DISSERTATION

THE INTEGRAL STRUCTURE OF HECKE ALGEBRAS FOR FINITE GENERALIZED POLYGONS

Submitted by

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In partial fulfillment of the requirements

for the degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Fall 2005

COLORADO STATE UNIVERSITY

August 22, 2005

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY ILHAN HACIOGLU ENTITLED "THE INTEGRAL STRUCTURE OF HECKE ALGEBRAS FOR FINITE GENERALIZED POLYGONS" BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

THE INTEGRAL STRUCTURE OF HECKE ALGEBRAS FOR FINITE GENERALIZED POLYGONS

Suppose (P, B, F) are the points, blocks and flags of generalized m-gon and $H(F)$ the associated rank 2 Iwahori-Hecke algebra. $H(F)$ acts naturally on the integral standard module ZF based on F. This work gives arithmetic conditions on subring R, where R contains the integers and is contained in the rationals, that insure the associated R-ary Iwahori-Hecke algebra is completely reducible on RF. The constituent multiplicities are related to the R-normal form of the incidence matrix of (P, B, F).

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ACKNOWLEDGEMENTS

I am very thankful to my adviser, Dr. R. A. Liebler, for all his help and patience.

I would like to thank all the other committee members for their useful comments on my thesis.

Teaching assistantship from the Department of Mathematics helped me to get an excellent teaching experience and financial support.

DEDICATION

To My Parents

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1 Introduction

Tits in [4] proved powerful generalization of fundamental theorem of geometry. However this theorem leaves open classification of B_3 geometries. It is built on the Feit-Higman theorem which relies ultimately on multiplicity conditions. Ott and Liebler used similar methods to B_3 geometries (see chapter 14 of [1]), but there are cases where many parameter families survive.

In fact multiplicity conditions can be refined by restricting coefficients from a field to a subring since non-isomorphic R-modules over an R-order may well be isomorphic over the associated F-algebra where F is the quotient field of R.

This thesis uses the rank 2 Hecke algebra and its action on the standard module (Chapter 2 page 7) to refine Feit-Higman's multiplicity conditions for a subring R where $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$.

Chapter 2 contains the main result which tells us the conditions necessary for the existence of the direct sum decomposition of the integral standard module whenever the coefficient field $\mathbb{C}$ is changed to a subring R. Moreover the inequivalent irreducible R-forms that arise are given. We set up the connection between multiplicity of an R-form appearing in the integral standard module and the multiplicity of a parameter appearing in the R-normal form of the incidence matrix M. The proof of the main theorem is spread over the remaining chapters.

Chapter 3 contains the detailed computations of the central primitive idempotents. They are used to identify arithmetic conditions (related to R) necessary for the existence of the direct sum decomposition of the integral standard module over R.

In order to understand the structure of the standard module better, a second module is introduced (point block module). The multiplicities of the irreducible

R-forms constituents of the standard module are obtained from the point block module by means of a combinatorial homomorphism.

2 Definitions and Preliminaries

An *incidence structure* is a triple (P, B, F) consisting of a set P of *points*, a set B of *blocks*, and a subset $F \subseteq P \times B$ called *flags* . Say a point p is *on* or *incident with* a block b (sometimes written pIb), if (p, b) is a flag.

We assume throughout that P and B are finite sets, and that there are $s+1$ points on every block and $t+1$ blocks incident to every point. The integers s, t are called the *parameters* of (P, B, F).

An *incidence matrix* M has rows indexed by points and columns indexed by blocks. The (p, b) entry of M is a 1 or 0 according as $(p, b) \in F$ or not.

For $a, b \in P \cup B$, a *chain of length* h from a to b is a sequence $a_0 = a, a_1, ..., a_h = b$ of elements from $P \cup B$ such that a_i and a_{i+1} are incident, for $0 \leq i \leq h-1$.

The least integer h (if one exists) for which there is a chain of length h from a to b is the *distance* $\rho(a, b)$ from a to b. The *diameter* of (P, B, F) is the maximum distance between elements of $P \cup B$ and if $\rho(a, b)$ equals the diameter we say a and b are *opposite*. Note that any element in $P \cup B$ can have more than one opposite.

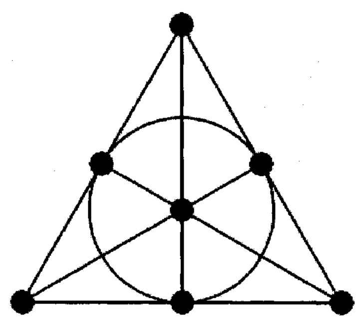
The incidence structure (P, B, F) is a *generalized m-gon* if it has diameter m, each element of $P \cup B$ has opposite, and whenever $\rho(a, b) < m$ there is a unique shortest chain from a to b. A *generalized polygon* is a generalized m-gon for some m. Moreover (P, B, F) is *thick* if the parameters s, t are both greater than 1.

For a generalized m-gon with parameters s, t, the number of flags is a polynomial function of s, t:

$$\begin{cases} g_3(s,t) = (1+s)(1+s+\ldots+s^{m-1}), & for\ m = odd \\ g_m(s,t) := |F| = (1+s)(1+st+\ldots+(st)^{\frac{m-2}{2}})(1+t), & for\ m = even \end{cases}$$

This polynomial is called the *Poincare polynomial.*

A square is a thin generalized 4-gon with parameters $(s,t) = (1,1)$. The smallest thick generalized m-gon is the *Fano plane* also known as $PG_2(2)$. It has 7 points (black dots), 7 blocks (6 lines and one circle), 21 flags and parameters $(2,2)$.



Fano plane

Note that interchanging points with blocks leads to the same incidence structure. Since the incidence structure is self-dual the generalized m-gon axioms need only be discussed for chains that start with a point.

Start with any point a. The elements at distance 1 from a are the 3 lines on a. The elements at distance 2 from a are the 6 other points on these lines. The elements at distance 3 from a are the 4 lines that do not contain a. Since we covered all the points and lines the incidence structure has diameter 3.

Each element has four opposites, for example the opposites of any point in the figure above are the 4 blocks that are left after removing all the blocks that contain the point.

The last axiom of a generalized 3-gon requires that if $\rho(a, b) < 3$, then there is a unique shortest chain of from a to b where $a \in P$. The uniqueness of the chains of length 2 follows from the fact that 2 poins uniquely determine a line.

Let R be a integral domain with field of quotients K of characteristic 0.

A *projective* R-module is a direct summand of a free module. The *torsion submodule* of an R-module M is defined by

$t(M) = \{n \in M : rn = 0 \text{ for some nonzero } r \in R\}$.

Call M R-torsionfree if this submodule is 0. By choice of R, every projective R-module is torsionfree.

An *R-lattice* is a finitely generated R-torsionfree R-module.

Definition 1 *An R-order is a ring Λ whose center contains R, and such that the additive structure of Λ is an R-lattice.*

Definition 2 *Let Λ be an R-order in a K-algebra A. A Λ-lattice is a (left) Λ-module which is also an R-lattice, that is a finitely generated and projective as R-module.*

Let M be a Λ-lattice, where Λ is an R-order. If M is R-free with a finite R-basis $\{m_1, \ldots, m_d\}$, then, relative to this basis, M affords a matrix representation $\mathbf{M}$ of Λ. For $x \in \Lambda$, let

$$xm_j = \sum_{i=1}^{d} \alpha_{ij} m_i, \ \alpha_{ij} \in R, \ 1 \le j \le d, \tag{1}$$

and put $\mathbf{M}(x) = (\alpha_{ij}) \in M_d(R)$. The map $x \to \mathbf{M}(x)$, $x \in \Lambda$, is then a representation of Λ by means of matrices with entries in R. We call $\mathbf{M}$ an *integral representation, R-representation* or *R-form* of Λ.

In order to describe how a change of R-basis of M affects the matrix representation of $\mathbf{M}$, it is convenient to write the R-basis $\{m_i\}$ as a formal row vector $\mathbf{m} = (m_1, \ldots, m_d)$. (Warning: The entries of $\mathbf{m}$ and $\mathbf{m}'$ are elements of the Λ-lattice M, not elements of R.) Then (1) may be written as

$$x\mathbf{m} = \mathbf{m}\mathbf{M}(x), \ x \in \Lambda.$$

Let $\{m_1', \ldots, m_d'\}$ be another free R-basis for M, and put $\mathbf{m}' = (m_1', \ldots, m_d')$. We may write $\mathbf{m}' = \mathbf{m}\mathbf{P}$ for some matrix $\mathbf{P} \in GL_d(R)$. For $x \in \Lambda$, we have

$$x\mathbf{m}' = x\mathbf{m}\mathbf{P} = \mathbf{m}\mathbf{M}(x)\mathbf{P} = \mathbf{m}'\mathbf{P}^{-1}\mathbf{M}(x)\mathbf{P}.$$

Thus, change of R-basis of M has the effect of replacing $\mathbf{M}$ by the *R-equivalent* representation $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$.

