INTEGRAL EQUATIONS FROM AN AMATEUR'S STNADPOINT

by

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TECHNICAL MEMORANDUM

TO: Files
FROM: H. J. Morel-Seytoux HJM March 8, 1965

SUBJECT: INTEGRAL EQUATIONS FROM AN AMATEUR'S STANDPOINT

ABSTRACT AND CONCLUSION

Methods of solution of linear integral equations are described. Methods are compared and ranked with respect to the ease with which they can yield exact analytical solutions in closed form. It is concluded that methods based on transform theory are most efficient. The transform method uses the transform property of the kernel operator itself, or that of an external transform operator, e.g., Laplace's transform. In the latter case the integral equation is Laplace-transformed, whereas in the former case the equation is "kernel"-transformed. The kernel-transform procedure yields immediate results whenever the n^{th}-iterated kernel operator is expressible as a linear combination of lower order iterated kernel operators. The Laplace transform method yields immediate results whenever the integral in the equation is a convolution of the kernel and the unknown function. Generalizations are possible.

INTRODUCTION

On the whole, the usual method in reservoir analysis to solve a problem can be curtly summarized in the following procedure. The problem is formulated in terms of a partial differential equation. Either this partial differential equation can be solved exactly or the equation is approximated by the method of finite difference. The solution is then obtained numerically from a computer, the problem having been reduced to the solution of a system of linear or nonlinear algebraic equations. Instead of formulating the problem in terms of differential equations, one might formulate the problem in terms of integral equations. Actually, the integral equation is more general than the differential equation. The integral equation is often still valid under conditions when the differential equation breaks down.

The use of integral equations raises several questions. First, how does one formulate a problem in terms of an integral equation? Second, when is it possible to formulate a problem in terms of an integral equation? Will this formulation be possible, whether the differential equation be of the elliptic,

parabolic, or hyperbolic type? Third, once a problem has been formulated in terms of an integral equation, how does one obtain a solution - either an exact solution or an approximate one? Fourth, if the solution is obtained exactly or approximately, is the formulation advantageous compared to the formulation of the problem in terms of a differential equation?

This particular memorandum is not concerned with the method of formulating a problem in terms of an integral equation. It is concerned with the methods of solutions once the problem has been formulated in terms of an integral equation. Even then, the integral equation considered here is not the most general integral equation. The integral equation considered is a linear one which can be put in the form of Eq. (1).

$$\delta\varphi(x) - \lambda\int_a^b K(x,s)\,\varphi(s)\,ds = f(x) \qquad (1)$$

The interval of integration in the integral is called the range of the integral equation. $\varphi(x)$ is the unknown function, $K(x,s)$ is called the kernel of the integral equation and $f(x)$ is known as the right-hand side or free term. The character of an integral equation is determined essentially by the properties of the its kernel. If the double integral, over the range, of the square of the modulus of the kernel is finite, the equation is said to be of the Fredholm type. If the kernel is of the form $\frac{H(x,s)}{|x-s|^{\alpha}}$ where α is less than 1, the integral equation is said to have a weak singularity. If the kernel is of the form $\frac{H(x,s)}{x-s}$, the integral equation is said to be singular. The integral in the equation must be understood in the Cauchy sense. In Eq. (1) if $\delta = 0$, the equation is said to be of first kind. If $\delta = 1$, the equation is said to be of the second kind. If the right-hand side of Eq. (1) is 0, the equation is said to be homogeneous.

REVIEW OF METHODS OF SOLUTIONS

Notations

In the following, we shall use an operator's notation. We shall call K an operator, defined by the equation

$$K(\,) = \int_a^b K(x,s)\,(\,)\,ds \qquad (2)$$

With that notation in the case $\delta = 1$, Eq. (1) transforms into (3)

$$\varphi - \lambda K\varphi = f \qquad (3)$$

KØ is called the kernel transform of Ø.

