DESIGN AND APPLICATION OF RECURSIVE DIGITAL FILTERS

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ABSTRACT

The main objective of seismic digital processing, is the improvement of the signal-to-noise ratio in the recorded data. Digital processes must be designed for the particular noise and signal present. Recursive digital filters are designed with the frequency spectrum as the mode of description of the signal and noise. The filters of this thesis were designed using different pole and zero configurations in the z-plane. These filters were applied to a seismic section of fifty records, recorded by the Department of Geophysics, Colorado School of Mines during the summer field session of 1968 at South Park Colorado. The results show an enhancement of the data characterized by an increase in the continuity of the reflections, at the expense of increased ringing.

Design procedures for bandpass, highpass, single-frequency rejection, and bandstop filters are illustrated. Both cascade and parallel type of difference equations and their particular advantages are described.

The application of the recursive digital filters to seismic recording, showed a great advantage in execution time with respect to the normal convolution filtering techniques. The recursive filters used were ten times faster than equivalent convolution filters.
INTRODUCTION

The use of digital computers to process digitized seismic recordings is now well established. A continuous seismic trace, or some other continuous recording of geophysical data versus time, may be converted into a sequence of numbers, each number represents the reading, or amplitude of the trace at a specific time. The time points are chosen to be equally spaced.

Conventional filtering is performed by means of electric networks, while digital filtering, on the other hand, is often applied by convolving the recorded time series with the unit impulse response, or weighting function of the filter. One advantage of digital narrow-band filters over their analog counterpart is the flexibility of design and applications. Another advantage is that digital filters have much greater dynamic range.

Because of the large number of operations involved in computing the output, filtering can consume a considerable fraction of the computer time involved in processing seismograms. The advantage of recursive filtering techniques, is that certain filtering operations can be performed much faster than by convolutions.

The heart of the design problem consists of assigning appropriate values to the coefficients in the filter
difference equations so as to achieve the desired frequency response. It is most convenient to design these equations using the theory and techniques of the z-transform. A recursive filter can be expressed as a ratio of two polynomials in the z-transform. Therefore we can design filters by specifying the roots of the polynomials, also we can use the relationship between the Laplace transform and the z-transform to convert desirable analog filter equations into their digital equivalents. Another possibility is to somehow establish criteria for the location of the poles and zeros in the z-plane. The remainder of the problem would be straightforward. In selecting pole and zero locations, two limitations must be imposed.

(1) The poles and zeros must either be real or they must be assigned in complex conjugate pairs, in order to yield real values for the coefficients.
(2) The poles must lie outside of the unit circle in order to produce a stable causal filter. Zeros inside of the unit circle should be avoided also, since they lead to nonminimum-phase systems.
DISCRETE TIME FUNCTION AND Z-TRANSFORM DEFINITION

The z-transform method constitutes one of the transform methods that can be applied to the solution of linear difference equations (Jury, 1964). It reduces the solutions of such equations into those of algebraic equations.

In discrete systems, the signals flowing are considered at discrete values of $t$, usually at $n\Delta t$, $n = 0,1,2, \ldots$ where $\Delta t$ is a fixed positive number usually referred to as the sampling period. The study of such discrete systems may be carried through by using the z-transform method.

Let $\Delta t$ be a fixed positive number (it could be taken as unity). Let $f(t)$ be defined for $t \geq 0$. The z-transform of $f(t)$ is the function:

$$Z[f] = F(z) = \sum_{n=0}^{\infty} f(n\Delta t)z^n$$

for $|z| < \rho$

where $\rho$ = radius of convergence of the series, and $z$ is the complex variable.

Properties of Z-Transform

In the following we shall show a few properties and theorems related to the z-transform. Their use will enable us to develop the z-transform, which is the main tool of the
present work.

**Linearity of the Z-Transform.** For all constants $c$, and $c_2$, the following property holds

$$Z(c \cdot f + c_2f_2) = c \cdot Z[f] + c_2Z[f_2] \quad (2)$$

Thus $Z$ is a linear operator on the linear space of all z-transformable functions $f(t), (t>0)$

**Shifting Theorem.** If $Z[f] = F(z)$

$$Z[f(t + \Delta t)] = z^{\prime} \left[F(z) - f(0)\right] \quad (3)$$

In general we can extend the procedure for any positive integer $m$.

$$Z[f(t + m\Delta t)] = z^{\prime m} \left[F(z) - \sum_{k=0}^{m\Delta t} f(k\Delta t)z^k\right] \quad (4)$$

**Complex Multiplication.** If $f_1$ and $f_2$ have the z-transform $F_1(z)$ and $F_2(z)$, then

$$F_1(z)F_2(z) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f_1(k\Delta t)f_2[(n-k)\Delta t]\right]z^n \quad (5)$$

**Inverse Z-Transform**

The discrete function at $t = n\Delta t$ or $f(n\Delta t)$ can be obtained from $F(z)$ by a process called the inverse z-transform.
This process is symbolically denoted as

\[ f(n\Delta t) = Z[F(z)] \quad (6) \]

where \( F(z) \) is the z-transform of \( f(t) \) or \( f(n\Delta t) \).

