GLOBAL MODEL OF LINEARIZED ATMOSPHERIC PERTURBATIONS: REVISIONS AND IMPROVEMENTS TO ORIGINAL MODEL

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Abstract

Subsequent to the original description of our linearized primitive equation model, several improvements to the its original design and corrections to its original description have been made. This manuscript represents a consolidation of these changes which include:

- Numerous corrections (typographic and otherwise);
- A provision for applying the horizontal boundaries at the poles. This document gives the form of these boundary conditions along with their derivation;
- An improved latitudinal finite differencing scheme in which the horizontal wind variables (u, v) are staggered midway between thermodynamic variables. This new grid structure eliminates grid scale numerical noise in the results and has the added benefit of increasing computational efficiency by a factor of two;
- Treatment of the case where both the longitudinal wavenumber and frequency of the perturbation approach zero. In the original version of the model, considerable difficulty was encounter in computing a solution for this case.

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1 Introduction

The original description of our linearized primitive equation model was first presented in Stevens and Ciesielski (1984; hereafter referred to as SC). The characteristics unique to this model and/or important for application include:

1. The specification of an 'arbitrary' mean zonal flow which can depend on both latitude and height;
2. Calculation of a mean meridional circulation which is dynamically consistent with the mean zonal flow (i.e., satisfies conservation of angular momentum, the balance approximation, the hydrostatic approximation, conservation of mass and energy);
3. Vertical transport of momentum by the deep convective clouds in the tropics in both the mean and perturbation circulations;
4. Spherical geometry;
5. Coordinate stretching in both the vertical and latitudinal coordinates, which is represented in the coupled differential equations;
6. Very fine vertical resolution: experiments have been run with 51 points in the vertical; computer processing increases only linearly with the number of grid points in the vertical;
7. Horizontal resolution of up to 31 points (square matrices with approximately five times the number of horizontal points must be inverted) at each vertical level;
8. Very economic computation; the global response in a single longitudinal wavenumber is obtained with approximately 13 seconds of NCAR CRAY1 time (using 31 points in the vertical direction and 21 in the horizontal).

Full spherical geometry enables applications to phenomena of middle latitudes, polar latitudes, and planetary scale as well as in tropics. Vertical and meridional stretched coordinates enable emphasis on specified regions of the atmosphere while retaining spherical geometry. Since computation time is linearly proportional to the number of vertical grid points, high vertical resolution is a fundamental asset. A major dynamical component excluded (to our knowledge) from other linear models is the specification of a consistent meridional circulation as part of the basic state.

In the course of using this model over the past several years, numerous improvements to the model's original design and corrections to its original description have been made. This manuscript represents a consolidation of these changes, the main ones of which are summarized below.

- Sections 2–11 contained numerous corrections (typographic and otherwise) to SC.

- Section 8.2.2 lists the boundary conditions to be imposed when the horizontal boundaries of the model are at the poles. These polar boundary conditions are derived in Appendix A. The earlier version of the model had no provision for applying the horizontal boundaries at the poles.

- In Section 9, we describe an improved latitudinal finite differencing scheme. Using a non-staggered grid with second-order centered differencing, the model results for certain choices of model parameters would contain numerical noise on the scale of the grid. Investigation of the one-dimensional shallow water equations (Appendix B) suggests that the numerical solution could be significantly improved (both in accuracy and efficiency) by changing to a staggered grid with the horizontal wind variables (u, v) spaced midway between thermodynamic variables.

- In the original version of the model, difficulty was encounter when computing a solution for the case of a steady, zonally symmetric heating (i.e., where both longitudinal wavenumber and frequency approached zero). This problem and its solution are treated in Section 12.

One must still refer to the original model write-up for details on the verification of the model (Section 12 of SC), its optimization on the CRAY (Section 13 of SC) and computation of the model's basic state (Section 14 of SC).

2 Full Primitive Equations and Model Parameters

For consistency all three components of the zonal mean circulation in the advection terms of the linearized perturbation equations are included. It is possible that some of the advective terms by the mean meridional cell $(\overline{v}, \overline{w})$ could be consistently scaled out for some problems, but in the interest of generality we have elected to leave them in. Following Holton (1975, p. 29) with slightly different notation, the (hydrostatic) primitive equations in $\log p$ coordinates on a sphere are written as follows:

Zonal-momentum

$$\frac{\partial u}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial u}{\partial \lambda^*} + \frac{v}{a}\frac{\partial u}{\partial \theta} - \frac{uv}{a}\tan\theta + w\frac{\partial u}{\partial z} - fv + \frac{1}{a\cos\theta}\frac{\partial \Phi}{\partial \lambda^*}$$

$$= \frac{g}{p}\frac{\partial}{\partial z}\left[M_c(u-u_c) + \frac{\mu\dagger}{H}\frac{\partial u}{\partial z}\right] - \alpha_R u \tag{2.1}$$

Meridional-momentum

$$\frac{\partial v}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial v}{\partial \lambda^*} + \frac{v}{a}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{u^2}{a}\tan\theta + fu + \frac{1}{a}\frac{\partial \Phi}{\partial \theta}$$

$$= \frac{g}{p}\frac{\partial}{\partial z}\left[M_c(v-v_c) + \frac{\mu\dagger}{H}\frac{\partial v}{\partial z}\right] - \alpha_R v \tag{2.2}$$

Continuity

$$\frac{1}{a\cos\theta}\frac{\partial u}{\partial \lambda^*} + \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{v}{a}\tan\theta + \frac{\partial w}{\partial z} - w = 0 \tag{2.3}$$

Thermodynamic

$$\frac{\partial T}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial T}{\partial \lambda^*} + \frac{v}{a}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z} + w\kappa T = \frac{Q}{c_p} + \frac{g}{p}\frac{\partial}{\partial z}\frac{\hat{\mu}\dagger}{H}\frac{\partial T}{\partial z} - \alpha_N T \tag{2.4}$$

Hydrostatic approximation

$$\frac{\partial \Phi}{\partial z} = RT \tag{2.5}$$

The vertical diffusion terms:

$$\frac{g}{p}\frac{\partial}{\partial z}\frac{\mu\dagger}{H}\frac{\partial u}{\partial z}, \quad \frac{g}{p}\frac{\partial}{\partial z}\frac{\mu\dagger}{H}\frac{\partial v}{\partial z} \quad \text{and} \quad \frac{g}{p}\frac{\partial}{\partial z}\frac{\hat{\mu}\dagger}{H}\frac{\partial T}{\partial z}$$

are required by the numerical integration scheme. As noted by Stevens *et al.* (1977), vanishing of the mass flux M_c at the cloud-top level gives singular solutions of the inviscid equations which can be avoided by inclusion of small vertical diffusion terms. The independent variables for this system of equations are:

$\lambda^* \equiv$ longitude

$\theta \equiv$ latitude

$t =$ time

$z = \ln\left(\frac{p_o}{p}\right)$

The dependent variables for this system of equations are:

$u = a\cos\theta\frac{d\lambda^*}{dt} \equiv$ horizontal velocity component in λ^*-direction

$v = a\frac{d\theta}{dt} \equiv$ horizontal velocity component in θ-direction

$w = \frac{dz}{dt} \equiv$ vertical velocity component

$T \equiv$ temperature

$\Phi \equiv$ geopotential height

Other specified variables and constants are:

$f \quad = \quad 2\Omega \sin\theta \equiv$ coriolis parameter

$p \quad = \quad p_o e^{-z} \equiv$ pressure

$p_o \quad \equiv$ surface pressure (10^5 N m^{-2})

$M_c \quad \equiv$ cumulus mass flux

$Q \quad \equiv$ diabatic heat source

$H \quad = \quad \frac{RT}{g} \equiv$ scale height

$g \quad \equiv$ gravitational acceleration (9.81 m s^{-2})

$R \quad \equiv$ gas constant for dry air (2.87×10^2 m^{2}s^{-2}K^{-1})

$\Omega \quad \equiv$ angular speed of rotation of earth (7.292×10^{-5} s^{-1})

$a \quad \equiv$ mean radius of earth (6.37×10^6 m)

$z_c \quad \equiv$ height of cloud base

$u_c \quad = \quad u(z_c) \equiv$ u-component of wind at cloud base

$v_c \quad = \quad v(z_c) \equiv$ v-component of wind at cloud base

$\alpha_R \quad \equiv$ Rayleigh friction ($2\Omega \times$ DISWIND), where DISWIND is the nondimensional dissipation coefficient of the horizontal wind

$\alpha_N \quad \equiv$ Newtonian cooling ($2\Omega \times$ DISTEMP), where DISTEMP is the nondimensional dissipation coefficient of temperature

$c_p \quad \equiv$ specific heat of air at constant pressure (1004 m^{2}s^{-2}K^{-1})

$\kappa \quad = \quad R/c_p$

$\mu\dagger \quad \equiv$ dynamic coefficient of viscosity

$\hat{\mu}\dagger \quad \equiv$ dynamic coefficient of thermal diffusion

Other constants used in this model but not explicitly appearing in equations 2.1 – 2.5 are listed below for the readers convenience.

$$
\begin{aligned}
\pi &= \cos^{-1}(-1) \\
\sigma &\equiv \text{angular frequency} \\
&= (\Omega/\text{PERIOD}) \text{ where PERIOD} = \text{period of disturbance in days} \\
&\quad \sigma > 0, \text{ for eastward propagating disturbance} \\
&\quad \sigma < 0, \text{ for westward propagating disturbance} \\
s &\equiv \text{longitudinal wavenumber} \\
c &\equiv \text{phase speed } (\sigma \cdot a \cdot \cos\theta / s) \\
T_o &\equiv \text{surface temperature (300 K)} \\
U_o &\equiv \text{horizontal wind scale used in non-dimensionalization (10 m s}^{-1}) \\
V_o &\equiv \text{gustiness factor (8 m s}^{-1}) \\
CD &\equiv \text{surface drag coefficient } ([1.0 + 0.07 \times V_o] \times 10^{-3}) \\
ST &\equiv \text{static stability (878 m)} \\
\Gamma_1 &= \tfrac{g \cdot ST}{R} \text{ (30.01 K)} \\
IZ &= \text{number of points in z-direction} \\
IY &= \text{number of points in } \theta\text{-direction} \\
ZT &= z(1) \equiv \text{value of } z \text{ at top of model} \\
ZTROP &\equiv \text{value of } z \text{ at tropopause}
\end{aligned}
$$

3 Linearized Equations

Using the perturbation method all variables are expanded into two parts: a basic state, which is assumed to be independent of time and longitude, and a perturbation, which is a local deviation of the field from the basic state. This expansion is shown below:

$$
\left.
\begin{array}{lcccc}
u(\lambda^*, \theta, z, t) & = & \overline{u}(\theta, z) & + & u'(\theta, z) \\
v(\lambda^*, \theta, z, t) & = & \overline{v}(\theta, z) & + & v'(\theta, z) \\
w(\lambda^*, \theta, z, t) & = & \overline{w}(\theta, z) & + & w'(\theta, z) \\
\Phi(\lambda^*, \theta, z, t) & = & \overline{\Phi}(\theta, z) & + & \Phi'(\theta, z) \\
T(\lambda^*, \theta, z, t) & = & \overline{T}(\theta, z) & + & T'(\theta, z) \\
M_c(\lambda^*, \theta, z, t) & = & \overline{M}_c(\theta, z) & + & M_c'(\theta, z) \\
u_c(\lambda^*, \theta, z, t) & = & \overline{u}_c(\theta, z) & + & u_c'(\theta, z) \\
v_c(\lambda^*, \theta, z, t) & = & \overline{v}_c(\theta, z) & + & v_c'(\theta, z) \\
Q(\lambda^*, \theta, z, t) & = & \overline{Q}(\theta, z) & + & Q'(\theta, z)
\end{array}
\right\} \times \quad e^{i(s\lambda^* - \sigma t)}
$$

where

$\overline{(\ \ )} \equiv$ basic state

$(\tilde{\ }) = (\ \ )' e^{i(s\lambda - \sigma t)} \equiv$ perturbation from basic state

To illustrate how Eqs. 2.1 - 2.5 are linearized we have shown below how this method works for the u-component of the advection term in the zonal-momentum equation. Upon expansion:

$$\frac{u}{a\cos\theta}\frac{\mu}{\partial\lambda^*} = \frac{1}{a\cos\theta}\left(\overline{u}\frac{\partial\overline{u}}{\partial\lambda^*} + \overline{u}\frac{\partial\tilde{u}}{\partial\lambda^*} + \tilde{u}\frac{\partial\overline{u}}{\partial\lambda^*} + \tilde{u}\frac{\partial\tilde{u}}{\partial\lambda^*}\right) \tag{3.1}$$

The assumption is made here that the basic state variables must themselves satisfy the governing equations so that the first term on the right hands side of (3.1) will cancel out with the other terms of the basic state equation. Secondly, we assume that terms which involve products of perturbation variables (e.g., last term on right-hand side of (3.1)) can be neglected since $(\ )' \ll \overline{(\ )}$. In addition,

$$\frac{\partial(\tilde{\ })}{\partial\lambda^*} = is(\tilde{\ })$$

and

$$\frac{\partial(\tilde{\ })}{\partial t} = -i\sigma(\tilde{\ })$$

so that

$$\frac{\partial\tilde{u}}{\partial\lambda^*} = \frac{\partial}{\partial\lambda^*}\left(u'(\theta,z)e^{i(s\lambda^*-\sigma t)}\right) = is\left[u'(\theta,z)e^{i(s\lambda^*-\sigma t)}\right] = is\tilde{u}$$

Therefore Eq. 3.1 can be simplified to:

$$\frac{u}{a\cos\theta}\frac{\partial u}{\partial\lambda^*} = \left[\frac{\overline{u}\ isu'}{a\cos\theta} + \frac{u'}{a\cos\theta}\frac{\partial\overline{u}}{\partial\lambda^*}\right]e^{i(s\lambda^*-\sigma t)}$$

By using this method to linearize the other terms in Eqs. (2.1-2.5), our system of equations now in their linearized perturbation form are given as follows where $e^{i(s\lambda^*-\sigma t)}$ has been factored out.

<u>Zonal-momentum</u>

$$\left[-i\sigma + \frac{is\overline{u}}{a\cos\theta} + \alpha_R + \frac{\overline{v}}{a}\frac{\partial}{\partial\theta} + \overline{w}\frac{\partial}{\partial z} - \frac{\overline{v}}{a}\tan\theta - \frac{g}{p}\frac{\partial}{\partial z}\overline{M_c} - \frac{g}{p}\frac{\partial}{\partial z}\frac{\mu\dagger}{H}\frac{\partial}{\partial z}\right]u'$$

$$+\left[\frac{1}{a}\frac{\partial\overline{u}}{\partial\theta} - \frac{\overline{u}}{a}\tan\theta - f\right]v' + \left[\frac{\partial\overline{u}}{\partial z}\right]w' + \left[\frac{is}{a\cos\theta}\right]\Phi' = -\frac{g}{p}\frac{\partial}{\partial z}\left[\overline{M_c}u_c' + M_c'(\overline{u}_c - \overline{u})\right] \quad (3.2)$$

Meridional-momentum

$$\left[\frac{2\overline{u}}{a}\tan\theta + f\right]u' + \left[-i\sigma + \frac{is\overline{u}}{a\cos\theta} + \alpha_r + \frac{\overline{v}}{a}\frac{\partial}{\partial\theta} + \overline{w}\frac{\partial}{\partial z} + \frac{1}{a}\frac{\partial\overline{v}}{\partial\theta} - \frac{g}{p}\frac{\partial}{\partial z}\overline{M_c} - \frac{g}{p}\frac{\partial}{\partial z}\frac{\mu\dagger}{H}\frac{\partial}{\partial z}\right]v'$$

$$+\left[\frac{\partial\overline{v}}{\partial z}\right]w' + \left[\frac{1}{a}\frac{\partial}{\partial\theta}\right]\Phi' = -\frac{g}{p}\frac{\partial}{\partial z}\left[\overline{M_c}v_c' + M_c'(\overline{v}_c - \overline{v})\right] \tag{3.3}$$

Continuity

$$\left[\frac{is}{a\cos\theta}\right]u' + \left[\frac{1}{a}\frac{\partial}{\partial\theta} - \frac{\tan\theta}{a}\right]v' + \left[\frac{\partial}{\partial z} - 1\right]w' = 0 \tag{3.4}$$

Thermodynamic

$$\left[\frac{1}{a}\frac{\partial\overline{T}}{\partial\theta}\right]v' + \left[\frac{\partial\overline{T}}{\partial z} + \overline{T}\kappa\right]w'$$

$$+\left[-i\sigma + \frac{is\overline{u}}{a\cos\theta} + \alpha_N + \frac{\overline{v}}{a}\frac{\partial}{\partial\theta} + \overline{w}\frac{\partial}{\partial z} + \overline{w}\kappa - \frac{g}{p}\frac{\partial}{\partial z}\frac{\hat{\mu}\dagger}{H}\frac{\partial}{\partial z}\right]T' = \frac{Q'}{c_p} \tag{3.5}$$

Hydrostatic approximation

$$\left[\frac{\partial}{\partial z}\right]\Phi' + [-R]T' = 0 \tag{3.6}$$

4 Coordinate Stretching

To allow us the capability of stretching the coordinates in certain regions of the model's domain, (e.g., to increase resolution in regions of interesting phenomena) we transformed the vertical (z) and horizontal (θ) coordinates of the model into the independent variables $\lambda(z)$ and $\eta(y)$, respectively. By defining:

$$\eta = \eta(y) \text{ is the stretched latitudinal coordinate}$$

$$y = a\theta \text{ is the latitudinal distance from the equator}$$

$$\eta' = \frac{d\eta}{dy}$$

$$\frac{\partial}{\partial\theta} = a\eta'\frac{\partial}{\partial\eta}$$

$$\lambda = \lambda(z) \text{ is the stretched vertical coordinate}$$

$$\lambda' = \frac{\partial\lambda}{\partial z}$$

$$\frac{\partial}{\partial z} = \lambda'\frac{\partial}{\partial\lambda}$$

$$k = \frac{s}{a\cos\theta}$$

$$\nu = \frac{\mu_t}{\overline{\rho}} \text{ is the kinematic coefficient of viscosity}$$

$$\hat{\nu} = \frac{\hat{\mu}_t}{\overline{\rho}} \text{ is the kinematic coefficient of thermal diffusion}$$

$$p = \overline{\rho}R\overline{T} = \overline{\rho}g\overline{H}$$

Eqs. 3.2–3.6 can then be written as:

Zonal-momentum

$$\left[-i\sigma + ik\overline{u} + \alpha_R - \frac{\overline{v}}{a}\tan\theta + \overline{v}\eta'\frac{\partial}{\partial\eta} + \overline{w}\lambda'\frac{\partial}{\partial\lambda} - \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\overline{M}_c - \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\frac{p\nu}{gH^2}\lambda'\frac{\partial}{\partial\lambda}\right]u'$$

$$+\left[\eta'\frac{\partial\overline{u}}{\partial\eta} - \frac{\overline{u}}{a}\tan\theta - f\right]v' + \left[\lambda'\frac{\partial\overline{u}}{\partial\lambda}\right]w' + [ik]\Phi' + \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\left(\overline{M}_c u_c'\right) = -\frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\left[M_c'(\overline{u}_c - \overline{u})\right] \tag{4.1}$$

Meridional-momentum

$$\left[\frac{2\overline{u}}{a}\tan\theta + f\right] u'$$

$$+\left[-i\sigma + ik\overline{u} + \alpha_R + \eta'\frac{\partial\overline{v}}{\partial\eta} + \overline{v}\eta'\frac{\partial}{\partial\eta} + \overline{w}\lambda'\frac{\partial}{\partial\lambda'} - \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\overline{M}_c - \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\frac{p\nu}{gH^2}\lambda'\frac{\partial}{\partial\lambda}\right] v' \quad (4.2)$$

$$+\left[\lambda'\frac{\partial\overline{v}}{\partial\lambda}\right] w' + \left[\eta'\frac{\partial}{\partial\eta}\right]\Phi' + \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\left(\overline{M}_c v_c'\right) = -\frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\left[M_c'(\overline{v}_c - \overline{v})\right]$$

Continuity

$$[ik]u' + \left[\eta'\frac{\partial}{\partial\eta} - \frac{\tan\theta}{a}\right] v' + \left[\lambda'\frac{\partial}{\partial\lambda} - 1\right] w' = 0 \quad (4.3)$$

Thermodynamic

$$\left[\eta'\frac{\partial\overline{T}}{\partial\eta}\right] v' + \left[\lambda'\frac{\partial\overline{T}}{\partial\lambda} + \overline{T}\kappa\right] w'$$

$$+\left[-i\sigma + ik\overline{u} + \alpha_N + \overline{v}\eta'\frac{\partial}{\partial\eta} + \overline{w}\lambda'\frac{\partial}{\partial z} + \overline{w}\kappa - \frac{g}{p}\lambda'\frac{\partial}{\partial\lambda}\frac{p\hat{\nu}}{gH^2}\lambda'\frac{\partial}{\partial\lambda}\right] T' = \frac{Q'}{c_p} \quad (4.4)$$

Hydrostatic approximation

$$\left[\lambda'\frac{\partial}{\partial\lambda}\right]\Phi' + [-R]T' = 0 \quad (4.5)$$

5 Flux Form of Equations

To satisfy general conservation properties in finite difference form and to place the vertical advection and cumulus friction terms in the same form, we have chosen at this time to rewrite the equations in flux form. The advection operator in the meridional plane becomes a flux operator when combined with the continuity equation for the basic state:

$$\overline{v}\eta'\frac{\partial}{\partial\eta} + \overline{w}\lambda'\frac{\partial}{\partial\lambda} \equiv \frac{\eta'}{\cos\theta}\frac{\partial}{\partial\eta}\overline{v}\cos\theta + \frac{\lambda'}{(p/p_o)}\frac{\partial}{\partial\lambda}\frac{p}{p_o}\overline{w}$$

Using this identity the equations in flux form become:

