R$, the *gradient of* G is the vector $\nabla G := \left(\frac{\partial G}{\partial X_0}, \ldots, \frac{\partial G}{\partial X_r}\right)$.

Lemma 5.4.2. *Suppose $g \in S$ is a homogeneous polynomial and write $f = \ell g$ for some linear form ℓ. Let $F \in R$ be homogeneous of degree $d \geq 1$. Then, if we abuse notation and write ℓ for the corresponding linear form in R, we have*

$$F \circ f = (\nabla F \cdot \nabla \ell) \circ g + \ell\,(F \circ g)\,.$$

(Here $\nabla F \cdot \nabla \ell$ denotes the dot product.) In particular, if $f = \ell_1 \ell_2 \cdots \ell_t$ is a product of $t \geq n$ linear forms, n of which are linearly independent, then there is an $\ell \in \{\ell_1, \ldots, \ell_t\}$ such that $\nabla F \cdot \nabla \ell$ is nonzero.

Proof. Write $\ell = a_0x_0 + \cdots + a_rx_r$. First, let F be a monomial of degree d, say $F = X_{i_1}^{d_1} \cdots X_{i_t}^{d_t}$, where $d_1, \ldots, d_t$ are positive. Then it is easy to see that $F \circ f$ is given by

$$\left(\sum_{j=1}^{t} d_j a_j X_{i_1}^{d_1} \cdots X_{i_j}^{d_j - 1} \cdots X_{i_t}^{d_t}\right) \circ g + \ell(F \circ g) =$$

$$(\nabla F \cdot \nabla \ell) \circ g + \ell(F \circ g) \quad (\star)$$

By linearity of the gradient, $(\star)$ holds for arbitrary polynomials F. The rest is clear. $\square$

Definition 5.4.3. If f is a form, the *kth order Jacobian* of f is the ideal generated by all partials of f of order k and is denoted by $J^k(f)$.

Remark 5.4.4. The Jacobian of f is $J^1(f)$; geometrically, $V(J^1(f))$ is the singular locus of f. Analogously, $V(J^k(f))$ is the set of singular points with multiplicity at least $k + 1$.

Remark 5.4.5. Since we assume f is homogeneous, the Euler identity $\sum x_i \frac{dg}{dx_i} = \deg(g) \cdot g$ applied repeatedly to f and its partials yields the containments $(f) \subset J^1(f) \subset J^2(f) \subset \cdots \subset J^k(f)$. Geometrically, this yields a nested sequence of subvarieties of the hypersurface $V(f)$ ordered according to the severity of the singularities.

Remark 5.4.6. If f is a form of degree d, the degree k component of the apolar algebra $(R_f)_k$, is isomorphic (as a vector space over $\mathbb{K}$) to $J^{d-k}(f)_k$ via apolarity. Hence $\mathrm{Ann}_R(f)_k = 0$ if and only if $J^{d-k}(f)$ is the kth power of the maximal ideal.

According to Remark 5.4.4, if f is a product of linear forms, then $V(J^k(f))$ is exactly those points which lie at the intersection of at least $k + 1$ of the hyperplanes defined by the linear forms whose product is f. Now we arrive at the crucial point: if $f = \mathcal{Q}(\mathcal{A})$ for a *generic* arrangement, $V(J^k(f))$ is precisely the union of all codimension $k + 1$ intersections of hyperplanes from $\mathcal{A}$. Thus $V(J^k(f))$ is a *star configuration* [25]; a star configuration is by definition the union of all codimension c intersections of a generic arrangement (in [25, Definition 2.1] the property of *meeting properly* is exactly what we mean by a *generic* arrangement). In [25] it is shown that the ideal of codimension c intersections of an arrangement of $|\mathcal{A}|$ hyperplanes is generated by all distinct products of $|\mathcal{A}| - c + 1$ of the linear forms defining $\mathcal{A}$.

