ASYMPTOTIC SOLUTIONS TO DIRECT
AND INVERSE SCATTERING
IN ANISOTROPIC ELASTIC MEDIA

by

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ABSTRACT

The level of sophistication in seismic data processing has increased tremendously over the past ten years, especially in the area of imaging complex structures. However, the earth's crust is only now starting to be considered for what it really is, an inhomogeneous, weakly anisotropic and anelastic medium, rather than a mere superposition of acoustic layers.

Anisotropy, which was long considered as a complicated lower order effect and a nuisance, has more recently been associated with the presence of aligned cracks in oil-producing reservoirs. This has generated a widespread interest in that subject, particularly in the field of enhanced oil recovery.

The propagation of seismic signals in anisotropic elastic media can efficiently be analyzed with asymptotic wave theory. Compared to purely numerical solutions, this approach has the advantage of yielding explicit representations of the field in terms of meaningful physical parameters.

In developing solutions to the direct scattering problem, the WKB or ray theoretical method is used to generate true-amplitude synthetic seismograms. A computer implementation is described in the case of tabular fractured media, that allows simulation of the most striking features associated with crack-induced anisotropy.
Another fundamental issue is that of solving the inverse scattering problem, that is, imaging reflectors in the subsurface in the presence of anisotropy. In effect, imaging under the assumption of isotropy may result in significant errors in locating layer boundaries. The proposed solution, which is based on a Kirchhoff-WKBJ representation of the field, generalizes the concept of Kirchhoff migration to anisotropic elastic media. The result of the inversion is a map of the unknown reflectors as well as their reflectivity to various wavetypes.
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INTRODUCTION

The last two years have seen a renewed interest from the exploration community in elastic anisotropy. Although anisotropy effects in geological formations have been extensively documented in the past (Love 1944, Stoneley 1949, White 1955 & 1981), they have been only recently studied in relation to a phenomenon of crucial importance to hydrocarbon exploration: the presence of natural fractures in reservoirs.

Recent work by Crampin (1982), Martin (1985), Thomsen & al. (1986), and others, has shown that when oriented in a preferred direction, microcracking of a rock results in global anisotropy of mechanical properties that affects wave propagation. The widely accepted model describing this situation is known as transverse isotropy and its main influence on wave propagation is referred to as shear wave birefringence.

Mechanical anisotropy in geological formations is conventionally classified into three major categories (Crampin, 1985): Intrinsic anisotropy arises from the presence of crystals of particular symmetries, within an isotropic matrix. PTL anisotropy on the other hand reflects the presence of thin laminations of well contrasted lithologies, while EDA is associated with microfracturing of rocks along the principal axes of present or past regional stresses.
Although all three anisotropy types may yield birefringence, fracture induced anisotropy is not only the most suitable for detection by reflection seismology, but also the most crucial to hydrocarbon exploration. In effect, fracture networks usually determine a preferred direction for fluid flow, and any information on that matter, that is, fracture orientation and density, can be extremely valuable to the recovery process.

To analyze the feasibility of fracture detection, one must understand the effects of anisotropy on wave propagation. The first chapter, entitled "Preliminaries", is an attempt to convey that understanding. Most of the derivations presented there may be found in some form in the literature of the past thirty years, but rarely in a form suited to our purpose. The mathematical description of the problem is complex because of the large number of parameters involved. The use of Cartesian tensor algebra is, for example, essential to a concise exposition. However, we emphasize as much as possible the physical significance behind the equations. A computer implementation of ray tracing in tabular, transversely isotropic media is described in much detail, and possible applications are discussed.

In the second chapter, we propose an asymptotic solution to the problem of reflector imaging in media made of a superposition of curved, homogeneous, anisotropic layers. There, the goal is to reconstruct a reflector located below an
otherwise known set of background layers. We first develop an asymptotic representation of the field of the unknown scatterer using a Kirchhoff-WKB integral. Using the theory of Fourier integrals, an operator is then devised which, when applied to that field, yields a bandlimited spatial delta function that peaks on the unknown reflector. Moreover, the peak amplitude on the output is proportional to the reflectivity of the scatterer to the wavetype considered for imaging.

Finally, that theory is specialized to the particular case of data recorded along a single line perpendicular to the strike of a cylindrical structure. With simple assumptions on the symmetry of the anisotropy, the resulting formula represents a simple two-dimensionnal process that still accounts for the full three-dimensionnal spreading of energy. The resulting images are cross-sections of the cylindrical structure in the vertical plane of the source-receiver line. That method is implemented for tabular background models, and several applications are discussed.
1. PRELIMINARIES

1-1. High Frequency Waves in Elastic Anisotropic Media

1-1.1 Introduction

Wave propagation in anisotropic media is a complex physical phenomenon, whose mathematical description may appear cumbersome. For example, the apparently simple problem of determining the field of a point force in a homogeneous, unbounded, anisotropic solid does not have a closed form solution. With such an example in mind, one might think that the representation of wavefields in heterogeneous anisotropic media is completely out of reach. Fortunately, when the mechanical excitation of the medium is "high frequency", the wavefield can be accurately described by a relatively simple asymptotic expression. The usefulness of asymptotics is apparent when one realizes that most of the physical concepts used in wave propagation, such as Huygen's principle, Ray theory, Fermat's principle or Snell's law, only hold in the asymptotic range.

The next natural question is therefore: how high is "high frequency"? In practical terms, high frequency asymptotics are valid if no significant changes in elastic parameters occur within a wavelength. Discontinuity surfaces in elastic properties can be handled insofar as their radius of curvature as well as the
distance between any two surfaces exceed several wavelengths. These assumptions seem to be generally met, although marginally at times, in exploration seismology. This is not only because seismic sources have a limited ability to produce low frequency signals, but also because the typical propagation medium is a juxtaposition of thick quasi-homogeneous layers.

The asymptotic representation that we use in this section is the classical WKB expansion, of which we keep but the leading order term. The asymptotic field is characterized by a displacement vector, and a traveltime function. The corresponding asymptotic wave equation is an algebraic equation that dictates the allowable values for the magnitude of the gradient of the traveltime function, or slowness, as well as for the orientation of the displacement vector or displacement polarization. As it turns out, there are only three different allowable slowness/polarization pairs at any point of an anisotropic medium. Those pairs are referred to as wavetypes or modes, and are equivalent to the $P$, $SV$ and $SH$ propagation modes in isotropic media. Because of anisotropy however, the velocity and polarization of each wavetype not only depend on the properties of the medium at the observation point, but also on the direction in which the wave is propagating.

Asymptotic elastic wavefields can be described using the familiar tools of geometrical optics: given initial values for the displacement amplitude, traveltime,
and propagation direction of a given wavetype at some point, a trajectory or ray can be found where the evolution of traveltime and amplitude is known. These rays are also the lines of energy transport within the medium. A remarkable consequence is that to compute the wavefield at a given observation point, one only needs to find the ray that links that point to the initial surface and solve the evolution equations along that ray. Considering that exploration seismology only uses a sparse set of observation points, this ray tracing technique is particularly efficient in computing synthetic seismograms, since it does not require the knowledge of the entire wavefield.

Next we review the mathematical bases of asymptotic wave propagation.

1-1.2 Fundamental Relations and Definitions:

In the following discussion, all the time-dependent quantities are implicitly expressed in the Fourier domain with the following conventions: if \( F(t) \) is a time-dependent field, then its Fourier domain equivalent

\[
f(\omega) = \int_{-\infty}^{+\infty} F(t) e^{i\omega t} \, dt,
\]

and, reciprocally:

\[
F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega) e^{-i\omega t} \, d\omega.
\]
Linear elastic anisotropic solids, are characterized by a linear stress-strain relation that can be considered as the most general form of Hooke's law. The stress tensor $\sigma_{ij}$ is expressed as the second order contraction of the deformation tensor $\epsilon_{kl}$ with a fourth rank tensor of elastic coefficients $c_{ijkl}$, that is, if we use Einstein's summation convention on repeated indices:

$$\sigma_{ij}(x) = c_{ijkl}(x) \epsilon_{kl}(x) \quad i,j,k,l=1,2,3$$ \hspace{1cm} (1.1)

It would be a mistake, however, to consider $c_{ijkl}$ as a general tensor because physical considerations of symmetry and energy conservation impose some strong constraints on the elastic coefficients, which are summarized in the following equations:

$$c_{ijkl}(x) = c_{jikl}(x) \quad , \hspace{2cm} (1.2)$$

$$c_{ijkl}(x) = c_{ijlk}(x) \quad , \hspace{2cm} (1.3)$$

$$c_{ijkl}(x) = c_{klij}(x) \quad . \hspace{2cm} (1.4)$$

The meaning of these relations is that 21 independent elastic constants are enough to describe the most general linear elastic anisotropic solid.

The second law of importance is Newton's law, which expresses the mechanical equilibrium of an elementary volume of solid submitted to a stress field, in the presence of a field of external forces $f$:

$$\sigma_{ij,j}(x) + f_i(x) = -\rho(x) \omega^2 u_i(x) \quad , \hspace{2cm} (1.5)$$

where $\rho$ is the density of the solid and $u$ the particle displacement.
Strain is defined from displacement as follows:

\[
\varepsilon_{kl}(x) = \frac{1}{2} \left[ u_{k,l}(x) + u_{l,k}(x) \right],
\]

so that stress can be directly related to displacement according to (1.1):

\[
\sigma_{ij}(x) = \frac{1}{2} c_{ijkl}(x) u_{k,l}(x) + \frac{1}{2} c_{ijkl}(x) u_{l,k}(x) .
\]

Next we notice that:

\[
c_{ijkl}(x) u_{k,l}(x) = c_{ijkl}(x) u_{l,k}(x) = c_{ijkl}(x) u_{k,l}(x) .
\]

Here we have made use of the symmetry of the elastic coefficients in (1.3) and then merely exchanged two pairs of dummy indices. Consequently (1.7) can be rewritten in the simplified form:

\[
\sigma_{ij}(x) = c_{ijkl}(x) u_{k,l}(x) ,
\]

and substituted into (1.5) to yield the elastic wave equation, which is a vector equation on displacement:

\[
\left[ c_{ijkl}(x) u_{k,l}(x) \right]_{,j} + f_i(x) = -\rho(x) \omega^2 u_i(x) .
\]

This equation holds in any domain where the displacement field \( u \) is twice differentiable with respect to both time and space, and where the elastic tensor is spatially differentiable.
A quantity that is particularly instrumental in developing solutions to the wave equation is the Green’s tensor, denoted by $g_{im}(\mathbf{z},\mathbf{z}')$, which describes the displacement field observed in the direction $i$ at a point $\mathbf{z}$ created by a unit impulsive point force in the direction $m$ at a point $\mathbf{z}'$. From this definition, it appears that $g_{im}$ is not a tensor of rank 2 in the classical sense of the word because its indices do not refer to a single location.

The defining equation for the Green’s tensor is:

$$\left[c_{ijkl}(\mathbf{z}) g_{km,l}(\mathbf{z},\mathbf{z}')\right]_{ij} + \rho \omega^2 g_{im}(\mathbf{z},\mathbf{z}') = -\delta_{im} \delta(\mathbf{z}-\mathbf{z}')$$

where $\delta_{im}$ is the Kronecker symbol and $\delta(\mathbf{z}-\mathbf{z}')$ the Dirac delta; moreover, the differentiations in (1.10) are carried out with respect to the observation variable $\mathbf{z}$.

Another useful quantity derived from the Green’s tensor is the Green’s traction $t_{im}$ defined as follows:

$$t_{im}(\mathbf{z},\mathbf{z}') = c_{ijkl}(\mathbf{z}) g_{km,l}(\mathbf{z},\mathbf{z}') \nu_j(\mathbf{z})$$

where $\nu$ is the unit normal to the surface over which the traction is evaluated.

The Green’s tensor possesses a remarkable symmetry, which is usually referred to as reciprocity (Morse and Feshbach). The reciprocity relation is formulated within a domain $D$ bounded by a surface $S$, as shown below:

$$g_{nm}(\mathbf{z}'',\mathbf{z}') - g_{mn}(\mathbf{z}',\mathbf{z}'') =$$
\[ \int_S dS(x) \left[ g_{in}(x,x^\prime) t_{im}(x,x^\prime) - g_{in}(x,x^\prime) t_{im}(x,x^\prime') \right] . \] (1.12)

The proof of (1.12) can be found in Appendix A.

1-1.3 WKBJ Ansatz and Ray Equations

We now assume the validity of a high frequency or WKBJ solution to equation (1.9); that is, we seek a representation of the displacement field as an algebraic series in increasing powers of \(1/i\omega\) (see Bleistein, 1985, for example):

\[ u_j(\omega, x) = e^{i\omega \tau(x)} \sum_{n=0}^{\infty} \frac{U_j^{(n)}(x)}{(i\omega)^n} . \] (1.13)

It is interesting to note here that the time domain equivalent for the leading term in the series is the singular distribution of the surface \(t = \tau(x)\), that is, the distribution \(\delta(t-\tau(x))\). This distribution delineates the front of the wave field as time progresses, and \(\tau\) should therefore be considered as the traveltime field. Lower order terms are progressively smoother terms corresponding to further integrations of the singular function and can be ignored for high frequency waves. If we consider a bandlimited signal, then the leading order term in the time domain can be seen to represent a bandlimited function of a variable normal to the surface \(t = \tau(x)\), which peaks on the wavefront.
Substituting expression (1.13) into (1.9) in the absence of body forces and ordering the resulting sum, we obtain:

\[
\sum_{n=0}^{\infty} \frac{e^{i\omega r}}{(i\omega)^{n-2}} \left\{ c_{ijkl} \left( U_i^{(n-2)} + U_k^{(n-1)} \tau_j + U_l^{(n-1)} \tau_i + U_i^{(n-1)} \tau_j \right. \\
+ U_j^{(n)} \tau_i \tau_j \left. \right) + c_{ijkl,j} U_i^{(n-1)} \tau_i - \rho U_i^{(n)} \right\} = 0 \quad (1.14)
\]

We must now solve (1.14) to “leading order”, that is, solve for the amplitude \( U_i^{(0)} \) and the phase (traveltime) \( \tau \) of the wavefield. By setting the coefficient of the leading order term in (1.14) to zero and renormalizing, we obtain the equation:

\[
\left[ \frac{c_{ijkl}}{\rho} d_l d_j - V^2 \delta_{ik} \right] U_i^{(0)} = 0 \quad , \quad (1.15)
\]

where

\[ d_i = \frac{p_i}{(p_n p_n)^{1/2}} \quad , \]

is the propagation direction,

\[ V = \frac{1}{(p_n p_n)^{1/2}} \quad , \]

is the phase velocity, and

\[ p_i = \tau_i \]

is the traveltime gradient or slowness vector.

Equation (1.15) is an algebraic equation that can be presented as an eigenvalue/eigenvector problem for the matrix \( M_{ik} \) defined by:
\[ M_{ik} = \frac{c_{ijkl}}{\rho} d_i d_j \]

That is, for a given propagation direction \( d \) in a medium characterized by rigidities \( c_{ijkl} \) and density \( \rho \), the phase velocity \( V \) of a wave as well as the direction of the associated displacement \( U^{(0)} \) are uniquely determined by eigenvalues/eigenvectors of \( M_{ik} \); the phase velocity amplitude is the square root of one of the eigenvalues of \( M_{ik} \), while the direction of displacement or displacement polarization is the associated unit eigenvector.

The characteristic equation is best rewritten as the following third degree polynomial equation in \( V^2 \) or eikonal equation:

\[
\det \left[ \frac{c_{ijkl}}{\rho} d_i d_j - V^2 \delta_{ik} \right] = 0 \quad , \tag{1.16}
\]

that can be formally factored to yield three implicit equations of the form:

\[
V^2 = f(c_{ijkl}, \rho, d) \quad . \tag{1.17}
\]

Each of these relations represents the dispersion equation for one of the three wavetypes that exist in a generally anisotropic medium.

Having solved (1.16) for \( V^2 \), one can determine the associated displacement polarization as follows: substituting (1.17) into (1.15) makes the matrix \( [M_{ik} - V^2 \delta_{ik}] \) singular, so that solutions for \( U_i^{(0)} \) are available. The displacement polarization is one of the two solutions that have unit amplitude. This result holds
insofar as there are no double or triple roots in (1.16). The double root case corresponds to the isotropic elastic solid, while the triple root case represents the fluid limit, that is, a solid without shear rigidity. In these situations, the multiple eigenvalues respectively span a two dimensional vector space and the entire space itself.

Although computing displacement polarizations can be cumbersome in general, we can always make the following statement: displacement polarizations for the three wavetypes are orthogonal to each other for a given propagation direction. This results from the matrix $M_{ik}$ being symmetric, which we show below:

$$\rho M_{ik} = c_{ijkl}d_l d_j = c_{iklj}d_j d_l = c_{kjil}d_i d_l = c_{kjil}d_i d_j = \rho M_{ki}$$

Here we have again permuted two pairs of dummy indices, used symmetry (1.4), and the commutativity of scalar multiplication. As a consequence, the eigenvectors of $M_{ik}$ must be orthogonal.

As we have just seen, the eikonal equation permits one to determine the dispersion relation and thereby the phase velocity of each wavetype, as well as the displacement polarization of the leading order term in the $WKBJ$ expansion. However, no further information has been obtained on the displacement amplitude. We therefore introduce energy density, a quantity proportional to the squared displacement amplitude, and for which we can write a conservation equation that
characterizes the amplitude distribution within the wavefield.

We start by defining *instantaneous energy* $I$ as the sum of potential (elastic) and kinetic energies per unit volume of solid:

$$I = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \rho \dot{u}_i \dot{u}_i \quad , \quad (1.19)$$

Dropping the superscript $^{(0)}$ for simplicity, the leading order displacement field is:

$$u_i = U_i(x) e^{i\omega r(x)} \quad .$$

Substituting this expression in (1.19) yields:

$$I = \frac{\omega^2}{2} \left[ \varepsilon_{ijkl} U_i U_k p_j p_l + \rho U_i U_i \right] e^{2i\omega r} \quad .$$

On the other hand, Newton's law in the absence of body forces simplifies to leading order into

$$\varepsilon_{ijkl} U_k p_j p_l = \rho U_i \quad ,$$

and may be rewritten

$$\frac{\omega^2}{2} \varepsilon_{ijkl} U_i U_k p_j p_l = \frac{\omega^2}{2} \rho U_i U_i \quad , \quad (1.20)$$

and interpret as follows: kinetic and potential energy are equal for high frequency waves in elastic anisotropic media. A quantity that proves useful in later developments is *energy density* $E$, which we define from instantaneous energy as:

$$E = 2 I e^{-2i\omega r} \quad .$$
Following (1.20), we may express energy density in two equivalent forms,

\[ E = \rho \omega^2 U_i U_i \quad , \]

or,

\[ E = \omega^2 c_{ijkl} U_i U_k p_j p_l \quad . \] (1.22)

Let us now develop a conservation equation from the second order term of the asymptotic wave equation (1.14). This equation is:

\[
\left[ c_{ijkl} p_l p_j - \rho \delta_{ik} \right] U_i^{(1)} - c_{ijkl,j} U_i p_l \\
- c_{ijkl} \left[ U_i, l p_j + U_i, j p_l + U_i p_l, j \right] = 0 \quad . \] (1.23)

This is a vector equation for \( U_i^{(1)} \) of the form:

\[ A_{ik} U_i^{(1)} = B_k \quad , \] (1.23)

where

\[ A_{ik} = c_{ijkl} p_l p_j - \rho \delta_{ik} \]

and

\[ B_k = c_{ijkl} U_i p_l + c_{ijkl} \left[ U_i, l p_j + U_i, j p_l + U_i p_l, j \right] = 0 \quad . \]

but where \( \det(A_{ik}) = 0 \) according to the eikonal equation (1.16). In order for (1.24) to have a solution, \( B_k \) must therefore be orthogonal to the eigenvector of \( A_{ik} \) whose eigenvalue is zero. According to the eikonal equation, that vector is nothing else but \( U_i \), and therefore the orthogonality condition is:
\[ B_k U_k = 0 \quad . \quad (1.25) \]

After substitution into (1.24), we obtain:

\[ c_{ijkl} U_i U_k p_l + c_{ijkl} U_{i,l} U_k p_j + c_{ijkl} U_{i,j} U_k p_l + c_{ijkl} U_i U_k p_{ij} = 0 \quad . \]

The second term can be rearranged by permutation of indices and the use of symmetry (1.4):

\[ c_{ijkl} U_{i,l} U_k p_j = c_{ilkj} U_{i,j} U_k p_l = c_{klji} U_{k,j} U_i p_l = c_{ijkl} U_{k,j} U_i p_l \quad , \]

making the left member of (1.25) an exact divergence:

\[ \left( c_{ijkl} U_i U_k p_l \right)_{,j} = 0 \quad . \]

This relation can in turn be expressed in the form:

\[ (E W_j)_{,j} = 0 \quad , \quad (1.26) \]

where \( E W_j \) is the energy density flux vector. The vector \( W_j \), which has the dimensions of velocity, is known as group velocity. Taking for \( E \) the expression (1.21) and identifying the terms in (1.26), we get the following expression for group velocity:

\[ W_j = \frac{c_{ijkl} U_i U_k p_l}{\rho U_n U_n} \quad . \quad (1.27) \]

At this point it is interesting to notice that phase and group velocity are not necessarily colinear. We can however prove that group velocity is always made up
of the vector sum of phase velocity and a vector perpendicular to phase velocity; from (1.27) we have:

\[
p_j W_j = \frac{c_{ijkl} U_i U_k p_i p_j}{\rho U_n U_n}
\]

Equations (1.21) and (1.22) show that the right hand term is the ratio of potential to kinetic energy, and therefore equal to unit, hence:

\[
p_j W_j = 1
\]

(1.28)

On the other hand, from the definitions of slowness and phase velocity, we have:

\[
p_j = \frac{V_j}{V_n V_n}
\]

that is, phase velocity and slowness are reciprocal vectors. Substituting this result into (1.28) yields:

\[
V_j W_j = V_n V_n
\]

which means that the orthogonal projection of the group velocity vector on the phase velocity vector equals the magnitude of phase velocity. In other words, group velocity is the sum of phase velocity and a vector perpendicular to phase velocity. It follows that group velocity exceeds phase velocity in magnitude.

At this point we have derived the two differential equations necessary to solve for high frequency wavefields in anisotropic media, namely the eikonal equation
(1.15) and the transport equation (1.26). The eikonal equation is in general a non-linear first order differential equation in $\tau$, which can be solved by the method of characteristics. For a given propagation direction, that equation involves $\tau$ through the squared phase velocity $V^2$ exclusively. However, the following canonical form is more suitable to our purpose:

$$F(x_i, \tau, p_i) = \left( c_{ijkl}p_ip_j - \rho \delta_{ik} \right) U_i U_k = 0 \quad .$$

where once again $p_i = \tau, n$. That equation is obtained by projecting (1.15) onto $U_k$. We solve it by rewriting it as a system of one dimensional first order differential equations along characteristic curves or rays (see Bleistein, 1984 for example). The rays are defined by:

$$\frac{dx_i}{F_{,p_i}} = \lambda d\sigma \quad ,$$

where $\lambda$ is any non-zero scalar, while $\sigma$ is the ray variable. Along a ray, the three unknown quantities $\tau, x_i, p_i$ satisfy the following differential equations with respect to the ray variable:

$$\frac{d\tau}{d\sigma} = \lambda p_n F_{,p_n} \quad ,$$

$$\frac{dx_n}{d\sigma} = \lambda F_{,p_n} \quad ,$$

$$\frac{dp_n}{d\sigma} = -\lambda (F_{,x_n} + p_n F_{,\tau}) \quad .$$

The starting point for a ray is determined by an initial value for $x_i$, where $\tau$ and $p_i$
are specified.