Successive Substitutions

It will be assumed that the single integral of the square of the modulus of f(x) is finite. Then we can write Eq. (2) as

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,s)\,\varphi(s)\,ds \qquad (4)$$

If we substitute under the integral sign the value Ø obtained from Eq. (4), we obtain

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,s) f(s)\,ds + \lambda^2 \int_a^b K(x,s) \left[\int_a^b K(s,t)\,\varphi(t)\,dt \right] ds \qquad (4)$$

We see that the kernel operator has been applied twice. The symbol for the product is the operator K^2. There are two ways of looking at this operation. We have either applied the kernel operator twice or we have applied the product of the two operators once. The kernel of the product operator is called the second iterated kernel.

$$K_2(x,s) = \int_a^b K(x,t)\,K(t,s)\,dt \tag{5}$$

The two ways are equivalent. Proceeding with the substitution n times, we obtain

$$\varphi^{(n)}(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,s) f(s)\,ds \tag{6}$$

where φ of order n is the n^{th} order approximation to the solution. This procedure will converge, provided that λ is small enough. Proceeding to the limit, Eq. (6) becomes (7)

$$\varphi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,s) f(s)\,ds \tag{7}$$

Inverting the order of summation and integration in Eq. (7), we obtain Eq. (8)

$$\varphi(x) = f(x) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,s) \right] f(s)\,ds \tag{8}$$

The expression in brackets in Eq. (8) is known as the resolvent.

$$\Gamma(x,s;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,s) \tag{9}$$

Substituting Eq. (9) into Eq. (8), we obtain Eq. (10)

$$\varphi(x) = f(x) + \lambda \int_a^b \Gamma(x,s;\lambda) f(s)\,ds \tag{10}$$

Formally speaking, if the resolvent is first calculated, the solution is obtained immediately by Eq. (10). Even though Eq. (1) was proved valid only for small values of λ, this suggests that the solution may always be of the form given in Eq. (10). Naturally for large values of λ the resolvent is yet unknown and would be given by a different expression than Eq. (9).

"The method of successive substitutions leads to a series which, as a rule, can not be summed in closed form. In practice, the method of successive substitutions can give only an approximate solution of the integral equation. As a rule in the cases when the series in Eq. (7) can be successfully summed in closed form, it turns out to be possible, with the help of some special trick or other, to solve the integral equation without having recourse to the general theory."[1]

A particular type of Fredholm's integral equation known as Volterra's equation corresponds to the case when the upper limit of the integral is the variable x.

$$\varphi(x) - \lambda \int_a^x K(x,s)\,\varphi(s)\,ds = f(x) \qquad (12)$$

The range of this integral equation can be reduced to that of the Fredholm type with the understanding that the kernel is equal to $K(x,s)$ for $s < x$ and 0 for $s > x$. If the free term in Volterra's equation is absolutely integrable then successive substitutions for this equation converge, for all values of λ. This property is also true for Volterra's equation with a weak singularity (convergence is slower).

Successive Approximations

In the previous method, the zeroth order approximation to the solution was chosen equal to the free term, but the zeroth order of approximation could have been chosen in a completely arbitrary manner. The procedure will converge.[2] This procedure may be advantageous, with an educated guess, to obtain an approximate numerical solution but from the form of solution in Eq. (10), it can be inferred that it will not be a good procedure to obtain an exact solution.

