SIGN PATTERNS THAT ALLOW A POSITIVE OR NONNEGATIVE LEFT INVERSE*

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Abstract. An m by n sign pattern S is an m by n matrix with entries in {+, −, 0}. Such a sign pattern allows a positive (resp., nonnegative) left inverse, provided that there exist an m by n matrix A with the sign pattern S and an n by m matrix B with only positive (resp., nonnegative) entries satisfying BA = In, where In is the n by n identity matrix. For m > n ≥ 2, a characterization of m by n sign patterns with no rows of zeros that allow a positive left inverse is given. This leads to a characterization of all m by n sign patterns with m ≥ n ≥ 2 that allow a positive left inverse, giving a generalization of the known result for the square case, which involves a related bipartite digraph. For m ≥ n, m by n sign patterns with all entries in {+, 0} and m by 2 sign patterns with m ≥ 2 that allow a nonnegative left inverse are characterized, and some necessary or sufficient conditions for a general m by n sign pattern to allow a nonnegative left inverse are presented.

Key words. bipartite digraph, nonnegative left inverse, positive left inverse, positive left null-vector, sign pattern, strong Hall

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1. Introduction. An m by n sign pattern S = [sij] is an m by n matrix with entries in {+, −, 0}. If a sign pattern S has all entries in {+, 0}, then S is a nonnegative sign pattern. A subpattern of S is an m by n sign pattern U = [uij] such that uij = 0 whenever sij = 0. If U is a subpattern of S, then S is a superpattern of U. The sign pattern class Q(S) of an m by n sign pattern S is the set of m by n matrices A = [aij] such that sgn(aij) = sij for all i, j. If A ∈ Q(S), then A is a realization of S.

Let A = [aij] be an m by n matrix. If each entry of A is positive (resp., nonnegative), then A is positive (resp., nonnegative), written A > 0 (resp., A ≥ 0). A left inverse of an m by n matrix A is an n by m matrix B such that BA = In, where In denotes the n by n identity matrix. If B > 0, then B is a positive left inverse (abbreviated as PLI) of A. If B ≥ 0, then B is a nonnegative left inverse (abbreviated as NLI) of A. In general, neither a PLI nor an NLI of A is unique. It is easily verified that A has a left inverse if and only if rank A = n; thus, if A has a left inverse, then necessarily m ≥ n. An m by n sign pattern S allows a positive (resp., nonnegative) left inverse, provided there exist A ∈ Q(S) and a matrix B > 0 (resp., B ≥ 0) such that BA = In. Note that if P1 and P2 are permutation matrices, then S allows a PLI (resp., an NLI) if and only if P1SP2 allows a PLI (resp., an NLI).

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A motivation for studying PLIs and NLIs comes from determining the qualitative behavior of solutions of $A^T x = b$ with $A$ an $m$ by $n$ matrix; see, for example, [2, Chapter 1] and [5] for applications in economics. Specifically, $A$ has a PLI (resp., an NLI) if and only if for each $n$ by 1 nonzero vector $b \geq 0$ there exists an $m$ by 1 vector $x > 0$ (resp., $x \geq 0$) satisfying $A^T x = b$ or equivalently $x^T A = b^T$; see Proposition 4.1 for a proof.

Square sign patterns with entries in \{+, -\} that allow a positive (left) inverse are characterized in [6], and this characterization is extended to arbitrary square sign patterns in [4]. These results are summarized in [2, section 9.2]. In section 2, we characterize nonsquare sign patterns that allow a PLI, and combine the square and nonsquare characterizations. In section 3, we discuss sign patterns that allow an NLI. More specifically, we characterize nonnegative sign patterns and $m$ by 2 sign patterns in [4]. These results are summarized in [2, section 9.2]. In section 2, we characterize nonnegative sign patterns and $m$ by 2 sign patterns with $m \geq 2$ that allow an NLI, and present some necessary or sufficient conditions for general $m$ by $n$ sign patterns with $m \geq n$ to allow an NLI. We conclude with some remarks in section 4.

**2. Positive left inverses.** We begin this section with a necessary and sufficient condition for a column sign pattern to allow a PLI or an NLI.

**Proposition 2.1.** Let $S = (s_1, s_2, \ldots, s_m)^T$ be an $m$ by 1 sign pattern. Then the following are equivalent:

(i) $S$ has a $+$ entry.

(ii) $S$ allows a PLI.

(iii) $S$ allows an NLI.

**Proof.** Suppose there is an index $i \in \{1, \ldots, m\}$ with $s_i = +$. For $j \in \{1, \ldots, m\}$, set

$$a_j = \begin{cases} 
1 & \text{if } j \neq i \text{ and } s_j = +, \\
-1 & \text{if } j \neq i \text{ and } s_j = -, \\
0 & \text{if } j \neq i \text{ and } s_j = 0, \\
1 + \sum_{k \neq i} |a_k| & \text{if } j = i.
\end{cases}$$

Then $A = (a_1, \ldots, a_m)^T \in Q(S)$, and $(1, 1, \ldots, 1)A = 1 + \sum_{k \neq i} (|a_k| + a_k) = c > 0$. This implies that $\frac{1}{c}(1, 1, \ldots, 1)$ is a PLI of $A$. Thus, $S$ allows a PLI.

It is clear that (ii) implies (iii). Next, suppose that the sign pattern $S$ allows an NLI. Then there exist $A = (a_1, \ldots, a_m)^T \in Q(S)$ and $B = (b_1, \ldots, b_m) \geq 0$ such that $BA = 1$, i.e., $\sum_{j=0}^m b_j a_i = 1 > 0$. This implies that there exists an $i$ with $b_i a_i > 0$. Since $b_i \geq 0$, it follows that $b_i > 0$; hence $a_i > 0$ and thus $s_i = +$. \(\square\)

We now consider $m \geq n \geq 2$. The following two lemmas give necessary conditions for a sign pattern to allow a PLI.