Let M be an R-lattice, and let L be submodule of M. L is an *R-pure* submodule of M if M/L is R-torsionfree.

The importance of purity is that there is a bijective and inclusion preserving correspondence between pure sublattices of M and the subspaces over $\mathbb{Q}$.

Example 1 *(Inequivalent R-forms) Let $G = Z_2 = \langle g \mid g^2 = 1 \rangle$ and $\varphi, \vartheta, \gamma : G \longrightarrow M_{2\times 2}(\mathbb{Z})$ be the homomorphisms such that*

$$\varphi(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vartheta(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma(g) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

then there exists $P \in M_{2\times 2}(\mathbb{Z})$ such that $P^{-1}\vartheta(g)P = \varphi(g), \forall g \in G$ however $P^{-1} \notin M_{2\times 2}(\mathbb{Z})$ because $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $det(P) = 2$. The eigenspaces of ϑ are spanned by the column vectors of P. Each of these eigenspaces is pure and they form a $\mathbb{Z}$-lattice but their span is not a pure lattice. Therefore φ and ϑ are not $\mathbb{Z}$-equivalent representations. Indeed they are $\mathbb{Q}$-equivalent representations *as $P^{-1} \in M_{2\times 2}(\mathbb{Q})$. Here φ and ϑ are the inequivalent $\mathbb{Z}$-forms of the $\mathbb{Q}$-matrix representation.*

On the other hand if we consider φ and γ we have $P = \vartheta(g)$ and $P^{-1}\gamma(g)P = \varphi(g)$ where $P, P^{-1} \in M_{2\times 2}(\mathbb{Z})$. So φ and γ are $\mathbb{Z}$-equivalent representation and they are the $\mathbb{Z}$-forms of the integral matrix representations.

Suppose (P, B, F) are the points, blocks and flags of generalized m-gon. The *integral standard module* $\mathbb{Z}F$ for (P, B, F) is the free $\mathbb{Z}$-module, $\mathbb{Z}F$ with distinguished basis (labeled by) F.

Informally we identify $\mathbb{Z}F$ with the set of all v by b matrices $\mathbf{X}$ with entries in R that satisfy $\mathbf{X} \circ M = \mathbf{X}$ where M is the incidence matrix and $\circ$ is the Hadamard product (defined by $(\mathbf{X} \circ M)_{ij} = x_{ij}m_{ij}$ where $x \in \mathbf{X}$ and $m \in M$). This is certainly a free R-module of rank equal to the number of 1's in M, i.e. the number of flags.

In this matrix model the points and blocks correspond to the rows and columns of the matrices and a single flag $f = (p, b)$ to the elementary matrix $E_{p,b}$ having a 1 in row p and column b and zeros elsewhere.

Let $f = (p, b) \in F$. Define

$$\Sigma_f = \{(q, b) |\ q \neq p\} \quad and \quad \mathsf{T}_f = \{(p, c) |\ c \neq b\}.$$

A flag g in Σ_f corresponds to an elementary matrix having 1 in the same column but a different row than that corresponding to f. Also a flag g in T_f corresponds to an elementary matrix having 1 in the same row but a different column than that corresponding to f.

The *integral Hecke algebra* $\mathcal{H}_{\mathbb{Z}}(F)$ (or simply $\mathcal{H}(F)$) is the $\mathbb{Z}$-order generated by $\sigma, \tau \in Hom_{\mathbb{Z}}(\mathbb{Z}F, \mathbb{Z}F)$ where

$$\sigma(p,b) := \sum_{g \in \Sigma_f} g = \sum_{p \neq q I b} (q,b), \qquad \tau(p,b) := \sum_{g \in \mathrm{T}_f} g = \sum_{p I c \neq b} (p,c).$$

Notice that

$$\tau\sigma(p,b) = \sum_{g \in \mathrm{T}\Sigma_f} g, \qquad \tau\sigma(p,b) = \sum_{g \in \Sigma\mathrm{T}_f} g \tag{2}$$

where

$$\mathrm{T}\Sigma_f = \{(q,c) |\ q \neq p, c \neq b \ and \ (q,b) \ is \ a \ flag\}$$

$$\Sigma\mathrm{T}_f = \{(q,c) |\ q \neq p, c \neq b \ and \ (p,c) \ is \ a \ flag\}$$

because, for example, elements of $\mathrm{T}\Sigma_f$ are obtained by a walk in M of length 2 which consists of vertical movement (starting from position (p,b) and ending at (p_1,b)) and then horizontal movement (starting from position (p_1,b) and ending at (p_1,b_1)).

We consider the following two types of chain:

Let $p_i \in P$ and $b_j \in B$. Then $p_1 = p, b_1, p_2, b_2 ..., p_m, b_m = b$ is an alternating sequence with elements points and blocks where every point (block) is on the blocks (points) that are located in the both sides of the point (block) in the sequence.

By grouping the pairs $(p_1,b_1), (b_1,p_2), (p_2,b_2), \ldots, (p_m,b_m)$ we obtain the second type of chain. This chain consists of sequences of flags that share a point or block.

We consider the second type of chains for the rest of this chapter.

Proposition 1 *Let $f = (p, b)$ be a flag of a generalized m-gon (P, B, F). Then*

$$\begin{cases} (1 + \sigma + \tau + \sigma\tau + \tau\sigma + \ldots + (\sigma\tau)^{\frac{m-1}{2}}\sigma)f = \sum_{g \in F} g, & if\ m\ is\ odd \\ (1 + \sigma + \tau + \sigma\tau + \tau\sigma + \ldots + (\sigma\tau)^{\frac{m}{2}})f = \sum_{g \in F} g, & if\ m\ is\ even \end{cases}$$

Proof: Let $f = (p, b)$ be a flag. Suppose $\alpha \in \{f, \mathsf{T}_f, \mathsf{T}\Sigma_f, \ldots\}$ and $\beta \in \{f, \Sigma_f, \Sigma\mathsf{T}_f, \ldots\}$ have been defined. Then recursively define

$$\Sigma\alpha := \{(r, d) |\ (r_1, d) \in \alpha, r_1 \neq r\} \text{ and } \mathsf{T}\beta := \{(r, d) |\ (r, d_1) \in \beta, d \neq d_1\}.$$

The generalized m-gon axioms require that the chains of length $\leq m - 1$ are uniquely determined by their terminal elements. In addition a chain of length m is uniquely determined by its penultimate element. Therefore the sets

$$\begin{cases} f, \Sigma_f, \mathsf{T}_f, \mathsf{T}\Sigma_f, \Sigma\mathsf{T}_f, (\Sigma\mathsf{T})^{\frac{m-1}{2}}\Sigma_f, & if\ m\ is\ odd \\ f, \Sigma_f, \mathsf{T}_f, \mathsf{T}\Sigma_f, \Sigma\mathsf{T}_f, \ldots, (\Sigma\mathsf{T})_f^{\frac{m}{2}}, & if\ m\ is\ even \end{cases}$$

are disjoint. In fact the indicated sets form a partition of all flags because the generalized m-gon has diameter m. The proposition follows from the obvious generalization of equation (2).

Note that

$$(\sigma + 1)(\sigma - s) = 0 = (\tau + 1)(\tau - t) \tag{3}$$

where s, t are the parameters of the generalized m-gon, because the relation on flags given by "share a block", respectively "share a point" is an equivalence relation.

Theorem 1 *Let $\mathcal{H}(F)$ be the integral Hecke algebra arising from the generalized m-gon (P, B, F). Then the generators σ and τ satisfy:*

$$(\sigma\tau)^k = (\tau\sigma)^k \quad if \quad m = 2k \quad is \quad even$$

$$(\sigma\tau)^k\sigma = (\tau\sigma)^k\tau \quad if \quad m = 2k + 1 \quad is \quad odd$$

and no further relations other than (3).

