DISSERTATION

NORMALIZING PARSEVAL FRAMES BY GRADIENT DESCENT

Submitted by

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ABSTRACT

NORMALIZING PARSEVAL FRAMES BY GRADIENT DESCENT

Equinorm Parseval Frames (ENPFs) are collections of equal-length vectors that form Parseval frames, meaning they are spanning sets that satisfy a version of the Parseval identity. As such, they have many of the desirable features of orthonormal bases for signal processing and data representation, but provide advantages over orthonormal bases in settings where redundancy is important to provide robustness to data loss. We give three methods for normalizing Parseval frames: that is, flowing a generic Parseval frame to an ENPF. This complements prior work showing that equal-norm frames could be "Parsevalized" and potentially provides new avenues for attacking the Paulsen problem, which seeks sharp upper bounds on the distance to the space of ENPFs in terms of norm and spectral data. This work is based on ideas from symplectic geometry and geometric invariant theory.

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Chapter 1

Introduction

1.1 What are Frames?

Frames were introduced in the context of non-harmonic Fourier series and infinite-dimensional Hilbert spaces by Duffin and Schaeffer [13] while investigating non-integer frequency Fourier analysis.

Definition 1.1.1. A *frame* on a Hilbert space, V, is a countable set of vectors $\{f_k\}_{k\in I}$ satisfying for any $v \in V$

$$A||v||^2 \leq \sum_{k\in I} |\langle f_k, v\rangle|^2 \leq B||v||^2, \tag{1.1}$$

where $0 < A \leq B < \infty$ are constants, referred to as *frame bounds*.

Today, frames find uses in image compression and reconstruction, data transfer, antenna networks and many other applied areas [11,18,36,40]. Though the definition was for Hilbert spaces in general, a frame on a finite-dimensional Hilbert space is simply a linear spanning set. Putting the frame vectors f_k into the columns of a matrix (allowing for countably many columns and rows), we can define

$$F = \begin{bmatrix} | & | & | \\ f_1 & f_2 & \cdots \\ | & | & | \end{bmatrix} \text{ and } F^* = \begin{bmatrix} -\overline{f_1}- \\ -\overline{f_2}- \\ \vdots \\ -\overline{f_k}- \\ \vdots \end{bmatrix}.$$

Definition 1.1.2. A frame F is called

- *Equinorm* if $||f_k|| = c$ for all k. *Unit norm* is when $c = 1$.
- *Tight* if $FF^* = AI$, where $A = B \in \mathbb{R}_+$ are the frame constants from Definition 1.1.1.

- *Parseval* if it satisfies the Parseval identity

$$||v|| = \sum_k |\langle f_k, v\rangle|^2.$$

- To be a Parseval frame is equivalent to

$$FF^* = I.$$

Definition 1.1.3. If the Hilbert space is finite-dimensional and the frame is finite, so $F \in \mathbb{C}^{d\times n}$ (or $\mathbb{R}^{d\times n}$), then F is called *full spark* if any d-tuple of the frame vectors form a basis for $\mathbb{C}^d$ (or $\mathbb{R}^d$).

In [20], being full spark is the sufficient condition for a collection of vectors to be put in radially isotropic position. In [30], full spark is the sufficient condition for a unit norm frame to flow to a unit norm tight frame (UNTF) via gradient decent. This paper adds to the pile of evidence that full spark frames are of great importance for producing equal norm Parseval frames (ENPFs).

Example 1.1.4. Let

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}.$$

P is equinorm and Parseval but is not full spark since the first two columns of P do not form a basis for $\mathbb{C}^2$. Whereas Q is equinorm, Parseval, and full spark, as any two columns form a basis for $\mathbb{C}^2$. (Notice $||q_k||^2 = \frac{2}{4} = \frac{d}{n}$ as promised by equation (2.15).)

1.2 Finding ENPFs and UNTFs

Finding ENPFs and UNTFs has proven to be rather challenging. The analysis operator $F^* : V \to \ell^2(I)$ is a bounded linear map. This type of operator is well-studied in functional analysis and lets one attack problems with tools from that area. For example, the *Schur–Horn Theorem* is about the existence of positive semidefinite self-adjoint matrices with specified spectrum and diagonal entries. This theorem is used in multiple frame theory applications [1, 28, 39] and in [8] their Theorem 2 is a "not easily implemented algorithm" to construct finite frames from a specified spectrum and magnitudes using spectral analysis. Aside from special cases, e.g. truncated discrete Fourier transform, producing ENPFs can only be done through numerical methods. In 2003, Benedetto and Fickus showed:

Theorem 1.2.1. *[5] Denote the set of unit norm frames by* $\mathcal{F}_1 = \{F \in \mathbb{C}^{d\times n} \mid ||f_k|| = 1\}$ *and define* $\Phi(F) = \sum_{i=1}^{n}\sum_{k=i}^{n} |\langle f_i, f_k\rangle|^2 = ||F^*F||_{Fr}^2$*. then* $\Phi : \mathcal{F}_1 \to \mathbb{R}$ *has no spurious local minimizers.*

Lacking spurious local minimizers means the critical points of Φ are either saddle points or global minima. The map Φ is called the *frame potential* and UNTFs minimize it. In [30], the authors use ideas from [24] to reprove and strengthen Theorem 1.2.1. They show the gradient flow defined by

$$\frac{d}{dt}F(t) = -\nabla\Phi(F(t))$$

limits to a UNTF, provided $F(0)$ is a unit-norm full spark frame. That is, beginning with a unit norm frame, it is "tightened" by gradient flow to a unit-norm tight frame. The main theorem of this paper flips this around. We show that the negative gradient of a certain potential function flows almost every tight frame to an equinorm tight frame. The tools of this paper are from symplectic and algebraic geometry.

The first explicit connections between symplectic geometry and frame theory seems to have been made in [32], leading to a plethora of recent work using symplectic geometry to find insights in frame theory [30, 33, 34]. All of these works use moment maps. Moment maps are special functions at the intersection of algebraic and symplectic geometry that are induced by group actions. The frame potential is an example of a function which arises as the squared norm of a moment map. In general, Kirwan [24] showed the squared norm of a moment map on a symplectic manifold tends to lack spurious minima, leading to results like Theorem 1.2.1. Furthermore, the Kempf–Ness Theorem 6.3.2 relates critical points of the squared norm of a moment map to group orbits of the Geometric Invariant Theory (GIT) quotient. The point being that moment maps can open up new avenues and approaches in frame theory and, indeed, they lurk in the background of this paper.

1.2.1 The Paulsen Problem

While work, including this one, has been done showing numerical methods converging to ENPFs, the Paulsen Problem asks the question: how far is the nearest ENPF?

Definition 1.2.2. $\{f_1, \cdots, f_n\} \subset \mathbb{C}^d$ is ϵ-equinorm Parseval (abbr. ϵ-ENPF) if

1. All eigenvalues of FF^* are in $(1-\epsilon, 1+\epsilon)$ and

2. For all $k \in \{1, \cdots, n\}$, $\frac{d}{n}(1-\epsilon) \leq ||f_k|| \leq \frac{d}{n}(1+\epsilon)$.

Given two frames $P, Q \in \mathbb{C}^{d\times n}$, define the squared distance between P and Q to be $\operatorname{dist}^2(P,Q) = ||P-Q||_{Fr}^2 = \sum_{k=1}^n ||p_k - q_k||^2$, where $||\cdot||_{Fr}$ is the Frobenius norm.

The *Paulsen Problem* simply asks, given an ϵ-ENPF frame P, is there an ENPF frame Q and a polynomial, f, such that $\operatorname{dist}(P,Q) \leq f(\epsilon, d)$, which is to say the distance is independent of the number of frame vectors. First answered in 2017 by Kwok et. al. [25] with a bound of $C\epsilon d^{\frac{13}{2}}$, this bound was tightened in 2018 by Hamilton and Moitra [20], showing $\operatorname{dist}^2(V,W) \leq 20\epsilon d^2$. While in 2011, Casazza and Cahill [7] showed $\mathcal{O}(\epsilon d) \leq \operatorname{dist}^2(V,W)$. Some speculate $\operatorname{dist}^2(V,W) \leq \mathcal{O}(\epsilon d)$ based on numerical results but as it stands the goal is to decrease the upper bound.

Remark 1.2.3. Implicitly, we require $\lim_{\epsilon\to 0} f(\epsilon, d) = 0$ for it is the case that for any ϵ-ENPF $P \in \mathbb{C}^{d\times n}$ and any ENPF $Q \in \mathbb{C}^{d\times n}$, we have $\|P-Q\|^2 \leq Cd$. This is simply because we can sum over rows instead of columns and the magnitude of any row is bounded by $\sqrt{1+\epsilon}$. Then, by possibly swapping a row of Q with its negative, we can assume the j^{th} row of P dots non-negatively with the j^{th} row of Q. Then we are simply adding hypotenuse lengths and have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{d}|P_{j,k}-Q_{j,k}|^2 &= \sum_{j=1}^{d}\sum_{k=1}^{n}|P_{j,k}-Q_{j,k}|^2\\ &\leq \sum_{j=1}^{d}(2+\epsilon)\\ &= (2+\epsilon)d.\end{aligned}$$

The struggle lies in showing the rows of Q are close to the rows of P, but hopefully it's now conceivable that $\mathrm{dist}^2(V,W) \leq \mathcal{O}(\epsilon d)$.

There is also interest in applying symplectic geometry to the Paulsen Problem [15]. In Section 6.5, we introduce yet another function whose minima are ENPFs and importantly this function is strictly convex. This may open techniques from convex optimization, giving new bounds. It appears a similar idea was attempted in [2] with a slightly different potential function and without explicit connection to symplectic or algebraic geometry. Hopefully this work helps bridge that gap.

1.3 Main Results and Outline of Proofs

The Stiefel manifold is the set of Parseval n-frames on $\mathbb{C}^d$: $St(d,n) = \{F \in \mathbb{C}^{d\times n} \mid FF^* = I_d\}$. Define the *normalizing potential* $\mu : St(d,n) \to \mathbb{R}$ by

$$\mu(F) = \sum_k ||f_k||^4.$$

The first main theorem of this paper is

Theorem 1.3.1. *Fix $d,n \in \mathbb{N}$ with $d \leq n$ and consider the gradient flow*

$$\Gamma : St(d,n) \times [0,\infty) \to St(d,n)$$

defined by the differential equation

$$\Gamma(F_0, 0) = F_0, \quad \frac{d}{dt}\Gamma(F, t) = -\nabla\mu(\Gamma(F, t)) \tag{1.2}$$

If F_0 is full spark, then $F_\infty = \lim_{t\to\infty} \Gamma(F_0, t)$ is an equinorm Parseval frame.

The path $\Gamma(F, t)$ on $St(d, n)$ is horizontal with respect to the projection to the Grassmannian, $Gr(d, n)$, which is both a smooth complex manifold and a complex algebraic variety. This allows us to use tools from algebraic geometry on $Gr(d, n)$ to understand the path on $St(d, n)$. The main tool for this problem is the GIT quotient. With this in mind the outline of the proof of Theorem 1.3.1 is as follows:

1. Show the tangent vector $\frac{d}{dt}\Gamma(F, t) = -\nabla\mu(\Gamma(F, t))$ is horizontal with respect to the projection $St(d, n) \to Gr(d, n)$. Thus projecting to the Grassmannian does not distort the path in any essential way. (Section 3.2.1.)

2. Show $G = \{(g_1, \cdots, g_n) \mid g_i \in \mathbb{C}^\times, \prod g_i = 1\}$ acts on $Gr(d, n)$ such that gradient flow is contained in the orbit of this action, thus the limit of the flow is in the closure of the orbit, $F_\infty \in \overline{GF_0}$. (Section 3.2.2.)

3. Identify the critical points of μ and show minima correspond to ENPFs. (Section 3.2.3.)

4. Use the Hilbert-Mumford Criterion (HMC) to show non-minimizing critical points (CPs) are *unstable* under G, while ENPFs are semistable. (Section 4.3.2.)

5. Use the HMC to show that every full spark frame is semi-stable under this action and therefore flows to minimizing CPs in the limit. (Section 5.)

The other results are empirical, and inspired by work of Kempf–Ness on special functions. Let $F \in \mathbb{C}^{d\times n}$ and $I \subset \{1, \cdots, n\}$ such that $|I| = d$. Define F_I to be the $d \times d$ submatrix of F using the columns f_i with $i \in I$. Define $P_F : \mathbb{R}^n \to \mathbb{R}$ by

$$P_F(t_1, \cdots, t_n) = \sum_{k=1}^{\binom{n}{d}} e^{2t_{I_k}} \det(F_{I_k} F_{I_k}^*),$$

where $e^{2t_{I_k}} = \prod_{i \in I_k} e^{2t_i}$ and the sum is over all subsets of $\{1, \ldots n\}$ of cardinality d.

Theorem 1.3.2. *(See Theorem 6.4.1) For a fixed full spark frame $F \in St(d, n)$, minimizing P_F subject to the constraint $\sum_k t_k = 0$ produces an ENPF.*

While we can prove this is true, there is a computational trade-off in that P_F has $\binom{n}{d}$ many terms.

Finally, we use a similar construction but with only n terms. Fix $F \in St(d, n)$ and define $N_F : \mathbb{R}^n \to \mathbb{R}$ by

$$N_F(t) = \sum_{k=1}^{n} e^{4t_k} ||f_k||^4$$

Thinking of t as an element of $\mathfrak{g}$, the lie algebra of $G = \{(g_1, \cdots, g_n) \mid g_i \in \mathbb{C}^\times, \prod g_i = 1\}$, we impose the constraint that $\sum_k t_k = 0$. Since N_F is convex, we can then use Lagrange Multipliers to find that the k^{th} component of the minimum of N_F is given by

$$\frac{1}{n}\left(\sum_j \ln ||f_j||\right) - \ln ||f_k|| = t_k^* \tag{1.3}$$

This gives us a new algorithm:

- Initialize a Parseval frame, F_0. Then for $k = \{0, 1, \cdots, \}$ do

    1. Find t^*, the minimum of p_{F_k}.

    2. Project $F_k e^{t^*}$ onto $St(d, n)$, call the projection F_{k+1}.

    3. Repeat steps 1 and 2 until F_k is suitably ENPF.

This method of finding ENPFs seems to converge exponentially as seen in Figure 6.1. While the details of the path remain unknown, we do compute the tangent vector corresponding to the group action by e^{t^*} and our hope is that this contributes to the body of research looking into the Paulsen problem and transformations that produce radially isotropic vectors.

Chapter 2

Background on Frames

The analysis operator $F^* : V \to \ell^2(I)$ given by $F^*(v) = (\langle f_n, v\rangle)_{n\in I}$ may be thought of as a tall skinny matrix, whose rows are the complex conjugates of the f_k. We use F interchangeably with $\{f_k\}$ as they convey the same data.

Recall that a frame satisfies

$$A||v||^2 \leq \sum_k |\langle f_k, v\rangle|^2 \leq B||v||^2,$$

where the constants A and B are called *frame bounds*. Let's briefly talk about what these bounds give us.

- The lower frame bound: The fact that $A > 0$ implies $\{f_k\}$ is a spanning set. (If $\langle f_k, v\rangle = 0$ for all k then $||v|| = 0$, so $v = 0$.). Furthermore, the map

  $$F^*(v) = (\langle f_1, v\rangle, \langle f_2, v\rangle, \langle f_3, v\rangle, \cdots) \in \ell^2(I)$$

  is injective for the same reason. We can thus form a bijection with the image of F^*. That is, we know there exists a way to reconstruct v from the data of $F^*(v)$.