We now demonstrate that rays are always directed along the lines of energy transport and therefore colinear to group velocity. To prove this, we compute the ray direction from (1.31) to obtain:

\[ F_{,p_n} = c_{ijkl}\delta_{nl}p_jU_iU_k + c_{ijkl}\delta_{jn}p_lU_iU_k + \left( c_{ijkl}p_l p_j - \rho \delta_{ik} \right) \left( U_k U_{k,p_n} + U_i U_{i,p_n} \right) \]

The last term is zero owing to (1.15) and to the symmetry of the eikonal matrix. Contracting the first two terms yields:

\[ F_{,p_n} = c_{ijkn} p_j U_i U_k + c_{inkl} p_l U_i U_k \]

The first term is identical to the second one as we show below using index permutations and, once again, symmetry (1.4):

\[ c_{ijkn} p_j U_i U_k = c_{knij} p_k U_i U_j = c_{inkj} p_j U_i U_k = c_{inkl} p_l U_i U_k \]

Hence from (1.31):

\[ \frac{dz_n}{d\sigma} = 2\lambda c_{inkl} U_i U_k p_l \]

Next we compute the scalar \( d\tau/d\sigma \) according to (1.30):

\[ \frac{d\tau}{d\sigma} = p_n F_{,p_n} = 2\lambda c_{inkl} U_i U_k p_l p_n \]

Using identity (1.20), this expression becomes:
\[
\frac{d\tau}{d\sigma} = 2\lambda \rho U_n U_n ,
\]
so that we can express the ray direction in terms of a velocity \(dx_n/d\tau\) by properly scaling \(dx_n/d\sigma\):

\[
\frac{dx_n}{d\tau} = \frac{dx_n/d\sigma}{d\tau/d\sigma} = \frac{c_{ikl} U_l U_k P_l}{\rho U_n U_n} .
\]

A comparison of the right hand term with that of (1.27) shows that:

\[
\frac{dx_n}{d\tau} = W_n ,
\] (1.33)

that is, the rays are directed along group velocity. This general result is of great importance in showing that energy is conserved along rays. In particular, it allows us to rewrite the transport equation as a differential equation along rays, as we show next.

Let us first define a ray tube as a 2-parameter family of rays: if a ray is indexed by the couple \((\gamma_1^0, \gamma_2^0)\), then the associated ray tube is a beam containing all rays indexed in the range:

\[
\frac{d\gamma_1}{2} < \gamma_1 < \frac{d\gamma_1}{2} ,
\]

\[
\frac{d\gamma_2}{2} < \gamma_2 < \frac{d\gamma_2}{2} ,
\]

where \(d\gamma_1\) and \(d\gamma_2\) are arbitrarily small quantities. Next we integrate the transport
equation (1.26) along a section of ray tube defined by:

$$\sigma_1 < \sigma < \sigma_2 \quad .$$

Such a section of ray tube is illustrated in Figure (1.1). The transport equation becomes:

$$\int_V (EW_i)_{,i} d^3 x = 0 \quad ,$$

and the divergence theorem allows us to rewrite it as:

$$\int_S EW_i \nu_i d^2 S = 0 \quad ,$$

(1.34)

where $\nu_i$ is the unit vector normal to the boundary $S$ of the tube section. On the walls of the tube, defined by:

$$\gamma_1 = \gamma_1^0 \pm d \gamma_1 \quad ,$$

and

$$\gamma_2 = \gamma_2^0 \pm d \gamma_2 \quad ,$$

the product $W_i \nu_i$ vanishes since the walls are by definition tangent to the rays, hence to group velocity $W_i$. In consequence (1.34) simplifies into:

$$\int_{\sigma=\sigma_1} EW_i \nu_i d^2 S - \int_{\sigma=\sigma_2} EW_i \nu_i d^2 S = 0 \quad .$$

Since the two integrals apply to infinitesimal surfaces, they can be computed using the average value of the integrand, so that we get:
Figure (1.1): A section of ray tube.
\[
\left[ EW_i n_i d^2 S \right]_{\sigma_1} = 0 .
\]

Next we notice that:
\[
n_i d^2 S = \epsilon_{ijk} \frac{\partial x_i}{\partial \gamma_1} \frac{\partial x_j}{\partial \gamma_2} d\gamma_1 d\gamma_2 ,
\]

and substitute for \( W_i \) according to (1.33) to obtain:
\[
\left[ E\epsilon_{ijk} \frac{\partial x_i}{\partial \tau} \frac{\partial x_j}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2} \right]_{\sigma_1} = 0 .
\]

Finally, substituting for \( E \) according to (1.21) and defining a ray Jacobian \( J \) as follows:
\[
J(\sigma, \gamma_1, \gamma_2) = \epsilon_{ijk} \frac{\partial x_i}{\partial \tau} \frac{\partial x_j}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2} ,
\]

we get the transport equation as a difference equation along rays:
\[
\left[ \rho U^2 J \right]_{\sigma_1} = 0 .
\]

This equation can also be expressed in differential form by making \( \sigma_1 \) and \( \sigma_2 \) arbitrarily close:
\[
\frac{d}{d\sigma} \left( \rho U^2 J \right) = 0 . \quad (1.35)
\]

Joined with the system (1.30) through (1.32), this equation allows a complete solution of the leading order elastic field.
At this point, we have formally solved the problem of determining a high frequency solution to the wave equation in (weakly) inhomogeneous linear elastic anisotropic media, by the method of characteristics. We have not, however, described practically how to factor (1.16), which is the starting point for the method, nor have we considered what happens at discontinuity surfaces for elastic parameters (reflectors), where the WKBJ expansion fails. It also remains to provide adequate initial conditions for the prescribed equations. In order to address these problems and others, we now go back to a specific case of interest to us, namely transverse isotropy.

1-2. Transversely Isotropic Media

1-2.1 Introduction

Mechanical anisotropy with hexagonal symmetry, or transverse isotropy, describes media whose rigidity depends on the direction of applied traction with respect to the symmetry axis or anisotropy axis.

Transverse isotropy appears in geologic formations in three major forms (Crampin, 1985): (i) Intrinsic anisotropy, (ii) Periodic Thin Lamination anisotropy (PTL), or (iii) Extensive Dilatancy Anisotropy (EDA). Intrinsic anisotropy arises from the presence of crystals of hexagonal symmetries, within an isotropic matrix.
PTL anisotropy on the other hand reflects the presence of thin laminations of well contrasted lithologies, while EDA is associated with microfracturing of rocks along the principal axes of present or past regional stresses.

The mathematical model for transverse isotropy was first proposed by Love (Love, 1944) as a modification of the linear isotropic case, and therefore much resembles Hooke's law, except for the introduction of three additional elastic constants.

If we consider a natural Cartesian system of coordinates \((x_i)\) where \(x_1\) is the symmetry axis, the stress-strain relation is written:

\[
\begin{align*}
\sigma_{11} &= C\epsilon_{11} + F\epsilon_{22} + F\epsilon_{33} \\
\sigma_{22} &= F\epsilon_{11} + A\epsilon_{22} + (A-2N)\epsilon_{33} \\
\sigma_{33} &= F\epsilon_{11} + (A-2N)\epsilon_{22} + A\epsilon_{33} \\
\sigma_{12} &= 2L\epsilon_{12} \\
\sigma_{13} &= 2L\epsilon_{13} \\
\sigma_{23} &= 2N\epsilon_{23}
\end{align*}
\]

(1.36)

In practical terms, the anisotropy axis \(x_1\) is the unit normal to the fracture planes (EDA) or to the lamination planes (PTL). The stress-strain relation would be much more complicated if the symmetry axis was not a coordinate axis.

We first notice that when restricted to the fracture plane \((x_2, x_3)\), the stress-strain relation is purely isotropic. In effect, with deformations allowed only in that plane, system (1.36) becomes:
\[
\sigma_{22} = A \epsilon_{22} + (A - 2N) \epsilon_{33} \\
\sigma_{33} = (A - 2N) \epsilon_{22} + A \epsilon_{33} \\
\sigma_{23} = 2N \epsilon_{23}
\]

These equations exactly describe a two dimensional isotropic elastic medium where \( A \) and \( N \) would be the compressional and shear stiffnesses, respectively. It is this property that suggested the name “transversely isotropic” for media described by system (1.1), with the meaning of “isotropic in a plane transverse to a symmetry axis” (here the \( x_1 \) axis).

The best way to understand how transverse isotropy differs from isotropy is to study the constraints which, when applied to the elastic constants in (1.36), yield Hooke’s law for isotropic media in terms of the Lame constants \( \lambda \) and \( \mu \). These constraints are:

\[
A = C = \lambda + 2\mu \\
L = N = \mu \\
F = \lambda
\]  

(1.37)

Therefore the degree of anisotropy is determined by how much the constants \( A \) and \( C \) on one hand, the constants \( L \) and \( N \) on the other hand, and finally the quantities \( A - 2N \) and \( F \), respectively differ.

It is an easy and useful exercise to give a physical interpretation to the five elastic parameters in terms of static experiments. This is done in Figure (1.2) where we have adopted Thomsen’s idea of the deck of cards: although the cards in
Figure (1.2): Physical interpretation of the five elastic parameters using Thomsen’s ideal model of the deck of cards. Dotted lines indicate shape of solid before deformation.
the deck should be somewhat bonded together for a more accurate description, the idea is that a transversely isotropic medium deforms more like a deck of cards than like a solid block of paper, which would be an adequate model for an isotropic material. The absence of bonds between successive cards is an exaggerated simulation of the weak mechanical bond that exists, for example, between the two walls of a microcrack in a real rock. The direction perpendicular to a card is therefore our anisotropy axis or $x_1$ direction. Deformations which involve sliding the cards with respect to each other require little applied stress. On the other hand, equal amounts of deformation within the plane of a card require a much larger stress. The elastic coefficients simply describe how much stress must be applied to obtain a standard deformation in a given direction. Figure (1.2a) shows for example that $N$ is the amount of shear stress required to create a unit shear deformation within a card, and is therefore larger than $L$, which is the amount of stress required for the same deformation but across the deck, where free sliding occurs (Figure (1.2b)). (Actually $L$ is zero if totally free sliding is allowed between cards). In the same line of thought, Figures (1.2c) and (1.2d) show that $A$ is the stress required for a unit extensional deformation of a card, while $C$ is merely the stress necessary to separate the cards a certain distance from each other. If the cards are not bonded in any way, $C$ is exactly zero, but for a real material $C$ is simply less than $A$ because the fractures yield more easily than the rock matrix.
To complete this preliminary picture, it is interesting to describe the propagation of plane waves in our model. We define the propagation direction of a wave at a given point as the direction of the normal to the wavefront at that point. That direction is also that of phase velocity, which is the velocity we refer to in this section. First let us consider waves propagating along the principal axes of the system. It is elementary to show that compressional waves can travel along fracture planes at speed $\sqrt{A/\rho}$, and at a slower $\sqrt{C/\rho}$ along the symmetry axis. Shear waves propagating in the fracture planes can travel at two different speeds depending on the shear deformation they produce: if the deformation is in the fracture plane, the speed is $\sqrt{N/\rho}$; if the deformation is perpendicular to the fracture plane, that is, along the anisotropy axis, the speed is a slower $\sqrt{L/\rho}$.

To summarize, three decoupled wavetypes can travel in a transversely isotropic medium:

- a *Quasi-*P (*QP*) wave with polarization roughly along propagation direction and velocity in the range $[\sqrt{C/\rho}, \sqrt{A/\rho}]$.

- a *Quasi-*S (*QS*) wave polarized almost transversely, traveling at speeds around $\sqrt{L/\rho}$.

- an *S-Parallel* (*SP*) wave, truly transverse and polarized in the fracture plane, propagating at velocities in the range $[\sqrt{L/\rho}, \sqrt{N/\rho}]$. 
Figures (1.3) and (1.4) illustrate the dependence of phase velocity and polarization on propagation direction for all three wavetypes. A systematic derivation of these quantities is presented in the next section.

1-2.2 Wavetypes Characterization

We now proceed to quantitatively describe high frequency wave propagation in transversely isotropic media. We start by rewriting the elastic tensor in terms of Love’s coefficients. This is done by term by term identification of (1.36) and (1.1), and yields:

\[
\begin{align*}
    c_{1111} &= C, \\
    c_{2222} &= c_{3333} = A, \\
    c_{2233} &= c_{3322} = A - 2N, \\
    c_{1133} &= c_{3311} = c_{1122} = c_{2211} = F, \\
    c_{1212} &= c_{2121} = c_{2112} = c_{1221} = L, \\
    c_{1313} &= c_{3131} = c_{3113} = c_{1331} = L, \\
    c_{2323} &= c_{3232} = c_{3223} = c_{2332} = N.
\end{align*}
\]

The eikonal equation (1.16) therefore becomes:
Phase Velocities

--- S-Parallel

----- Quasi-S

-------- Quasi-P

Figure (1.3): Cross-section of the velocity surfaces for the three modes in an axial plane, showing the dependence of phase velocity on propagation direction. The actual surface has a symmetry of revolution around the anisotropy axis, here shown vertical \((A=10, C=3, N=4, L=1, F=1, \rho=1)\).
Polarization Effects

Figure (1.4): Polarization anomalies due to anisotropy: only the S-parallel mode is shown to be a truly transverse mode. Propagation azimuth is the angle between propagation direction (slowness) and anisotropy axis (the "1" axis). Same elastic parameters as in Figure (1.3).
\[ \begin{vmatrix} \nu^2 \frac{Ld_2^2 + Ld_3^2 + Cd_1^2}{\rho} & \frac{-(F+L)d_1 d_2}{\rho} & \frac{-(F+L)d_1 d_3}{\rho} \\ -\frac{(F+L)d_1 d_2}{\rho} & \nu^2 \frac{Nd_3^2 + Ad_2^2 + Ld_1^2}{\rho} & -\frac{(A-N)d_2 d_3}{\rho} \\ -\frac{(F+L)d_1 d_3}{\rho} & -\frac{(A-N)d_2 d_3}{\rho} & \nu^2 \frac{Ad_3^2 + Nd_2^2 + Ld_1^2}{\rho} \end{vmatrix} = 0. \]

To factor this determinant, we first multiply column 2 by \(d_3\) and subtract it from column 3 multiplied by \(d_2\), then multiply row 3 by \(d_3\) and add it to row 2 multiplied by \(d_2\). Next, expanding along the third column yields the following equation:

\[ \begin{vmatrix} \nu^2 \frac{Ld_1^2 + N(d_2^2 + d_3^2)}{\rho} \end{vmatrix} \begin{vmatrix} \frac{Cu^2 + L(d_2^2 + d_3^2)}{\rho} & \frac{-(F+L)d_1}{\rho} \\ -\frac{(F+L)d_1 (d_2^2 + d_3^2)}{\rho} & \nu^2 \frac{Ld_1^2 + A(d_2^2 + d_3^2)}{\rho} \end{vmatrix} = 0. \]

We finally factor the 2x2 determinant to get the desired form:

\[ \begin{vmatrix} \nu^2 \frac{Ld_1^2 + N(d_2^2 + d_3^2)}{\rho} \end{vmatrix} \begin{vmatrix} \nu^2 \frac{(A+L)(d_2^2 + d_3^2) + (C+L)d_1^2 + \sqrt{\Delta}}{2\rho} \end{vmatrix}. \]

\[ \begin{vmatrix} \nu^2 \frac{(A+L)(d_2^2 + d_3^2) + (C+L)d_1^2 - \sqrt{\Delta}}{2\rho} \end{vmatrix} = 0, \]

where:

\[ \Delta = \left[(A-L)(d_2^2 + d_3^2) - (C-L)d_1^2\right]^2 + 4d_1^2(d_2^2 + d_3^2)(L+F)^2. \]

As could be expected from the hexagonal symmetry of the model, the eikonal
equation only depends on the propagation direction relative to the anisotropy axis \( a \). Therefore if we define the axial and radial vector components of propagation direction:

\[
d_A = (d \cdot a) \ a
\]

and

\[
d_R = d - d_A
\]

the dispersion relation somewhat simplifies into:

\[
\left[ V^2 - \frac{Ld_A^2 + Nd_R^2}{\rho} \right] \left[ V^2 - \frac{(A+L)d_R^2 + (C+L)d_A^2 + \sqrt{\Delta}}{2\rho} \right] = 0 \quad (1.38)
\]

where

\[
\Delta = \left[ (A-L)d_R^2 - (C-L)d_A^2 \right]^2 + 4d_A^2 d_R^2 (L+F)^2
\]

\[
d_A^2 = d_A \cdot d_A
\]

and

\[
d_R^2 = d_R \cdot d_R
\]

This relation illustrates once again that to leading order in frequency, three decoupled wavetypes, or modes, can simultaneously exist in an transversely isotropic medium. Each mode can therefore be considered separately, since coupling only appears in the scattering at interfaces.
The amplitude of phase velocity can instantly be written for each mode as a function of the propagation direction $d$:

for the $Quasi-P$ mode:

$$V_{QP}(d) = \left[ \frac{\alpha + \beta}{2 \rho} \right]^{1/2},$$

for the $Quasi-S$ mode:

$$V_{QS}(d) = \left[ \frac{\alpha - \beta}{2 \rho} \right]^{1/2}, \quad (1.39)$$

for the $S-Parallel$ mode:

$$V_{SP}(d) = \left[ \frac{Nd_R^2 + Ld_A^2}{\rho} \right]^{1/2},$$

where:

$$\alpha = (A+L)d_R^2 + (C+L)d_A^2,$$

$$\beta = \left[ \left( (A-L)d_R^2 - (C-L)d_A^2 \right)^2 + 4d_R^2d_A^2(L+F)^2 \right]^{1/2}.$$ 

These results show that for $S-Parallel$ and $Quasi-P$ waves, phase velocity is minimum along the symmetry axis and increases as propagation departs from that direction to reach a maximum with in-plane propagation. In contrast, the $Quasi-S$ phase velocity must exhibit at least one extremum as one goes from axial to radial propagation, since its magnitude is identical in these two directions.
We can now compute the displacement polarizations by making use of equation (1.15). Rewriting that equation in the transversely isotropic case, we obtain:

\[
(V^2 - \frac{Ld_2^2 - Ld_3^2 - Cd_1^2}{\rho})U_1 - \frac{(F+L)d_1 d_2}{\rho} U_2 - \frac{(F+L)d_1 d_3}{\rho} U_3 = 0, \]

\[
- \frac{(F+L)d_1 d_2}{\rho} U_1 + (V^2 - \frac{Nd_2^2 - Ad_2^2 - Ld_1^2}{\rho})U_2 - \frac{(A-N)d_2 d_3}{\rho} U_3 = 0, \tag{1.40}
\]

\[
- \frac{(F+L)d_1 d_3}{\rho} U_1 - \frac{(A-N)d_2 d_3}{\rho} U_2 + (V^2 - \frac{Ad_3^2 - Nd_3^2 - Ld_1^2}{\rho})U_3 = 0.
\]

To solve for the eigenvectors, we consecutively substitute the eigenvalues determined by (1.39) into the system (1.40) which then becomes singular of order one. The solutions are therefore found in the form of one-dimensional sets which are the eigenspaces associated with the eigenvalues. Let us start with the \textit{S-Parallel} mode, that is, substitute:

\[
V^2 = \frac{Ld_1^2 + N(d_2^2 + d_3^2)}{\rho}
\]

into (1.40). This makes the last two equations dependent so that the system reduces to:

\[
\frac{(C-L)d_1^2 + (N-L)(d_2^2 + d_3^2)}{F+L} d_1 U_1 + d_2 U_2 + d_3 U_3 = 0,
\]

\[
\frac{F+L}{A-N} d_1 U_1 + d_2 U_2 + d_3 U_3 = 0.
\]
These two equations can be satisfied simultaneously only if $U_1$ vanishes. This means that the $S$-Parallel mode is polarized within the fracture planes, hence its name. Another consequence is that we can write:

$$0 = d_2 U_2 + d_3 U_3 = d_1 U_1 + d_2 U_2 + d_3 U_3 = d_i U_i,$$

that is, the parallel mode is a truly transverse mode since it is polarized perpendicular to propagation direction, and we can write:

$$U_{SP} = \frac{a \times d}{|a \times d|},$$

As was proved earlier, the other two eigenvectors must be orthogonal to $U_{SP}$, and we therefore define them in the form:

$$U_{QP} = \frac{\epsilon_P d_A + d_R}{\sqrt{\epsilon_P^2 d_A^2 + d_R^2}}, \quad (1.41)$$

$$U_{QS} = \frac{\epsilon_S d_A + d_R}{\sqrt{\epsilon_S^2 d_A^2 + d_R^2}},$$

Then substituting:

$$V^2 = \frac{(A+L)(d_A^2 + d_R^2) + (C+L)d_1^2 + \sqrt{\Delta}}{2\rho},$$

into (1.40) we obtain:

$$\epsilon_P = \frac{(F+L)d_R^2}{\rho V_{QP}^2 - Ld_R^2 - Cd_A^2},$$

while by substituting:
\[ v^2 = \frac{(A+L)(d_2^2 + d_3^2) + (C+L)d_1^2 - \sqrt{\Delta}}{2\rho}, \]

we obtain:

\[ \epsilon_s = \frac{(F+L)d_R^2}{\rho V_{QS}^2 - Ld_R^2 - Cd_A^2}. \]

There are two special cases where these formulas do not hold. First, for propagation along the anisotropy axis, we have \( U_{QP}=a \) while \( U_{QS} \) and \( U_{SP} \) are any two mutually orthogonal unit vectors perpendicular to \( a \). Second, for propagation perpendicular to the anisotropy axis, \( \epsilon_s \) becomes negative infinite, so that \( U_{QS}=-a \).

It is of interest here to remark that polarizations are independent of density. Figure (1.5) shows the geometrical relationships between displacement polarizations, propagation direction, and anisotropy axis.

We presented in (1.27) an expression for group velocity involving displacement polarizations, slowness vector, and elastic coefficients. It is simpler, and therefore preferable, to use a direct calculation based on (1.30) and (1.31). In effect, we have proved that:

\[ W_n = \frac{dx_n}{d\tau} = \frac{F_{p_n}}{p_i F_{p_i}^{\prime}} \quad , \tag{1.42} \]

where \( F(p_n) = 0 \) is the eikonal equation, that we take in the form (1.17). Again we consider each mode separately; for the Quasi-\( P \) wave, we have:
Figure (1.5): Geometrical construction of some essential vector quantities.
\[ F(p_n) = \frac{1}{p_i p_i} - \frac{\alpha(p) + \beta(p)}{2\rho} \ . \]

The computation of group velocity is straightforward if one recognizes that:

\[ p_n F_{,p_n} = \alpha + \beta \ , \]

After some algebra, we obtain:

\[ W_{QP} = \frac{(a_1 + a_2) d_A + (a_3 + a_4) d_R}{\beta \sqrt{2\rho (\alpha + \beta)}} \ , \]

where

\[ a_1 = \beta (C + L) \ , \quad a_2 = 2 (F + L)^2 d_R^2 - \gamma (C - L) \ , \]

and

\[ a_3 = \beta (A + L) \ , \quad a_4 = 2 (F + L)^2 d_A^2 - \gamma (A - L) \ . \]

In a similar way, we obtain the group velocity for the Quasi-S mode:

\[ W_{QS} = \frac{(a_1 - a_2) d_A + (a_3 - a_4) d_R}{\beta \sqrt{2\rho (\alpha - \beta)}} \ , \quad (1.43) \]

and finally for the S-Parallel mode:

\[ W_{SP} = \frac{L d_A + N d_R}{\sqrt{\rho (N d_R^2 + L d_A^2)}} \ . \]

Those results agree with those found in the literature (see Buchwald, for example).