Lemma 5.4.7. *Suppose f decomposes non-trivially as a product $f = gh$; write $I = Ann_R(h)$ and $I' = Ann_R(f) = Ann_R(gh)$. If $D \in I'_k \setminus I_k$, then $g \in J^{k-1}(h) : (D \circ h)$.*

Proof. Repeatedly using the product rule yields that $D \circ gh = g(D \circ h) + T$, where $T \in J^{k-1}(h)$. Since $D \circ gh = 0$, this gives the result. $\square$

Corollary 5.4.8. *Suppose f is a product of at least $n+2$ distinct linear forms defining a generic arrangement $\mathcal{A}$ in $\mathbb{P}^r$. Factor f as a product $f = gh$ so that $\deg(h) \geq n+1$. Write $I = Ann_R(h)$ and $I' = Ann_R(f) = Ann_R(gh)$. If $I_k = 0$ for any $k \leq n$ then $I'_k = 0$.*

Proof. Suppose to the contrary that $D \in I'_k$ and $D \neq 0$. By Lemma 5.4.7, $g \in J^{k-1}(h) : (D \circ h)$. Write $h = \ell_1 \ell_2 \cdots \ell_t$, where $t \geq n+1$; then $V(J^{k-1}(h))$ is the union of linear spaces which are the intersections of at least k of the hyperplanes $V(\ell_1), \cdots, V(\ell_t)$. This is nonempty since $k \leq n < t$. As $\mathcal{A}$ is a generic arrangement, none of the factors of g vanish along any component of $V(J^{k-1}(h))$; in other words g is not in any prime ideal that comprises the intersection that is the radical of $J^{k-1}(h)$. This means that $g \in J^{k-1}(h) : (D \circ h)$ only if $D \circ h$ is in every minimal prime of $J^{k-1}(h)$. In other words, $D \circ h$ is in the radical of $J^{k-1}(h)$. Let $K = \sqrt{J^{k-1}(h)}$; this is the ideal of the union of linear spaces which are the intersections of k of the hyperplanes $V(\ell_1), \cdots, V(\ell_t)$. As previously noted, this is a star configuration, and by [25, Proposition 2.9], K is generated by all possible products of $t-k+1$ of the linear forms $\ell_1, \ldots, \ell_t$. On the other hand $D \circ h$ has degree $t-k$, so $D \circ h \notin K$. With this contradiction, we must have $I'_k = 0$. $\square$

Remark 5.4.9. Consider the A_3 arrangement in $\mathbb{P}^2$, defined by $f = xyz(x-y)(x-z)(y-z)$. Write $f = gh$ with $g = y - z$ and $h = xyz(x-y)(x-z)$. Set $I' = \mathrm{Ann}_R(f)$ and $I = \mathrm{Ann}_R(h)$. Then $I_2 = 0$ but $I'_2 \neq 0$. Thus the hypothesis that $\mathcal{A}$ is generic in Corollary 5.4.8 is necessary.

Now we give the main result of this section, which is a a bound on the initial degree of the apolar ideal of a generic arrangement. For an ideal $I \subset R$ we will denote by $\alpha(I)$ its initial degree, that is, the smallest degree d for which $I_d \neq 0$.

Proposition 5.4.10. *Suppose $\mathcal{A}$ is a generic arrangement of at least $r+1$ hyperplanes in $\mathbb{P}^r$ and $f = \mathcal{Q}(\mathcal{A})$. Then $\alpha(Ann_R(f)) \geq \min\{|\mathcal{A}| - r + 1, r + 1\}$.*

Proof. We first prove by induction on $|\mathcal{A}|$ that if $r+1 \leq |\mathcal{A}| \leq 2r$, then $\alpha(\mathrm{Ann}_R(f)) \geq |\mathcal{A}| - r + 1$. If $|\mathcal{A}| = r + 1$ then without loss of generality, $f = x_0 x_1 \cdots x_r$ and $\mathrm{Ann}_R(f) = (x_0^2, \ldots, x_r^2)$, so $\alpha(\mathrm{Ann}_R(f)) = 2 = |\mathcal{A}| - r + 1$.