- The upper frame bound: The upper bound shows the map $F^* : V \to \ell^2(I)$ is a bounded linear map.

- Tight frames: If $A = B$ we say the frame is *tight*. Tight frames are of special importance because $\frac{1}{\sqrt{A}}F^* : V \to \ell^2(I)$ is an isometry onto its image. Moreover, preserving norms will also preserve inner products via the polarization identity, that is:

$$\langle v, y\rangle_V = \frac{1}{4}\left(||v+y||_V^2 - ||v-y||_V^2 - i||v+iy||_V^2 + i||v-iy||_V^2\right) \tag{2.1}$$

$$= \frac{1}{4A}\Bigg(||F^*(v) + F^*(y)||_{\ell^2}^2 - ||F^*(v) - F^*(y)||_{\ell^2}^2 + \cdots \tag{2.2}$$

$$- i||F^*(v) + iF^*(y)||_{\ell^2}^2 + i||F^*(v) - iF^*(y)||_{\ell^2}^2 \Bigg) \tag{2.3}$$

$$= \frac{1}{A}\left\langle F^*(v), F^*(y)\right\rangle_{\ell^2}. \tag{2.4}$$

The fact that tight frames preserve inner products is the machinery behind the proof of Proposition 2.0.1. In 1986 Daubechies, Grossmann, and Meyer applied frames to signal processing [12] and emphasized the use of overcomplete systems $\{f_k\}$ satisfying

$$v = \sum_k \langle f_k, v\rangle f_k \tag{2.5}$$

for all v in some Hilbert space H. As Proposition 2.0.1 implies, frames satisfying equation (2.5) are tight frames with frame constant $A = 1$. As we said previously, these are called Parseval frames. Let's understand equation (2.5) as a matrix equation:

$$v = \sum_k \langle f_k, v\rangle f_k = \begin{bmatrix} f_1 & f_2 & \cdots \end{bmatrix} \begin{bmatrix} \langle f_1, v\rangle \\ \langle f_2, v\rangle \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} f_1 & f_2 & \cdots \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} -\overline{f_1}- \\ -\overline{f_2}- \\ \vdots \end{bmatrix}}_{F^*} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}.$$

So we find that a Parseval frame is one for which

$$FF^* = I. \tag{2.6}$$

By equation (2.6), or equivalently equation (2.5), it's often said that Parseval frames are generalizations of orthonormal bases.[1] By the same logic, looking at Proposition 2.0.1, a tight frame is one satisfying

$$FF^* = AI \tag{2.7}$$

and with a simple rescaling of $f'_k = \frac{1}{\sqrt{A}} f_k$ we see that all tight frames may be rescaled to Parseval frames.

Proposition 2.0.1. *If $\{f_k\}$ is a finite tight frame such that*

$$||v||^2 = \frac{1}{A} \sum_k | \langle f_k, v \rangle |^2, \tag{2.8}$$

then

$$v = \frac{1}{A} \sum_{k \in I} \langle f_k, v \rangle f_k.$$

Proof. By simply rescaling $f'_k = \frac{1}{\sqrt{A}} f_k$ we may, without loss of generality, assume $A = 1$. The chain of equalities

$$\sup_{||g||=||v||} \langle v, g \rangle = ||v||^2 = \sum_k | \langle f_k, v \rangle |^2 \tag{2.9}$$

hints at defining $g = \sum_{k \in I} \langle f_k, v \rangle f_n$. For, if such a g exists, then

$$\begin{aligned} \langle v, g \rangle &= \sum_{k \in I} \langle v, f_k \langle f_n, v \rangle \rangle \\ &= \sum_{k \in I} \langle v, f_k \rangle \langle f_k, v \rangle \\ &= \sum_{k \in I} \overline{\langle f_k, v \rangle} \langle f_k, v \rangle \\ &= \sum_k | \langle f_k, v \rangle |^2 \\ &= ||v||^2. \end{aligned}$$

[1]A word of caution about that colloquialism, $||f_k|| \neq 1$ in most cases and so the word orthonormal may be somewhat misleading.

It remains to show $g = v$. By the Reisz Representation Theorem, if $\langle x, v\rangle = \langle x, g\rangle$ for all $x \in V$ then we know $g = v$, and this will be our plan of attack. Rather than compute $\langle x, g\rangle$ for an arbitrary x, we use the fact that $\{f_n\}$ is a spanning set and compute

$$\langle f_m, g\rangle = \left\langle f_m, \sum_{k\in I} \langle f_k, v\rangle f_n \right\rangle \tag{2.10}$$

$$= \sum_{k\in I} \langle f_k, v\rangle \langle f_m, f_k\rangle \tag{2.11}$$

$$= \sum_{k\in I} \langle f_k, v\rangle \overline{\langle f_k, f_m\rangle} \tag{2.12}$$

$$= \langle F(f_m), F(v)\rangle \tag{2.13}$$

$$= \langle f_m, v\rangle . \tag{2.14}$$

Equality (2.13) comes from the ℓ^2 inner product, which we define as conjugate linear in the first term, i.e. $\langle a, b\rangle = \sum_i \overline{a_i} b_i$. Since this holds for all f_m we get that $\langle x, v\rangle = \langle x, g\rangle$ for all $x \in V$ and so $g = v$. □

Remark 2.0.2. While we stated Proposition 2.0.1 for finite frames, it does hold in general Hilbert spaces [11, 42]. For the rest of the paper we will concern ourselves with finite frames, and so that was emphasized here.

2.1 Finite Frames

A *frame* on $\mathbb{C}^d$ is a linear spanning set $\{f_1, \cdots, f_n\}$ (notice that $d \leq n$ by necessity). By spanning $\mathbb{C}^d$ we satisfy the lower frame bound and by only having finitely many frame vectors we satisfy the upper frame bound. The definitions from before all carry over in matrix form. Thus, a frame is *tight* if $FF^* = AI$. Parseval frames are tight frames with $A = 1$. If all the f_n have the same norm, $||f_k|| = c$, then $\{f_k\}$ is an *equinorm frame*. The main theorem of this paper is about producing equinorm Parseval frames (ENPFs), frames that are both Parseval and equinorm. In the context of ENPFs, the norm c is completely determined in the finite dimensional case by a simple computation:

$$d = tr(I_d) = tr(FF^*) = tr(F^*F) = \sum_{i=1}^{n} ||f_i||^2 = nc. \tag{2.15}$$

If we desire a *unit norm tight frame* (UNTF), that is a tight frame for which $||f_i|| = 1$ for all i, then it follows that $A = \frac{n}{d}$ by an analogous computation to eq. (2.15). The quantity of $\frac{n}{d}$ is called the *redundancy* of a given frame for obvious reasons. If a frame is both unit norm and Parseval, it is an orthonormal basis.

Using the notation from [33], let $\mathcal{F}_{\boldsymbol{\lambda}}^{n,d}(\mathbf{r}) \subset \mathbb{C}^{n\times d}$ be the set of frames $F \in \mathbb{C}^{n\times d}$ such that

1. The eigenvalues of F^*F are $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_d)$.

2. The list of column magnitudes $(||f_1||, \cdots, ||f_n||) = \mathbf{r}$.

We allow the possibility that $\mathcal{F}_{\boldsymbol{\lambda}}^{n,d}(\mathbf{r}) = \emptyset$. As equation (2.15) shows, there is a balancing act between the eigenvalues of FF^* and the sum of the norms $\sum_{i=1}^{n} ||f_i||^2$. For F to be a frame it is necessary that

$$\sum_{i=1}^{n} r_i = \sum_{k=1}^{d} \lambda_k, \tag{2.16}$$

In [33] $\boldsymbol{r}$ and $\boldsymbol{\lambda}$ are called *admissible* when equation (2.16) and equation (2.17) are satisfied

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} \lambda_i, \quad \forall k \tag{2.17}$$

Equations (2.16) and (2.17) are equivalent to saying $\mathcal{F}_{\boldsymbol{\lambda}}^{n,d}(\mathbf{r}) \neq \emptyset$. In this notation, the set of all ENPFs is thus $\mathcal{F}_{(1,1,\cdots,1)}^{n,d}(\frac{d}{n}, \frac{d}{n}, \cdots, \frac{d}{n})$ is clearly admissible and thus non-empty.

A frame is said to be *full spark* if every collection of d-many f_i form a basis of $\mathbb{C}^d$. In [33], the authors show full spark frames have full measure within all admissible sets $\mathcal{F}_{\boldsymbol{\lambda}}^{n,d}(\mathbf{r})$. For the purposes of this paper we use the fact that the space $\mathcal{F}_{(1,1,\cdots,1)}^{n,d}(\frac{d}{n}, \frac{d}{n}, \cdots, \frac{d}{n})$ of ENPFs is full spark almost everywhere.

2.2 Frames Applications

Frames were introduced by [13] while investigating $\int f(x)e^{\lambda_n x}dx$ for $\lambda_n \neq 2\pi n$. Which is to say that frames began their journey in non-uniform sampling [3,13]. Non-uniform sampling allows one to sample more intensely on higher detail regions, useful in image compression and texture classification [40]. For these reasons the use of frames in discrete wavelet transforms make up the largest chapter of Daubechies' later book on wavelets [11], demonstrating the vital role frames play in image compression and analysis.

Not requiring linear independence is a strength of frames, as this enables frames to better pick up on important characteristics within data [4,6]. This is because a vector representing desirable traits can be added to any frame without having to remove any others, not true of bases. Redundancy in frames also makes them more stable to data reconstruction in the presence of erasures and noise [17–19]. Different types of frames are more suited to their various applications.

Unit-norm tight frames (UNTFs) and Equal-Norm Parseval Frames (ENPFs) are of particular interest as [18] showed equal-norm frames minimize mean squared error when encoding data with additive noise if and only if they are tight. ENPFs are also useful in multiple antenna code design [36], a technique which can have high channel capacities [16,37]. In the literature UNTFs are also called radially isotropic vectors (RIVs) which are used in statistics and probability [2,20].

Chapter 3

Geometric Background

3.1 Frame Spaces as Manifolds

The Stiefel manifold is defined by

$$St(d,n) = \{F \in \mathbb{C}^{d\times n} \mid FF^* = I_d\} \subset \mathbb{C}^{d\times n}.$$

This is clearly equivalent to the set of all Parseval n-frames on $\mathbb{C}^d$. Alternatively the Stiefel manifold has a quotient representation as $St(d,n) \cong \mathcal{U}(n)/\mathcal{U}(n-d)$, where $\mathcal{U}(n)$ is the group of $n\times n$ unitary matrices [14]. In the quotient representation, $\overline{F} \in \mathcal{U}(n)/\mathcal{U}(n-d)$ is the set of all possible $\mathcal{F} \in \mathcal{U}(n)$ whose first d rows are F, i.e. if $F = \left(\begin{array}{c|c} I_d & 0 \end{array} \right) \mathcal{F}$ then

$$\overline{F} = \left\{ \left(\begin{array}{c|c} I_d & 0 \\ \hline 0 & Q \end{array} \right) \mathcal{F} \;\middle|\; Q \in \mathcal{U}(n-d) \right\}. \tag{3.1}$$

Notice that if F is a Parseval frame then so is UF, where U is a unitary $d\times d$ matrix, since $(UF)(UF)^* = UFF^*U^* = UI_dU^* = UU^* = I_d$. Looking at the algebra, UF is an operation on F that preserves row-orthogonality and thus $(UF)(UF)^* = I_d$ is obvious. Geometrically, the map $f_i \mapsto Uf_i$ is a unitary transformation of our frame. It thus preserves all the relative positions of the frame vectors and their magnitudes and these are the key geometric feature for a statement like $x = \sum_k \langle f_k, x\rangle f_k$ to hold. Indeed, since $x = \sum_k \langle f_k, x\rangle f_k$ holds for all x, in particular it holds for U^*x:

$$U^*x = \sum_k \langle f_k, U^*x\rangle f_k \quad \leftrightarrow \quad x = \sum_k \langle Uf_k, x\rangle Uf_k.$$

Geometrically, the relative position of x to Uf_i is the same as the relative position of f_i to U^*x because U is unitary. So F being Parseval implies UF is Parseval. Looking at $\{UF \mid U \in \mathcal{U}(d)\}$ naturally leads us to the Grassmannian.

Let $Gr(d,n)$ denote the Grassmannian of d-dimensional subspaces of $\mathbb{C}^n$. Elements of $Gr(d,n)$ may be thought of as rank d projectors, or the d-dimensional subspace they project onto. It is known in the frame community that F is Parseval iff its Gramian, F^*F, is an orthogonal projection [42]. In this case we are projecting onto the rowspace of F (or rather its complex conjugate). The quotient map from the Stiefel manifold to the Grassmannian can be defined in multiple ways. Notice that left multiplication by $U \in \mathcal{U}(d)$ preserves the row space and preserves the Gramian, $F^*F = F^*U^*UF = (UF)^*UF$. The map $F \mapsto [F] \in Gr(d,n)$ can be thought of as $[F] = F^*F$, i.e. "projection onto the rowspace of F" or $[F] = \{UF \mid U \in \mathcal{U}(d)\}$. Each interpretation gives different insight. The latter leads to the quotient representation of $Gr(d,n)$:

$$Gr(d,n) \cong St(d,n)/\mathcal{U}(d) \cong \mathcal{U}(n)/(\mathcal{U}(d) \times \mathcal{U}(n-d)).$$

Analogous to (3.1), let $\mathcal{F} \in \mathcal{U}(n)$ be any unitary matrix whose first d rows span the row space of F. Then

$$[F] = \left\{ \left(\begin{array}{c|c} Q_d & 0 \\ \hline 0 & Q_{n-d} \end{array} \right) \mathcal{F} \;\middle|\; Q_d \in \mathcal{U}(d),\ Q_{n-d} \in \mathcal{U}(n-d) \right\}. \tag{3.2}$$

We thus use F^*F and $[F]$ interchangeably, as both are ultimately referring to the d-dimensional row space of F, i.e. the point on the Grassmannian associated to F.

3.2 Gradient on Frame Manifolds

Remark 3.2.1. This section heavily relies on computations outlined in [14]. However, their convention was $F \in \mathbb{R}^{n \times d}$ while we use $F \in \mathbb{C}^{d \times n}$, thus all formulas seen here are the conjugate transpose of the equivalent formula in [14].