Proof: The case $m =$ *is odd* is very similar to the case $m = 2k$ *is even*. We discuss only the case m is even. An expression that is a linear combination of products of σ and τ (called terms) that equals 0 is called a relation. By equation (3) there is no loss of assuming that, for $s > 1$, neither σ^s nor τ^s appears in any term of relations discussed. Call the terms in such an expressions reduced terms. Define the length of a relation to be the number of times the generators appear in its longest (reduced) term.

To prove the theorem we need to show the following:

1. $(\sigma\tau)^k = (\tau\sigma)^k$ is the only relation of length m

2. We do not have any shorter relations.

3. We do not have any longer relations.

Proof of 1: Since the incidence structure of the generalized m-gon is self-dual we can interchange the generators σ and τ. Then we have the equality

$$1 + \sigma + \tau + \sigma\tau + \tau\sigma +, \ldots, +(\sigma\tau)^{\frac{m}{2}} = 1 + \tau + \sigma + \tau\sigma + \sigma\tau +, \ldots, +(\tau\sigma)^{\frac{m}{2}}.$$

Since each of these corresponds to a partition of F, it follows that $(\Sigma\mathrm{T})^k = (\mathrm{T}\Sigma)^k$ and so $(\sigma\tau)^k = (\tau\sigma)^k$. By proposition 1, no reduced term of length m can be expressed as a linear combination of shorter terms. This implies the uniqueness of the relation of length m.

Proof of 2: Distinct reduced terms of length $\leq m-1$ correspond to disjoint sets of flags in the proof of proposition 1. Therefore there are no shorter relations of length $\leq m - 1$.

Proof of 3: Suppose there is a relation containing a reduced term of length $> m$. Use the relation in part 1 to transform it to a non-reduced term. This contradiction completes the proof.

Theorem 2 *Let $\mathcal{H}$ $(= \mathcal{H}(F) \otimes_{\mathbb{Z}} \mathbb{C})$ be the complex Hecke algebra defined by the relations*

$$(\sigma\tau)^k = (\tau\sigma)^k \quad \text{if} \quad m = 2k \in \mathbb{Z} \quad \text{is} \quad \text{even.}$$

$$(\sigma\tau)^k\sigma = (\tau\sigma)^k\tau \quad \text{if} \quad m = 2k+1 \in \mathbb{Z} \quad \text{is} \quad \text{odd.}$$

and

$$(\sigma+1)(\sigma-s) = 0 = (\tau+1)(\tau-t) \text{ where } s, t \in \mathbb{Z}.$$

Then $\mathcal{H}$ is isomorphic to the complex group algebra of the dihedral group of order $2m$. It has irreducible representations of dimension 1 or 2 only.

Its one dimensional representations are

Name	σ	τ
index	s	t
Steinberg	-1	-1
first one dimensional	s	-1
second one dimensional	-1	t

when m is even. If m is odd, the only one dimensional representations are index and Steinberg.

For $1 \leq j \leq m/2$, $j \in \mathbb{Z}$ take $\{c_j\}, \{d_j\} \in \mathbb{C}$ such that $c_j d_j = s + t + 2\sqrt{st}\cos 2\pi j/m$. Then $\{T_j \mid j \in \mathbb{Z},\ 1 \leq j \leq m/2\}$ are a full set of inequivalent irreducible two-dimensional representation of $\mathcal{H}$ where

$$T_j(\sigma) = \begin{pmatrix} -1 & 0 \\ c_j & s \end{pmatrix}, \quad T_j(\tau) = \begin{pmatrix} t & d_j \\ 0 & -1 \end{pmatrix}.$$

Proof:(Curtis and Reiner [[2] p.620])

The fact that the constituent multiplicities of $\mathcal{H}_{\mathbb{C}}(F)$ acting on the standard module $\mathbb{C}F$ must be integers forms the basis for the Kilmoyer-Solomon proof of the Feit-Higman theorem [9].

Theorem 3 (Feit-Higman) *Finite thick generalized m-gons can exist only if $m = 3, 4, 6$ or 8.*

As already mentioned there are two proofs of the Feit-Higman theorem. The original proof considers an adjacency matrix A, whose rows and columns are indexed by the points and whose entry at (p,q) is the number of lines incident both with p and q. As we have shown, axioms for a generalized m-gon are sufficiently strong to describe all of the chains of given length r from point p to a point q. This provides information about the powers A^r. The minimal polynomial of A and the traces of the powers A^r are determined. Finally the eigenvalue multiplicities of A are calculated as rational functions of the parameters s, t and the fact that these must be integers leads to the theorem. In this proof there is only one matrix A.

A second proof due to Kilmoyer-Solomon [5] , introduces the Hecke algebra $\mathcal{H}(F)$ described in Theorem 1. Their proof uses the complex character theory of $\mathcal{H}(F)$ and the irreducible character multiplicities leads to diophantine conditions equivalent to those of Feit and Higman.

The relationship between these two proofs is exploited in chapter 4 to relate the irreducible R-forms of Hecke algebra to the arithmetic invariants of the incidence matrix.

This relationship is hinted by the following identity, Liebler [7]

$$|F| = |P| + |B| + m_{st} - m_{ind} \tag{4}$$

where m_{st} and $m_{ind} = 1$ are respectively the multiplicities of the Steinberg and index representation in the complex standard module.

One of the other major ideas in this thesis is to follow through the complex theory of the Hecke algebra $\mathcal{H}_{\mathbb{C}}(F)$ of a generalized m-gon and obtain the central primitive idempotents (See Chapter 3 for definitions). Then change the field to a ring R. By observing exactly what denominators are required in a ring R where $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ so that $\mathcal{H}_R(F)$ contains the idempotents, the first part of our main theorem is obtained.

Theorem 4 (Main Theorem) *Suppose $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$. Consider the R-standard module associated with a thick generalized m-gon (P, B, F) having parameters s, t and Poincare polynomial $g_m(s,t)$. Suppose the expressions $g_m(s,t)$; $(s+t)$ if $m=4$; (s^2+st+t^2) if $m=6$; $(s+t)(s^2+t^2)$ if $m=8$ are units in R. Then the R-ary Hecke algebra $\mathcal{H}_R(F)$ acting on the R-ary standard module $RF := R \otimes_{\mathbb{Z}} \mathbb{Z}F$ has the following structure:*

i. RF is a direct sum of $\mathcal{H}_R(F)$-submodules $RF = \sum \oplus M_i$ where $\mathbb{C} \otimes M_i$ is an $\mathcal{H}_{\mathbb{C}}(F)$ direct sum of isomorphic irreducible summands of $\mathbb{C}F$.

ii. The $\mathcal{H}_R(F)$-modules M_i are completely reducible and the one dimensional irreducible R-forms that arise are also $\mathbb{C}$-forms. However some two dimensional $\mathbb{C}$-form corresponds to several inequivalent R-forms. Each two dimensional R-form is equivalent to one of the form:

$$\sigma = \begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}, \quad \tau = \begin{pmatrix} t & b \\ 0 & -1 \end{pmatrix}$$

where $a, b \in R$ and $ab = s = t$ if $m = 3$; $ab = s+t$ if $m = 4$; $ab = s \pm \sqrt{st} + t$ if $m = 6$; and $ab = s+t, s \pm \sqrt{2st} + t$ if $m = 8$.

iii. The multiplicity of an R-form appearing in ii. equals the multiplicity of the parameter a in the R-normal form of the generalized m-gon's point-block incidence matrix M.

The proof of the theorem will be given at the end of chapter 4.