Degenerate Kernels

A degenerate kernel is a kernel which is constituted of a finite sum of products of a function of x times a function of s.

$$K(x,s) = \sum_{i=1}^{n} a_i(x)\, b_i(s) \tag{13}$$

where the functions $a_i(x)$ and $b_i(s)$ are linearly independent. This class of integral is simply solved by reduction to a system of algebraic equations. The integral equation takes the form

$$\varphi(x) - \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(s)\, \varphi(s)\, ds = f(x) \tag{14}$$

Define the n scalar product operators $B_j = \int_a^b b_j(x)\{\}\, dx$ and the scalar product $(b_j,u) = \int_a^b b_j(x)\, u(x)\, dx$. With this notation, Eq. (14) becomes (15)

$$\varphi(x) - \lambda \sum_{i=1}^{n} (b_i, \varphi)\, a_i(x) = f(x) \tag{15}$$

Apply the operator B_j once for all j's to Eq. (15).

$$B_j \varphi - \lambda \sum_{i=1}^{n} (b_i, \varphi)(B_j a_i) = B_j f \tag{16}$$

We obtain a system of n equations for n unknown scalar products $(b_j,\varnothing)$. A solution of this system of equations is possible only if the determinant, which is a function of λ , is different from 0.

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda(b_1,a_1) & -\lambda(b_1,a_2) & \cdots & -\lambda(b_1,a_n) \\ -\lambda(b_2,a_1) & 1-\lambda(b_2,a_2) & \cdots & -\lambda(b_2,a_n) \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda(b_n,a_1) & \cdots & & 1-\lambda(b_n,a_n) \end{vmatrix} \tag{17}$$

$\Delta(\lambda)$ is a polynomial in λ of order at most n. The solution for the (b_i, φ) is always possible except for possibly n values of λ. For the values of λ which make $\Delta(\lambda) = 0$, the integral equation is either insoluble or has an infinite number of solutions. This follows from the theory of equations.

Partition of the Kernel

The solution of Fredholm's equation in the general case can be reduced to the solution of an equation with a degenerate kernel. This can be achieved by many methods. For instance, the kernel can be expanded in a double Fourier cosine series.

$$K(x,s) = \sum_{i,k=0}^{\infty}{}' A_{ik} \cos\left(\frac{i\pi x}{b-a}\right)\cos\left(\frac{k\pi s}{b-a}\right) \qquad (18)$$

Let

$$P(x,s) = \sum_{i,k=0}^{n}{}' A_{ik} \cos\left(\frac{i\pi x}{b-a}\right)\cos\left(\frac{k\pi s}{b-a}\right) \qquad (19)$$

then

$$K(x,s) = P(x,s) + K'(x,s) \qquad (20)$$

Provided n is large enough and the single integral of the square of the modulus of the kernel K' is less than a constant C', the solution will be fairly accurate. Any partition of the kernel into a degenerate kernel and a small kernel is applicable. The method accordingly reduces to this. Without changing the right-hand side of the equation, the kernel is replaced by an approximate degenerate kernel. The degree of the approximation to the kernel is determined by the value of C'.

Application of Approximate Formulae of Integration

The approximation of a kernel by a degenerate one, as well as the method of substitution, requires a number of quadratures. Instead, let us replace the integrand in the equation by a finite sum of values of the function at regular values of x in the interval (the rectangle formula)

$$\int_a^b K(x,s)\varphi(s)\,ds \approx h\sum_{k=1}^{n} K(x,x_k)\varphi(x_k) \qquad h=\frac{b-a}{n} \qquad x_k = a+kh \tag{21}$$

In the approximate equation,

$$\varphi(x) - \lambda h \sum_{k=1}^{n} K(x,x_k)\varphi(x_k) = f(x) \tag{22}$$

let us replace successively, x by x_1, x_2, etc., x_n. Then we obtain a system of linear algebraic equations in $\varphi(x_1)$, etc., $\varphi(x_n)$. Solving, we obtain $\varphi(x_1)$, etc., $\varphi(x_n)$, i.e., the value of the solution for discrete values of the variable. Interpolation for the complete solution is best obtained between these points by the very integral equation formula (22) because $\varphi(x_k)$ is known

$$\varphi(x) = f(x) + \lambda h \sum_{k=1}^{n} K(x,x_k)\varphi(x_k) \tag{23}$$

More accurate results would be obtained, for example, using Simpson's Rule instead of using the rectangle formula.