**The Power Series Method**

When \( F(z) \) is given as an analytic function for \( z < \beta \), the value of \( f(n\Delta t) \) can be readily obtained as the coefficient of \( z^n \) in the power series expansion of \( F(z) \).

From equation (1) it is observed that

\[ F(z) = f(0\Delta t) + f(1\Delta t)z + \ldots + f(n\Delta t)z^n + \ldots \quad (7) \]

Thus it is noticed that \( f(n\Delta t) \) can be read off as the coefficient of \( z^n \).

If \( F(z) \) is given as a ratio of two polynomials in \( z \)

\[ F(z) = \frac{a_0 + a_1z + a_2z^2 + \ldots + a_nz^n}{b_0 + b_1z + b_2z^2 + \ldots + b_nz^n} \quad (8) \]

The coefficients \( f(0\Delta t), \ldots, f(n\Delta t) \) can be obtained from the simultaneous equations

\[ a_0 = f(0\Delta t)b_0 \]

\[ a_n = f(n\Delta t)b_n + f(0\Delta t)b_n \]
It is also observed that \( f(n\Delta t) \) can be obtained by a synthetic division of the numerator by the denominator.

**Partial Fraction Expansion**

If \( F(z) \) is a rational function of \( z \) analytic at \( \infty \), it can be expressed by a partial fraction expansion

\[
F(z) = F_1(z) + F_2(z) + F_3(z) + \ldots \quad (10)
\]

The inverse of this equation \( f(n\Delta t) \) can be obtained as the sum of the individual inverses obtained from the expansion, that is

\[
f(n\Delta t) = Z^{-1}F(z) = Z^{-1}[F_1(z)] + Z^{-1}[F_2(z)] + \ldots \quad (11)
\]
THE Z-TRANSFORM AND DIFFERENCE EQUATION

Let a function of time \( f(t) \) be sampled in the interval from 0 to \( \infty \), the sampled function \( \tilde{f}(t) \) may be written as

\[
\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\delta(t - n\Delta t) \tag{12}
\]

Since

\[
\mathcal{L}[\delta(t - n\Delta t)] = \int_{-\infty}^{\infty} \delta(t - n\Delta t)e^{-st}dt = e^{-s\Delta t} \tag{13}
\]

taking the Laplace transform of (12), we obtain

\[
\tilde{F}(s) = \mathcal{L}[\tilde{f}(t)] = \sum_{n=-\infty}^{\infty} f(n\Delta t)e^{-s\Delta t} = \sum_{n=-\infty}^{\infty} f(n\Delta t)z^{-n} \tag{14}
\]

where

\[
z = e^{-s\Delta t} \tag{15}
\]

Let the output of the filter \( \tilde{F}(s) \) be \( \tilde{f}(t) \) when the input is \( \delta(t) \) then the impulse response of the filter is

\[
\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\delta(t - n\Delta t) \tag{16}
\]

and

\[
\tilde{F}(s) = \sum_{n=-\infty}^{\infty} f(n\Delta t)e^{-s\Delta t} = \sum_{n=-\infty}^{\infty} f(n\Delta t)z^{-n} \overset{\Delta}{=} \tilde{F}(z). \tag{17}
\]

It is well known that the Laplace transform of the output of a linear system, is the product of the transform of the input time function and the transform of the system impulse response. Thus if \( \tilde{x}(t) \) is a discrete input, and
\( y(t) \) is the corresponding output time series

\[
\tilde{Y}(s) = \tilde{\mathcal{F}}(s)\tilde{X}(s) = \mathcal{F}(z)X(z) = Y(z)
\]  

(18)

Often, \( \mathcal{F}(z) \) may be approximated by the ratio of two polynomials, as shown (Shanks, 1967).

\[
\mathcal{F}(z) = \frac{a_0 + a_1z + a_2z^2 + \cdots + a_kz^k}{1 + b_1z + b_2z^2 + \cdots + b_mz^m}
\]  