Suppose now that $n + 1 < |\mathcal{A}| \leq 2r$, and additionally suppose for a contradiction that there is some $D \in \mathrm{Ann}_R(f)_{|\mathcal{A}|-r}$. Since $\mathcal{A}$ is defined by more than r linearly independent linear forms, by Lemma 5.4.2 there is some $\ell \in \mathcal{A}$ so that $\nabla \ell \cdot \nabla D \neq 0$. Writing $f = g\ell$, with $\deg(g) = r$, and using Lemma 5.4.2 again, we have

$$0 = D \circ f = (\nabla \ell \cdot \nabla D) \circ g + \ell(D \circ g).$$

Suppose $D \circ g = 0$, so that $(\nabla \ell \cdot \nabla D) \circ g = 0$. Now $\deg(\nabla \ell \cdot \nabla D) = |\mathcal{A}| - r - 1$, and by induction $\alpha(\mathrm{Ann}_R(g)) \geq |\mathcal{A}| - 1 - r + 1 = |\mathcal{A}| - r$. With this contradiction, $D \circ g \neq 0$.

With the above, $\ell(D \circ g)) = -(\nabla \ell \cdot \nabla D) \circ g$, so $\ell(D \circ g) \in J^{|\mathcal{A}|-r-1}(g)$. Write $K = \sqrt{J^{|\mathcal{A}|-r-1}(g)}$, so that K is the ideal defining all possible intersections of $|\mathcal{A}| - r$ hyperplanes of g; by [25], $\alpha(K) = (|\mathcal{A}| - 1) - (|\mathcal{A}| - r) + 1 = r$. Since $\deg(D \circ g) = (|\mathcal{A}| - 1) - (|\mathcal{A}| - r) = r - 1$, $D \circ g \notin K$. Since K is radical, ℓ must be in at least one minimal prime of K. This would imply that $V(\ell)$ passes through a codimension $|\mathcal{A}| - r$ intersection of $\mathcal{A}$. As $|\mathcal{A}| \leq 2r$, K is not the homogeneous maximal ideal, so that this contradicts that $\mathcal{A}$ is a generic arrangement. Hence no such D can exist, and it follows that $\alpha(\mathrm{Ann}_R(f)) \geq |\mathcal{A}| - r + 1$.

If $|\mathcal{A}| \geq 2r$ we prove by induction on $|\mathcal{A}|$ that $\alpha(\mathrm{Ann}_R(f)) \geq r + 1$. The base case $|\mathcal{A}| = 2r$ has already been shown. If $|\mathcal{A}| > 2r$ then the result follows from Corollary5.4.8. □

Corollary 5.4.11. *If $\mathcal{A}$ is a generic arrangement of at least $r+2$ hyperplanes in $\mathbb{P}^r$ whose apolar ideal is a complete intersection, then $|\mathcal{A}| \geq r(r+1)$.*

Proof. Put $f = \mathcal{Q}(\mathcal{A})$. If $\operatorname{Ann}_R(f)$ is a complete intersection generated in degrees $d_0 \leq \ldots \leq d_r$, then $(d_0 - 1) + (d_1 - 1) + \cdots + (d_r - 1) = |\mathcal{A}|$, so $d_0 + \cdots + d_r = |\mathcal{A}| + r + 1$. With this notation, $\alpha(\operatorname{Ann}_R(f)) = d_0$, and this gives $d_0 \leq (|\mathcal{A}| + r + 1)/(r + 1)$.

It is straightforward to check that if $r + 1 < |\mathcal{A}| \leq 2r$ then the lower bound for $\alpha(\operatorname{Ann}_R(f))$ from Proposition 5.4.10 is strictly larger than $(|\mathcal{A}| + r + 1)/(r + 1)$, so $\operatorname{Ann}_R(f)$ cannot be a complete intersection.