Fredholm's Method of Solution

It was seen earlier that the solution did not exist or was not unique for a number of values of λ in the case of a degenerate kernel. For these values of λ, then the resolvent in Eq. (10) must not exist. The values of λ for which the resolvent of Fredholm's equations exist will be called regular and the values of λ for which the resolvent does not exist will be called eigenvalues.

Fredholm's alternative: either the inhomogeneous equation is soluble whatever its right-hand side may be, or else the corresponding homogeneous equation has a nontrivial solution. Nontrivial solutions of the homogeneous integral equation are called eigenfunctions of the kernel, or of the equation, corresponding to the given eigenvalue. Based on the idea that the method of approximate formulae of integration will yield the exact solution in the limit, Fredholm was able to obtain a solution in the form

$$\varphi(x) = f(x) + \lambda \int_a^b \frac{D(x,s;\lambda)}{D(\lambda)} f(s)\, ds \qquad (24)$$

and gave expressions for $D(x,s;\lambda)$ and $D(\lambda)$. The eigenvalues are the solution of $D(\lambda) = 0$. $D(\lambda)$ is a power series of λ, valid for all λ. $D(x,s;\lambda)$ is an integral function of λ. The entire dependence of the resolvent on x and s is in $D(x,s,\lambda)$. Expressions for $D(x,s,\lambda)$ and $D(\lambda)$ are given in Mikhlin,[1] p. 54, 55 and in Lovitt[2] p. 32, .33.

Hilbert-Schmidt Theory for Symmetric Kernels

K(x,s) was defined earlier as the kernel. The conjugate kernel, K* is the complex conjugate of K(x,s) after the variables x,s have been inverted.

$$K^* = \overline{K(s,x)} \qquad (25)$$

In the paragraph on degenerate kernels, the scalar product of two functions, f and g, was defined as

$$\int_a^b f(x)\, g(x)\, dx = (f,g) \qquad (26)$$

This definition applies for real functions. For complex functions, the scalar product of f and g is defined as

$$\int_a^b f(x)\, \overline{g(x)}\, dx = (f,g) \qquad (27)$$

It is easily shown that $(g,f) = \overline{(f,g)}$ (28)

Note that the symbol $K\varphi$ means $\int_a^b K(x,s)\,\varphi(s)\, ds$ and is different from $(K,\varphi) = \int_a^b K(x,s)\, \overline{\varphi(s)}\, ds$. The kernel is said to be symmetric if it is equal to its conjugate. The Hilbert-Schmidt method of solution of an integral equation with a symmetric kernel is based on the property that the

eigenfunctions of a symmetric equation corresponding to different eigenvalues are orthogonal. Then both the unknown function and the right-hand side are expanded in a Fourier series of the eigenfunctions $\phi_n(s)$, where $\phi_n(x)$ is the eigenfunction corresponding to the eigenvalue λ_n. They satisfy the equation

$$\lambda_n K\varphi_n = \varphi_n \tag{29}$$

$$\varphi = \Sigma a_n \varphi_n \qquad f = \Sigma' f_n \varphi_n$$

Substituting into the integral equation, we obtain

$$\Sigma' a_n \varphi_n - \lambda \Sigma' a_n K\varphi_n = \Sigma' f_n \varphi_n \tag{30}$$

Using Eq. (29), it follows that

$$a_n = \frac{\lambda_n f_n}{\lambda_n - \lambda} \tag{31}$$

Finally

$$\varphi(x) = f(x) + \lambda \sum_{n=1}^{\infty}{}' \frac{f_n \varphi_n(x)}{\lambda_n - \lambda} \tag{32}$$

Integral Equations with a Fourier Kernel

An integral operator is said to have a Fourier kernel[3] if the product of two operations is equivalent to the identity operation times a constant.