Zonal-momentum

$$(-i\sigma + ik\overline{u} + \alpha_R - \frac{\overline{v}}{a}\tan\theta)u' + \frac{\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(\overline{v}\cos\theta u') + \left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\left[\left(\frac{p}{p_o}\overline{w} - \frac{g\overline{M_c}}{p_o}\right)u'\right]$$

$$-\left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\frac{p}{p_o}\frac{\nu}{H^2}\lambda'\frac{\partial u'}{\partial\lambda} + \left(\eta'\frac{\partial\overline{u}}{\partial\eta} - \frac{\overline{u}}{a}\tan\theta - f\right)v' + \left(\lambda'\frac{\partial\overline{u}}{\partial\lambda}\right)w' + (ik)\Phi' \tag{5.1}$$

$$+\left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\left(\frac{g\overline{M_c}}{p_o}\right)u'_c = -\left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\left[\frac{gM'_c}{p_o}(\overline{u}_c - \overline{u})\right]$$

Meridional-momentum

$$\left(\frac{2\overline{u}}{a}\tan\theta + f\right)u' + \left(-i\sigma + ik\overline{u} + \alpha_R + \eta'\frac{\partial\overline{v}}{\partial\eta}\right)v' + \frac{\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(\overline{v}\cos\theta v')$$

$$+\frac{\lambda'}{p/p_o}\frac{\partial}{\partial\lambda}\left[\left(\frac{p}{p_o}\overline{w} - \frac{g\overline{M}_c}{p_o}\right)v'\right] - \left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\frac{p}{p_o}\frac{\nu}{H^2}\lambda'\frac{\partial v'}{\partial\lambda} + \left(\lambda'\frac{\partial\overline{v}}{\partial\lambda}\right)w' \tag{5.2}$$

$$+\eta'\frac{\partial\Phi'}{\partial\eta} + \frac{\lambda'}{p/p_o}\frac{\partial}{\partial\lambda}\left(\frac{g\overline{M}_c}{p_o}\right)v'_c = -\left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\left[\frac{gM'_c}{p_o}(\overline{v}_c - \overline{v})\right]$$

Continuity

$$(ik)u' + \frac{\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(v'\cos\theta) + \left(\frac{\lambda'}{p/p_o}\right)\frac{\partial}{\partial\lambda}\left(\frac{p}{p_o}w'\right) = 0 \tag{5.3}$$

Thermodynamic

$$\left(\eta'\frac{\partial\overline{T}}{\partial\eta}\right)v' + \left(\lambda'\frac{\partial\overline{T}}{\partial\lambda} + \overline{T}\kappa\right)w' + (-i\sigma + ik\overline{u} + \alpha_N)T'$$

$$\overline{v}\eta'\frac{\partial T'}{\partial\eta} + e^{-\kappa z}\overline{w}\lambda'\frac{\partial}{\partial\lambda}(e^{\kappa z}T') - \frac{\lambda'}{p/p_o}\frac{\partial}{\partial\lambda}\frac{p}{p_o}\frac{\hat{\nu}}{H^2}\lambda'\frac{\partial T'}{\partial\lambda} = \frac{Q'}{c_p} \tag{5.4}$$

Hydrostatic approximation

$$\left(\lambda'\frac{\partial}{\partial\lambda}\right)\Phi' + (-R)T' = 0 \tag{5.5}$$

6 Non-dimensional Form of Equations

It is now convenient to non-dimensionalize the equations so that the solutions and the coefficients of the terms in the equations are O(1). The following variables in our system of equations are non-dimensionalized as follows:

$$
\begin{aligned}
f &\quad \text{by} \quad 2\Omega \\
\sigma &\quad \text{by} \quad U_o/L \\
t &\quad \text{by} \quad L/U_o \\
x, y &\quad \text{by} \quad L = a/s' \text{ where } s' = \begin{cases} 1 \text{ for } s = 0 \\ s \text{ for } s \neq 0 \end{cases} \\
z &\quad \text{by} \quad 1 \\
\overline{u}, u', \overline{v}, v' &\quad \text{by} \quad U_o \\
\overline{w}, \frac{g\overline{M}_c}{p_o}, \frac{gM'_c}{p_o}, w' &\quad \text{by} \quad \frac{U_o}{L} \\
\Phi' &\quad \text{by} \quad 2\Omega U_o L \\
T' &\quad \text{by} \quad 2\Omega U_o L/R
\end{aligned}
$$

To simplify the form of the non-dimensional equations we define the following quantities:

$$
\begin{aligned}
R_o &\equiv \frac{U_o}{2\Omega L} \text{ is the Rossby number} \\
R_i &= \frac{R\Gamma_1}{U_o^2} \text{ is the Richardson number} \\
Fr &= \frac{U_o^2}{R\Gamma_1} = Ro^2\epsilon = Ri^{-1} \text{ is the Froude number} \\
\xi &\equiv \frac{p}{p_o} = e^{-z} \text{ is non-dimensional pressure} \\
E_o &\equiv \frac{\nu}{2\Omega H^2} = E_o(\eta, \lambda) \\
\hat{E}_o &\equiv \frac{\hat{\nu}}{2\Omega H^2} = \hat{E}_o(\eta, \lambda) \\
\epsilon &= \frac{4\Omega^2 L^2}{R\Gamma_1} = \frac{L^2}{a^2}\frac{4\Omega^2 a^2}{R\Gamma_1} = \left(\frac{2\Omega L}{U_o}\right)^2 \frac{U_o^2}{R\Gamma_1} = R_o^{-2} Fr = R_o^{-2} R_i^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\delta &= \tfrac{L}{a} = \tfrac{1}{s'} \\
f^* &= \tfrac{f}{2\Omega} \\
\sigma^* &= \tfrac{\sigma}{U_o/L} \\
\overline{u}^*, u'^*, \overline{v}^*, v'^* &= \tfrac{\overline{u}}{U_o}, \tfrac{u'}{U_o}, \tfrac{\overline{v}}{U_o}, \tfrac{v'}{U_o} \\
\overline{w}^*, w'^* &= \tfrac{\overline{w}}{U_o/L}, \tfrac{w'}{U_o/L} \\
\overline{M}_c^*, M_c'^* &= \tfrac{g\overline{M}_c}{p_oU_o/L}, \tfrac{gM_c'}{p_oU_o/L} \\
\Phi'^* &= \tfrac{\Phi'}{2\Omega LU_o} \\
T'^* &= \tfrac{T'}{2\Omega LU_o/R} \\
Q'^* &= \tfrac{L}{U_o\Gamma_1}\tfrac{Q'}{c_p}, \text{ where } \Gamma_1 \text{ is a typical stability}
\end{aligned}
$$

where the ( )* represents a non-dimensional quantity. Using these definitions and notation the linearized system of equations in flux form are non-dimensionalized as follows. The zonal momentum equation is multiplied by $(1/2\Omega U_o)$, and thus can be written as

$$
\frac{U_o}{2\Omega L}\left(\frac{-i\sigma + \frac{is\overline{u}}{a\cos\theta} + \alpha_R}{U_o/L} - \frac{\overline{v}/U_o}{a/L}\tan\theta\right)\frac{u'}{U_o} + \frac{U_o}{2\Omega L}\frac{L\eta'}{\cos\theta}\frac{\partial}{\partial\eta}\left(\frac{\overline{v}}{U_o}\cos\frac{u'}{U_o}\right)
$$

$$
+\frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[\left(\frac{\xi\overline{w}}{U_o/L} - \frac{g\overline{M}_c}{p_oU_o/L}\right)\frac{u'}{U_o}\right] - \frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\xi\frac{\nu}{2\Omega H^2}\lambda'\frac{\partial(u'/u_o)}{\partial\lambda}
$$

$$
+\left(\frac{U_o}{2\Omega L}L\eta'\frac{\partial\overline{u}/U_o}{\partial\eta} - \frac{U_o}{2\Omega L}\frac{L}{a}\frac{\overline{u}}{U_o}\tan\theta - \frac{f}{2\Omega}\right)\frac{v'}{U_o} + \frac{U_o}{2\Omega L}\left(\lambda'\frac{\partial\overline{u}/U_o}{\partial\lambda}\right)\frac{w'}{U_o/L}
$$

$$
+\frac{is}{a\cos\theta}\frac{2\Omega U_oL}{2\Omega U_o}\frac{\Phi'}{2\Omega U_oL} + \frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left(\frac{g\overline{M}_c}{p_oU_o/L}\right)\frac{u_c'}{U_o}
$$

$$
= -\frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[\frac{gM_c'}{p_oU_o/L}\left(\frac{\overline{u}_c}{U_o} - \frac{\overline{u}}{U_o}\right)\right]
$$

Rewriting this equation in an alternate form yields

$$R_o\left(-\frac{i\sigma}{U_o/L}+\frac{is/s'}{\cos\theta}\overline{u}^*+\frac{\alpha_R}{U_o/L}-\frac{1}{s'}\overline{v}^*\tan\theta\right)u'^*+R_o\frac{L\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(\overline{v}^*\cos\theta u'^*)$$

$$+R_o\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[(\xi\,\overline{w}^*-\overline{M}_c^*)u'^*\right]-\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\xi E_o\lambda'\frac{\partial u'^*}{\partial\lambda}$$

$$+\left(R_oL\eta'\frac{\partial\overline{u}^*}{\partial\eta}-R_o\frac{\overline{u}}{s'}\tan\theta-f^*\right)v'^*+R_o\left(\lambda'\frac{\partial\overline{u}^*}{\partial\lambda}\right)w'^*$$

$$+\frac{is/s'}{\cos\theta}\Phi'^*+R_o\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}(\overline{M}_c^*)u_c'^*=-R_o\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[M_c'^*(\overline{u}_c^*-\overline{u}^*)\right] \tag{6.1}$$

Similarly, by multiplying the v-momentum equation by $(1/2\Omega U_o)$, we can write

$$\left(\frac{U_o}{2\Omega L}\frac{L}{a}\frac{2\overline{u}}{U_o}\tan\theta+\frac{f}{2\Omega}\right)\frac{u'}{U_o}+\frac{U_o}{2\Omega L}\left(\frac{-i\sigma+\frac{is\overline{u}}{a\cos\theta}+\alpha_R}{U_o/L}+L\eta'\frac{\partial\overline{v}_i/U_o}{\partial\eta}\right)\frac{v'}{U_o}$$

$$+\frac{U_o}{2\Omega L}\frac{L\eta'}{\cos\theta}\frac{\partial}{\partial\eta}\left(\frac{\overline{v}}{U_o}\cos\theta\frac{v'}{U_o}\right)+\frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[\left(\xi\frac{\overline{w}}{U_o/L}-\frac{g\overline{M}_c}{p_oU_o/L}\right)\frac{v'}{U_o}\right]$$

$$-\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\xi\frac{\nu}{2\Omega H^2}\lambda'\frac{\partial}{\partial\lambda}\left(\frac{v'}{U_o}\right)+\frac{U_o}{2\Omega L}\left(\lambda'\frac{\partial\overline{v}/U_o}{\partial\lambda}\right)\frac{w'}{U_o/L}$$

$$+\frac{2\Omega U_oL\eta'}{2\Omega U_o}\frac{\partial}{\partial\eta}\frac{\Phi'}{2\Omega U_oL}+\frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left(\frac{g\overline{M}_c}{p_oU_o/L}\right)\frac{v_c'}{U_o}$$

$$=-\frac{U_o}{2\Omega L}\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[\frac{gM_c'}{p_oU_o/L}\left(\frac{\overline{v}_c}{U_o}-\frac{\overline{v}}{U_o}\right)\right]$$

Alternately

$$\left(R_o\frac{2\overline{u}^*}{s'}\tan\theta+f^*\right)u'^*+R_o\left(-\frac{i\sigma}{U_o/L}+\frac{is/s'}{\cos\theta}\overline{u}^*+\frac{\alpha_R}{U_o/L}+L\eta'\frac{\partial\overline{v}^*}{\partial\eta}\right)v'^*$$

$$+R_o\frac{L\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(\overline{v}^*\cos\theta v'^*)+R_o\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[\left(\xi\,\overline{w}^*-\overline{M}_c^*\right)v'^*\right]$$

$$-\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\xi E_o \lambda' \frac{\partial v'^*}{\partial\lambda} + R_o\left(\lambda'\frac{\partial\overline{v}^*}{\partial\lambda}\right) w'^*$$

$$+L\eta'\frac{\partial\Phi'^*}{\partial\eta} + R_o\frac{\lambda'}{\xi}\frac{\partial\overline{M}_c^*}{\partial\lambda}v_c'^* = -R_o\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left[M_c'^*(\overline{v}_c^* - \overline{v}^*)\right] \tag{6.2}$$

The continuity equation is multiplied by (L/U_o)

$$\frac{is}{a\cos\theta}\frac{L}{U_o}u' + \frac{L}{U_o}\frac{\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(v'\cos\theta) + \frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\left(\xi w'\frac{L}{U_o}\right) = 0$$

Alternately,

$$\left(\frac{is/s'}{\cos\theta}\right)u'^* + \frac{L\eta'}{\cos\theta}\frac{\partial}{\partial\eta}(v'^*\cos\theta) + \frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}(\xi w'^*) = 0 \tag{6.3}$$

The thermodynamic equation is multiplied by $(L/U_o\Gamma_1)$

$$\frac{L}{\Gamma_1}\left(\eta'\frac{\partial\overline{T}}{\partial\eta}\right)\frac{v'}{U_o} + \frac{L}{\Gamma_1}\left(\lambda'\frac{\partial\overline{T}}{\partial\lambda} + \kappa\overline{T}\right)\frac{w'}{U_o} + \frac{L}{U_o\Gamma_1}\left[\left(-i\sigma + \frac{is\overline{u}}{a\cos\theta} + \alpha_N\right)T'\right.$$

$$\left.+\overline{v}\eta'\frac{\partial T'}{\partial\eta} + e^{-\kappa z}\overline{w}\lambda'\frac{\partial}{\partial\lambda}\left(e^{\kappa z}T'\right)\right] - \frac{L}{U_o\Gamma_1}\frac{\lambda'}{p/p_o}\frac{\partial}{\partial\lambda}\frac{p}{p_o}\frac{\overline{\nu}}{H^2}\lambda'\frac{\partial T'}{\partial\lambda} = \frac{L}{U_o\Gamma_1}\frac{Q'}{c_p}$$

Alternately,

$$\left[L\eta'\frac{\partial}{\partial\eta}\left(\frac{\overline{T}}{\Gamma_1}\right)\right]v'^* + \left(\frac{\lambda'\frac{\partial\overline{T}}{\partial\lambda} + \kappa\overline{T}}{\Gamma_1}\right)w'^* + R_o\epsilon\left[\left(\frac{-i\sigma}{U_o/L} + \frac{is/s'}{\cos\theta}\overline{u}^* + \frac{\alpha_N}{U_o/L}\right)\right.$$

$$\left.+\overline{v}^*L\eta'\frac{\partial}{\partial\eta} + e^{-\kappa z}\overline{w}^*\lambda'\frac{\partial}{\partial\lambda}e^{\kappa z}\right]T'^* - \epsilon\frac{\lambda'}{\xi}\frac{\partial}{\partial\lambda}\xi\hat{E}_o\lambda'\frac{\partial T'^*}{\partial\lambda} = Q'^* \tag{6.4}$$

Finally, the hydrostatic approximation is multiplied by $(1/2\Omega LU_o)$ which yields

$$-\frac{RT'}{2\Omega LU_o} + \lambda' \frac{\partial}{\partial \lambda}\left(\frac{\Phi'}{2\Omega LU_o}\right) = 0$$

Alternately,

$$-T'^{*} + \lambda' \frac{\partial \Phi'^{*}}{\partial \lambda} = 0 \tag{6.5}$$

7 Equations Written with Coefficients

One can readily note from the previous section that the equations in their linearized non-dimensional flux form are rather lengthy and obviously would be cumbersome to work with. In view of this difficulty we have chosen to rewrite Eqs. 6.1–6.5 with coefficients which operate upon the non-dimensional dependent variables. With this strategy the appearance of the equations is simplified, and the programming aspects of the problem become more tractable. These coefficients are defined as follows:

$$
\begin{aligned}
A(\eta,\lambda) &= R_o\left(\frac{-i\sigma}{U_o/L}+\frac{is/s'}{\cos\theta}\overline{u}^*\right)=\frac{U_o}{2\Omega L}\left(\frac{-i\sigma}{U_o/L}+\frac{is/s'}{cos\theta}\frac{\overline{u}}{U_o}\right)\\
&= \frac{-i\sigma}{2\Omega}+\frac{i(s/s')\overline{u}}{2\Omega L\cos\theta}=\frac{-i\sigma}{2\Omega}+\frac{is}{\cos\theta}\frac{\overline{u}}{2\Omega a}
\end{aligned}
$$

$$
AR(\eta,\lambda) = A+R_o\frac{\alpha_R}{U_o/L}=A+\frac{U_o}{2\Omega L}\left(\frac{\alpha_R}{U_o/L}\right)=A+\frac{\alpha_R}{2\Omega}
$$

$$
AN(\eta,\lambda) = A+R_o\frac{\alpha_N}{U_o/L}=A+\frac{U_o}{2\Omega L}\left(\frac{\alpha_N}{U_o/L}\right)=A=\frac{\alpha_N}{2\Omega}
$$

$$
Q9(\eta,\lambda) = AR-R_o\frac{\overline{v}^*}{s'}\tan\theta=AR-\frac{U_o}{2\Omega L}\frac{\overline{v}}{s'U_o}\tan\theta=AR-\frac{\overline{v}}{2\Omega a}\tan\theta
$$

$$
P1(\eta) = R_o\frac{L\eta'}{\cos\theta}=\frac{U_o}{2\Omega L}\frac{L\eta'}{\cos\theta}=\frac{U_o}{2\Omega}\frac{\eta'}{\cos\theta}
$$

$$
P2(\eta,\lambda) = \overline{v}^*\cos\theta=\frac{\overline{v}}{U_o}\cos\theta
$$

$$
P3(\eta,\lambda) = R_o(\xi\overline{w}^*-\overline{M}_c^*)=\frac{U_o}{2\Omega L}\left(\frac{\xi\overline{w}}{U_o/L}-\frac{g\overline{M}_c}{p_oU_o/L}\right)=\frac{\xi\overline{w}}{2\Omega}-\frac{g\overline{M}_c}{2\Omega p_o}
$$

$$D(\lambda) = \frac{\lambda'}{\xi}$$

$$E(\eta, \lambda) = \xi E_o \lambda'$$

$$F(\eta, \lambda) = \xi \hat{E}_o \lambda'$$

$$Q10(\eta, \lambda) = R_o L\eta' \frac{\partial \overline{u}^*}{\partial \eta} - R_o \frac{\overline{u}^*}{s'} \tan\theta - f^* = \frac{L\eta'}{2\Omega L} \frac{\partial \overline{u}}{\partial \eta} - \frac{\overline{u}}{2\Omega a} \tan\theta - \frac{f}{2\Omega}$$

$$Q3(\eta, \lambda) = R_o \lambda' \frac{\partial \overline{u}^*}{\partial \lambda} = \frac{\lambda'}{2\Omega L} \frac{\partial \overline{u}}{\partial \lambda}$$

$$Q2(\eta) = i \frac{s/s'}{\cos\theta}$$

$$AL2(\eta, \lambda) = R_o \frac{\lambda'}{\xi} \frac{\partial}{\partial \lambda} \overline{M}_c^* = D \frac{\partial}{\partial \lambda} \frac{U_o}{2\Omega L} \frac{g\overline{M}_c}{p_o U_o / L} = D \frac{\partial}{\partial \lambda} \left(\frac{g\overline{M}_c}{2\Omega p_o} \right)$$

$$B1(\eta, \lambda) = -R_o \frac{\lambda'}{\xi} \frac{\partial}{\partial \lambda} \left[M_c'^* (\overline{u}_c^* - \overline{u}^*) \right] = D \frac{\partial}{\partial \lambda} \frac{U_o}{2\Omega L} \left[\frac{gM_c'}{p_o U_o / L} \left(\frac{\overline{u}}{U_o} - \frac{\overline{u}_c}{U_o} \right) \right]$$

$$= D \frac{\partial}{\partial \lambda} \left[\frac{gM_c'}{2\Omega p_o} \left(\frac{\overline{u}}{U_o} - \frac{\overline{u}_c}{U_o} \right) \right]$$

$$B2(\eta, \lambda) = -R_o \frac{\lambda'}{\xi} \frac{\partial}{\partial \lambda} \left[M_c'^* (\overline{v}_c^* - v^*) \right] = D \frac{\partial}{\partial \lambda} \frac{U_o}{2\Omega L} \left[\frac{gM_c'}{p_o U_c / L} \left(\frac{\overline{v}}{U_o} - \frac{\overline{v}_c}{U_o} \right) \right]$$

$$= D \frac{\partial}{\partial \lambda} \left[\frac{gM_c'}{2\omega p_o} \left(\frac{\overline{v}}{U_o} - \frac{\overline{v}_c}{U_o} \right) \right]$$

$$B3(\eta, \lambda) = Q'^* = \frac{L}{U_o \Gamma_1} \frac{Q'}{c_p}$$

$$Q11(\eta,\lambda) = R_o \frac{2\overline{u}^*}{s'} \tan\theta + f^* = \frac{U_o}{2\Omega L} \frac{2\overline{u}}{s' U_o} \tan\theta + \frac{f}{2\Omega} = \frac{2\overline{u}\tan\theta}{2\Omega a} + \frac{f}{2\Omega}$$

$$Q13(\eta,\lambda) = AR + R_o L\eta' \frac{\partial\overline{v}^*}{\partial\eta} = AR + \frac{L\eta'}{2\Omega L} \frac{\partial\overline{v}}{\partial\eta}$$

$$Q5(\eta,\lambda) = R_o \lambda' \frac{\partial\overline{v}^*}{\partial\lambda} = \frac{\lambda'}{2\Omega L} \frac{\partial\overline{v}}{\partial\lambda}$$

$$C1(\eta) = L\eta' = \frac{a\eta'}{s'}$$

$$Q22(\eta) = \frac{is/s'}{\cos\theta}$$

$$C2(\eta) = \frac{L\eta'}{\cos\theta}$$

$$C3(\eta) = \cos\theta$$

$$C5(\lambda) = \xi$$

$$D2(\lambda) = \lambda'$$

$$Q6(\eta,\lambda) = \frac{L\eta'}{\Gamma_1} \frac{\partial\overline{T}}{\partial\eta}$$

$$Q7(\eta,\lambda) = \frac{1}{\Gamma_1} \left(\lambda' \frac{\partial\overline{T}}{\partial\lambda} + \kappa\overline{T} \right)$$

$$C7(\lambda) = e^{\kappa z}$$

$$
\begin{aligned}
C8 &= \epsilon = \frac{4\Omega^2 L^2}{R\Gamma_1} \\
C8D(\lambda) &= C8 \cdot D \\
CC(\eta,\lambda) &= R_o \overline{v}^* L\eta' = \frac{U_o}{2\Omega L}\frac{\overline{v}}{U_o} L\eta' = \frac{\overline{v}}{2\Omega L} L\eta' \\
C6(\eta,\lambda) &= R_o e^{-\kappa z}\overline{w}^* \lambda' = \frac{U_o}{2\Omega L} e^{\kappa z}\frac{\overline{w}L}{U_o}\lambda' = \frac{\overline{w}}{2\Omega} e^{-\kappa z}\lambda' \\
C8AN(\eta,\lambda) &= C8 \cdot AN \\
C8CC(\eta,\lambda) &= C8 \cdot CC \\
C8C6(\eta,\lambda) &= C8 \cdot C6
\end{aligned}
$$

By defining these coefficients, our systems of equations can now be written as:

Zonal-momentum

$$\left(Q9+P1\frac{\partial}{\partial\eta}P2+D\frac{\partial}{\partial\lambda}P3-D\frac{\partial}{\partial\lambda}E\frac{\partial}{\partial\lambda}\right)u'^{*}+(Q10)v'^{*}+(Q3)w'^{*}$$

$$+(Q2)\phi'^{*}+(AL2)u_c'^{*}=(B1) \tag{7.1}$$

Meridional-momentum

$$(Q11)u'^{*}+\left(Q13+P1\frac{\partial}{\partial\eta}P2+D\frac{\partial}{\partial\lambda}P3-D\frac{\partial}{\partial\lambda}E\frac{\partial}{\partial\lambda}\right)v'^{*}+(Q5)w'^{*}$$

$$+(C1)\frac{\partial\phi'^{*}}{\partial\eta}+(AL2)v_c'^{*}=(B2) \tag{7.2}$$

Continuity

$$(Q22)u'^{*}+\left(C2\frac{\partial}{\partial\eta}C3\right)v'^{*}+\left(D\frac{\partial}{\partial\lambda}C5\right)w'^{*}=0 \tag{7.3}$$

Thermodynamic

$$(Q6)v'^{*}+(Q7)w'^{*}+\left(C8AN+C8CC\frac{\partial}{\partial\eta}+C8C6\frac{\partial}{\partial\lambda}C7-C8D\frac{\partial}{\partial\lambda}F\frac{\partial}{\partial\lambda}\right)T'^{*}=(B3) \tag{7.4}$$

Hydrostatic approximation

$$-T'^{*}+(D2)\frac{\partial\phi'^{*}}{\partial\lambda}=0 \tag{7.5}$$

8 Boundary Conditions

In this section we consider the boundary conditions used with our model. These conditions will be expressed in non-dimensional form, and where necessary to simplify the equations, coefficients will be used.