**Lemma 2.2.** Let $S$ be an $m$ by $n$ sign pattern with $n \geq 2$. If $S$ allows a PLI, then each column of $S$ has a $+$ and a $-$ entry.

**Proof.** Suppose that there exist $A \in Q(S)$ and an $n$ by $m$ positive matrix $B$ such that $BA = I_n$. Let $i \in \{1, 2, \ldots, n\}$. Since the $(i, i)$-entry of $BA$ is 1 and each entry of $B$ is positive, it follows that some entry in column $i$ of $A$ is positive. Hence, column $i$ of $S$ has a $+$ entry.

Since $n \geq 2$, there exists $j \in \{1, \ldots, n\}$ with $j \neq i$. The $(j, i)$-entry of $BA$ is 0, so since $B > 0$ and at least one entry of column $i$ of $A$ is positive, it follows that at least one entry of column $i$ of $A$ must be negative. Thus, column $i$ of $S$ has a $-$ entry. \(\square\)

An $m$ by $n$ sign pattern $S$ with $n \geq 2$ is strong Hall, provided that for every nonempty proper subset $\gamma$ of $\{1, 2, \ldots, n\}$ the submatrix of $S$ consisting of the columns...
indexed by $\gamma$ has nonzero entries in at least $|\gamma| + 1$ rows (see [3]). Note that if $\mathcal{S}$ is strong Hall, then necessarily $m \geq n$. Also, for $m \geq n$, $\mathcal{S}$ is not strong Hall if and only if there exist permutation matrices $P_1$ and $P_2$ such that

$$P_1SP_2 = \begin{bmatrix} S_{11} & S_{12} \\ O & S_{22} \end{bmatrix},$$

where $S_{11}$ is a $k$ by $\ell$ sign pattern for some integers $k$, $\ell$ with $n > \ell \geq 1$ and $k \leq \ell$.

**Lemma 2.3.** Let $\mathcal{S}$ be an $m$ by $n$ sign pattern with $n \geq 2$. If $\mathcal{S}$ allows a PLI, then $\mathcal{S}$ is strong Hall.

**Proof.** To prove the contrapositive, assume that $\mathcal{S}$ is not strong Hall. If $m < n$, then it is clear that $\mathcal{S}$ does not allow a PLI. Otherwise, without loss of generality, we may assume that $\mathcal{S}$ has the form (2.1). If $k < \ell$, then the first $\ell$ columns of each realization of $\mathcal{S}$ are linearly dependent, and hence $\mathcal{S}$ does not allow a PLI.

Otherwise, $k = \ell < n$. Suppose that there exists a matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix} \in Q(\mathcal{S})$ with a left inverse $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $B_{11}$ is an $\ell$ by $\ell$ matrix. Clearly, the $\ell$ by $\ell$ matrix $A_{11}$ is invertible, and by $BA = I_n$, it follows that $B_{21}A_{11} = O$. Thus, $B_{21}A_{11} = O$, and since $A_{11}$ is invertible, the $(n - \ell)$ by $\ell$ matrix $B_{21} = O$. Since $n - \ell \geq 1$ and $\ell \geq 1$, every left inverse of a matrix in $Q(\mathcal{S})$ has a zero entry, and hence $\mathcal{S}$ does not allow a PLI.

Note that if $\mathcal{S}$ is a square sign pattern of order $n \geq 2$, then $\mathcal{S}$ is strong Hall if and only if $\mathcal{S}$ is fully indecomposable (see [3]), and $\mathcal{S}$ allows a PLI if and only if $\mathcal{S}$ allows a positive inverse. The next theorem, first proved in [4], provides a characterization of square sign patterns that allow a positive inverse. In order to recall this characterization, we use the following definition as in [1] and [2]. Let $S = [s_{ij}]$ be an $m$ by $n$ sign pattern. The **bipartite digraph** $D(\mathcal{S})$ of $\mathcal{S}$ is the digraph with row vertices $u_1, \ldots, u_m$, column vertices $v_1, \ldots, v_n$, an arc $u_i \rightarrow v_j$ if $s_{ij} = +$, and an arc $v_j \rightarrow u_i$ if $s_{ij} = -$. Note that there exists at most one arc between $u_i$ and $v_j$.

**Theorem 2.4** (see [2, Theorem 9.2.1]). An $m$ by $n$ square sign pattern $\mathcal{S}$ with $n \geq 2$ allows a positive (left) inverse if and only if $\mathcal{S}$ is strong Hall and the bipartite digraph $D(\mathcal{S})$ of $\mathcal{S}$ is strongly connected.

Let $\mathcal{S}$ be an $m$ by $n$ sign pattern and let $D(\mathcal{S})$ be its bipartite digraph. A **strong component** of $D(\mathcal{S})$ is a maximal strongly connected subdigraph of $D(\mathcal{S})$. If $\alpha$ is a strong component of $D(\mathcal{S})$, then $|\alpha|$ denotes the number of vertices in $\alpha$.

**Remark 2.5.** Let $\alpha$ be a strong component of $D(\mathcal{S})$. Since $D(\mathcal{S})$ is a bipartite digraph with no cycles of length 2, it follows that if $|\alpha| \geq 2$, then $\alpha$ has at least two row vertices and at least two column vertices.

Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be the strong components of $D(\mathcal{S})$. The **condensed digraph** $CD(\mathcal{S})$ of $\mathcal{S}$ has vertices $\alpha_1, \alpha_2, \ldots, \alpha_t$ and an arc $\alpha_i \rightarrow \alpha_j$ if and only if $i \neq j$ and $D(\mathcal{S})$ has at least one arc from a vertex in $\alpha_i$ to a vertex in $\alpha_j$. A strong component $\alpha_i$ of $D(\mathcal{S})$ is a **source** if there is no arc coming into $\alpha_i$ in $CD(\mathcal{S})$ and there is at least one arc coming out of $\alpha_i$ in $CD(\mathcal{S})$; $\alpha_i$ is a **sink** if there is no arc coming out of $\alpha_i$ in $CD(\mathcal{S})$ and there is at least one arc coming into $\alpha_i$ in $CD(\mathcal{S})$; and $\alpha_i$ is **isolated** if there are no arcs coming into or out of $\alpha_i$ in $CD(\mathcal{S})$.

**Lemma 2.6.** Let $\mathcal{S}$ be an $m$ by $n$ sign pattern which has a $+$ and a $-$ entry in each column and no rows of zeros. Then the following hold for $D(\mathcal{S})$:

(i) Each sink and source strong component of $D(\mathcal{S})$ has at least one row vertex.

(ii) Each isolated strong component has at least two row vertices.

**Proof.** (i) Let $\alpha$ be a sink or source strong component. If $|\alpha| = 1$, then since each column of $\mathcal{S}$ has a $+$ and a $-$ entry, it follows that no sink or source strong component
consists of exactly one column vertex. Hence, $\alpha$ is a row vertex. If $|\alpha| \geq 2$, then Remark 2.5 implies that $\alpha$ has at least one row vertex.

(ii) By the assumptions on the rows and columns of $S$, there is no isolated strong component with exactly one vertex. Hence, by Remark 2.5, each isolated strong component has at least two row vertices. \hfill $\square$

Let $A$ be an $m$ by $n$ matrix with $m \geq n$. If there exists an $m$ by 1 vector $y > 0$ satisfying $y^T A = 0$, then $y^T$ is a positive left nullvector of $A$. The following theorem gives a characterization of nonsquare sign patterns with no rows of zeros that allow a PLI. Note that conditions for such a sign pattern to allow a PLI are weaker than those for square sign patterns (Theorem 2.4), although the bipartite digraph is used in our proof for a nonsquare sign pattern.

**Theorem 2.7.** For $m > n \geq 2$, let $S$ be an $m$ by $n$ sign pattern with no rows of zeros. Then the following are equivalent:

(i) There exists a matrix $A \in Q(S)$ with a PLI and a positive left nullvector.

(ii) $S$ allows a PLI.

(iii) Each column of $S$ has a $+$ and a $-$ entry, and $S$ is strong Hall.

**Proof.** Clearly, (i) implies (ii). By Lemmas 2.2 and 2.3, (ii) implies (iii).

To prove that (iii) implies (i), assume that $S$ is strong Hall and that $S$ has a $+$ and a $-$ entry in each column. We claim that it suffices to show that there exists an $m$ by $(m-n)$ sign pattern $C$ so that the $m$ by $m$ sign pattern $[S | C]$ allows a positive (left) inverse. To prove this claim, suppose there exists an $m$ by $m$ matrix $[A | C] \in Q([S | C])$ with a positive (left) inverse $[B_1 | B_2]$, where $B_1$ is an $n$ by $m$ positive matrix and $B_2$ is an $(m-n)$ by $m$ positive matrix. Then $B_1 A = I_n$ and hence $B_1$ is a PLI of $A$, implying that $S$ allows a PLI. In addition, since $B_2 A = O$ and $B_2$ has at least one positive row, $A$ has a positive left nullvector. Therefore, by Theorem 2.4, it suffices to find an $m$ by $(m-n)$ sign pattern $C$ such that the $m$ by $m$ sign pattern $[S | C]$ is strong Hall and its bipartite digraph $D([S | C])$ is strongly connected.

Consider the bipartite digraph $D(S)$ of $S$. Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be its strong components, where $\alpha_1, \ldots, \alpha_k$ are sinks, $\alpha_{k+1}, \ldots, \alpha_{k+\ell}$ are sources, $\alpha_{k+\ell+1}, \ldots, \alpha_{k+\ell+r}$ are isolated, and $\alpha_{k+\ell+r+1}, \ldots, \alpha_t$ are neither sinks, sources, nor isolated strong components. By Lemma 2.6 (i), each sink and source strong component has a row vertex. Let $r_i$ be a fixed row vertex of $\alpha_i$ for each $i \in \{1, \ldots, k+\ell\}$. Also, by Lemma 2.6 (ii), each isolated strong component has at least two row vertices. Let $r_i^+, r_i^-$ be distinct fixed row vertices of $\alpha_i$ for each $i \in \{k+\ell+1, \ldots, k+\ell+r\}$. Let $C_{n+1}$ be the $m$ by 1 column sign pattern with nonzero $j$th coordinate:

\[
\begin{cases}
  + & \text{if } u_j \in \{r_1, \ldots, r_k\} \cup \{r_{k+\ell+1}, \ldots, r_{k+\ell+r}\}, \\
  - & \text{if } u_j \in \{r_{k+1}, \ldots, r_{k+\ell}\} \cup \{r_{k+\ell+1}, \ldots, r_{k+\ell+r}\}, \\
  + & \text{otherwise.}
\end{cases}
\]

Then $D([S | C_{n+1}])$ is obtained from $D(S)$ by appending a new column vertex $c_{n+1}$, and arcs $r_j \to c_{n+1}$ if $r_j$ is in a sink component; $c_{n+1} \to r_j$ if $r_j$ is in a source component; $r_j^+ \to c_{n+1}$ and $c_{n+1} \to r_j^-$ if $r_j^+$ and $r_j^-$ are in the same isolated component; as well as some additional arcs coming into vertex $c_{n+1}$.