Figure (1.6) shows a polar plot of group velocity for the three wavetypes for a particular choice of parameters. The driving parameter for these curves is the angle between propagation direction and anisotropy axis, and we note that the Quasi-S
Figure (1.6): Cross-section of the wave surfaces of the three modes in an axial plane, showing triplication effects associated with the *Quasi*–*S* mode. Same parameters as in Figure (1.3).
velocity exhibits singular points. These singularities are related to the phenomenon of triplication, which arises because energy and wavefronts do not propagate at the same velocity in anisotropic media.

Let us now describe the phenomenon of triplication. For this, we use the property that group velocity is perpendicular to the slowness surface, which is the surface formed by the extremity of the slowness vector as its direction is varied through all possible azimuths within a homogeneous medium. The equation of this surface is nothing else but the eikonal equation for a given mode, and we rewrite it as in section (1-2) in the form:

\[ F(p_i) = 0 \]

On the other hand, according to equations (1.42), group velocity can be expressed in the form:

\[ W_i = \frac{F,_{p_i}}{p_n F,_{p_n}} \]

This shows that group velocity is aligned on the p-gradient of the function \( F(p_i) \), and hence perpendicular to the slowness surface \( F(p_i) = 0 \).

Whenever the slowness surface is convex, the ray network emanating from it is singlefold. It is only when the slowness surface is concave that this network becomes multifold, as illustrated in Figure (1.7). Typically, each concave cusp in the surface is source to a three-fold ray family limited by two surface caustics,
Figure (1.7): Symbolic illustration of triplication: The ray network becomes three-fold at each cusp in the slowness surface.
whereby the name "triplication". Only the \( Quasi-S \) mode occasionally exhibits triplications, when severe anisotropy is considered.

Since the slowness surface has a symmetry of revolution around the anisotropy axis, triplication points can be determined from the analysis of the trace of that surface in an axial plane. In such a plane, the equation of the slowness surface for a given mode \( M \) can be expressed in polar coordinates \((p, \phi)\) in the form:

\[
p = p^{(M)}(\phi),
\]

where \( \phi \) is the angle between propagation direction and anisotropy axis. For example, in the \( Quasi-S \) case, we have according to (1.39):

\[
p^{(M)}(\phi) = \left[ \frac{2p}{\alpha(\phi) - \beta(\phi)} \right]^{1/2},
\]

where \( \alpha \) and \( \beta \) are the functions of \( \phi \) through the quantities \( d_R^2 = \cos^2 \phi \) and \( d_A^2 = \sin^2 \phi \).

For the curve to exhibit concavity, its curvature must vanish whenever concave and convex arcs meet together. Hence triplication is present if and only if the equation of vanishing curvature:

\[
\frac{d^2p^{(M)}}{d\phi^2} - \frac{2}{p^{(M)}} \left( \frac{dp^{(M)}}{d\phi} \right)^2 - p^{(M)} = 0
\]

admits real solutions. Finding the possible triplication angles from this formulation is not trivial however, since it amounts to finding the roots of a higher order real
polynomial.

Finally, the hexagonal symmetry of the problem makes it easy to express group velocity as the vector sum of phase velocity and a vector perpendicular to phase velocity:

\[ W^{(M)} = \frac{1}{p^{(M)}} \left[ p^{(M)} + \frac{dp^{(M)}}{d\phi} u_\phi \right] , \]  

(1.43)

where \( u_\phi \) is the unit vector perpendicular to \( p^{(M)} \) that lies in the axial plane of \( p^{(M)} \) and oriented in the sense of increasing \( \phi \).

1-2.3 Scattering at Interfaces

So far we have considered the propagation of elastic waves in media where elastic properties vary "smoothly," that is, where the relative change of parameters within a wavelength remains small. This excludes, of course, proper handling of sharp discontinuities in elastic parameters or reflectors, which must be dealt with separately. The classical and efficient way to do this is to formally express the wave field on both sides of the reflector and to apply some boundary conditions. When considering two rock layers, these conditions are the continuity of displacement and traction at the interface, which describe a welded contact. From them, one obtains the scattering directions and amplitudes for the various wavetypes.
The incident medium is the side of the interface where the incident field, that is, the field that would exist without the presence of the reflector, is known. The transmission medium is thus the other side of the interface. In each region, all three wavetypes should a priori be present, and the displacement field is therefore represented as the superposition of all modes. The continuity of displacement at the interface is then written as follows:

$$
\sum_{M} U_{M} e^{i\omega \tau_{M}} = \sum_{M'} U_{M'} e^{i\omega \tau_{M'}} ,
$$

(1.44)

where $M$ and $M'$ stand for the various modes in the incident and transmission media. From this we can conclude that travel times have to be equal for incident and scattered events at the interface:

$$
\tau_{M} = \tau_{M'} 
$$

(1.45)

This equality can be supported by noting that (1.44) holds for any value of angular frequency $\omega$. Substituting this identity in (1.44), we get a second important continuity condition on amplitudes:

$$
\sum_{M} U_{M} = \sum_{M'} U_{M'} ,
$$

(1.46)

Let us then assume that the interface is locally continuous, and that the reflector is parametrized by two independent spatial variables $\sigma_1$ and $\sigma_2$. Expanding (1.45) along the interface, we obtain:
\[ \nabla_{\sigma} \tau_M = \nabla_{\sigma} \tau_{M'} \quad , \quad (1.47) \]

which is Snell's law in its general form. An equivalent statement is that the component of slowness tangent to the interface is identical for incident and scattered waves.

Let us now briefly discuss how to apply this result to our problem; contrarily to the isotropic case, Snell's law does not reduce to a simpler form, and the scattered slownesses must be evaluated numerically. We start by computing the component of slowness tangent to the interface for the incident field. According to Snell's law, all the scattered waves share the same tangential slowness as the incident one, hence the problem reduces to finding the normal component of slowness for each scattered mode.

For the SP mode, this amounts to solving a second degree real polynomial equation, which is done explicitly. If the roots are real, one distinguishes the reflected solution from the transmitted one by computing the associated group velocity vector and determining whether it points towards the desired scattering medium. Note that in general the sign of the normal slowness is not enough to discriminate between reflection and transmission. In particular, the two real roots may have the same sign and the corresponding slownesses point towards the same medium; in such case, it is the group velocity that unambiguously differentiates between reflected and transmitted solutions (Figure (1.8)). If the roots are
Figure (1.8): Separation between reflection and transmission is based on group velocity (ray direction) rather than on the direction of slowness.
complex, one must choose the solution that is evanescent in the scattering medium.

For the QS and QP modes, the scattering problem reduces to a fourth degree real polynomial equation, which must be solved numerically. The roots are then classified according to the mode they represent (two QP and two QS), which is a simple task unless all four roots are complex. Then the adequate solution is selected for each mode separately exactly as for the SP mode.

Knowing the slownesses associated with both incident and scattered waves, we can determine the associated displacement polarizations from equations (1.41). We then project those onto the principal directions of anisotropy (one axial and two radial), to obtain the strain tensor to leading order,

$$\varepsilon_{ij} = \frac{1}{2} \left( U_i p_j + U_j p_i \right).$$

Following this operation, we are able to compute the stresses according to equations (1.1). We then obtain the tractions at the interface by contracting the stress tensor with the unit vector normal to the reflector previously expressed in the principal axes. Finally, the tractions are projected back on the reference coordinate system so that continuity conditions can be applied. The whole procedure is independent of the reference system, which allows us to treat the most general situations, and, although many intermediate operations are involved, it is not computationally intensive; the use of trigonometric functions can, in particular, be entirely avoided. The scattering coefficients are defined here as the ratios of
scattered displacement amplitudes to the incident amplitude. The continuity of
displacement and traction therefore result in a 6x6 system of linear equations
which may be solved using Gaussian elimination.

To conclude this chapter, we have joined a few examples illustrating how the
scattering angles relate to the incident ones for various modes. Figure (1.9)
emphasizes the effect of fracture azimuth; scattering angles do not seem to be
seriously affected by the fracture azimuth in this example. Figure (1.10) shows the
effect of fracture dip on the scattering angle; as can be expected, incidence and
scattering angles are identical whenever the fracture network is parallel or
perpendicular to the interface. Intermediate fracture orientations however show a
significant deviation between these angles, especially at large incidence angles.
Figure (1.11) presents an example of reflection and transmission coefficients at the
interface between two transversely isotropic solids (only the amplitudes are shown).

1-2.4 Free Surface Radiation

So far we have discussed the equations of evolution for wavefields along
characteristic lines or rays, but we have not addressed the matter of initial
conditions for these equations. The object of this section is to determine the
asymptotic field of a point force located at the surface of a homogeneous
transversely isotropic half space or free—surface Green’s tensor. That solution
Effect of Fracture Azimuth

Figure (1.9): Effect of fracture azimuth on scattering at an interface. Azimuth is the angle between fracture direction and tangential slowness. Here Quasi-$P$ to Quasi-$P$ reflection is shown for the same set of parameters as in Figure (1.3), and the dip of the fractures relative to the reflector is $80^\circ$. 
Figure (1.10): Effect of fracture dip on scattering at an interface. Dip is the relative angle between the plane tangent to the reflector and the fracture planes. Quasi—S to Quasi—S reflection is shown here for the same set of parameters as in Figure (1.3), and fracture azimuth is N45° E.
Figure (1.11): Scattering amplitudes of the \textit{S-Parallel} mode. Incident medium parameters are: $A=10$, $C=3$, $N=4$, $L=1$, $F=1$, $\rho=1$, fractures at $N45^\circ E$. Transmission medium parameters are: $A=8$, $C=4$, $N=2$, $L=1$, $F=1$, $\rho=1$, fractures at $N45^\circ W$. Fractures are vertical in both media.
can in turn be used as a good approximation to the field of a point force at the surface of a (weakly) inhomogeneous half space, at least in the neighborhood of the source. Thus it is natural to choose the initial surface just around the source of the field.

Computing the free-surface Green’s tensor $g_{im}^{Free}$ proceeds in two steps: first, evaluate the asymptotic homogeneous Green’s tensor $g_{im}^{Hom}$, that is, the far field of a point force in a homogeneous, unbounded transversely isotropic medium; second, use free-surface scattering conditions and the reciprocity relation.

The first step is described in detail in appendix B, and an excellent discussion of the problem can be found in Buchwald (Buchwald, 1959). The main conclusion is that the homogeneous asymptotic Green’s tensor in transversely isotropic media exhibits features similar to that in isotropic media, that is: an algebraic decay inversely proportional to the distance from the source, and a radiated amplitude inversely proportional to the rigidity of the medium for the wave under consideration.

In the second step, we consider a homogeneous half space limited by a traction-free surface. We place a unit source in direction $m$ at a point $x$ below the surface, and an observation point $x'$ on the surface. The total displacement $g_{im}^{Free}(x',x)$ at $x'$ is the superposition of the incident displacement $g_{im}^{Hom}(x',x)$, which is known, and of the three reflected displacements $u_{im}^{(M)}$, $M=1,2,3$. The
reflected amplitudes are computed from the incident ones with the help of the traction-free condition. In consequence, we have:

\[ g_{im}^{Free}(x', z) = g_{im}^{Hom}(x', z) + \sum_{M=1}^{3} u_{im}^{(M)}(x', z) \]

Then, according to the reciprocity relation, we can write:

\[ g_{mi}^{Free}(z, x') = g_{im}^{Free}(x', z) = g_{im}^{Hom}(x', z) + \sum_{M=1}^{3} u_{im}^{(M)}(x', z) \]

where \( g_{mi}^{Free}(z, x') \) is the displacement in direction \( m \) at \( z \) (below the surface) created by a unit force in direction \( i \) at \( x' \) (on the surface); that quantity is the one we need to initialize the ray tracing process.

The free-surface Green's tensor is also an essential tool in designing seismic experiments over anisotropic media, because it predicts how much energy is radiated into each wavetype for a given single or multi-source configuration. A radiation pattern is a polar plot of the amplitude radiated in a vertical plane, at a constant distance from the source. Figure (1.12) compares the \( P - Wave \) amplitude radiated by a vertical force at the surface of an isotropic half space (dashed line) to that radiated into a transversely isotropic medium with a vertical axis of symmetry (solid line). It is interesting to notice that although \( P - Waves \) propagate at the same velocity in the vertical direction, they do not have the same amplitude. In Figure (1.13a,b,c), the radiation patterns of the \( SP \) and \( QS \) waves are given for a vertical force on top of a vertically fractured medium.
Figure (1.12): Comparison of the free surface radiation pattern of a vertical force for an isotropic and a transversely isotropic medium.
Figure (1.13): Free surface radiation of a vertical force on top of a vertically fractured medium.
Figure (1.13a) shows the radiation in a vertical plane parallel to the fractures, while Figures (1.13b) and (1.13c) show the radiation in vertical planes making angles of 90° and 45° respectively with the fracture planes. These pictures predict that shear-wave energy may be observed at smaller offsets for a shot profile perpendicular to fracture strike than for a survey parallel to fracture strike.

1-3. Ray Tracing in Tabular Fractured Media

1-3.1 Model Parameterization

This section describes an implementation of true amplitude ray tracing in horizontally layered, transversely isotropic media. The modeling algorithm requires three types of inputs: the geometry and elastic properties of the medium, the acquisition geometry and source characteristics, and a description of the events to be traced.

The medium parameters are given layer by layer, and include the thickness of the layer, five elastic constants, density, as well as two angles describing the anisotropy direction. The orientation of the anisotropy axis may be chosen arbitrarily in each layer. For the elastic parameters, it is preferable to use Thomsen's notations (Thomsen, 1986) rather than Love's impractical elastic rigidities. The five independent constants are: the (slow) $P$-wave velocity along the
anisotropy axis $V_p$, the (slow) $S$-wave velocity along the anisotropy axis $V_s$, the relative $P$-wave velocity anisotropy $\varepsilon$, the relative $S$-wave velocity anisotropy $\gamma$, and the anellipticity parameter $\delta$ (in the sense that when $\delta=\varepsilon$, the $QP$ slowness surface becomes an ellipsoid of revolution, while the $QS$ surface becomes a sphere).

The relations between Thomsen's and Love's parameters are as follows:

$$V_p = \sqrt{C/\rho} \quad , \quad V_s = \sqrt{L/\rho} \quad ,$$

$$\gamma = \frac{N-L}{2L} \quad , \quad \delta = \frac{(F+L)^2-(C-L)^2}{2C(C-L)} \quad , \quad \varepsilon = \frac{A-C}{2C} \quad .$$

Isotropic layers can be introduced freely in the model, in which case the layer is simply characterized by its $P$ and $S$ velocities, density, and thickness.

The acquisition geometry is restricted to a single source and a line of receivers at the surface of the medium. The source location, the location of the first receiver, the azimuth of the receiver line, and the spacing between receivers can be arbitrarily specified. In particular, the source need not coline with the receiver line. The receivers have three components, one vertical, and two horizontal that can be oriented along any azimuth. The source is assimilated to a point force with arbitrary direction, characterized by an azimuth and a dip angle. Its signature is a Klauder wavelet generated by the autocorrelation of a linear vibrator sweep whose cutoff frequencies, length and taper are adjustable.
As in any ray tracing algorithm, one must somehow specify the events to be traced; this is because it is impractical to trace all possible waves (primary and multiple reflections of any order, all converted waves) whenever more than a few layers are present. Moreover, most of the energy in the wavefield is usually concentrated in a few events. To ease the characterization of an event, the layers are indexed top to bottom starting at zero (the surface). An event is parametrized by a succession of layer indices starting and ending at the surface. Since each new index represents the next ray segment in the path, it must be accompanied by the type of propagation along that segment, that is, $QP$, $QS$, or $SP$ in an anisotropic layer, and $P$ or $S$ in an isotropic layer. This means for example that for a model composed of four anisotropic layers (three reflectors), the total number of primary reflections from the lowermost reflector is $3^6 = 729$ instead of $2^6 = 64$ if the model were isotropic!

1-3.2 Algorithm Description

Our implementation of ray tracing may be split in three phases: for each event, a bundle of kinematic rays (i.e. without amplitude attributes) is shot from the source for a coarse set of slowness take-off angles, and their emergence points at the surface are stored. The resulting two dimensional map (see example in Figure (1.14)) is then used to linearly interpolate for the take-off angles associated
Figure (1.14): Example of interpolation map used in determining ray take-off angles. Cross hair shows source location, and dots represent ray emergence points for a coarse set of take-off angles.
with each receiver location. Finally, these angles are used to shoot true amplitude rays to each of the receivers.

Given that the medium is piecewise homogeneous, tracing a particular ray is straightforward once the take-off slowness angle is known. The process can be separated into four elementary modules, namely: initialization of ray parameters at the source; tracing between two interfaces; continuation of ray parameters through an interface; ending of the ray at the receiver, accounting for the free surface correction. Each module keeps track of all the ray attributes, which are: slowness $\mathbf{p}$, amplitude $A$, location $\mathbf{z}$, traveltime $\tau$, ray Jacobian $J$, plus the kinematic attributes ($\mathbf{p}$, $\mathbf{z}$, $\tau$) of four neighboring rays used in computing the geometric spreading. We now proceed to describe how each module affects the various attributes.

The initialization module requires two take off angles (azimuth $\gamma_1$ and dip $\gamma_2$) for the initial slowness $\mathbf{p}_0$, and an initial traveltime $\tau_0$ that should be chosen as a small fraction of the minimum expected traveltime within the first layer. The starting point $\mathbf{z}_0$ is determined according to:

$$\mathbf{z}_0(\gamma_1, \gamma_2) = W_0(\gamma_1, \gamma_2) \tau_0 .$$

The initial amplitude $A_0(\gamma_1, \gamma_2)$ is given by the radiation pattern of the source in the desired mode, as discussed in Section 1.2.4. The four neighboring rays are defined by the take-off angles $(\gamma_1 \pm \Delta \gamma_1, \gamma_2)$ and $(\gamma_1, \gamma_2 \pm \Delta \gamma_2)$, where $\Delta \gamma_1$ and $\Delta \gamma_2$ have been optimally determined during the interpolation of $\gamma_1$ and $\gamma_2$. Those rays
are traced to traveltime \( \tau_0 \) according to:

\[
x_{0i}(\gamma_1 \pm \Delta \gamma_1, \gamma_2) = W_{0i}(\gamma_1 \pm \Delta \gamma_1, \gamma_2) \tau_0 \quad i=1,2 \quad ,
\]

and

\[
x_{0i}(\gamma_1, \gamma_2 \pm \Delta \gamma_2) = W_{0i}(\gamma_1, \gamma_2 \pm \Delta \gamma_2) \tau_0 \quad i=3,4 \quad .
\]

From these points, the initial ray Jacobian can be computed by finite difference as follows,

\[
J_0(\tau, \gamma_1, \gamma_2) \approx W_0 \cdot \left[ \frac{x_{01} - x_{02}}{2 \Delta \gamma_1} \times \frac{x_{03} - x_{04}}{2 \Delta \gamma_2} \right] \quad .
\]

The tracing module between two interfaces is simple because the layers are homogeneous. If we elect to choose traveltime to be the ray variable (\( \sigma=\tau \)), equations (1.30) through (1.32) simplify into:

\[
\frac{\partial x(\tau, \gamma_1, \gamma_2)}{\partial \tau} = W(\tau, \gamma_1, \gamma_2) \quad ,
\]

(1.48)

\[
\frac{\partial p(\tau, \gamma_1, \gamma_2)}{\partial \tau} = 0 \quad ,
\]

(1.49)

while the transport equation (1.35) becomes:

\[
\frac{\partial}{\partial \tau} \left[ A^2(\tau, \gamma_1, \gamma_2)J(\tau, \gamma_1, \gamma_2) \right] = 0 \quad .
\]

(1.50)

Equation (1.49) shows that within a layer the slowness is constant that is:

\[ p = p_0 \quad . \]
Therefore the rays are straight lines directed along group velocity $W_0(p_0)$ and starting at the prescribed initial point $x_0$. To continue a ray to an interface at depth $x_3$ from initial ray attributes, we first compute the arrival time $\tau$ at that interface according to:

$$\tau = \tau_0 + \frac{x_3 - x_{30}}{W_{30}}$$

Here we have assumed the $x_3$ direction to be downward vertical. The horizontal coordinates of the ray on the interface can then be calculated as follows:

$$x_i = x_{i0} + W_{i0}(\tau - \tau_0) \quad i=1,2$$

The four extra rays are traced to traveltime $\tau$:

$$x_i(\tau) = x_{i0} + W_0(\tau - \tau_0) \quad i=1,4$$

so that the ray Jacobian at the interface can be approximated once again by finite difference:

$$J(\tau) \approx W_0 \cdot \left[ \frac{x_1(\tau) - x_2(\tau)}{2 \Delta \gamma_1} \times \frac{x_3(\tau) - x_4(\tau)}{2 \Delta \gamma_2} \right]$$

The four rays must then be continued to the interface itself so that they can later be used in the next layer (i.e. the ray tube must be kept continuous). Finally, the amplitude of the ray is only affected by the geometric spreading, and according to the transport equation, we have:
\[ A = A_0 \left( \frac{J_0}{J(\tau)} \right)^{1/2} . \]

The module that continues rays through an interface uses the procedure discussed in Section 1-2.3: the scattered slownesses are determined from the incident ones for the central ray and the four auxiliary rays by means of Snell’s law. Scattering coefficients are computed from the incident slowness and the medium parameters on both sides of the interface using a system of Zoeppritz equations. The amplitude of the scattering mode of interest is obtained by multiplying the incident amplitude by the corresponding scattering coefficient. On the other hand, the location and traveltime of the ray are continuous through interfaces. The only remaining computation concerns the scattered ray Jacobian. Here the boundary condition is that the areas of the surfaces formed by the intersection of the incident and scattered ray tubes be equal. Expressing this continuity condition is straightforward as in the isotropic case. We first define the elementary area of wavefront \( a_0 \) as:

\[ a_0 = \left| \frac{\partial z}{\partial \gamma_1} \times \frac{\partial z}{\partial \gamma_2} \right|_{\tau} d\gamma_1 d\gamma_2 . \]

Then, according to the definition of the ray Jacobian and Figure (1.15) we recognize that:

\[ J d\gamma_1 d\gamma_2 = W a_0 \cos \theta_0 , \]
Figure (1.15): Schematic illustration of the parameters used in section I-2.3 in determining a continuation formula for the ray Jacobian.
that is, the quantity $J d\gamma_1 d\gamma_2$ represents the cross-sectional area of the ray tube, scaled by group velocity. On the other hand, the area $a$ of the intersection of the ray tube with the reflector is:

$$a = \frac{a_0 \cos \theta_0}{\cos \theta_1},$$

and substituting for $J$, we have:

$$a = \frac{J d\gamma_1 d\gamma_2}{W \cos \theta_1}.$$

Then $\cos \theta_1$ can be expressed as:

$$\cos \theta_1 = \frac{W \cdot n}{W},$$

so that we finally obtain:

$$a = \frac{J d\gamma_1 d\gamma_2}{W \cdot n}.$$

Since $a$ must be continuous at the interface, we can formulate a continuation equation for $J$ in the following way:

$$\frac{J_{\text{inc.}}}{W_{\text{inc.}} \cdot n} = \frac{J_{\text{scat.}}}{W_{\text{scat.}} \cdot n} \quad \text{at the interface}.$$

The last of the four modules assembles the final displacement field $u$ at the free surface; that vector is made of the sum of the incident and the three reflected displacements at the free surface. The incident displacement is simply the product of the incident polarization $U$ and the amplitude $A$, while the reflected displacements are the product of the incident amplitude, a reflection coefficient,
and the reflected polarization. The event is then incorporated to the synthetic vector seismogram $s(t)$ at the traveltime $\tau$ with the signature $w(t)$ of the source, that is:

$$s(t) = s(t) + \Re(u)w(t) + \Im(u)h(t)$$

where $h(t)$ is the Hilbert transform of $w(t)$.