(19)

From (18) we obtain:

\[
Y(z) = \sum_{n=0}^\infty y(nAt)z^n = \sum_{n=0}^\infty \frac{a_0 + a_1z + a_2z^2 + \cdots + a_kz^k}{1 + b_1z + b_2z^2 + \cdots + b_mz^m}x(nAt)z^n
\]  

(20)

or

\[
\left(\sum_{n=0}^\infty y(nAt)z^n\right)(1+b_1z+\cdots+b_mz^m) = \left(\sum_{n=0}^\infty x(nAt)z^n\right)(a_0 + a_1z + \cdots + a_kz^k)
\]  

(21)

For every \( n \) the coefficients of \( z \) on each side of (21) should be equal. Thus

\[
y(nAt) + b_1y[(n-1)At] + b_2y[(n-2)At] + \cdots + b_my[(n-m)At] \\
= a_0x(nAt) + a_1x[(n-1)At] + \cdots + a_kx[(n-k)At]
\]  

(22)

In other words, the difference equation giving the \( n \)th coefficient in the series expansion of \( Y(z) \) is
\[ y_n = a_0 x_n + a_1 x_{n-1} + \ldots + a_k x_{n-k} - b_1 y_{n-1} - \ldots - b_m y_{n-m} \]  \hspace{1cm} (23)

Equation (23) may be used to find the output of the filter \( F(z) \).

\( F(z) \) may be written as a sum of partial fractions as

\[ F(z) = \sum_{n=1}^{N} F_n(z) \]  \hspace{1cm} (24)

and the entire filter can be visualized as the parallel connection of simpler filters \( F_n(z) \) of lower order. In this case, the filter is said to be realized in the parallel form. Similarly, if \( F(z) \) is written as a product of factors

\[ F(z) = \prod_{n=1}^{N} G_n(z) \]  \hspace{1cm} (25)

the filter may be visualized as a cascade of lower order filters. The original filter is then said to be realized in cascade form.

For the filters realized in parallel form numerical error problems have been studied extensively (Knowles, 1965, Lien and Kaneko, 1964). This effect is particularly serious for a narrow-band filters. It is suggested (Golden and Kaiser, 1964) that when we apply high order recursive filters, we express the polynomials of (19) in quadratic factors, and use the cascade or parallel two-pole-filter combination.
If we want to design a recursive digital filter to reject some particular frequency \( f_0 \), we place a single pole and a single zero on the line \( \arg z = w_0 \Delta t \). The zero should be placed on the unit circle and the pole slightly outside.

To make the results generally applicable, it is desirable to represent these filters in dimensionless form. A convenient dimensionless quantity proportional to frequency is

\[
w_\Delta t = 2\pi f \Delta t \left( \frac{180^\circ}{\pi} \right)
\]

where \( \Delta t \) is the sampling interval. Note that the Nyquist frequency corresponds to \( 180^\circ \).

Suppose, for example, that we desire to reject 60 cps interference in data which has been sampled at one-ms intervals. The \( z \)-plane points which correspond to 60 cps occur on the unit circle at angles of \( \pm 21.6^\circ \). Therefore we shall design the filter to have zeros at \( R_0 = 1.0 \), and poles just outside of the unit circle close to the zeros. We have located the poles at \( R_p = 1.005, 1.009, 1.01, 1.03, 1.05, \) and \( 1.08 \).

If we call \( x_n \) the input and \( y_n \) the output of the system function, then methods used to develop equation (23) show
that the recursive equation is of the following form

\[ y_n = x_n - a_1 x_{n-1} + a_2 x_{n-2} + b_1 y_{n-1} - b_2 y_{n-2} \]  

(26)

where \( a_1 = \frac{2 \text{Re}(z_o)}{|z_o|^2} \), \( a_2 = \frac{1}{|z_o|^2} \), \( b_1 = \frac{2 \text{Re}(z_p)}{|z_p|^2} \), \( b_2 = \frac{1}{|z_p|^2} \)

The poles and zeros of the filter are plotted on Figure 1.

Figure 1. Plot of z-plane poles and zeros of 60 cps rejection filter using a one-ms sample interval.

Figure 2 shows the amplitude and phase characteristics of this single frequency rejection filter.