$$K^2 = C \tag{33}$$

To solve

$$\varphi - \lambda K\varphi = f \tag{34}$$

it suffices to kernel-transform Eq. (34), which gives

$$K\varphi - \lambda K^2\varphi = Kf \tag{35}$$

Using Eq. (33), we get (36)

$$K\varphi - \lambda C\varphi = Kf \tag{36}$$

Elimination of KØ between Eq. (34) and (36) yields the solution

$$\varphi = \frac{f + \lambda Kf}{1 - \lambda^2 C} \tag{37}$$

and explicitly,

$$\varphi(x) = \frac{f(x) + \lambda \int_a^b K(x,s) f(s)\, ds}{1 - \lambda^2 C} \tag{38}$$

Examples of Fourier Kernels[3]

Range	Kernel	Transform
$(0, \infty)$	$\cos(xs)$	Fourier cosine transform
$(0, \infty)$	$\sin(xs)$	Fourier sine transform
$(0, \infty)$	$\sqrt{xs}\, J_\nu(xs)$	Hankel transform
$(-\infty, +\infty)$	$\frac{1}{x-s}$	Hilbert transform
$(0, \infty)$	$\frac{1}{1-(xs)^2}$	

Note that if k(x) is a Fourier kernel function, so are k(ax) and $x^{\frac{1}{2}(b-1)} k(x^b)$. The resultant m(x)

$$m(x) = \int_0^\infty k(xs)\, l(s)\, ds \tag{39}$$

of two Fourier kernels, k(x) and $l(x)$, is a Fourier kernel. Thus

$$\int_0^\infty \cos(xs)\, \frac{1}{a^2 - s^2}\, ds = \frac{\sin(ax)}{2\pi a} \tag{40}$$

Integral Equations with a Generalized Fourier Kernel

The operator with such a kernel is such that the product of n operations can be expressed as a linear combination of lower order operations

$$K^n = a_{n-1} K^{n-1} + \cdots + a_1 K + a_0 \tag{41}$$

Say $K^3 = a_2 K^2 + a_1 K + a_0$. Then the integral equation is kernel-transformed twice

$$\varphi - \lambda K \varphi = f \tag{42}$$

$$K\varphi - \lambda K^2 \varphi = Kf \tag{43}$$

$$K^2\varphi - \lambda K^3 \varphi = K^2 f \tag{44}$$

Between the above equations, $K^3\emptyset$, $K^2\emptyset$ and $K\emptyset$ can be eliminated, with the result

$$\varphi = \frac{(1 - \lambda a_2 - \lambda^2 a_1) f + \lambda (1 - \lambda a_2) Kf + \lambda^2 K^2 f}{1 - \lambda a_2 - \lambda^2 a_1 - \lambda^3 a_0} \tag{45}$$

For a generalized Fourier kernel of order n,

$$\varphi = \frac{(1 - \lambda a_{n-1} \cdots - \lambda^{n-1} a_1) f + \cdots + \lambda^{n-2} (1 - \lambda a_{n-1}) K^{n-2} f + \lambda^{n-1} K^{n-1} f}{1 - \lambda a_{n-1} - \lambda^2 a_{n-2} - \cdots - \lambda^{n-1} a_1 - \lambda^n a_0} \tag{46}$$

Integral Equation of the Convolution Type

$$\varphi - \lambda K \varphi = f \tag{47}$$

where

$$K = \int_0^x k(x-s)\{\}\, ds \tag{48}$$

or $K = \int_0^\infty k(x-s)\{\}\,ds$ where k of negative argument is identically 0. Take the Laplace transform[4] of Eq. (47)

$$\mathcal{L}\varphi - \lambda(\mathcal{L}\varphi)(\mathcal{L}k) = \mathcal{L}f \tag{49}$$

The solution of Eq. (49) is immediate

$$\mathcal{L}\varphi = \frac{\mathcal{L}f}{1-\lambda\mathcal{L}k} \tag{50}$$

The solution is obtained by taking the inverse transform of LØ

$$\varphi = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}f}{1-\lambda\mathcal{L}k}\right\} \tag{51}$$