Example 2 *Consider the unique projective plane of order 3. The structure is a generalized 3-gon with parameters* $(s,t) = (3,3)$. *By [[3] p.180] the Smith normal form of the incidence matrix is a diagonal matrix with diagonal entries* $1,1,1,1,1,1,3,3,3,3,3,3,12$. *The standard module over* $\mathbb{Z}$ *has rank*

$$|F| = (1+s)(1+s+s^2) = 4.13 = 52.$$

If $R = \mathbb{Z}_{(3)}$ *is the 3-adic integers, then* $\mathcal{H}_R(F)$ *is an R-order and the R-standard module RF has the structure:*

$$RF \simeq 1 \cdot index + 27 \cdot steinberg + 6 \cdot T_1 + 6 \cdot T_2$$

where, as in Theorem 2,

Name	σ	τ
index	s	t
steinberg	-1	-1
T_1	$\begin{pmatrix} -1 & 0 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 3 \\ 0 & -1 \end{pmatrix}$
T_2	$\begin{pmatrix} -1 & 0 \\ 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}$

Note that T_1 *and* T_2 *are* $\mathbb{Q}$ *equivalent since*

$$P^{-1}T_1(\sigma)P = T_2(\sigma) \text{ and } P^{-1}T_1(\tau)P = T_2(\tau)$$

where P *is a diagonal matrix with entries 1and 1/3.*

Part (iii) of the main theorem tells us that the multiplicity of Φ_1 *and* Φ_2 *equals the multiplicity of the parameter* $a = 1$ *and* $a = 3$ *respectively in the Smith normal form of the generalized 3-gon's point-block incidence matrix which in our case is 6.*

3 Central Primitive Idempotents

Let (P, B, F) be the points, blocks and flags of the generalized m-gon and σ, τ be as in theorem 1. Consider $\mathcal{H}_R(F)$ where R is a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$.

Let H, soon to be $\mathcal{H}_{\mathbb{Q}}(F)$, be a semisimple algebra over $\mathbb{Q}$. Let

$$H = \bigoplus_V H(V) \tag{5}$$

be a direct sum decomposition of H where V runs over the simple H-modules (up to isomorphism), and each $H(V)$ is a simple $\mathbb{Q}$-algebra. This decomposition is known as the Wedderburn decomposition, and $H(V)$'s are known as the Wedderburn components of H. Set

$$1_H = \sum_V e_V, \ e_V \in H(V). \tag{6}$$

Then for $x \in H$, we have

$$x = x1_H = x\sum_V e_V = \sum_V xe_V, \ e_V \in H(V). \tag{7}$$

The set $\{e_V \mid e_V \in H(V)\}$ is the set of *central primitive idempotents* of H. It is unique and completely determined by the decomposition (6).

Since $H(V)$ is a simple algebra, we have an isomorphism $\rho_V : H(V) \to M_{n_V}(D_V)$ onto a full matrix algebra, where $D_V \cong End_H(V)$ is a division algebra over $\mathbb{Q}$ and n_V is the multiplicity of V as a composition factor of $H(V)$ regarded as a module over itself.

A simple module V is *split simple* if $dim_K D_V = 1$, and that H is *split* if all simple modules are split simple.

Proposition 2 *Let $\{e_1, ..., e_n\}$ be a set of the central idempotents of H such that $1 = \sum_{i=1}^n e_i$ and let W be an H-module. Then each e_iW's is H-invariant and $W = \sum_{i=1}^n \oplus e_iW$.*

Proof: Let $h \in H$. We have $h(e_i W) = e_i(hW) \subseteq e_i W$. This shows that $e_i W$ is H-invariant. Since $e_i e_j = e_j e_i = 0$ for $i \neq j$ the $Im(e_i) \cap Im(e_j) = 0$. Therefore $Im(e_i)$ forms a basis for $e_i W$. Because we have a direct sum, the union of the images of the central idempotents form a basis for W.

Lemma 1 *Suppose $e \in H$ is a central primitive idempotent associated with a linear representation λ i.e. the equation $he = \lambda(h)e$ holds. Suppose V is an H-module. Then $v \in eV$ if and only if $hv = \lambda(h)v$ for all $h \in H$.*

Proof: Suppose $v \in eV$. Then $v = ev_1$ for some $v_1 \in V$ and $hv = hev_1 = \lambda(h)ev_1 = \lambda(h)v$. Conversely assume v is in the λ-component of V. Then $fv = 0$ for every central primitive idempotent $f \neq e$ and $v = 1v = (1-e)v+ev = fv + ev = 0 + ev = ev$ by equation (6).

Definition 3 *A trace function on H is an R-linear map $\mu : H \rightarrow R$ such that $\mu(hh') = \mu(h'h)$ for all h,h' $\in$ H. The set of trace functions on H is an R-module, with pointwise defined operations. We say that a trace function μ is a symmetrizing trace if the bilinear form*

$$H \times H \rightarrow R, \qquad (h, h') \mapsto \mu(hh'),$$

is non degenerate. This means that the determinant of the matrix $(\mu(bb'))_{(b,b')\in B}$ is not only not zero but also a unit in R for some (and hence every) R-basis B of H.

If μ is a symmetrizing trace on H and B is a basis for H, we denote by $B^\vee = \{b^\vee | b \in B\}$ the dual basis; it is uniquely determined by the equation $\mu(b^\vee b') = \delta_{bb'}$ for all b,b' $\in B$.

Definition 4 *Let V, V' be (left) H-modules and B be a basis of H, with dual basis $B^\vee$. For any $\varphi \in Hom_R(V, V')$, define $I(\varphi) \in Hom_R(V, V')$ by*

$$I(\varphi)(v) := \sum_{b \in B} b^\vee \varphi(bv) \qquad (v \in V). \tag{8}$$

Theorem 5 *Let V be a split simple H-module. Then there is a unique element $c_V \in K$ such that*

$$I(\varphi) = c_V Tr(\varphi) id_V \qquad \text{for all } \varphi \in End_K(V).$$

Furthermore, the constant c_V only depends on the isomorphism class of V.

Proof: (Geck and Pfeiffer [6] p.223)

The element $c_V \in K$ is called the Schur element associated with V.

Proposition 3 *Let H be split semisimple module and $H = \oplus_V H(V)$ be the Wedderburn decomposition of H, where V runs over the simple H-modules. Let B be any basis of H. Let $1_H = \sum_V e_V$ with $e_V \in H(V)$. Then each e_V is a central primitive idempotent and, denoting by χ_V the characters of V, we have*

$$e_V = \frac{1}{c_V} \sum_{b \in B} \chi_V(b) b^\vee. \tag{9}$$

Proof: (Geck and Pfeiffer [6] p.227)

Let $e_{Ind}, e_S, e_1, e_2, f$ denote the central idempotents corresponding to the index, Steinberg, first one dimensional, second one dimensional and all the two dimensional representations together respectively.

Proposition 4 *If $\mathcal{H}(F)$ is the complex Hecke algebra for the generalized m-gon (P, B, F) where $m \geq 3$ then $1 = e_{Ind} + e_S + e_1 + e_2 + f$ and the different factors of the denominators of the idempotents are*

$$\begin{cases} (1+s), (1+s+s^2) & for\ m = 3 \\ (1+s), (1+t), (1+st), (s+t) & for\ m = 4 \\ (1+s), (1+t), (1+st+s^2t^2), (s^2+st+t^2) & for\ m = 6 \\ (1+s), (1+t), (1+st), (1+s^2t^2), (s+t), (s^2+t^2) & for\ m = 8 \end{cases}$$

Proof: We will prove the case where $m = 3$ and the other cases are done in an analogous way. In this proof we start with $R = \mathbb{Z}$ and then extend our ring whenever a division is required, but in the end restrict R to be just big enough to contain the resulting central idempotents.

Define the symmetrizing trace of generalized 3-gon by

$$\mu : H \to \mathbb{Z}, \qquad h \mapsto coefficient(constant)$$

and compute the corresponding dual basis. If the basis of generalized 3-gon is $B = \{1, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma\}$ then the corresponding dual basis for it is $B^\vee = \{1, \frac{\sigma}{s}, \frac{\tau}{s}, \frac{\tau\sigma}{s^2}, \frac{\sigma\tau}{s^2}, \frac{\sigma\tau\sigma}{s^3}\}$ and now we need to replace R with $\mathbb{Z}[\frac{1}{s}]$.

By theorem 2 there are two one dimensional and one two dimensional representations for generalized 3-gon.

The computation of equation (8) for the representations is as follows:

In the following three cases let φ be the identity homomorphism. So for the index representation we get

$$\begin{aligned} I(\varphi)(v) &:= \sum_{b\in B} b^\vee \varphi(bv) = \sum_{b\in B} b^\vee(bv) \\ &= 1^\vee \cdot (1 \cdot v) + \sigma^\vee \cdot (\sigma \cdot v) + \tau^\vee \cdot (\tau \cdot v) \\ &\quad +(\sigma\tau)^\vee \cdot (\sigma\tau \cdot v) + (\tau\sigma)^\vee \cdot (\tau\sigma \cdot v) + (\sigma\tau\sigma)^\vee \cdot (\sigma\tau\sigma \cdot v) \\ &= 1 \cdot v + \frac{s}{s} \cdot s \cdot v + \frac{s}{s} \cdot s \cdot v + \frac{s^2}{s^2} \cdot s^2 \cdot v + \frac{s^2}{s^2} \cdot s^2 \cdot v \\ &\quad + \frac{s^3}{s^3} \cdot s^3 \cdot v \\ &= (1 + 2s + 2s^2 + s^3) \cdot v \end{aligned}$$

Now use theorem 5 to compute the c_V (the Schur element) of the index representation of the generalized 3-gon. The equation $I(\varphi) = c_V Tr(\varphi) id_V$ reads $(1+2s+2s^2+s^3) = c_V \cdot 1 \cdot 1$ which implies $c_V = (1+2s+2s^2+s^3) = (1+s)(1+s+s^2)$.