If $|\mathcal{A}| > 2r$ then we obtain from Proposition 5.4.10 that $r + 1 \leq (|\mathcal{A}| + r + 1)/(r + 1)$ or equivalently $r(r + 1) \leq |\mathcal{A}|$, proving the corollary. $\square$

Recall that the *Waring rank* of a form $f \in S$ is the smallest integer k for which there exist linear forms $\ell_1, \ldots, \ell_r$ so that $f = \ell_1^d + \cdots + \ell_k^d$.

Corollary 5.4.12. *The Waring rank of a generic arrangement $\mathcal{A} \subset \mathbb{P}^r$ with at least $r + 1$ hyperplanes is at least* $\min\{\binom{|\mathcal{A}|}{r}, \binom{2r}{r}\}$.

Proof. Put $f = \mathcal{Q}(\mathcal{A})$. By Proposition 5.4.10, $\alpha(\operatorname{Ann}_R(f)) \geq \min\{|\mathcal{A}| - r + 1, r + 1\}$. Suppose $f = \sum_{i=1}^{k} \ell_i^{|\mathcal{A}|}$, and let $X = \{P_i\}_{i=1}^{k}$ be the dual points in $\mathbb{P}^r$ found by stripping off the coordinates of the linear forms ℓ_i. By Lemma 5.2.1, $I_X \subset \operatorname{Ann}_R(f)$. For this to happen, X must impose independent conditions on forms of degree $d = \alpha(\operatorname{Ann}_R(f)) - 1$. In other words, X must consist of at least as many points as the dimension of the vector space S_d, where $S = \mathbb{K}[x_0, \ldots, x_r]$. Since $\dim S_d = \binom{r+d}{r}$, this gives the result. $\square$

Remark 5.4.13. As Corollary 5.4.12 does not account for the degree of $\mathcal{Q}(\mathcal{A})$, we suspect that Corollary 5.4.12 is not optimal.

5.5 Conclusions and Further Questions

There are two main results of this paper. The first is a bound on the initial degree of the apolar ideal of a generic arrangement, attained using defining equations of star configurations from [25]. From this we obtained a necessary condition on the size of a generic arrangement with a complete intersection apolar algebra, as well as a lower bound on the Waring rank of a generic arrangement. A subsequent question raised by Wakefield [67] remains wide open: is the apolar algebra of a generic arrangement ever a complete intersection? To this we add two additional questions concerning the optimality of Proposition 5.4.10 and Corollary 5.4.12. First, are there arbitrarily large generic arrangements in $\mathbb{P}^r$ whose apolar ideals have initial degree $r+1$? Second, are there arbitrarily large generic arrangements in $\mathbb{P}^r$ whose Waring rank is $\binom{2r}{r}$?

The general problem of determining the degree d irreducible multi-arrangements in $\mathbb{P}^r$ that have minimal Waring rank is currently out of reach but we leave it as a suggestion for a further path of research. It is worth noting that each of the extremal examples we found has interesting combinatorial properties. In particular, after a change of coordinates, one is the defining ideal of the A_3 braid arrangement. Another is half of the Hessian arrangement. Perhaps there is a clue in the structure of these examples that can help one search for higher degree extremal examples. One promising avenue is to look for extremal behavior among the simplicial line arrangements catalogued in [30]; such arrangements have recently led to interesting examples for the *containment problem* between regular and symbolic powers [64]. For now, we leave this as an open problem for the interested reader.