8.1 Top and bottom boundaries

Since our system of equations can be reduced to an eighth order differential equation in the vertical, the continuous solution requires eight boundary conditions at the top and bottom of the model. These boundary conditions in the stretched coordinate system (η, λ) are given as follows.

At the upper boundary

$$\frac{\partial u'}{\partial \lambda} = \frac{\partial v'}{\partial \lambda} = \frac{\partial T'}{\partial \lambda} = 0 \text{ (due to large dissipation)} \qquad (8.1) - (8.3)$$

$$w' = 0 \text{ (rigid upper lid)} \qquad (8.4)$$

In non-dimensional units these equations appear in the same form as in (8.1)–(8.4) except now the dependent variables are replace by $(u'^*, v'^*, w'^*$ and $T'^*)$.

At the lower boundary

$$\left.\begin{aligned} \frac{\partial u'}{\partial \lambda} &= BCU\ u' \\ \frac{\partial v'}{\partial \lambda} &= BCV\ v' \\ \frac{\partial T'}{\partial \lambda} &= BCT\ T' \end{aligned}\right\} \text{ bulk aerodynamic parameterization}$$

where

$$\begin{pmatrix} BCU \\ BCV \\ BCT \end{pmatrix} = \frac{\overline{H}_o \cdot CD \cdot V_o}{\lambda'} \begin{pmatrix} \nu_o^{-1} \\ \nu_o^{-1} \\ \hat{\nu}_o^{-1} \end{pmatrix}$$

$$\overline{H}_o = \frac{R\overline{T}_o}{g}$$

$$\nu_o = \nu(z=0)$$

$$\hat{\nu}_o = \hat{\nu}(z=0)$$

The constants T_o, CD, v_o are defined is Section 2.

In non-dimensional units (8.5) - (8.7) can be written as

$$\frac{\partial}{\partial\lambda}\begin{pmatrix} u'^* \\ v'^* \\ T'^* \end{pmatrix} - (BCU \quad BCV \quad BCT)\begin{pmatrix} u'^* \\ v'^* \\ T'^* \end{pmatrix} = 0 \qquad (8.8)-(8.10)$$

The fourth boundary condition at the bottom is given as:

$$\tilde{w}' = 0\left(\text{where } g\tilde{w}' = \frac{d\phi'}{dt}\right)$$

Expanding out the total derivative $\left(\frac{d}{dt}\right)$ we find

$$\frac{d\phi'}{dt} = i\left(-\sigma + \frac{s\overline{u}}{a\cos\theta}\right)\phi' + \eta'\frac{\partial\phi'}{\partial\eta}v' + \overline{v}\eta'\frac{\partial\phi'}{\partial\eta} + w'\lambda'\frac{\partial\overline{\Phi}}{\partial\lambda} = 0 \qquad (8.11)$$

or in non-dimensional units

$$B35\phi'^* + B32v'^* + AC35\frac{\partial\phi'^*}{\partial\eta} + B33w'^* = 0 \qquad (8.12)$$

where the coefficients in (8.12) are defined as

$$\begin{aligned}
B35(\eta) &= i\left(\frac{s\overline{u}}{s'U_o\cos\theta} - \frac{\sigma}{U_o/L}\right) \\
B32(\eta) &= \eta' L\frac{\partial\overline{\phi}/(2\Omega U_o L)}{\partial\eta} \\
AC35(\eta) &= \frac{\overline{v}}{U_o}\eta' L \\
B33(\eta) &= \lambda'\frac{\partial\overline{\phi}/(2\Omega U_o L)}{\partial\lambda} = R\overline{T}/(2\Omega L U_o)
\end{aligned}$$

8.2 Horizontal boundaries

8.2.1 Boundaries not at poles

In the horizontal direction the system of equations is second order, so that two boundary conditions are required on the sides of the model. These boundary conditions were chosen as $v = 0$ which inhibits flow through the side boundaries of the model. With this condition the v-momentum equation on the sides is replaced with $v' = 0$. In addition since $\overline{v} = 0$ on boundaries, coefficients involving $\overline{v}$ (*i.e.*, P2 and C8CC) are also zero. This will be shown explicitly in Section 9.2.2.1 when the discretized form of the equation is presented.

8.2.2 Boundaries at poles

When a horizontal boundary is extended to the pole, the boundary conditions which are imposed at that point are shown in the following table. The subscript p in this table implies that the condition is imposed at the pole. Details on the formulation of these boundary conditions are given in Appendix A.

Table 8.2.2

variable	$s = 0$	$s = \pm 1$	$\lvert s \rvert > 1$
u'	$u'_p = 0$	$\left(\frac{\partial u'}{\partial \theta}\right)_p = 0$	$u'_p = 0$
v'	$v'_p = 0$	$\left(\frac{\partial v}{\partial \theta}\right)_p = 0$	$v'_p = 0$
w'	$(\nabla \cdot \mathbf{v}')_p + \frac{\partial w'}{\partial z} - w' = 0$	$w'_p = 0$	$w'_p = 0$
T'	$\left(\frac{\partial T}{\partial \theta}\right)_p = 0$	$T'_p = 0$	$T'_p = 0$
ϕ'	$\left(\frac{\partial \phi'}{\partial \theta}\right)_p = 0$	$\phi'_p = 0$	$\phi'_p = 0$

For the $s = 0$ case, the condition on w' involves the horizontal divergence at the pole. This can be expressed as

$$(\nabla \cdot \mathbf{v})_p = \frac{1}{a \cos\theta_p}\left(isu'_p + \frac{\partial v'_p \cos\theta_p}{\partial\theta_p}\right)$$

since $u'_p = 0$ in this case

$$(\nabla \cdot \mathbf{v})_p = \left(\frac{\partial v'}{a\partial\theta}\right)_p - \frac{v'_p \sin\theta_p}{a\cos\theta_p}$$

Since in the second term on the right-hand side of the above equation both $v'_p = 0$ and $\cos\theta_p = 0$, we use L'Hospital's Rule to evaluate it. This is shown below.

$$-\frac{v'_p \sin\theta_p}{a\cos\theta_p} = \frac{\lim_{\theta\to\theta_p}\frac{\partial}{\partial\theta}\left(-v'\sin\theta\right)}{\lim_{\theta\to\theta_p}\frac{\partial}{\partial\theta}\left(a\cos\theta\right)} = \frac{-v'_p\cos\theta_p - \sin\theta_p\left(\frac{\partial v'}{\partial\theta}\right)_p}{-a\sin\theta_p} = \left(\frac{\partial v'}{a\partial\theta}\right)_p$$

Thus $(\nabla \cdot \mathbf{v}')_p = \left(\frac{2}{a}\frac{\partial v'}{\partial\theta}\right)_p$. Using a standard one-sided, first-order difference formula at the poles to approximate this term in the continuity equation is equivalent to assuming that the area average vertical motion, ω' (i.e., $-pw'$), between the pole and the first grid point equals the average divergence over this same area.

The continuity equation in flux form with coordinate stretching can be written as

$$\frac{2}{a}\frac{\partial v'}{\partial\theta} + \frac{\lambda'}{p/p_o}\frac{\partial}{\partial\lambda}\left(\frac{p}{p_o}w'\right) = 0$$

Note: the first term here was not expressed in stretched coordinates as $2\eta'\frac{\partial v'}{\partial\eta}$ because when the stretched coordinate is $\eta = \sin\theta$, it follows that $\eta' = \cos\theta$. Thus at the poles $\eta'_p = \cos\theta_p = 0$. For this reason the boundary conditions in table 8.2.2, which involve horizontal derivatives at the poles, are left expressed in the unstretched coordinate, θ.

In non-dimensional form with coefficients this equation becomes

$$2\frac{\partial v'^*}{\partial\theta} + \left(D\frac{\partial}{\partial\lambda}C5\right)w'^* = 0$$

In non-dimensional units the other polar boundary conditions are in the same form as in Table 8.2.2, except now the dependent variables are replaced by (u'^*, v'^*, w'^*, T'^* and ϕ'^*).

9 Discretized Equations

The equations 7.1-7.5 are finite differenced in the latitudinal and vertical directions. To do this we define the following:

$$\eta_i = \eta_S + (i-1)\Delta\eta \text{ where } i \to 1, IY$$

$$\lambda_j = \lambda_1 + (j-1)\Delta\lambda \text{ where } j \to 1, IZ$$

where

$$\begin{aligned}
\Delta\eta &= (\eta_N - \eta_S)/(IY-1) \\
\eta_S &\equiv \eta \text{ at southern boundary of model} \\
\eta_N &\equiv \eta \text{ at northern boundary of model} \\
IY &= \text{number of nodes in the latitudinal direction} \\
\Delta\lambda &= (\lambda_{IZ} - \lambda_1)/(IZ-1) \\
\lambda_{IZ} &\equiv \lambda \text{ at bottom of model atmosphere} \\
\lambda_1 &\equiv \lambda \text{ at top of model atmosphere} \\
IZ &= \text{number of levels in the vertical direction}
\end{aligned}$$

For brevity we drop the $(\ )^*$ notation, but it must be realized that the dependent variables are still non-dimensional.

In the revised version of the model we elected to use a staggered grid in the latitudinal direction. This decision was motivated by the following observations. (1) For certain choices of model parameters (e.g., $s = 0$, frequency small), adjacent perturbation variables appeared to be decoupled from each other in the latitudinal direction by exhibiting a large $2\Delta\eta$ oscillation. (2) Arakawa and Lamb (1977) have noted that proper finite differencing (i.e., staggering of the variables) is needed to properly maintain the geostropic adjustment process. For the one-dimensional shallow water equations the distribution of dependent variables which results in the best simulation of the geostropic adjustment process locates the u and v variables midway between the grid points where ϕ is carried. (3) Use of a properly staggered grid is computationally efficient in that it requires half the number of grid points as the non-staggered grid to achieve the same level of accuracy. For more details on these last two observations one should refer to Appendix B.

Based on the analysis in Appendix B, the latitudinal distribution of variables chosen for our staggered grid is shown in Fig. 9.1. In this configuration the variables w', T', and Φ' are located at whole grids points with u' and v' half way in between. The u' and v' variables are also carried at the horizontal boundaries where they are needed for computation in the thermodynamic and continuity equations.

When a coefficient or a dependent variable is required at a certain grid point but is not explicitly carried there, it is computed using linear interpolation. The example below shows how a field g, which is defined only at half grid points, is computed at a whole grid point.

$$
\begin{aligned}
g_{i,j} &= g_{i-\frac{1}{2},j} + \frac{g_{i+\frac{1}{2},j} - g_{i-\frac{1}{2},j}}{\eta_{i+\frac{1}{2}} - \eta_{i-\frac{1}{2}}} \left(\eta_{i+\frac{1}{2}} - \eta_i\right) \\
&= g_{i-\frac{1}{2},j} + \frac{g_{i+\frac{1}{2},j} - g_{i-\frac{1}{2},j}}{\Delta\eta} \left(\frac{\Delta\eta}{2}\right) \\
&= \frac{g_{i+\frac{1}{2},j} + g_{i-\frac{1}{2},j}}{2}
\end{aligned}
$$

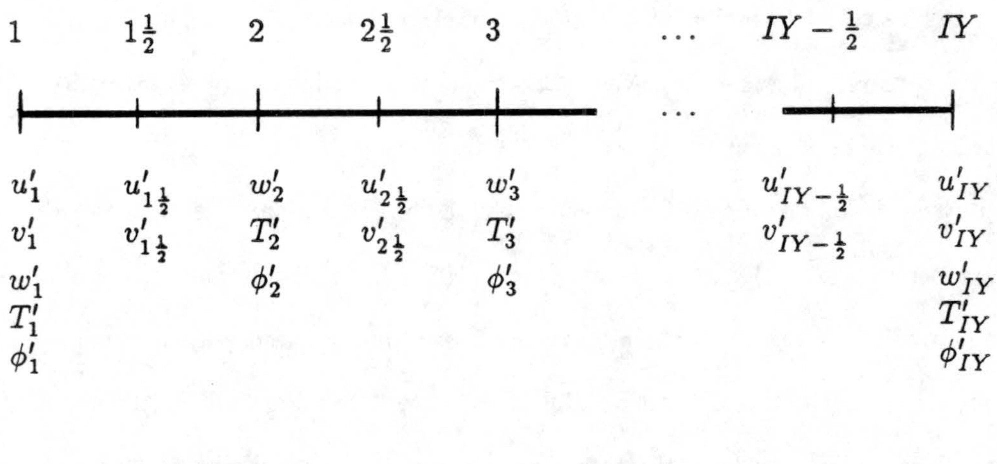


Fig. 9.1 Latitudinal distribution of perturbation variables.

9.1 Discretized equations at interior points

Using grid structure shown in Fig. 9.1 the discretized equations at the interior points are shown below.

Zonal-momentum at half grid points $(\frac{3}{2}, IY-\frac{1}{2})$: $i \rightarrow 2, IY$

$$
\begin{aligned}
& Q9_{i-\frac{1}{2},j}u'_{i-\frac{1}{2},j} + P1_{i-\frac{1}{2}}\left(P2D\ u'\right)_{i-\frac{1}{2},j} + D_j\left[\frac{(P3\ u')_{i-\frac{1}{2},j+1} - (P3\ u')_{i-\frac{1}{2},j-1}}{2\Delta\lambda}\right] \\
& -\frac{D_j}{\Delta\lambda}\left[E_{i-\frac{1}{2},j+\frac{1}{2}}\frac{u'_{i-\frac{1}{2},j+1} - u'_{i-\frac{1}{2},j}}{\Delta\lambda} - E_{i-\frac{1}{2},j-\frac{1}{2}}\frac{u'_{i-\frac{1}{2},j} - u'_{i-\frac{1}{2},j-1}}{\Delta\lambda}\right] + Q10_{i-\frac{1}{2},j}v'_{i-\frac{1}{2},j} \\
& +\frac{Q3_{i-\frac{1}{2},j}}{2}\left[w'_{i-1,j} + w'_{i,j}\right] + \frac{Q2_{i-\frac{1}{2}}}{2}\left[\Phi'_{i-1,j} + \Phi'_{i,j}\right] + AL2_{i-\frac{1}{2},j}u'_{ci-\frac{1}{2}} = B1_{i-\frac{1}{2},j}
\end{aligned}
\tag{9.1}
$$

where

$$
(P2D\ u')_{i-\frac{1}{2},j} = \begin{cases} \left[(P2\ u')_{i+\frac{1}{2},j} + 3\,(P2\ u')_{i-\frac{1}{2},j}\right]/3\Delta\eta & \text{if } i = 2 \\ -\left[(P2\ u')_{i-\frac{3}{2},j} + 3\,(P2\ u')_{i-\frac{1}{2},j}\right]/3\Delta\eta & \text{if } i = IY \\ \left[(P2\ u')_{i+\frac{1}{2},j} - (P2\ u')_{i-\frac{3}{2},j}\right]/2\Delta\eta & \text{if } 2 < i < IY \end{cases}
$$

These formulas assume that the coefficient P2, which is a function of $\bar{v}$ is zero on the boundaries. To see where these formulas for $(P2Du')$ come from, let us consider the zonal momentum equation at the first half grid point in from the southern boundary (i.e., $i = \frac{3}{2}$). Due to the distribution of the dependent variables (see Fig. 9.1), horizontal derivatives of quantities involving u' and v' at this point cannot be represented as centered differences. Thus to compute a horizontal derivative of some function, $f(\eta)$, at $i = \frac{3}{2}$, the following second-order difference formula is used:

$$
f'_{\frac{3}{2}} = \left(-4f_1 + 3f_{\frac{3}{2}} + f_{\frac{5}{2}}\right)/3\Delta\eta
$$

A similar argument is used to derive the difference formula shown above at a half grid point in from the northern boundary (i.e, $i = IY$).

Meridional-momentum at half grid points $(\frac{3}{2}, IY - \frac{1}{2})$: $i \to 2, IY$

$$Q11_{i-\frac{1}{2},j}u'_{i-\frac{1}{2},j} + Q13_{i-\frac{1}{2},j}v'_{i-\frac{1}{2},j} + P1_{i-\frac{1}{2}}\left(P2D\ v'\right)_{i-\frac{1}{2},j} + D_j\left[\frac{(P3\ v')_{i-\frac{1}{2},j+1} - (P3\ v')_{i-\frac{1}{2},j-1}}{2\Delta\lambda}\right]$$

$$-\frac{D_j}{\Delta\lambda}\left[E_{i-\frac{1}{2},j+\frac{1}{2}}\frac{v'_{i-\frac{1}{2},j+1} - v'_{i-\frac{1}{2},j}}{\Delta\lambda} - E_{i-\frac{1}{2},j-\frac{1}{2}}\frac{v'_{i-\frac{1}{2},j} - v'_{i-\frac{1}{2},j-1}}{\Delta\lambda}\right] \tag{9.2}$$

$$+\frac{Q5_{i-\frac{1}{2},j}}{2}\left[w'_{i-1,j} + w'_{i,j}\right] + C1_{i-\frac{1}{2}}\left[\frac{\Phi'_{i,j} - \Phi'_{i-1,j}}{\Delta\eta}\right] + AL2_{i-\frac{1}{2},j}v'_{ci-\frac{1}{2}} = B2_{i-\frac{1}{2},j}$$

where $(P2D\ v')_{i-\frac{1}{2},j}$ is defined by substituting v' for u' in the formulas for $(P2D\ u')_{i-\frac{1}{2},j}$.

Continuity at whole grid points: $i \to 2, IY - 1$

$$\frac{Q22_i}{2}\left[u'_{i+\frac{1}{2},j} + u'_{i-\frac{1}{2},j}\right] + C2_i\left[\frac{(C3\ v')_{i+\frac{1}{2},j} - (C3\ v')_{i-\frac{1}{2},j}}{\Delta\eta}\right]$$

$$+D_j\left[\frac{(C5\ w')_{i,j+1} - (C5\ w')_{i,j-1}}{2\Delta\lambda}\right] = 0 \tag{9.3}$$

Thermodynamic at whole grid points: $i \to 2, IY - 1$

$$\frac{Q6_{i,j}}{2}\left(v'_{i+\frac{1}{2},j} + v'_{i-\frac{1}{2},j}\right) + Q7_{i,j}w'_{i,j} + C8AN_{i,j}T'_{i,j}$$

$$+C8CC_{i,j}\left[\frac{T'_{i+1,j} - T'_{i-1,j}}{2\Delta\eta}\right] + C8C6_{i,j}\left[\frac{(C7\ T')_{i,j+1} - (C7\ T')_{i,j-1}}{2\Delta\lambda}\right] \tag{9.4}$$

$$-\frac{C8D_j}{\Delta\lambda}\left[F_{i,j+\frac{1}{2}}\frac{T'_{i,j+1} - T'_{i,j}}{\Delta\lambda} - F_{i,j-\frac{1}{2}}\frac{T'_{i,j} - T'_{i,j-1}}{\Delta\lambda}\right] = B3_{i,j}$$

Hydrostatic approximation at whole grid points: $i \to 2, IY$

$$-T'_{i,j} + D2_j\left[\frac{\Phi'_{i,j+1} - \Phi'_{i,j-1}}{2\Delta\lambda}\right] = 0 \tag{9.5}$$

9.2 Discretized equations at horizontal boundaries

In this section the horizontal boundary conditions are applied to vertical levels $j \rightarrow 2, IZ-1$.