The z-plane can be used to make a rapid graphical evaluation of the amplitude and phase response of a filter whose pole and zero locations are known. Thus, by graphical
analysis of the pole-zero locations, we can use the z-plane to design simple digital filters by appropriately locating the filter poles and zeros.
Figure 2. Amplitude and phase response of 60 cps rejection filter.
BANDPASS RECURSIVE FILTER WITH RIGHT POLES IN THE Z-PLANE

The purpose of the present and the following sections is to show that certain classes of recursive digital filters can be designed in a simple, almost intuitive form. This is based upon the relationship between response and location of the poles and zeros of the transfer function.

If we establish a criteria for the location of poles and zeros in the z-plane the rest of the problem is easy.

Suppose for the moment that we want to design a bandpass recursive filter having lower and upper cutoff frequencies of $f_1$ and $f_2$ respectively, and having eight poles in the z-plane (four conjugate pairs). In addition to maximum attenuation outside of the passband, we wish to approach flat response within the passband.

Recent papers (Golden, 1968, Mooney, 1967), provide new methods for the design of recursive digital filters, and suggest that for bandpass filters the poles and zeros be equally-spaced between the lower and upper cutoff frequencies. The interval suggested is between 0.5° and 1°. Therefore if $f_1 = 20$ cps and $f_2 = 40$ cps then since

$$\omega_{\Delta t} = 2\pi f_1 \Delta t \left(\frac{180°}{\pi}\right) = 7.2°$$

and

$$\omega_{\Delta t} = 2\pi f_2 \Delta t \left(\frac{180°}{\pi}\right) = 14.4°$$
we would need 32 poles and zeros for the 0.5° interval and 16 poles and zeros for the 1° interval.

As we can see with this method the only possibility to have cutoff frequencies of 20 cps and 40 cps would be placing the poles at 2° intervals in order to have eight poles in the z-plane. I tried this approach but the result was unsatisfactory. Instead of having flat response within the passband, it showed undesirable peaking, and the attenuation characteristics outside the passband were not satisfactory. Therefore I had to abandon these methods and develop one of my own.

Working in the z-plane my first guess was to place poles in the intervals between \( w_\Delta t \) and \( w_\Delta t \) and spaced along a small circle with center near to the corresponding center frequency. Four zeros were placed at \( z = 1 \) and another four at \( z = -1 \). Figures 3 and 4 show the position of poles and zeros in the z-plane.

Figures 5 and 6 show the amplitude and phase responses obtained with this method. The impulse responses of this two filters are shown in Figures 7 and 8 for pole and zero positions described in Figures 3 and 4 respectively.

As we can see from Figures 5 and 6 the results are, flat response within the passband and a satisfactory attenuation outside of the passband.

To use these filters we must assess the effect upon filter response produced by varying each of the parameters,
the small radius of the circle and the location of the poles on this small circle. We notice that as the center of the small circle approaches the unit circle in the z-plane, the attenuation outside of the passband increases, but an undesirable peaking is produced. On the other hand, as we move the center of the small circle away from the unit circle we produce flat response within the passband, but the attenuation outside of the passband is poorer.

If the poles are symmetric with respect to the center frequency (line at \((w_c + w_s)\Delta t/2\)), the frequency response will be symmetric. If the poles are moved toward one side closer to the unit circle the cutoff is steeper on that side.

Figure 3. Pole and zero diagram for a bandpass filter of eight order. Circular form. \(R_c = 1.0\), \(w_c\Delta t = 11°\) \(R_p = 1.02, 1.05, 1.06, 1.03, w\Delta t = 7°, 8°, 11°, 14°\).
Figure 4. Pole and zero diagram for a bandpass filter of eight order. Circular form. $R_c = 0.995$, $\omega_c \Delta t = 11^\circ$ $R_p = 1.015, 1.045, 1.055, 1.025$, $\omega_p \Delta t = 7^\circ, 8.5^\circ, 11^\circ, 14^\circ$.

Note that the filter might be approximated satisfactorily with about 150 to 200 terms at one-ms intervals. Thus, simple convolution would require 150 to 200 multiplications per output point. The recursive equations require only eight multiplications per output point in each filter.
Figure 5
My second guess was to place poles and zeros in the same interval between \( w_{Δ t} \) and \( w_{2Δ t} \), but now placing the poles on a small ellipse. As before four zeros were placed at \( z = 1 \) and four more at \( z = -1 \). Figures 9 and 10 show the positions for the poles and zeros in the \( z \)-plane.

![Diagram of pole and zero positions](image)

Figure 9. Pole and zero diagram for a bandpass filter of eight order. Elliptical form. \( R_c = 0.95, w_{Δ t} = 11.5° \), \( R_p = 1.016, 1.05, 1.06, 1.03, w_{Δ t} = 7°, 8°, 11°, 14° \).