To prove that $D([S | C_{n+1}])$ is strongly connected, we show that for each vertex $w$ of $D(S)$ there exists in $D([S | C_{n+1}])$ a walk from $c_{n+1}$ to $w$ and a walk from $w$ to $c_{n+1}$. Note that if $w$ is not in an isolated strong component of $D(S)$, then there is a walk from $w$ to a vertex in a sink strong component $\alpha_i$ of $D(S)$ ($i \in \{1, \ldots, k\}$). Since $\alpha_i$ is strongly connected, this walk from $w$ can be extended to the fixed row vertex $r_i$ of $\alpha_i$. By (2.2), there is an arc $r_i \to c_{n+1}$ in $D([S | C_{n+1}])$. Hence, there is
a walk from $w$ to $c_{n+1}$. Similarly, there is a walk from $c_{n+1}$ to $w$.

Next, suppose that $w$ is a vertex in an isolated strong component $\alpha_i$ in $D(S)$ ($i \in \{k + \ell + 1, \ldots, k + \ell + r\}$). Since $\alpha_i$ is strongly connected, there is a walk from $w$ to the fixed row vertex $r_i^-$ of $\alpha_i$. By (2.2), there are arcs $r_i^- \to c_{n+1}$ and $c_{n+1} \to r_i^+$ in $D([S | C_{n+1}])$. Since $\alpha_i$ is strongly connected, there is a walk from $r_i^+$ to $w$. Thus, there exist a walk from $w$ to $c_{n+1}$ and a walk from $c_{n+1}$ to $w$.

Finally, define $C_{n+2}, \ldots, C_m$ to be $m$ by 1 column sign patterns, each having no zeros, at least one $+$, and at least one $-$ entry. Then it is easily verified that $D([S | C_{n+1} | \ldots | C_m])$ is strongly connected. Since $S$ is strong Hall and $[C_{n+1} | \ldots | C_m]$ has no zeros, it is clear that $[S | C_{n+1} | \ldots | C_m]$ is strong Hall, completing the proof.

Example 2.8. Consider the 6 by 4 sign pattern

$$S = \begin{bmatrix} + & - & 0 & 0 \\ - & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & - \\ 0 & 0 & - & + \\ 0 & 0 & 0 & - \end{bmatrix}$$

with

$$D(S)$$

Each column of $S$ has a $+$ and a $-$ entry, and $S$ is strong Hall. Thus, by Theorem 2.7, $S$ allows a PLI. However, $D(S)$ is not strongly connected, illustrating a distinction between the nonsquare and square cases (see Theorem 2.4). In fact, $D(S)$ has one sink strong component $\alpha_1$ that consists of vertex $u_6$, one source strong component $\alpha_2$ with vertices $u_4, u_5, v_3, v_4$, and one isolated strong component $\alpha_3$ with vertices $u_1, u_2, u_3, v_1, v_2$. Taking $r_1 = u_6$, $r_2 = u_5$, $r_3^+ = u_1$, and $r_3^- = u_2$ in the proof of Theorem 2.7, it follows that

$$C_5 = \begin{bmatrix} - \\ + \\ + \\ - \\ + \end{bmatrix}.$$  

The last column $C_6$ can be taken to be any 6 by 1 column having a $+$ and a $-$ entry, and no zeros. Let $C = [C_5 | C_6]$. In order to determine a matrix $[A | C] \in Q([S | C])$ with a positive (left) inverse $[B_1 | B_2]$, the algorithm described in the proof of [2, Theorem 9.2.1] can be used. Then $B_1$ is a PLI of $A$, and the rows of $B_2$ are positive left nullvectors of $A$.

The next lemma is used to prove Theorem 2.10, in which square and nonsquare cases are combined.
LEMMA 2.9. Let $S$ be an $m$ by $n$ sign pattern with $n \geq 2$ and let $T$ be the sign pattern obtained from $S$ by deleting the rows of zeros in $S$. Then

(i) $S$ is strong Hall if and only if $T$ is strong Hall, and

(ii) $S$ allows a positive (nonnegative) left inverse if and only if $T$ allows a positive (nonnegative) left inverse.

Proof. Without loss of generality, assume that $S = [O]$. The proof of (i) follows immediately from the definition of strong Hall.

To prove (ii), suppose first that $S$ allows a PLI. Let $A_1 \in Q(T)$ and $A = [A_1 O] \in Q(S)$ have $B = [B_1 B_2]$ as a PLI. Then $B_1 A_1 = I_n$ and hence $T$ allows a PLI. Next, suppose that $T$ allows a PLI. Let $A_1 \in Q(T)$ have $B_1$ as a PLI. With $J$ denoting the all 1’s matrix, it is easily verified that $B = [B_1 J]$ is a PLI for $A = [A_1 O] \in Q(S)$. Hence, $S$ allows a PLI. The nonnegative case can be shown by a similar argument to that above.

THEOREM 2.10. Let $m \geq n \geq 2$. The $m$ by $n$ sign pattern $S$ allows a PLI if and only if

(i) each column of $S$ has a + and a − entry;

(ii) $S$ is strong Hall; and

(iii) the bipartite digraph $D(S_1)$ of $S_1$ is strongly connected whenever $S$ is permutationally equivalent to $[S_1 O]$ and $S_1$ is an $n$ by $n$ sign pattern.

Proof. For the necessity, suppose that $S$ allows a PLI. Then (i) and (ii) follow from Lemmas 2.2 and 2.3, and (iii) follows from Theorem 2.4 and Lemma 2.9 (ii).