Is the convolution property a property of the Laplace transform only? Let us consider the more general transform

$$\Lambda = \int_0^\infty \lambda(s,t)\{\}\,dt \tag{52}$$

and define the same convolution as before

$$c(t) = \int_0^\infty f(t-\tau)\,g(\tau)\,d\tau \tag{53}$$

Then it is possible to take the transform of the convolution and as for the usual Laplace transform, we invert the order of integration

$$\Lambda c = \int_0^\infty \lambda(s,t)\left\{\int_0^\infty f(t-\tau)g(\tau)\,d\tau\right\}dt \tag{54}$$

$$\Lambda c = \int_0^\infty g(\tau)\left\{\int_0^\infty \lambda(s,t)\,f(t-\tau)\,dt\right\}d\tau \tag{55}$$

Let $t - \tau = u$ and $dt = du$. Then the transform of the convolution becomes $\int_0^\infty g(\tau)\left\{\int_0^\infty \lambda(s, u+\tau) f(u)\,du\right\} d\tau$

The transform will have the Laplace convolution property if

$$\Lambda c = \int_0^\infty g(\tau)\,\lambda(s,\tau)\,d\tau \int_0^\infty f(u)\,\lambda(s,u)\,du \qquad (56)$$

which requires that the kernel of the transform satisfies the functional relationship $\lambda(s, u+\tau) = \lambda(s,u)\,\lambda(s,\tau)$ (57)

The solution of this equation is $\lambda(s,t) = e^{\nu(s)t}$

where $\nu(s)$ can be any arbitrary function provided that the integral in the transform converges. In the case of the Laplace transform, $\nu(s) = -s$. The transform $\int_0^\infty e^{(\log s)t}\{\}\,dt = \int_0^\infty s^t\{\}\,dt$

is another such transform. However, the usefulness of such a transform is conditional to obtaining its inverse.

Integral Equations with a
Product Kernel (That is not a Fourier Kernel)

Is there a transform to give an appropriate convolution property? The convolution is of the type

$$c(t) = \int_0^\infty k(t\tau)\,\varphi(\tau)\,d\tau \qquad (58)$$

and the transform must be such that the transform of the convolution be equal to the transform of the unknown function times any function.

$$\Lambda c = \int_0^\infty \lambda(s,t)\left\{\int_0^\infty k(t\tau)\,\varphi(\tau)\,d\tau\right\} dt$$

$$= \int_0^\infty \varphi(\tau)\left\{\int_0^\infty \lambda(s,t)\,k(t\tau)\,dt\right\} d\tau \qquad (59)$$

Let $t\tau = u$ $\quad dt = \frac{du}{\tau}$

$$\Lambda c = \int_0^\infty \varphi(\tau) \left\{ \int_0^\infty \lambda(s, \frac{u}{\tau})\, k(u)\, \frac{du}{\tau} \right\} d\tau \qquad (60)$$

The transform will have the required property if its kernel satisfies the functional relationship

$$\lambda(s, \frac{u}{\tau}) = \tau\, \lambda(s,\tau)\, \mu(s,u) \qquad (61)$$

where $\mu(s,u)$ is an arbitrary function. The solution for this equation is

$$\lambda(s,t) = t^{\nu(s)-1} \qquad \mu(s,t) = t^{-\nu(s)} \qquad (62)$$

If $\nu(s) = s$, then the transform is the Mellin transform.

But any $\int_0^\infty t^{\nu(s)} \{\}\, dt$ will do.