The amplitude and phase response for the filters represented in the diagrams of Figures 9 and 10 are shown in Figures 11 and 12, respectively. As we can see from Figures 11 and 12, the results are better than the results of the first guess. The attenuation is sharper outside of
the passband, and the response is flatter within the passband.

Figure 10. Pole and zero diagram for a bandpass filter of eight order. Elliptical form. $R_c = 0.93$, $w_c \Delta t = 11^\circ$ $R_p = 1.021, 1.05, 1.06, 1.0283$, $w \Delta t = 7^\circ, 8^\circ, 11^\circ, 14^\circ$.

As before, we notice that as we move the center of the ellipse toward the unit circle we produce sharper attenuation outside of the passband, but this is at the expense of a decrease in the length of the flat response in the passband.
Figure 11
Figure 12
As we can see the main problem is to adjust the parameters in the z-plane in order to achieve the desirable amplitude and phase response. There is still much work to be done in order to get better amplitudes and phases, but for the purpose of the present work, the results obtained are satisfactory. One might use parabolic and hyperbolic configurations but the results should probably be similar to those obtained with the two configurations described before.

Parallel Form

The eight-pole bandpass filter was applied by both methods, parallel and cascade. The block diagram for the parallel method is shown in Figure 13 and it was obtained by making a partial fraction expansion of (19) as:

\[
F(z) = k_i \frac{1}{1 - z/z_i} + \frac{k^*_i}{1 - z^*/z_i^*} + \ldots + \frac{k^*_n}{1 - z^*/z_n^*} 
\]

\[
F(z) = \frac{a_1 + a_2 z}{1 + b_1 z + b_2 z^2} + \frac{a_1 + a_2 z}{1 + b_3 z + b_4 z^2} + \ldots + \frac{a_1 + a_2 z}{1 + b_7 z + b_8 z^2} 
\]

\[
a_i = k_i + k^*_i \\
a_2 = - \frac{k_i z_i + k^*_i z^*_i}{|z_i|^2} \\
b_i = - \frac{z_i + z^*_i}{|z_i|^2} \\
b_2 = \frac{1}{|z_i|^2} \\
extc.
\]
where \( z^* \) is the complex conjugate of \( z \).

The realization of the eight-pole filter was performed with four block diagrams in parallel as we can see in Figure 13, \( x_n \) represents the input and \( y_n \) the output of the filter.

The difference equation for the parallel realization are:

\[
\begin{align*}
   c_n &= x - b_1 c_{n-1} - b_2 c_{n-2} \\
   d_n &= x - b_3 d_{n-1} - b_5 d_{n-2} \\
   e_n &= x - b_5 e_{n-1} - b_6 e_{n-2} \\
   f_n &= x - b_7 f_{n-1} - b_8 f_{n-2} \\
   y_n &= a_1 c_n + a_2 c_{n-1} + a_3 d_n + a_4 d_{n-1} + a_5 e_n + a_6 e_{n-1} + a_7 f_n + a_8 f_{n-1}.
\end{align*}
\]

**Cascade Form**

The cascade combination was realized by factoring the denominator of (19) into four quadratic factors. The recursive operations indicated by the factors were then performed successively. The block diagram is shown in Figure 14.

The sample interval used in this computations was one-ms, and the general system function for both configurations, circular and elliptical was

\[
F(z) = \frac{(1 - z^*)^n}{B_1(z)B_2(z)B_3(z)B_4(z)}
\]
where \( B_n(z) \) \( n = 1, 2, 3, 4 \) is a quadratic factor.

If we call \( z_n \) the position of the \( n \)th zero and \( z_{p_n} \)
the position of the \( n \)th pole in the \( z \)-plane, and \( z_n^* \), \( z_{p_n}^* \)
their complex conjugates, \( B_n(z) \) can be written as follows

\[
B_n(z) = 1 - \frac{z_{p_n} + z_{p_n}^*}{z_{p_n} z_{p_n}^*} z + \frac{1}{z_{p_n} z_{p_n}^*} z^2
\]

for \( n = 1, 2, 3, 4 \)

\[
P(z) = \frac{(1 - z^2)^n}{(1-a_1 z+b_1 z^2)(1-a_2 z+b_2 z^2)(1-a_3 z+b_3 z^2)(1-a_4 z+b_4 z^2)}
\]

where
\[
a_n = \frac{2 \Re[z_{p_n}]}{|z_{p_n}|^2} \quad \text{and,} \quad b_n = \frac{1}{|z_{p_n}|^2}
\]