For the sufficiency, first assume $m = n$. Then $S$ is permutationally equivalent to $S_1$, and by Theorem 2.4 the result follows from (ii) and (iii). Next, suppose that $m > n$. If $S$ has no rows of zeros, then, by Theorem 2.7, the result follows from (i) and (ii). Otherwise, without loss of generality, assume that $S = [S_1 O]$, where $T$ has no rows of zeros. By Lemma 2.9 (i), it follows from (ii) that $T$ is strong Hall. Thus, if $T$ is an $n$ by $n$ sign pattern, then (iii) and Theorem 2.4 imply that $T$ allows a PLI. By Lemma 2.9 (ii), this implies that $S$ allows a PLI. Otherwise, since it follows from (i) that each column of $T$ has a + and a − entry, Theorem 2.7 implies that $T$ allows a PLI. Therefore, by Lemma 2.9 (ii), $S$ allows a PLI.

Remark 2.11. For $m \geq n \geq 2$, let $S$ be an $m$ by $n$ sign pattern. Then the following hold:

(i) If $S$ satisfies (i), (ii), and (iii) in Theorem 2.10, then so does every superpattern of $S$. Hence, if $S$ allows a PLI, then every superpattern of $S$ allows a PLI.

(ii) Suppose that $S = [S_1 O]$, where $S_1$ is a square sign pattern, satisfies (iii) in Theorem 2.10. Then, in contrast with Theorem 2.7 (i), there is no matrix $A = [A_1 O] \in Q(S)$ with a PLI that also has a positive left nullvector, since the equation $[y^T \ z^T] [A_1 O] = 0$ and the fact that $A_1$ is nonsingular together imply that $y = 0$.

The following theorem gives sufficient conditions for an $m$ by $n$ sign pattern with $m > n \geq 1$ to have a realization with a PLI and a positive left nullvector.

THEOREM 2.12. Let $S$ be an $m$ by $n$ sign pattern with $m > n$ and let $T$ be the $n$ by $n$ sign pattern obtained from $S$ by deleting the rows of zeros in $S$. Then

(i) If $n = 1$ and $T$ has a + and a − entry, then there exists a matrix in $Q(S)$ with a PLI and a positive left nullvector.

(ii) If $t > n \geq 2$ and $T$ allows a PLI, then there exists a matrix in $Q(S)$ with a PLI and a positive left nullvector.

Proof. (i) By Proposition 2.1, a + entry implies the existence of $A \in Q(S)$ with a
by induction, it can be shown that $S$ decomposable sign patterns that allow a nonnegative (left) inverse.

Clearly, $S$ has a nonnegative column having only + or 0 entries. For ease of notation, we sometimes use $(M)_{ij}$ to denote the $(i, j)$-entry of a matrix $M$.

Proposition 3.2. For $m > n \geq 2$, let $S$ be an $m$ by $n$ strong Hall sign pattern with a + and a − entry in each column, and let $T$ be the $t$ by $n$ sign pattern obtained from $S$ by deleting the rows of zeros in $S$. If $t > n$, then $S$ allows an NLI. If $t = n$, then $S$ allows an NLI if and only if $D(T)$ is strongly connected.

Proof. The result follows directly from Theorem 2.10 and the fact that if $S$ allows a PLI, then $S$ allows an NLI.

Let $I_n$ denote the $n$ by $n$ sign pattern with $I_n$ as a realization, i.e., $I_n \in Q(I_n)$. Clearly, $I_n$ allows an NLI. Thus, in order to allow an NLI, an $m$ by $n$ sign pattern with $m \geq n$ need not have a − entry in each column as is required to allow a PLI (see Lemma 2.2), but clearly must have a + entry in each column. We first consider the case that $S$ has a nonnegative column having only + or 0 entries. For ease of notation, we sometimes use $(M)_{ij}$ to denote the $(i, j)$-entry of a matrix $M$.

Proposition 3.3. For $m \geq n \geq 2$, let $S$ be an $m$ by $n$ sign pattern with at least one nonnegative column. If $S$ allows an NLI, then each nonnegative column has at most $m - n + 1$ positive entries.


Proof. Let $B$ be an NLI of $A \in Q(S)$, and let $t$ be the number of positive entries in any nonnegative column of $A$. Without loss of generality, assume that the first column of $A$ is a nonnegative column with its first $t$ entries positive. Since $(BA)_{h1} = 0$ for each $h \in \{2, \ldots, n\}$, it follows that $B$ has the block form $B = [B_{ij}]$ with $1 \leq i, j \leq 2$, where the $(2,1)$-block $B_{21}$ is the $(n-1)$ by $t$ zero matrix. Hence, the equality rank $B = n$ implies that the rank of the $(n-1)\times(m-t)$ matrix $B_{22}$ is $n-1$. Thus, $n-1 \leq m-t$ and the result follows.

If all columns are nonnegative, then the following result gives a necessary and sufficient condition for such a sign pattern to allow an NLI.

Theorem 3.4. For $m \geq n \geq 1$, let $S$ be an $m \times n$ nonnegative sign pattern. Then $S$ allows an NLI if and only if $S$ is permutationally equivalent to

\[
\begin{bmatrix}
I_n \\
T
\end{bmatrix},
\]

where $T$ is an $(m-n) \times n$ nonnegative sign pattern.

Proof. The case $n = 1$ follows directly from Proposition 2.1. Suppose that $n \geq 2$.

For the sufficiency, assume without loss of generality that $S = \begin{bmatrix} I_n \\ T \end{bmatrix}$. Since $[I_n \mid O]A = I_n$, it follows that $S$ allows an NLI.