9.2.1 Boundaries not at poles

Zonal-momentum at: $i = 1$ and $i = IY$

$$
\begin{aligned}
Q9_{i,j}u'_{i,j} + P1_i\,(P2D\ u')_{i,j} + D_j \left[\frac{(P3\ u')_{i,j+1} - (P3\ u')_{i,j-1}}{2\Delta\lambda}\right] \\
-\frac{D_j}{\Delta\lambda}\left[E_{i,j+\frac{1}{2}}\frac{u'_{i,j+1} - u'_{i,j}}{\Delta\lambda} - E_{i,j-\frac{1}{2}}\frac{u'_{i,j} - u'_{i,j-1}}{\Delta\lambda}\right] \\
+Q3_{i,j}w'_{i,j} + Q2_i\Phi'_{i,j} + AL2_{i,j}u'_{ci} = B1_{i,j}
\end{aligned}
$$

where

$$
(P2D\ u')_{i,j} = \begin{cases} \frac{2}{\Delta\eta}\,(P2\ u')_{\frac{3}{2},j} & \text{if } i = 1 \\ \frac{-2}{\Delta\eta}\,(P2\ u')_{IY-\frac{1}{2},j} & \text{if } i = IY \end{cases}
$$

These formulas for $(P2D\ u')_{i,j}$ assume that $P2_1 = P2_{IY} = 0$ where $P2$ is a function of $\bar{v}$ and $\bar{v} = 0$ on the horizontal boundaries. Also the term involving v' (i.e., $Q10_{i,j}v'_{i,j}$) is zero and has been dropped from the above expression.

Meridional-momentum at $i = 1$ and $i = IY$

$$
v'_{i,j} = 0
$$

Continuity at $i = 1$ and $i = IY$

$$
Q22_iu'_{i,j} + C2_i\,(C3D\ v')_{i,j} + D_j \left[\frac{(C5\ w')_{i,j+1} - (C5\ w')_{i,j-1}}{2\Delta\lambda}\right] = 0
$$

$$
(C3D\ v')_{i,j} = \begin{cases} \frac{2}{\Delta\eta}\,(C3\ v')_{\frac{3}{2},j} & \text{if } i = 1 \\ \frac{-2}{\Delta\eta}\,(C3\ v')_{IY-\frac{1}{2},j} & \text{if } i = IY \end{cases}
$$

These formulas for $(C3D\ v')_{i,j}$ assume that $C3_1 = C3_{IY} = 0$ where $C3$ is a function of $\bar{v}$ and $\bar{v} = 0$ on the horizontal boundaries.

Thermodynamic at $i = 1$ and $i = IY$

$$Q7_{i,j}w'_{i,j} + C8AN_{i,j}T'_{i,j} + C8C6_{i,j}\left[\frac{(C7\,T')_{i,j+1} - (C7\,T')_{i,j-1}}{2\Delta\lambda}\right]$$

$$-\frac{C8D_j}{\Delta\lambda}\left[F_{i,j+\frac{1}{2}}\frac{T'_{i,j+1} - T'_{i,j}}{\Delta\lambda} - F_{i,j-\frac{1}{2}}\frac{T'_{i,j} - T'_{i,j-1}}{\Delta\lambda}\right] = B3_{i,j}$$

The terms is this expression involving v' (i.e., $Q6\ v'$) and $\bar{v}$ (i.e. $C8CC\frac{\partial T'}{\partial \eta}$) are zero on the horizontal boundaries and have dropped accordingly.

Hydrostatic approximation at $i = 2$ and $i = IY$

$$-T'_{i,j} + D2_j\left[\frac{\Phi'_{i,j+1} - \Phi'_{i,j-1}}{2\Delta\lambda}\right] = 0$$

9.2.2 Boundaries at the poles

If $s = 0$

$$u'_{i,j} = v'_{i,j} = 0 \quad \text{at } i = 1 \text{ and } i = IY$$

$$(Dv)_{i,j} + D_j\left[\frac{(C5w')_{i,j+1} - (C5w')_{i,j-1}}{2\Delta\lambda}\right] = 0 \quad \text{at } i = 1 \text{ and } i = IY$$

where

$$(Dv)_{i,j} = \begin{cases} \frac{2}{(\theta_{\frac{3}{2}}-\theta_1)}v'_{\frac{3}{2},j} = DVS\ v'_{\frac{3}{2},j} & \text{at } i = 1 \\ \frac{-2}{(\theta_{IY}-\theta_{IY-\frac{1}{2}})}v'_{IY-\frac{1}{2},j} = DVN\ v'_{IY-\frac{1}{2},j} & \text{at } i = IY \end{cases}$$

These formulas for $(Dv)_{i,j}$ assume that $v' = 0$ at the poles.

$$T'_{2,j} - T'_{1,j} = 0, \quad \Phi'_{2,j} - \Phi'_{1,j} = 0 \quad \text{at } i = 1$$

$$T'_{IY,j} - T'_{IY-1,j} = 0, \quad \Phi'_{IY,j} - \Phi'_{IY-1,j} = 0 \quad \text{at } i = IY$$

If $s = \pm 1$

$$u'_{\frac{3}{2},j} - u'_{1,j} = 0, \quad v'_{\frac{3}{2},j} - v'_{1,j} = 0 \quad \text{at } i = 1$$

$$u'_{IY,j} - u'_{IY-\frac{1}{2},j} = 0, \quad v'_{IY,j} - v'_{IY-\frac{1}{2},j} = 0 \quad \text{at } i = IY$$

$$w'_{1,j} = T'_{1,j} = \Phi'_{1,j} = 0 \quad \text{at } i = 1$$

$$w'_{IY,j} = T'_{IY,j} = \Phi'_{IY,j} = 0 \quad \text{at } i = IY$$

If $|s| > 1$

$$u'_{1,j} = v'_{1,j} = w'_{1,j} = T'_{1,j} = \Phi'_{1,j} = 0 \quad \text{at } i = 1$$

$$u'_{IY,j} = v'_{IY,j} = w'_{IY,j} = T'_{IY,j} = \Phi'_{IY,j} = 0 \quad \text{at } i = IY$$

9.3 Discretized equations at vertical boundaries and corner points

9.3.1 At upper boundary ($j = 1$)

$$u'_{i,2} - u'_{i,1} = 0 \quad \text{for } i = 1 \text{ and } i = IY$$

$$u'_{i-\frac{1}{2},2} - u'_{i-\frac{1}{2},1} = 0 \quad \text{for } i \to 2, IY$$

$$v'_{i,1} = 0 \quad \text{for } i = 1 \text{ and } i = IY$$

$$v'_{i-\frac{1}{2},2} - v'_{i-\frac{1}{2},1} = 0 \quad \text{for } i \to 2, IY$$

$$w'_{i,1} = 0 \quad \text{for } i \to 1, IY$$

$$T'_{i,2} - T'_{i,1} = 0 \quad \text{for } i \to 1, IY$$

$$-T'_{i,1} + D2_1 \left[\frac{\Phi'_{i,2} - \Phi'_{i,1}}{\Delta\lambda} \right] = 0 \quad \text{for } i \to 1, IY$$

9.3.2 At lower boundary ($j = IZ$)

$$\left(\frac{u'_{i,IZ}-u'_{i,IZ-1}}{\Delta\lambda}\right) \quad -BCU_i u'_{i,IZ} \quad = 0 \quad \text{for } i = 1 \text{ and } i = IY$$

$$\left(\frac{u'_{i-\frac{1}{2},IZ}-u'_{i-\frac{1}{2},IZ-1}}{\Delta\lambda}\right) \quad -BCU_{i-\frac{1}{2}} u'_{i-\frac{1}{2},IZ} \quad = 0 \quad \text{for } i \to 2, IY$$

$$v'_{i,IZ} \quad = 0 \quad \text{for } i = 1 \text{ and } i = IY$$

$$\left(\frac{v'_{i-\frac{1}{2},IZ}-v'_{i-\frac{1}{2},IZ-1}}{\Delta\lambda}\right) \quad -BCV_{i-\frac{1}{2}} v'_{i-\frac{1}{2},IZ} \quad = 0 \quad \text{for } i \to 2, IY$$

$$\left(\frac{T'_{i,IZ}-T'_{i,IZ-1}}{\Delta\lambda}\right) \quad -BCT_i\, T'_{i,IZ} \quad = 0 \quad \text{for } i \to 1, IY$$

$$B35_i\Phi'_{i,IZ} + AC35_i\,(D\Phi')_{i,IZ} + B33_i w'_{i,IZ} \quad = 0 \quad \text{for } i = 1 \text{ and } i = IY$$

$$\text{where } (D\Phi')_{i,IZ} = \begin{cases} \frac{\Phi_{i+1,IZ}-\Phi_{i,IZ}}{\Delta\eta} & \text{if } i = 1 \\ \frac{\Phi_{i,IZ}-\Phi_{i-1,IZ}}{\Delta\eta} & \text{if } i = IY \end{cases}$$

$$B35_i\Phi'_{i,IZ} + \frac{B32_i}{2}\left(v'_{i-\frac{1}{2},IZ} + v'_{i+\frac{1}{2},IZ}\right)$$

$$+AC35_i\left(\frac{\Phi'_{i+1,IZ}-\Phi'_{i-1,IZ}}{2\Delta\eta}\right) + B33_i w'_{i,IZ} \quad = 0 \quad \text{for } i \to 2, IY-1$$

10 Matrix Form of the Equations

To aid us in solving the system of discretized equations given in the previous section, it is convenient to conceptualize these equations at each point in the model's domain in the form of 10.1 given below

$$LL_{i,j}\chi_{i,j-1} + AK_{i,j}\chi_{i-1,j} + BK_{i,j}\chi_{i,j} + CK_{i,j}\chi_{i+1,j} + UU_{i,j}\chi_{i,j+1} + JJ_{i,j}\chi_{ci} = BB_{i,j} \quad (10.1)$$

where

$$\chi_{i,j} = \begin{cases} \left(u'_{1,j},\ v'_{1,j},\ w'_{1,j},\ T'_{1,j},\ \Phi'_{1,j}\right)^T & \text{if } i = 1 \\ \left(u'_{i-\frac{1}{2},j},\ v'_{i-\frac{1}{2},j},\ w'_{i,j},\ T'_{i,j},\ \Phi'_{i,j}\right)^T & \text{if } 2 \le i \le IY \\ \left(u'_{IY,j},\ v'_{IY,j},\ 0,\ 0,\ 0\right)^T & \text{if } i = IY+1 \end{cases}$$

represents perturbation variables at the model grid points and $LL_{i,j}$, $UU_{i,j}$, $JJ_{i,j}$, $BB_{i,j}$, $AK_{i,j}$, $BK_{i,j}$ and $CK_{i,j}$ are defined respectively below. In these matrices the row index corresponds to the equation (1 - zonal momentum, 2 - meridional momentum, 3 - continuity, 4 - thermodynamic, 5 - hydrostatic approximation) and the column index corresponds to the variable operated on (1 - u', 2 - v', 3 - w', 4 - T', 5 - Φ'). For example, the matrix element (2,3) is the coefficient of w' in the meridional momentum equation.

$$LL_{i,j} = \begin{pmatrix} D_j\left[\frac{-P3_{i-\frac{1}{2},j-1}}{2\Delta\lambda} - \frac{E_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta\lambda^2}\right] & 0 & 0 & 0 & 0 \\ 0 & D_j\left[\frac{-P3_{i-\frac{1}{2},j-1}}{2\Delta\lambda} - \frac{E_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta\lambda^2}\right] & 0 & 0 & 0 \\ 0 & 0 & -D_j\left(\frac{C5_{j-1}}{2\Delta\lambda}\right) & 0 & 0 \\ 0 & 0 & 0 & \left[-C8C6_{i,j}\left(\frac{C7_{j-1}}{2\Delta\lambda}\right) - \frac{C8D_j}{\Delta\lambda^2}F_{i,j-\frac{1}{2}}\right] & 0 \\ 0 & 0 & 0 & 0 & \frac{-D2_j}{2\Delta\lambda} \end{pmatrix}$$

$$UU_{i,j} = \begin{pmatrix} D_j\left[\frac{P3_{i-\frac{1}{2},j+1}}{2\Delta\lambda} - \frac{E_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta\lambda^2}\right] & 0 & 0 & 0 & 0 \\ 0 & D_j\left[\frac{P3_{i-\frac{1}{2},j+1}}{2\Delta\lambda} - \frac{E_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta\lambda^2}\right] & 0 & 0 & 0 \\ 0 & 0 & D_j\left(\frac{C5_{j+1}}{2\Delta\lambda}\right) & 0 & 0 \\ 0 & 0 & 0 & \left[C8C6_{i,j}\left(\frac{C7_{j+1}}{2\Delta\lambda}\right) - \frac{C8D_j}{\Delta\lambda^2}F_{i,j+\frac{1}{2}}\right] & 0 \\ 0 & 0 & 0 & 0 & \frac{D2_j}{2\Delta\lambda} \end{pmatrix}$$

$$
JJ_{i,j} = \begin{pmatrix} AL2_{i-\frac{1}{2},j} & 0 & 0 & 0 & 0 \\ 0 & AL2_{i-\frac{1}{2},j} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
BB_{i,j} = \begin{pmatrix} B1_{i-\frac{1}{2},j} \\ B2_{i-\frac{1}{2},j} \\ 0 \\ B3_{i,j} \\ 0 \end{pmatrix}
$$

$$
AK_{i,j} = \begin{pmatrix} P1_{i-\frac{1}{2}}\left(\frac{-P2_{i-\frac{3}{2},j}}{2\Delta\eta}\right) & 0 & \frac{Q3_{i-\frac{1}{2},j}}{2} & 0 & \frac{Q2_{i-\frac{1}{2},j}}{2} \\ 0 & P1_{i-\frac{1}{2}}\left(\frac{-P2_{i-\frac{3}{2},j}}{2\Delta\eta}\right) & \frac{Q5_{i-\frac{1}{2},j}}{2} & 0 & \frac{-C1_{i-\frac{1}{2}}}{\Delta\eta} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-C8CC_{i,j}}{2\Delta\eta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
BK_{i,j} = \begin{pmatrix} \left[Q9_{i-\frac{1}{2},j} + \frac{D_j}{\Delta\lambda^2}\tilde{E}_{i-\frac{1}{2},j}\right] & Q10_{i-\frac{1}{2},j} & \frac{Q3_{i-\frac{1}{2},j}}{2} & 0 & \frac{Q2_{i-\frac{1}{2}}}{2} \\ Q11_{i-\frac{1}{2},j} & \left[Q13_{i-\frac{1}{2},j} + \frac{D_j}{\Delta\lambda^2}\tilde{E}_{i-\frac{1}{2},j}\right] & \frac{Q5_{i-\frac{1}{2},j}}{2} & 0 & \frac{C1_{i-\frac{1}{2},j}}{\Delta\eta} \\ \frac{Q22_i}{2} & C2_i\frac{-C3_{i-\frac{1}{2}}}{\Delta\eta} & 0 & 0 & 0 \\ 0 & \frac{Q6_{i,j}}{2} & Q7_{i,j} & \left[C8AN_{i,j} + \frac{C8D_j}{\Delta\lambda^2}\tilde{F}_{i-\frac{1}{2},j}\right] & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}
$$

where $\tilde{E}_{i,j} = E_{i,j+\frac{1}{2}} + E_{i,j-\frac{1}{2}}$ and $\tilde{F}_{i,j} = F_{i,j+\frac{1}{2}} + F_{i,j-\frac{1}{2}}$

$$
CK_{i,j} = \begin{pmatrix} P1_{i-\frac{1}{2}}\left(\frac{P2_{i+\frac{1}{2},j}}{2\Delta\eta}\right) & 0 & 0 & 0 & 0 \\ 0 & P1_{i-\frac{1}{2}}\left(\frac{P2_{i+\frac{1}{2},j}}{2\Delta\eta}\right) & 0 & 0 & 0 \\ \frac{Q22_i}{2} & C2_i\frac{C3_{i+\frac{1}{2}}}{\Delta\eta} & 0 & 0 & 0 \\ 0 & \frac{Q6_{i,j}}{2} & 0 & \frac{C8CC_{i,j}}{2\Delta\eta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

To implement the finite difference formulas following 9.1 and 9.2 in the previous section, the matrices BK and CK at $i = 2$ are altered as follows:

$$
BK_{2,j}(1,1) \quad = \text{old } BK_{2,j}(1,1) + P1_{\frac{3}{2}}\left(\frac{P2_{\frac{3}{2},j}}{\Delta\eta}\right)
$$

$$
\begin{aligned}
BK_{2,j}(2,2) &= \text{old } BK_{2,j}(2,2) + P1_{\frac{3}{2}}\left(\frac{P2_{\frac{3}{2},j}}{\Delta\eta}\right)\\
CK_{2,j}(1,1) &= \tfrac{2}{3}\left[\text{old } CK_{2,j}(1,1)\right] = P1_{\frac{3}{2}}\left(\frac{P2_{\frac{5}{2},j}}{3\Delta\eta}\right)\\
CK_{2,j}(2,2) &= \tfrac{2}{3}\left[\text{old } CK_{2,j}(2,2)\right] = P1_{\frac{3}{2}}\left(\frac{P2_{\frac{5}{2},j}}{3\Delta\eta}\right)
\end{aligned}
$$

Likewise the matrices AK, and BK at $i = IY$ are altered as follows:

$$
\begin{aligned}
AK_{IY,j}(1,1) &= \tfrac{2}{3}\left[\text{old } AK_{IY,j}(1,1)\right] = P1_{IY-\frac{1}{2}}\left(\frac{-P2_{IY-\frac{3}{2},j}}{3\Delta\eta}\right)\\
AK_{IY,j}(2,2) &= \tfrac{2}{3}\left[\text{old } AK_{IY,j}(2,2)\right] = P1_{IY-\frac{1}{2}}\left(\frac{-P2_{IY-\frac{3}{2},j}}{3\Delta\eta}\right)\\
BK_{IY,j}(1,1) &= \text{old } BK_{2,j}(1,1) + P1_{IY-\frac{1}{2}}\left(\frac{-P2_{IY-\frac{1}{2},j}}{\Delta\eta}\right)\\
BK_{IY,j}(2,2) &= \text{old } BK_{2,j}(2,2) + P1_{IY-\frac{1}{2}}\left(\frac{-P2_{IY-\frac{1}{2},j}}{\Delta\eta}\right)
\end{aligned}
$$

In computing the above matrices the following restrictions apply. At $i = 1$ certain formulas (e.g., $LL_{1,j}(1,1)$) contain coefficients at $i = \frac{1}{2}$; in this case the coefficients should be computed instead at $i = 1$. Likewise, at $i = IY + 1$ coefficients which appear in formulas at $i = IY + \frac{1}{2}$ should be evaluated instead at $i = IY$. In addition, elements in the last three rows of the matrices at $i = IY + 1$ are always zero. The remainder of this section deals with how the above matrices are altered to implement the boundary conditions.

10.1 Horizontal boundaries

The horizontal boundary conditions discussed in this section are applied at vertical levels $j \to 2, IZ-1$.