If, as before, we call \( X_n \) the input and \( Y_n \) the output,
the recursive equations are:

\[
c_n = x_n - x_{n-1} + a_1 c_{n-1} - b_1 c_{n-2}
\]
\[
d_n = c_n - c_{n-2} + a_2 d_{n-1} - b_2 d_{n-2}
\]
\[
e_n = d_n - d_{n-2} + a_3 e_{n-1} - b_3 e_{n-2}
\]
\[
y_n = e_n - e_{n-2} + a_4 y_{n-1} - b_4 y_{n-2}
\]

The difference equations (31) having this form require
two-thirds of the computation required by the parallel
combination.
The impulse response of the recursive filter described by equations (31) is shown in Figure 15, for the pole and zero positions of Figure 9, and the impulse response of the recursive filter described by the same equations, but with pole and zero positions described in the diagram of Figure 10 is shown in Figure 16.

We note as before that the filter might be approximated satisfactory with about 150 to 200 terms.

Following the same procedure as in the eight pole bandpass recursive filter, I used the circular and elliptical configurations for six, ten, and twelve poles in the z-plane in order to get a more exact knowledge of the effects of the number of poles and zeros in the z-plane on the impulse response, and therefore, on the amplitude and phase response of the digital filter.

Comparing the results, I found the following differences: as the number of poles and zeros is increased the length of the flat response is increased, and the slope outside of the passband is increased.
Figure 13. Realizing the eight-pole digital filter with four parallel two-pole filters.
Figure 14. Block diagram of eight-pole bandpass filter with four cascade two-pole filters.
Figure 15

Figure 16
HIGH-PASS RECURSIVE FILTER WITH
FIFTEEN POLES IN THE Z-PLANE

Cascade Form

Suppose now that we want to construct a high-pass filter to attenuate ground roll for all frequencies below 20 cps. This can be accomplished by taking poles and zeros at equally spaced intervals. All zeros are placed at $Ro = 1.0$. A satisfactory approach for a one millisecond sample interval will be to take poles and zeros at either one-half or one-degree intervals from 0 to 7 degrees.

Figure 17. Pole and zero diagram for a high-pass recursive digital filter.
If we call \( z_p \) the positions of the poles and \( z_o \) the positions of the zeros in the z-plane, taking one degree interval as shown in Figure 17 and \( R_p = 1.03 \), gives the position of the poles and zeros shown in Table I.

**TABLE I**

| \( z_{p1} \) | \( 1.022326 \pm j 0.125526 \) |
| \( z_{p2} \) | \( 1.024355 \pm j 0.107666 \) |
| \( z_{p3} \) | \( 1.026075 \pm j 0.089775 \) |
| \( z_{p4} \) | \( 1.027487 \pm j 0.071853 \) |
| \( z_{p5} \) | \( 1.028589 \pm j 0.053910 \) |
| \( z_{p6} \) | \( 1.029372 \pm j 0.035947 \) |
| \( z_{p7} \) | \( 1.029845 \pm j 0.017974 \) |
| \( z_{p8} \) | \( 1.03 \) |

| \( z_{o1} \) | \( 0.99255 \pm j 0.12187 \) |
| \( z_{o2} \) | \( 0.99452 \pm j 0.10453 \) |
| \( z_{o3} \) | \( 0.99619 \pm j 0.08716 \) |
| \( z_{o4} \) | \( 0.99756 \pm j 0.06976 \) |
| \( z_{o5} \) | \( 0.99863 \pm j 0.05234 \) |
| \( z_{o6} \) | \( 0.99939 \pm j 0.03490 \) |
| \( z_{o7} \) | \( 0.99985 \pm j 0.01745 \) |
| \( z_{o8} \) | \( 1.0 \) |