Appendix C summarizes the algorithm for ray tracing in tabular fractured media. The next section presents a discussion of some particular aspects of wave propagation in fractured media based on the analysis of synthetic seismograms.

1-3.3 Examples

The first example illustrates birefringence on a typical model: a fractured layer embedded in two isotropic shoulders (Figure (1.16)). In the anisotropic layer, the fractures are oriented $N 45^\circ W$ and dip at $70^\circ$ to the $SW$. The source simulates a horizontal vibrator polarized $NS$. The source signature after autocorrelation is a Klauder wavelet obtained from an 8 second linear sweep with a half second taper and a 5–40 Hz bandpass. The recording array is a split spread $NS$ line of 61 three component geophones ($NS$, $EW$ and vertical components). Five events were traced: the $S-S$ reflection at the base of the first layer, and the four possible combinations of $S$ modes reflected at the bottom of the fractured formation.
Figure (1.16): Description of the three layer model discussed in section I-3.3.
These last events can be identified by the chronology of their propagation modes from source to receiver; they are: $S-SP-SP-S$, $S-SP-QS-S$, $S-QS-QS-S$, $S-QS-SP-S$. Note that $S-SP-SP-S$ and $S-QS-QS-S$ represent the two split shear waves associated with birefringence. Figure (1.17) is a perspective plot of the raypaths for the five events. A striking feature is that the ray segments do not remain in the vertical plane containing the source-receiver line.

The seismic sections in Figures (1.18) through (1.20) show the displacement field as recorded in the NS, EW, and vertical geophones respectively (please note that the scaling varies from plot to plot). The first arrival around 3 seconds is of course the $S-S$ reflection. Although that event propagates exclusively in an isotropic medium, the scattering conditions at the interface with the underlying fractured layer significantly affect its polarization. If the second layer were isotropic, the $S-S$ arrival would have a horizontal polarization in the same direction as that of the source. Instead here, polar plots showing horizontal displacement versus time, or hodograms, demonstrate that the $S-S$ event has a non zero $EW$ component upon arrival at the surface (Figure (1.21)). Unfortunately, this polarization "anomaly" strongly depends on the scattering coefficients at the interface, which are offset-dependent. Therefore it is not in itself an indicator of fracture orientation.
Figure (1.17): Perspective ray tracing plot for the example discussed, showing an outline of the layer boundaries. Note the out of plane propagation of energy.
In Line Displacement

Figure (1.18): Shot profile of the in-line horizontal component of the displacement field for the example discussed in section I-3.3.
Cross Line Displacement

Figure (1.19): Shot profile of the cross-line horizontal component of the displacement field for the example discussed in section I-3.3.
Vertical Displacement

Figure (1.20): Shot profile of the vertical component of the displacement field for the example discussed in section I-3.3.
Figure (1.21): Horizontal hodograms of the reflection at the interface between the first, isotropic layer, and the second, fractured layer. Note that the polarization of the arrival does not follow that of the source (North). Also note that the polarization “anomaly” is offset-dependant.
Now to the reflections from the base of the fractured layer. At small offsets, the prevailing events energywise are the split $S-SP-SP-S$ and $S-QS-QS-S$ waves. The converted events can however be seen at wider offsets, especially on the in-line and vertical displacement sections (Figures (1.18) and (1.20)). Actually, the latest event at large negative offsets is $S-QS-SP-S$, while the latest one at large positive offsets is $S-SP-QS-S$. Figure (1.22) shows perspective hodograms in the time window 4.9–5.4 seconds, for a set of receiver locations (please note again that the scaling varies from plot to plot). It is remarkable that these plots exhibit the same pattern throughout the section: the first trend is $N45^\circ W$ and corresponds to the $S-SP-SP-S$ wave, which travel fastest. The second trend is $N45^\circ E$ and represents the slower $S-QS-QS-S$ reflection. This suggests that the study of polarization patterns may indeed be a straight forward way of determining fracture orientation. If the fractured layer is thinner or its anisotropy weaker however, the two split shear arrivals partly superpose in time; this interaction, which can also be made worse by lowering the frequency content of the signal, eventually seriously degrades the polarization patterns. Figure (1.23) illustrates what happens to the zero-offset hodogram as the thickness of the fractured layer progressively decreases: the trends described earlier can be inferred for thicknesses down to 500m, which is about eight wavelengths in this example: a rather large number. This gives some idea of the limitations of direct birefringence observation. In the presence of noise, more robust techniques must be used in order to
Figure (1.22): Perspective hodograms in the window 4.9s-5.4s showing birefringence. Note that the polarization patterns are essentially offset-independant.
Figure (1.23): Zero offset perspective hodograms for the example discussed: here the thickness of the fractured layer is decreased from 3000m down to 200m.
determine fracture azimuth, as demonstrated in the recent work of Lewis (Lewis, 1989), Gurch (Gurch, 1989), and Schipperijn (Schipperijn, 1989).

The second example shows an effect of PTL anisotropy on amplitude-versus-offset (AVO) analysis. The model contains three layers of which the top two are isotropic. The source is a vertical force at the surface, and the event of interest is the P-Wave reflected at the bottom of the second layer (depth 1500m) and recorded in vertical receivers. In the first case, the bottom layer is also isotropic, and Figure (1.24) shows the corresponding seismogram and the associated amplitudes. One can note the presence of a critical angle at an offset of about 4200m, far beyond the reach of conventional surveys. In the second case, the bottom layer is made weakly anisotropic (γ=.165, δ=.090, ε=.100). This does not affect the traveltime of the event at all, because its propagation still remains limited to the top two, isotropic layers. However, Figure (1.25) shows a drastic change in the reflected amplitude: although it remains the same near zero offset (note that the scales are different), the AVO curve exhibits a critical angle at an offset of only 2500m, much more within the range of an exploration survey. Considering that PTL anisotropy is presumably common in sedimentary basins (particularly in shaly formations), this example demonstrates that robust AVO analyses cannot dispense from investigating anisotropy as a contributing factor. This is experimentally demonstrated by Lu (Lu, 1988) on compressional data from the Silo field in Wyoming.
Figure (1.24): Shot record and AVO curve for the isotropic model discussed at the end of section I-3.3.
Figure (1.25): Shot record and AVO curve for the anisotropic model discussed at the end of section I-3.3. Compare the curve to that in Figure (1.24) and note how the critical angle has moved to near offsets.
2. KIRCHHOFF-WKBJ INVERSION IN LAYERED ANISOTROPIC MEDIA

2-1. Kirchhoff- *WKBJ* scattering

2-1.1 Introduction

The Kirchhoff scattering formula allows one to compute the field reflected by a surface of known reflectivity as a function of the incident field on that surface. The incident field is, by definition, the field that exists in the absence of the scattering surface. The modeling formula is derived using Green's theorem for an observation point within a domain bounded by the scatterer. In our case, this domain is a multilayered medium limited above by a free surface and below by the reflector of interest (i.e. what is later considered as the "target" layer for inversion). The incident field is that of a point source at the free surface, that takes into account all but the lowermost reflector.

The field scattered by the target reflector is expressed as an integral over the reflector that involves the incident field, a reflection coefficient, and the Green's tensor from the observation point (Sumner, 1988). The reflection coefficient is that of the plane wave that propagates from the source to the interface and reflects to the observation point according to Snell's law. In order to obtain an explicit representation for the scattered field on which to base the derivation of an
inversion operator, we use a \textit{WKBJ} expansion of both incident and scattered fields, whereby the name Kirchhoff-\textit{WKBJ} formula.

2-1.2 The modeling integral

Let us consider a piecewise homogeneous medium $V$ bounded by a surface $B$. $B$ is formed by a free surface $F$ of infinite lateral extent on the top, a scattering surface $S$ of infinite lateral extent on the bottom, and by a closing piece $C$ at infinity joining those two surfaces. This geometry is illustrated in Figure (2.1).

We define the scattered displacement field $u_i(x,\omega)$ as the field generated by the presence of the scattering surface; that is, $u_i$ satisfies a homogeneous wave equation inside $V$ with some boundary conditions on $B$ that we review later:

\begin{equation}
(c_{ijkl}u_{k,l})_{,j} - \rho\omega^2 u_i = 0 \quad x \in V . \tag{2.1}
\end{equation}

We also define the Green’s tensor $g_{im}(x,x')$ as the displacement field at $x$ created by a point force in the direction $m$ at $x'$, in the absence of the scatterer $S$:

\begin{equation}
(c_{ijkl}g_{km,l})_{,j} - \rho\omega^2 g_{im} = -\delta_{im}\delta(x-x') \quad x,x' \in V . \tag{2.2}
\end{equation}

Let us now multiply (2.2) by $u_i$, subtract it from (2.1) previously multiplied by $g_{im}$, and integrate over all $x$ in $V$ to get:

\[ \int_V \left[ g_{im}(c_{ijkl}u_{k,l})_{,j} - u_i(c_{ijkl}g_{km,l})_{,j} \right] dV = u_m(x',\omega) . \]
Figure (2.1): A domain $V$ limited by a free surface $F$ and a scatterer $S$. All surfaces are meant to extend indefinitely in the lateral direction.
Rearranging the left hand side to form exact divergences, we obtain:

\[ u_m(x', \omega) = \int_V \left[ (g_{im} c_{ijkl} u_{k,l})_{,j} - (u_i c_{ijkm,l})_{,j} + u_{i,j} c_{ijkl} g_{km,l} - g_{im} c_{ijkl} u_{k,l} \right] dV. \]

Next we use the symmetry of the elastic coefficient tensor to eliminate the last two terms in the previous integral

\[ u_{i,j} c_{ijkl} g_{km,l} = u_{i,j} c_{klij} g_{km,l} = u_{k,l} c_{ijkl} g_{im,j}, \]

and apply the divergence theorem on the remaining terms which are exact divergences to obtain:

\[ u_m(x', \omega) = \int_B \left[ g_{im} c_{ijkl} u_{k,l} \nu_j - u_i c_{ijkm,l} \nu_j \right] dB, \]

where \( \nu_j \) is the unit vector normal to \( B \).

At this point, it is convenient to introduce the scattered and Green's tractions respectively,

\[ t_i = c_{ijkl} u_{k,l} \nu_j, \]

and

\[ f_{im} = c_{ijkl} g_{km,l} \nu_j. \]

Substituting these expressions in the integral representation above, we obtain

\[ u_m(x', \omega) = \int_B \left[ g_{im}(x, x', \omega) t_i(x, \omega) - u_i(x, \omega) f_{im}(x, x', \omega) \right] dB. \]
This is an integral relation to compute the scattered field anywhere inside $V$ from its value and the value of its spatial derivatives on $B$.

We are now ready to discuss the boundary conditions to be applied on $B$. We first impose that the top surface $F$ be traction-free, because it corresponds in practice to the earth/air boundary which is traction-free (i.e. the atmospheric pressure can be neglected). That is, the integral on $F$ vanishes because both $f_{im}$ and $t_i$ vanish there. On the other hand, both the Green's tensor and the scattered field satisfy radiation conditions that insure that the integral on the closing surface $C$ vanishes in the limit. Therefore the value of the scattered field need only be specified on the scattering surface $S$ to be evaluated anywhere else in $V$:

$$u_m(x^r, \omega) = \int_S \left[ g_{im}(x, x^r, \omega) t_i(x, \omega) - u_i(x, \omega) f_{im}(x, x^r, \omega) \right] dS . \quad (2.3)$$

In following developments, we consider the Green's tensor when $x^r$ is located on the otherwise free surface $F$. In that case, (2.2) does not hold strictly and we have to formulate the problem slightly differently, that is, we introduce the source as a point traction on $F$ and define the Green's tensor as

$$(c_{ijkl}g_{km,l})_{,j} - \rho \omega^2 g_{im} = 0 \quad x, x^r \in V ,$$

and

$$f_{im}(x, x^r) = -\delta_{im} \delta(x - x^r) \quad x, x^r \in F .$$

Following the same steps as before, we obtain
\[
0 = \int_S \left[ g_{im}(x, x', \omega) t_i(x, \omega) - u_i(x, \omega) f_{im}(x, x', \omega) \right] dS \\
+ \int_F \left[ g_{im}(x, x', \omega) t_i(x, \omega) - u_i(x, \omega) f_{im}(x, x', \omega) \right] dF
\]

Substituting the boundary conditions on \( t_i \) and \( f_{im} \), we get:

\[
0 = \int_S \left[ g_{im}(x, x', \omega) t_i(x, \omega) - u_i(x, \omega) f_{im}(x, x', \omega) \right] dS - u_m(x', \omega)
\]

which is exactly (2.2). Consequently we can extend the validity of (2.2) to sources on the free surface.

2-1.3 The Kirchhoff-WKBJ Approximation

The Kirchhoff approximation consists in expressing the scattered field on \( S \) as a function of a known incident field. That is, we substitute for \( u_i \) and \( t_i \) on the scatterer some known functions \( u_i^{Scat} \) and \( t_i^{Scat} \) respectively:

\[
u_i(x, \omega) = u_i^{Scat} \quad x \in S
\]
\[
t_i(x, \omega) = t_i^{Scat} \quad x \in S
\]

We elect to represent the incident field as the WKBJ Green's displacement of a specified point force at \( x' \) within \( V \), computed as if the scatterer were absent and the lowermost homogeneous medium extended to infinity. The incident field is therefore a superposition of modes that we denote by the letter \( M \). There is one
mode for each event that is generated in \( V \) by the source including all multiply scattered and converted waves:

\[
u_{i}^{nc}(z,z',\omega) \sim S(\omega) \sum_{M} u_{i}^{M}(z,z') e^{i\omega \tau(z,z')}
\]

Here \( S(\omega) \) is the spectrum of the source, which we assume to be a high bandpass filter.

On the other hand, the Green's tensor \( g_{im} \) is also a superposition of modes denoted by \( L \). Here again there is one mode for each possible event generated at \( z \) by a point force at \( z' \) including multiple scattering and conversion of any order:

\[
g_{im}(z,z',\omega) \sim \sum_{L} G_{lm}^{L}(z,z') e^{i\omega \tau(z,z')}
\]  

(2.4)

Accordingly, the Green's traction may be expressed as

\[
f_{im}(z,z',\omega) = i\omega \sum_{L} c_{ijkl} G_{klm}^{L}(z,z') p_{j}^{\hat{r}}(z,z') \nu_{j}(z) e^{i\omega \tau(z,z')}
\]  

(2.5)

where

\[
p_{j}^{\hat{r}}(z,z') = \nabla_{z} \tau_{L}(z,z')
\]

We now introduce the Kirchhoff approximation for the scattered field on \( S \): for each incident mode \( M \), there are three possible scattered modes, one for each wavetype. However, only one of these wavetypes can be selected: the one that is consistent with the mode \( L \) of the Green's tensor in the medium just adjacent to the scatterer. The scattered displacement amplitude can be presented as the
product of the incident displacement amplitude $u^M(x, x')$ and a scattering coefficient $R_{ML}$, which is determined by the usual continuity equations on displacement and traction. The phase $\tau_M$ of the scattered displacement on $S$ is the same as that of the incident field and its polarization $v^{ML}$ can be obtained from the scattered slowness $p^{ML}$ which is itself computed according to Snell's law. The Kirchhoff-\textit{WKBJ} approximation to the scattered displacement is therefore:

$$u_i^{Scat}(x, x', \omega) \sim S(\omega) \sum_M R_{ML}(x, x') u^M(x, x') v^{ML}_i(x, x') e^{i\omega \tau_M(x, x')}, \quad x \in S. \quad (2.6)$$

Correspondingly, the Kirchhoff-\textit{WKBJ} approximation to the scattered traction is

$$t_i^{Scat}(x, x', \omega) \sim i\omega S(\omega) \sum_M \left\{ R_{ML}(x, x') u^M(x, x') c_{ijkl} v^{ML}_k(x, x') p^{ML}_l(x, x') \nu_j(x) \right\} e^{i\omega \tau_M(x, x')}, \quad x \in S. \quad (2.7)$$

### 2.1.4 Modeling Formula

We are now ready to substitute expressions (2.4) through (2.7) into the modeling integral in (2.3) to obtain

$$u_m(x^r, x^s, \omega) \sim$$

$$i\omega S(\omega) \sum_{M, L} \int_S u^M(x, x') R_{ML}(x, x') I^{ML}_m (x, x', x^s) e^{i\omega \left[ \tau_M(x, x') + \tau_L(x, x') \right]} dS. \quad (2.8)$$

where $I^{ML}_m$ is an 	extit{interaction vector} defined by:
\begin{equation}
I_{m}^{ML}(z, z', z^r) = c_{ijkl}(z) \nu_j(z) \left[ G_{lm}^k(z, z') v_{k}^{ML}(z, z^r) p_{l}^{ML}(z, z^r) - v_{l}^{ML}(z, z^r) G_{km}^l(z, z') p_{l}^{ML}(z, z^r) \right].
\end{equation}

(2.9)

In this formulation of the scattered field, we have explicitly assumed a high frequency solution by selecting a WKBJ Green's tensor. Therefore we expect that solution (2.8) is equivalent to a ray theoretical solution. To confirm this point, we notice that at high frequencies, the integral in (2.8) has an oscillatory kernel and its behavior is therefore dominated by stationary points. To leading order asymptotically, the integral can be reduced to a discrete sum of contributions:

\begin{equation}
u_m(z^r, z^s, \omega) \sim 2\pi S(\omega) \sum_{z_0} \sum_{M, L} C_m^{ML}(z_0, z^r, z^s) e^{i\omega \left[ \tau_M(z_0, z^r) + \tau_L(z_0, z^s) \right]}, \tag{2.10}
\end{equation}

where \( C_m^{ML} \) is an amplitude term proportional to the interaction vector, reflection coefficient, and incident amplitude, and \( z_0 \) are the points on \( S \) that satisfy the stationarity condition,

\begin{equation}
\frac{\partial \tau_M}{\partial \eta_j} \bigg|_{z_0} + \frac{\partial \tau_L}{\partial \eta_j} \bigg|_{z_0} = 0, \quad j = 1, 2. \tag{2.11}
\end{equation}

Here \( \eta_j \) are two arclength variables parametrizing \( S \).

Not surprisingly, condition (2.11) is just another statement of Snell's law. If, for a given pair \((M, L)\) there is no point on \( S \) satisfying (2.11), then the scattered field is asymptotically lower order in \( \omega \). On the other hand, (2.10) means that the
scattered field is a superposition of all the ray theoretical (WKB\textsuperscript{J}) solutions traced from \( z' \) to \( z' \) through \( S \), that are specular on \( S \). In particular, \( R_{ML} \) in (2.10) is the usual specular plane wave reflection coefficient. This result is fundamental to the development of an inversion theory.

2-2. Kirchhoff-WKB\textsuperscript{J} Inversion

2-2.1 Introduction

In this section we investigate the reconstruction of a reflectivity map of the subsurface from surface measurements of scattered fields, as well as elastic parameter estimation. For this we make the assumption that the field is adequately represented by the Kirchhoff scattering described earlier. We also assume that the scattering medium is piecewise homogeneous (formed by a juxtaposition of homogeneous layers). This assumption is not strictly necessary, and weak inhomogeneity (i.e. consistent with the WKB\textsuperscript{J} representation) may be allowed.

Although we do not put any restriction on the source-receiver configurations used in generating/measuring the scattered field, one can see that this constitutes an important issue in itself. In particular, those portions of scatterer that are in a geometrical shadow zone for the source-receiver configuration cannot be reconstructed. This is, however, a limitation of all seismic imaging techniques. It is also understood that measurements corresponding to a limited set of source-
receiver pairs can only yield partial images of reflectors, and that many such images must be superposed to obtain a complete picture of the subsurface.

Nonetheless, the procedure is as follows: we assume that the scattering medium is specified all the way down to some reflector, whose location is unknown and below which the elastic parameters are unknown as well. From this input, we are able to determine the location of the unknown reflector as well as its reflectivity to various wavetypes. This in turn permits us to estimate the elastic properties of the medium below this reflector. Therefore we are able to reconstruct the successive reflectors by downward recursion of this procedure insofar as the parameters of the top layer are known.

It is essential that more than one mode be considered in the processing of each scatterer. This is because the reflection coefficients for several wavetypes are needed to solve for the elastic coefficients below a reflector. An added advantage of considering several modes is that several images of the same reflector are obtained. Testing if those images match is a good indicator of whether the background parameters are correct.

2-2.2 Mode Decoupling

Let us now consider imaging a particular reflector $S$ and from a scattered field $u_m(x', x^*)$ as defined in (2.8). This field is a superposition of many modes, while
our inversion technique operates on only one at a time. Consequently, some way must be found to discriminate between modes in the inversion process. This is done at least partially by matching the phase (traveltime) and polarization of the inversion operator to that of the event selected for processing. Polarization matching is achieved by projecting the recorded displacement along the polarization of the selected wavetype. Equation (2.8) shows that the polarization direction of the signal is that of the interaction vector. Therefore by projecting the scattered field along the proper interaction vector, we maximize all the modes that reach the receiver with the desired wavetype. Moreover, if all the modes reached the receiver along the same propagation direction, this projection would also exactly eliminate the modes that reach the receiver with a polarization other than the selected one. This is because of the orthogonality property of displacement polarizations. Although it is true that different modes propagate along different paths in elastic media, the frequent presence of a low velocity top layer (water and unconsolidated sediments in marine seismic, or weathered zone in land seismic) tends to gather near surface propagation directions towards the vertical, making the filtering efficient.

Polarization matching discriminates based only on the polarity of an event at the receiver, so that events from many different scatterers as well as multiple reflections and converted waves are retained. This is why phase-matching is needed to further discriminate against all but the selected mode.
Equation (2.10) provides a clue to phase matching; let us rewrite it in the simplified form

\[ u_m(x', x^s, \omega) \sim S(\omega) \sum_{z_{0, M, L}} C_{mL}^{ML}(x_0, x', x^s) e^{i\omega \phi^{ML}(x_0, x', x')} \]  \hspace{1cm} (2.12)

where

\[ \phi^{ML}(x_0, x', x^s) = \tau_M(x_0, x^s) + \tau_L(x_0, x') \] .

That is, to leading order asymptotically, the Fourier-domain scattered field is a superposition of harmonic functions, with phases equal to the sum of travel times from source to receiver through specular points on a scatterer. If we multiply \( u \) by a harmonic function whose phase is exactly opposite to that of one harmonic in the signal, and take a high frequency bandpass inverse Fourier transform, the resulting quantity only depends on that specific harmonic. Unfortunately, the phases in the signal cannot be matched \textit{a priori}, because the specular scattering points are unknown. However, we can proceed as follows: we select a mode \((N, Q)\) to be processed, and compute the traveltime from source to receiver through some output point \( x' \) below the last known scatterer:

\[ \phi^{NQ}(x', x^s, x') = \tau_N(x', x^s) + \tau_Q(x', x') \] .