Applying the Mellin transform to the integral equation it transforms into

$$\phi(s) - \lambda\, \phi(s) K(1-s) = F(s) \qquad (63)$$

Capital letters are used to denote the transform, The solution is easily obtained

$$\phi(s) = \frac{F(s)}{1 - \lambda K(1-s)} \qquad (64)$$

Finally, the solution of the integral equation is obtained by taking the inverse Mellin transform of Eq. (64)

$$\varphi(x) = \mathcal{M}^{-1}\left\{\frac{F(s)}{1-\lambda K(1-s)}\right\} \tag{65}$$

Integral Equation with a Ratio Kernel

An integral equation with a ratio kernel is of the form

$$\varphi(x) - \lambda\int_0^\infty k\left(\frac{x}{s}\right)\varphi(s)\,ds = f(x) \tag{66}$$

Following the procedure of the previous paragraphs, we look for a general transform that would have the appropriate convolution property. The kernel of such a transform must satisfy the functional relationship

$$\lambda(s,u\tau) = \frac{1}{\tau}\,\lambda(s,\tau)\,\mu(s,u) \tag{67}$$

where $\mu(s,u)$ is an arbitrary function. Solution of that equation is

$$\lambda = \tau^{\nu(s)} \qquad \mu = u^{\nu(s)-1} \tag{68}$$

If $\nu(s) = s-1$, then we obtain again the Mellin transform. The transformed integral equation is

$$\phi(s) - \lambda\,\phi(s)K(s+1) = F(s) \tag{69}$$

from which the solution to the integral equation is obtained by inverse transform

$$\varphi(x) = \mathcal{M}^{-1}\left\{\frac{F(s)}{1-\lambda K(s+1)}\right\} \tag{70}$$

RELATIVE MERITS OF THE METHODS

The relative merits of the various methods reviewed in this memorandum can be deduced from the form of the resolvent as obtained by these methods.

Generalities

Integral equation:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,s)\,\varphi(s)\,ds \qquad (71)$$

Solution:

$$\varphi(x) = f(x) + \lambda \int_a^b \Gamma(x,s;\lambda) f(s)\,ds \qquad (72)$$

The analogy between Eq. (71) and (72) is self-evident. That Eq. (72) is the general form of the solution was proved by Fredholm. It must be noted that the operator Γ may not always be an integral operator. Thus Eq. (72) may be rewritten symbolically and more generally.

$$\varphi = f + \lambda \Gamma f \qquad (73)$$

Successive Substitutions

In this instance

$$\Gamma = \sum_{m=1}^{\infty} \lambda^{m-1} K^m \qquad (74)$$

Thus Γ is an infinite sum of integral transform operators K^m

$$K^m = \int_a^b K_m(x,s)\{\}\,ds$$

Note that symbolically it follows from Eq. (71) and (73) that

$$\varphi = \frac{1}{1-\lambda K} f = (1+\lambda\Gamma) f$$

Expanding $\frac{1}{1-\lambda K}$ in power series (for small λ K)

it follows that: $1 + \lambda K + \lambda^2 K^2 + \cdots + \lambda^m K^m + \cdots = 1 + \lambda \Gamma$

or $$\Gamma = \sum_{m=1}^{\infty} \lambda^{m-1} K^m$$

Degenerate Kernels

$$\Gamma = \sum_{i=1}^{n} \frac{a_i(x) \Delta_i}{\Delta(\lambda)} \qquad (75)$$

where $C_i = \frac{\Delta_i f}{\Delta(\lambda)}$ is the solution of the system

$$c_j - \lambda \sum_{i=1}^{n} c_i (b_j, a_i) = (b_j, f) \quad j = 1, 2 \ldots n$$

The operator Δ_i is a finite sum of scalar product operators as is seen from the solution for the C_j by Kramer's rule.

Approximate Formulae of Integration

$$\Gamma = h \sum_{k=1}^{n} K(x, x_k) \frac{\Delta_k}{\Delta(\lambda)} \qquad (76)$$

where $C_k = \frac{\Delta_k f}{\Delta(\lambda)}$ is the solution of the system:

$$c_k - \lambda h \sum_{i=1}^{n} K(x_i, x_k) c_i = f(x_k)$$

The operator Δ_k is a finite sum of "discretization"[5] operators because the C_k are linear forms of the values of $f(x)$ at discrete points $x_1\ x_2 \ldots x_k \ldots x_n$. No quadratures are required to obtain the solution.

