DIMER MONOMER

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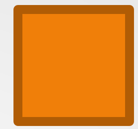
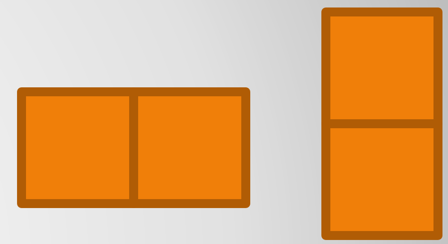
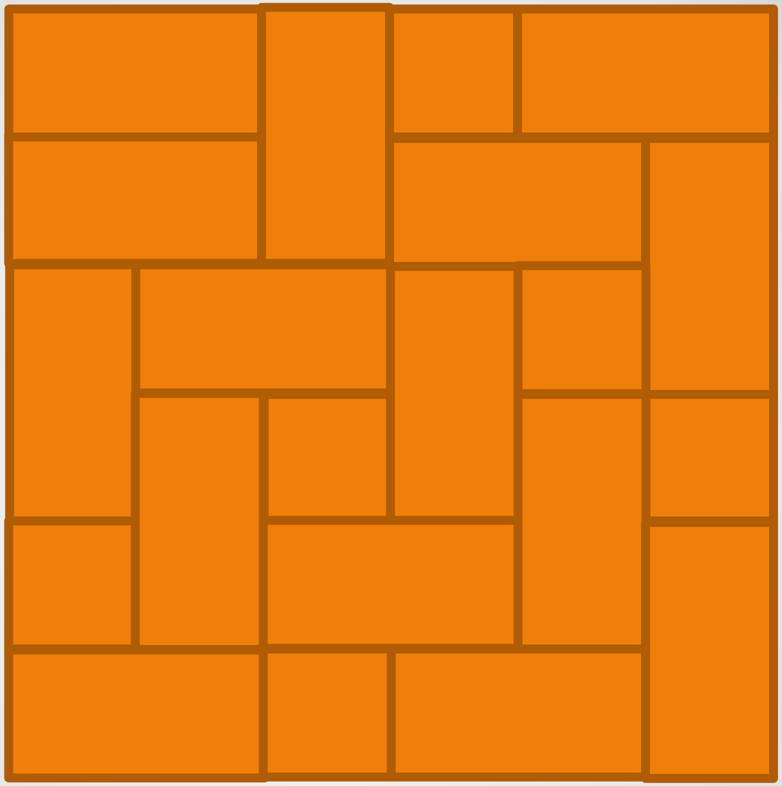
Monomer-Dimer Tilings

- Planar monomer-dimer tilings
- Monomer – 1 by 1 tile 
- Dimer – 2 by 1 tile or 1 by 2 



Monomer-Dimer Tilings

- What properties does a “random” monomer-dimer tiling of a region have?
  - Expected number of Dimers?
  - Is a given cell more likely to be a monomer?
- How can we generate a uniform, random monomer-dimer tiling of a region?

Fundamental Questions:

- Methods
  - Computer-based simulations
    - Kastelyn’s Theorem
  - Probabilistic Models (matrices)
  - Analytical Models
- Sage Math (Python) was used throughout the year to aid us

Monomer-Dimer Tilings

Beginning Work

- N by N grids w/ 2 monomers, rest dimers
- 4 by 4
  - 552 possible tilings
  - Monomer's probable locations: corners, sides, center
- 6 by 6
  - 363,536 possible tilings
  - Probable Locations:
- 8 by 8
  - 1,760,337,760 tilings

1	6	2	2	6	1
6	5	4	4	5	6
2	4	3	3	4	2
2	4	3	3	4	2
6	5	4	4	5	6
1	6	2	2	6	1

- Recursive Formulas:
  - $F_N = 1 + \frac{2^{N-2}+1}{2^{N-1}}E_N + \frac{1}{8}F_{N-3} + \frac{N-2}{2^{N-1}}(F_0 + E_{N-2}) + \sum_{k=1}^{N-4} \frac{N-k-2}{2^{N-k}}(F_k + E_{N-k-2})$
  - $E_N = \sum_{k=0}^{N-3} \frac{1}{2^{k+1}}(F_{N-k-2} + E_k) + \frac{1}{2^{N-2}}(F_0 + E_{N-2})$
- E_N = E(Dimers)
- E_{20000} /20000 = 0.43344653343386
- Bounded … Must Converge

1 by N : Mathematical Discoveries

- Matrix A_8:
  - "Characteristic matrix" for monomer dimer tilings of 8 X 1 region
  - A_8^N represents "characteristic matrix" for 8 X N region
  - Top-left entry represents the number of tilings possible
  - 8 X N plane: 256 X 256 matrix
  - Created using numerical methods/coding

Enumerating 8 X N Tilings

- Formula for total number of tilings
- This is a 136th degree recursive formula:

$$T_N = 133^* T_{N-1} + 9461^* T_{N-2} - 444475^* T_{N-3} + \ldots - 23797\, T_{N-133} + 1027\, T_{N-134} + 21\, T_{N-15} - T_{N-136}$$

- $\lim_{N \to \infty} T_N / T_{N-1} \approx 170.8771576$

Recursive Formula for 8XN

Implications

- Cayley-Hamilton theorem for a square matrix
- Satisfies a linear recurrence
- Could do this for any kXN - first to do 8XN and find the recurrence for it
- Counting problem with computational difficulties for high N

Extension

- Use the same “characteristic matrix” for discerning vertical and horizontal dimers
- Able to characterize our tilings with more detail

- Original 3X3

$$\begin{bmatrix} 3 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Updated 3X3

$$\begin{bmatrix} 2v+1 & hv+h & h & h^2 & hv+h & h^2 & h^2 & h^3 \\ v+1 & 0 & h & 0 & h & 0 & h^2 & 0 \\ 1 & h & 0 & 0 & h & h^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v+1 & h & h & h^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & h & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Extension

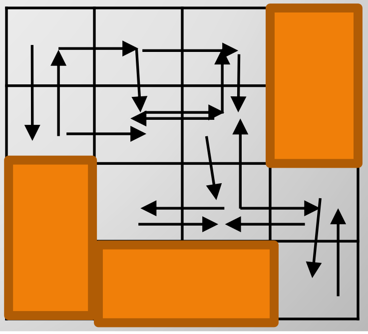
- Found a relationship for A_N :

$$A_k = \begin{bmatrix} A_{k-1} & h * A_{k-1} \\ A_{k-1} & 0 \end{bmatrix} + \begin{bmatrix} v * A_{k-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Can extend this to our simple numerical example
- Create more characteristic equations for k X N regions

Extension

- Static -> Dynamic
- Explored various regions: N by N, 2 by N, 1 by N
- Began as all monomers, “paired” to become dimers.
- Pair until “frozen”



Generating tilings

- Creating probability distributions for any MXN area (E(dimers))
  - Created every possible tiling by coding it
    - Represented by matrix of 0’s, 1’s, 2’s
  - Use these tilings to create probability distibutions
- Computational issues:
  - 6 X 6 possibilities aren’t able to be calculated in less than 2 hours
  - Probabilities require ‘permanent’ calculation – even more difficult

Final work

- Finding tiling distribution – need to find probabilities for ending with each
- Finding a formula for E(dimers) as a recursion? Possibly the same recurrence with different initial conditions?
- Applications in diatomic nuclear bonding and ice-formations

Future Potential Research

Thank you.