For the necessity, suppose that $S = [s_{ij}]$ allows an NLI; i.e., there exist $A = [a_{ij}] \in Q(S)$ and an $n \times m$ nonnegative matrix $B = [b_{ij}]$ such that $BA = I_n$. Let $i \in \{1, \ldots, n\}$. Since $(BA)_{ii} = 1$, there exists $j_i \in \{1, \ldots, m\}$ such that $b_{i j_i} a_{j_i i} > 0$. This implies that $s_{j_i i} = +$. Also, for each $k \in \{1, \ldots, n\} \setminus \{i\}$, $(BA)_{ik} = 0$ implies that $b_{ij_k} a_{k j_i} = 0$. Thus, row $j_i$ of $S$ is equal to row $i$ of $I_n$. As this holds for each $i \in \{1, \ldots, n\}$, the result follows.

Remark 3.5. Let $S = \begin{bmatrix} I_p \\ T \end{bmatrix}$ be the $m \times n$ nonnegative sign pattern with $m \geq n \geq 2$, where $T$ is the sign pattern with all entries positive. Then, by Theorem 3.4, $S$ allows an NLI. However, in contrast with Remark 2.11 (i), Theorem 3.4 implies that no nonnegative superpattern of $S$ (except $S$ itself) allows an NLI.

Next, we consider sign patterns that have at least one nonnegative column and at least one column with a $+$ and a $-$ entry. We use $e_i$ to denote the $i$th column vector of an identity matrix.

Theorem 3.6. For $m \geq n \geq 2$, let $S$ be an $m \times n$ sign pattern that has $p \geq 1$ nonnegative columns and $n-p \geq 1$ columns with a $+$ and a $-$ entry. Suppose that $S$ allows an NLI. Then $S$ is permutationally equivalent to a matrix of the form

\[
\begin{bmatrix}
I_p & S_{12} \\
S_{21} & S_{22} \\
O & S_{32}
\end{bmatrix},
\]

where $S_{21}$ is an $r \times p$ nonnegative sign pattern with no rows of zeros, $O$ is an $s \times p$ zero matrix with $s \geq 1$, and each of the last $n-p$ columns of $S$ has a $+$ and a $-$ entry. Furthermore, if $S$ is strong Hall, then $S_{21}$ is not vacuous and has no column of zeros.

Proof. Without loss of generality, we may assume that the first $p$ columns of $S$ are nonnegative, and that each of the last $n-p$ columns of $S$ has a $+$ and a $-$ entry.
Since $S$ allows an NLI, so does the $m$ by $p$ nonnegative sign pattern consisting of the first $p$ columns of $S$. Therefore, by Theorem 3.4, we may permute the rows of $S$ to obtain a matrix of the form (3.2), where $S_{21}$ is a nonnegative matrix with no row of zeros, $O$ is an $s$ by $p$ zero matrix with $s \geq 0$, and each of the last $n - p$ columns has a + and a − entry.

Let $A$ be a matrix in $Q(S)$ that has an NLI, say $B$. Since $BA = I_n$, each of the vectors $e_1^T, \ldots, e_n^T$ is a nontrivial, nonnegative linear combination of the rows of $A$. Since the first $p$ columns of $A$ are nonnegative and $n > p$, this requires that $s \geq 1$, and we conclude that $S$ has the desired form.

If $S_{21}$ is vacuous or has a column of zeros, then $S$ has an $(m - 1)$ by 1 zero submatrix. Hence $S$ is not strong Hall, and the result follows by taking the contrapositive. \qed

**Proposition 3.7.** For $m \geq n \geq 2$, let $S$ be an $m$ by $n$ sign pattern that has $p \geq 1$ nonnegative columns and $n - p \geq 1$ columns with a + and a − entry. Suppose that $S$ allows an NLI and has the form (3.2). Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ O & A_{32} \end{bmatrix} \in Q(S)$$

have an NLI $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$, where each of $A_{21}, A_{22}, B_{12}$, and $B_{22}$ may be vacuous if $S$ is not strong Hall. Then the following hold:

1. $B_{11}$ is a diagonal matrix, and $B_{21}$ and $B_{22}$ are zero matrices.
2. The sign pattern $S_{21}$ allows an NLI.
3. If row $q$ of $S_{21}$ has at least two positive entries, then column $q$ of $B_{12}$ is a zero column.
4. Each column of $B_{12}$ has at most one positive entry. Furthermore, the sign pattern of $B_{12}$ is a subpattern of $S_{21}$.

**Proof.** Assume that $S_{21}$ is not vacuous. Since $BA = I_n$, it follows that $B_{21}A_{11} + B_{22}A_{21} = O$. Moreover, since $B_{21}, B_{22}, A_{11},$ and $A_{21}$ are nonnegative, and no row of $A_{11}$ or $A_{21}$ is all zeros, $B_{21} = O$ and $B_{22} = O$. Also, $BA = I_n$ implies that $B_{11}A_{11} + B_{12}A_{21} = I_p$. Since $B_{11}, B_{12}, A_{11},$ and $A_{21}$ are nonnegative, both $B_{11}A_{11}$ and $B_{12}A_{21}$ are diagonal matrices. Since $A_{11} \in Q(I_p)$, $A_{11}$ is an invertible diagonal matrix, and hence $B_{11}$ is a diagonal matrix. Thus, (i) is proven.

Since $B_{21}$ and $B_{22}$ are zero matrices, and $BA = I_n$, $B_{23}$ is an NLI of $A_{32}$, and (ii) is proven.