10.1.1 Boundaries not at poles

At the southern boundary (i.e., $i = 1$), the discretized boundary conditions given in Section 9.2 are implemented as follows. The matrices LL, UU, JJ and BB at $i = 1$ are defined in a similar manner to that shown earlier with exceptions noted below. To set $v' = 0$ the elements of the second row of the following matrices (i.e., $LL_{1,j}$, $UU_{1,j}$, $AK_{1,j}$, $BK_{1,j}$, $CK_{1,j}$, $JJ_{1,j}$, $BB_{1,j}$) are set to zero, then as shown below set $BK_{1,j}(2,2) = 1$. Furthermore, since $\bar{v} = 0$ and horizontal differences are one sided, the matrices AK, BK and CK are defined as follows: The elements of $AK_{1,j}$ are set to zero.

$$BK_{1,j} = \begin{pmatrix} \left[Q9_{1,j} + \frac{D_j}{\Delta\lambda^2}\tilde{E}_{1,j}\right] & 0 & Q3_1 & 0 & Q2_1 \\ 0 & 1 & 0 & 0 & 0 \\ Q22_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q7_1 & \left[C8AN_{1,j} + \frac{C8D_j}{\Delta\lambda^2}\tilde{F}_{1,j}\right] & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$CK_{1,j} = \begin{pmatrix} 2\left[P1_1\left(\frac{P2_{\frac{3}{2},j}}{\Delta\eta}\right)\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2\left[C2_1\frac{C3_{\frac{3}{2}}}{\Delta\eta}\right] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

At the northern boundary (i.e., $i = IY$), the discretized boundary conditions given in Section 9.2 are implemented as follows. To set $v' = 0$ the elements of the second row of the following matrices (i.e., $LL_{IY+1,j}$, $UU_{IY+1,j}$, $AK_{IY+1,j}$, $BK_{IY+1,j}$, $CK_{IY+1,j}$, $JJ_{IY+1,j}$, $BB_{IY+1,j}$) are set to zero, then as shown below set $BK_{IY+1,j}(2,2) = 1$. Furthermore, since $\bar{v} = 0$ and horizontal differences are one sided, the matrices AK, BK and CK at $i = IY + 1$ are altered as follows:

$$AK_{IY+1,j} = \begin{pmatrix} 2\left[P1_{IY}\left(\frac{-P2_{IY-\frac{1}{2},j}}{\Delta\eta}\right)\right] & 0 & Q3_{IY} & 0 & Q2_{IY} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$BK_{IY+1,j} = \begin{pmatrix} \left[Q9_{IY,j} + \frac{D_j}{\Delta\lambda^2}\tilde{E}_{IY,j}\right] & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The elements of $CK_{IY+1,j}$ are set to zero. In addition, the matrices AK, BK and CK at

$i = IY$ are altered as follows:

$$\begin{aligned} BK_{IY,j}(3,1) &= 0, \text{ since } w'_{IY} \text{ operates on } CK_{IY,j}(3,1) \\ CK_{IY,j}(3,1) &= Q22_{IY} \\ CK_{IY,j}(3,2) &= 2\left[C2_{IY}\frac{C3_{IY-\frac{1}{2}}}{\Delta\eta}\right] \end{aligned}$$

10.1.2 Boundaries at poles

Initialize the following matrix elements:

- set $LL_{1,j}$, $UU_{1,j}$, $AK_{1,j}$, $BK_{1,j}$, $CK_{1,j}$, $JJ_{1,j}$, $BB_{1,j}$, to be 0
- set $LL_{IY+1,j}$, $UU_{IY+1,j}$, $AK_{IY+1,j}$, $BK_{IY+1,j}$, $CK_{IY+1,j}$, $JJ_{IY+1,j}$, $BB_{IY+1,j}$ to be 0
- set row 3 of $LL_{IY,j}$, $UU_{IY,j}$, $AK_{IY,j}$, $BK_{IY,j}$, $CK_{IY,j}$, $JJ_{IY,j}$, $BB_{IY,j}$, to be 0
- set row 4 of $LL_{IY,j}$, $UU_{IY,j}$, $AK_{IY,j}$, $BK_{IY,j}$, $CK_{IY,j}$, $JJ_{IY,j}$, $BB_{IY,j}$, to be 0
- set row 5 of $LL_{IY,j}$, $UU_{IY,j}$, $AK_{IY,j}$, $BK_{IY,j}$, $CK_{IY,j}$, $JJ_{IY,j}$, $BB_{IY,j}$, to be 0

<u>If $s = 0$:</u>

at south pole

$$BK_{1,j}(1,1) = 1 \qquad u'_{1,j} = 0$$

$$BK_{1,j}(2,2) = 1 \qquad v'_{1,j} = 0$$

$$\left.\begin{aligned} CK_{1,j}(3,2) &= DVS \\ LL_{1,j}(3,3) &= -D2_j\left(\tfrac{C5_{j-1}}{2\Delta\lambda}\right) \\ UU_{1,j}(3,3) &= D2_j\left(\tfrac{C5_{j+1}}{2\Delta\lambda}\right) \end{aligned}\right\} \qquad DVS v'_{\frac{3}{2}} + D2_j\left[\tfrac{(C5w')_{i,j+1}-(C5w')_{i,j-1}}{2\Delta\lambda}\right] = 0$$

$$\left.\begin{aligned} BK_{1,j}(4,4) &= -1 \\ CK_{1,j}(4,4) &= 1 \end{aligned}\right\} \qquad T'_{2,j} - T'_{1,j} = 0$$

$$\left.\begin{aligned} BK_{1,j}(5,5) &= -1 \\ CK_{1,j}(5,5) &= 1 \end{aligned}\right\} \qquad \Phi'_{2,j} - \Phi'_{1,j} = 0$$

at north pole

$$BK_{IY+1,j}(1,1) = 1 \qquad u'_{IY,j} = 0$$

$$BK_{IY+1,j}(2,2) = 1 \qquad v'_{IY,j} = 0$$

$$\left.\begin{array}{l} BK_{IY,j}(3,2) = DVN \\ LL_{IY,j}(3,3) = -D2_j\left(\frac{C5_{j-1}}{2\Delta\lambda}\right) \\ UU_{IY,j}(3,3) = D2_j\left(\frac{C5_{j+1}}{2\Delta\lambda}\right) \end{array}\right\} \qquad DVN\,v'_{IY-\frac{1}{2}} + D2_j\left[\frac{(C5w')_{i,j+1}-(C5w')_{i,j-1}}{2\Delta\lambda}\right] = 0$$

$$\left.\begin{array}{l} AK_{IY,j}(4,4) = -1 \\ BK_{IY,j}(4,4) = 1 \end{array}\right\} \qquad T'_{IY,j} - T'_{IY-1,j} = 0$$

$$\left.\begin{array}{l} AK_{IY,j}(5,5) = -1 \\ BK_{IY,j}(5,5) = 1 \end{array}\right\} \qquad \Phi'_{IY,j} - \Phi'_{IY-1,j} = 0$$

<u>If $s = \pm 1$:</u>

at south pole

$$\left.\begin{array}{l} BK_{1,j}(1,1) = -1 \\ CK_{1,j}(1,1) = 1 \end{array}\right\} \qquad u'_{\frac{3}{2},j} - u'_{1,j} = 0$$

$$\left.\begin{array}{l} BK_{1,j}(2,2) = -1 \\ CK_{1,j}(2,2) = 1 \end{array}\right\} \qquad v'_{\frac{3}{2},j} - v'_{1,j} = 0$$

$$BK_{1,j}(3,3) = 1 \qquad w'_{1,j} = 0$$

$$BK_{1,j}(4,4) = 1 \qquad T'_{1,j} = 0$$

$$BK_{1,j}(5,5) = 1 \qquad \Phi'_{1,j} = 0$$

at north pole

$$\left.\begin{array}{l} AK_{IY+1,j}(1,1) = -1 \\ BK_{IY+1,j}(1,1) = 1 \end{array}\right\} \qquad u'_{IY,j} - u'_{IY-\frac{1}{2},j} = 0$$

$$\left.\begin{array}{l} AK_{IY+1,j}(2,2) = -1 \\ BK_{IY+1,j}(2,2) = 1 \end{array}\right\} \qquad v'_{IY,j} - v'_{IY-\frac{1}{2},j} = 0$$

$$BK_{IY,j}(3,3) = 1 \qquad w'_{IY,j} = 0$$

$$BK_{IY,j}(4,4) = 1 \qquad T'_{IY,j} = 0$$

$$BK_{IY,j}(5,5) = 1 \qquad \Phi'_{IY,j} = 0$$

<u>If $|s| > 1$:</u>

at south pole

$$BK_{1,j}(1,1) = 1 \quad u'_{1,j} = 0$$

$$BK_{1,j}(2,2) = 1 \quad v'_{1,j} = 0$$

$$BK_{1,j}(3,3) = 1 \quad w'_{1,j} = 0$$

$$BK_{1,j}(4,4) = 1 \quad T'_{1,j} = 0$$

$$BK_{1,j}(5,5) = 1 \quad \Phi'_{1,j} = 0$$

at north pole

$$BK_{IY+1,j}(1,1) = 1 \quad u'_{IY,j} = 0$$

$$BK_{IY+1,j}(2,2) = 1 \quad v'_{IY,j} = 0$$

$$BK_{IY,j}(3,3) = 1 \quad w'_{IY,j} = 0$$

$$BK_{IY,j}(4,4) = 1 \quad T'_{IY,j} = 0$$

$$BK_{IY,j}(5,5) = 1 \quad \Phi'_{IY,j} = 0$$

10.2 Vertical boundaries and corner points

At top boundary (i.e., $j = 1$), the discretized boundary conditions given in the previous section are implemented by setting all the elements of the matrices (i.e., LL, UU, AK, BK, CK, JJ, BB) to be zero, then altering BK and UU as follows:

$$BK_{i,1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{D2_1}{\Delta\lambda} \end{pmatrix} UU_{i,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{D2_1}{\Delta\lambda} \end{pmatrix}$$

At corner point (1,1) the matrices are defined in a similar manner to those above with the following alterations to impose the condition $v' = 0$.

$$BK_{1,1}(2,2) = 1$$

$$UU_{1,1}(2,2) = 0.$$

Likewise, to impose $v' = 0$ at corner point $(IY, 1)$ the last three rows of matrices $BK_{IY+1,j}$ and $UU_{IY+1,j}$ are set to zero and

$$BK_{IY+1,j}(2,2) = 1$$

$$UU_{IY+1,j}(2,2) = 0.$$

At the bottom boundary (i.e., $j = IZ$), the discretized boundary conditions given in the previous section are implemented by setting all the elements of the matrices (i.e., LL, UU, AK, BK, CK, JJ, BB) to be zero, then altering BK, UU, AK and CK as follows:

$$BK_{i,IZ} = \begin{pmatrix} (1 - \Delta\lambda BCU_{i-\frac{1}{2}}) & 0 & 0 & 0 & 0 \\ 0 & (1 - \Delta\lambda BCV_{i-\frac{1}{2}}) & 0 & 0 & 0 \\ 0 & \frac{B32_i}{2} & B33_i & 0 & B35_i \\ 0 & 0 & 0 & (1 - \Delta\lambda BCT_i) & 0 \\ 0 & 0 & 0 & -\Delta\lambda & D2_{IZ} \end{pmatrix}$$

$$LL_{i,IZ} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -D2_{IZ} \end{pmatrix}$$

$$\begin{aligned} AK_{i,IZ}(3,5) &= \frac{-AC35_i}{2\Delta\eta} \\ CK_{i,IZ}(3,2) &= \frac{B32_i}{2} \\ CK_{i,IZ}(3,5) &= \frac{AC35_i}{2\Delta\eta} \end{aligned}$$

At corner point $(1, IZ)$ the condition $v' = 0$ and one-sided horizontal derivatives are used by making the following alterations to the matrices defined above.

$$BK_{1,IZ} = \begin{pmatrix} (1 - \Delta\lambda BCU_1) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & B33_1 & 0 & B35_1 - \frac{AC35_1}{\Delta\eta} \\ 0 & 0 & 0 & (1 - \Delta\lambda BCT_1) & 0 \\ 0 & 0 & 0 & -\Delta\lambda & D2_{IZ} \end{pmatrix}$$

$$LL_{1,IZ} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -D2_{IZ} \end{pmatrix}$$

$$\begin{aligned} AK_{1,IZ}(3,5) &= 0 \\ CK_{1,IZ}(3,5) &= \tfrac{AC35_1}{\Delta\eta} \\ CK_{1,IZ}(3,2) &= 0 \end{aligned}$$

Analogously, similar conditions are used at corner point (IY, IZ) by making the following alterations to the matrices defined above.

$$BK_{IY+1,IZ} = \begin{pmatrix} (1-\Delta\lambda BCU_{IY}) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$LL_{IY+1,IZ} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} AK_{IY,IZ}(3,5) &= \frac{-AC35_{IY}}{\Delta\eta} \\ BK_{IY,IZ}(3,5) &= B35_{IY} + \frac{AC35_{IY}}{\Delta\eta} \\ BK_{IY,IZ}(3,2) &= 0 \\ CK_{IY,IZ}(3,5) &= 0 \\ CK_{IY,IZ}(3,2) &= 0 \end{aligned}$$

11 Algorithm for Solving Problem

For a specific heating function (Q'), the response in the perturbation fields of the three-dimensional wind (u', v', w'), geopotential (ϕ'), and temperature (T') are calculated from Eqs. 9.1–9.5. The algorithm which solves for these perturbation fields can be divided into the following three sections.

11.1) Filling of matrices

11.2) Gaussian elimination

11.3) Backsubstitution

Each of these sections will be discussed in the order that they appear in our computer algorithm which is flowcharted in the Appendix of SC.

11.1 Filling of matrices

By combining Equation 10.1 for all the horizontal nodes on a level, we can write an equation for each level in the model as follows:

$$L_j X_{j-1} + D_j X_j + U_j X_{j+1} + J_j X_c = B_j \tag{11.1}$$

where $j \equiv$ vertical level of model. The X_j is a column vector which consists of the $IY + 1$ grid point vectors $\chi_{i,j}$ in sequence, where $\chi_{i,j}$ is defined following Equation 10.1. B_j is a similar column vector

$$X_j \equiv \begin{pmatrix} (\chi_{1,j}) \\ (\chi_{2,j}) \\ (\chi_{3,j}) \\ \vdots \\ (\chi_{IY+1,j}) \end{pmatrix} \qquad B_j \equiv \begin{pmatrix} (BB_{1,j}) \\ (BB_{2,j}) \\ (BB_{3,j}) \\ \vdots \\ (BB_{IY+1,j}) \end{pmatrix}$$

The L_j, U_j, and J_j matrices are block diagonal, with the i^{th} block sub–matrix being $LL_{i,j}$, $UU_{i,j}$, $JJ_{i,j}$, respectively. From Section 10, $LL_{i,j}$, $UU_{i,j}$, and $JJ_{i,j}$ are themselves diagonal 5 by 5 sub–matrices. For example,

$$L_j \equiv \begin{pmatrix} (LL_{i,j}) & & & \\ & (LL_{2,j}) & & \\ & & \ddots & \\ & & & (LL_{IY+1,j}) \end{pmatrix}$$

U_j and J_j are similarly diagonal matrices. L_j, U_j and J_j each involves storage of $5 \times IY + 1$ elements on the diagonal. The D_j matrix, which is a combination of the $AK_{i,j}$, $BK_{i,j}$, and $CK_{i,j}$ operators from Equation 10.1, is a block tri-diagonal matrix, with each block a 5 by 5 submatrix. At some vertical level j:

$$D_j = \begin{bmatrix} BK_{1,j} & CK_{1,j} & & & & & & & \\ AK_{2,j} & BK_{2,j} & CK_{2,j} & & & 0's & & & \\ & AK_{3,j} & BK_{3,j} & CK_{3,j} & & & & & \\ & & AK_{4,j} & BK_{4,j} & CK_{4,j} & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & AK_{IY-2,j} & BK_{IY-2,j} & CK_{IY-2,j} & & & \\ & & & & AK_{IY-1,j} & BK_{IY-1,j} & CK_{IY-1,j} & & \\ & 0's & & & & AK_{IY,j} & BK_{IY,j} & CK_{IY,j} \\ & & & & & & AK_{IY+1,j} & BK_{IY+1,j} \end{bmatrix}$$

Since D_j is a block tri-diagonal matrix, in storing D_j we have taken advantage of the fact that most of its elements are zero. In storage the compressed matrix D at any vertical level appears as follows:

$$D_j = \begin{bmatrix} BK_{1,j} & CK_{1,j} & CK_{2,j} & \dots & CK_{i-1,j} & \dots & CK_{IY-2,j} & CK_{IY-1,j} & 0 \\ AK_{2,j} & BK_{2,j} & BK_{3,j} & \dots & BK_{i,j} & \dots & BK_{IY-1,j} & BK_{IY,j} & CK_{IY,j} \\ 0 & AK_{3,j} & AK_{4,j} & \dots & AK_{i+1,j} & \dots & AK_{IY,j} & AK_{IY+1,j} & BK_{IY+1,j} \end{bmatrix}$$

where D is needed for computation, it is reformed into its original sparse structure. The memory requirements for storing the compressed version of D over its sparse structure are reduced by a factor of $(IY/3)$. For example with typical values needed to resolve wave structure of $IY = 21$ and $IZ = 31$, the storage of the full matrix D for all levels in the models would require $\sim 10^6$ words of memory. On the other hand, the compressed D matrixes would require only $\sim 1.3 \times 10^5$ words of memory or a factor of 8 less! By storing the matrices in Equation 11.1 as outlined above, the CRAY computer can easily contain in its memory these matrices for all the levels in the model simultaneously.

Filling the matrices in Equation 11.1 at a specific vertical level, requires that the component sub–matrices: UU, LL, JJ, BB, AK, BK, and CK be filled first at each horizontal node on that level. Once these sub– matrices are defined they are used to form the L, U, J, B, and D matrices at one vertical level. By repeating this process at each level in the model, our system of equations can now be represented in the form of the linear matrix equation, $AX = B^*$ which is shown schematically below.

$$\begin{bmatrix} D_1 & U_1 & & & & J_1 & & & \\ L_2 & D_2 & U_2 & & & J_2 & & & \\ & L_3 & D_3 & U_3 & & J_3 & & & \\ & & & \vdots & & \vdots & & & \\ & & & L_{\ell-1} & (U+J)_{\ell-1} & & & & \\ & & & & L_\ell & (D+J)_\ell & U_\ell & & \\ & & & & & (L+J)_{\ell+1} & D_{\ell+1} & U_{\ell+1} & \\ & & & & & \vdots & & \vdots & \\ & & & & & 0 & & L_{IZ} & D_{IZ} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{\ell-1} \\ X_\ell \\ X_{\ell+1} \\ \vdots \\ X_{IZ} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{\ell-1} \\ B_\ell \\ B_{\ell+1} \\ \vdots \\ B_{IZ} \end{bmatrix}$$

We have assumed here that $J_j = 0$ (i.e. no cloud mass flux) for $j > \ell+1$. The parameter $'\ell'$ as it appears here is defined as the level of cloud base and is computed from the following equation:

$$\ell = \mathrm{IFIX}(\lambda(ZC)/\Delta\lambda + 0.5) \tag{11.2}$$

Because the vertical differentiation is at most second–order in the five variables, only vertical grid levels separated by $\Delta\lambda$ and $2\Delta\lambda$ are related in the finite difference scheme. Thus the matrix A is a block tridiagonal matrix, with blocks of dimension $5 \times IY$ by $5 \times IY$.

11.2 Gaussian elimination

To reduce the linear matrix system $AX = B^*$ to an upper triangular matrix, we employ a Gaussian elimination scheme, slightly modified for the cumulus friction terms, from a version suggested by Lindzen and Kuo (1969). In this scheme, IZ matrices ($5 \times IY$ by

$5 \times IY$) must be inverted in full–storage (non–sparse) mode. The procedure for using this scheme at the various levels in the model is shown below.

$$\boxed{\text{For } j = 1}$$

$$\begin{array}{rcrcrcr}
D_1X_1 & + & U_1X_2 & + & J_1X_\ell & = & B_1 \\
X_1 & + & D_1^{-1}U_1X_2 & + & D_1^{-1}J_1X_\ell & = & D_1^{-1}B_1 \\
X_1 & + & \alpha_1X_2 & + & \beta_1X_\ell & = & \gamma_1
\end{array}$$

where

$$\begin{array}{rcl}
\delta_1 & = & D_1^{-1} \\
\alpha_1 & = & \delta_1U_1 \\
\beta_1 & = & \delta_1J_1 \\
\gamma_1 & = & \delta_1B_1
\end{array}$$

$$\boxed{\text{For } j = 2, 3, \ldots, \ell - 2}$$

$$\begin{array}{rcrcrcrcr}
-L_j(X_{j-1} & + & \alpha_{j-1}X_j & & & + & \beta_{j-1}X_\ell & = & \gamma_{j-1}) \\
L_jX_{j-1} & + & D_jX_j & + & U_jX_{j+1} & + & J_jX_\ell & = & B_j \\
\hline
 & & (D_j - L_j\alpha_{j-1})X_j & + & (U_j)X_{j+1} & + & (J_j - L_j\beta_{j-1})X_\ell & = & (B_j - L_j\gamma_{j-1}) \\
 & & X_j & + & (\delta_jU_j)X_{j+1} & + & \delta_j(J_j - L_j\beta_{j-1})X_\ell & = & \delta_j(B_j - L_j\gamma_{j-1}) \\
 & & X_j & + & \alpha_jX_{j-1} & + & \beta_jX_\ell & = & \gamma_j
\end{array}$$

where

$$\begin{array}{rcl}
\delta_j & = & (D_j - L_j\alpha_{j-1})^{-1} \\
\alpha_j & = & \delta_jU_j \\
\beta_j & = & \delta_j(J_j - L_j\beta_{j-1}) \\
\gamma_j & = & \delta_j(B_j - L_j\gamma_{j-1})
\end{array}$$

$$\boxed{\text{For } j = \ell - 1}$$

$$\begin{array}{rcrcrcr}
-L_{\ell-1}(X_{\ell-2} & + & \alpha_{\ell-2}X_{\ell-1} & + & \beta_{\ell-2}X_\ell & = & \gamma_{\ell-} \\
L_{\ell-1}X_{\ell-2} & + & D_{\ell-1}X_{\ell-1} & + & (U+J)_{\ell-1}X_\ell & = & B_\ell \\
\hline
 & & (D_{\ell-1} - L_{\ell-1}\alpha_{\ell-2})X_{\ell-1} & + & [(U+J)_{\ell-1} - L_{\ell-1}\beta_{\ell-2}]X_\ell & = & (B_{\ell-1} - L_{\ell-1}\gamma_{\ell-} \\
 & & X_{\ell-1} & + & \alpha_{\ell-1}X_\ell & = & \gamma_\ell
\end{array}$$

where

$$\begin{array}{rcl}
\delta_{\ell-1} & = & (D_{\ell-1} - L_{\ell-1}\alpha_{\ell-2})^{-1} \\
\alpha_{\ell-1} & = & \delta_{\ell-1}[(U+J)_{\ell-1} - L_{\ell-1}\beta_{\ell-2}] \\
\gamma_{\ell-1} & = & \delta_{\ell-1}(B_{\ell-1} - L_{\ell-1}\gamma_{\ell-2})
\end{array}$$

$$\boxed{\text{For } j = \ell}$$

$$\begin{array}{rcrcrcr}
-L_\ell(X_{\ell-1} & + & \alpha_{\ell-1}X_\ell & & & = & \gamma_{\ell-1}) \\
L_\ell X_{\ell-1} & + & (D+J)_\ell X_\ell & + & U_\ell X_{\ell+1} & = & B_\ell \\
\hline
\end{array}$$

$$\begin{array}{rcrcr}
[(D+J)_\ell - L_\ell \alpha_{\ell-1}]X_\ell & + & U_\ell X_{\ell+1} & = & B_\ell - L_\ell \gamma_{\ell-1} \\
X_\ell & + & \alpha_\ell X_{\ell+1} & = & \gamma_\ell
\end{array}$$

where

$$\begin{aligned}
\delta_\ell &= [(D+J)_\ell - L_\ell \alpha_{\ell-1}]^{-1} \\
\alpha_\ell &= \delta_\ell U_\ell \\
\gamma_\ell &= \delta_\ell (B_\ell - L_\ell \gamma_{\ell-1})
\end{aligned}$$

$$\boxed{\text{For } j = \ell + 1}$$

If $\ell + 1 < IZ$:

$$\begin{array}{rcrcrcr}
-(L+J)_{\ell+1}(X_\ell & + & \alpha_\ell X_{\ell+1} & & & = & \gamma_\ell) \\
(L+J)_{\ell+1}X_\ell & + & D_{\ell+1}X_{\ell+1} & + & U_{\ell+1}X_{\ell+2} & = & B_{\ell+1} \\
\hline
\end{array}$$

$$\begin{array}{rcrcr}
[D_{\ell+1} - (L+J)_{\ell+1}\alpha_\ell]X_{\ell+1} & + & U_{\ell+1}X_{\ell+2} & = & B_{\ell+1} - (L+J)_{\ell+1}\gamma_\ell \\
X_{\ell+1} & + & \alpha_{\ell+1}X_{\ell+2} & = & \gamma_{\ell+1}
\end{array}$$

where

$$\begin{aligned}
\delta_{\ell+1} &= [D_{\ell+1} - (L+J)_{\ell+1}\alpha_\ell]^{-1} \\
\alpha_{\ell+1} &= \delta_{\ell+1}U_{\ell+1} \\
\gamma_{\ell+1} &= \delta_{\ell+1}[B_{\ell+1} - (L+J)_{\ell+1}\gamma_\ell]
\end{aligned}$$