As written, the normalizing potential, $\mu(F) = \sum_{k=1}^{n} ||f_k||^4$, is a map $\mu : \mathbb{C}^{d\times n} \to \mathbb{R}$. For a function $f : \mathbb{C}^n \to \mathbb{R}$, the Wirtinger gradient $\frac{\partial f}{\partial \overline{z}} : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\left(\frac{\partial f}{\partial \overline{z_i}}, \cdots, \frac{\partial f}{\partial \overline{z_n}}\right)$. Used in the signal processing community by Candés et. al. to analyze the phase retrieval problem [9] and again by Mixon et. al. [30] to analyze gradient flow of the frame potential, one can check that

$$\nabla f = 2\frac{\partial f}{\partial \overline{z}}$$

where ∇f is the best real linear approximation to

$$f(z + \Delta z) - f(z) = \textbf{Re} \left\langle \nabla f(z), \Delta z \right\rangle .$$

For our problem, this implies the gradient of the normalizing potential is

$$\nabla \mu(F) = 2\frac{\partial \mu}{\partial \overline{M}}(F). \tag{3.3}$$

Computing $\frac{\partial \mu}{\partial \overline{M}}(F)$, we begin with a single elementary matrix. Denote by E_{kj} the matrix with 1 in the k, j entry and 0's else, then

$$\begin{aligned}
\frac{\partial \mu}{\partial \overline{E}_{kj}}(F) &= \frac{\partial}{\partial \overline{E}_{kj}} \sum_i ||f_i||^4 \\
&= \frac{\partial}{\partial \overline{E}_{kj}} \left\langle f_j, \overline{f_j} \right\rangle^2 \\
&= \frac{\partial}{\partial \overline{E}_{kj}} \left(\sum_p F_{pj}\overline{F_{pj}} \right)^2 \\
&= 2||f_j||^2 \cdot \frac{\partial}{\partial \overline{E}_{kj}} \left(\sum_p F_{pj}\overline{F_{pj}} \right) \\
&= 2||f_j||^2 \cdot F_{kj}.
\end{aligned}$$

We can see that varying the k, j entry only affects the k, j entry, but the appearance of $||f_j||^2$ hints that we should consider varying the j-th column. Indeed,

$$\sum_{k=1}^{d} \frac{\partial \mu}{\partial \overline{E_{k,j}}}(F) = 2||f_j||^2 \cdot f_j. \tag{3.4}$$

Seeing this behavior on the columns of F suggests we should define

$$D = \begin{pmatrix} ||f_1||^2 & 0 & \cdots & 0 \\ 0 & ||f_2||^2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & ||f_n||^2 \end{pmatrix}$$

and using Equation (3.3) we may write

$$\frac{\partial \mu}{\partial \overline{M}}(F) = 2FD. \tag{3.5}$$

By equations (3.5) and (3.3) we have

$$\nabla \mu(F) = 4FD. \tag{3.6}$$

Even though, via restriction, we may think of $\mu : St(d, n) \to \mathbb{R}$, $\nabla \mu(F)$ is likely not to be a valid tangent vector in $T_F St(d, n)$. Nonetheless, because $St(d, n) \subset \mathbb{C}^{d \times n}$, work in [14] gives a formula for projection onto $T_F St(d, n)$:

$$\nabla \mu_S(F) = \nabla \mu(F) - F \left(\nabla \mu(F)\right)^* F, \tag{3.7}$$

where $\nabla \mu_S(F) \in T_F St(d, n)$. Substituting $\nabla \mu(F) = 4FD$ into Equation (3.7) we get

$$\nabla \mu_S(F) = 4FD - F4DF^*F = 4FD \left(I_n - F^*F\right). \tag{3.8}$$

This is the gradient of μ on $St(d,n)$ at F.

3.2.1 The Stiefel Gradient is also the Grassmannian Gradient

Remark 3.2.2. For $F \in St(d,n)$ we use $[F] \in Gr(d,n)$ to be the corresponding equivalence class on the Grassmannian. We use $\nabla\mu_S(F)$ when referring to $\nabla\mu_S(F) \in T_F St(d,n)$ and $\nabla_G \mu([F])$ when referring to $\nabla\mu_G([F]) \in T_{[F]}Gr(d,n)$. This section is about the fact that $\nabla\mu_S(F) = \nabla\mu_G([F])$, when thought of as elements of $\mathbb{C}^{d\times n}$.

Since $||Uf_i|| = ||f_i||$ for all $U \in \mathcal{U}(d)$ we have $\mu_G([F]) = \mu_S(F)$ is well defined and so μ_G is a function on the Grassmannian. Said with a diagram, we have

$$\begin{array}{ccc} St(d,n) & \xrightarrow{\;q\;} & Gr(d,n) \\ & \searrow^{\mu_S} & \downarrow{\scriptstyle \mu_G} \\ & & \mathbb{R} \end{array}$$

which is to say $\mu_G = \mu_S \circ q$. It follows that $q_*(\nabla\mu_S(F)) = \nabla\mu_G(F)$ and $q_*\left(-\nabla\mu_S(\Gamma(F,t))\right) = -\nabla\mu_G(\Gamma(F,t))$. That is, our flow on $St(d,n)$ has an identical copy on $Gr(d,n)$, of which we may use algebraic properties on the Grassmannian to deduce truths about the flow on the Stiefel manifold. In some sense, this is enough to move on to the next section. However, for those who prefer something more tangible, there is an equation in [14] for $\nabla\mu_G([F]) \in T_{[F]}Gr(d,n)$ which is the first equality in (3.9) below.

$$\nabla\mu_G([F]) = \nabla\mu(F)\left(I_n - F^*F\right) = 4FD\left(I_n - F^*F\right) = \nabla\mu_S(F). \tag{3.9}$$

Thus, no diagrams are needed and we can directly see $\nabla\mu_G([F]) = \nabla\mu_S(F)$ using equations from [14].

Remark 3.2.3. From here on out we avoid subscripts and just write $\nabla\mu(F) \in T_F St(d,n)$ as this tangent vector is what really matters. Therefore moving forward

$$\nabla\mu(F) = 4FD(I - F^*F).$$

3.2.2 Gradient Flow is Captured by Group Actions

Let $G = \{(g_1, \cdots, g_n) \mid g_i \in \mathbb{C}^\times, \prod g_i = 1\}$. Thinking of elements of G as diagonal matrices, G acts on $Gr(d,n)$ via $g \cdot [F] = [Fg]$. Notice that $G \cdot [F]$ is a manifold because G is. It therefore makes sense to talk about $T_{[F]}(G \cdot F)$, the tangent space to $[F]$ on $G \cdot [F] \subset Gr(d,n)$. We will show the entire flow $\Gamma([F_0], t)$ on the Grassmannian stays in the group orbit $G \cdot [F_0]$ and by continuity $\lim_{t\to\infty} \Gamma([F_0], t) = [F_\infty] \in \overline{G \cdot [F_0]}$. Furthermore, this action lifts to a path on the Stiefel manifold in a nice way.

Proposition 3.2.4. *For G acting on $Gr(d,n)$ as defined previously, the entire path $\Gamma([F_0], t)$, where $\frac{d}{dt}\Gamma(F,t) = -\nabla\mu(\Gamma(F_0,t))$, is contained in $G \cdot [F]$.*

Proof. We will show $\nabla\mu(F) \in T_{[F]}(G \cdot [F])$, which implies $\Gamma(F,t) \in G \cdot F$ for all t.

Recall that on $Gr(d,n)$, $[UF] = [F]$ for all $U \in \mathcal{U}(d)$ and

$$\frac{1}{4}\nabla\mu([F]) = FD - FDF^*F = (-FDF^*)F + FD.$$

Notice that because D is real valued, FDF^* is Hermitian and hence $U(t) = ie^{i(FDF^* - \frac{d}{n}I)t} \in \mathcal{U}(d)$. By equation (2.15), $\operatorname{tr}(D) = d$ and we define $R(t) = e^{-i(D - \frac{d}{n}I)t} \in G$. Now the path defined by $[U(t)FR(t)]$ on $Gr(d,n)$ is the same as the path defined by $[FR(t)]$ on $Gr(d,n)$. Furthermore, using [14] we may compute the derivative of $U(t)FR(t)$ as a matrix in $\mathbb{C}^{d\times n}$ and so long as it is of the correct form, it will be in $T_{[F]}Gr(d,n)$. So we compute

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} ie^{i(FDF^*-\frac{d}{n}I)t}Fe^{-i(D-\frac{d}{n}I)t} &= \left[\frac{d}{dt}\left[ie^{i(FDF^*-\frac{d}{n}I)t}\right]Fe^{-iDt} + ie^{iFDF^*t}F\frac{d}{dt}\left[e^{-i(D-\frac{d}{n}I)t}\right]\right]_{t=0} \\
&= \left[-(FDF^* - \frac{d}{n}I)e^{i(FDF^*-\frac{d}{n}I)t}Fe^{-i(D-\frac{d}{n}I)t} + e^{i(FDF^*-\frac{d}{n}I)t}F(D-\frac{d}{n}I)e^{-i(D-\frac{d}{n}I)t}\right]_{t=0} \\
&= -(FDF^* - \frac{d}{n}I)F + F(D - \frac{d}{n}I) \\
&= -FDF^*F + FD + \frac{d}{n}IF - F\frac{d}{n}I \\
&= \frac{1}{4}\nabla\mu(F).
\end{aligned}$$

Hence, $\nabla\mu(F) = \frac{d}{dt}\Big|_{t=0} U(4t)FR(4t)$ and therefore $\nabla\mu(F) \in T_{[F]}(G \cdot [F])$. $\square$

3.2.3 Critical Points and Extrema

When $\nabla\mu = 0$ we have $FD(I_n - F^*F) = 0$ or equivalently

$$FD = FDF^*F. \tag{3.10}$$

Defining $FDF^* = M$, equation (3.10) says $FD = MF$. Looking column-wise, equation (3.10) is an eigenvalue/eigenvector equation:

$$||f_k||^2 f_k = Mf_k. \tag{3.11}$$

The f_k are thus eigenvectors for M with eigenvalues given by their squared norm, $||f_k||^2$. Since $M = M^*$, eigenspaces of M are orthogonal. We can thus permute columns and choose a suitable basis to assume F and D have block diagonal structure:

$$F = \begin{pmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_p \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_p I_{n_p} \end{pmatrix},$$

where $F_i \in \mathbb{C}^{d_i \times n_i}$ are the frame vectors in the i^{th} eigenspace. Since $FF^* = I_d$ and F is block diagonal, it follows that $F_i F_i^* = I_{d_i}$ and $\lambda_i = \frac{d_i}{n_i}$. That is, each F_i is an ENPF on $\mathbb{C}^{d_i}$.

Remark 3.2.5. Critical points of μ therefore correspond to block-equinorm Parseval frames. We want to emphasize that equinormality only occurs on blocks of F, i.e. $||f_i|| = ||f_j||$ when $f_i,\ f_j \in F_k$ for some block F_k. While $F = \operatorname{diag}(F_1, \cdots, F_p)$ is Parseval, it is not ENPF unless there is only one block, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_p$.

At critical points, using the notation $F = \operatorname{diag}(F_1, \cdots, F_p)$ and $D = \operatorname{diag}(\lambda_1 I_{n_1}, \cdots, \lambda_p I_{n_p})$, we get many equivalent ways to write $\mu(F)$. The one of interest to us is

$$\mu(F) = \sum_{i=1}^{p} n_i \lambda_i^2. \tag{3.12}$$

Dividing $\mu(F) = \sum_{i=1}^{p} n_i \lambda_i^2$ by $n = \sum_i n_i$ and applying Jensen's inequality we have

$$\frac{\mu(F)}{n} = \sum_{i=1}^{p} \frac{n_i}{n} \lambda_i^2 \geq \left(\sum_{i=1}^{p} \frac{n_i \lambda_i}{n} \right)^2 = \left(\frac{1}{n} \sum_{i=1}^{p} d_i \right)^2 = \left(\frac{d}{n} \right)^2,$$

where equality is achieved exactly when $\lambda_1 = \lambda_2 = \cdots = \lambda_p$, i.e. when F is ENPF. It follows that ENPFs minimize μ.

Chapter 4

Algebraic Geometry

Remark 4.0.1. Given the many applications of frames, the reader is not expected to know algebraic geometry and in what follows we attempt to give enough background to introduce the concepts of the geometric invariant theory at the level of polynomial rings. Standard material on what follows can be found in [21, 41]. Basic knowledge of $\mathbb{CP}^n$ and familiarity with some terminology of abstract algebra is assumed. We also use without proof the first isomorphism theorem.

Theorem 4.0.2 (First Isomorphism Theorem)**.** *If $f : R \to S$ is a homomorphism of rings (or groups) then*

- $\ker(f)$ *is an ideal (or normal subgroup) of* R*.*
- *Im*(f) *is a subring (subgroup) of* R*.*
- $R/\ker(f) \cong$ ***Im***(f).

While the language of category theory cannot be avoided, as use of the Spec functor must appear in certain definitions, we try to give a running example of how maximal ideals of polynomial rings correspond to points of our space. The proof of Theorem 1.3.1 relies on showing full spark frames are semi-stable, and thus limit to semi-stable limit points. Semi-stability is a notion from geometric invariant theory and it is a sufficient and necessary condition on $v \in V$ for the set $\overline{G \cdot v}$ to be an element of the quotient $V//G$.

4.1 Brief Introduction to Algebraic Geometry

4.1.1 Affine Coordinate Rings and Spec

Everything we do is over $\mathbb{C}$ unless stated otherwise. Let $\mathbb{C}[x_1, \cdots, x_n]$ denote the ring of polynomials over $\mathbb{C}$ in n variables. If $f_i \in \mathbb{C}[x_1, \cdots, x_n]$ for $i \in \{1, \cdots, k\}$, then $(f_1, \cdots, f_k)$ will be used to denote the ideal generated by $f_1, \cdots, f_k$. We also use $\{f = 0\}$ to mean the zero set of the polynomial f. We may think of an *affine variety* as the common zero set of some collection of polynomials,

$$V = \{x \in \mathbb{C}^n \mid f_m(x) = 0,\ m = 1, \cdots, k\} = \bigcap_{m=1}^{k} \{f_m = 0\}.$$

The *coordinate ring* [21] associated to V, denoted $\mathbb{C}(V)$, is defined by

$$\mathbb{C}(V) = \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_1, \cdots, f_k)}.$$

Notice that since $\mathbb{C}[x_1, \cdots, x_n]$ is a ring and $(f_1, \cdots, f_k)$ is an ideal, $\mathbb{C}(V)$ is a ring.

Example 4.1.1. Consider the curve $y = \frac{1}{x}$, call this curve V. $V \subset \mathbb{C}^2$ is an affine variety because it is the zero set of the polynomial $xy - 1$. Since $y = 1/x$ on V, any polynomial $p \in \mathbb{C}[x, y]$ may be written as $p \in \mathbb{C}[x, \frac{1}{x}]$ by substituting $y = 1/x$. Indeed, $\mathbb{C}(V) = \mathbb{C}[x, x^{-1}] = \mathbb{C}[x, y]/(xy - 1)$.

Moreover, given a coordinate ring $\mathbb{C}(V)$ there exists a way to recover V via the Spec functor (though technically we just look at $\max \operatorname{Spec}$). The definition being $\operatorname{Spec} \mathbb{C}(V) := \{\mathfrak{p} \subset \mathbb{C}(V) \mid \mathfrak{p} \text{ is a prime ideal}\}$. We do not need Spec in full rigor nor abstraction. We will briefly say how to recover V and then move on to the projective case, which is more relevant to our problem.

Notice that evaluation at a point, $(a_1, a_2, \cdots, a_n) \in \mathbb{C}^n$, is a ring homomorphism $ev_a : \mathbb{C}(V) \to \mathbb{C}$. Define $\ker(ev_a) = \mathfrak{m}_a$. By the first isomorphism theorem, $\mathfrak{m}_a$ is an ideal. We can see $\mathfrak{m}_a$ is a maximal ideal by the fact that $\mathbb{C}(V)/\mathfrak{m}_a \cong \mathbb{C}$ and $\mathbb{C}$ is a field. In words, the kernel of ev_a is the set of all polynomials that evaluate to zero at a. Thus, $p \in \mathfrak{m}_a$ iff some $x_i - a_i$ divides p. It follows that $\mathfrak{m}_a$ is the ideal generated by $x_1 - a_1, \cdots, x_n - a_n$. We write this fact symbolically as $\mathfrak{m}_a = (x_1 - a_1, \cdots, x_n - a_n)$.