If we express the digital filter as a product of ratios...
of first or second degree polynomials in $z$; that is,

$$ F_n(z) = \frac{A_n(z)}{B_n(z)} \quad (33) $$

where

$$ A_n(z) = 1 - a_n z + b_n z^2 $$

$$ B_n(z) = 1 - a_n z + b_n z^2 \quad n = 1, 2, 3, \ldots, 7 $$

$$ A_8(z) = 1 - z \quad B_8(z) = 1 - a_8 z $$

$$ a_{ni} = \frac{2 \text{Re}[z_{en}]}{|z_{en}|^2} \quad b_{ni} = \frac{1}{|z_{en}|^2} $$

$$ a_{n2} = \frac{2 \text{Re}[z_{en}]}{|z_{en}|^2} \quad b_{n2} = \frac{1}{|z_{en}|^2} $$

$$ a_{82} = \frac{\text{Ro}}{\text{Rp}} \quad n = 1, 2, 3, \ldots, 7 $$

then

$$ F(z) = \prod_{n=1}^{8} F_n(z) $$

The recursive equations of the filter given by equation (33), if $x_n$ is the input and $y_n$ the output to the system

$$ c_n = x_n - a_{11} x_{n-1} + b_{11} x_{n-2} + a_{12} c_{n-1} - b_{12} c_{n-2} $$

$$ d_n = c_n - a_{21} c_{n-1} + b_{21} c_{n-2} + a_{22} d_{n-1} - b_{22} d_{n-2} $$

$$ e_n = d_n - a_{31} d_{n-1} + b_{31} d_{n-2} + a_{32} e_{n-1} - b_{32} e_{n-2} $$

$$ f_n = e_n - a_{41} e_{n-1} + b_{41} e_{n-2} + a_{42} f_{n-1} - b_{42} f_{n-2} \quad (34) $$

$$ g_n = f_n - a_{51} f_{n-1} + b_{51} f_{n-2} + a_{52} g_{n-1} - b_{52} g_{n-2} $$

$$ r_n = g_n - a_{61} g_{n-1} + b_{61} g_{n-2} + a_{62} r_{n-1} - b_{62} r_{n-2} $$

$$ s_n = r_n - a_{71} r_{n-1} + b_{71} r_{n-2} + a_{72} s_{n-1} - b_{72} s_{n-2} $$

$$ y_n = s_n - s_{n-1} + a_{81} y_{n-2} $$
Pole radii used for the above filter were \( R_p = 1.0025, 1.008, 1.01, 1.03, 1.05, 1.09, 1.12 \). The amplitude responses corresponding to three of these radii are shown in Figure 18.

We notice from Figure 18 that as we move the poles close to the unit circle the cutoff becomes sharper, but the rejection inside of the bandstop decreases. On the other hand, as we move the poles away from the unit circle the rejection inside of the bandstop increases, but the cutoff becomes less sharp.

We can have better rejection inside of the rejection-band, and a sharper cutoff if poles nearest the ends of the rejection-band are displaced toward the unit circle.

Figure 19 shows a block diagram of the realization of the high-pass recursive filter given by equations (34).
Figure 18. Amplitude response of high-pass recursive digital filters. (A) Rp = 1.003
(B) Rp = 1.05. (C) Rp = 1.12
Figure 19. Block diagram of a fifteen-pole bandstop filter in cascade form.
**Parallel Form**

If we want to design a recursive digital filter that rejects all frequencies between \( f_1 \) and \( f_2 \), the choice of which of the two forms we should use, depends on the complexity of the filter function. Generally, simulation by the parallel form requires considerably greater accuracy in the determination of the filter parameters than in the cascade form. This choice also depends largely on how that parallel method is applied.

If we express the digital filter as a ratio of two polynomials, considerable care must be taken in the computation of the poles and zeros, as this computation involves subtraction of nearly equal numbers. Another way would be to design the filter in the z-plane, placing poles and zeros in the appropriate places.

For realization in the parallel form we express \( F(z) \) as a partial fraction expansion

\[
F(z) = \sum_{n=1}^{N} \frac{a_{zn} + a_{zn}z}{1 + b_{zn}z + b_{zn}z^2}
\]  

(35)

Each rational function in the above summation can be synthesized by the recursive structure shown in functional
block diagram form in Figure 20. This recursive structure uses only two delays, four multiplications, and five additions. The complete realization of (35) is shown in Figure 21.

As an example, consider the design of a particular bandstop filter. The filter is to exhibit at least 75 db loss in the rejection band which extends between 30 cps and 330 cps.

The poles and zeros for the transfer function of this bandstop recursive digital filter are listed in Table II. The bandstop filter has been normalized, and the sample interval for this example was 30 ms.