After finding the Schur element of the index representations of the generalized 3-gon we find the central primitive idempotents by using proposition 3. The primitive idempotent corresponding to the index representation of the generalized 3-gon is

$$e_{Ind} = \frac{(1+\sigma)(1+\tau+\tau\sigma)}{(1+s)(1+s+s^2)}.$$

Therefore e_{Ind} exists if $R = \mathbb{Z}[\frac{1}{1+s}, \frac{1}{1+s+s^2}]$.

For the Steinberg representation we have:

$$\begin{aligned}
I(\varphi)(v) &:= \sum_{b\in B} b^{\vee}\varphi(bv) = \sum_{b\in B} b^{\vee}(bv) \\
&= 1^{\vee}\cdot(1\cdot v) + \sigma^{\vee}\cdot(\sigma\cdot v) + \tau^{\vee}\cdot(\tau\cdot v) \\
&\quad +(\sigma\tau)^{\vee}\cdot(\sigma\tau\cdot v) + (\tau\sigma)^{\vee}\cdot(\tau\sigma\cdot v) + (\sigma\tau\sigma)^{\vee}\cdot(\sigma\tau\sigma\cdot v) \\
&= (1\cdot v) + \frac{(-1)}{s}\cdot(-1)\cdot v + \frac{(-1)}{s}\cdot(-1)\cdot v \\
&\quad + \frac{1}{s^2}\cdot 1\cdot v + \frac{1}{s^2}\cdot 1\cdot v + \frac{(-1)}{s^3}\cdot(-1)\cdot v \\
&= (1+\frac{2}{s}+\frac{2}{s^2}+\frac{1}{s^3})\cdot v
\end{aligned}$$

Now use theorem 5 to compute the c_V of the Steinberg representation of the generalized 3-gon. The equation $I(\varphi) = c_V Tr(\varphi) id_V$ reads $(1+\frac{2}{s}+\frac{2}{s^2}+\frac{1}{s^3}) = c_V\cdot 1\cdot 1$ which implies $c_V = 1+\frac{2}{s}+\frac{2}{s^2}+\frac{1}{s^3}$. After finding the Schur element of the Steinberg representations of the generalized 3-gon we find the central primitive idempotents by using proposition 3. The primitive idempotent corresponding to the Steinberg representation of the generalized 3-gon is

$$e_S = \frac{(s-\sigma)(s^2-s\tau+\tau\sigma)}{(1+s)(1+s+s^2)}.$$

Therefore e_S exists if $R = \mathbb{Z}[\frac{1}{1+s}, \frac{1}{1+s+s^2}]$.

For the two dimensional representation we get

$$
\begin{aligned}
I(\varphi)(v) &:= \sum_{b\in B} b^\vee \varphi(bv) = \sum_{b\in B} b^\vee(bv) \\
&= 1^\vee\cdot(1\cdot v) + \sigma^\vee\cdot(\sigma\cdot v) + \tau^\vee\cdot(\tau\cdot v) \\
&\quad +(\sigma\tau)^\vee\cdot(\sigma\tau\cdot v) + (\tau\sigma)^\vee\cdot(\tau\sigma\cdot v) + (\sigma\tau\sigma)^\vee\cdot(\sigma\tau\sigma\cdot v) \\
&= \Bigg[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{s}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}^2 + \frac{1}{s}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix}^2 \\
&\quad +\frac{1}{s^2}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix} \\
&\quad +\frac{1}{s^2}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix} \\
&\quad +\frac{1}{s^3}\cdot\left(\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}\cdot\begin{pmatrix} s & s/a \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix}\right)^2\Bigg]\cdot v \\
&= \begin{pmatrix} \frac{2s^2+2s+2}{s} & 0 \\ 0 & \frac{2s^2+2s+2}{s} \end{pmatrix}\cdot v
\end{aligned}
$$

The equation $I(\varphi) = c_V Tr(\varphi) id_V$ reads

$$
\begin{pmatrix} \frac{2s^2+2s+2}{s} & 0 \\ 0 & \frac{2s^2+2s+2}{s} \end{pmatrix} = c_V\cdot 2\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

which implies that the Schur element for the two dimensional representation is $c_V = \frac{s^2+s+1}{s}$.

The primitive idempotent corresponding to the two dimensional representation of the generalized 3-gon is

$$
e_f = \frac{(2s + (s-1)(\sigma+\tau) - \sigma\tau - \tau\sigma)}{s^2+s+1}.
$$

Therefore e_f exists if $R = \mathbb{Z}[\frac{1}{1+s+s^2}]$.

Since $1 = e_{Ind} + e_S + e_f$ we can write V as

$$
V = V.1 = Ve_{Ind} \oplus Ve_S \oplus Ve_f.
$$

Therefore this decomposition is defined over R if $R = \mathbb{Z}[\frac{1}{1+s}, \frac{1}{1+s+s^2}]$ i.e. if the Poincare polynomial is a unit in R.

Remark 1 *We have used the techniques of [6] for the computations of the Schur elements. The assumption that the parameters s and t are units in R is needed for the computation of the dual basis. However this is not necessary for the existence of the central primitive idempotents.*

Proposition 5 *If the expressions in the hypothesis of theorem 4 are units in R then the R-ary standard module RF can be written as a direct sum of $\mathcal{H}_R F$-submodules $RF = \sum \oplus M_i$ where $\mathbb{C} \otimes M_i$ is an $\mathcal{H}_{\mathbb{C}}(F)$ direct sum of irreducibles of $\mathbb{C}F$ corresponding to the one dimensional representations separately and all two dimensional representations together.*

Proof: Apply proposition 3 and the result follows.

4 The Point Block Module

Assume the expressions listed in proposition 4 are units in R and $\mathcal{H}_R(F)$ contains the central idempotents appearing in proposition 4. Define RP and RB to be a free R-module with distinguished basis labeled by the points and blocks respectively. Define the maps:

$$\phi_{PB} : RP \to RB(p \mapsto \sum_{pIb} b); \quad \phi_{BP} : RB \to RP(b \mapsto \sum_{pIb} p);$$

$$\phi_{PF} : RP \to RF(p \mapsto \sum_{pIb} (p,b)); \quad \phi_{BF} : RB \to RF(b \mapsto \sum_{pIb} (p,b));$$

$$\phi : RP \oplus RB \to RF(\phi = \phi_{PF} \oplus \phi_{BF}).$$

Let $\alpha_1 = \sum_{p \in P} p \in RP \oplus RB$, $\beta_{|B|} = \sum_{b \in B} b \in RP \oplus RB$, $L =< \alpha_1, \beta_{|B|} >$ and $RPB := (RP \oplus RB)/L$.

Let $\sigma', \tau' \in Hom_R(RP \oplus RB, RP \oplus RB)$ be defined by

$$\sigma'(p) = -p + \phi_{PB}(p), \qquad \sigma'(b) = sb,$$

$$\tau'(p) = tp, \qquad \tau'(b) = \phi_{BP}(b) - b.$$

With respect to the basis labeled by the points and blocks respectively σ' and τ' have the following form:

$$\sigma' = \begin{pmatrix} -I & 0 \\ M^T & s * I \end{pmatrix} \text{ and } \tau' = \begin{pmatrix} t * I & M \\ 0 & -I \end{pmatrix}$$

where M is the incidence matrix.

We need to go back and forth between $RP \oplus RB$ and RPB so we write $\bar{x}$ for $x + L \ mod \ L$ when $x \in RP \oplus RB$ and extend this notation to include submodules writing $\overline{A}$ for $A + L \ mod \ L$ when A is a submodule of $RP \oplus RB$. In addition, if $\phi \in End(RP \oplus RB)$ leaves invariant L then write $\overline{\phi} \in End(RPB)$ for the induced map.