We then form the quantity

\[ C_{m}^{NQ}(x', x^s, x') = \frac{1}{2\pi} \int d\omega \, u_m(x', x^s, \omega) e^{-i\omega \phi^{NQ}(x', x^s, x')} \] .

Substituting (2.12) in this expression, we obtain
\[ O_{m}^{NQ}(x', z', x') = \frac{1}{2\pi} \sum_{x_0, M, L} \int d\omega S(\omega) C_{m}^{ML} e^{i\omega \left[ \phi^{ML}(z_0, x', z') - \phi^{NQ}(z', x', x') \right]} . \]

The integral \( O_{m}^{NQ} \) is apparently a bandlimited delta function whose support is the set of points \( x' \) where the phase of the integrand vanishes. This happens whenever \((M, L)\) is identical to \((N, Q)\) and \( x' \) is a specular points \( x_0 \), that is, \( O_{m}^{NQ} \) maps specular points on the unknown scatterer using mode \((N, Q)\). There are also, of course, non-specular points where the phase may be zero, but by repeating the phase matching process on a dense set of source-receiver pairs, we expect the contours of the scatterer to become continuous while other contributions form no coherent image. Furthermore, the partial coherence of mismatched modes is further attenuated by post-inversion stacking of outputs.

Phase matching is therefore not only an efficient way to isolate the contribution of a selected mode, but also a method for imaging unknown scatterers. We make these heuristic statements rigorous in the following section.

2-2.3 Inversion Formula

Following Sumner (1988), we assume that all sources and receivers belong to an acquisition surface \( S_{\xi} \) parametrized with a 2-D vector \( \xi \), that is:

\[ x' = x'(\xi_1, \xi_2) , \]
\[ z' = z'(\xi_1, \xi_2) . \]
We then seek an inversion formula that incorporates polarization and phase matching for a selected mode \((N,Q)\). Since each output point \(z'\) must be tested for specularity against the scattered field for every source-receiver pair, the formula is an integral over the entire acquisition surface,

\[
\beta^{NQ}(z') = \int d\omega \int d^2 \xi \left\{ u_m(z'^*,z^*,\omega) I_m^{NQ}(z',x'^*,x^*) \right. \\
A^{NQ}(z'^*,z^*,x'^*,\omega) \left. \right\} e^{-i\omega [r_0(z'^*,x'^*)+r_0(z^*,x^*)]} ,
\]

(2.13)

where \(A^{NQ}\) is an amplitude term that we determine next. The projection on \(I^{NQ}\) insures the proper polarization match whenever the output point \(z'\) is a specular point on the scatterer. The phase of the integrand matches that of the selected mode at specular points as well.

To determine \(A^{NQ}\), we impose that the peak values of the output (i.e. the values of \(\beta^{NQ}\) on the reflector) be equal to the specular reflection coefficient \(R_{NQ}\). The computation is achieved by substituting the Kirchhoff-WKBJ expression of the scattered field into (2.13). That field can be written as a superposition of many modes \((M,L)\) as follows,

\[
u_m(z'^*,z^*,\omega) = \sum_{M,L} U_m^{ML}(z'^*,z^*,\omega) .
\]

(2.14)

According to (2.8) and (2.9), the displacement for each mode is defined by
\[ \begin{align*}
U^M_m(x',x^s,\omega) \sim \int_{S_n} d^2 \eta \sqrt{|g_{\eta}|} u^M(x,x^s) R^M_{nL}(x,x^s) I^M_m(x,x'^r,x^s) e^{i\omega \phi_{ML}(x,x',x^s)},
\end{align*} \]

where \( S_{\eta} \) is the scattering surface parametrized by a 2-D vector \( \eta = (\eta_1, \eta_2) \); \( \sqrt{|g_{\eta}|} \) is the determinant

\[ \sqrt{|g_{\eta}|} = \left| \frac{\partial x}{\partial \eta_1}, \frac{\partial x}{\partial \eta_2} \right| = \nu \left( \frac{\partial x}{\partial \eta_1} \times \frac{\partial x}{\partial \eta_2} \right), \]

and \( \phi_{ML} \) is the traveltime from source to receiver through a point \( z \) on the scatterer

\[ \phi_{ML}(x,x'^r,x^s) = r_M(x,x^s) + r_L(x,x'^r). \]

As emphasized earlier, the polarization and phase matching in (2.13) discriminate against all but the selected mode \( (N,Q) \) within the signal. We therefore proceed forward and substitute \( U^N_m \) for \( u_m \) in (2.13) to obtain

\[ \begin{align*}
\beta^{NQ}(x') &\sim \int_{S_\xi} d\omega i\omega S(\omega) \int_{S_n} d^2 \eta \sqrt{|g_{\eta}|} u^N(x,x^s) R^{NQ}(x,x^s) I^N_m(x,x'^r,x^s) \\
&\quad I^N_m(x',x'^r,x^s) A^{NQ}(x',x'^r,x^s,\omega) e^{i\omega \left[ \phi^{NQ}(x,x',x^s) - \phi^{NQ}(z',z^s,z^s) \right]}.
\end{align*} \]

At this point, we use our high frequency assumption to asymptotically evaluate the quadruple spatial integral by the method of stationary phase; this assumption is valid whenever the bandlimit of the signal is high enough. The
stationarity conditions for the phase in (2.16) are

\[ \frac{\partial \Phi_{NQ}^k}{\partial \eta_k} = 0 \quad , \quad k = 1,2 \quad , \quad (2.17) \]

and

\[ \frac{\partial \Phi_{NQ}^l}{\partial \xi_l} = 0 \quad , \quad l = 1,2 \quad , \quad (2.18) \]

where

\[ \Phi_{NQ}^N(x,x',x',x) = \phi_{NQ}^N(x,x',x) - \phi_{NQ}^N(x',x',x) \]

\[ = \tau_N(x,x') + \tau_Q(x,x') - \tau_N(x',x') - \tau_Q(x',x') \quad . \]

Next we define the slowness vector for mode \( M \),

\[ p^M(y,z) = \nabla_y r_M(y,z) \quad , \]

and use the chain rule to rewrite the stationarity conditions as follows:

\[ \left[ p^N(x,x') + p^Q(x,x') \right] \cdot \frac{\partial x}{\partial \eta_k} = 0 \quad , \quad k = 1,2 \quad , \quad (2.19) \]

and

\[ \left[ p^N(x,x') - p^N(x',x') \right] \cdot \frac{\partial x'}{\partial \xi_l} + \]

\[ \left[ p^Q(x,x') - p^Q(x',x') \right] \cdot \frac{\partial x'}{\partial \xi_l} = 0 \quad , \quad l = 1,2 \quad . \quad (2.20) \]

Not surprisingly, equation (2.19) is the specularity condition (2.11) that we encountered with the Kirchhoff-WKB integral, and another statement of Snell’s law.
The second condition is more complex, although one can check that it is met when \( x' = x \) (Bleistein, 1989); this means that \( \beta^{NQ} \) is asymptotically large when the output point \( x' \) is located on the scatterer \( S_n \) and at a specular point for a source receiver pair within the integration range. Other stationary points may exist, but we show next why they do not usually contribute significantly to the output, that is, their contribution is asymptotically zero. From the multidimensional stationary phase formulae (Bleistein, 1985), we obtain

\[
\beta^{NQ}(x') \sim (2\pi)^2 \int d\omega \, i\omega S(\omega) \left\{ \frac{\sqrt{|g_n|} \, u^N(x,x') \, R^{srcc}_{NQ}(x,x') \, I^{NQ}_{m}(x,x',x')}{\omega^2} \right. \\
\left. \frac{I^{NQ}_{m}(x',x',x') \, A^{NQ}(x',x',x',\omega)}{\sqrt{|H_{NQ}(x,x',x',x')|}} \right\} \, e^{i\omega \Phi^{NQ}(x,x',x',x') + \frac{i}{4} \text{sgn}(\omega) \text{sgn} H_{NQ}}, \quad (2.21)
\]

where \( |H_{NQ}| \) and \( \text{sgn} H_{NQ} \) are respectively the determinant and signature of the 4×4 Hessian of the phase, which is itself defined by

\[
H_{NQ}(x,x',x',x') = \begin{bmatrix}
\frac{\partial^2 \Phi^{NQ}}{\partial \xi_k \partial \xi_l} & \frac{\partial^2 \Phi^{NQ}}{\partial \eta_k \partial \xi_l} \\
\frac{\partial^2 \Phi^{NQ}}{\partial \xi_k \partial \eta_l} & \frac{\partial^2 \Phi^{NQ}}{\partial \eta_k \partial \eta_l}
\end{bmatrix}, \quad k,l = 1,2.
\]

All quantities are subject to the stationarity conditions (2.19) and (2.20). In particular, \( R^{srcc}_{NQ} \) is the specular reflection coefficient for the specular source-receiver pair. If we elect to choose the yet unknown coefficient \( A^{NQ} \) to be
proportional to frequency, then $\beta^{NQ}$ is a band-limited delta function that peaks where $\Phi^{NQ}$ vanishes. The output is therefore maximum where the phase-matching condition,

$$\Phi^{NQ}(x, x', x', x') = 0$$

is met. This happens once again on the reflector, and possibly at a few other points. However, it is unlikely that there would exist a point $x'$ not on the reflector, that would fulfill both stationarity conditions (2.19) and (2.20) and phase-matching condition (2.22). This is why the following discussion ignores such points and focuses exclusively on stationary points belonging to the reflector $S_\eta$.

We follow Bleistein (1987) in computing $|H_{NQ}|$. Since neither $x$ nor $x'$ depend on $\xi$, we can set the stationarity condition $x = x'$ before differentiating the phase with respect to $\xi$. Therefore all orders of partial derivatives of $\Phi^{NQ}$ with respect to $\xi$ vanish at stationary points, and in particular the determinant of the Hessian simplifies to

$$|H_{NQ}| = \left[ \left| \frac{\partial^2 \Phi^{NQ}}{\partial \xi_i \partial \eta_k} \right| \right]^2 = \left[ \left| \frac{\partial P^{NQ}}{\partial \xi_i} \frac{\partial x}{\partial \eta_k} \right| \right]^2$$

where

$$P^{NQ}(x, x', x') = p^N(x, x') + p^Q(x, x')$$

Next we notice that
\[ \sqrt{H_{NQ}} = \left| \frac{\partial P^{NQ}}{\partial \xi_1} \frac{\partial x}{\partial \eta_k} \right| = \left| \begin{bmatrix} \frac{\partial P^{NQ}}{\partial \xi_1} \\ \frac{\partial P^{NQ}}{\partial \xi_2} \end{bmatrix} \right| \cdot \left| \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial x}{\partial \eta_2} \end{bmatrix} \right| \]

\[ = \left| \begin{bmatrix} \nu \\ \frac{\partial P^{NQ}}{\partial \xi_1} \frac{\partial x}{\partial \eta_1} \frac{\partial x}{\partial \eta_2} \end{bmatrix} \right| = \nu \left| \begin{bmatrix} \frac{\partial P^{NQ}}{\partial \xi_1} \\ \frac{\partial P^{NQ}}{\partial \xi_2} \end{bmatrix} \right| \nu \left| \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial x}{\partial \eta_2} \end{bmatrix} \right| , \quad (2.25) \]

where \( \nu \) is a vector satisfying to the following conditions:

\[ \nu \nu = 1 \]
\[ \nu \frac{\partial x}{\partial \eta_1} = 0 \]
\[ \nu \frac{\partial x}{\partial \eta_2} = 0 \]

that is, \( \nu \) is the unit normal to the scatterer \( S_\eta \). Although that vector is unknown in general, we can build it by simply normalizing \( P^{NQ} \), which is, according to (2.19), perpendicular to \( S_\eta \) at stationary points:

\[ \nu = \frac{P^{NQ}}{P^{NQ}} \quad \text{at stationary points} \]

Substituting this expression into (2.24), and noting that

\[ \left| \begin{bmatrix} \nu, \frac{\partial x}{\partial \eta_1}, \frac{\partial x}{\partial \eta_2} \end{bmatrix} \right| = \nu \left| \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \times \frac{\partial x}{\partial \eta_2} \end{bmatrix} \right| = \sqrt{|g_\eta|} , \]

we obtain:
\[ \sqrt{|H_{NQ}|} = \frac{1}{P^{NQ}} \left| B_{NQ} \right| \left| g_n \right|, \]

where \( |B_{NQ}| \) is the determinant of Beylkin (1985) and Bleistein (1987):

\[
|B_{NQ}| = \begin{vmatrix} P^{NQ} & p^{N+p^{Q}} \\ \partial P^{NQ} & \partial p^{N+p^{Q}} \\ \partial \xi_1 & \partial \xi_1 \\ \partial P^{NQ} & \partial p^{N+p^{Q}} \\ \partial \xi_2 & \partial \xi_2 \end{vmatrix} = \begin{vmatrix} p^{N+p^{Q}} \\ \partial p^{N+p^{Q}} \\ \partial \xi_1 \\ \partial p^{N+p^{Q}} \\ \partial \xi_2 \end{vmatrix}. \tag{2.26}
\]

The signature of the Hessian is twice that of the 2×2 submatrix of mixed second partial derivatives, which can itself only take the values 0, ±2. Consequently we have

\[
e^{-\frac{i}{4} \text{sgn}(\omega) \text{sign} H_{QN}} = \pm 1 \quad \text{at stationary points},
\]

Bleistein (1987) shows however that the signature is zero for \( z' \) in the vicinity of the scatterer, hence,

\[
\beta^{NQ}(z') \sim (2\pi)^2 \int d\omega \ i \omega S(\omega) \left\{ \frac{u^N(z, z') R^{spec.}_{NQ}(z, z') I^{NQ}_{m}(z, z', z^1)}{\omega^2} \frac{I^{NQ}_{m}(z', z', z^2)}{\left| B_{NQ}(z, z', z', z^2) \right|} \right\} e^{i \omega \Phi^{NQ}(z, z', z', z^2)}. \]

At this point, we are able to determine the unknown amplitude \( A^{NQ} \) and fulfill the objectives of inversion; if we define \( A^{NQ} \) as follows,
\[ A^{NQ}(z', z', z^s, \omega) = \frac{-i\omega |B_{NQ}(z', z', z', z^s)|}{(2\pi)^3 P^{NQ}(z', z', z^s) \left(I^{NQ}(z', z', z^s)\right)^2 u^N(z', z^s)}, \]

then \( \beta^{NQ} \) reduces to

\[ \beta^{NQ}(z') \sim \frac{1}{2\pi} \int d\omega S(\omega) R^{spec}_{NQ}(z, z^s) e^{i\omega \Phi^{NQ}(z, z', z', z^s)}. \]

That is, the inversion output is a bandlimited delta function that peaks on the reflector \((z' = z)\) and whose peak value is proportional to the specular reflection coefficient

\[ \left[ \beta^{NQ} \right]^{peak} = R^{spec}_{NQ} \frac{1}{2\pi} \int S(\omega) d\omega. \]

Therefore the reflectivity of the scatterer can be retrieved insofar as the spectrum of the source is known. Finally substituting for \( A^{NQ} \) into (2.13), we obtain the desired inversion formula:

\[
\beta^{NQ}(z') = \frac{1}{(2\pi)^3} \int d\omega i\omega \int d^2 \xi \left\{ u_m(z', z^s, \omega) I^N_m(z', z', z^s) \right. \\
\left. \frac{|B_{NQ}(z', z', z', z^s)|}{P^{NQ}(z', z', z^s) \left[I^{NQ}(z', z', z^s)\right]^2 u^N(z', z^s)} \right\} e^{-i\omega \left[r_n(z', z^s) + r_q(z', z^s)\right]}.
\]

The various quantities are defined in equations (2.9), (2.24), and (2.26). As it stands, the inversion formula requires to Fourier-transform the input and perform
the receiver integration for every \( \omega \). There is, however, a much simpler way to proceed. Considering that

\[
\frac{1}{2\pi} \int d\omega \, i\omega \, u_m(z',z',\omega) \, e^{-i\omega \left[ \tau_N(z',z') + \tau_Q(z',z') \right]} = \dot{U}_m(z',z',\tau_N + \tau_Q)
\]

where \( \dot{U}_m(z',z',t) \) is the time derivative of the recorded signal at time \( t \), the \( \omega \) integration can be dispensed with if the input is replaced by \( \dot{U}_m \). This requires that the field records be preprocessed with a derivative operator, which is a simple operation. That preprocessed input is then interpolated to the desired traveltime during the inversion. The inversion formula is therefore best rewritten

\[
\beta^{NQ}(z') = \frac{1}{(2\pi)^2} \int_{S_\zeta} d^2 \xi \left\{ \dot{U}_m \left[ x',x',\tau_N(x',x') + \tau_Q(x',x') \right] I_{m}^{NQ}(z',x',z') \right\} \frac{|B_{NQ}(x',z',x',z')|}{P^{NQ}(x',z',z')} \left( I_{NQ}(z',z',z') \right)^2 u^N(z',z')
\]

(2.27)

2.2.4 Parameter estimation

Formula (2.27) allows us to retrieve the fully non-linear reflection coefficient of the scatterer \( S_\eta \) at the specular angle for a particular source-receiver pair. From the value of this coefficient for different modes and at different incidence angles, we can hope to retrieve the parameters of the medium below the scatterer, as well as the orientation of the anisotropy there. This could be achieved by using one of the
numerous data-fitting techniques available, such as non-linear least square inversion. Such methods can take advantage of the redundancy of the data and incorporate constraints on parameter values. The object of this section is not to discuss these methods, but to indicate how to assemble the data necessary for their implementation.

As we pointed out earlier, the reflection coefficient for a given mode \((N,Q)\) depends on the parameters above and below the scatterer (including the orientation of anisotropy), as well as on the direction of the incident slowness and the normal to the reflector. The only real unknowns in the inversion problem are the parameters below the scatterer, while the parameters above are given. We show next how the incident slowness and normal directions can be retrieved as part of the inversion process.

Let us first generalize the definition of the inversion output, where we have included an extra amplitude term:

\[
\beta^{NQ}[x',f] = \frac{1}{(2\pi)^2} \int_{S_{\xi}} d^2\xi \left\{ \tilde{U}_m \left[ z',x',z'_{N}(x',x') + r_Q(x',x') \right] I_{mQ}^{NQ}(x',x',x') \right. \\
\left. \frac{f(x',x',x')}{{P}_{NQ}(x',x',x')} \left[ I_{NQ}^{NQ}(x',x',x') \right]^2 u^N(x',x') \right\}, \tag{2.28}
\]

where \(f(x',x',x')\) is an arbitrary known smooth function. Comparing this
expression to (2.27), it appears that

\[ \beta^{NQ}(z') = \beta^{NQ}[z',1] \]

Considering that one can carry exactly the same asymptotic analysis on (2.28) as we did on (2.27), one sees that \( \beta[z',f] \) is a bandlimited delta function that peaks on the scatterer. Moreover, the specular value of \( f \) can be computed at any output point by simply taking the ratio of two outputs, that is

\[ \left[ f(z') \right]^{\text{spec.}} = \frac{\beta[z',f]}{\beta[z',1]} \]

except at points where the reflectivity vanishes.

Equation (2.29) can be used in extracting the incidence and normal directions associated with a particular reflection coefficient. For example, suppose that we want to evaluate the components of the incident slowness \( p^N \) in an orthogonal Cartesian system \( (x_j) \), then we simply choose

\[ f(z',x'^i) = p^N_j(z',x') \], \( j = 1,2,3 \)

On the other hand, by choosing:

\[ f(z',x'^i,z'^j) = \frac{p^N_j(z',x') + p^Q_j(z',x')}{|p^N_j(z',x') + p^Q_j(z',x')|} \], \( j = 1,2,3 \)

we obtain the three Cartesian components of the unit vector normal to the scatterer. Of course, the values of \( f \) are only needed on the reflector, that is, where
\( \beta^Q[z', f] \) peaks.

Now it is clearer how parameter estimation may be organized. Equation (2.29) allows us to evaluate the normal to the reflector, as well as the incident slowness vector at any illuminated point on the scatterer. The parameters above the scatterer are, of course, already known. We then solve for parameters below. For example, guessing a set of parameters below the reflector, we compute a trial reflection coefficient. We also estimate the true reflection coefficient and the corresponding incidence angle from the data, insofar as we know the area under the source spectrum. By comparing the measured and trial values at several points and for various modes, we can iteratively improve our guess of the parameters below the scatterer. In practice, the method is only limited by the presence of noise, or an insufficient coverage of the scatterer in terms of range of incidence angles and modes. Some sophisticated techniques, such as singular value decomposition (Lanczos, 1961), may be used to evaluate the performance of the parameter estimation when such complications are expected.

2-3. Computational Overview

2-3.1 Introduction

Equation (2.28) allows one to image a reflector and extract, from the data, the quantities needed for estimating the parameters below that reflector. The
computation of the various outputs must be made simultaneously, since the quasi-totality of the calculations are common to all outputs. There are, however, many possible variants in implementing (2.28) in an optimum way. The notion of optimality is itself dependent on whether priority is put on memory economy, or computational economy, which are the two areas where processing speed can be gained. In what follows, we restrict ourselves to the processing of a single common shot data set, which consists of the signal recorded at many receivers for a single source location. Consequently, \( x' \) is a constant in the processing. The mode \((N,Q)\) is also fixed in the processing.

The common shot configuration has several advantages. First, this is the way the data is recorded in the field, and therefore data sorting can be reduced to a minimum; second, the data corresponds to a single physical experiment, so that there are no difficulties associated with source variability (which is often high in land seismic); third, the inversion process is simplified by having a fixed point source and the \( \zeta \) integration is helped by the existence of a dense and uniform set of receiver points.

We propose below three algorithms to perform the inversion, based on different optimization criteria. We then point out some additional computational simplifications.
2.3.2 Algorithm 1: Minimizing Computation.

To analyze algorithm 1, we break up the computation of $\beta^{NQ}$ into four parts, as illustrated below:

$$\beta^{NQ}(z', f) = \int_{S_t} d^2 \xi \, b^{NQ}(z', z^r, z^s, f) \left| B_{NQ}(z', z^r) \right|$$  \hspace{1cm} (2.30)

where

$$b^{NQ}(z', z^r, z^s, f) = c_m^{NQ}(z', z^r, z^s, f) \dot{U}_m(z^r, z^s, \tau_N + \tau_Q)$$ \hspace{1cm} (2.31)

$$\left| B_{NQ}(z', z^r) \right| = \left| \begin{array}{c} p^Q(z', z^r) \\ \frac{\partial p^Q(z', z^r)}{\partial \xi_1} \\ \frac{\partial p^Q(z', z^r)}{\partial \xi_2} \end{array} \right|$$ \hspace{1cm} (2.32)

and

$$c_m^{NQ}(z', z^r, z^s, f) =$$

$$\frac{I_m^{NQ}(z', z^r, z^s) f(z', z^r, z^s)}{2\pi P^{NQ}(z', z^r, z^s) \left[ I^{NQ}(z', z^r, z^s) \right]^2 u^N(z', z^s)}$$ \hspace{1cm} (2.33)

Step (2.30) is an integration over all receiver locations ($z^r = z^r(\xi)$). The computation of $c_m^{NQ}$ involves dynamic ray tracing down from source and receiver to the output point; The determinant $\left| B_{NQ} \right|$ can be evaluated in many ways, and requires the knowledge of how the slowness varies at the output point when the receiver location is perturbed. Evaluating that determinant makes the algorithm significantly more complex than that of a regular Kirchhoff migration.
In our first algorithm, $|B_{NQ}|$ is computed by using the kinematic information obtained when shooting rays from all the receivers. By differentiating the slowness $p^Q$ at an output point using neighboring receivers, a good estimate of $|B_{NQ}|$ is at hand with practically no extra computation. The algorithm is therefore divided into two main phases: in the first one, dynamic rays are traced from source and all receivers to all output points and $p^Q$ is stored. Considering that this requires the storage on disk of at least four 5-dimensional scalar data sets for a 3-D, 3-C survey, we can hardly recommend this algorithm but for the largest computer systems currently available. In the second phase, the stored values of $p^Q$ are recalled for all receivers at each output point, and $B_{NQ}$ is evaluated using the method described above. It is during this phase that the receiver integration is performed.