If $\ell + 1 = IZ$:

$$\begin{array}{rcrcr}
-(L+J)_{\ell+1}(X_\ell & + & \alpha_\ell X_{\ell+1} & = & \gamma_\ell) \\
(L+J)_{\ell+1}X_\ell & + & D_{\ell+1}X_{\ell+1} & = & B_{\ell+1} \\
\hline
\end{array}$$

$$\begin{array}{rcr}
[D_{\ell+1} - (L+J)_{\ell+1}\alpha_\ell]X_{\ell+1} & = & B_{\ell+1} - (L+J)_{\ell+1}\gamma_\ell \\
X_{\ell+1} & = & \gamma_{\ell+1}
\end{array}$$

where

$$\begin{aligned}
\delta_{\ell+1} &= [D_{\ell+1} - (L+J)_{\ell+1}\alpha_\ell]^{-1} \\
\gamma_{\ell+1} &= \delta_{\ell+1}[B_{\ell+1} - (L+J)_{\ell+1}\gamma_\ell]
\end{aligned}$$

$$\boxed{\text{For } j = \ell + 2, \ell + 3, \ldots, IZ - 1}$$

If $\ell + 1 < IZ$:

$$\begin{array}{rcrcrcr}
-L_j(X_{j-1} & + & \alpha_{j-1}X_j & & & = & \gamma_{j-1}) \\
L_jX_{j-1} & + & D_jX_j & + & U_jX_{j+1} & = & B_j \\
\hline
\end{array}$$

$$\begin{array}{rcrcr}
(D_j - L_j\alpha_{j-1})X_j & + & U_jX_{j+1} & = & B_j - L_j\gamma_{j-1} \\
X_j & + & \alpha_jX_{j+1} & = & \gamma_j
\end{array}$$

where

$$\begin{aligned}
\delta_j &= (D_j - L_j\alpha_{j-1})^{-1} \\
\alpha_j &= \delta_j U_j \\
\gamma_j &= \delta_j(B_j - L_j\gamma_{j-1})
\end{aligned}$$

$$\boxed{\text{For } j = IZ}$$

If $\ell + 1 < IZ$:

$$\begin{array}{rcrcr}
-L_{IZ}(X_{IZ-1} & + & \alpha_{IZ-1}X_{IZ} & = & \gamma_{IZ-1}) \\
L_{IZ}X_{IZ-1} & + & D_{IZ}X_{IZ} & = & B_{IZ} \\
\hline
 & & (D_{IZ} - L_{IZ}\alpha_{IZ-1})X_{IZ} & = & (B_{IZ} - L_{IZ}\gamma_{IZ-1}) \\
 & & X_{IZ} & = & \gamma_{IZ}
\end{array}$$

where

$$\begin{aligned}
\delta_{IZ} &= (D_{IZ} - L_{IZ}\alpha_{IZ-1})^{-1} \\
\gamma_{IZ} &= \delta_{IZ}(B_{IZ} - L_{IZ}\gamma_{IZ-1})
\end{aligned}$$

The table at the end of Section 11 summarizes the form of the operators that are used in the Gaussian elimination scheme at the various levels (j) of the model.

In an elegant extension of the Lindzen–Kuo method, Professor Paul Duchateau of the Mathematics Department of CSU developed a scheme to solve the linear matrix equation $AX = B^*$ with a considerable reduction in computer time. He observed that each of the five equations involves a vertical derivative in a single variable, and this variable is different for each equation: specifically, u', v', w', T', ϕ' in that order. These equations are not all second order in the vertical, so we cannot directly use the Lindzen–Kuo scheme. However, if we finite difference them as they appear, without combining them, and retain all five variables as unknowns, we find again a block tridiagonal structure; but the off–diagonal blocks are themselves diagonal matrices! This is precisely the matrix structure outlined above.

Duchateau noted that with non–vanishing viscosity and thermal diffusivity, the off–diagnonal block matrices are guaranteed to be trivially invertable. Consequently, the algorithm can be modified so that only a single $5 \times IY$ by $5 \times IY$ dense matrix need be inverted. In the standard method, such a matrix inversion must be accomplished at each vertical level. Thus Duchateau's scheme reduces the matrix inversion workload, which constitutes the primary computational burden, by a factor of IZ which is typically a factor of 30 or more.

In testing this scheme, we determined that its usefulness is limited to cases where viscosity is rather large (e.g. 100 m^2 s^{-1}) throughout the model's domain. The restriction of this scheme results from using the L^{-1} matrix, which is inversely related to viscosity and diffusivity, to operate on a row of matrix A in reducing it to an upper triangular system (refer to schematic of $AX = B^*$ in Section 11.1). Apparently when the magnitude of L is small (due to a small value of viscosity; e.g. $\nu \leq 50$m^2 s^{-1}), the condition number of the matrix to be inverted increases so that for all practical purposes the matrix is not invertible.

Heuristically, all the ill–behaved aspects with small dissipation are collected into a single matrix inversion stage, which the algorithm cannot properly handle. When the ill behavior is distributed over many (time–consuming) matrix inversions, the algorithm works quite adequately. This result is apparently an application of the computer proverb, 'You don't get something for nothing'. Note, however, that Duchateau's scheme may be useful in second–order, dissipation–dominated problems. Unfortunately, that is not our area of interest.

11.3 Backsubstitution

Once the system $AX = B^*$ has been reduced to upper triangular form as shown below, it becomes a trivial matter to solve for the solution matrix X.

$$\begin{bmatrix} I & \alpha_1 & & & \beta_1 & & & \\ & I & \alpha_2 & & \beta_2 & & & \\ & & I & \alpha_3 & \beta_3 & & & \\ & & & \ddots & \vdots & & & \\ & & & I & \alpha_{\ell-1} & & & \\ & & & & I & \alpha_\ell & & \\ & & & & & I & \alpha_{\ell+1} & \\ & & & & & & & I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{\ell-1} \\ X_\ell \\ X_{\ell+1} \\ X_{IZ} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{\ell-1} \\ \gamma_\ell \\ \gamma_{\ell+1} \\ \gamma_{IZ} \end{bmatrix}$$

In the backsubstitution, we compute

$$\begin{aligned} X_{IZ} &= \gamma_{IZ} \\ \text{and } X_j &= \gamma_j - \alpha_j X_{j+1} - \beta_j X_\ell \text{ for } j = IZ-1, 1 \end{aligned}$$

where $\beta_j = 0$ for $j > \ell - 2$

Since the γ, α, and β matrices are needed in the backsubstitution process, they are temporarily stored on a random access file and recalled as needed. This was done because the size of these matrices prohibited storing them for all levels simultaneously. For example at each level γ and α consist of $50 \times IY^2$ words and β of $10 \times IY$ words.

This completes the description of the model formulation. In the next section we describe our effort to understand and fix the problems our model has as both s and $\sigma \to 0$.

Summary of the operators that are used in the Gaussian elimination scheme at various levels (j) of the model

j	δ_j	α_j	β_j	γ_j
1	D_j^{-1}	$\delta_j U_j$	$\delta_j J_j$	$\delta_j B_j$
$[2, \ell - 2]$	$(D_j - L_j \alpha_{j-1})^{-1}$	$\delta_j U_j$	$\delta_j (J_j - L_j \beta_{j-1})$	$\delta_j (B_j - L_j \gamma_{j-1})$
$\ell - 1$	$(D_j - L_j \alpha_{j-1})^{-1}$	$\delta_j [(U + J)_j - L_j \beta_{j-1}]$	—	$\delta_j (B_j - L_j \gamma_{j-1})$
ℓ	$[(D + J)_j - L_j \alpha_{j-1}]^{-1}$	$\delta_j U_j$	—	$\delta_j (B_j - L_j \gamma_{j-1})$
$[\ell + 1, IZ - 1]$	$[D_j - (L_j + \Delta_{j,\ell-1} J_{\ell+1} \alpha_{j-1}]^{-1}$	$\delta_j U_j$	—	$\delta_j [B_j - (L_j + \Delta_{j,\ell+1} J_{\ell+1}) \gamma_{j-1}]$
IZ	$[D_j - (L_j + \Delta_{j,\ell+1} J_{\ell+1}) \alpha_{j-1}]^{-1}$	—	—	$\delta_j [B_j - (L_j + \Delta_{j,\ell+1} J_{\ell+1}) \gamma_{j-1}]$

$$\text{where} \quad \Delta_{j,\ell+1} = \begin{cases} 0 & \text{if } j \neq \ell + 1 \\ 1 & \text{if } j = \ell + 1 \end{cases}$$

12 Steady, Symmetric Response to Convective Heating

Lim and Chang (1983) pointed out the importance of the zonally symmetric response to tropical heating, particularly for barotropic modes. Their conclusions involved dynamical arguments based on a shallow water system with constant coefficients (10 days) for the dissipation time scale of both Rayleigh friction and Newtonian cooling.

On the other hand, studies of the Walker circulation using primitive equation models (e.g. Geisler, 1981, and Rosenlof et al., 1985) have generally avoided the steady, zonally symmetric response to tropical heating by formulating the model forcing so as to exclude the zonal average component. Investigation of the symmetric response has been avoided because an inadequacy in the primitive equation model occurs when both longitudinal wavenumber (k) and frequency (σ) approach zero. However, in problems where this cannot be avoided (e.g., an isolated, stationary heat source over South America), this inadequacy in the model must be resolved.

To better understand the problem as k and $\sigma \to 0$, we can separate the primitive equations into their horizontally and vertically varying parts. For the inviscid primitive equations with a basic state at rest, the horizontal structure equation is given as:

$$\left[D\nabla^2 - \frac{df^2}{dy}\frac{\partial}{\partial y} + \frac{D^2 L}{R\Gamma}\frac{\partial}{\partial z}\right] i\sigma\phi' + \left[f\left(\frac{df^2}{dy}\right) - \frac{1}{\cos\theta}\frac{d}{dy}\left(f\cos\theta\right)D\right] ik\phi' = D^2 L\left(\frac{Q'}{\Gamma}\right) \tag{12.1}$$

where $D = (f^2 - \sigma^2)$ and $L = \frac{1}{p}\frac{\partial}{\partial z}p$. For the $k = 0$, $\sigma = 0$ case, the dynamical adjustment on the left hand side of (12.1) is zero. Thus for a zonally symmetric ($k = 0$), stationary ($\sigma = 0$) heat source, (12.1) implies that a steady, inviscid response is prohibited.

Within the context of the primitive equation model presented in this document the difficulty with the steady symmetric response is manifested primarily in the geopotential field. For example, in most of our runs with $k = 0$ and $\sigma \approx 0$, the velocity fields are qualitatively correct but the geopotential field is represented by a large constant value. The aberrant behavior of the perturbation ϕ field can be explained in part by considering equations (4.1)-(4.5) when k and $\sigma \to 0$. For this case all the terms in these equations which involve the parameters k and σ can be neglected. In this form the equations involve only derivatives of ϕ', and not ϕ' itself. Thus, no constraint exists in these equations to determine the overall amplitude of ϕ'.

To examine the behavior of ϕ' as $\sigma \to 0$, Fig. 12.1 shows the product $\sigma\phi'_{max}$ as a function of wave period obtained from several model runs with $k = 0$. In addition, these model runs used a dissipation time scale of 20 days for Rayleigh friction and Newtonian cooling, a heat source with a half width of $\sim 9°$ centered on the equator and a resting basic state. The variable ϕ'_{max} represents the maximum value of ϕ' over the model domain. We note from this plot that $\sigma\phi'_{max}$ asymptotes to a constant value ($\sim 9.5 \times 10^{-4}$) as the wave period $\to \infty$ (or $\sigma \to 0$). Further investigation showed that this value is determined directly from the lower boundary condition (8.11) which in this case simplifies to $i\sigma\phi' = w'R\overline{T}$. Since the value of w' at the bottom boundary varies little as the wave period increases beyond 100 days (see Fig. 12.2), the amplitude of $\sigma\phi'$ becomes fixed. Thus at large wave periods, ϕ' at the bottom boundary is inversely proportional to σ so that as $\sigma \to 0$, the variable $\phi' \to \infty$. This effect propagates throughout the domain in the back substitution process described in Section 11.3 via the hydrostatic equation.

The horizontal and vertical variation in the ϕ' field is concealed when the value of ϕ' introduced at the bottom boundary, as discussed above, becomes too large. In these cases, which occur when the wave period exceeds 100 days, we propose the following mechanical fix so that real physical variation in the ϕ' field can be observed. This fix, outlined in the steps below, is equivalent to assuming that there is no net accumulation of mass in the model domain (i.e., the average ϕ' along the lower boundary of the model is zero).

1. After the first step in back substitution process we have $\phi'_{IZ}(\eta) = \hat{\phi}'_{IZ}(\eta)$+constant.

2. Use mean value theorem to compute this constant. In our case this theorem takes the following form:

$$\text{constant} = \frac{1}{\eta_N - \eta_S} \int_{\eta_S}^{\eta_N} \phi'_{IZ}(\eta) d\eta.$$

3. Subtract the constant from $\phi'_{IZ}(\eta)$:

$$\hat{\phi}'_{IZ}(\eta) = \phi'_{IZ}(\eta) - \text{constant}$$

4. Continue back substitution process as before using adjusted value, $\hat{\phi}'_{IZ}$.

We also have observed in the case $k = 0$ and $\sigma \rightarrow 0$, that the linear matrix problem, $AX = B^*$, becomes ill-posed when $w' = 0$ is imposed as the upper boundary condition and thus cannot be solved. As an alternate upper boundary condition we set, $\frac{\partial w'}{\partial z} - w' = 0$, which is equivalent setting divergence to zero at the upper boundary. This however produces a solution in w' and the thermodynamic fields which increases with height as e^z. A simple mechanical fix which does not allow w' to increase as rapidly in height is to impose $\frac{\partial w'}{\partial z} = 0$ as the upper boundary condition.

Finally, it is worth noting that the solution becomes increasingly sensitive to the parameterization of dissipation as $\sigma \rightarrow 0$. This is physically reasonable since in such cases the time scale of dissipation is short relative to the time scale of the wave. For example, when the specified wave period is 10,000 days, increasing the dissipation time scale by a factor of four *reduces* the amplitude of the ϕ' field by a similar factor.

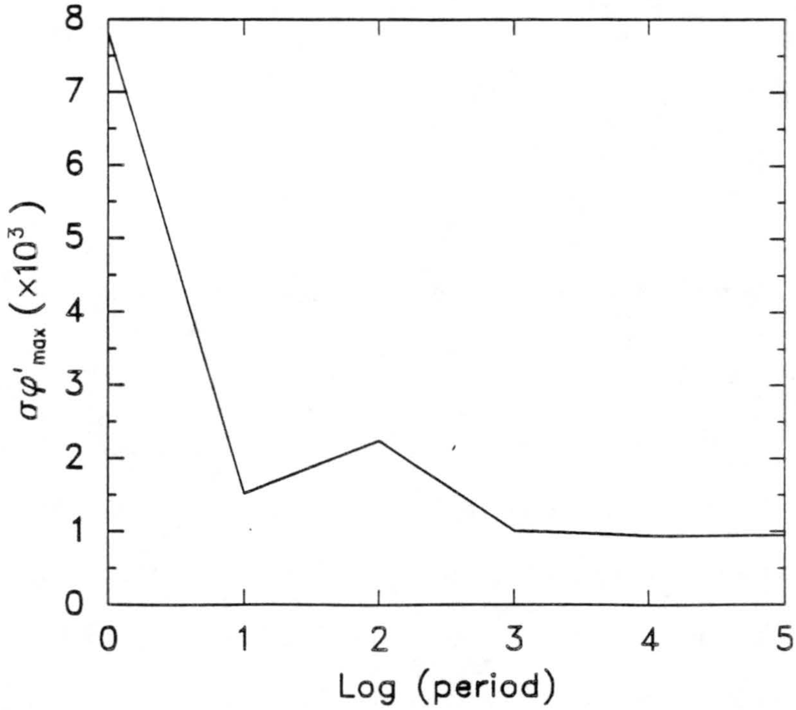


Fig. 12.1. The product $\sigma\phi'_{max}$ (m^2s^{-3}) plotted as a function of wave period (days), where ϕ'_{max} repres the maximum value of ϕ' over the model domain.

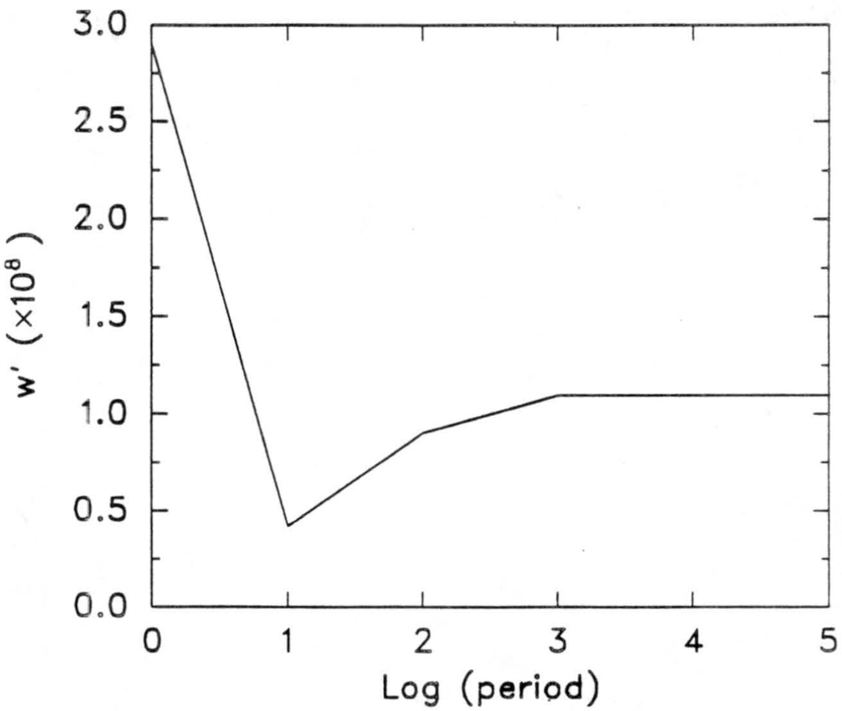


Fig. 12.2. Perturbation vertical motion at the equator and bottom boundary of the model, w' (s plotted as a function of wave period (days).

13 Concluding Remarks

Due to the wide range of problems to which our model is applicable, several researchers in the field of atmospheric science have sought to use it on their problems. For these researchers to correctly use and understand the model in all its complexity, it is important that a current and accurate documentation of the model be maintained. This manuscript documents the numerous corrections and improvements that have been made to the model since its original design and description were presented in Stevens and Ciesielski (1984). The main improvements presented here include: (1) the capability of placing the horizontal boundaries at the poles, (2) an improved latitudinal differencing scheme, and (3) treatment of the $k = 0$, $\sigma = 0$ case.

To date we have successfully used our primitive equation wave model on several research problems. Of the problems completed, two have resulted in refereed publications. The first of these (Rosenlof et al., 1985) examined the effects of a Hadley cell and cumulus friction upon the Walker circulation. The second (Shapiro et al., 1988) studied the differences in the structure and dynamics of easterly propagating tropical waves in the context of different mean zonal wind profiles. Ongoing projects are using the wave model to study atmospheric circulations driven by heat sources over South America, and to examine the effects of the quasi-biennial oscillation on structure of easterly waves.

Recently we have developed and successfully tested a time integration (TINTG) version of our primitive equation model. In the frequency version (FREQ) of the model, the system of equations is represented in the form of a linear matrix equation $AX = B^*$, where A, X and B^* are matrices defined in Section 11. In the FREQ code the solution matrix, X, is obtained for a specified frequency with a Gaussian elimination scheme suggested by Lindzen and Kuo (1969). Alternately, we can write the system as

$$AX = \frac{\partial}{\partial t}(X) + LX = B^*$$

or

$$\frac{\partial X}{\partial t} = B^* - LX$$

In the TINTG code we solve this latter equation with a second-order Runge-Kutta time integration scheme. Since the matrix L is only a slight modification to the matrix A, a significant portion of the FREQ code was easily adapted to use in the TINTG version of the model. We plan to modify the linear TINTG code to a nonlinear version where only the zonally symmetric flow is considered, and with it study nonlinear, nonsteady Hadley circulations.

Appendix A

Formulation of Polar Boundary Conditions

At the singular points of the spherical coordinate system – *i.e.*, the North and South Poles – we require all true scalar fields to be continuous variables that remain finite at the pole. This applies to the geopotential, vorticity, divergence, and deformation fields. The horizontal vector velocity field must also be continuously varying; otherwise infinite vorticity, divergence, and/or deformation would result. However, the two spherical components (u, v) of velocity need not, in general, be continuous across the poles because the corresponding unit vectors $(\hat{\imath}, \hat{\jmath})$ change discontinuously at the pole.