All of the above notation will be used throughout this section.

4.1 The $RP \oplus RB$ module and $Im(\phi)$

Proposition 6 *The map $\phi : RP \oplus RB \to RF$ satisfies the following relations:*

i. $\phi\sigma' = \sigma\phi$ and $\phi\tau' = \tau\phi$ where σ, τ act on the standard module as in theorem 1.

ii. $Ker(\phi) = R(\alpha_1 - \beta_{|B|})$ is invariant under σ' and τ'.

iii. L is invariant under σ' and τ' and their matrices with respect to the basis $\{\alpha_1, \beta_{|B|}\}$ are

$$\sigma'|_L = \begin{pmatrix} -1 & 0 \\ s+1 & s \end{pmatrix} \quad \text{and} \quad \tau'|_L = \begin{pmatrix} t & t+1 \\ 0 & -1 \end{pmatrix}.$$

iv. RPB is an $\mathcal{H}(\sigma, \tau)$-module with $\sigma \to \sigma'$ mod L and $\tau \to \tau'$ mod L. Moreover $Im(\bar{\phi})$ is isomorphic to RPB.

Proof: Liebler ([7] p.586) shows (i), which implies that ϕ is an $\mathcal{H}(\sigma, \tau)$-homomorphism and so the first part of (iv) follows from (ii). $Ker(\bar{\phi}) = 0$ since $Ker(\phi) \leq L$. This shows the isomorphism claimed in the second part of (iv). Only (ii) and (iii) need proof.

To prove (ii) suppose $v = \sum_{p \in P} c_p p + \sum_{b \in B} d_b b$ and $\phi(v) = 0$. It is sufficient to show that $c_p = -d_b$ independent of $p \in P$ and $b \in B$.

Let C_p be the diagonal matrix with diagonal entries $c_p : p \in P$ and similarly let D_b be the diagonal matrix with diagonal entries $d_b : b \in B$. Then $C_p M$ corresponds to $\phi(\sum_{p \in P} c_p p)$ in the matrix model introduced after example 1 and $M_d B$ correspond to $\phi(\sum_{b \in B} d_b b)$. Note that the i'th row of $D_p M$ is a scalar multiple of the i'th row of M, and similarly the j'th column of $M D_b$ is a scalar multiple of the j'th column M. Since $\phi(v) = 0$ we have

$$D_p M + M D_b = 0.$$

This matrix equation can only hold if both D_p and D_b are scalar matrices. Suppose by way of contradiction $c_{p_1} \neq c_{p_2}$ for points $p_1 \neq p_2$. Among all such pairs, choose p_1 and p_2 so that the chain joining them $p_1 = a_0, \ldots, a_h = p_2$ where $h \geq 3$ (as defined at the beginning of chapter 2) is as short as possible. If we remove p_1 we obtain a shorter chain where all the coefficients c_i are equal to c_{p_2}. Similarly removing p_2 gives us a shorter chain where all the coefficients c_i are equal to c_{p_1}. Thus $c_{p_1} = c_{p_2}$. This is a contradiction. Therefore all the coefficients of the points has to be the same. Similar argument holds for points. Since the incidence graph of generalized m-gon is connected the result follows.

To prove (iii) we apply σ' to α_1 and $\beta_{|B|}$ and we get

$$\sigma'(\alpha_1) = \sigma'(\sum_{p \in P} p) = -\sum_{p \in P} p + (s+1)\sum_{b \in B} b = -1\alpha_1 + (s+1)\beta_{|B|}$$

and

$$\sigma'(\beta_{|B|}) = \sigma'(\sum_{b \in B} b) = s\sum_{b \in B} b = s\beta_{|B|}.$$

The result follows.

Proposition 7 *Let $e_{Ind}, e_S, e_1, e_2, f \in \mathcal{H}(F)$ be the central idempotents appearing in proposition 4. Then*

i. $e_S Im(\phi) = 0$.

ii. $\phi(L) = e_{Ind} Im(\phi) = e_{Ind} RF$.

Proof : Note that $\phi(p) = (\sigma + 1)(f)$ for any flag f on a point p. Let $v = \phi(p)$ and apply lemma 1. Then

$$e_S v = e_S(\sigma + 1) = (\sigma + 1)e_S = \sigma e_S + e_S = -e_S + e_S = 0.$$

Similar argument holds if we replace p with b and σ with τ. Part (i) follows.

By proposition 6-(ii) $\phi(\alpha_1 - \beta_{|B|}) = 0$, so $\phi(\alpha_1) = \phi(\beta_{|B|})$ and $\phi(L) = \langle \phi(\alpha_1) \rangle = \langle \sum_{f \in F} f \rangle$. Note that $\phi(L) \leq RF$ is pure because $RF/\phi(L)$ is torsionfree. Since

$$\sigma(\sum_{f \in F} f) = s(\sum_{f \in F} f), \quad \tau(\sum_{f \in F} f) = t(\sum_{f \in F} f),$$

and $m_{Ind} = 1$ (after equation 4) we get $\phi(L) = e_{Ind}RF$. This completes part (ii) of the proposition.

4.2 RPB and the Smith Normal Form

Recall that $RPB := (RP \oplus RB)/L$.

Proposition 8 *Let $e_{Ind}, e_S, e_1, e_2, f \in \mathcal{H}(F)$ be the central idempotents appearing in proposition 4. Then $Im(\bar{\phi}) \cong (e_1 + e_2 + f)RF$.*

Proof: By proposition 7-(ii), $\phi(L) = e_{Ind}RF$. We have

$$\begin{aligned} Im(\bar{\phi}) &= Im(\phi)/\phi(L) = Im(\phi)/e_{Ind}RF \\ &= (e_{Ind} + e_S + e_1 + e_2 + f)Im(\phi)/e_{Ind}RF \\ &\cong (e_1 + e_2 + f)Im(\phi). \end{aligned}$$

The only thing left to show is $(e_1+e_2+f)Im(\phi) = (e_1+e_2+f)RF$. Since $Im(\bar{\phi})$ has rank $|P|+|B|-2$, equation (4) implies this result if coefficients are extended to $\mathbb{Q}$. Therefore $(e_1 + e_2 + e_f)RF/(e_1 + e_2 + e_f)Im(\phi)$ is torsion module.

It remains to show $Im(\bar{\phi})$ is pure. By definition of pure, we must show that if $\alpha \in R$ and $\alpha x \in Im(\bar{\phi})$ then $x \in Im(\bar{\phi})$. Suppose $\alpha \in R$, $\alpha x \in Im(\bar{\phi})$ and

$$\alpha x = \bar{\phi}(w) \ for \ w \in RPB. \tag{10}$$

Define an R-homomorphism $\lambda : RF \to RPB$ by $\lambda((p,b)) = (\overline{p,b})$. Suppose that $(\lambda\bar{\phi})^{-1}$ exists. Apply $(\lambda\bar{\phi})^{-1}\lambda$ to both sides of equation (10)

$$(\lambda\bar{\phi})^{-1}\lambda(\alpha x) = (\lambda\bar{\phi})^{-1}\lambda\bar{\phi}(w) = w.$$

Finally substituting for w in equation (10) we obtain

$$\alpha x = \bar{\phi}(w) = \bar{\phi}((\lambda\bar{\phi})^{-1}\lambda(\alpha x)) = \alpha(\bar{\phi}((\lambda\bar{\phi})^{-1}\lambda(x))) = \alpha\bar{\phi}(z)$$

where $z = (\lambda\bar{\phi})^{-1}\lambda(x)$. Now $\alpha \neq 0$ implies that $x = \bar{\phi}(z)$. This shows that $x \in Im\bar{\phi}$. This shows that $Im(\bar{\phi})$ is pure provided $(\lambda\bar{\phi})^{-1} exists$.

To show $(\lambda\bar{\phi})^{-1}$ exists provided $(\bar{\phi}\lambda|_{Im(\bar{\phi})})^{-1}$ exists, let ρ be the map from RPB to the $Im(\bar{\phi})$ such that $\rho(\overline{p,b}) = \bar{\phi}(\overline{p,b})$. Note that ρ differs from $\bar{\phi}$ only in codomain. Because $Ker(\bar{\phi}) = 0$, ρ is an isomorphism and has an inverse. Now $\rho^{-1}(\bar{\phi}\lambda|_{Im(\bar{\phi})})\rho = \rho^{-1}(\rho\lambda|_{Im(\bar{\phi})})\rho = \lambda|_{Im(\bar{\phi})}\rho = \lambda|_{Im(\bar{\phi})}\bar{\phi} = \lambda\bar{\phi}$ shows that $(\lambda\bar{\phi})^{-1}$ exists provided $(\bar{\phi}\lambda|_{Im(\bar{\phi})})^{-1}$ exists.