Although it might seem too complicated to be efficient, algorithm 1 has a definite advantage related to the way ray tracing algorithms work. Since in the first phase, rays must be traced from source and receivers to all output points, we can make use of the following trick: instead of tracing entire rays, we first shoot a dense bundle of rays that ends at the lowermost interface in the background medium, and store the initial values for tracing beyond that interface, into the output region. Each of the rays in the bundle can then be continued down to many points into the output region at little extra computational cost. The ray attributes can be accurately interpolated among these points whenever needed during the inversion process. Appendix D shows a sketch of algorithm 1 that emphasizes the
various points described earlier.

2-3.3 Algorithm 2: Mixed Priorities

In this algorithm, we decrease the memory requirements of algorithm 1 by computing $|B_{NQ}|$ at a some extra cost, using a kinematic version of the reciprocity relation. For each receiver point $x'$, we know the slowness $p^Q(x', x')$ at any output point $x'$ directly from the ray tracing. Reciprocally, if we shoot a ray from $x'$ with slowness $-p^Q(x', x')$, this ray emerges on the acquisition surface at $x'$.

The idea is then to perturb $p^Q$ by $\Delta p^Q(1)$, change the sign of the resulting vector, and shoot the corresponding ray up to the acquisition surface, where it emerges at a location $(\xi_1 + \Delta \xi_1^{(1)}, \xi_2 + \Delta \xi_2^{(1)})$. We then repeat this operation with an independent perturbation $\Delta p^Q(2)$ and measure the corresponding $(\Delta \xi_1^{(2)}, \Delta \xi_2^{(2)})$ to form the non-singular system:

$$\Delta p^Q(1) \sim \frac{\partial p^Q}{\partial \xi_1} \Delta \xi_1^{(1)} + \frac{\partial p^Q}{\partial \xi_2} \Delta \xi_2^{(1)},$$

and

$$\Delta p^Q(2) \sim \frac{\partial p^Q}{\partial \xi_1} \Delta \xi_1^{(2)} + \frac{\partial p^Q}{\partial \xi_2} \Delta \xi_2^{(2)},$$

which we invert to get the desired quantities:
\[
\frac{\partial p^Q}{\partial \xi_1} \sim \frac{\Delta p^Q(1) \Delta \xi_2^{(2)} - \Delta p^Q(2) \Delta \xi_2^{(1)}}{\Delta \xi_1^{(1)} \Delta \xi_2^{(2)} - \Delta \xi_1^{(2)} \Delta \xi_2^{(1)}}
\]

and

\[
\frac{\partial p^Q}{\partial \xi_2} \sim \frac{\Delta p^Q(2) \Delta \xi_1^{(1)} - \Delta p^Q(1) \Delta \xi_1^{(2)}}{\Delta \xi_1^{(1)} \Delta \xi_2^{(2)} - \Delta \xi_1^{(2)} \Delta \xi_2^{(1)}}
\]

There remains the question of how to perturb \( p^Q \) so that \( B_{NQ} \) is not artificially made zero. For this, one can take the perturbations along two independent directions perpendicular to \( P^{NQ} \). That way, one insures that \( B_{NQ} \) is only singular when the output point is on a caustic.

The advantage of this technique is that \( |B_{NQ}| \) is computed within phase 1 of the algorithm described earlier, which makes it possible to increment the receiver stacks \( \beta^{NQ}(x',f) \) as soon as the attributes for a particular receiver are computed. The memory savings are significant, since the storage of \( b_{NQ} \) is no longer required. Pretracing can still be used, although it now becomes the most memory intensive operation in the algorithm, with the storage of about 25 3-dimensional scalar sets for a 3-D, 3-C survey. If retained, the added computational cost in algorithm 2 compared to algorithm 1 is reasonable, because shooting just two kinematic rays is sufficient to compute \( |B_{NQ}| \).

A sketch of this algorithm is presented in Appendix E.
2.3.4 Algorithm 3: Minimizing Storage

In this version, we use the reciprocity relation derived in Chapter I, that is, we trace rays from output points to the set of receivers. The resulting algorithm is much more compact than the previous ones, and has the advantage that inversion outputs are generated one by one as the algorithm runs, making the memory requirements minimum (i.e. the partial receiver integrations need not be stored). The determinant $|B_{NQ}|$ is computed as a by-product of dynamic ray tracing: we use the initial slowness and emergence point of the rays forming the ray tube to compute the quantities $\partial p^Q/\partial \xi_1$ and $\partial p^Q/\partial \xi_2$. Pretracing can only be retained for the source point, and this results in a significant increase in computational intensity. A rule of thumb is that the amount of dynamic ray tracing involved in this algorithm is that required in algorithm 1 multiplied by the cubic root of the number of output points for a 3-D inversion, and by the square root of the number of output points for a 2-D inversion. However, that algorithm permits to focus the inversion on a reduced target area, because the output is computed pointwise. Although it is the only viable alternative on systems where little disk memory is available, algorithm 3 is also particularly well suited to parallel processing. A sketch for computer implementation is given in appendix F.
2-3.4 Further Simplifications

For each output point, dynamic ray tracing must be performed from both source and receiver locations to the output point. In particular, all components of the Green’s tensor must be known. Fortunately, this does not mean that rays must be traced for all three directions of the point force at \( \mathbf{x}' \); in effect, the asymptotic Green’s tensor can be written as the dyadic

\[
G_m^Q(\mathbf{z}', \mathbf{x}') = a_m^Q(\mathbf{x}') g_m^Q(\mathbf{z}', \mathbf{x}') \ ,
\]

(2.34)

where \( a_m^Q \) is a radiation vector that represents the amplitude radiated into mode \( Q \) by a force in the direction \( m \). That vector can easily be evaluated from the parameters of the medium around the receiver point and the take off angle for the ray. Consequently, the computation of the Green’s tensor is practically reduced to that of \( g_m^Q \), and is therefore as fast as that of the incident field.

Another simplification concerns the interaction vector \( I_m^{NQ} \) because the value of the amplitude terms in (2.27) are only relevant at stationary points, where the scattered slowness and polarization are known. Let us substitute (2.34) into (2.9) to obtain the following expression for the \( N \) to \( Q \) interaction vector:

\[
I_m^{NQ}(\mathbf{z}', \mathbf{z}'', \mathbf{z}^s) = a_m^Q(\mathbf{z}', \mathbf{z}'') c_{ijkl} \epsilon_j \left[ g_k^Q(\mathbf{z}', \mathbf{z}'') v_l^{NQ}(\mathbf{z}', \mathbf{z}^s) p_i^{NQ}(\mathbf{z}', \mathbf{z}^s) - v_l^{NQ}(\mathbf{z}', \mathbf{z}^s) g_k^Q(\mathbf{z}', \mathbf{z}'') p_i^{NQ}(\mathbf{z}', \mathbf{z}^s) \right] .
\]

(2.35)

The scattered slowness \( p^{NQ} \) must be computed from the incident one \( p^N \) according
to Snell's law, that is:

\[ p^{NQ}(x', x^s) \frac{\partial x'}{\partial \eta_j} = p^N(x', x^s) \frac{\partial x'}{\partial \eta_j} \quad j = 1, 2. \]

On the other hand, stationarity condition (2.19) states:

\[ p^N(x', x^s) \frac{\partial x'}{\partial \eta_j} = -p^Q(x', x^r) \frac{\partial x'}{\partial \eta_j} \quad j = 1, 2. \]

Consequently, \( p^{NQ} \) and \( p^Q \) have opposite tangential components at specular points. Moreover, they must satisfy the same eikonal equation (mode \( Q \)); it is easy to see that if the two vector are simply opposite, they do satisfy the same eikonal equation and therefore we have:

\[ p^{NQ}(x', x^s) = -p^Q(x', x^r) \quad \text{at stationary points} \quad (2.36) \]

From the eigenvector equations defining displacement polarizations, it is easy to show that (2.36) in turn implies:

\[ v^{NQ}(x', x^s) = -v^Q(x', x^r) = \frac{-g^Q(x', x^r)}{g^Q(x', x^r)} \quad (2.37) \]

Finally, the unit vector normal to the scatterer \( \nu \) can be expressed at specular points as the sum of the slownesses from source and receiver to the output point:

\[ \nu = \frac{p^N(x', x^s) + p^Q(x', x^r)}{|p^N(x', x^s) + p^Q(x', x^r)|} \quad (2.38) \]

Substituting (2.36) through (2.38) into (2.29), the interaction coefficient finally reduces to:
\[ I_m^{NQ}(x', x'', x^*) = 2 \epsilon_{ijkl} a_m^Q(x', x'') g_k^Q(x', x'') p_l^Q(x', x') \]
\[ \frac{g_i^Q(x', x'')}{g^Q(x', x'')} \left( \frac{p_f^N(x', x^*) + p_f^Q(x', x'')}{|p^N(x', x^*) + p^Q(x', x'')|} \right). \] (2.39)

Alternatively, we can express the interaction coefficient in terms of tractions:

\[ I_m^{NQ}(x', x'', x^*) = a_m^Q(x', x'') \frac{t_i^Q(x', x'') g_i^Q(x', x'')}{g^Q(x', x')} \]
3. 2.5D SHOT PROFILE INVERSION

3-1. Inversion of Dip Lines over Cylindrical Models

3-1.1 Introduction

Seismic data is in the majority of cases recorded along lines, that is, a source mechanism (vibrator, explosive, air gun, ...) is activated and the seismic field recorded along a line of receivers, usually aligned with the source. We showed earlier that to image a piece of reflector, one needs to integrate observations collected over a two-dimensional acquisition surface. Therefore it is not surprising if the data gathered by a single receiver line can only reconstruct a curvilinear segment of reflector. However, seismic methods have long been successful in imaging the subsurface from line data. This is probably because sediments often exhibit cylindrical structures, which are completely characterized by their cross-section. Prior knowledge of the structural trend of an area usually allows to position seismic surveys perpendicularly to the structural direction (strike). Processing the data yields a cross-sectional image of the cylindrical reflectors that can be extrapolated by invariance in the strike direction, at least over some distance.
Line recording has obvious economical advantages, on land as well as at sea. On land, a real 3-D survey would for instance involve setting a receiver line, and moving the source perpendicularly to the line, rather than a simple trailing of the receiver line behind the source on a preset and surveyed path. At sea, recording from a large number of parallel streamers would be ideal, but is practically extremely difficult. This is why the cylindrical model is always implicitly used in processing conventional seismic surveys to make for the lack of subsurface coverage inherent to line recording. Moreover, processing under this model is far less costly than full 3-D processing, both computationally and in terms of memory requirements. In inverting for a line of data in the strike direction, it is usually possible to fully account for three-dimensional propagation in the treatment of amplitudes. That treatment is the only one that is truly consistent with the cylindrical model, and is referred to as 2.5-D inversion (Bleistein 1987, Docherty 1987). The images obtained are cross sections of the reflectors in the vertical plane containing the recording line. This, of course, fails to be true if the subsurface geology is not cylindrical or if the line is not perpendicular to the strike.

3-1.2 Inversion Formula

We now assume that the cylindrical model is valid and that a seismic line is recorded perpendicular to the strike of an anisotropic structure (Figure (3.1)).
Figure (3.1): A typical cylindrical model: the geometry is invariant in the strike direction.
We show that in contrast to the isotropic case, processing with the cylindrical assumption does not in general reduce to a two-dimensional problem, that is, the output (image) is not located in the vertical plane of the data line. This stems from the peculiar properties of energy propagation in anisotropic media.

We choose the $z_2$ direction to be along the strike, and parametrize the recording line with the variable $\xi_1$, that is:

$$z^s = (x_1^s, 0, x_3^s),$$

$$z^r = (x_1^r(\xi_1), 0, x_3^r(\xi_1)),$$

where $x_1$ is the horizontal coordinate along the receiver line and the $z_2$-direction is along strike. We seek an inversion formula in the same form as (2.13), except that the integration over the acquisition surface becomes a simple integral along the receiver line:

$$\beta_2^{NQ}(z') = \int d\omega \int d\xi_1 \left\{ u_m(z^r, z^s, \omega) I_m^{NQ}(z', z^r, z^s) \right. \right.$$  

$$A_2^{NQ}(z', z^r, z^s, \omega) \left. e^{-i\omega \left[ r_N(z', z^s) + r_2(z', z^s) \right]} \right\}, \quad (3.1)$$

where $A_2^{NQ}$ is the 2.5-D amplitude coefficient that must make the peak value of $\beta_2^{NQ}$ equal to the specular reflection coefficient times the area under the source spectrum. Proceeding as in (2.15), we substitute the matching mode of the Kirchhoff representation into (3.1) to get:
\[ \beta_2^{NQ}(z') = \int d\omega i \omega S(\omega) \int d\xi_1 \int d\eta_1 d\xi_2 \left\{ \sqrt{|g_\eta|} u^N(z,z^s) R_{NQ}(z,z^s) \right\} \]

\[ \begin{aligned} &I_{m}^{NQ}(z,z',z^s) \quad I_{m}^{NQ}(z',z^r,z^s) \quad A_{NQ}^{NQ}(z',z^r,z^s,\omega) \quad e^{i\omega \Phi_{NQ}(z,z',z^s,z^r)}, \end{aligned} \]

where

\[ \Phi_{NQ}(z,z',z^r,z^s) = \tau^N(z,z^s) + \tau^N(z,z^r) - \tau^Q(z^s,z^s) - \tau^Q(z^r,z^r), \]

\[ z = (x_1(\eta_1), x_2, x_3(\eta_1)), \]

\[ \sqrt{|g_\eta|} = \left| \frac{\partial z}{\partial \eta_1} \right|. \]

Note that the cylindrical reflector is parametrized by \( \eta_1 \) and the strike variable \( z_2 \).

As before, we evaluate (3.2) by the method of stationary phase; the stationarity conditions are the same as (2.19) and (2.20), and can be specialized to our particular case to yield:

\[ p_2^N(z,z^s) + p_2^Q(z,z^r) = 0, \]

(3.3)

\[ \left[ p^N(z,z^s) + p^Q(z,z^r) \right] \frac{\partial z}{\partial \eta_1} = 0, \]

(3.4)

and

\[ \left[ p^N(z,z^r) - p^N(z^s,z^r) \right] \frac{\partial z^r}{\partial \xi_1} = 0. \]

(3.5)

Once again, the stationary points are specular points on the reflector and (3.2)
asymptotically reduces to:

\[ \beta_2^{NQ}(x') \sim (2\pi)^{3/2} \int d\omega \, i\omega \, S(\omega) \left\{ \frac{\partial x}{\partial \eta_1} \right\} \left\{ \frac{u^N(x, x') R^{NQ}_{NQ}(x, x') I_m^{NQ}(x, x', x')}{|\omega|^{3/2}} \right\} \]

\[ \frac{I_m^{NQ}(x', x', x') A_2^{NQ}(x', x', x', \omega)}{\sqrt{|H_{NQ}(x, x', x', x')|}} \right\}

\[ e^{i\omega \psi(x, x', x', x') + i\frac{x}{4} \text{sgn}(\omega) \text{sgn} H_{NQ}} \]

where \( H_{NQ} \) is now the \( 3 \times 3 \) Hessian of the phase:

\[ H_{NQ}(x, x', x', x') = \begin{bmatrix}
\frac{\partial^2 \Phi^{NQ}}{\partial x_2^2} & \frac{\partial^2 \Phi^{NQ}}{\partial x_2 \partial x_1} & \frac{\partial^2 \Phi^{NQ}}{\partial x_2 \partial \eta_1} \\
\frac{\partial^2 \Phi^{NQ}}{\partial x_2 \partial x_1} & \frac{\partial^2 \Phi^{NQ}}{\partial x_1^2} & \frac{\partial^2 \Phi^{NQ}}{\partial x_1 \partial \eta_1} \\
\frac{\partial^2 \Phi^{NQ}}{\partial x_2 \partial \eta_1} & \frac{\partial^2 \Phi^{NQ}}{\partial x_1 \partial \eta_1} & \frac{\partial^2 \Phi^{NQ}}{\partial \eta_1^2}
\end{bmatrix}. \]

In this form, \( H_{NQ} \) is not easily computable; however, using the stationarity conditions, and after some algebra, we can rewrite its determinant as follows,

\[ |H_{NQ}(x, x', x', x')| = \left| \frac{\partial x}{\partial \eta_1} \right|^2 |D_{NQ}(x, x', x', x')|, \]

where
\[ |D_{NQ}(x,x',x',x^s)| = \begin{vmatrix} 0 & \frac{\partial P_{NQ}}{\partial t_1} & \frac{\partial P_{NQ}}{\partial t_2} \\ \frac{\partial P_{NQ}}{\partial t_1} & \left(\nabla_{x'} P_{NQ} \cdot t_1\right) & \left(\nabla_{x'} P_{NQ} \cdot t_2\right) \\ \frac{\partial P_{NQ}}{\partial t_2} & \left(\nabla_{x'} P_{NQ} \cdot t_1\right) & \left(\nabla_{x'} P_{NQ} \cdot t_2\right) \end{vmatrix}, \quad (3.6) \]

\[ P_{NQ}(x,x',x^s) = p^N(x,x^s) + p^Q(x,x') \]

\[ t_1 = \frac{(P^1_{NQ}, 0, -P^1_{NQ})}{|P_{NQ}|} \]

and

\[ t_2 = (0, 1, 0) \]

On the other hand, the signature of the Hessian can only assume the following values:

\[ \text{sig} H_{NQ} = \pm 1, \pm 3 \]

Consequently, within a constant, we can determine \( A^2_{NQ} \) in the same way as before:

\[ A^2_{NQ}(x',x',x^s) = \frac{\sqrt{i\omega} \sqrt{|D_{NQ}(x',x',x',x^s)|}}{(2\pi)^{5/2} \left[I^{NQ}(x',x',x^s)\right]^2 u^N(x',x^s)} \]

so that the output becomes:

\[ \beta^2_{NQ}(x') = \frac{1}{2\pi} \int d\omega S(\omega) R^\text{spec}_{NQ}(x,x^s) e^{i\omega \Phi_{NQ}(x,x',x',x^s)} \]

and its peak value, reached at \( x' = x \), is:
\[
\left[ \beta_2^{NQ}(z') \right]_{\text{peak}} = R_{NQ}^{\text{spec}} \frac{\int d\omega S(\omega)}{2\pi}
\]

Substituting for \( A_2^{NQ} \) into (3.1), we finally obtain the 2.5-D common-shot inversion formula:

\[
\beta_2^{NQ}(z') = \frac{1}{(2\pi)^{5/2}} \int d\omega \sqrt{i\omega} \int d\xi \left\{ u_m(z',z',\omega) I_m^{NQ}(z',z',z') \right. \\
\left. \frac{\sqrt{|D_{NQ}(z',z',z',z')|}}{|I_m^{NQ}(z',z',z')|^2} u_N(z',z') \right\} e^{-i\omega [r_N(z',z') + r_Q(z',z')]}.
\]

3-1.3 Discussion

As in the 3-D case, it is still possible to rewrite the inversion operator in the time domain. However, this requires a preprocessing of the data that is not as simple as a time derivative anymore. Instead, one has to take a half derivative of the signal, that is, multiply the data by \( \sqrt{i\omega} \) in the frequency domain. This operation is a separate processing step to be applied to the data before the inversion itself. The expression for the preprocessed input \( ^\wedge U_m(z',z',t) \) is as follows

\[
^\wedge U_m(z',z',t) = \frac{1}{2\pi} \int d\omega \sqrt{i\omega} e^{-i\omega t} \int dt' U_m(z',z',t') e^{i\omega t'} ,
\]

where \( U_m \) is, once again, the raw data. With this definition, the time domain
The inversion operator is

$$\beta_2^{NQ}(z') = \frac{1}{(2\pi)^{3/2}} \int d\xi_1 \left\{ \hat{U}_m \left[ x', z', r_N(x', z') + r_Q(x', x') \right] I_m^{NQ}(z', x', z') \right\} \frac{\sqrt{|D_{NQ}(z, z', z', z')|}}{\left[ I^{NQ}(z', x', z') \right]^2 u^N(z', z')} \right\}.$$

This expression does not significantly differ from the full 3-D formula, and indeed except for the absence of the $\xi_2$ integral, this inversion usually requires three-dimensional ray tracing. This is because the stationary points for a common shot experiment perpendicular to the strike are usually not located in the dip plane (i.e., the vertical plane containing the receiver line). This is well illustrated by the example raypath perspective plot of Figure (1.17). As a consequence, the output $\beta_2^{NQ}$ must be computed on a complete 3-D mesh. The determinant $|D_{NQ}|$ now involves derivatives of the slowness vector $P^{NQ}$ both with respect to the receiver location and the output point.

This complication occurs whenever the symmetry of the anisotropy does not match that of the cylindrical model. In those cases, the cost of inversion is in many respects comparable to that of a 3-D inversion; the 3-D segment of reflector imaged in this manner can be projected on the $z_2 \equiv 0$ plane to get the desired cross-sectional image of the reflector, and parameter estimation can be carried out as
previously described.

However, for isotropic media or anisotropic media that possess the cylindrical symmetry of the model, energy propagation remains within the dip plane and inversion reduces to a much simpler two-dimensional process. This is described in Docherty 1987 for the acoustic case, and Sumner 1988 for the isotropic elastic case. Next we extend their approach to the case of transversely isotropic media with a symmetry axis in the dip plane.

3-2. 2.5-D fractured media

3-2.1 Introduction

The two-and-one-half dimensional fractured model is designed to represent cylindrical media that are fractured along the strike direction. Considering that the regional stresses are usually aligned along the axis of existing geological structures, this model is probably representative of many real situations. On the other hand, this model can be rightfully used whenever a single line of multi-component data is acquired and that a polarization analysis of the shear waves reveals that the recording line is along a principal axis of anisotropy. For our purpose, the 2.5-D fractured medium is a superposition of cylindrical transversely isotropic layers with their anisotropy axis within the dip plane. This means in practice that the various layers may be fractured in different directions, insofar as the fracture planes always
contain the strike axis.