Bibliography

[1] Abe T., Terao H., and Yoshinaga M. Totally Free Arrangements of Hyperplanes. *Proc. Amer. Math. Soc.*, 137(4),1405-1410, 2009

[2] Barth W.,Some Properties of stable Rank-2 Vector Bundles on $\mathbb{P}^n$, *Mathematische Annalen*, 226 (2), 125-150

[3] Bass H., *Algebraic K-Theory*, W.A. Benjamin, Inc. New York, New York, 1968

[4] Bates D., Hauenstein J., Peterson C., and Sommese A., Numerical Decomposition of the Rank-Deficiency Set of a Matrix of Multivariate Polynomials, *Approximate Commutative Algebra*, 55–77, Springer, 2009

[5] Boij M., Components of the Space Parametrizing Graded Gorenstein Artin Algebras with a Given Hilbert Function, *Pacific Journal of Mathematics*, 187 (1), 1-11, 1999

[6] Boij M., Artin Level Modules, *Journal of Algebra*, 226(1), pp. 361-374, 2000

[7] Boij M., Migliore J., Miró-Roig R.M., and Nagel U. The non-Lefschetz Locus, *Journal of Algebra* 505: 288-320, 2018

[8] Brenner, H. and Kaid, A., Syzygy Bundles on $\mathbb{P}^2$ and the Weak Lefschetz Property, *Illinois Journal of Mathematics*, 51 (4), 1299-1308, 2007

[9] Buchsbaum, D., A Generalized Koszul Complex, *Transactions of the American Mathematical Society*, 111 (2), 183-196, 1964

[10] Buchsbaum, D., A generalized Koszul complex II. Depth and multiplicity, *Transactions of the American Mathematical Society*, 111 (2), 197-224, 1964

[11] Carlini E., Catalisano M., and Geramita A.V., The Solution to the Waring Problem for Monomials and the Sum of Coprime Monomials. *Journal of Algebra*, 370, 5-14, 2012

[12] Cook II, D. and Nagel, U., The Weak Lefschetz Property, Monomial Ideals, and Lozenges, *Illinois Journal of Mathematics*, 55 (1), 337-395, 2011

[13] Dao, H. and Huneke, C., Vanishing of Ext, Cluster Tilting modules and Finite Global Dimension of Endomorphism Rings, *American Journal of Mathematics*, 135, 561-578, 2013

[14] Dao, H., Faber, E., Ingalls, C., Noncommutative (Crepant) Desingularizations and the Global Spectrum of Commutative Rings. *Algebras and Representation Theory*, 18, 633-664, 2015

[15] De Paris A., Seeking for the Maximum Symmetric Rank *Mathematics*, 6 (11), 247, 2018

[16] DiPasquale M., Flores Z., and Peterson C., The Apolar Algebra of a Product of Linear Forms, https://arxiv.org/abs/2002.04818

[17] Doherty B., Faber E., C. Ingalls, *Computing Global Dimension of Endomorphism Rings via Ladders*, *Journal of Algebra*, 458, 307-350, 2016

[18] Dugger D. and Shipley B., K-theory and Derived Equivalences, *Duke Math Journal*, 124, 587-617, 2004

[19] Eisenbud, D., *Commutative Algebra with a View Toward Algebraic Geometry*. Springer-Verlag New York, Inc. New York, New York, 1995.

[20] Eisenbud, D., *The Geometry of Syzygies*. Springer Science+Business Media, Inc. New York, New York, 2005.

[21] Failla G., Flores Z., Peterson Z. On the Weak Lefschetz Property for Vector Bundles on $\mathbb{P}^2$, arXiv:1803.10337

[22] Favacchio, G., Thieu, P.D., On the Weak Lefschetz Property for Graded Modules over $k[x, y]$. *Matematiche (Catania* 67 (1), pp. 223-235, 2012

[23] Flores Z., G-Groups of Cohen-Macaulay Rings with n-Cluster Tilting Objects, *Algebras and Representation Theory* (2019), https://doi.org/10.1007/s10468-019-09876-6

[24] Flores Z., Symmetry, Unimodality, and Lefschetz Properties for Graded Modules, https://arxiv.org/abs/1908.03648

[25] Geramita A.V., Harbourne B., and Migliore J., Star Configurations in $\mathbb{P}^n$. *Journal of Algebra*, 376, 279-299, 2013