Fredholm's Resolvent

$$\Gamma = \int_a^b \frac{D(x, s; \lambda)}{D(\lambda)} \{\}\, ds \qquad (77)$$

This resolvent is obtained as the limit of Eq. (76) as $h \longrightarrow 0$ and $n \longrightarrow \infty$. The limit of an infinite sum of discretization operators being an integral operator.

Hilbert-Schmidt Resolvent

$$\Gamma = \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{\lambda_n - \lambda} \int_a^b \varphi_n(s) \{\} ds \qquad (78)$$

This resolvent is more explicit than Fredholm's but eigenvalues and eigenfunctions must be known to apply Eq. (78). In Eq. (77) it is not necessary to calculate the roots of $D(\lambda)$ nor the eigenfunctions.

Fourier Kernels

$$\Gamma = \frac{\lambda C + K}{1 - \lambda^2 C} \qquad (79)$$

Note the equivalence of Eq. (79) and (77). Because the solution is unique $D(\lambda) = 1 - \lambda^2 C$ and

$$\lambda C + K = \lambda C \{\} + \int_a^b K(x,s) \{\} ds = \int_a^b D(x,s;\lambda) \{\} ds$$

Obviously $D(x,s;\lambda) = K(x,s) + \lambda C\, R(x,s)$ where $R(x,s)$ is a reproducing kernel, i.e., a kernel such that

$$\int_a^b R(x,s) f(s) ds = f(x) \quad \text{for arbitrary } f(x)$$

(Example of such operator is limit $\lim_{\delta \to 0} \int_{-\infty}^{+\infty} \frac{\delta}{\delta^2 + (x-s)^2} \{\} ds$

Though the same resolvents should be obtained by the Fourier kernel-method or by Fredholm's method, the resolvent obtained by the latter will look formally more complicated.

The method of successive substitutions would have given a resolvent equivalent to (79) but valid for small λ only. It would be the power series expansion of (79) for small λ

$$\Gamma = \lambda C (1 + \lambda^2 C + \lambda^4 C^2 + \cdots) + (1 + \lambda^2 C + \lambda^4 C^2 + \cdots) K \qquad (80)$$

Proceeding in reverse order, it would have taken more steps to realize that the power series representation of Γ obtained by the method of successive substitutions would converge to the closed form (79).

Generalized Fourier Kernels

$$\Gamma = \frac{\lambda^{n-1} a_0 + (1 - \lambda a_{n-1} + \cdots - \lambda^{n-2} a_2) K + \cdots + \lambda^{n-3}(1 - \lambda a_{n-1}) K^{n-2} + \lambda^{n-2} K^{n-1}}{1 - \lambda a_{n-1} - \lambda^2 a_{n-2} - \cdots - \lambda^{n-1} a_1 - \lambda^n a_0} \quad (81)$$

What was said about the Fourier kernel-method is applicable. The more complicated Eq. (81) is, the more involved would be the application of Fredholm's method and the more difficult it would be to realize that the power series expansion of Γ for small λ converges to Eq. (81).

External Transform with Proper Convolution Property

(e.g., Laplace transform)

$$\Gamma = \left[L^{-1} \left\{ \frac{L}{1 - \lambda L k} \right\} - 1 \right] \quad (82)$$

The use of the generalized Fourier-kernel method is possible (exactly) only if the K, K^2 .. K^{n-1} transforms can be obtained explicitly. This restriction applies to Fredholm's method and to the method of successive substitutions. The 1st, 2nd ... n^{th} order transform may still be linearly dependent, or may not. It is under these circumstances that the external transform method will be most useful.

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