In that case, the propagation of energy remains in the dip plane, although ray direction (i.e. that of group velocity) and slowness direction (i.e. that of phase velocity) remain distinct. This is easily verified by studying the expression of group velocity for each mode given in equations (1.43); all three expressions are in the form:

\[ W = w_a p_a + w_r p_r \]

where \( p_a \) and \( p_r \) are the axial and radial vector components of the slowness \( p \), that is:

\[ p = p_a + p_r \quad p_r \cdot a = 0 \]

where \( a \) is the anisotropy axis which, according to our assumption and the strike axis being \( x_2 \), may be written as:

\[ a = (a_1, 0, a_3) \]

Alternatively, group velocity can be decomposed as:

\[ W = W_{1,3} + W_2 \]

where \( W_{1,3} \) and \( W_2 \) are respectively the in-plane and out-of-plane vector components of group velocity:

\[ W_2 = w_r p_2 x_2 \quad W_{1,3} = W - W_2 \quad (3.7) \]

Now consider shooting a ray from the source with an in-plane slowness (i.e. \( p_2 = 0 \));
according to Snell's law and the cylindrical nature of the model, the slowness must remain in-plane because $p_2$ is continuous across reflectors, that is:

$$p_2 = 0 \quad \text{along the ray}$$

Therefore, according to (3.7), we have:

$$W_2 = 0 \quad \text{along the ray}$$

which means that rays that start in-plane remain in-plane.

3-2.2 Forward Problem

As we just showed, the kinematics of a shot-profile across the strike are confined to the dip plane. On the other hand, a proper handling of amplitudes in the forward problem requires to evaluate scattering at interfaces as well as geometric spreading. Interface scattering may be treated as described in Chapter I, but there is an additional simplification owing to the complete decoupling of the $SP$ mode on one hand, and the $QP$ and $QS$ modes on the other hand, in the case of in-plane propagation. To be more specific, there is no conversion of energy between $SP$ and $QP/QS$ modes at interfaces; the $SP$ displacement is polarized exclusively in the strike direction, and in that sense is an $SH$ mode, while $QP$ and $QS$ waves are polarized in the dip plane. In that situation, scattering breaks into two simpler parts: scalar scattering for the $SP$ wave, and 2-D in-plane vector scattering for $QP/QS$ waves.
There remains to compute the geometric spreading along rays. Here again, the problem can be decomposed into in-plane and out-of-plane spreading; the in-plane part may be computed as usual from neighboring in-plane rays and, fortunately, the out-of-plane part can be explicitly expressed from in-plane kinematic parameters, as we show below.

Let us recall the expression of the ray Jacobian:

\[
J(\tau, \gamma_1, \gamma_2) = W \left[ \frac{\partial x}{\partial \gamma_1} \times \frac{\partial x}{\partial \gamma_2} \right].
\]

For convenience, we take \( \gamma_2 = p_2 \) as the out-of-plane ray parameter, so that \( \partial x/\partial \gamma_1 \) and \( \partial x/\partial p_2 \) are orthogonal, and \( J \) can be rewritten as:

\[
J(\tau, \gamma_1, p_2) = J(\tau, \gamma_1, 0) \left| \frac{\partial x_2}{\partial p_2} \right|,
\]

where \( J(\tau, \gamma_1, 0) \) is the 2-D in-plane Jacobian:

\[
J(\tau, \gamma_1, 0) = \left| W(\tau, \gamma_1, 0) \times \frac{\partial x}{\partial \gamma_1} \right|,
\]

which is computed from in-plane kinematic parameters only. On the other hand, the out-of-plane coordinate of a ray after passing through the \( i^{th} \) layer is:

\[
(x_2)_{i+1} = (x_2)_i + (W_2)_{i+1} (\tau_{i+1} - \tau_i),
\]

where \((\tau_{i+1} - \tau_i)\) is the traveltime in the \( i^{th} \) layer and \((x_2)_i\) the out-of-plane coordinate of the ray upon entering that layer.
Now we introduce the out-of-plane component of slowness for the ray, \( p_2 \), which according to Snell's law is identical in all layers, and rewrite (3.8) as follows:

\[
(x_2)_{i+1} = (x_2)_i + (w_r)_{i+1} p_2 (\tau_{i+1} - \tau_i) \quad i = 0, ..., n
\]

By adding all these identities together, we obtain:

\[
(x_2)_n - (x_2)_0 = p_2 \sum_i (w_r)_{i+1} (\tau_{i+1} - \tau_i)
\]

Taking the limit of vanishing \( p_2 \), we get a closed form expression for the out-of-plane spreading at the \( n^{th} \) interface:

\[
\left( \frac{\partial x_2}{\partial p_2} \right)_n = \lim_{p_2 \to 0} \frac{(x_2)_n - (x_2)_0}{p_2} = \sum_i (w_r)_{i+1} (\tau_{i+1} - \tau_i)
\]

(3.9)

where \( \tau \) and \( w_r \) are now evaluated in the dip plane, that is, (3.9) provides the means of evaluating a full three-dimensional spreading from in-plane ray parameters only.

3-2.3 Inverse Problem

The inverse problem is also subject to significant simplifications in 2.5-D fractured media. First of all, the stationary points in the inversion formula are all located in the dip plane \( x_2' = 0 \), so that we need only compute the output \( \beta_2^{NQ} \) on a two-dimensional grid. The image so obtained is a cross-section of the cylindrical reflector on which the processing is targeted.
The next simplification occurs in the determinant $|D_{NQ}|$ of (3.6); in effect, the geometry of the propagation and to the common shot configuration allow us to write the following identities:

$$\frac{\partial P^{NQ}}{\partial \xi_1} \cdot t_1 = 0$$
$$\left( \nabla_x P^{NQ} \cdot t_2 \right) \cdot t_2 = 0$$

$$\left( \nabla_x P^{NQ} \cdot t_2 \right) \cdot t_2 = \frac{\partial P^2_{NQ}}{\partial x_2}$$
and
$$\frac{\partial P^{NQ}}{\partial \xi_1} = \frac{\partial p^Q}{\partial \xi_1}$$

so that $|D_{NQ}|$ simply reduces to:

$$|D_{NQ}| = \left| \frac{\partial p^Q}{\partial \xi_1} \cdot t_1 \right|^2 \left| \frac{\partial P^2_{NQ}}{\partial x_2} \right|$$

The first factor only involves in-plane parameters; we show next that the second one can be computed without out-of-plane ray tracing as well. For this, we recall that $P_2$ is just a sum of slownesses at the output point:

$$P_{2}^{NQ}(x', x', x^*) = p_2^N(x', x^*) + p_2^Q(x', x^*)$$

We then rewrite equation (3.9) in the case of downward propagation from source and receiver to a virtual interface within the output region:

$$\frac{\partial p_2^N}{\partial x_2} = \frac{1}{\Sigma (w_i^N)_{i+1} (r_i^N - r_{i+1}^N)}$$

and

$$\frac{\partial p_2^Q}{\partial x_2}$$
\[
\frac{\partial p_Q^2}{\partial x_2} = \frac{1}{\Sigma(w_i^Q)_{i+1}(\tau_i^Q - \tau_i^Q)} ,
\]

These two relations permit us to compute \(\partial P_{2}^{NQ}/\partial x_2\) as a function of known in-plane variables. Consequently, inversion formula (3.6) simplifies into:

\[
\beta_{2}^{NQ}(x') = \frac{1}{(2\pi)^{5/2}} \int d\omega \sqrt{i\omega} \int d\xi_1 \frac{u_m(x', x'; \omega) \int_{m}^{NQ}(x', x'; x')}{\left(\int_{NQ}(x', x'; x')\right)^2 \int_{NQ}(x', x'; x')} \left. \frac{\partial p_Q^2}{\partial \xi_1} \right|_t \frac{\partial p^N_2}{\partial x_2} + \frac{\partial p^Q_2}{\partial x_2} \right|^{1/2} e^{-i\omega [\tau_N(x'; x') + \tau_Q(x'; x')]}. \tag{3.12}
\]

In that expression, all the quantities are computed in-plane, including the out-of-plane spreading coefficients implicitly involved in \(u^N\) and \(I^{NQ}\). In the time domain, the formulation is even simpler:

\[
\beta_{2}^{NQ}(x') = \frac{1}{(2\pi)^{3/2}} \int d\xi_1 \hat{U}_m \left[ x', x'; \tau_N(x', x') + \tau_Q(x', x') \right] \int_{m}^{NQ}(x', x'; x') \left(\int_{NQ}(x', x'; x')\right)^2 \int_{NQ}(x', x'; x') \left. \frac{\partial p_Q^2}{\partial \xi_1} \right|_t \frac{\partial p^N_2}{\partial x_2} + \frac{\partial p^Q_2}{\partial x_2} \right|^{1/2}. \tag{3.13}
\]

The next section presents an implementation of (3.13) for horizontal layers.
3-3 Implementation in Tabular Media

3-3.1 Introduction

The implementation of formula (3.13) in the case of tabular media is straightforward. In effect, the elastic field is invariant by translation in the horizontal direction, and the receiver Green’s tensor need only be computed from a single receiver.

To test our algorithm, we first consider the simplest possible example: a horizontal source is placed perpendicular to the recording line on top of an isotropic medium containing but one reflector. In that case, only $SH$ waves are generated and the shot record shows a single event (Figure (3.2)). The reflector depth is 1000m; the parameters of the top layer are $V_s = 1000m/s$, $\rho = 2g/cm^3$; the parameters of the bottom layer are $V_s = 500m/s$, $\rho = 2g/cm^3$; there are 100 geophones in a split-spread configuration with 50m receiver spacing; the source signature, after autocorrelation, is a 10-30Hz Klauder wavelet.

The inversion is carried out with the velocity of the top layer, and the resulting image is presented in Figure (3.3). Comparing that result to the ray diagram in Figure (3.4), we see that we correctly imaged the segment of reflector that is illuminated by the source-receiver array (offsets between -1250m and 1250m). There are endpoint effects at the edge of the image due to the limited
Figure (3.2): Shot record and amplitude plot for the test case discussed in section III-3.1.
Figure (3.3): Inversion result for the example discussed in section III-3.1. Note that the reflector location and reflectivity are correctly recovered within the illuminated area.
Figure (3.4): Ray diagram for the example of section III-3.1 showing the portion of reflector actually illuminated by the survey.
aperture of the survey and its abrupt truncation at offsets smaller than -2500m or larger than 2500m. However, within the illuminated area, the reflectivity function is recovered within five percent, as can be seen by comparing the amplitude in Figure (3.3) to the theoretical reflection coefficient in Figure (3.5).

Figure (3.6) shows two other by-products of the inversion that can be used in conjunction with the reflectivity function to estimate the elastic parameters of the lower medium; on the top is the direction cosine between the incident slowness and the vertical, and on the bottom is the direction cosine between the normal to the reflector and the vertical. Both quantities are computed at the reflector depth and are relevant in the illuminated zone only. From them, one can compute the incidence angle at any illuminated point on the reflector. Knowing the associated reflectivity, one can therefore retrieve the angular dependance of the reflection coefficient and thereby estimate the elastic parameters of the lower medium.

To obtain a complete image of the reflector, as well as to reduce endpoint effects, one needs to superpose or stack several shot inversions. Figure (3.7) shows such a stack, computed using ten shots and a 500m move-up between shots. Note that although the amplitude information was lost in this process, the resulting image is now completely free from artifacts.
Figure (3.5): Theoretical reflection coefficient to be compared to the amplitude plot in Figure (3.3) (horizontal variables are different).
Figure (3.6): Direction cosines for the incident slowness and the reflector normal as retrieved by the inversion at the reflector depth. Values are relevant only within the illuminated zone (-1250m to 1250m).
Stack of 10 Shot Inversions

Figure (3.7): Stack of 10 shot inversions. Although the reflectivity information is lost, the image shows no edge effects.
3.3.2 Imaging Curved Reflectors

Although our computer implementation allows for stratified backgrounds only, this does not exclude inverting for a curved reflector insofar as it is overlayed by a tabular sequence. To prove our point, we consider such a model and generate nine shot profiles using an acoustic ray-tracing program. The model comprises two flat reflectors on top of a curved reflector which constitutes the target to be imaged. The velocities increase from 3000m/s in the upper layer, to 6000m/s in the lower layer. The depth of the first and second reflectors are 500m and 1500m respectively, and the average depth to the curved reflector is 2500m. Each shot profile includes 41 split-spread receivers spaced 20m. The move-up between shots is 500m. Figure (3.8) shows the geometry of the model as well as the ray diagram for all nine shots (only 40% of the rays are plotted for clarity). Figure (3.9) shows all nine shot records from left to right; for the purpose of display, only one out of four traces is plotted.

To invert this data, we use isotropic elastic layers with shear velocities matching those of the acoustic model. We then simulate acoustic waves using \( SH \) waves, which are also scalar waves in 2.5D media. All shot records are inverted separately, and then stacked together to produce the final image (Figure (3.10)). Comparing Figures (3.8) and (3.10), which are plotted to the same scale, we conclude that the reflector has been accurately imaged everywhere but in the two
Figure (3.8): Model and ray diagram discussed in section III-3.2. Note that some portions of the curved reflector are not illuminated by the survey.
Figure (3.9): Nine shot records corresponding to the survey discussed in section III-3.2. One out of four traces shown.
Figure (3.10): Inversion of the shot records shown in Figure (3.9). The curved reflector is accurately imaged except in the areas not illuminated by the survey.
regions not illuminated by any shot profile. In order to image these areas as well, one would need to extend the survey to the sides of the model.

3-3.3 Imaging Weak Reflectors

So far we have discussed examples where the input data is generated from a single reflector with a single wavetype. These are not necessarily realistic conditions, and we now examine a rather pathological model, where the signal of the target reflector is weak and superposed to that of other, stronger reflectors. Figure (3.11) describes the model in question. A vertically polarized source is placed at the surface, and all 64 primary reflections are traced to a set of 100 split-spread receivers. Figure (3.12) shows the corresponding shot records for the horizontal (in-line) and vertical components of displacement. The three events that dominate those records are all generated by the first reflector; from fastest to slowest, these events are $P-P$, $P-SV$, and $SV-SV$. After Automatic Gain Control (AGC) is applied (Figure (3.13)), other events become visible. In particular, the $P-P-QP-QP-P-P$ event from the target reflector is visible on the vertical component at about 1.5s zero offset arrival time.

Figure (3.14) shows a direct inversion of the data using the correct background velocities. Although the reflector is imaged at the correct depth of 2000m, there is an unacceptable level of artifacts in the image.
Figure (3.11): Four layer model discussed in section III-3.3.
Figure (3.12): Ungained shot records for the horizontal (left) and vertical (right) components of displacement for the example discussed in section III-3.3.
Gained Input (AGC)

Figure (3.13): Gained shot records. Note the target event on the vertical component (right) at about 1.5s zero-offset time.
Figure (3.14): Direct inversion of the input of Figure (3.12). The reflector is correctly positioned at a depth of 2000m, but the image is dominated by some "noise" at wide offsets, and a "smile" above the reflector.
Those spurious effects are due to the interference of strong events during the inversion. Actually, the "noise" at large offsets comes from the $SV-SV$ event. In effect, for an output point on the boundary of the survey, the sum of traveltimes to the source and to the farthest receiver becomes comparable to the actual $SV-SV$ travelt ime at that receiver. Although the receiver integration tends to cancel such contributions, the spatial aliasing of the $SV-SV$ event and its relative strength with respect to the target event result in large artifacts. The "smile" above the reflector is, on the other hand, typical of an "overmigrated" event. The event in question is the strong $P-SV$ converted arrival clearly visible on the horizontal component around 1.5s zero-offset travelt ime. Although that event does not fit the polarization of the inversion operator, its strength relative to the target event results, once again, in a significant amount of artifacts.

To illustrate those points, we present two "quick and dirty" remedies to these problems. To minimize the interference of the $SV-SV$ event, we first truncate the data to keep only the 40% nearest offsets. Figure (3.15) shows that the $SV-SV$ arrival does not dominate the section anymore. In particular, the target event is now visible without any gain on the vertical component, around 1.5s zero-offset time. Inverting this data, we obtain the image in Figure (3.16); most of the noise at large offsets has disappeared, but we are still left with the overmigrated $P-SV$ arrival. In truncating the input data, we have also considerably limited the illuminated portion of the reflector so that the image only survives between offsets
Figure (3.15): Raw shot records truncated to the 40% nearest offsets. The target event can be seen as the weak event at 1.5s zero-offset time on the vertical component (right).
Inversion of Near Offsets

Figure (3.16): Inversion output for the data of Figure (3.15). The reflector image is confined to near offsets, but the "noise" at wide offsets is practically gone.
of -600m and 600m. To eliminate the smile artifact, we reprocess the data without the horizontal component, and obtain the image in Figure (3.17). The reflector image is now strong, and the only artifact left is actually an overmigrated $SV-SV$ still present on the vertical component. To get rid of the last artifact, we truncate the vertical component to traveltimes below 1.8s, and finally obtain a decent, though restricted, image of the reflector (Figure (3.18)).

There are other alternatives to enhancing image quality while keeping the complete angular aperture provided by the survey. The first one that we present here is based once again on the elimination of the undesirable $SV-SV$ and $P-SV$ events, but this time by mean of $\tau-p$ filtering. Without entering into the details of this technique, let us simply say that it allows to separate events according to their velocity and zero-offset travelt ime, and thus makes it easy to select or reject particular events. Figure (3.19) shows the shot records after $\tau-p$ filtering: the $SV-SV$ and $P-SV$ events have been totally removed, and the target $P-P-QP-QP-P-P$ event can be guessed on the vertical component. Figure (3.20) presents the image obtained by inverting that data; the reflector is correctly mapped within the illuminated region, although the amplitudes are somewhat altered.

The second alternative we present for improving the inversion output consists in imposing limits on the reflector dip within the inversion algorithm.
Inversion of Vertical Component

Figure (3.17): Inversion output for the data of Figure (3.15), but using the vertical component only. The "smile" associated with the $P-SV$ event is eliminated.
Figure (3.18): Inversion output when considering only the first 1.8 seconds of the near offset traces of the vertical component. All undesirable events being removed from the data, the image is good although limited to a small portion of the reflector.
Figure (3.19): Shot records after $\tau-p$ filtering of the undesirable events. The target event can (barely) be guessed on the vertical component.
Inversion of Tau–P Filtered Input

Figure (3.20): Inversion of the \( r-p \) filtered input of Figure (3.19). The image quality is good despite of altered amplitudes.
In effect, at an output point located on a reflector, the sum of slownesses from source and receiver defines the direction of the normal to the reflector. A priori information can therefore be included in the inversion process to retain only those reflector dips that fall in the expected range. That procedure has the advantage of not requiring any additional processing, and its success lies in an adequate restriction on the range of the receiver integration. Figures (3.21) and (3.22) show results of this technique when applied to the raw data with dip limits of 25° and 10° respectively. With the 10° dip limit, the image is virtually noise-free, and the full aperture of the survey has been used.
Figure (3.21): Inversion of the raw input of Figure (3.12) with a $25^\circ$ dip limit. Only the $P-SV$ "smile" is still partly visible.
Inversion with 10 deg. Dip Constraint

Figure (3.22): Inversion of the raw input of Figure (3.12) with a 10° dip limit. The image is virtually artifact-free and the amplitudes as well as the full aperture provided by the data are preserved.
CONCLUSIONS

We have reviewed the propagation of high frequency waves in anisotropic elastic media. The major contrast from the isotropic case is the existence of three wavetypes, instead of two, and the dependance of both velocities and polarizations on propagation direction. Moreover, group and phase velocities are distinct both in intensity and direction, with the meaning that energy does not propagate perpendicular to the wavefronts. When severe anisotropy is considered, the classical *WKBJ* asymptotic expansion may fail along certain propagation directions, like, for example, along the cuspidal edges of triplication in transversely isotropic media. Such problems are most likely to have solutions through uniform asymptotic expansions, and should be the focus of future work.

A detailed discussion of ray tracing in fractured media has been presented, and typical applications discussed, including fracture detection through the study of shear-wave birefringence. Many issues remain to be explored in the area of direct birefringence observation from reflection data. The author has in particular observed that hodograms generated from field data may reveal complex and sometimes ambiguous polarization patterns, because of the nature of the source signature, of the interference between events, and probably of near surface effects as well.
In solving the inverse scattering problem, we have generalized the approach of Bleistein and Cohen, which stands to this day as a most versatile and elaborated seismic imaging techniques. It seems that complex structure imaging in the presence of anisotropy in now mostly a matter of developing adequate ray tracing codes. With such a general scope however, many practical aspects of the method need to be studied. The determination of background velocities is one of them, which can probably be solved by conventional velocity analysis. Reducing the artifacts occasionally produced in addition to the correct image may require the introduction of constraints and/or a-priori information into the processing scheme. Fortunately, this task is made easy because the algorithm is composed of many explicit steps, which can be separately tuned. Some conventional preprocessing, such as $\tau-p$ filtering, may also be successfully used in conjunction with the inversion. Another advantage to the Kirchhoff approach is that it may be targeted on a specific region of the subsurface, with a substantial gain in processing time.

Finally, we have presented a specialization of the algorithm to 2.5-D media, to be used in processing lines of data recorded perpendicular to the strike of a cylindrical structure. Such recording configurations are used in the majority of seismic surveys. The algorithm was implemented for tabular media to analyze a few examples. Numerous interesting issues, such as parameter estimation, will make the object of future work.
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APPENDIX A: Reciprocity Relation

This appendix presents a derivation of the reciprocity relation in inhomogeneous layered anisotropic media.

We first define two Green’s tensors with sources at \( x' \) and \( x'' \) within a volume \( V \):

\[
c_{ijkl}(x)g_{km,l}(x,x'), j + \rho \omega^2 g_{im}(x,x') = -\delta_{im} \delta(x-x') ,
\]

and

\[
c_{ijkl}(x)g_{kn,l}(x,x''), j + \rho \omega^2 g_{in}(x,x'') = -\delta_{in} \delta(x-x'') ,
\]

Next we multiply the first equation by \( g_{in} \), subtract from this the second equation previously multiplied by \( g_{im} \), and integrate over \( V \) to form the new equation:

\[
\int_V dV(x) \left\{ g_{in}(x,x'') \left[ c_{ijkl}(x) \ g_{km,l}(x,x') \right]_{,j} + g_{in}(x,x'') \rho(x) \omega^2 g_{im}(x,x') - g_{im}(x,x') \left[ c_{ijkl}(x) \ g_{kn,l}(x,x'') \right]_{,j} - g_{im}(x,x') \rho(x) \omega^2 g_{in}(x,x'') \right\} =
\]

\[
\int_V dV(x) \left[ -g_{in}(x,x'') \delta_{im} \delta(x-x') + g_{in}(x,x') \delta_{in} \delta(x-x'') \right] .
\]

The right-hand side \( R \) can be simplified by contraction and sifting into:

\[ R = -g_{mn}(x',x'') + g_{nm}(x'',x') \]

Two of the terms in the integrand of the left-hand side \( L \) cancel, and the remaining
terms can be rewritten:

\[
L = \int_V dV(x) \left\{ \left[ g_{in}(x, x'') \ c_{ijkl}(x) \ g_{km,l}(x,x') \right]_{,j} - g_{in,j}(x, x''') \ c_{ijkl}(x) \ g_{km,l}(x,x') - \right. \\
\left. \left[ g_{im}(x, x') \ c_{ijkl}(x) \ g_{kn,l}(x,x'') \right]_{,j} + g_{im,j}(x, x'') \ c_{ijkl}(x) \ g_{kn,l}(x,x'') \right\} .
\]

We now use dummy index permutations followed by symmetry (1.4) to show that:

\[
g_{im,j}(x, x') \ c_{ijkl}(x) \ g_{kn,l}(x,x'') = g_{km,l}(x, x') \ c_{klj}(x) \ g_{in,j}(x, x''') = g_{km,l}(x, x') \ c_{ijkl}(x) \ g_{in,j}(x, x''') .
\]

In consequence, \( L \) simplifies into:

\[
L = \int_V dV(x) \left[ g_{in}(x, x'') \ c_{ijkl}(x) \ g_{km,l}(x,x') - g_{im}(x, x') \ c_{ijkl}(x) \ g_{kn,l}(x,x'') \right]_{,j} .
\]

Using the divergence theorem, we transform \( L \) into a surface integral on the boundary \( S \) of \( V \):

\[
L = \int_S dS(x) \left[ g_{in}(x, x'') \ c_{ijkl}(x) \ g_{km,l}(x,x') - g_{im}(x, x') \ c_{ijkl}(x) \ g_{kn,l}(x,x'') \right] \nu_j(x) ,
\]

where \( \nu(x) \) is the unit vector normal to \( S \) at \( x \). Finally we recognize from (1.11) that some terms in \( L \) can be grouped in the integrand to form Green tractions at the surface:

\[
L = \int_S dS(x) \left[ g_{in}(x, x'') \ t_{im}(x,x') - g_{im}(x, x') \ t_{in}(x,x'') \right] .
\]
The equality between $L$ and $R$ therefore proves (1.12).