What we see here is that maximal ideals of $\mathbb{C}(V)$ correspond to points in V. So, it is conceivable that one may recover V through a collection of maximal (and therefore prime) ideals of $\mathbb{C}(V)$. We are glossing over a lot of intricacies because they will not apply to this paper. The motivation to sharing this is to show that knowing the polynomial ring that defines your affine variety is as good as knowing your affine variety thanks to the Spec functor, $V \leftrightarrow \operatorname{Spec}\mathbb{C}(V)$.

Definition 4.1.2. A function[2] $f : V \to \mathbb{C}$ is *G-invariant* if $f(g \cdot x) = f(x)$ for all $g \in G$. With a slight abuse of notation we abbreviate this as $f(G \cdot x) = f(x)$, emphasizing that f is well defined on G-orbits.

Remark 4.1.3. Assuming f is G-invariant then, by continuity, $f(w) = f(x)$ for all $w \in \overline{G \cdot x}$, since

$$f^{-1}(\{f(x)\}) = \{w | f(w) = f(x)\}$$

is a closed set containing $G \cdot x$, and therefore containing $\overline{G \cdot x}$.

4.1.2 GIT on Affine Varieties

We use $\mathbb{C}(V)^G = \{f \in \mathbb{C}(V) \mid f(G \cdot x) = f(x)\}$ to denote the ring of G-invariant maps on V. If V is a finite-dimensional complex vector space and G a reductive group acting linearly on V (e.g. $GL(V)$, $SL(V)$, and $(\mathbb{C}^\times)^n$), Geometric Invariant Theory constructs a type of quotient space denoted $V//G = \operatorname{Spec}\mathbb{C}(V)^G$. Again, if we think of the kernel of $ev_a : \mathbb{C}(V)^G \to \mathbb{C}$ then G-invariance implies $\ker(ev_a) = \ker(ev_{\overline{G \cdot a}})$ and so points in $V//G$ correspond to orbit closures in V. That is $q : V \to V//G$ is essentially the map $q(x) = \overline{G \cdot x}$.

Remark 4.1.4. This is the main idea to the GIT quotient but projective varieties will require a little more work. Similar to manifolds, functions (and therefore quotient maps) on projective varieties are defined on local charts with conditions about agreement/transition on the overlap. While locally the quotient maps appear to work the same as the affine case, there is a formal machinery we need to build up to make sure all the gluing works out. One could imagine this detail getting rather tedious. The work around is: Instead of defining functions on the variety directly, we define functions on global sections of line bundles. The global sections take care of the necessary gluing behind the scenes.

[2]This is all taking place in the land of algebra and so the word "function" may be replaced with polynomial throughout this section.

4.1.3 Projective Varieties

Polynomials on projective coordinates are not typically well defined because in projective space we identify $(x_0, \cdots, x_n) \sim (\lambda x_0, \cdots, \lambda x_n) = \lambda(x_0, \cdots, x_n)$. Consider the polynomial x_0 evaluated at $(1, 0, \cdots, 0)$. What does this even mean when $(1, 0, \cdots, 0) \sim (\lambda, 0, \cdots, 0)$? However, $\{f = 0\}$ is well defined for homogeneous polynomials because homogeneous polynomials satisfy $f(\lambda x) = \lambda^d f(x)$. Indeed, the zero set $\{x_0 = 0\}$ makes complete sense,

$$\{x_0 = 0\} = \{[0 : x_1 : \cdots : x_n] \mid x_i \in \mathbb{C}, \text{ at least one } x_j \neq 0\} = \mathbb{CP}^{n-1}.$$

Thus $\{x_0 = 0\}$ certainly should be a projective variety.

As an initial definition we will say a projective variety is the common zero set of a collection of homogeneous polynomials [21]. If

$$\left(\bigcap_{i=1}^{k} \{f_i = 0 \mid f_i \text{ is homogeneous}\}\right) = V \subset \mathbb{CP}^n$$

is a projective variety and $I = (f_1, \cdots, f_k)$ is the ideal generated by the f_i, then we define the coordinate ring $\mathbb{C}(V) = \frac{\mathbb{C}[x_0, \cdots, x_n]}{I}$. What is happening here is, we identify the projective variety V with its *affine cone* $\tilde{V} \subset \mathbb{C}^{n+1}$. That is, if $f \in \mathbb{C}(V)$ then $f(\lambda x_0, \cdots, \lambda x_n) = 0$ and so each point $[x_0 : \cdots : x_n] \in V \subset \mathbb{CP}^n$ corresponds to a line $\{\lambda(x_0, \cdots, x_n) \mid \lambda \in \mathbb{C}\} \subset \mathbb{C}^{n+1}$. The collection of these lines form the affine cone $\tilde{V} \subset \mathbb{C}^{n+1}$, and $\mathbb{C}(V) = \mathbb{C}(\tilde{V})$.

While this is great if V is explicitly defined as zero sets of homogeneous polynomials, sometimes we have to discover/create these relations. For example, $Gr(d, n)$ is not a priori defined by some collection $\{f_i = 0\}$, but the Plücker embedding $\rho : Gr(d, n) \to \mathbb{CP}^N$ allows us to identify $Gr(d, n)$ with its image $\rho(Gr(d, n))$, which is the zero set of the homogeneous polynomials called the *Plücker relations*. For projective varieties the coordinate ring depends on the embedding of the variety.

Example 4.1.5. If we denote points in $\mathbb{CP}^2$ with the tuple $[a : b : c]$, then $A = \{b^2 - ac = 0\} \subset \mathbb{CP}^2$ is a projective variety with coordinate ring $\mathbb{C}(A) = \frac{\mathbb{C}[a,b,c]}{(b^2-ac)}$. If we denote points in $V = \mathbb{CP}^1$ by

$[x : y]$ then $\mathbb{C}(V) = \mathbb{C}[x, y]$. One can check

$$\phi([x : y]) = [x^2 : xy : y^2] \text{ and } \phi^{-1}([a : b : c]) = \begin{cases} [1 : \frac{b}{a}] & a \neq 0 \\ [\frac{b}{c} : 1] & c \neq 0 \end{cases}$$

defines a homeomorphism $V \cong A$. However, $\mathbb{C}(V) = \mathbb{C}[x, y]$ is not isomorphic as a graded ring to $\mathbb{C}(A) = \frac{\mathbb{C}[a,b,c]}{(b^2-ac)}$. To see this, consider that the degree 1 polynomials in $\mathbb{C}(A)$ are generated by $a,\ b,$ and c with no degree 1 relation between $a,\ b$ and c. If $\psi : \mathbb{C}(A) \to \mathbb{C}(V)$ were an isomorphism then

$$\psi(a) = \alpha_a x + \beta_a y$$

$$\psi(b) = \alpha_b x + \beta_b y$$

$$\psi(c) = \alpha_c x + \beta_c y$$

and one may write $\psi(c) = \alpha\psi(a) + \beta\psi(b)$. However, applying ψ^{-1} yields the degree 1 relation $c = \alpha a + \beta b \in \mathbb{C}(A)$, a contradiction, so $\mathbb{C}(A) \ncong \mathbb{C}(V)$.

While it may be disheartening to see projective coordinate rings are not preserved by variety isomorphisms, the Proj construction (defined below) will still recover isomorphic varieties from non-isomorphic coordinate rings. Given a graded[3] ring $R = \bigoplus_d R_d$, the Proj construction is defined by

$$\operatorname{Proj} R = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a homogeneous prime ideal not containing } \bigoplus_{d>0} R_d\}.$$

The reason for not letting $\mathfrak{p}$ contain $\bigoplus_{d>0} R_d$ is because $[0 : \cdots : 0] \notin \mathbb{CP}^n$ and

$$\bigoplus_{d>0} R_d = (x_0, \cdots, x_n) = \mathfrak{m}_{[0:\cdots:0]} = \ker(ev_{[0:\cdots:0]}).$$

[3]All polynomial rings have a natural grading by degree.

Example 4.1.6. To build on Example 4.1.5, let $\mathbb{C}_d[x, y]$ denote the degree d polynomials in $\mathbb{C}[x, y]$. Then $\mathbb{C}[x, y] = \bigoplus_{d \geq 0} \mathbb{C}_d[x, y]$. Just as with Spec, we hope to recover V via the kernel of an evaluation map. If, as an initial guess, we define $\mathfrak{p} = \ker(ev_{(x_0,y_0)})$, we need to remember that $\mathfrak{p}$ needs to be a *homogeneous* ideal and therefore generated by homogeneous polynomials. Since $x - x_0$ and $y - y_0$ are not homogeneous, they won't do. Homogeneity implies $(-y_0 x + x_0 y) \subseteq \mathfrak{p}$. That is, the zero set containing the point (x_0, y_0) must contain the line through the origin that point lies on, so again we see the affine cone appear. Thus, "maximal" homogeneous ideals $\mathfrak{p} = (\alpha x + \beta y)$ correspond to lines through the origin and we can see $\mathfrak{p} = (\alpha x + \beta y)$ does not contain $\bigoplus_{d>0} \mathbb{C}_d[x, y]$. So $\mathrm{Proj}\, \mathbb{C}[x, y]$ seems to recover $\mathbb{CP}^1$ via maximal homogeneous prime ideals.

As for $\mathrm{Proj}\, \frac{\mathbb{C}[a,b,c]}{(b^2-ac)}$, we will again recover lines through the origin except primality of $\mathfrak{p}$ and the fact that $b^2 = ac$ will require these lines to obey $b^2 = ac$. For example, if $a \in \mathfrak{p}$ then $ac = b^2 \in \mathfrak{p}$ implies $b \in \mathfrak{p}$ by primality. In that case $(a, b) \subset \mathfrak{p}$ and this means $a = 0$ and $b = 0$ if we want $\mathfrak{p}$ to be maximal and homogeneous then c must be free and we are looking at the line $[0 : 0 : 1]$. Doing a thorough analysis like this would recover the image $\phi(V) = A$. So, despite non-isomorphic coordinate rings, we still recover the isomorphic varieties A and V through the Proj construction.

The non-isomorphic coordinate rings in Example 4.1.5 are a result of different line bundles over V. We elaborate on this in the next section but the reader may also jump to Section 4.1.5 to get a sufficient background on GIT for the purposes of this paper.

4.1.4 Line Bundles and Coordinate Rings of Projective Varieties

Remark 4.1.7. Our goal is to make sense of the following statement: *The coordinate ring of a projective variety is generated by symmetric powers of global sections of an ample line bundle over that variety.* (See [21] Chapter 2, Section 7.)

As a result, the GIT quotient for projective varieties depends on how the group acts on these global sections, whereas in the affine case the action on the variety was enough. In our case, all the formalism falls right out from the fact that we are using the Plücker embedding. Being an embedding, it captures everything we need and each of its coordinate functions are *G-equivariant*. Thus, for a reader less interested in the formalism, they may skip to Section 4.2.

Definition 4.1.8. [41] The *total space* L of a *line bundle* (over V) is a complex manifold with an open cover $\{U_i \times \mathbb{C} \mid U_i \subset V,\ i \in I\}$. Implicit in this definition is the fact that L has dimension one higher than V because $U_i \subset V$ and $L \subset \bigcup_i (U_i \times \mathbb{C})$. For a point $(u, z) \in U_i \times \mathbb{C}$, the last coordinate, z, may appear different on different charts, while u is something fixed in V. We thus require

- Transition functions f_{ij} on $(U_i \cap U_j) \times \mathbb{C}$ that are analytic, nowhere zero, satisfying $f_{ij} \circ f_{jk} = f_{ik}$, which implies $f_{ii} = Id_{U_i}$ and $f_{ij} = f_{ji}^{-1}$.
- A (projection) morphism $\pi : L \to V$.

In practice, the transition function f_{ij} tells you how to change z going from $U_j \times \mathbb{C}$ to $U_i \times \mathbb{C}$. So, if $u \in U_i \cap U_j$, then the point $(u, z) \in U_j \times \mathbb{C}$ is identified with the point $(u, f_{ij} z) \in U_i \times \mathbb{C}$.

Definition 4.1.9. A *global section* is a map $s : V \to \mathbb{C}$ such that $s|_{U_i}(p) = f_{ij} s|_{U_j}(p)$ for $p \in U_i \cap U_j$. The collection of all global sections of L on V is denoted $H^0(V, L)$.

Example 4.1.10. Consider $V = \mathbb{CP}^1$ and let $U_0 = \{[x : y] \mid x \neq 0\}$ and $U_1 = \{[x : y] \mid y \neq 0\}$. Define L via the transition functions $f_{10} z = \frac{y}{x} z$ and $f_{01} z = \frac{x}{y} z = f_{10}^{-1} z$. A global section is a function $s : V \to \mathbb{C}$ such that

$$s|_{U_0}(\underbrace{[1 : \frac{y}{x}]}_{p \in U_0}) = f_{10} s|_{U_1}(\underbrace{[\frac{x}{y} : 1]}_{p \in U_1}) \quad \text{equivalently} \quad \sum_{p \geq 0} \alpha_p (\frac{y}{x})^p = \frac{y}{x} \sum_{p \geq 0} \beta_p (\frac{x}{y})^p$$

Multiplying both sides of the last equality by x and we get

$$\alpha_0 x + \alpha_1 y + \frac{y}{x}(\text{stuff}) = \beta_0 y + \beta_1 x + \frac{x}{y}(\text{other stuff})$$

For this to hold for all $x, y \in \mathbb{C} \setminus \{0\}$, the terms involving $\frac{x}{y}$ or $\frac{y}{x}$ need to be zero while $\alpha_1 = \beta_0$ and $\beta_1 = \alpha_0$. This means $s([x : y]) = \alpha x + \beta y$. So, $H^0(V, L) = \{\alpha x + \beta y \mid \alpha, \beta \in \mathbb{C}\}$ consists of the degree 1 homogeneous polynomials and is generated by x and y. This line bundle is called the hyperplane bundle, also called the twisting sheaf, $\mathcal{O}_V(1)$, where we use subscripts to indicate the base space.