[26] Geramita A.V, Inverse systems of Fat Points: Waring's Problem, Secant Varieties of Veronese Varieties and Parameter Spaces for Gorenstein Ideals, In *The Curves Seminar at Queen's, Vol. X (Kingston, ON, 1995)*, volume 102 of *Queen's Papers in Pure and Appl. Math.*, pg. 2-114, Queen's Univ., Kingston, ON, 1996

[27] Gersten, S.M., *Algebraic K-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 - September 8, 1972*, Higher K-theory of Rings, 3-42, Springer-Verlag, Berlin, 1973

[28] Giusti M. and Merle M., *Singularités isolées et sections planes de variétés déterminantielles. II. Sections de variétés déterminantielles par les plans de coordonnées, Algebraic geometry* (La Rábida, 1981), 1982, pp.103-118

[29] Grauert H. and Mülich G., Vektorbündel vom Rang 2 über dem n-dimensionalen Komplex-projektiven Raum,*Manuscripta Mathematica*, 16 (1), 75-100, 1975

[30] Grünbaum B., A Catalogue of Simplicial Arrangements in the Real Projective Plane, *Ars Math. Contemp.*, 2(1), 1-25, 2009

[31] Holm, H., K-Groups for Rings of Finite Cohen-Macaulay Type, *Forum Mathematicum*, 27, 2413-2452, 2015

[32] Harima T., Characterization of Hilbert functions of Gorenstein Artin Algebras with the Weak Stanley Property, *Proceedings of the American Mathematical Society*, 123 (12), 3631-3638, 1995

[33] Harima T., Maeno T., Morita, H., Numata, Y., Wachi, A., and Watanabe, J., *Lefschetz Properties*, Springer, 2013

[34] Harima, T., Migliore, J.C., Nagel, U., Watanabe, J., The Weak and Strong Lefschetz Properties for Artinian K-algebras. *Journal of Algebra* 262(1), pp. 99-126, 2003

[35] Harima T. and Watanabe J., The Strong Lefschetz property for Artinian Algebras with non-Standard Grading, *Journal of Algebra*, 311 (2), 511-537, 2007

[36] Hartshorne, R., *Algebraic Geometry*, Springer Verlag, New York, 1977

[37] Iarrobino, A., *Associated graded algebra of a Gorenstein Artin Algebra, Memoirs of the American Mathematical Society*, 1994

[38] Iarrobino A. and Kanev V., *Power Sums, Gorenstein algebras, and Determinantal Loci*, Volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999, Appendix C by A. Iarrobino and S. Kleiman

[39] Iyama O. *Trends in Representation Theory of Algebras and Related Topics*, Auslander-Reiten Theory Revisited, European Mathematical Society, Zürich, 349-398, 2008

[40] Iyama O., Higher-Dimensional Auslander Reiten Theory on Maximal Orthogonal Subcategories, *Advances in Mathematics*, 210, 22-50, 2007

[41] Kane R., *Reflection groups and invariant theory*, volume 5 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*, Springer-Verlag, New York, 2001

[42] Kreuzer, M., Migliore, J.C, Peterson, C., and Nagel, U., Determinantal Schemes and Buchsbaum-Rim Sheaves, *Journal of Pure and Applied Algebra*, 150 (2), 155-174, 2000

[43] Kunte, M., Gorenstein Modules of Finite length. *Mathematische Nachrichten* 284(7), pp. 899–919, 2011

[44] Kustin A.R., Ulrich B., If the Socle Fits, *Journal of Algebra*, 147(1), pp. 63-80, 1992.