Let us consider the linearized shallow water equations with a geostrophic zonal flow $\overline{u}$ as a basic state.

$$\left(\frac{\partial}{\partial t} + \frac{\overline{u}}{a\cos\theta}\frac{\partial}{\partial\lambda}\right) u' - \left(f - \frac{\partial\overline{u}\cos\theta}{a\cos\theta\partial\theta}\right) v' = -\frac{\partial\Phi'}{a\cos\theta\partial\lambda}$$

$$\left(\frac{\partial}{\partial t} + \frac{\overline{u}}{a\cos\theta}\frac{\partial}{\partial\lambda}\right) v' + \left(f + \frac{2\overline{u}}{a}\tan\theta\right) u' = -\frac{\partial\Phi'}{a\partial\theta}$$

$$\left(\frac{\partial}{\partial t} + \frac{\overline{u}}{a\cos\theta}\frac{\partial}{\partial\lambda}\right) \Phi' + \overline{\Phi}\left(\frac{\partial u'}{a\cos\theta\partial\lambda} + \frac{\partial v'\cos\theta}{a\cos\theta\partial\theta}\right) + v'\frac{\partial\overline{\Phi}}{a\partial\theta} = 0$$

with

$$\frac{\overline{u}^2}{a}\tan\theta + f\overline{u} = -\frac{\partial\overline{\Phi}}{a\partial\theta}$$

Writing a single Fourier component for each perturbation field

$$(\ )' = (\ )e^{i(s\lambda-\sigma t)}$$

we obtain

$$-i\hat{\sigma}u - f_1 v = \frac{-is}{a\cos\theta}\Phi$$

$$-i\hat{\sigma}v + f_2 u = -\frac{\partial\Phi}{a\partial\theta}$$

$$-i\hat{\sigma}\Phi + v\frac{\partial\overline{\Phi}}{a\partial\theta} + \overline{\Phi}\left(\frac{isu}{a\cos\theta} + \frac{\partial v\cos\theta}{a\cos\theta\partial\theta}\right) = 0$$

where

$$\begin{aligned}
\overline{\omega} &\equiv \frac{\overline{u}}{a\cos\theta} \quad \text{is the relative angular velocity of the mean flow} \\
\hat{\sigma} &\equiv \sigma - s\overline{\omega} \\
f_1 &\equiv f + \overline{\zeta} = f - \frac{\partial\overline{u}\cos\theta}{a\cos\theta\partial\theta} \\
f_2 &\equiv f + \frac{2\overline{u}}{a}\tan\theta
\end{aligned}$$

For the basic state, continuity of the wind at the pole implies $\overline{u}_p = 0$. Thus the angular velocity $\overline{\omega}_p$ is finite and well- behaved at the pole. Suppose we focus on the North Pole and take $\alpha \equiv \frac{\pi}{2} - \theta$ to be the angular difference between the pole and latitude θ. [Corresponding relationships apply to the South Pole.] We write $\overline{u}$ as a Taylor series in α:

$$\overline{u}(\alpha) = \overline{u}_o + \overline{u}_1\alpha + \overline{u}_2\alpha^2 + \ldots$$

Thus

$$\overline{u}\left(\theta = \frac{\pi}{2}\right) = \overline{u}_o = 0.$$

Also,

$$\cos\theta = \sin\alpha = \alpha - \frac{\alpha^3}{6} + O(\alpha^5)$$

$$\sin\theta = \cos\alpha = 1 - \frac{\alpha^2}{2} + O(\alpha^4)$$

Expanding in α near the pole,

$$\overline{u} = \overline{\omega} a \cos\theta = \overline{\omega} a \sin\alpha$$

$$\overline{u}_1\alpha + \overline{u}_2\alpha^2 + \ldots = (\overline{\omega}_o + \overline{\omega}_1\alpha + \overline{\omega}_2\alpha^2 + \ldots)a\left(\alpha - \frac{\alpha^3}{6} + \ldots\right)$$

Therefore:

$$\overline{\omega}_o = \frac{\overline{u}_1}{a},\ \overline{\omega}_1 = \frac{\overline{u}_2}{a}$$

Geostrophic balance on the sphere implies

$$\begin{aligned}
-\frac{\partial\overline{\Phi}}{a\partial\theta} = \frac{\partial\overline{\Phi}}{a\partial\alpha} &= \frac{\overline{u}}{a\cos\theta}\overline{u}\sin\theta + 2\Omega\sin\theta\overline{u} = \overline{u}\sin\theta(\overline{\omega} + 2\Omega) \\
\frac{1}{a}\left(\overline{\Phi}_1 + 2\overline{\Phi}_2\alpha + \ldots\right) &= \overline{u}\cos\alpha(\overline{\omega} + 2\Omega) \\
&= (\overline{u}_1\alpha + \overline{u}_2\alpha^2 + \ldots)\left(1 - \frac{\alpha^2}{2} + \ldots\right)(2\Omega + \overline{\omega}_o + \overline{\omega}_1\alpha + \ldots) \\
&= (2\Omega + \overline{\omega}_o)\overline{u}_1\alpha + \alpha^2[\overline{u}_1\overline{\omega}_1 + (2\Omega + \overline{\omega}_o)\overline{u}_2] + \ldots
\end{aligned}$$

Therefore,

$$\overline{\Phi}_1 = 0,\ \ \frac{2}{a}\overline{\Phi}_2 = (2\Omega + \overline{\omega}_o)\overline{u}_1$$

Vorticity and divergence in spherical coordinates can be expressed as

$$\begin{aligned}
\zeta \equiv \hat{k}\cdot\nabla\times\mathbf{v} &= \frac{1}{a\cos\theta}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial u\cos\theta}{\partial\theta}\right) \\
&= \frac{1}{a\cos\theta}\left(isv + u\sin\theta - \cos\theta\frac{\partial u}{\partial\theta}\right) \\
\nabla\cdot\mathbf{v} &= \frac{1}{a\cos\theta}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\theta}{\partial\theta}\right) \\
&= \frac{1}{a\cos\theta}\left(isu - v\sin\theta + \cos\theta\frac{\partial v}{\partial\theta}\right)
\end{aligned}$$

Taylor series expansions for u, v, $\cos\theta$ and $\sin\theta$ near the North Pole ($\theta = \frac{\pi}{2}, \alpha = \frac{\pi}{2} - \theta = 0$) give

$$\begin{aligned}
\zeta &= \tfrac{1}{a\,\sin\alpha}\left(isv + u\cos\alpha + \sin\alpha\tfrac{\partial u}{\partial\alpha}\right) \\
&= \tfrac{1}{a}\tfrac{1}{\alpha-\frac{\alpha^3}{6}+\ldots}\Big[is(v_o + \alpha v_1 + \ldots) + (u_o + \alpha u_1 + \ldots)(1 - \tfrac{\alpha^2}{2} + \ldots) \\
&\quad +(\alpha - \tfrac{\alpha^3}{6} + \ldots)(u_1 + 2u_2\alpha + \ldots)\Big] \\
&= \tfrac{1}{a}\tfrac{1}{\alpha-\frac{\alpha^3}{6}+\ldots}\left[(isv_o + u_o) + \alpha(isv_1 + 2u_1) + \alpha^2(isv_2 + 3u_2 - \tfrac{u_o}{2}) + \ldots\right]
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \tfrac{1}{a\sin\alpha}(isu - v\cos\alpha - \sin\alpha\tfrac{\partial v}{\partial\alpha}) \\
&= \tfrac{1}{a}\tfrac{1}{\alpha-\frac{\alpha^3}{6}+\ldots}\Big[is(u_o + \alpha u_1 + \ldots) - (v_o + \alpha v_1 + \ldots)(1 - \tfrac{\alpha^2}{2} + \ldots) \\
&\quad -(\alpha - \tfrac{\alpha^3}{6} + \ldots)(v_1 + \alpha 2v_2 + \ldots)\Big] \\
&= \tfrac{1}{a}\tfrac{1}{\alpha-\frac{\alpha^3}{6}+\ldots}\left[(isu_o - v_o) + \alpha(isu_1 - 2v_1) + \alpha^2(isu_2 - 3v_2 + \tfrac{v_o}{2}) + \ldots\right]
\end{aligned}$$

For $s = 0$:

Finite vorticity at the pole implies $u_o = 0 \;\Rightarrow\; \boxed{u_p = 0}$

Finite divergence at the pole implies $v_o = 0 \;\Rightarrow\; \boxed{v_p = 0}$

where the subscript p implies that the condition is imposed at the pole.

Perturbation equations: $-i\sigma u - f_1 v = 0$

$$-\,i\sigma v + f_2 u = -\tfrac{1}{a}\tfrac{\partial\Phi}{\partial\theta} = \tfrac{1}{a}\tfrac{\partial\Phi}{\partial\alpha}$$

$$-\,i\sigma\Phi + v\tfrac{\partial\overline{\Phi}}{a\partial\theta} + \overline{\Phi}(\nabla\cdot\mathbf{v}) = 0$$

0th order in α: $0 = 0$

$$0 = \tfrac{1}{a}\Phi_1 \;\Rightarrow\; \boxed{\Phi_1 = 0}$$

$$-i\sigma\Phi_o + \overline{\Phi}_o\cdot\tfrac{1}{a}(-2v_1) = 0 \;\Rightarrow\; \boxed{-i\sigma\Phi_o = \overline{\Phi}_o\tfrac{2v_1}{a}}$$

or

$$\boxed{-i\sigma\Phi_p = \overline{\Phi}_p\left(-\tfrac{2}{a}\tfrac{\partial v}{\partial\theta}\right)_p = \overline{\Phi}_p(-\nabla\cdot\mathbf{v})_p}$$

The 0th order perturbation equations for $s \neq 0$:

$-i\hat{\sigma}u_o - f_1 v_o = \frac{-is}{a\alpha}(\Phi_o + \alpha\Phi_1 + \ldots) = \frac{-is}{a}\Phi_1$, with $\boxed{\Phi_o = 0}$ or $\boxed{\Phi_p = 0}$

is required for continuity of Φ across the pole

$$-i\hat{\sigma}v_o + f_2 u_o = -\frac{1}{a}\frac{\partial\Phi}{\partial\theta} = \frac{1}{a}\frac{\partial\Phi}{\partial\alpha} = \frac{\Phi_1}{a}$$

$$-i\hat{\sigma}\Phi_o + v_o\frac{1}{a}(-\overline{\Phi}_1) + \overline{\Phi}_o(\nabla\cdot\mathbf{v})_o = 0$$

Since

$$\Phi_o = 0, \;\; \overline{\Phi}_1 = 0, \;\; \overline{\Phi}_o \neq 0 \;\; \Rightarrow \;\; (\nabla\cdot\mathbf{v})_o = 0$$

For $s \neq 0$, continuous and finite vorticity and divergence at the pole implies

$$(\nabla\cdot\mathbf{v})_o = 0, \qquad \zeta_o = 0$$

Therefore,

$$isu_o - v_o = 0 \qquad isv_o + u_o = 0 \qquad \text{for fitness}$$

$$isu_1 - 2v_1 = 0 \qquad isv_1 + 2u_1 = 0 \qquad \text{for continuity}$$

$$\begin{pmatrix} is & -1 \\ 1 & is \end{pmatrix}\begin{pmatrix} u_o \\ v_o \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 1 - s^2 = 0 \;\text{ or }\; u_o = v_o = 0$$

Therefore:

For $s \neq \pm 1 : \begin{cases} u_o = 0 & u_p = 0 \\ v_o = 0 & v_p = 0 \end{cases}$

For $s = \pm 1 : \boxed{v_o = isu_o}$

$$\begin{pmatrix} is & -2 \\ 2 & is \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4 - s^2 = 0 \;\text{ or }\; u_1 = v_1 = 0$$

Therefore:

$$\text{For } s \neq \pm 2 : \begin{cases} u_1 = 0 & \left(\frac{\partial u}{\partial \theta}\right)_p = 0 \\ v_1 = 0 & \left(\frac{\partial v}{\partial \theta}\right)_p = 0 \end{cases}$$

For $s \neq \pm 1, u_o = v_o = 0$ implies through the momentum equations:

$$\boxed{\Phi_1 = 0} \quad \text{or} \quad \boxed{\left(\frac{\partial \Phi}{\partial \theta}\right)_p = 0}$$

Summary of polar boundary conditions for linearized shallow water equations

variable	$s = 0$	$s = \pm 1$	$\lvert s \rvert > 1$
Φ	$-i\sigma\Phi_p = \overline{\Phi}_p(-\nabla \cdot \mathrm{v})_p$	$\Phi_p = 0$	$\Phi_p = 0$
u	$u_p = 0$	$\left(\frac{\partial u}{\partial \theta}\right)_p = 0$	$u_p = 0$
v	$v_p = 0$	$\left(\frac{\partial v}{\partial \theta}\right)_p = 0$	$v_p = 0$

Additional characteristics that have been determined:

For $s = \pm 1 :$ $\quad v_o = isu_o$

For $s \neq \pm 1 :$ $\quad \left(\frac{\partial \Phi}{\partial \theta}\right)_p = 0$

For $s \neq \pm 2$ or 0: $\quad \left(\frac{\partial u}{\partial \theta}\right)_p = 0 = \left(\frac{\partial v}{\partial \theta}\right)_p$

Appendix B

Motivation for Using a Staggered Grid in Horizontal Direction

A1. Introduction

Energy propagation by small-scale dispersive gravity waves, excited by a local breakdown of geostrophy, is an important mechanism in restoring quasi-geostrophic flow by the geostrophic adjustment process. Previous attempts (Winninghoff, 1968; Arakawa and Lamb, 1977) to numerically simulate geostrophic adjustment have shown the propagation of energy to be highly dependent upon the manner in which the dependent variables are distributed over the grid. In Section A2 the dispersion properties for the simplest fluid in which geostrophic adjustment can occur, namely the linearized shallow water equations, are analytically derived for several different arrangements of the discretized variables. For completeness, numerical results are presented (in Section A3) for a shallow water model using a staggered and a non-staggered grid.

A2. Analytical analysis of energy propagation

Consider the simplest fluid in which geostrophic adjustment can occur – namely a rotating fluid which is incompressible, homogeneous in the vertical (z) direction, nonviscous, hydrostatic, and has a flat bottom and a free top surface. The basic equations which govern such a fluid are the so–called shallow water equations.

$$\begin{aligned} u_t - fv + gh_x &= 0 \\ v_t + fu + gh_y &= 0 \\ h_t + hu_x + hv_y &= 0 \end{aligned}$$

where the subscripts denote a derivative with respect to that variable. In these equations t is time, x and y the horizontal cartesian coordinates, g the gravitational acceleration, f a constant coriolis parameter, u and v the velocity components in the x and y directions, respectively, and h the depth of the fluid.

Using the perturbation method the dependent variables are expanded into two parts: a basic state $\overline{(\ )}$ which is assumed to be independent of time and the x–direction, and a perturbation $(\ )'$ which is a local deviation of the field from the basic state. This expansion is shown below

$$
\begin{aligned}
u(x,y,t) &= \overline{u}(y) + u'(x,y,t) \\
v(x,y,t) &= \overline{v}(y) + v'(x,y,t) \\
h(x,y,t) &= \overline{h}(y) + h'(x,y,t)
\end{aligned}
$$

To simplify the analysis even further, consider the case with a resting basic state (i.e. $\overline{u} = \overline{v} = 0$ and $\overline{h}(y) = H$) and perturbations that are independent of the y–direction, that is $(\ )'_y = 0$. With these assumptions the equations can be written as:

$$u'_t - fv' + gh'_x = 0 \tag{1}$$

$$v'_t + fu' = 0 \tag{2}$$

$$h'_t + Hu'_x = 0 \tag{3}$$

Assuming the perturbation solutions to be proportional to $e^{i(kx-\sigma t)}$ as shown below:

$$\left.\begin{aligned} u' &= \hat{u} \\ v' &= \hat{v} \\ h' &= \hat{h} \end{aligned}\right\} \times e^{i(kx-\sigma t)}$$

Equations (1) - (3) can be written as:

$$
\begin{aligned}
-i\sigma\hat{u} &- f\hat{v} + ikg\hat{h} = 0 \\
-i\sigma\hat{v} &+ f\hat{u} = 0 \\
-i\sigma\hat{h} &+ Hik\hat{u} = 0
\end{aligned} \tag{4}
$$

In order for these equations to have a nontrival solution for $\hat{u}$, $\hat{v}$, and $\hat{h}$, the determinant of (4) must equal 0. This condition leads to the following cubic frequency equation:

$$(-i\sigma)^3 + f^2(-i\sigma) - ikg(-i\sigma)ikH = 0 \tag{5}$$

Equation 5 contains a geostrophic mode

$$(\sigma/f)^2 = 1 + \Gamma^2 k^2 \tag{6}$$

where $\Gamma = \sqrt{gH/f}$ is the Rossby radius of deformation. From (6) one can note that the frequency of the gravity–inertial waves increases monotonically with wavenumber k, unless Γ is zero. In addition, the group velocity $(d\sigma/dk)$, which is the velocity at which energy propagates, is never zero except for case $\Gamma = 0$.

The effect of the space discretization error on the frequency is now considered for the distributions of the dependent variables shown in Fig. A1. For scheme A, equations 1–3 are finite differenced in the x–direction as shown below:

$$(u'_t)_j - (f/2)(v'_{j+\frac{1}{2}} + v'_{j-\frac{1}{2}}) + (g/d)(h'_{j+\frac{1}{2}} - h'_{j-\frac{1}{2}}) = 0 \tag{7}$$

$$(v'_t)_j + (f/2)(u'_{j+\frac{1}{2}} + u'_{j-\frac{1}{2}}) = 0 \tag{8}$$

$$(h'_t)_j + (H/d)(u'_{j+\frac{1}{2}} - u'_{j-\frac{1}{2}}) = 0 \tag{9}$$

where $x_j = d \times j$. For the discrete grids shown in Fig. A1 the solutions are now assumed proportional to $e^{i(kx_j - \sigma t)}$. With this assumption equations 7–9 become:

$$\begin{aligned}
-i\sigma\hat{u} \quad & -(f/2)[e^{ikd/2} + e^{-ikd/2}]\hat{v} \quad +(g/d)[e^{ikd/2} - e^{-ikd/2}]\hat{h} &= 0 \\
-i\sigma\hat{v} \quad & +(f/2)[e^{ikd/2} + e^{-ikd/2}]\hat{u} &= 0 \\
-i\sigma\hat{h} \quad & +(H/d)[e^{ikd/2} + e^{-ikd/2}]\hat{u} &= 0
\end{aligned} \tag{10}$$

To understand how (10) was arrived at, shown below is the derivation of a perturbation term at the $(j + \frac{1}{2})$ point.

$$\begin{aligned}
(\,)' &= (\hat{\,})e^{i(kx_{j+\frac{1}{2}} - \sigma t)} \\
&= (\hat{\,})e^{i(kd(j+\frac{1}{2}) - \sigma t)} \\
&= (\hat{\,})e^{i(kdj - \sigma t)}e^{ikd/2}
\end{aligned}$$

Setting the determinant of (10) equal to zero results in the following frequency relationship for gravity–inertial waves for scheme A:

$$\sigma^2 = (f^2/4)(e^{ikd/2} + e^{-ikd/2})^2 - (gH/d^2)(e^{ikd/2} - e^{-ikd/2})^2$$

which when simplified yields the following equation

$$\text{Scheme A}: \ (\sigma/f)^2 = \cos^2(kd/2) + 4(\Gamma/d)^2(kd/2) \tag{11}$$

In a similar manner, frequency relationships can be obtained for schemes B through D as given below:

$$\text{Scheme B:} \quad (\sigma/f)^2 = 1 + 4(\Gamma/d)^2 \sin^2(kd/2) \tag{12}$$

$$\text{Scheme C:} \quad (\sigma/f)^2 = 1 + (\Gamma/d)^2 \sin^2(kd) \tag{13}$$

$$\text{Scheme D:} \quad (\sigma/f)^2 = \cos^2(kd/2) + (\Gamma/d)^2 \sin^2(kd/2) \tag{14}$$

These frequencies (11) – (14) are compared to the differential frequency in Figs. A2 and A3 for the values of (Γ/d) equal to 2.0 and 0.2, respectively. Since the shortest wavelength resolvable is $2d$ (i.e. $k = \pi/d$), it is sufficient to consider frequencies over the range $0 \leq kd/\pi \leq 1$.

The results shown in Figs. A2 and A3 indicate that at small wavenumbers the difference schemes approximate well the differential frequency. However, for shorter waves the error in the group velocities becomes increasingly large and in some cases even spuriously negative. Scheme B results in the best simulation of the geostrophic adjustment process as described by continuous theory, while for (Γ/d) sufficiently larger than 0.5 Scheme A is nearly as good. At wavenumbers where the group velocity is zero (i.e. $d\sigma/dk = 0$), energy from gravity–inertia waves excited somewhere in the domain would stay there. For example in Fig. A2 zero group velocity occurs at $kd/\pi = 0.5$ for Scheme C, at $kd/\pi = 0.48$ for Scheme D, and at $kd/\pi = 1.0$ for Schemes A and B.

Cahn (1945) gave the solution of an initial value problem for which (1)–(3) are the governing equations. At the initial time he let h = constant, $u = 0$, $v = V_o$ in the domain from $x = -b$ to $x = b$, and $v = 0$ outside this domain. Some results of these calculations with $b/d = 1$ and $(\Gamma/d) = 2$ are shown in Figs. A4 and A5. Figure A4 shows the time variation of h at $x = b$ for the differential case and for each of the difference schemes. In a similar fashion Fig. A5 gives the space variation of h at $t = 80$ hours. As expected, Schemes A and B simulate the geostrophic adjustment better than the other schemes.

It is of interest to note from Figs. A4 and A5 that even in the case of the best difference scheme, there remains a significant error in the solution when compared to the differential case. A further improvement in the accuracy of the solution without increasing the number of degrees of freedom, requires that a higher order difference scheme (e.g. 4th order) be used or a spectral method (Fulton, 1984) be employed. A remarkable difference between the spectral and finite difference discretizations is the behavior of the error as the number of degrees of freedom is increased (Schubert *et al.*, 1984). For problems with smooth (infinitely differentiable) solutions, the error in the finite difference discretization decreases slowly (algebraically) while that of spectral discretization decreases rapidly (exponentially).