Since $\bar{\phi}\lambda \in \mathcal{H}_{\mathbb{C}}(\sigma,\tau)$ the complex theory of this algebra applies. It remains to show $det(\Theta(\bar{\phi}\lambda)) = det(\Theta(\sigma + \tau + 2 * I))$ is either 0 or invertible over R for every complex irreducible representation Θ of $\mathcal{H}(F)$. The calculations of the $det(\bar{\phi}\lambda)$ are as follows:

For the Steinberg representation we get

$$\sigma + \tau + 2 * I = -1 - 1 + 2 = 0.$$

For the one dimensional representations we get for example:

$$\sigma + \tau + 2 * I = s - 1 + 2 = s + 1.$$

For the two dimensional representation we get

$$\sigma + \tau + 2 * I = \begin{pmatrix} -1 & 0 \\ a & s \end{pmatrix} + \begin{pmatrix} t & b \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} t+1 & b \\ a & s+1 \end{pmatrix}$$

So $det(\sigma+\tau+2*I) = st+s+t+1-ab$. The product ab for the different two dimensional representations is a factor of the expressions appearing in proposition 4 and we have assumed these to be units in R at the beginning of this chapter.

We need to understand what happens to the incidence matrix M and M^T when τ' and σ' are regarded *mod* L. We only discuss τ' because σ' is handled in an analogous way.

Consider the square matrices

$$E_1 = \left(\begin{array}{c|c} 1 & 1...1 \\ \hline 0 & -I \end{array}\right)_{|P|\times|P|} \quad and \;\; E_2 = \left(\begin{array}{c|c} 1 & 1...1 \\ \hline 0 & -I \end{array}\right)_{|B|\times|B|}$$

where $|P|$ and $|B|$ denote the number of points and blocks respectively. Identify these as transition matrices. Then $E_1 M E_2 = \left(\begin{array}{c|c} s+1 & 0...0 \\ \hline A & M_1 \end{array}\right)$. Since we assumed that the Poincare polynomial is a unit in R, $s+1$ and $t+1$ are units in R. Let $E_3 = \left(\begin{array}{c|c} (s+1)^{-1} & 0 \\ \hline 0 & I \end{array}\right)$. Then $E_3 E_1 M E_2 = \left(\begin{array}{c|c} 1 & 0 \\ \hline A & M_1 \end{array}\right)$. Finally, there is a matrix E_4 invertible over R such that $E_4 E_3 E_1 M E_2 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M_1 \end{array}\right)$.

Let E to be a 2 by 2 block diagonal matrix with diagonal blocks $(E_4 E_3 E_1)^{-1}$ and E_2. Then the first and the $p+1^{th}$ row of E corresponds to α_1 and $\beta_{|B|}$ respectively. Moreover

$$E^{-1}\tau' E = \left(\begin{array}{c|c} t*I & E_4 E_3 E_1 M E_2 \\ \hline 0 & -I \end{array}\right)$$

and $E^{-1}\tau' E$ *mod* L corresponds to removing the the first and the $p+1^{th}$ row and column of $E^{-1}\tau' E$. This proves

Lemma 2 *All the nonzero entries in the R-normal form of M_1 appear in the R-normal form of M.*

Proposition 9 *We have the following:*

i. *The first one dimensional representation is afforded by*

$e_1(RPB) = \overline{(0 \oplus Ker(M))} = \overline{(0 \oplus ker(\phi_{PB})} \subset RPB$ *and has multiplicity* $m_1 = |B| - rank(M)$.

ii. *The second one dimensional representation is afforded by*

$e_2(RPB) = \overline{(Ker(M^T) \oplus 0)} = \overline{(ker(\phi_{BP}) \oplus 0)} \subset RPB$ *and has multiplicity* $m_2 = |P| - rank(M)$.

iii. *Two dimensional* $\mathcal{H}(F)$ *irreducible direct summands have the same multiplicity in* RPB *as in* RF.

Proof: Suppose $\bar{v}$ affords the first one dimensional representation. Then $\sigma(\bar{v}) = s\bar{v}$ and $\tau(\bar{v}) = -\bar{v}$ i.e.

$$\sigma^{'}(v) = sv + l_\sigma, \quad \tau^{'}(v) = -1v + l_\tau, \quad where \;\; l_\sigma, l_\tau \in L$$

Write $v = v_1 \oplus v_2$ where $v_1 \in RP$ and $v_2 \in RB$. Then

$$\begin{pmatrix} sv_1 + l_{\sigma_p} \\ sv_2 + l_{\sigma_b} \end{pmatrix} = \sigma^{'}(v) = \begin{pmatrix} -I & 0 \\ M^T & s*I \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ M^T v_1 + sv_2 \end{pmatrix}$$

by comparison of "first coordinates" it follows that $(s+1)v_1 = -l_{\sigma_p}$. Similarly

$$\begin{pmatrix} -1v_1 + l_{\tau_p} \\ -1v_2 + l_{\tau_b} \end{pmatrix} = \tau^{'}(v) = \begin{pmatrix} t*I & M \\ 0 & -I \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} tv_1 + Mv_2 \\ -v_2 \end{pmatrix}$$

implies $Mv_2 = l_{\tau_p} - (t+1)v_1 = l_{\tau_p} + \frac{t+1}{s+1} l_{\sigma_p}$. This shows $\bar{v} \in \overline{(0 \oplus Ker(M))}$.

The multiplicity of the first one dimensional representation in RPB equals the rank of $\overline{(0 \oplus Ker(M))}$, which, by a fundamental isomorphism, equals the rank of $(0 \oplus Ker(M))/(L \cap (0 \oplus Ker(M)))$. But $L \cap (0 \oplus RB) = R_{\beta_{|B|}}$ and $\beta_{|B|} \notin Ker(\phi)$. The rest of part (i)follows by lemma 1.

The proof of (ii) is similar to (i) and will be omitted.

Property (iii) follows from proposition 6.

Now construct a new basis of RPB. Let $r = Rank(M)$, $|P|$ be the number of points and $|B|$ be the number of blocks.

Let $\Gamma_{e_1} = \{\bar{\beta}_{|P|+r+1}, \bar{\beta}_{|P|+r+2}, \ldots, \bar{\beta}_{|B|-1}\}$ and $\Gamma_{e_2} = \{\bar{\alpha}_2, \bar{\alpha}_3 \ldots \bar{\alpha}_{|P|-r-1}\}$ be an R-basis for the $\overline{(0 \oplus Ker(M))}$ and $\overline{(Ker(M^T) \oplus 0)}$ respectively.

Because $fRPB$ and $\overline{RP}$ are both pure in RPB, their intersection is again pure and $(fRPB \cap \overline{RP}) \cap (fRPB \cap \overline{RB}) = 0$ which implies

$$fRPB = (fRPB \cap \overline{RP}) \oplus (fRPB \cap \overline{RB}). \tag{11}$$

Pick an R-bases

$$\Gamma_p = \{\bar{\delta}_{|P|-r}, \bar{\delta}_{|P|-r+1}, \ldots, \bar{\delta}_r\} \; of \; fRPB \cap \overline{RP}$$

and

$$\Gamma_b = \{\bar{\delta}_{r+1}, \bar{\delta}_{r+2}, \ldots, \bar{\delta}_{|P|+r}\} \; of \; fRPB \cap \overline{RB}.$$

Proposition 2 implies that $\Gamma = \{\Gamma_{e_2} \cup \Gamma_p \cup \Gamma_b \cup \Gamma_{e_1}\}$ is an R-basis for RPB.

Needless to say, we need the matrices of $\bar{\sigma}$ and $\bar{\tau}$ with respect to Γ. The next three paragraphs explain their form when blocked according to Γ.

For each $\bar{\alpha} \in \Gamma_{e_2}$ let $\alpha \in RP \oplus RB$ such that $\bar{\alpha} = \alpha + L \; mod \; L$. Then $\sigma'\alpha = -\alpha$. Therefore $\bar{\sigma}\bar{\alpha} = -\bar{\alpha}$. This explains the first column of $(\bar{\sigma})_\Gamma$.