[45] Lam T.Y., *A First Course in Noncommutative Rings*, Springer-Verlag, New York 2001

[46] Leuschke G.J., Endomorphism Rings of Finite Global Dimension, *Canadian Journal of Mathematics*, 59, 332-342, 2007

[47] Leuschke G.J. and Wiegand R., *Cohen-Macaulay Representations*, American Mathematical Society, Providence, RI, 2012

[48] Li, J. and Zanello, F., Monomial Complete Intersections, the Weak Lefschetz Property and Plane Partitions, *Discrete Mathematics*, 310 (24), 3558-3570, 2010

[49] Mezzetti, E., Miró-Roig, Rosa and Ottaviani, G., Laplace Equations and the Weak Lefschetz Property, *Canadian Journal of Mathematics*, 65 (3), 634-654, 2013

[50] Migliore J., Nagel U., A Tour of the Weak and Strong Lefschetz Properties. *Journal of Commutative Algebra* Vol.5 No. 3, 2013.

[51] Migliore J.C., Miró-Roig R.M., and Nagel U. . Monomial Ideals, Almost Complete Intersections and the Weak Lefschetz Property, *Trans. Amer. Math. Soc.*, 363(1), 229–257, 2011.

[52] Migliore, J.C., Nagel, U., and Peterson, C., Buchsbaum-Rim Sheaves and their Multiple Sections, *Journal of Algebra*, 219 (1), 378-420, 1999

[53] Navkal V. (2013), *K'-theory of a Cohen-Macaulay Local Ring with n-Cluster Tilting Object* (Doctoral dissertation), Retrieved from ProQuest Dissertations and Theses (Order No. 3563356)

[54] Okonek C., Schneider M., Spindler, H., *Vector Bundles on Complex Projective Space*. Birkhäuser Boston. Boston, MA, 1980

[55] Orlik P. and Terao H. *Arrangements of Hyperplanes*, Volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1992

[56] Peng Y. and Guo X, The K_1-group of tiled orders, *Communications in Algebra*, 41, 3739-3744, 2013

[57] Quillen D., *Algebraic K-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 - September 8, 1972*, Higher Algebraic K-theory I, 85-147, Springer-Verlag, Berlin, 1973

[58] Rosenberg J., *Algebraic K-theory and its Applications*, Springer-Verlag, New York, 1994

[59] Sherman C., *Algebraic K-Theory. Proceedings of a Conference Held at Oberwolfach, June 1980. Part 1*, Group Representations and Algebraic K-theory, 208-243, Springer-Verlag, Berlin, 1982

[60] Srinivas V., *Algebraic K-theory*, Birkhäuser, Boston, MA, 1996

[61] Stanley, R., The Number of Faces of a Simplicial Convex Polytope, *Advances in Mathematics*, 35 (3), 236-238, 1980

[62] Stanley, R.,Weyl Groups, the Hard Lefschetz theorem, and the Sperner Property, *SIAM Journal on Algebraic Discrete Methods*, 1 (2), 164-184, 1980

[63] Swanson I. and Huneke C., *Integral Closure of Ideals, Rings and Modules*, Cambridge University Press, Cambridge, 2006

[64] Szpond J. and Malara G., The Containment Problem and a Rational Simplicial Arrangement, *Electron. Res. Announc. Math. Sci.*, 24, 123-128, 2017

[65] Teitler Z. and Woo A., Power Sum Decompositions of Defining Equations of Reflection Arrangements, *J. Algebraic Combin.*, 41(2), 365-383, 2015

[66] Vaserstein L.N., On the Whitehead Determinant for Semi-local Rings, *J. of Algebra*, 283, 690-699, 2005

[67] Wakefield M., *On the Derivation Module and Apolar Algebra of an Arrangement of Hyperplanes*, ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–University of Oregon.

[68] Watanabe, J., A Note on Complete Intersections of Height Three, *Proc. Amer. Math. Soc.* 126(1998) 3161-3168

[69] Yoshino Y., *Cohen-Macaulay Modules Over Cohen-Macaulay Rings*, London Mathematical Society Lecture Note Series, Cambridge University Press, Vol. 146, Cambridge, 1990.

[70] Zanello, F., A non-Unimodal Codimension 3 Level h-Vector *Journal of Algebra*, 305 (2), 949-956, 2006