<table>
<thead>
<tr>
<th>TABLE II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles</td>
</tr>
<tr>
<td>Rp = 1.03</td>
</tr>
<tr>
<td>( \omega \Delta t = 10.8 \quad 21.6 \quad 32.4 \quad 43.2 \quad 54.0 )</td>
</tr>
<tr>
<td>64.8 \quad 75.6 \quad 86.4 \quad 97.2 \quad 108.0 \quad 118.8</td>
</tr>
</tbody>
</table>

| Zeros    |
| Ro = 1.0 |
| \( \omega \Delta t = 10.8 \quad 21.6 \quad 32.4 \quad 43.2 \quad 54.0 \) |
| 64.8 \quad 75.6 \quad 86.4 \quad 97.2 \quad 108.0 \quad 118.8 |

If we use the poles and zeros of the Table II in
Figure 20. Block diagram of one term of the partial fraction expansion.

Figure 21. Realization of a bandstop recursive digital filter in parallel form.
equation (35), and apply this equation as recursive equations on the block diagram of Figure 21, we obtain the frequency response shown in Figure 22.
Figure 22. Frequency response characteristics of the sampled data bandstop recursive filter
PHASE RESPONSE

We now consider the phase characteristics of recursive digital filters. In general these filters do not have zero or linear phase spectra. However, one of the advantages of digital filters lies in the possibility of developing filtering operations with zero-phase-shift, if the input time series is finite in length. Using recursive digital filters, two techniques can be used to produce zero phase filters:

The first technique consists in filtering the input data with a recursive filter \( F(z) \) in the normal manner to produce a first output. This output is then reversed in time and filtered with the same recursive filter to produce a second output. Then this second output is reversed in time to obtain the final output. The phase response of this filtering operation is zero for all frequencies, while the amplitude spectrum is the square of the spectrum of \( F(z) \).

It is not actually necessary to reverse the data to achieve reverse time filtering. Reversing in time the input to the filter \( F(z) \), is equivalent to replace \( z \) in \( F(z) \) with \( 1/z \). Therefore, the above filtering operation can be performed by passing the input time function through the filter \( F(z) \), and then passing the output through the
The second technique consists in filtering the input time function with the filter \( F(z) \), and adding to this result the input filtered separately with the reverse time filter \( F(1/z) \).

As an example, consider the filter given by equation (29)

\[
F(z) = \frac{(1 - z^\ell)^q}{B(z)B(z)B(z)B(z)} \tag{29}
\]

Replacing \( z \) with \( 1/z \) we have

\[
F(1/z) = \frac{[1 - (1/z)^\ell]^q}{B_1(1/z)B_2(1/z)B_3(1/z)B_4(1/z)} \tag{36}
\]

If \( X(z) \) is the input and \( Y(z) \) is the output of the filter

\[
Y(z) = F(1/z)X(z)
\]

Since \( 1/z \) represents a time advance, we may use the same argument as we did for equations (32), to write the recursive equations

\[
c_n = x_n - x_{n+2} + a_1 c_{n+1} - b_1 c_{n+2}

d_n = c_n - c_{n+2} + a_2 d_{n+1} - b_2 d_{n+2}

e_n = d_n - d_{n+2} + a_3 e_{n+1} - b_3 e_{n+2} \tag{37}
\]
\[ y_n = e_n - e_{n+2} + a_n y_{n+1} - b_n y_{n+2} \]

If we take pole and zero positions given in Figure 10, as an example; the zero phase response we get is as shown in Figure 23.
Figure 23
APPLICATION OF RECURSIVE DIGITAL FILTERS TO SEISMOLOGY

The objective of geophysical data processing is to enhance data which has already been acquired. It is important to note that processing techniques can add no information to data which is not already present in the data.

The actual process of filtering must be designed for the particular noise and signal characteristics of the seismic records. The design of a filter involves the use of the frequency spectrum to describe the signal and noise. Once having described the signal and noise, we use the results to design a filter to enhance the data.

I analysed the frequency spectrum of three seismic traces of different records, and the results were similar. The frequency of the main signals were at about 25 cps. Ground roll was not evident in these records.

Having the frequency spectra of the signal and noise determined, I designed different bandpass recursive filters with cutoff frequencies between 20 cps and 36 cps, for filters of eight, ten, and twelve poles. Figure 12 shows the amplitude of one eight-pole filter with the above characteristics.

Eight, ten and twelve pole bandpass filters were
applied to a seismic section of fifty records, recorded by the Department of Geophysics in South Park, Colorado.

The execution time of the computer in filtering the central traces and preparing the plot tape was 9 min. 3 sec.; 10 min. 43 sec.; and 12 min. 23 sec.; for the eight, ten, and twelve pole filters, respectively.

Figure 24 shows the central traces of the seismic section without filtering. Figure 25 shows an enhancement of this section after the eight-pole recursive filter has been applied.
BIBLIOGRAPHY