In Example 4.1.5 the hyperplane bundle showed up over V and over $A = \frac{\mathbb{CP}^2}{(b^2-ac)}$. While $\mathcal{O}_V(1) = \{\alpha x + \beta y\}$ is generated by x and y, $\mathcal{O}_A(1) = \{\alpha a + \beta b + \gamma c\}$ is generated by the monomials a, b, and c. We also had the map $\phi([x : y]) = [x^2 : xy : y^2] = [a : b : c]$, which gave rise to the relation $b^2 - ac = 0$. We can pullback $\mathcal{O}_A(1)$ by pulling back its generators: $\phi^*(a) = x^2$, $\phi^*(b) = xy$, $\phi^*(c) = y^2$. This produces a different line bundle over V, namely $\phi^*\mathcal{O}_A(1)$ which happens to be $\mathcal{O}_V(2) := \mathcal{O}_V(1) \otimes \mathcal{O}_V(1)$. These different line bundles gave rise to different coordinate rings on V. Namely

$$\begin{aligned} \mathcal{O}_V(1) &\leftrightarrow \mathbb{C}[x, y] \\ \mathcal{O}_V(2) &\leftrightarrow \frac{\mathbb{C}[x^2, xy, y^2]}{\underbrace{((xy)^2 - x^2y^2)}_{\text{Relations From Pullback}}} \cong \frac{\mathbb{C}[a, b, c]}{(b^2 - ac)} \end{aligned}$$

The point of the last paragraph is: to define the coordinate ring over a projective variety V, we need an embedding of $V \hookrightarrow \mathbb{CP}^n$. Such embeddings are of the form

$$x \overset{s}{\mapsto} [s_0(x) : \cdots : s_n(x)].$$

Define the pullback of $\mathcal{O}_{\mathbb{CP}^n}(1)$ to be $L = s^*\mathcal{O}_{\mathbb{CP}^n}(1)$. Then, by construction, $s_0, \cdots, s_n$ generate $H^0(V, L)$. As we saw in Example 4.1.5, there may be relations between the $s_0(x), \cdots, s_n(x)$. So, we define "the" coordinate ring of a projective variety V to be

$$\mathbb{C}(V) = \frac{\mathbb{C}[s_0, \cdots, s_n]}{(\text{polynomial relations in } s_i)}. \tag{4.1}$$

In general, if $\phi : V \to \mathbb{CP}^n$ is a morphism then the pullback $L = \phi^*\mathcal{O}_{\mathbb{CP}^n}(1)$ will be a line bundle on V (see [21] chapter 2 Theorem 7.1). If ϕ is an embedding then generators of $H^0(V, L) = \phi^*(\mathcal{O}_{\mathbb{CP}^n}(1))$ are the sections used to create $\mathbb{C}(V)$ defined above.

Remark 4.1.11. In the GIT literature, when $\phi : V \to \mathbb{CP}^n$ is an embedding and $L = \phi^*\mathcal{O}_{\mathbb{CP}^n}(1)$, instead of equation (4.1) the coordinate ring of V is often denoted something like

$$\bigoplus_{d \geq 0} H^0(V, L^{\otimes d}), \tag{4.2}$$

where $L^{\otimes d}$ denotes taking *symmetric powers* of the generators of L. The implicit relations imposed by taking symmetric powers of L will coincide with the explicit relations in the quotient of equation (4.1). We can see this in Example 4.1.5. When $\phi([x : y]) = [x^2 : xy : y^2]$ and the generators of L were $s_0 = x^2,\ s_1 = xy$, and $s_2 = y^2$, in building $L^{\otimes 2}$, the fact that we take symmetric powers imposes[4] $s_1^2 - s_0 s_2 = 0$ and so

$$\frac{\mathbb{C}[s_0, s_1, s_2]}{(s_1^2 - s_0 s_2)} = \bigoplus_{d \geq 0} H^0(V, L^{\otimes d}).$$

Moving forward, we will use the more typical notation of $\bigoplus_{d \geq 0} H^0(V, L^{\otimes d})$. We would like to remind the reader of Remark 4.1.7 where we stated: *The coordinate ring of a projective variety is generated by symmetric powers of global sections of an ample line bundle over that variety.* This phrase is captured by the standard notation of $\bigoplus_{d \geq 0} H^0(V, L^{\otimes d})$.

4.1.5 GIT On Projective Varieties

The classic work to reference for Geometric Invariant Theory is [31]. This section is a very condensed version of work by Thomas [38] and Hoskins [22] meant to briefly summarize GIT in the context of our problem.

Definition 4.1.12. A *character* is a group homomorphism $\chi : G \to \mathbb{C}^\times$.

[4] $(x \otimes y) \otimes (x \otimes y) = (x \otimes x) \otimes (y \otimes y)$.

Definition 4.1.13. Let G act on a variety V, $\chi : G \to \mathbb{C}^\times$ a character, and $f : V \to \mathbb{C}$ a continuous function. Then f is said to be *G-equivariant* with respect to χ if $f(g \cdot x) = \chi(g) \cdot f(x)$.

To do a GIT quotient on a projective variety V, it suffices[5] to have the following:

- A reductive group, G, acting linearly on V.
- An ample line bundle L.
- L is generated by global sections $s_0, \cdots, s_n$, such that each s_i is G-equivariant with respect to some character χ_i. In this setting,

$$gx \mapsto [\chi_0(g)s_0(x) : \cdots : \chi_n(g)s_n(x)] =: g \cdot [s_0(x) : \cdots : s_n(x)].$$

Bullet 3 above is an example of a *linearization*[6] of G. The group orbit in the image is then

$$G \cdot s(x) = \{[\chi_0(g)s_0(x) : \cdots : \chi_n(g)s_n(x)] \mid g \in G\}.$$

For $f \in \bigoplus_{r\geq 0} H^0(V, L^{\otimes r})$, i.e. a polynomial in the s_i, we have a natural G action defined by

$$g \cdot f(s_0, \cdots, s_n) = f(\chi_0(g)s_0, \cdots, \chi_n(g)s_n).$$

We write $\bigoplus_{r\geq 0} H^0(V, L^{\otimes r})^G = \{f \in \bigoplus_{r\geq 0} H^0(V, L^{\otimes r}) \mid g \cdot f = f\}$ to denote the G-invariant polynomials. There is a natural inclusion

$$\bigoplus_{r\geq 0} H^0(V, L^{\otimes r})^G \hookrightarrow \bigoplus_{r\geq 0} H^0(V, L^{\otimes r}).$$

Essentially we want $V//G = \mathrm{Proj}\left(H^0(V, L^{\otimes r})^G\right)$ but there is an issue and it revolves around the fact that $[0 : \cdots : 0] \notin \mathbb{CP}^n$.

---

[5]We are not doing GIT quotients in full generality. A linearization of G may be different than specified here.

[6]For the purpose of this paper we call an embedding $\iota : V \to \mathbb{CP}^n$ a *linearization* of G if it is G-equivariant.

Definition 4.1.14. A nonzero vector $x \in V$ is said to be *unstable* under the action of G if $0 \in \overline{G \cdot s(x)}$. Otherwise x is *semi-stable* under the action of G.

Define $V^{ss} = \{x \in V \mid x \text{ is semi-stable}\}$. The GIT quotient will be denoted[7] and defined as

$$V//G = \operatorname{Proj} H^0(V^{ss}, L)^G.$$

Implicit in this definition is that we do not quotient all of V. We only quotient the subset V^{ss}. The quotient map on V^{ss} is $x \mapsto \overline{G \cdot x}$. Finding the set of all semi-stable points is a key part of GIT quotients. To do this we will use the Hilbert-Mumford Criterion, discussed in section 4.3. For the purposes of this paper $V = Gr(d,n)$, $G = \{(g_1, \cdots, g_n) \mid g_i \in \mathbb{C}^\times,\ \prod g_i = 1\}$ is the set of diagonal matrices with determinant 1, and the embedding used is the Plücker embedding.

4.2 Plücker Embedding

Let $N = \binom{n}{d}$. The Plücker embedding $\rho : Gr(d,n) \hookrightarrow \mathbb{CP}^{N-1}$ may be written succinctly and in a coordinate-free way as $[F] \mapsto [\bigwedge^d F]$. In coordinates we need to order the subsets $J \subset \{1, \cdots, n\}$ with $|J| = d$. There are clearly $N = \binom{n}{d}$ many sets J. Let F be a Parseval frame and denote by F_J the $d \times d$ submatrix of F whose columns are the f_i with $i \in J$, for example

$$F_{\{1,\cdots,d\}} = \left[f_1 \middle| f_2 \middle| \cdots \middle| f_d \right].$$

Choosing an ordering on the index sets J we define the Plücker embedding:

$$\rho([F]) = [\det(F_{J_1}) : \det(F_{J_2}) : \cdots : \det(F_{J_N})], \quad N = \binom{n}{d}. \tag{4.3}$$

[7] In other texts it may be written as $\operatorname{Proj} \bigoplus_{r \geq 0} H^0(V^{ss}, L^{\otimes r})^G$.

4.2.1 Pushing Forward the Group Action

Being defined via determinants, our embedding is G-equivariant, which is to say we have a *linearization* of the G action. With $g \in G$ we define

$$g \cdot \rho(F) := \rho(g \cdot [F]) = [\det g_{J_1} \det(F_{J_1}) : \det g_{J_2} \det(F_{J_2}) : \cdots : \det g_{J_N} \det(F_{J_N})]. \quad (4.4)$$

For brevity we use the notation g^{J_i} and $\prod_{k \in J_i} g_k$ interchangeably with $\det g_{J_i}$, depending on the context, different forms hopefully feel natural while reading.

Notice $g = \lambda I$ iff the g action is trivial on $\mathbb{CP}^{N-1}$. The forward implication is obvious. To show the converse we write $g = (g_1, \cdots, g_n)$ and assume g acts trivially. This implies that for all index sets J_i, $g^{J_i} = c$ for some constant c. In particular $g^{\{1,\cdots,d\}} = g^{\{2,\cdots,d+1\}}$, dividing out $g^{\{2,\cdots,d\}}$ yields $g_1 = g_{d+1}$. Similarly for each index set replacing 1 with $k > d$ we get $g_1 = g_k$. There is clearly nothing special about 1. Replacing any $j \in \{1, \cdots, d\}$ with $k > d$ and dividing the common terms yields $g_j = g_k$, so $g = \lambda I$.

By Proposition 3.2.4, group actions capture flow directions. Recall, the group element that flowed in the direction of $\nabla\mu(F)$ was $R(t) = e^{i(D - \frac{d}{n}I)t}$, where $D = \mathrm{diag}(||f_1||^2, ||f_2||^2, \cdots, ||f_n||^2)$. For an ENPF, $D = \frac{d}{n}I$ and so $R = I$. It is reassuring and a good reality check to see trivial group action exactly when our flow is stationary.

4.3 Hilbert-Mumford Criterion

4.3.1 Finding the Semi-Stable Locus

Since embeddings preserve all the necessary algebraic and topological features, the task of knowing if $0 \in \overline{G \cdot F}$ is analogous to knowing if $0 \in \overline{\rho(G \cdot F)}$. Frames being spanning sets imply that for a frame F, there exists a nonzero Plücker coordinate $\det(F_{J_i})$. However, at present whether $[0 : \cdots : 0]$ is a limit point in some equivalence class

$$[g^{J_1} det(F_{J_1}) : \cdots : g^{J_N} det(F_{J_N})] \sim [\frac{g^{J_1} det(F_{J_1})}{g^{J_i}} : \cdots : \det(F_{J_i}) : \cdots : \frac{g^{J_N} det(F_{J_N})}{g^{J_i}}]$$

is not obvious. The Hilbert-Mumford Criterion (HMC), Theorem 4.3.2 helps us tackle this obstacle.

Definition 4.3.1. A *1-parameter subgroup* (1-PS) of G is an algebraic group homomorphism $\gamma : \mathbb{C}^\times \to G$.

Theorem 4.3.2. *[Hilbert-Mumford criterion] [31] A point $v \in V$ is unstable if and only if there exists a 1-PS γ such that* $\lim_{t\to 0} \gamma(t) \cdot s(x) = 0$.

The HMC is useful because it says that if $0 \in \overline{G \cdot s(x)}$ then there is an "open path" $\gamma : (0,1) \to G$ such that $\gamma(t) \cdot s(x)$ goes from 0 to $s(x)$. Furthermore, if we compose $\rho \circ \gamma$ then each $\rho_i \circ \gamma : \mathbb{C}^\times \to \mathbb{C}^\times$ is a homomorphism. By Lemma 4.3.3, $\rho_i \circ \gamma(t) = t_i^w$.

Lemma 4.3.3. If $h : \mathbb{C}^\times \to \mathbb{C}^\times$ is an algebraic homomorphism then $h(t) = t^w$ for some integer w.

Proof. Being an *algebraic* homomorphism means $h(t)$ is a polynomial. Since $h(st) = h(s)h(t)$, it must be a monomial with coefficient 1. □

4.3.2 Classifying Stability Type Using HMC

Lemma 4.3.4. Suppose F is not ENPF and is a CP of μ. Then F is unstable.

Proof. Using the notation from section 3.2.3, let $F = \operatorname{diag}(F_1, \cdots, F_p)$ where each $F_i \in \mathbb{C}^{d_i \times n_i}$ is ENPF, thus each column of F_i has norm squared $\lambda_i = \frac{d_i}{n_i}$. Let $\gamma(t) \in G$ also have this block structure:

$$\gamma(t) = \operatorname{diag}(t^{x_1} I_{n_1}, t^{x_2} I_{n_2}, \cdots, t^{x_p} I_{n_p}), \quad \sum_i n_i x_i = 0.$$

Now, due to the block structure of F, rescaling the columns of the i^{th} block by t^{x_i} acts the same as rescaling the rows of the i^{th} block by t^{x_i}. As an example, consider the two block computation:

$$\left[\begin{array}{c|c} F_1 & \mathbf{0} \\ \hline \mathbf{0} & F_2 \end{array}\right] \underbrace{\left[\begin{array}{c|c} t^{x_1} I_{n_1} & \mathbf{0} \\ \hline \mathbf{0} & t^{x_2} I_{n_2} \end{array}\right]}_{n \times n \text{ matrix}} = \underbrace{\left[\begin{array}{c|c} t^{x_1} I_{d_1} & \mathbf{0} \\ \hline \mathbf{0} & t^{x_2} I_{d_2} \end{array}\right]}_{d \times d \text{ matrix}} \left[\begin{array}{c|c} F_1 & \mathbf{0} \\ \hline \mathbf{0} & F_2 \end{array}\right]. \tag{4.5}$$

This fact is useful because $\rho(F)$ is the d^{th} wedge product, $\rho(F) = [\bigwedge^d F]$. Thus, multiplying by a $d \times d$ square matrix just scales the Plücker coordinates by the determinant of that matrix, whereas multiplying by an $n \times n$ would require $d \times d$ minors. By Equation (4.5) we get

$$\begin{aligned}\rho(\gamma(t)F) &= \det\left(\mathrm{diag}(t^{x_1} I_{d_1}, \cdots, t^{x_p} I_{d_p})\right)\rho(F)\\ &= t^{x_1 d_1 + \cdots x_p d_p}\rho(F)\\ &= t^w \rho(F).\end{aligned}$$

Thus the weight of every coordinate function is $\sum d_i x_i = \sum n_i \lambda_i x_i$. Without loss of generality we may assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$, and since we are not at an ENPF at least one inequality is strict. Recall, $\gamma(t) \in G$ means $\sum_i n_i x_i = 0$. So, choose any $(x_1, \cdots, x_p)$ satisfying $\sum_i n_i x_i = 0$ such that only $x_p < 0$. Then we have

$$w = \sum n_i \lambda_i x_i > \lambda_p \sum n_i x_i = 0.$$

Since $w > 0$, $t^w \to 0$ as $t \to 0$. Thus F is unstable. □

Lemma 4.3.5. Suppose F is a full spark Parseval frame, then F is semi-stable.

Proof. Full spark implies $\det(F_{J_i}) \neq 0$ for all J_i. Thus

$$\begin{aligned}\rho(g \cdot [F]) &= \left[g^{J_1} det(F_{J_1}) : \ g^{J_2} det(F_{J_2}) : \cdots : \ g det(F_{J_N})\right]\\ &= \left[1 : \frac{g^{J_2} det(F_{J_2})}{g^{J_1} det(F_{J_i})} : \frac{g^{J_3} det(F_{J_3})}{g^{J_1} det(F_{J_1})} : \cdots : \frac{g^{J_N} det(F_{J_N})}{g^{J_1} det(F_{J_1})}\right].\end{aligned}$$

In particular, for all 1-PS, $\gamma(t)$, $\lim_{t\to 0}\rho(\gamma(t) \cdot F) = [1 : \star : \cdots : \star] \neq 0$. Thus F is semi-stable. □

Chapter 5

Putting Everything Together

We can now put all the pieces together to prove Theorem 1.3.1, which we restate for convenience.