Since $B_{12}A_{21}$ is a diagonal matrix and $B_{12}$ is nonnegative, the $i$th row of $B_{12}A_{21}$ is a nonnegative linear combination of the rows of $A_{21}$ (weighted by the entries of the $i$th row of $B_{12}$). As the $i$th row of $B_{12}A_{21}$ is a nonnegative multiple of $e_i^T$, and $A_{21}$ is a nonnegative matrix with no row of zeros, it follows that if the $(i, j)$-entry of $B_{12}$ is nonzero, then the $j$th row of $A_{21}$ is a multiple of $e_i^T$. In particular, this implies that each column of $B_{12}$ has at most one nonzero entry. If the $j$th row of $A_{21}$ has at least two positive entries, then column $j$ of $B_{12}$ is a column of zeros, proving (iii). If the $(i, j)$-entry of $B_{12}$ is nonzero, then the $(j, i)$-entry of $A_{21}$ is nonzero, completing the proof of (iv).

If $S_{21}$ is vacuous, then $A_{21}, A_{22}, B_{12}$, and $B_{22}$ are vacuous, in which case the proofs of (i) for $B_{11}, B_{21}$ and (ii) are similar, but statements (i) for $B_{22},$ (iii), and (iv) are vacuous. \qed
For \( m \geq 2 \), Proposition 3.2, Theorem 3.4, and the following theorem completely characterize the \( m \) by 2 sign patterns that allow an NLI.

**Theorem 3.8.** For \( m \geq 2 \), let \( S \) be an \( m \) by 2 sign pattern such that the first column is nonnegative and the second column has a + and a − entry. Then \( S \) allows an NLI if and only if the first column of \( S \) has a + entry and \([0 +]\) is a row of \( S \).

**Proof.** Suppose that \( S \) allows an NLI. Then the first column of \( S \) also allows an NLI. Hence, Theorem 3.4 implies that the first column of \( S \) has a + entry. By Theorem 3.6, we may assume without loss of generality that \( S \) is of the form (3.2). Since \( S_{32} \) is a column sign pattern, Propositions 3.7 (ii) and 2.1 imply that \( S_{32} \) has a + entry. Hence, \([0 +]\) is a row of \( S \).

For the converse, suppose that the first column of \( S \) has a + entry and \([0 +]\) is a row of \( S \). Suppose that \([+ −]\) is also a row of \( S \). Then without loss of generality, \( A \in S \) has the form

\[
\begin{bmatrix}
a & −b \\
u & v \\
0 & c
\end{bmatrix},
\]

where \( a, b, c > 0 \), and \( u \) and \( v \) are \((m − 2)\) by 1 vectors. It is easy to verify that

\[
\begin{bmatrix}
1/a & 0 & b/ac \\
0 & O & 1/c
\end{bmatrix}
\]

is an NLI of \( A \).

Next suppose that \([+ −]\) is not a row of \( S \). Then without loss of generality, \( A \in S \) has the form

\[
\begin{bmatrix}
a & b \\
u & v \\
0 & −c \\
0 & d
\end{bmatrix},
\]

where \( a, c, d > 0 \), \( b \geq 0 \), and \( u \) and \( v \) are \((m − 3)\) by 1 vectors. It is easy to verify that

\[
\begin{bmatrix}
1/a & 0 & b/ac & 0 \\
0 & O & 1/c & 2/d
\end{bmatrix}
\]

is an NLI of \( A \).

Hence, \( S \) allows an NLI. □

Note that the proof of Theorem 3.8 actually shows that if \( S \) is an \( m \) by 2 matrix whose first column is nonnegative, second column has a + and a − entry, and \([0 +]\) is one of its rows, then each matrix with sign pattern \( S \) has an NLI.

**Example 3.9.** The strong Hall sign pattern

\[
S = \begin{bmatrix}
+ & − \\
+ & − \\
0 & +
\end{bmatrix}
\]

does not allow a PLI (by Lemma 2.2), but does allow an NLI (by Theorem 3.8) since

\[
\begin{bmatrix}
1 & 0 & 1/2 \\
0 & 0 & 1/2
\end{bmatrix}
\begin{bmatrix}
1 & −1 \\
1 & −1/2
\end{bmatrix} = I_2.
\]
In general (as noted in the introduction) an NLI is not unique. For instance,

$$\begin{bmatrix}
1/2 & 1/2 & 1/2 \\
0 & 0 & 1/2
\end{bmatrix}$$

is another NLI of the above matrix.

In the next theorem, it is shown that if a sign pattern $S$ of the form (3.2) has a $(3,2)$-block $S_{32}$ that allows an NLI or PLI, then some conditions on the negative entries in $S_{12}$ insure that $S$ allows an NLI.

**Theorem 3.10.** For $m \geq n \geq 2$, let $S$ be an $m \times n$ sign pattern of the form (3.2) with $p \geq 1$, $n - p \geq 1$, and $S_{21}$, $S_{22}$ arbitrary. Then the following hold:

(i) If $S_{32}$ allows an NLI and $S_{12}$ has only 0 or $-$ entries, then $S$ allows an NLI.
(ii) If $S_{32}$ allows a PLI and each row of $S_{12}$ has a $-$ entry, then $S$ allows an NLI.

**Proof.** (i) Let

$$A = \begin{bmatrix}
I_p & A_{12} \\
A_{21} & A_{22} \\
O & A_{32}
\end{bmatrix} \in Q(S),$$

where $-A_{12} \geq 0$ and $A_{32}$ has $B_{23}$ as an NLI. Let

$$B = \begin{bmatrix}
I_p & O & B_{13} \\
O & O & B_{23}
\end{bmatrix}$$

with $B_{13} = -A_{12}B_{23}$, which is a nonnegative matrix. Then $B \geq 0$, $BA = I_n$, and hence the result follows.

(ii) Let $A \in Q(S)$ be of the form (3.3) and let $B$ be of the form (3.4). If $B_{23}$ is a PLI of $A_{32}$ and $B_{13} = -A_{12}B_{23}$, then $B_{13} > 0$, provided that the negative entries of $A_{12}$ are sufficiently large in magnitude, and $BA = I_n$ as required.