For each $\bar{\delta} \in \Gamma_p$ let $\delta \in RP \oplus RB$ such that $\bar{\delta} = \delta + L \; mod \; L$. Then $\sigma'\delta = -\delta + M^T\delta$. Therefore $\bar{\sigma}\bar{\delta} = -\bar{\delta} + \overline{M^T\delta}$. Moreover $\bar{\delta} \in fRPB$ so $\bar{\sigma}(\bar{\delta})$ is too by lemma 1. This implies that $\overline{M^T\delta} \in fRPB \cap RB$ and therefore is a linear combination of elements of Γ_b. Because $Ker(M^T)$ is accounted for by the first column of $(\bar{\sigma})_\Gamma$ below, the matrix G of $\overline{f(0 \oplus M^T)}$ has full rank equal to $r-1$. This explains the second column of $(\bar{\sigma})_\Gamma$.

For each $\bar{\delta} \in \Gamma_b \cup \Gamma_{e_1}$ let $\delta \in RP \oplus RB$ such that $\bar{\delta} = \delta + L \; mod \; L$. Then $\sigma^{'}\delta = s\delta$. Therefore $\bar{\sigma}\bar{\delta} = s\bar{\delta}$. This explains the third and fourth column of $(\bar{\sigma})_\Gamma$.

We obtain $(\bar{\tau})_\Gamma$ in a similar way and N also has full rank equal to $r-1$.

Therefore with respect to the basis Γ:

$$(\bar{\sigma})_\Gamma = \left(\begin{array}{c|c|c|c} -I & 0 & 0 & 0 \\ \hline 0 & -I & 0 & 0 \\ \hline 0 & G & s*I & 0 \\ \hline 0 & 0 & 0 & s*I \end{array}\right) \quad (\bar{\tau})_\Gamma = \left(\begin{array}{c|c|c|c} t*I & 0 & 0 & 0 \\ \hline 0 & t*I & N & 0 \\ \hline 0 & 0 & -I & 0 \\ \hline 0 & 0 & 0 & -I \end{array}\right)$$

Theorem 6 *Let $(\bar{\sigma})_\Gamma$ and $(\bar{\tau})_\Gamma$ be the above matrices. Then the following equations have to be satisfied:*

$$sG = GNG \quad for \quad m = 3 \tag{12}$$

$$(s+t)G = GNG \quad for \quad m = 4$$

$$((s+t) - GN)^2 G = (st)G \quad for \quad m = 6$$

$$(s+t)G = GNG \quad for \quad m = 8 \quad or$$

$$((s+t) - GN)^2 G = (2st)G \quad for \quad m = 8$$

Proof: The generators $(\bar{\sigma})_\Gamma$ and $(\bar{\tau})_\Gamma$ have to satisfy the relations stated in theorem 1. For $m = 3$ we have $\bar{\sigma}_\Gamma \bar{\tau}_\Gamma \bar{\sigma}_\Gamma = \bar{\tau}_\Gamma \bar{\sigma}_\Gamma \bar{\tau}_\Gamma$. Since $(\bar{\sigma})_\Gamma$ and $(\bar{\tau})_\Gamma$ are square block diagonal matrices we work with the submatrices

$$\Sigma = \left(\begin{array}{c|c} -I & 0 \\ \hline G & s*I \end{array}\right) \; and \; \mathsf{T} = \left(\begin{array}{c|c} t*I & N \\ \hline 0 & -I \end{array}\right).$$

So we have

$$\left(\begin{array}{cc} t*I - N*G & -s*I*N \\ -(s+t)*I*G + G*N*G & s*I*G*N - s^2*I \end{array}\right) = \Sigma\mathsf{T}\Sigma =$$

$$= \mathsf{T\Sigma T} = \begin{pmatrix} -t^2 * I + N * G & -(s+t) * I * N + N * G * N \\ -t * I * G & s * I - G * N \end{pmatrix}$$

Compare the second row and first column entries of matrices then we get

$-(s+t) * I * G + G * N * G = -t * I * G$ which is equivalent to equation (12).

The other cases are done in an analogous way and are omitted.

By the fundamental theorem for modules over a principal ideal domain there exist $S_1, S_2 \in Mat_{m_1 \times m_1}(R)$ such that $det(S_1)$ and $det(S_2)$ are units in R and $D_1 = S_1 G S_2$ is the R-normal form of G.

Let $\Delta_p = \{\gamma_1, \gamma_2, \ldots, \gamma_{r-1}\}$ and $\Delta_b = \{\gamma_1', \gamma_2', \ldots, \gamma_{r-1}'\}$ be a new R-basis of fRB where

$$\gamma_i = \sum_{j=|P|-r}^{r} u_{ki}\bar{\delta}_j, \quad \gamma_i' = \sum_{j=r+1}^{|P|+r} v_{ki}\bar{\delta}_j, \quad for\ i,k = 1,2,\ldots,r-1$$

where the coefficients u_{ki} and v_{ki} are the (i,k)-entries of the matrices S_1^{-1} and S_2 respectively. Let $\Delta = \{\Delta_p \cup \Delta_b\}$.

Theorem 7 *With respect to basis Δ , $\bar{\sigma}$ and $\bar{\tau}$ get the following form:*

$$(\bar{\sigma})_\Delta = \left(\begin{array}{c|c} -I & 0 \\ \hline D_1 & s * I \end{array}\right), (\bar{\tau})_\Delta = \begin{pmatrix} t * I & D_2 \\ 0 & -1 \end{pmatrix}$$

and $D_2 = S_2^{-1} N S_1^{-1}$ is a diagonal matrix.

Proof: The only thing that has to be shown is that D_2 is diagonal. This follows from theorem 6. For example in case $m = 3$. Multiply equation (12) from the left and from the right by S_1 and S_2 respectively. The left side of the resulting equation is diagonal and equal to sD_1. Since

$$sD_1 = S_1 G S_2 S_2^{-1} N S_1^{-1} S_1 G S_2 = D_1 S_2^{-1} N S_1^{-1} D_1$$

is diagonal, $D_2 = S_2^{-1} N S_1^{-1}$ has to be diagonal.

The other cases are handled in a similar way and are omitted.

Proof of Theorem 4:

We start by proving the result for the one dimensional representations of $\mathcal{H}(F)$. Then we handle the two dimensional representations.

By proposition 4 page 17 we have

$$RF = e_{Ind}RF \oplus e_S RF \oplus e_1 RF \oplus e_2 RF \oplus fRF.$$

By proposition 7-(iii) page 24 we have $Im(\bar{\phi}) = (e_1 + e_2 + f)RF$. Thus $RF = e_{Ind}RF \oplus e_S RF \oplus Im(\bar{\phi})$. By proposition 6 page 23 $Im(\bar{\phi})$ is $\mathcal{H}(F)$-isomorphic to RPB. By proposition 2 page 15 and proposition 5 page 21, we have $RPB = e_1 RPB \oplus e_2 RPB \oplus fRPB$. Since the one dimensional R-forms are just 1 by 1 integer matrices (entries are -1, s or t), they are exactly the same with the one dimensional $\mathbb{C}$-forms. This completes the proof relative to the one dimensional $\mathbb{C}$-forms.

In order to deal with the two dimensional $\mathbb{C}$-forms it is sufficient to show $fRPB$ is a direct sum of rank 2 $\mathcal{H}_R(F)$-modules and to establish the incidence matrix connection.

Reorder the basis Δ appearing in theorem 7 on page 31 by listing γ_i immediately following γ_i'. Denote this new basis by ε.

With respect to the basis ε, $(\bar{\sigma}_\Delta)_\varepsilon$ and $(\bar{\tau}_\Delta)_\varepsilon$ become block diagonal matrices which consists of 2 by 2 blocks of the form

$$\sigma = \begin{pmatrix} -1 & 0 \\ d_1 & s \end{pmatrix} \quad and \quad \tau = \begin{pmatrix} t & d_2 \\ 0 & -1 \end{pmatrix}$$

where d_1 and d_2 are diagonal entries of D_1 and D_2 respectively.

The product $d_1 d_2$ must satisfies the conditions given in theorem 2 page 11 because they are the complex two dimensional representations of $\mathcal{H}(F)$. This proofs part (ii) of the main theorem.

By Lemma 2 page 27 we see that d_1's come from the incidence matrix M. Therefore part (iii) of the main theorem follows.

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