Fix $d, n \in \mathbb{N}$ *with* $d \leq n$ *and consider the gradient flow*

$$\Gamma : St(d, n) \times [0, \infty) \to St(d, n)$$

defined by the differential equation

$$\Gamma(F_0, 0) = F_0, \quad \frac{d}{dt}\Gamma(F, t) = -\nabla\mu(\Gamma(F, t)). \tag{5.1}$$

If F_0 *is full spark then* $F_\infty = \lim_{t\to\infty} \Gamma(F_0, t)$ *is an equinorm Parseval frame.*

Proof. If F is full spark then by lemma 4.3.5 F is semi-stable under the G action. By Proposition 3.2.4 the orbit $G \cdot F$ contains the gradient flow defined by equation (5.1). It follows that $\overline{G \cdot F}$ contains its limit points and therefore its critical points. Since F is semi-stable, so are the critical points in $\overline{G \cdot F}$. By Lemma 4.3.4 the critical points must be Equinorm Parseval Frames. Thus $\lim_{t\to\infty} \Gamma(F, t)$ is an Equinorm Parseval Frame. □

5.1 Results

The code below is a basic implementation of the gradient descent proposed.

```
function F=ENPF(n,d,dx, tol)
%% Initialize Parseval Matrix F and Diagonal norm matrix D
F=randn(d,n);
F=F+i*randn(d,n);
[F,r]=qr(F');
```

```
F=F(:,1:d)';
D=diag(diag(F'*F));
err=norm(diag(D-d/n*eye(n)));
%% Gradient Flow
while err > tol
    gf=F*D*(eye(n)-F'*F);
    F=F-dx*gf;
    [F,r]=qr(F');
    F=F(:,1:d)';
    %F*F'
    D=diag(diag(F'*F));
    err=norm(diag(D-d/n*eye(n)));
end
end
```

Figure 5.1 shows run-time (in seconds) on the y-axis with redundancy $\frac{n}{d}$ on the x-axis. It is interesting that "optimal redundancy" appears to be around 1.5 for the trials shown. I wonder if this trend continues.

In Figure 5.2 we can see $\mu(F)$ rapidly decrease until we are near the critical point, at which point the gradient descent slowly continues until we are within 10^{-12} of an ENPF. Figure 5.2 was generated using $d = 50$, $n = 200$, and a random Parseval frame as F_0. In Figure 5.3 we also plot $\mu(F)$ v. iterations with $d = 50$ and $n = 200$. However, the initial F_0 had 3 blocks of ϵ-ENPF of size $10 \times 50,\ 20 \times 50,\ 20 \times 100$. Note, this F_0 is not full spark as one may take any collection of 50 frame vectors from the same block and it will not span $\mathbb{C}^{50}$. It appears the blocks merge and thus we flow near other non-minimal CPs. In hindsight this makes sense as $\nabla\mu$ will act on blocks

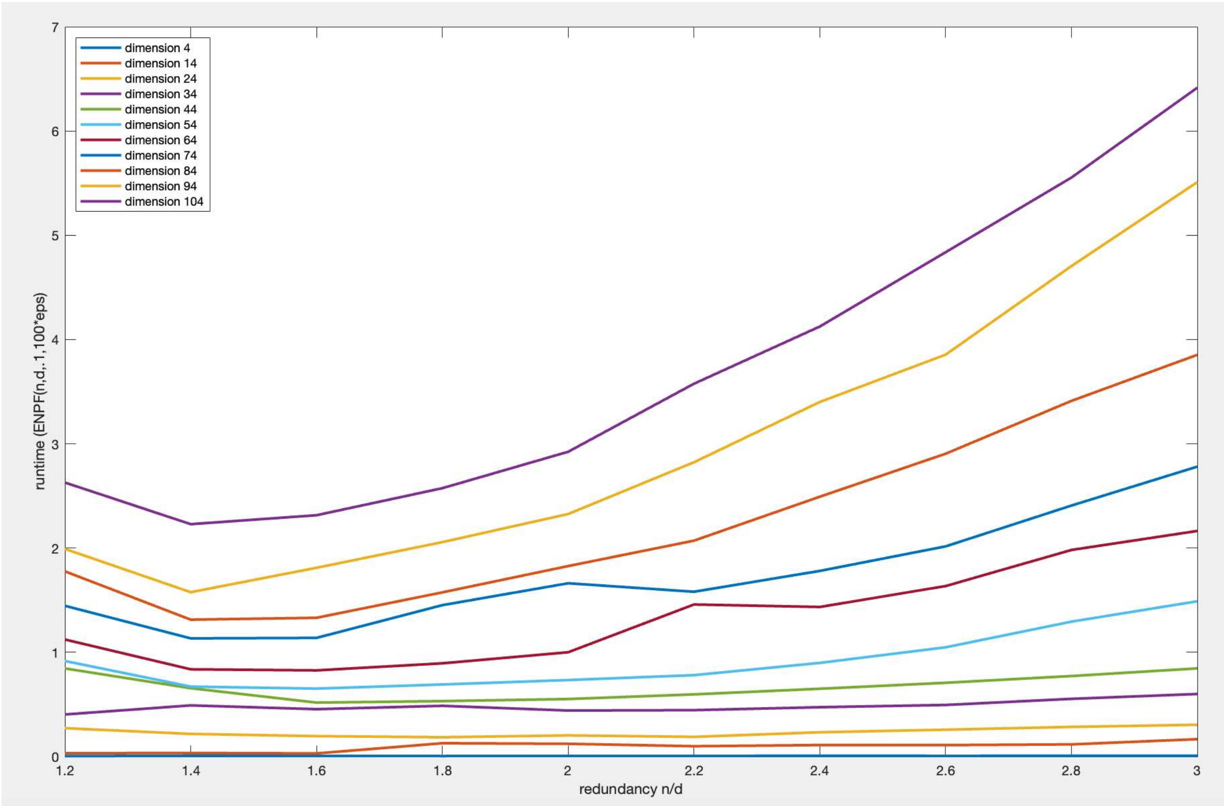

Figure 5.1: Runtime (in seconds) on the y-axis with redundancy $\frac{n}{d}$ on the x-axis.

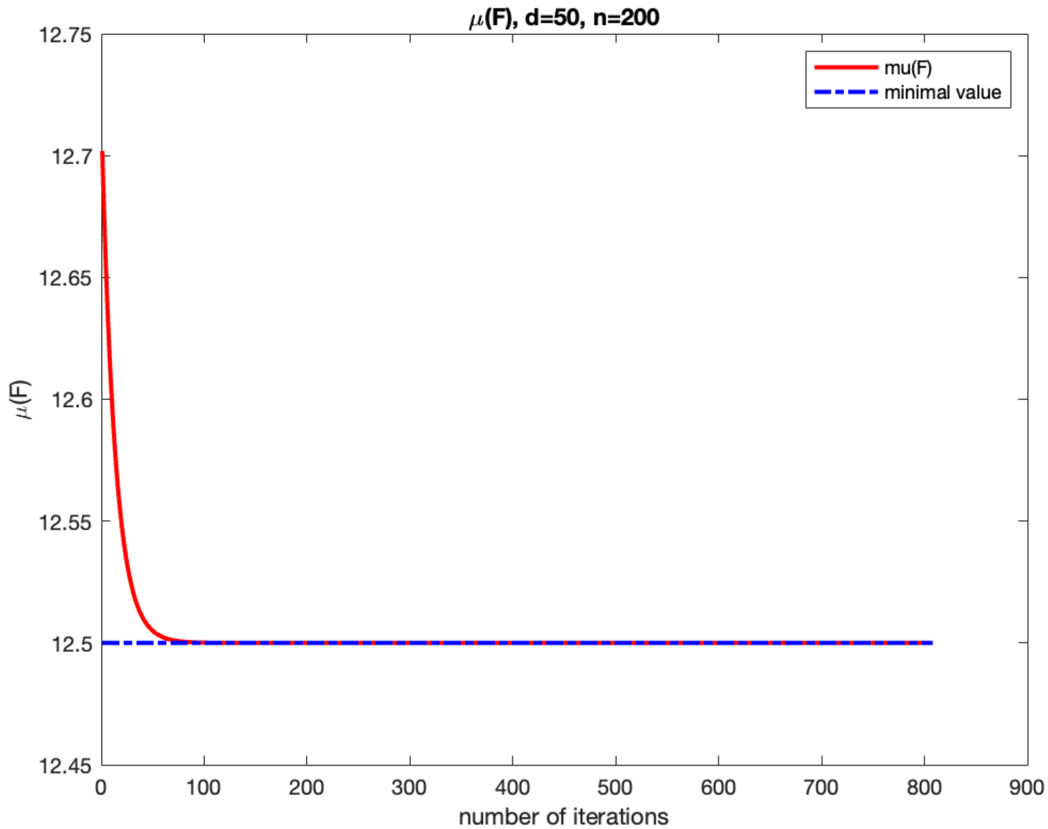

Figure 5.2: $\mu(F)$ on the y-axis with number of iterations on the x-axis. We can see that after approx 100 iterations the scale of the graph would have to change to see any change in $\mu(F)$.

and D essentially looks like

$$D = \left[\begin{array}{c|c|c} \frac{1}{5}I_{50} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{2}{5}I_{50} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \frac{1}{5}I_{100} \end{array}\right],$$

meaning each block gets scaled in the same way. Thus each block moves as a whole. Eventually two merge and then we have two blocks and so on. This may shed some light on the intuition behind why all non-minimal CPs exist in orbits disjoint from minimal CPs: it's because blocks become larger blocks.

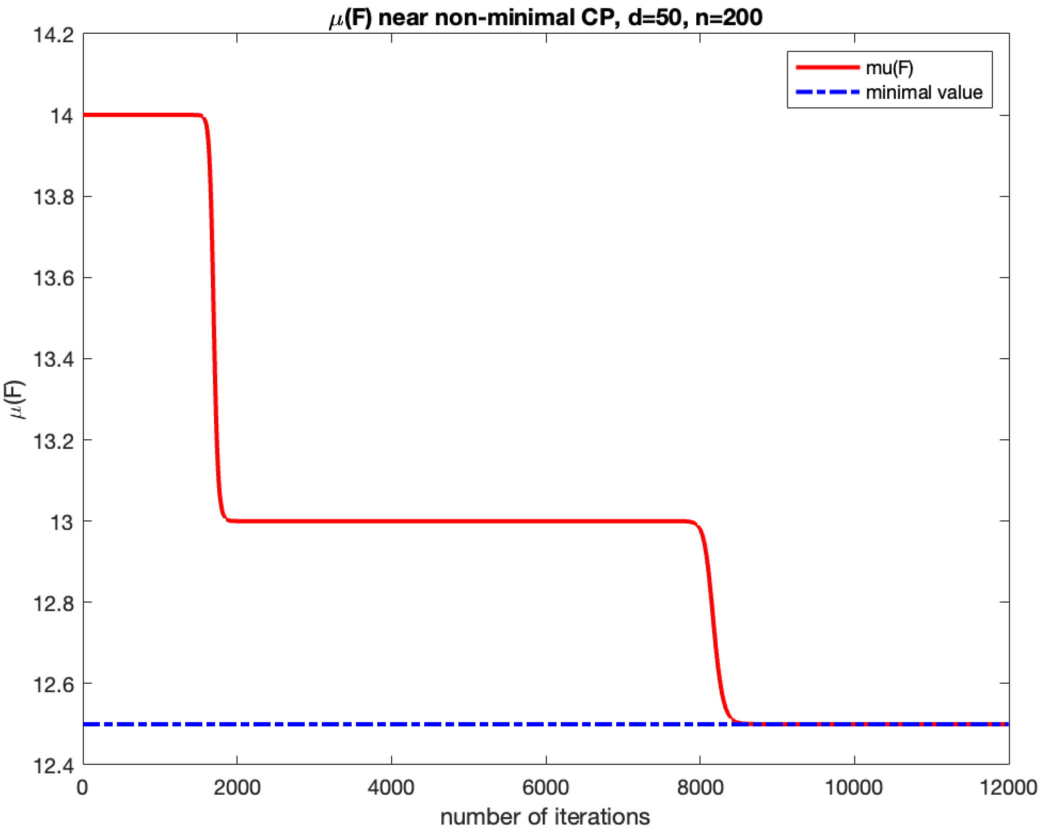

Figure 5.3: Beginning near a non full spark F_0, within 10^{-12} of a non-minimal critical point, we plot $\mu(F)$ on the y-axis with number of iterations on the x-axis.

Chapter 6

Kempf–Ness Approach

The Kempf–Ness Theorem 6.3.2 relates the GIT quotient $V//G$ to the symplectic reduction $V//K$, where K is a connected maximal compact subgroup of G. This relation will give us yet another approach to finding ENPFs, which ultimately amounts to iteratively minimizing a strictly convex function. New approaches to finding ENPFs hopefully will help tighten existing bounds on the Paulsen problem, which we restate here for convenience:

Definition 6.0.1. $\{f_1, \cdots, f_n\} \subset \mathbb{C}^d$ is ϵ-equinorm Parseval (abbr. ϵ-ENPF) if

1. All eigenvalues of FF^* are in $(1-\epsilon, 1+\epsilon)$ and
2. For all $k \in \{1, \cdots, n\}$, $\frac{d}{n}(1-\epsilon) \leq ||f_k|| \leq \frac{d}{n}(1+\epsilon)$.

Given two frames $P, Q \in \mathbb{C}^{d\times n}$, define the squared distance between P and Q to be $\operatorname{dist}^2(P,Q) = ||P-Q||_{Fr}^2 = \sum_{k=1}^n ||p_k - q_k||^2$, where $||\cdot||_{Fr}$ is the Frobenius norm. The *Paulsen Problem* simply asks, given an ϵ-ENPF frame P, is there an ENPF frame Q and a polynomial, f, such that $\operatorname{dist}(P,Q) \leq f(\epsilon, d)$. Currently the bound to tighten is $\operatorname{dist}(P,Q) \leq 20\epsilon d^2$ [20].

To begin this discussion we have to introduce some notions from symplectic geometry and Lie theory.

6.1 Differential Geometry and Lie Algebras.

In general, if G is a Lie group, with Lie algebra $\mathfrak{g} = T_eG$, and G acts on a manifold M, then each $X \in \mathfrak{g}$ determines a vector field $X^\#$ on M as follows. For $p \in M$ and $g \in G$, let $g \cdot p$ denote the action of g on p. Then we define

$$X^\#(p) := \frac{d}{dt}\bigg|_{t=0} \exp(tX)\cdot p,$$

where $\exp : \mathfrak{g} \to G$ is the exponential map of G. Since G actions and $\exp$ are defined as smooth maps, this is indeed a smooth vector field (see [26] chapter 20).