As stated so far, the reciprocity relation only holds for a domain where elastic coefficients remains differentiable. It is however easy to extend it to layered media insofar as the successive layers be in rigid contact. Let us consider $N$ consecutive layers $V_1$ through $V_N$; the $l_{th}$ layer $V_l$ is limited on the top by an interface $S_{l-1}$ and below by an interface $S_l$. All interfaces are assumed to be of infinite lateral extent. The points $x'$ and $x''$ are located in layers $p$ and $q$ respectively. The reciprocity relation can be written for each layer as follows,

$$g_{nm}(x'', x') \delta_{lq} - g_{mn}(x', x'') \delta_{lp} =$$

$$\int_{S_{l-1}} dS(x) \left[ g_{in}^l(x, x'') t_{im}^l(x, x') - g_{im}^l(x, x') t_{in}^l(x, x'') \right] +$$

$$\int_{S_l} dS(x) \left[ g_{in}^l(x, x'') t_{im}^l(x, x') - g_{im}^l(x, x') t_{in}^l(x, x'') \right]. \quad (A1)$$

On the other hand, rigid contact at the interfaces implies the continuity of displacement and traction; considering that the normal unit vectors are directed outward, the continuity equations are:

$$g_{in}^l(x, x'') = g_{in}^{l+1}(x, x'') \quad ,$$

$$g_{im}^l(x, x') = g_{im}^{l+1}(x, x') \quad ,$$

$$t_{in}^l(x, x'') = -t_{in}^{l+1}(x, x'') \quad ,$$

$$x \in S_l$$
\[ t_{lm}^{l}(x,x') = -t_{lm}^{l+1}(x,x') \]

Therefore by summing equation (A1) from \( l=1,\ldots,N \), all but the uppermost and lowermost surface integrals survive, yielding:

\[
g_{nm}(x'',x') - g_{mn}(x',x'') = \\
\int_{S_0} dS(x) \left[ g_{in}(x',x'') t_{im}(x,x') - g_{im}(x,x') t_{in}(x,x'') \right] + \\
\int_{S_N} dS(x) \left[ g_{in}(x',x'') t_{im}(x,x') - g_{im}(x,x') t_{in}(x,x'') \right]. \tag{A2}
\]

A special case of (A2) of interest in geophysics is when \( S_0 \) is the earth’s free surface, and \( S_N \) a lower hemispherical surface whose radius is arbitrarily large. In that situation, the reciprocity relation simplifies remarkably because the integrand of the \( S_0 \) integral vanishes identically, and because the integral over \( S_N \) vanishes in the limit under a Sommerfeld-type radiation condition, that is:

\[
g_{nm}(x'',x') - g_{mn}(x',x'') = 0. \tag{A3}
\]

Another case that is commonly encountered is one where one or both of \( x' \) and \( x'' \) is on the free surface \( S_0 \). Let us for example assume that \( x' \) is located on \( S_0 \). The corresponding Green’s tensor must then be redefined as the displacement field of a unit impulsive traction (i.e. a surfacic point force) at the free surface, that is:
\[
\left[ \epsilon_{ijkl}(x) g_{km,l}(x,x') \right]_{,j} + \rho \omega^2 g_{im}(x,x') = 0 ,
\]

and

\[
t_{im}(x,x') = \delta_{im} \delta(x-x') \quad x \in S_0 .
\]

Following the same logic as before, the reciprocity relation becomes:

\[
g_{nm}(x''',x') = \int_{S_0} dS(x) \left[ g_{in}(x,x''') \delta_{im} \delta(x-x') \right] ,
\]

and evaluating the integral yields (A3) once again, that is, we can extend the validity of (A3) to points on the free surface.
APPENDIX B: Homogeneous Asymptotic Green's Tensor

This appendix presents a derivation of the asymptotic Green's tensor in a homogeneous, unbounded transversely isotropic medium.

The wave equation for the homogeneous Green's tensor is:

\[ c_{ijkl} g_{km,lj}(x,\omega) + \rho \omega^2 g_{im}(x,\omega) = -\delta_{im} \delta(x) \ , \]

where \( \text{Im}(\omega) > 0 \). We introduce the three dimensional spatial Fourier transform of \( g_{jm} \) with the following conventions:

\[ G_{jm}(k,\omega) = \int \int \int d^3 x \ g_{jm}(x,\omega) \ e^{-ik \cdot x} \ , \]

and reciprocally,

\[ g_{jm}(x,\omega) = \frac{1}{(2\pi)^3} \int \int \int d^3 k \ G_{jm}(k,\omega) \ e^{ik \cdot x} \ . \quad (B1) \]

In the transformed domain, the wave equation turns into the following algebraic equation:

\[ c_{ijkl} k_i k_j G_{km}(k,\omega) - \rho \omega^2 G_{im}(k,\omega) = \delta_{im} \ . \]

Defining the slowness vector \( \mathbf{p} \) as:

\[ \mathbf{p} = \frac{k}{|\omega|} \ , \]

that equation can be put in the form:
\[ A_{ik}(p,\omega) \, G_{km}(p,\omega) = \delta_{im} \quad , \] 
where

\[ A_{ik}(p,\omega) = |\omega|^2 \left[ c_{ijkl} p_l p_j - \rho e^{2i\text{Arg}(\omega)} \delta_{ik} \right] \]
is nothing but the eikonal matrix presented in section 2.1. That matrix is symmetric, so that its eigenvectors can be used as an orthonormal basis that represents the polarization of each wavetype. Let \( U_i^{(M)} \) \( (M=1,2,3) \) be these eigenvectors, and \( \lambda^{(M)} \) the associated eigenvalues, that is:

\[ A_{ik} \, U_k^{(M)} = \lambda^{(M)} \, U_i^{(M)} \quad , \]

and

\[ U_i^{(M)} \, U_i^{(Q)} = \delta^{MQ} \quad . \]

Next we formally decompose the solution \( G_{im} \) on the eigenbasis to obtain:

\[ G_{km} = \sum_{M=1}^{3} \alpha_m^{(M)} \, U_k^{(M)} \quad . \]

Substituting this expression into (B2) yields:

\[ \sum_{M=1}^{3} \alpha_m^{(M)} \, \lambda^{(M)} \, U_i^{(M)} = \delta_{im} \quad , \]

and contracting with \( U_i^{(Q)} \) we get:

\[ \sum_{M=1}^{3} \alpha_m^{(M)} \, \lambda^{(M)} \, \delta^{MQ} = U_i^{(Q)} \quad , \]

that is,
\[ \alpha_m^{(Q)} = \frac{U_m^{(Q)}}{\lambda^{(Q)}}. \]

Consequently, we obtain an eigenvector expansion of \( G_{im} \) in the form:

\[ G_{im} = \sum_{M=1}^{3} \frac{U_i^{(M)} U_m^{(M)}}{\lambda^{(M)}}. \]

The Green's tensor may now be expressed in the spatial domain as the triple inverse Fourier transform of \( G_{im} \) according to (B1):

\[ g_{im}(x,\omega) = \frac{|\omega|^3}{(2\pi)^3} \sum_{M=1}^{3} \int \int \int d^3p \frac{U_i^{(M)} U_m^{(M)}}{\lambda^{(M)}} e^{i|\omega|p \cdot x}. \quad (B3) \]

In the transversely isotropic case, the integrand is best parametrized with a spherical coordinate system \((p, \phi, \theta)\) centered at the source, and with its polar axis along the anisotropy axis \((\phi \text{ is the polar angle})\). In that system, the eigenvalues can be presented in the general form:

\[ \lambda^{(M)} = \frac{|\omega|^2 \rho}{\left(p^{(M)}(\phi)\right)^2} \left[p^2 - \left(p^{(M)}(\phi)\right)^2\right], \]

where \( \text{Im}(p^{(M)}) > 0 \). Accordingly, (B3) is rewritten as:

\[ g_{im}(x,\omega) = \frac{|\omega|}{(2\pi)^3 \rho} \sum_{M=1}^{3} \int d\theta \int d\phi \sin\phi U_i^{(M)}(\theta, \phi) U_m^{(M)}(\theta, \phi) \left(p^{(M)}(\phi)\right)^2 \]

\[ \times \int \int \int d^3p \frac{U_i^{(M)} U_m^{(M)}}{\lambda^{(M)}} e^{i|\omega|p \cdot x}. \]
\[ \int dp \frac{p^2}{p^2 - \left[p^{(M)}(\phi)\right]^2} e^{i|\omega|bp} \quad (B4) \]

We now evaluate the \( p \)-integral, which is in the form:

\[ I = \int_0^\infty dp \frac{p^2}{p^2 - a^2} e^{i|\omega|bp} \]

where \( b \) is real positive, and \( Im(a)>0 \). We first regularize \( I \) as follows:

\[ I = \int_0^\infty dp \frac{a^2}{p^2 - a^2} e^{i|\omega|bp} + \int_0^\infty dp e^{i|\omega|bp} \]

The second integral only exists in the distributional sense:

\[ \int_0^\infty dp e^{i|\omega|bp} = \frac{1}{2\pi |\omega|} \left[ \pi \delta(b) + \frac{i}{b} \right] \]

On the other hand, the first integral can be evaluated using Cauchy's theorem on a contour that includes the positive real axis, a quarter of circle counterclockwise at infinity, and the positive imaginary axis. Considering that the integral over the arc at infinity vanishes, and that the contour includes the pole at \( p=a \), we are left with:

\[ \int_0^\infty dp \frac{a^2}{p^2 - a^2} e^{i|\omega|bp} = i\pi a e^{i|\omega|ba} - \int_0^\infty dp \frac{ia^2}{p^2 + a^2} e^{-|\omega|bp} \]

Grouping all expressions together, we obtain:
\[ I = i \pi a e^{i \omega |b|} - \int_0^\infty dp \frac{ia^2}{p^2 + a^2} e^{-|\omega|b} + \frac{1}{2\pi |\omega|} \left[ \pi \delta(b) + \frac{i}{b} \right]. \]

At this point we make use of the high frequency assumption: the first term is order zero in \( \omega \), the second one is order \( 1/\omega \) (i.e. the integral is bounded up by \( i/|\omega|b \)), and the third one is order \( 1/\omega \) as well. Therefore to leading order in frequency we have:

\[ I \approx i \pi a e^{i \omega |b|}. \]

Substituting this result into (B4) we get:

\[ g_{im}(x, \omega) = \frac{i |\omega|}{2(2\pi)^2 \rho} \sum_{M=1}^3 \int d\theta \int d\phi \sin \phi \left( p^{(M)}(\phi) \right)^3 U_i^{(M)}(\theta, \phi) U_m^{(M)}(\theta, \phi) e^{i |\omega| p^{(M)*}}. \]

where \( p^{(M)} \) is a vector in the direction \( (\theta, \phi) \) with modulus \( p^{(M)}(\phi) \). It is remarkable that the equation that defines the pole at \( p^{(M)} \) is the polar equation for the slowness surface:

\[ p = p^{(M)}(\phi). \]

Consequently, \( p^{(M)} \) is nothing but the vector centered at the source point whose extremity sits on the slowness surface.

Next we evaluate the remaining integrals by the method of stationary phase.

The stationarity conditions are:
\[ \frac{\partial p^{(M)}}{\partial \theta} \cdot z = 0, \quad (B5) \]

and

\[ \frac{\partial p^{(M)}}{\partial \phi} \cdot z = 0, \quad (B6) \]

Since \( \partial p^{(M)} / \partial \theta \) and \( \partial p^{(M)} / \partial \phi \) are two vectors tangent to the slowness surface, the stationarity conditions impose that \( \theta \) and \( \phi \) be such that \( z \) be normal to the slowness surface. Moreover, we showed in section (1-2.2) that group velocity \( W^{(M)} \) is perpendicular to the slowness surface for given propagation mode and direction; therefore the stationary points are at \((\theta, \phi)\) such that:

\[ z = r \frac{W^{(M)}(\theta, \phi)}{|W^{(M)}(\theta, \phi)|}, \quad (B7) \]

where \( r = |z| \); this means that the phase at a stationary point is:

\[ \tau^{(M)} = p^{(M)} \cdot z = \frac{r}{W^{(M)}} p^{(M)} \cdot W^{(M)} \]

where \( W^{(M)} = |W^{(M)}| \). Moreover, since we showed in (1.28) that:

\[ p^{(M)} \cdot W^{(M)} = 1 \]

the phase simplifies into:

\[ \tau^{(M)} = p^{(M)} \cdot z = \frac{r}{W^{(M)}} \]

that is, the traveltime (phase) at the stationary point corresponds to a wave traveling from source to observation point at the group velocity. The asymptotic
evaluation of $g_{im}$ reduces to algebraic contributions from stationary points defined by (B5) and (B6):

$$
g_{im}(z, \omega) = \frac{i |\omega|}{2(2\pi)^2 \rho} \sum_{M=1}^{3} \sum_{\text{stat.}} \sin\phi \left( p^{(M)} \right)^3 U_{i}^{(M)} U_{m}^{(M)} \frac{2\pi}{|\omega|} \frac{e^{i\pi/4 \sin H}}{\sqrt{|\det H|}} e^{i|\omega| \tau^{(M)}},
$$

where $H$ is the Hessian of the phase:

$$
H = \left[ \frac{\partial^2 r^{(M)}}{\partial (\theta, \phi)^2} \right].
$$

We now introduce the spherical coordinates $(r, \alpha, \beta)$ of the observation point $z$, and express the phase as follows:

$$
r^{(M)} = r p^{(M)}(\phi) \left[ \cos\phi \cos\alpha + \sin\phi \sin\alpha \cos(\theta - \beta) \right].
$$

Cast in these notations, equation (B5) becomes:

$$
-r p^{(M)} \sin\phi \sin\alpha \sin(\theta - \beta) = 0,
$$

that is, $p^{(M)}$ and $z$ must lie in the same axial plane ($\theta = \beta$). Using this fact, we may write:

$$
\frac{\partial^2 r^{(M)}}{\partial \theta^2} = -r p^{(M)} \sin\phi \sin\alpha,
$$

$$
\frac{\partial^2 r^{(M)}}{\partial \theta \partial \phi} = 0, \quad \text{at stationary points}
$$

$$
\frac{\partial^2 r^{(M)}}{\partial \phi^2} = r \left[ \frac{d^2 p^{(M)}}{d\phi^2} \cos(\phi - \alpha) - 2 \frac{dp^{(M)}}{d\phi} \sin(\phi - \alpha) - p^{(M)} \cos(\phi - \alpha) \right],
$$
where $\phi - \alpha$ is the angle between slowness and group velocity vectors. According to (1.43), we have:

$$\sin(\phi - \alpha) = \frac{1}{p^{(M)}} \frac{dp^{(M)}}{d\phi} \cdot$$

and

$$\cos(\phi - \alpha) = \frac{1}{p^{(M)} W^{(M)}} \cdot$$

so that:

$$\frac{\partial^2 r^{(M)}}{\partial \phi^2} = \frac{r}{W^{(M)} p^{(M)}} \left[ \frac{d^2 p^{(M)}}{d\phi^2} - \frac{2}{p^{(M)}} \left( \frac{dp^{(M)}}{d\phi} \right)^2 - p^{(M)} \right].$$

Interestingly, $\partial^2 r^{(M)}/\partial \phi^2$ is proportional to the Gaussian curvature of the slowness surface in an axial plane. Whenever that quantity vanishes, which happens at triplication cusps, the asymptotic expansion is not valid and must be modified (i.e. there is a higher order stationary point). Otherwise, the curvature is either negative, where the slowness surface is convex, or positive, where it is concave.

Focusing our attention on the case of convex slowness surfaces (weak anisotropy), the signature of the Hessian is -2, and there is only one stationary point per mode so that the asymptotic Green's tensor reduces to:

$$g_{lm}(x, \omega) = \frac{1}{4\pi p|x|} \sum_{M=1}^{3} p^{(M)} \left( \frac{U_i^{(M)} U_m^{(M)}}{|K^{(M)}(\phi)| \sin\alpha} \right)^{1/2} \left( \frac{\sin \phi}{|K^{(M)}(\phi)| \sin \alpha} \right)^{1/2} e^{i|\omega| \theta/w^{(M)}(\phi)},$$

where $K$ is a dimensionless curvature factor:
\[
K^{(M)}(\phi) = \frac{p^{(M)} d^2 p^{(M)}}{d\phi^2} - 2 \left( \frac{dp^{(M)}}{d\phi} \right)^2 - \left( \frac{p^{(M)}}{d\phi} \right)^2.
\]

Unfortunately, there is no explicit way to compute the stationary value of \( \phi \) given \( \alpha \). Therefore, it is preferable to proceed as follows: one specifies a mode \( M \), a propagation direction \((\theta, \phi)\), and a traveltime \( t \) at which the observation is to be made. One then *computes* the observation point \( x^{(M)} \), which sits on the wave surface of mode \( M \) at traveltime \( t \), that is:

\[
x^{(M)}(\theta, \phi) = W^{(M)}(\theta, \phi) t.
\]

The curvature term \( K^{(M)} \) can then be evaluated at \( \phi \) by taking derivatives of \( p^{(M)} \), while \( \alpha \) is just the angle between anisotropy axis and \( W^{(M)} \). Finally, the Green's tensor is best expressed separately for each mode as follows:

\[
g^{(M)}(t, \theta, \phi, \omega) = \frac{\left( p^{(M)}(\phi) \right)^2}{4\pi \rho |W^{(M)}| t} U_i^{(M)}(\theta, \phi) U_m^{(M)}(\theta, \phi) \left[ \frac{\sin \phi}{K^{(M)}(\phi) \sin \alpha} \right]^{1/2} e^{i \omega |t|}.
\]

This expression differs from the isotropic case only by the term in the square root. In particular, the field decays in inverse proportion to the distance from the source. There is one case where this formula needs to be modified, when \( \alpha = 0 \). In that case \( \phi \) vanishes as well and the ratio \( \sin \alpha / \sin \phi \) is finite. We give the value of this ratio for each wavetype using Love's parameters:
\[
\frac{\sin \alpha}{\sin \phi} \to 2 - \frac{N}{L} \quad (S-\text{Parallel mode})
\]

\[
\frac{\sin \alpha}{\sin \phi} \to 2 - \left[ \frac{L}{C} + \frac{(F+L)^2}{C(C-L)} \right] \quad (\text{Quasi-S mode})
\]

\[
\frac{\sin \alpha}{\sin \phi} \to 2 - \left[ \frac{A}{L} - \frac{(F+L)^2}{L(C-L)} \right] \quad (\text{Quasi-P mode})
\]
APPENDIX C : Ray Tracing Algorithm

This appendix presents a sketch of the modeling algorithm discussed in (1-3.2).

input model, survey parameters, events description

initialize synthetic traces to zero

for all events {
    trace kinematic interpolation map
    for all receivers {
        interpolate take-off angles from map
        compute optimum ray tube aperture from map
        initialize ray amplitudes
        for all segments in raypath {
            trace to next boundary
            compute scattering at boundary
            initialize ray amplitude and direction of next segment
        } compute free-surface correction
        output to trace
    }
}

}
APPENDIX D: Inversion Algorithm \#1

This appendix presents a sketch of the inversion algorithm discussed in (2-3.2).

Compute $\hat{U}_m(z', t)$ from $U_m(z', t)$

Pretrace ray bundle from $z^s$ to lowermost reflector

For many $z$ in output medium {
    Continue ray bundle to $z$
    Store ray attributes at $z$
}

For all $z'$ {
    Pretrace ray bundle from $z'$ to lowermost reflector
    For many $z$ in output medium {
        Continue ray bundle to $z$
        Store ray attributes at $z$
    }
    For all $z'$ {
        Interpolate ray attributes at $z'$ from $z'$ and $z^s$
        Assemble $c_m^{NQ}(z', z', z^s, f)$
interpolate $\hat{U}_m(x', t)$ at $t=r_N+r_Q$

assemble $b^{NQ}(x', x', x^s, f)$

store $p^Q(x', x')$

store $b^{NQ}(x', x', x^s, f)$

}]

for all $x'$ {

initialize $\beta^{NQ}[x', f]$ to zero

for all $x'$ {

recall $p^Q(x', x')$ and neighbors

compute $B_{NQ}(x', x')$

recall $b^{NQ}(x', x', x^s, f)$

increment $\beta^{NQ}[x', f]$

}

store $\beta^{NQ}[x', f]$

}
APPENDIX E: Inversion Algorithm #1

This appendix presents a sketch of the inversion algorithm discussed in (2-3.3).

compute $\hat{U}_m(z', t)$ from $U_m(z', t)$

pretrace ray bundle from $z^d$ to lowermost reflector

for many $z$ in output medium {
    continue ray bundle to $z$
    store ray attributes at $z$
}

initialise $\beta^{NQ}(z', f)$ to zero

for all $z'$ {
    pretrace ray bundle from $z'$ to lowermost reflector
    for many $z$ in output medium {
        continue ray bundle to $z$
        store ray attributes at $z$
    }
    for all $z'$ {

interpolate ray attributes at $x'$ from $x'$ and $x^s$

assemble $c_m^{NQ}(x', x', x^s, f)$

interpolate $\hat{U}_m(x', t)$ at $t = T_N + T_Q$ 

assemble $b^{NQ}(x', x', x^s, f)$

compute $B_{NQ}(x', x')$

increment $\beta^{NQ}[x', f]$

}
APPENDIX F: Inversion Algorithm # 1

This appendix presents a sketch of the inversion algorithm discussed in (2-3.4).

compute $\hat{U}_m(x', t)$ from $U_m(x', t)$

pretrace ray bundle from $x^o$ to lowermost reflector

for many $x$ in output medium {
    continue ray bundle to $x$
    store ray attributes at $x$
}

for all $x'$ {
    recall ray attributes from $x^o$ at many $x$
    interpolate ray attributes at $x'$ from $x^o$
    initialise $\beta^{NQ}(x', f)$ to zero
    for all $x''$ {
        compute ray attributes at $x'$ from $x''$
        interpolate $\hat{U}_m(x', t)$ at $t=r_N+r_Q$
        assemble $c^{NQ}_m(x', x'', x^o, f)$
        assemble $b^{NQ}(x', x'', f)$
        compute $B_{NQ}(x', x'')$
    }
}
increment $\beta^{NQ}(x', f)$

} 

store $\beta^{NQ}(x', f)$

}