A3. Numerical Results

In this section the energy propagation of gravity–inertial waves in numerically investigated on a staggered grid (SG) and a non–staggered grid (NSG) using a linearized shallow water model on a sphere. These grids correspond to finite difference schemes B and C, respectively, in Fig. A1. A description of the model used here can be found in Stevens *et al.*, (1984). In short, the solutions are assumed to be of the form $e^{-i(s\lambda-\sigma t)}$, where s is the (integral) zonal wavenumber and λ is longitude. In the latitudinal (θ) direction, derivatives are approximated with second–order finite differences. The model computes a set of eigenvalues σ_n and corresponding eigenvectors (u_n, v_n and h_n) for a specified basic state $u(\theta)$, equivalent depth (H) and zonal wavenumber (s).

In order to have a theoretical comparison for the numerical results, the model described above was run with a resting basic state and a small equivalent depth. Under these conditions, according to β–plane theory (Lindzen, 1967), the solution begins to decay within θ_d degrees of the equator. The parameter θ_d is defined as:

$$\theta_d = \epsilon^{-\frac{1}{4}}(2n+1)^{\frac{1}{2}} \tag{15}$$

where $\epsilon = (2\Omega a)^2/gH$, Ω is the earth's rotation rate, a is the radius of the earth and n is the number of nodal crossings in the v–eigenvector. For this investigation H was prescribed to be 0.088 meters (or $\epsilon = 10^6$), so that for modes $n \leq 15$ the solutions begin to decay within ten degrees of the equator. Under this condition of strongly equatorially trapped waves, the spherical equations asymptotically approximate those on a β–plane for which the following dispersion relationship for gravity–inertia waves exists:

$$\sigma^2 = gHk^2 + (2n+1)\beta\sqrt{gH} \tag{16}$$

where $\beta = 2\Omega/a$.

In order to adequately resolve the equatorially trapped waves in the model, an arbitrary stretching of the θ–coordinate was incorporated. This stretching was set up so that half the grid points were located between the latitudes $+\theta_d$ and $-\theta_d$.

In Fig. A6 the non–dimensional frequency of the gravity–inertia waves is plotted as a function of the number of nodal crossing of the v–eigenvector for a theoretical β–plane (solid line), the model described above with a SG (dotted line) and a NSG (dashed line). The numerical results shown here were computed using 40 grid points in the θ–direction. The range of numerical results as a function of n reflects the limits of resolvable modes. Although only lower order modes were resolvable in this case, one can note that the behavior of the numerical solutions for the SG and NSG is similar to the analytical results presented in Fig. A2. The frequencies computed using the SG provide a much better estimate to the theoretical values than the computations made with the NSG which produces a zero group velocity near $n = 10$. In addition, the eigenvectors computed with the NSG contained a large $2\Delta x$ oscillation in which their amplitudes reverse sign at every other grid point.

The error (e) in a finite difference scheme of order p and grid spacing Δx has the following asymptotic form:

$$e = c(\Delta x)^P \tag{17}$$

where $\Delta x = L/N$, L being the length of the domain and N the number of grid points. By defining e as the difference between the differential and computed frequencies, Fig. A7 shows for the SG and NSG results the behavior of e as a function of N for the three gravest easterly gravity modes. From this figure and (17) one can note that as $\Delta x \to 0$, the error likewise approaches zero. By substituting for Δx in (17) one obtains:

$$e = \overline{c}N^{-P} \tag{18}$$

where $\overline{c} = cL^P$. Taking the log of (18) yields:

$$\log e = \log \overline{c} - p \log N. \tag{19}$$

Equation (19) represents an equation for a line with slope ($-p$) and offset ($\log \overline{c}$). Table A1 lists the values of p and Σ corresponding to the modes in Fig. A7, where Σ is defined as follows:

$$\Sigma \equiv \log \overline{c}_{NSG} - \log \overline{c}_{SG} \tag{20}$$

As one would expect for a second order finite difference scheme, Table A1 shows that $p \approx 2$. Since the slopes of lines for the SG and NSG cases are nearly identical, their distinction must lie in the difference of the offsets (Σ). Using 0.6 as the average value of Σ in Table A1, (17) can be approximated for the SG and NSG cases, respectively, as:

$$e_{SG} = c(\Delta x_{SG})^2$$

$$e_{NSG} = 10^{0.6} c(\Delta x_{NSG})^2 \tag{21}$$

or

$$e_{NSG} = 4c(\Delta x_{NSG})^2 \tag{22}$$

Thus if the grid intervals are equal in the two cases (i.e. $\Delta x_{SG} = \Delta x_{NSG}$), e_{NSG} will be approximately four times as large as e_{SG}. Viewed in a different sense, to achieve the same accuracy in both grid schemes (i.e. $e_{SG} = e_{NSG}$), the grid interval of the NSG must be about half that of the SG (i.e. $\Delta x_{NSG} = 0.5\Delta x_{SG}$). In summary, *use of a properly staggered grid is computationally efficient since it requires half the number of grid points as the non-staggered grid to achieve the same level of accuracy.*

A4. Concluding Remarks

In this appendix the one-dimensional shallow water equations were investigated to examine the effects of different grid structures on the energy dispersion by gravity-inertial waves. Both numerical and analytical results support the premise that proper simulation of energy propagation will occur only with an appropriate distribution of dependent variables. For the one-dimensional case studied here the distribution which resulted in the best simulation located the u and v variables midway between the h grid points (i.e. Scheme B in Fig. A2). In addition it was demonstrated that a properly staggered grid can greatly improve computational efficiency. To obtain the best grid structure for problems which involve finite differences in two dimensions or where non-linearities are dominant, one should refer to Arakawa and Lamb's (1977) treatment of these cases. Finally, the results presented in this appendix are used as a guide for the design of the horizontal finite difference scheme described in section 9 of this manuscript.

TABLE A1

Values of p and Σ (defined in equation 20)

for the gravity modes shown in Figure A7

Order (n) of gravity mode	Grid structure	p	Σ	10^{Σ}
0	SG	2.07	0.60	3.98
	NSG	2.09		
1	SG	1.97	0.60	3.98
	NSG	2.10		
2	SG	2.00	0.61	4.12
	NSG	2.04		

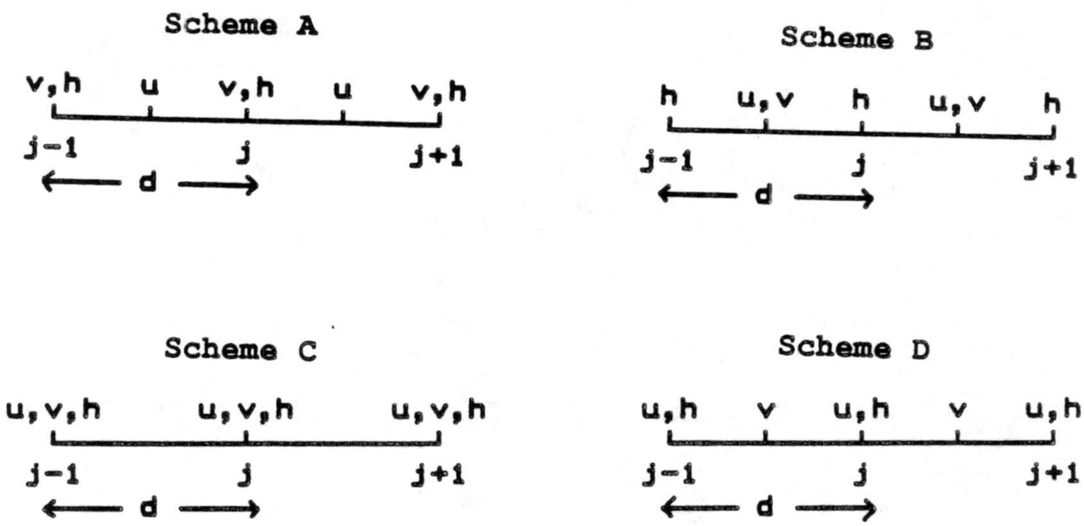
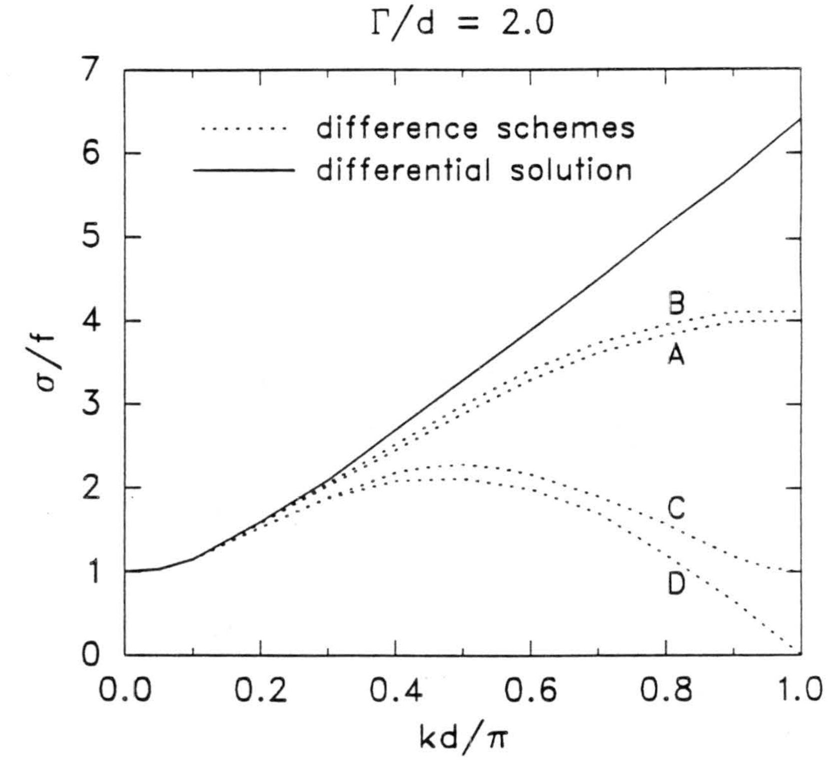


Fig. A1. Distribution of dependent variables for difference schemes A - D on a one-dimensional grid with mesh spacing d.



'ig. A2. Non-dimensional frequency (σ/f) plotted as a function of non-dimensional wavenumber (kd/π) for the shallow water equations with $\Gamma/d = 2.0$. Solid line corresponds to differential case, whereas dotted lines correspond to difference schemes A - D which are labeled accordingly.

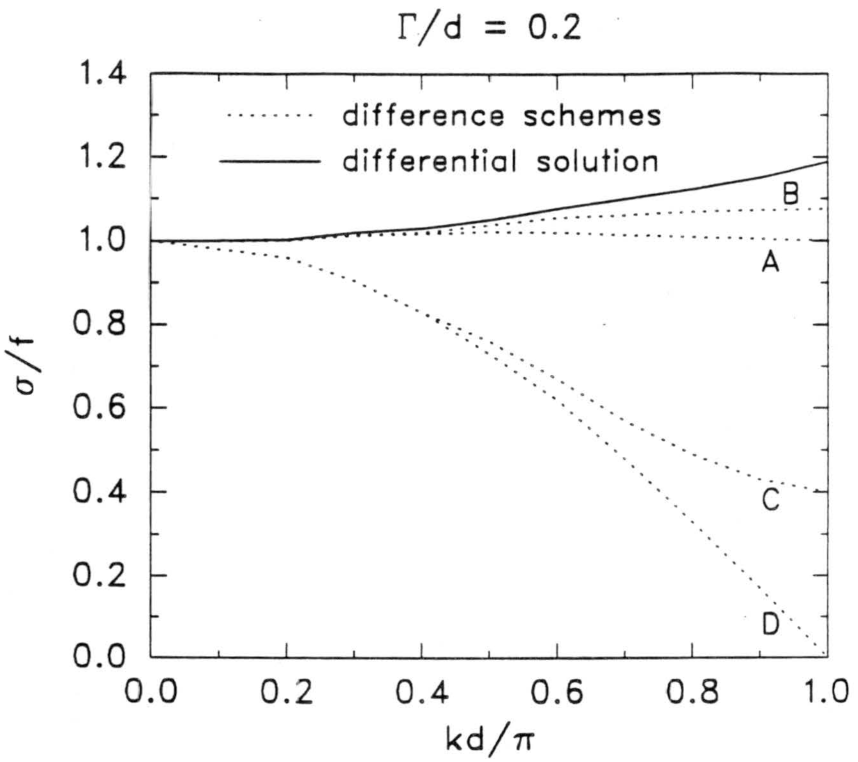


Fig. A3. Same as figure 2, except with $\Gamma/d = 0.2$.

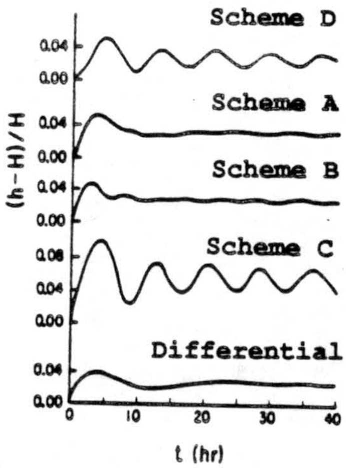


Fig. A4. Time variation of the (non-dimensional) height perturbation at $x = b$ for the initial value problem posed by Cahn (1945) with $\Gamma/d = 2.0$; comparison of differential results to those from difference schemes A - D (from Arakawa and Lamb, 1977).

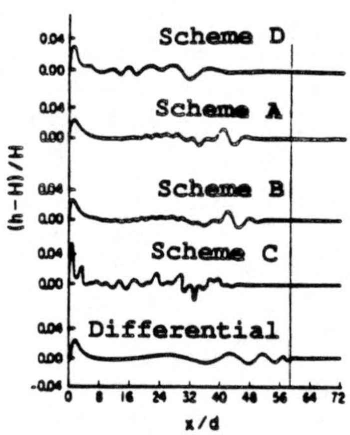
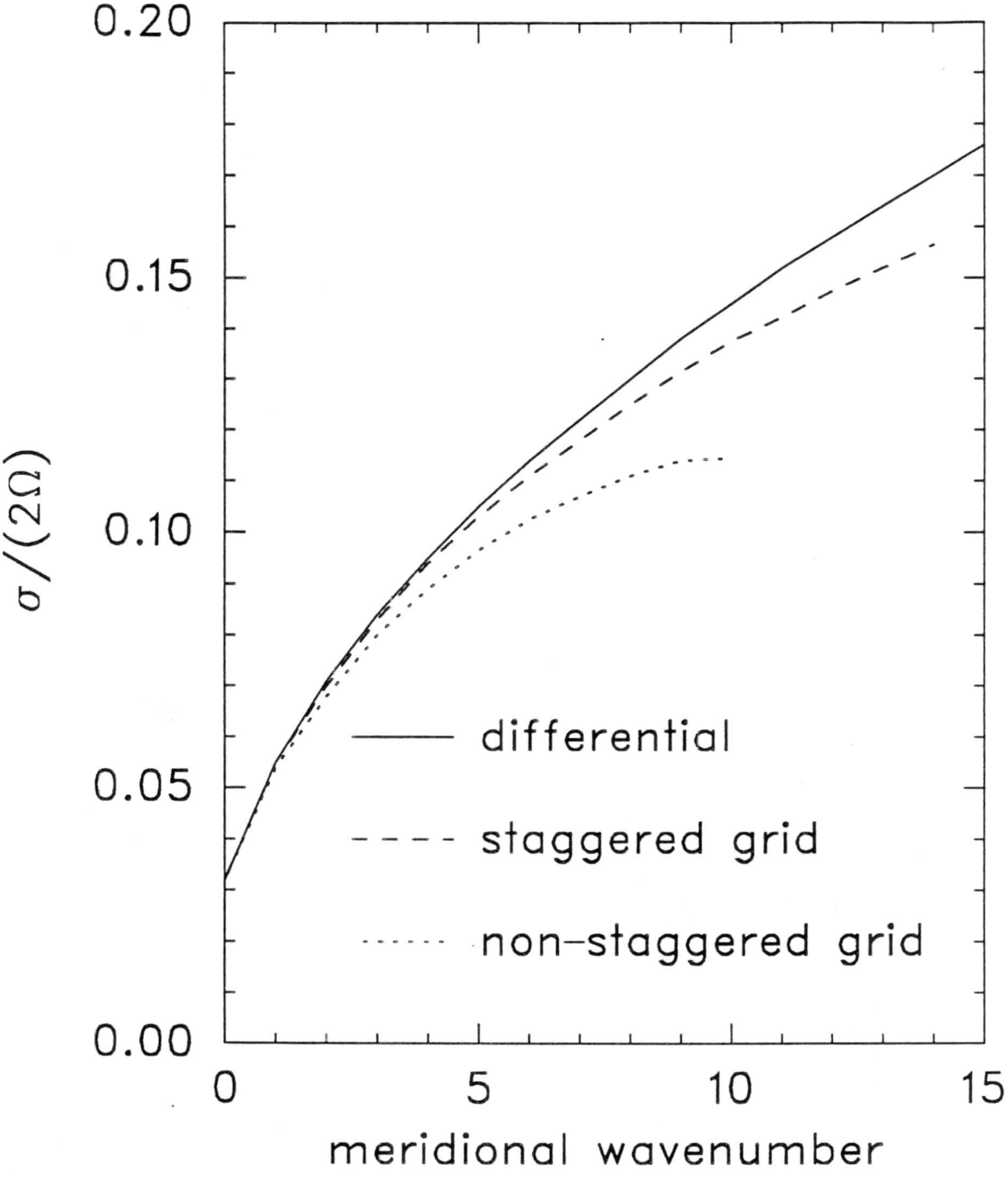


Fig. A5. The spatial variation of the (non-dimensional) height perturbation at $t = 80$ hours for the same initial value problem as in figure 4; comparison of differential results to those from difference schemes A - D. The thin vertical line at $x/d \approx 59$ indicates the theorectical limit of influence (from Arakawa and Lamb, 1977).



'ig. A6. Non-dimensional frequency of gravity-inertia waves plotted as function of meridional wavenumber (indicated by number of nodal crossings of v-eigenvector) for a theorectical β-plane (solid line), a shallow water model with a staggered grid (dashed line) and with a non-staggered grid (dotted line). Results shown here are for $s = 1$ and $h = 0.088$ meters.

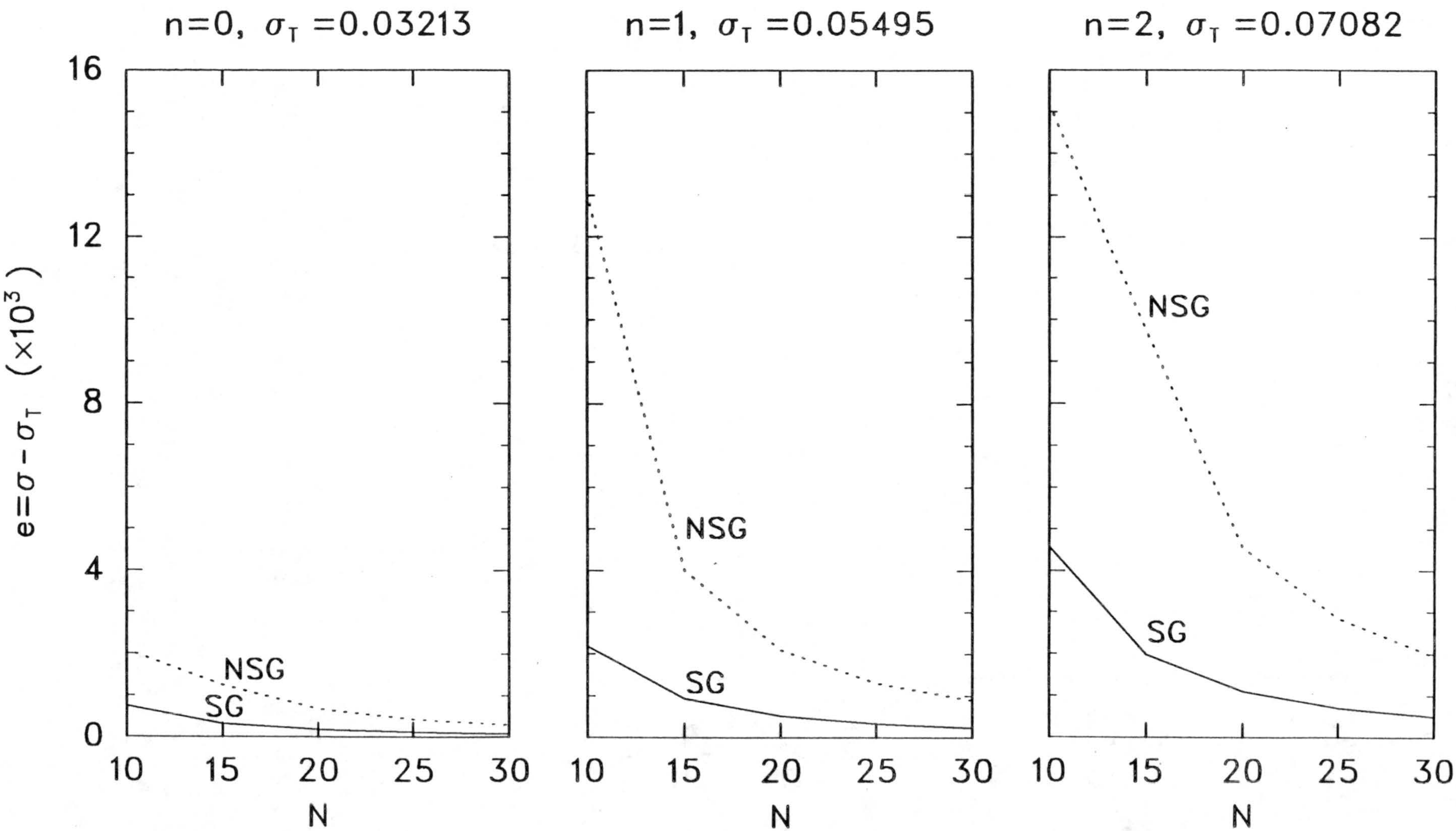


Fig. A7. The error ($e = \sigma_T - \sigma$, where σ_T and σ are respectively, the frequency computed theorectically using a β-plane and numerically using a shallow water model) as a function of the number of grid points (N) for the three gravest easterly gravity waves. Solid line is for staggered grid results and dashed line for non-staggered grid results.

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