For each $g \in G$ the *adjoint action of* G *on* $\mathfrak{g}$ is the map

$$Ad_g(X) = \frac{d}{dt}\bigg|_{t=0} g \exp(tX) g^{-1} \in T_e G = \mathfrak{g}.$$

Since we will be dealing with matrix groups, i.e. $G \subseteq GL_n(\mathbb{C})$, it follows that $Ad_g(X) = gXg^{-1}$ (see [26] chapter 20, exercise 21). Similarly we have the notion of the *coadjoint action of* G *on* $\mathfrak{g}^*$, denoted $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$, and defined via

$$(Ad_g^* \xi)(X) := \xi(Ad_{g^{-1}}(X)), \quad \xi \in \mathfrak{g}^*, X \in \mathfrak{g}. \tag{6.1}$$

Remark 6.1.1. For us $G = \{g \in (\mathbb{C}^\times)^n \mid \prod g_i = 1\}$ and $K = \{g \in \mathcal{U}(1)^n \mid \prod g_i = 1\}$. The Lie algebra of K is $\mathfrak{k} = \{iX \mid X \in \mathbb{R}^n,\ \sum X_i = 0\} \cong \mathbb{R}^{n-1}$ and therefore $\mathfrak{k}^* \cong \mathbb{R}^{n-1}$. In our setting, the adjoint action of $\mathfrak{k}$ on K is trivial because $\frac{d}{dt}\big|_{t=0} g \exp{(itX)} g^{-1} = giXg^{-1}$ is the component-wise multiplication of complex numbers, so it commutes. The vector field generated by $X = (X_1, \cdots, X_n) \in \mathfrak{k}$ can be found using the formula in [14] to project from $\mathbb{C}^{d\times n}$ to $T_{[F]}Gr(d,n)$. It's defined by

$$X^{\#}(F) = F \, \frac{d}{dt}\bigg|_{t=0} \operatorname{diag}(e^{itX_1}, \cdots, e^{itX_n})(I - F^*F) = iF \operatorname{diag}(X)(I - F^*F),$$

where $F \in St(d,n)$ has rowspace $[F] \in Gr(d,n)$.

6.2 Symplectic Manifolds

A symplectic manifold is a pair (M, ω), where M is a (smooth, real) manifold and ω is a closed[8], nondegenerate[9] 2-form on TM. We write $\omega_x(X,Y) \in \mathbb{R}$ for the evaluation of ω at the

---

[8]Being *closed* means that $d\omega = 0$, where d is the exterior derivative on M.

[9]Being *nondegenerate* means that, for each $x \in M$ and each $X \in T_xM$, there exists $Y \in T_xM$ so that $\omega_x(X,Y) \neq 0$.

point $x \in M$ on the pair of tangent vectors $X, Y \in T_xM$. A 2-form is a skew symmetric bilinear form. Since skew symmetric matrices are singular in odd dimension, nondegeneracy implies that a symplectic manifold must be even-dimensional over $\mathbb{R}$ [10].

Example 6.2.1. [10] The simplest example of a symplectic manifold is $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with real basis $\{x_1, \cdots, x_n, y_1 \cdots, y_n\}$ and $\omega = \sum_i dx_i \wedge dy_i$. We can express this 2-form in a couple of ways. A concise matrix expression for $\omega(u, v)$ with $u, v \in \mathbb{C}^n$ is

$$\omega(u, v) = (u_{1x}, \cdots, u_{nx}, u_{1y}, \cdots, u_{ny}) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} v_{1x} \\ \vdots \\ v_{ny} \end{pmatrix}.$$

Alternatively, identifying $\mathbb{C}^n$ with matrices $\mathbb{R}^{2\times n}$, we have $\omega(u, v) = -i \cdot \mathbf{Im}(\mathrm{tr}\left(uv^T\right))$. This follows from the computation

$$\begin{aligned} \omega(u, v) &= \mathrm{tr}\left(\begin{bmatrix} u_{1x} & \cdots & u_{nx} \\ u_{1y} & \cdots & u_{ny} \end{bmatrix} \begin{bmatrix} v_{1y} & -v_{1x} \\ \vdots & \vdots \\ v_{ny} & -v_{nx} \end{bmatrix} \right) \\ &= \mathrm{tr}\left(\begin{bmatrix} u_{1x} & \cdots & u_{nx} \\ u_{1y} & \cdots & u_{ny} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_{1x} & v_{1y} \\ \vdots & \vdots \\ v_{nx} & v_{ny} \end{bmatrix} \right) \\ &= -i \cdot \mathbf{Im}(\mathrm{tr}\left(uv^T\right)). \end{aligned}$$

It is known (see [26] proposition 22.7) that with a suitably chosen basis one may assume ω has this form. While algebraically this is the canonical symplectic form, one may be interested in a more geometric understanding. When $n = 1$, $\omega(u, v) = \det[u\ v]$. For the general case, when $u,\ v \in \mathbb{C}^n \cong \mathbb{R}^{2\times n}$, we have $\omega(u, v) = \sum_i \det[u_i\ v_i]$, where u_i and v_i are the i^{th} components of u and v, thought of as column vectors in $\mathbb{R}^2$. Thus, in some sense, ω sums a measure of the orthogonality of u_i and v_i, or the curl of u_i and v_i.

6.2.1 Moment Maps

Let G be a Lie group, with associated Lie algebra $\mathfrak{g}$, that acts on a symplectic manifold (M, ω). Let $\mathcal{M} : M \to \mathfrak{g}^*$ with derivative at x denoted $d\mathcal{M}_x : T_xM \to T_{\mathcal{M}(x)}\mathfrak{g}^*$. Since $\mathfrak{g}^*$ is a vector space, there is a natural isomorphism $T_{\mathcal{M}(x)}\mathfrak{g}^* \cong \mathfrak{g}^*$, so it is commonly stated $d\mathcal{M}_x : T_xM \to \mathfrak{g}^*$ or $d\mathcal{M}(X) : \mathfrak{g} \to \mathbb{R}$. With this in mind, $\mathcal{M}$ is called a *moment map* for the G-action if it satisfies the following two properties (see [10] chapter 22)

$$d\mathcal{M}_x(X)(Y) = \omega_x(Y^{\#}(x), X) \tag{6.2}$$

$$\mathcal{M}(x)(gXg^{-1}) = \mathcal{M}(g \cdot x)(X). \tag{6.3}$$

Equation (6.3) is often stated $Ad_g^*\mathcal{M}(x) = \mathcal{M}(g \cdot x)$. When a G-action admits a *moment map*, the action is called *Hamiltonian* and the tuple (M, ω, G, μ) is a *Hamiltonian G-space*.

Remark 6.2.2. Recall from example 6.2.1 that $\omega(u, v)$ sort of measures the orthogonality of u and v. So, intuitively, equation (6.2) says "the amount $\mathcal{M}$ increases in the X direction is proportional to magnitude of the part of X orthogonal to $\mathfrak{k}$." This agrees with equation (6.3) which, for an abelian group, says $\mathcal{M}$ is constant on K orbits. The Kempf–Ness function (see Section 6.3) uses this to stratify M.

6.2.2 Moment Maps on $Gr(d, n)$

Let $F \in St(d, n)$ correspond to $[F] \in Gr(d, n)$ and define $\mathcal{M}_k([F]) = ||f_k||^2$ and $\mathcal{M}([F]) = (||f_1||^2, \cdots, ||f_n||^2)$. Using the relation from Equation (2.15), we know $\|f_1\|^2 + \cdots + \|f_n\|^2 = d$ and so the image of $\mathcal{M}$ can be identified with a subset of $\mathbb{R}^{n-1} \cong \mathfrak{k}^*$. Our maximal compact subgroup is $K = \{g \in \mathcal{U}(1)^n \mid \prod g_k = 1\}$ acting by $k \cdot [F] = [Fk]$, where $[F] \in Gr(d, n)$ represents the row space of $F \in St(d, n)$. Needham and Shonkwiler have shown $\mathcal{M}$ is a moment map for the action of K on $Gr(d, n)$ [32].

6.3 Kempf–Ness Special Functions

Let a connected reductive algebraic group G act on M, a complex smooth projective variety. Let $K \leq G$ be a maximal compact subgroup of G. Given a Hamiltonian K-space (M, ω, K, μ) we can form a quotient space called the *symplectic reduction* of M by K defined in the classic theorem by Marsden, Weinstein, and Meyer.

Theorem 6.3.1. *[27, 29] Let (M, ω) be a symplectic manifold with a Hamiltonian action of a Lie group K and $\mathcal{M} : M \to \mathfrak{k}^*$ be the moment map for this action. For any regular value $\xi \in \mathfrak{k}^*$ of $\mathcal{M}$ which is fixed by the coadjoint action of K and so that K acts freely on $\mathcal{M}^{-1}(\xi)$, the space*

$$M//_{\xi}K := \mathcal{M}^{-1}(\xi)/K$$

has a natural symplectic structure ω_{red} satisfying

$$\iota^*\omega = \pi^*\omega_{red},$$

where $\iota : \mathcal{M}^{-1}(\xi) \to M$ and $\pi : \mathcal{M}^{-1}(\xi) \to \mathcal{M}^{-1}(\xi)/K$ are the inclusion and projection maps, respectively.

Since our manifold is also a smooth variety there is a homeomorphism between the symplectic reduction over K and the GIT quotient by G

Theorem 6.3.2 (The Kempf–Ness Theorem)**.** *[35] There is an inclusion $\mathcal{M}^{-1}(0) \subset V^{ss}$ which induces a homeomorphism between the symplectic reduction and the GIT quotient*

$$\mathcal{M}^{-1}(0)/K \cong M//G.$$

Define the sets

$$[\mathfrak{k}, \mathfrak{k}] := \{\text{linear combinations of } [X, Y] \text{ for any } X, Y \in \mathfrak{k}\}$$

and $[\mathfrak{k},\mathfrak{k}]^0 \subseteq \mathfrak{k}^*$ is the annihilator of $[\mathfrak{k},\mathfrak{k}]$. Although Theorem 6.3.2 is stated using $\mathcal{M}^{-1}(0)$, it turns out we can use $\mathcal{M}^{-1}(c)$ for any $c \in [\mathfrak{k},\mathfrak{k}]^0$.

Theorem 6.3.3. *[10] If $\mathcal{M} : M \to \mathfrak{k}^*$ is a moment map, then given any $c \in [\mathfrak{k},\mathfrak{k}]^0$, $\tilde{\mathcal{M}} = \mathcal{M} + c$ is another moment map,*

In our case, $\mathfrak{k}$ is commutative and so $[\mathfrak{k},\mathfrak{k}]^0 = \mathfrak{k}^*$ and so

$$\tilde{\mathcal{M}}([F]) = \left(||f_1||^2 - \frac{d}{n}, \cdots, ||f_n||^2 - \frac{d}{n} \right)$$

is still a moment map and one where $\tilde{\mathcal{M}}^{-1}(0) = \mathcal{M}^{-1}\left(\frac{d}{n}, \cdots, \frac{d}{n}\right)$ corresponds to ENPFs.

6.3.1 Squared Norm of Moment Maps

If $\{a_1, \cdots, a_n\}$ is an orthonormal basis (ONB) for $\mathfrak{k}$, with corresponding dual ONB $\{a_1^*, \cdots, a_n^*\}$ then for a given moment map $\mathcal{M} : M \to \mathfrak{k}^*$ we can define its component functions

$$\mathcal{M}_i(x) = \langle a_i^*, \mathcal{M}(x) \rangle$$

so that $\mathcal{M}(x) = \sum_i \mathcal{M}_i(x) a_i^*$. The squared norm of the moment map is then

$$f(x) = \|\mathcal{M}(x)\|^2 = \sum_i |\mathcal{M}_i(x)|^2$$

and has a lot of special properties [23, 24, 35]. Intuitively, because f is K-invariant, we can think of M as being partitioned by K orbits, each orbit corresponding to a level set of f. Furthermore, critical points of f correspond to critical points of $\mathcal{M}$, i.e. points in $\mathcal{M}^{-1}(0)$ [24]. We can more or less see this by taking a derivative. To be a critical point means $df_x : T_x M \to \mathbb{R}$ is the zero map. This means

$$df_x(X) = \sum_i 2\mathcal{M}_i(x) d\mathcal{M}_{i,x}(X) = 0.$$

Since this holds for all $X \in T_xM$, it actually implies[10] $\mathcal{M}_i(x) = 0$ for all i, which is to say $\mathcal{M}(x) = 0$. For a more thorough proof see Kirwan [24]. The main take away, for our purposes, is that minima of f will correspond to ENPFs. This function f is very similar to Kempf–Ness special functions, just with a different domain.

6.4 Using the Squared Norm of Plücker coordinates

We can see the magnitude of the Plücker coordinates remain fixed under the K action, so $||\rho(g \cdot F)||^2$ is K invariant. Based on [23] we define the *Kempf–Ness special function* $P_F : \mathbb{R}^n \to \mathbb{R}$ by fixing a representative $F \in St(d, n)$ then

$$P_F(t) = ||\rho(\exp{(t)} \cdot F)||^2 = \sum_{k=1}^{N} e^{2t_{I_k}} |\det(F_{I_k})|^2 = \sum_{k=1}^{N} e^{2t_{I_k}} \det(F_{I_k} F_{I_k}^*),$$

where $N = \binom{n}{d}$. We want the domain to correspond to $\mathfrak{g}/\mathfrak{k} = \{t \in \mathbb{R}^n | \sum_k t_k = 0\}$, so we impose this constraint. As a sum of convex functions, this function is convex. To find the minima, we can simply apply Lagrange multipliers.

We know our critical points occur when

$$\nabla P_F = \lambda \cdot (1,\ 1, \cdots,\ 1),$$

which is to say the i-th partial of P_F is the same constant for all i. In writing the i-th partial we use the notation

$$\sum_{i \in I} |\det(F_I)|^2$$

to mean summing $|\det(F_I)|^2$ over all index sets I which contain i. Then

$$\frac{\partial}{\partial t_i} P_F = \sum_{i \in I} e^{t_I} |\det(F_I)|^2.$$

[10]This is because $d\mathcal{M}_x(X)(Y) = \sum_i d\mathcal{M}_{i_x}(X)(Y) = \omega_x(Y^{\#}(x), X)$ and ω is non-degenerate.

Since the i^{th} and j^{th} partial all equal λ, we necessarily have

$$\sum_{i\in I} e^{t_I} |\det(F_I)|^2 = \sum_{j\in J} e^{t_J} |\det(F_J)|^2 \tag{6.4}$$

for a critical point. Thus critical points coorespond to frames whose Plücker coordinates satisfy this relation.

Theorem 6.4.1. *If* $[F] \in Gr(d,n)$ *is such that* $\sum_{i\in I} |\det(F_I)|^2 = \sum_{j\in J} |\det(F_J)|^2$ *then* $[F] = [Q]$ *where* $Q \in St(d,n)$ *is ENPF.*

Proof. Performing Gram–Schmidt on the rows of F to write $F = LQ$ where $Q \in St(d,n)$ and L is lower triangular one finds by Lemma 6.4.2 that

$$\sum_{i\in I} |\det(F_I)|^2 = \det(L^* L) \|q_i\|^2 .$$

Thus $\sum_{i\in I} |\det(F_I)|^2 = \sum_{j\in J} |\det(F_J)|^2$ if and only if Q is ENPF. Which is to say, the point on $Gr(d,n)$ corresponding to F is represented by Q, an ENPF, on $St(d,n)$. □

Lemma 6.4.2. $F \in St(d,n)$ if and only if $\sum_{i\in I} |\det(F_I)|^2 = \|f_i\|^2$.