4. **Concluding remarks.** In section 3, we have characterized nonnegative sign patterns, strong Hall sign patterns with each column having a + and a $-$ entry, and $m \times 2$ sign patterns that allow an NLI. For other cases, we have given some necessary or sufficient conditions for $S$ to allow an NLI. A characterization for the blocks of the last column of a sign pattern $S$ of the form (3.1) with $k \geq 2$ that allows an NLI remains open. We conclude by showing (in Theorem 4.2) that some conditions on the submatrix $S_{kk}$ of a sign pattern $S$ of the form (3.1) with $k \geq 2$ insure that $S$ allows an NLI for arbitrary $S_{1k}, \ldots, S_{k-1,k}$.

Let $S$ allow a PLI and $A \in Q(S)$. The following proposition, which is used to prove Theorem 4.2, describes a relation between a PLI of $A$ and the qualitative behavior of solutions of $x^T A = b^T$. The latter equation is given in the introduction as motivation for studying PLIs and NLIs.

**Proposition 4.1.** For $m \geq n$, let $A$ be an $m \times n$ matrix. Then $A$ has a PLI if and only if for each $n$ by 1 nonzero vector $b \geq 0$ there exists an $m$ by 1 vector $x > 0$ satisfying $x^T A = b^T$.

**Proof.** Suppose that an $n$ by $m$ matrix $B > 0$ is a PLI of $A$. For an $n$ by 1 nonzero vector $b \geq 0$, it is clear that $(b^T B)A = b^T$ and $b^T B > 0$. Hence, the result follows.

Next, suppose that for each $n$ by 1 nonzero vector $b \geq 0$ there exists an $m$ by 1 vector $x > 0$ satisfying $x^T A = b^T$. Take $b$ to be the $i$th column $e_i$ of $I_n$ and let $x_i > 0$.
be a solution of \( x^T A = e_i^T \). Then the matrix

\[
B = \begin{bmatrix}
x_1^T \\
\vdots \\
x_n^T
\end{bmatrix}
\]

is a PLI of \( A \).

**Theorem 4.2.** For \( m > s \geq 1 \), \( n > t \geq 1 \), and \( m > n \), let \( S_{11} \) be an \( s \times t \) sign pattern that allows an NLI and let \( S_{22} \) be an \( (m - s) \times (n - t) \) sign pattern that allows a PLI. Suppose that if \( n - t = 1 \), then \( S_{22} \) has a 0 entry, and if \( n - t \geq 2 \), then \( S_{22} \) is not permutationally equivalent to the sign pattern \( \left[ \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \right] \) in which \( T \) is a square sign pattern. Then, for an arbitrary \( s \) by \((n - t) \) sign pattern \( S_{12} \), the sign pattern \( S = \left[ \begin{smallmatrix} S_{11} & S_{12} \\ \tilde{O} & S_{22} \end{smallmatrix} \right] \) allows an NLI.

**Proof.** Let \( A_{11} \) be a matrix in \( Q(S_{11}) \) with \( B_{11} \) as an NLI. By Theorem 2.12, there exists \( A_{22} \in Q(S_{22}) \) that has a PLI \( B_{22} \) and a positive left nullvector \( y^T \). Let \( A_{12} \in Q(S_{12}) \). Then \( A_{12} \) can be written as \( A_{12} = V_1 - V_2 \), where \( V_1, V_2 \geq 0 \) and the entrywise (Hadamard) product \( V_1 \circ V_2 = O \). Let \( v_i^T \geq 0 \) for \( 1 \leq i \leq s \) denote row \( i \) of \( V_1 \). If \( v_i \neq 0 \), then by Proposition 4.1 there exists an \((m - s)\) by 1 vector \( x_i > 0 \) such that \( x_i^T A_{22} = v_i^T \). If \( v_i = 0 \), then \( x_i^T A_{22} = v_i^T = 0 \) when \( x_i^T = y^T \). Thus, \( K_1 = [x_1, \ldots, x_s]^T > 0 \) and \( K_1 A_{22} = V_1 \). Similarly, there exists \( K_2 \geq 0 \) such that \( K_2 A_{22} = V_2 \).

Let \( A_{12}(\epsilon) = \epsilon V_1 - V_2 = (\epsilon K_1 - K_2) A_{22} \) for a sufficiently small \( \epsilon > 0 \) such that \( K_2 - \epsilon K_1 > 0 \). Note that \( V_1 \circ V_2 = O \) implies that \( A_{12}(\epsilon) \in Q(S_{12}) \). Let \( B_{12} = B_{11}(K_2 - \epsilon K_1) \). Since \( K_2 - \epsilon K_1 > 0 \) and \( B_{11} \geq 0 \) with no rows of zeros, it follows that \( B_{12} > 0 \). It can be easily verified that \( \left[ \begin{smallmatrix} B_{11} & B_{12} \\ \tilde{O} & B_{22} \end{smallmatrix} \right] \left[ \begin{smallmatrix} A_{11} & A_{12}(\epsilon) \\ \tilde{O} & A_{22} \end{smallmatrix} \right] = I_n \). Hence, the result follows.

**Remark 4.3.** Take \( S_{11} \) and \( S_{22} \) in Theorem 4.2 to be \( S' \) in Remark 3.1 and \( S_{kk} \) in the form (3.1) with \( k \geq 2 \), respectively. Then the conditions on \( S_{kk} \) in Theorem 4.2 insure that the sign pattern \( S \) of the form (3.1) with \( k \geq 2 \) allows an NLI for arbitrary \( S_{1k}, \ldots, S_{k-1,k} \).

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**References**