Proof. First, notice $\sum_{i\in I} |\det(F_I)|^2 = \sum_{i\in I} \det(F_I F_I^*)$, as the latter form will be used throughout. Now, assume $FF^* = I_d$. Let F_i be F with the i^{th} column removed and $\sigma(F)$ denote the spectrum of F. Then $\sigma(I - f_i f_i^*) = \{(1 - \|f_i\|^2), 1\}$, which can be realized by creating an orthogonal basis of $\mathbb{C}^d$ of the form $\{f_i, u_2, \cdots u_d\}$ with $\|u_j\| = 1$. It follows that

$$\det(F_i F_i^*) = \det\left(\sum_j f_j f_j^*\right) \tag{6.5}$$

$$= \det(I - f_i f_i^*) \tag{6.6}$$

$$= \prod_{\lambda\in\sigma(I - f_i f_i^*)} \lambda \tag{6.7}$$

$$= (1 - \|f_i\|^2). \tag{6.8}$$

Alternatively, notice $1 = \det(FF^*)$ and use the Cauchy–Binet Formula to get

$$\begin{aligned}
1 &= \det(FF^*) && (6.9)\\
&= \sum_{|I|=d} \det(F_I F_I^*) && (6.10)\\
&= \sum_{i\notin I} \det(F_I F_I^*) + \sum_{i\in I} \det(F_I F_I^*) && (6.11)\\
&= \det(F_i F_i^*) + \sum_{i\in I} \det(F_I F_I^*) && (6.12)\\
&= 1 - \|f_i\|^2 + \sum_{i\in I} \det(F_I F_I^*). && (6.13)
\end{aligned}$$

Solving for $\sum_{i\in I} \det(F_I F_I^*)$ yields

$$\sum_{i\in I} |\det(F_I)|^2 = \|f_i\|^2 .$$

Conversely, use Gram–Schmidt on the rows of F to write $F = LQ$ where L is lower triangular and Q has orthonormal rows. Now $\det(F_I) = \det(L)\det(Q_I)$ and similarly for F_I^*. Thus

$$\begin{aligned}
\sum_{i\in I} \det F_I F_I^* &= \sum_{i\in I} \det(LQ_I)\det(Q_I^* L^*)\\
&= \sum_{i\in I} \det(LL^*)\det(Q_I Q_I^*)\\
&= \det(LL^*) \sum_{i\in I} \det(Q_I Q_I^*)\\
&= \det(LL^*)\|q_i\|^2 .
\end{aligned}$$

The last equality holds by the fact that $Q \in St(d,n)$ and we have already shown the forward direction. Now, if $\sum_{i\in I} \det F_I F_I^* = \|f_i\|^2$ then $\|f_i\|^2 = c\|q_i\|^2$ where $c = \det(LL^*)$. However $\|f_i\|^2 = \|Lq_i\|^2$ and so $c\|q_i\|^2 = \|Lq_i\|^2$, which holds for all i. Since the q_i span $\mathbb{C}^d$, it follows that $\|Lv\|^2 = c\|v\|^2$ for all $v \in \mathbb{C}^d$. This implies $L = cU$, for a unitary matrix U. We thus chose the "wrong Q" and should update $Q = U^*Q$ so we may say $F = cI_dQ$. This implies F is tight[11] since $FF^* = c^2I_d = AI_d$. This lets us run an analogous computation to Equations (6.5) and (6.9) where one finds

$$\det(F_iF_i^*) = A^{d-1}(A - ||f_i||^2)$$

and

$$\begin{aligned} A^d &= \det(FF^*) && (6.14)\\ &= \det(F_iF_i^*) + \sum_{i\in I}\det(F_IF_I^*) && (6.15)\\ &= A^{d-1}(A - ||f_i||^2) + \sum_{i\in I}\det(F_IF_I^*), && (6.16) \end{aligned}$$

concluding

$$A^{d-1}||f_i||^2 = \sum_{i\in I}\det(F_IF_I^*).$$

Recalling that A would be a real positive number corresponding to a frame bound, we see that assuming $||f_i||^2 = \sum_{i\in I}\det(F_IF_I^*)$ implies $A = 1$ and so $F \in St(d, n)$. □

Remark 6.4.3. We now have yet another algorithm for finding ENPF. Namely, fix a fullspark $F \in St(d, n)$ and minimize the strictly convex function that is $P_F(g)$. The computational trade-off is that $P_F(g)$ has $\binom{n}{d}$ terms.

6.5 Normalizing Potential Applied to Kempf–Ness Functions

In a very similar vein to the previous section for a fixed $F \in St(d, n)$ we define the function

$$N_F : G \to \mathbb{R} \text{ by } N_F(g) = \sum_k \mathcal{M}_k(g\cdot F)^2 = \sum_k |g_k|^4||f_k||^4.$$

To find minima of N_F it suffices to consider the domain as $\mathbb{R}^n$, since N_F is K invariant and $g \in G/K$ may be identified with a list of moduli, $g = (e^{t_1}, \cdots, e^{t_n})$. So, for $t \in \mathbb{R}^n$ define the

[11] We write $c^2 = A$ to decrease the exponent bookkeeping and to promote recall of how we defined tight frames.

diagonal matrix $g_{k,k} = \exp(t_k) \in G/K$. We can now write $N_F : \mathbb{R}^n \to \mathbb{R}$ as

$$N_F(t) = \sum_k e^{4t_k} ||f_k||^4.$$

We can then compute that

$$\nabla_k N_F = 4e^{4t_k} ||f_k||^4.$$

With our constraint $\sum_k t_k = 0$, an application of Lagrange multipliers shows our critical points occur when

$$\nabla N_F = \lambda \cdot (1,\ 1, \cdots,\ 1),$$

which is to say

$$e^{4t_j} ||f_j||^4 = e^{4t_k} ||f_k||^4.$$

Thus at a critical point t^* we have

$$\frac{e^{4t_j^*}}{e^{4t_k^*}} = \frac{||f_k||^4}{||f_j||^4}.$$

We can then write all $e^{t_j^*}$ in terms of $e^{t_1^*}$ as

$$e^{4t_j^*} = \frac{||f_1||^4}{||f_j||^4} e^{4t_1^*}.$$

Notice this holds for all $j \in \{1, \cdots, n\}$. Taking the product over all j and using $\sum_j t_j = 0$ we get

$$\begin{aligned}
\prod_j \left(\frac{||f_1||^4}{||f_j||^4} e^{4t_1^*} \right) &= \prod_j e^{4t_j^*} \\
e^{4nt_1^*} \prod_j \frac{||f_1||^4}{||f_j||^4} &= e^{4\sum_j t_j^*} \\
e^{4t_1^*} \left(\prod_j \frac{||f_1||^4}{||f_j||^4} \right)^{\frac{1}{n}} &= 1 \\
\left(\prod_j \frac{||f_j||}{||f_1||} \right)^{\frac{1}{n}} &= e^{t_1^*}.
\end{aligned}$$

So we see $e^{t_1^*}$ is the geometric mean of the magnitude ratios $||f_j||/||f_1||$, and clearly there is nothing special about t_1^*. In general we get

$$\left(\prod_{j=1}^{n} \frac{||f_j||}{||f_k||}\right)^{\frac{1}{n}} = e^{t_k^*}. \tag{6.17}$$

Solving for the exponents we get

$$\frac{1}{n}\left(\sum_j \ln ||f_j||\right) - \ln ||f_k|| = t_k^*. \tag{6.18}$$

Looking at equation (6.17), we can see this group element decreases/increases $||f_k||$ when $||f_k||$ is larger/smaller than the average respectively, thus pulling $||f_k||$ towards an equinorm frame.

Since the minimum of N_F is easy to find, this gives us a new algorithm:

1. Initialize a Parseval frame, F_0. Then for $k = \{0, 1, \cdots, \}$:

2. Find t^*, the minimum of N_{F_k}.

3. Project $F_k e^{t^*}$ onto $St(d, n)$, call the projection F_{k+1}.

4. Repeat steps 2 and 3 until F_k is suitably ENPF.

This algorithm seems to converge exponentially fast. We can see in Figure 6.1 that every 2 to 3 iterations makes $\operatorname{diag}(F^*F)$ an order of magnitude closer to $\frac{d}{n}I_n$. Running the same experiment as Figure 5.1, we can see in Figure 6.2 an order of magnitude decrease in runtime.

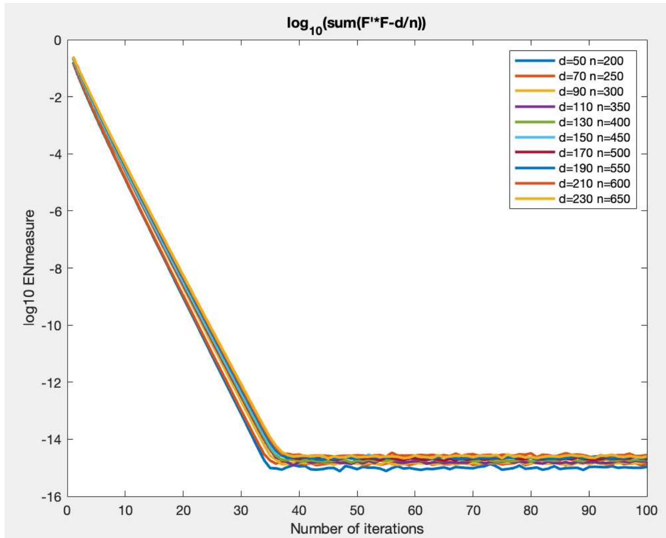

Figure 6.1: y-axis is $\log_{10}\left(\mathrm{tr}\left(F^*F - \frac{d}{n}I\right)\right)$. We can see exponential convergence with each iteration of this algorithm, with mild fluctuations for variation of d and n.

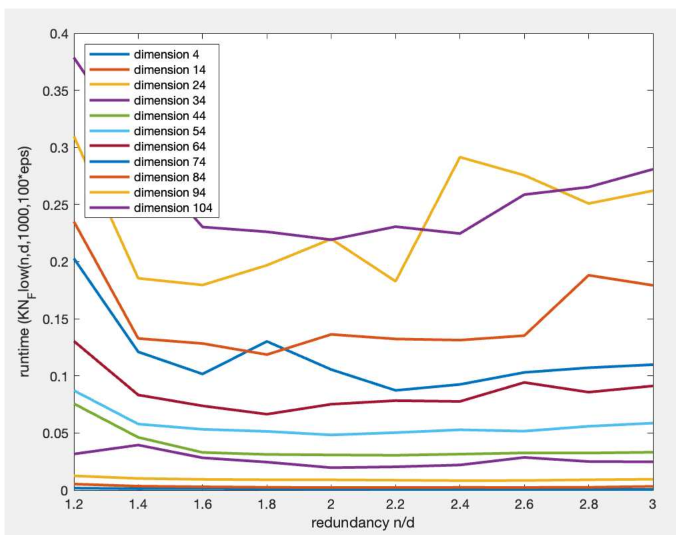

Figure 6.2: Runtine (in seconds) on the y-axis with redundancy $\frac{n}{d}$ on the x-axis.

6.6 Critical Points of N_F

At a critical point t^* we have

$$N_F(t^*) = \sum_{k=1}^{n} \left(\prod_{j=1}^{n} \frac{||f_j||}{||f_k||} \right)^{\frac{4}{n}} ||f_k||^4 \tag{6.19}$$

$$= \sum_{k=1}^{n} \left(\prod_{j=1}^{n} ||f_j||^4 \right)^{\frac{1}{n}} \tag{6.20}$$

$$= n \cdot \mathrm{gm}(\{||f_1||^4, \cdots, ||f_n||^4\}), \tag{6.21}$$

where $\mathrm{gm}(\{||f_1||^4, \cdots, ||f_n||^4\})$ is the geometric mean of the set $\{||f_1||^4, \cdots, ||f_n||^4\}$. The arithmetic-geometric mean inequality says

$$\frac{x_1 + \cdots + x_n}{n} \geq \left(\prod_j x_j \right)^{\frac{1}{n}}$$

with equality iff $x_1 = x_2 = \cdots = x_n$. Hence

$$\mu(F) \geq N_F(t^*) \tag{6.22}$$

with equality iff $||f_1|| = ||f_2|| = \cdots = ||f_n||$ and thus iff $t^* = 0$.

6.7 What Direction is Fe^{t^*}?

While the algorithm still remains a bit of a mystery, we can investigate the direction of Fe^{t^*} by projecting our movement in $\mathbb{C}^{d\times n}$ to $Gr(d,n)$ using [14]. Our motion in $\mathbb{C}^{d\times n}$ is $\left(Fe^{t^*} - F\right)$ and projecting to $T_F Gr(d,n)$ requires we compute

$$\left(Fe^{t^*} - F\right)(I - F^* F).$$

Let's introduce some notation first. Let $\nu_i = ||f_i||$ and $\alpha = \text{gm}(\{\nu_1, \cdots, \nu_n\})$ so that e^{t^*} correspond to the diagonal matrix

$$D = \begin{pmatrix} \frac{\alpha}{\nu_1} & 0 & \cdots & 0 \\ 0 & \frac{\alpha}{\nu_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\alpha}{\nu_n} \end{pmatrix}.$$

Then, as before we have

$$(FD - F)(I - F^*F) = FD - FDF^*F$$

because $F(I - F^*F) = 0$. Now, $FD - FDF^*F$ is a $d \times n$ matrix, let's look at its k^{th} column.

$$(FD - FDF^*F)_k = \frac{\alpha}{\nu_k} f_k - \sum_{l=1}^{n} \frac{\alpha}{\nu_l} \langle f_l, f_k \rangle f_l \tag{6.23}$$

$$= \frac{\alpha}{\nu_k} \sum_{l=1}^{n} \langle f_l, f_k \rangle f_l - \sum_{l=1}^{n} \frac{\alpha}{\nu_l} \langle f_l, f_k \rangle f_l \tag{6.24}$$

$$= \sum_{l=1}^{n} \left(\frac{\alpha}{\nu_k} - \frac{\alpha}{\nu_l} \right) \langle f_l, f_k \rangle f_l. \tag{6.25}$$

It follows that $FD - FDF^*F = FSF^*F$ where S is the skew symmetric matrix $S_{k,l} = \frac{\alpha}{\nu_k} - \frac{\alpha}{\nu_l}$,

$$S = \begin{pmatrix} 0 & \frac{\alpha}{\nu_1} - \frac{\alpha}{\nu_2} & \frac{\alpha}{\nu_1} - \frac{\alpha}{\nu_3} & \cdots & \frac{\alpha}{\nu_1} - \frac{\alpha}{\nu_n} \\ \frac{\alpha}{\nu_2} - \frac{\alpha}{\nu_1} & 0 & \frac{\alpha}{\nu_2} - \frac{\alpha}{\nu_3} & \cdots & \frac{\alpha}{\nu_2} - \frac{\alpha}{\nu_n} \\ & & 0 & & \\ \vdots & & & \ddots & \vdots \\ \frac{\alpha}{\nu_n} - \frac{\alpha}{\nu_1} & \cdots & & & 0 \end{pmatrix}.$$

Notice S satisfies the identity $S_{i,j} = S_{1,j} - S_{1,i}$, so S is completely determined by the $n-1$ nonzero entries in the first row.[12] Hopefully this computation can lead to further insights with regard to the Paulsen problem.

[12] $n-1$ is the number of degrees of freedom in G.

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