A THREE-DIMENSIONAL PHOTOELASTIC
STRESS ANALYSIS OF AN UNDERGROUND CRUSHER CHAMBER

by

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A Thesis submitted to the Faculty and the Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science.

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Climax Molybdenum Company has used a two-dimensional finite element method of stress analysis in designing an underground crusher chamber. To check their solution a three-dimensional stress analysis has been obtained by the use of a photoelastic model.

A scale model of the crusher chamber was constructed of photoelastic material. It was then loaded using the stress-freezing method and then sliced to obtain critical planes of interest. The critical planes were then analyzed on a comparator-polariscope using the shear difference method and the results obtained from the analysis agree well with those obtained by Climax.
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INTRODUCTION

The mining engineers of Climax Molybdenum Company have developed a computerized two-dimensional finite element method to determine the state of stress in the rock surrounding underground openings.

The method was applied to designing the underground primary crushing station on the 600 level to determine the state of stress in the rock surrounding underground openings. To determine this, a series of experimental designs was investigated by analyzing cross sections with the finite element method. After the analysis the cross section with minimum stress concentrations was chosen, and the design was developed from it.

The crusher chamber being studied is presently under construction. The engineers at Climax want to check their finite element method results with a three-dimensional method to ascertain whether or not their results are within an allowable design error of approximately ±20%. If the
results of this analysis verify results obtained by Climax, the engineers will have more confidence in the use of the finite element method for future design purposes.
RESEARCH AND DEVELOPMENT

A visit was made to the Industrial Engineering Department of Climax Molybdenum Company to confer with engineers in order to correlate research, obtain construction plans, and examine the actual crusher chamber while under construction.

Stress-Freezing Method

A detailed study was made of three-dimensional analysis methods applicable to the problem. Time, apparatus available, and funds available were considerations in determining the method of analysis used. The stress-freezing method was chosen primarily due to the fact that a large high-temperature oven was available in the mining research laboratory.

The stress-freezing method of three-dimensional photoelasticity is based upon the process of permanently locking induced deformations into a model.

As described by Dally (1956) some polymeric materials such as epoxy resins exhibit a diphase behavior upon heating. The polymeric materials are composed of long-chain hydrocarbon molecules of which some of the molecules are bonded into a strong network of three-dimensional bonds, and the remainder are weakly bonded into shorter secondary chains.
As the temperature of the polymer is increased, the secondary bonds break down and the entire load is carried by the primary bonds, which are elastically deflected due to the load.

As the temperature is lowered, the secondary bonds reform between the deformed primary bonds and thus lock the primary bonds into their deformed positions. When the load is removed, the elastic deformation of the primary bonds is permanently locked into the model by the deformed secondary bonds.

These deformations are locked in on a molecular scale so that the model can be sliced to remove planes of interest. These slices can be examined individually without changing the stress state in order to determine the state of stress existing in that particular plane by using a two-dimensional method such as the shear difference method.

**Testing Epoxy Resins**

It was necessary to decide which photoelastic materials would be best suited for this application. There are many important properties of a photoelastic material which must be considered when selecting a material for stress-freezing. Material properties are listed in Appendix I.

After a survey of available photoelastic materials, it was decided to use either Ciba "Araldite 502" resin plus "Araldite 591" hardener or Union Carbide "Bakelite 2774" resin plus Baker phthallic anhydride hardener. The
"Araldite 502" plus "Araldite 591" combination proved to be too exothermic for three-dimensional application -- the exothermic reaction causing large thermal stresses in the interior of the castings with a thickness greater than one inch. Testing details are given in Appendix II.

The "Bakelite 2774" resin plus Baker phthallic anhydride hardener proved to be a much better combination. Eastman dibutyl phthalate plasticizer was added to make the final material more easily machineable.

Several tests were then performed to determine the best ratio of resin/hardener/plasticizer necessary to obtain the best material properties for this particular model application. The proportion 100/55/15 by weight in parts per hundred was determined to be the optimum.

Other tests were conducted to determine the minimum critical temperature of 100° C. necessary for secondary-bond breakdown. The material fringe value $f_\sigma$ was calibrated to be two psi/fringe/inch as shown in Appendix II.

Modification of Temperature Controller

In using the stress-freezing method it is imperative that an oven be used in which the temperature can be accurately controlled for extended periods of time. The mining research laboratory is equipped with a Gehnrich 500° F. oven controlled by a Bristol manual temperature controller shown in Figure 1. Since required temperature increase and
decrease rates of 2° C./hour for 70 hours could not be obtained with the existing unit, it was converted to cam control by using modified parts from other controllers and by fabricating the remaining necessary parts. Control cams were cut from .0625" thick plexiglass.
Figure 1  Temperature controller, large oven, and pressure apparatus

Figure 2  Small model, mold, and loading frame
SMALL MODEL

In order to develop a procedure and determine the problems that would exist with the crusher chamber model (14"x8"x8"), a model was made of a spherical cavity in a 4"x4"x4" cube of epoxy resin as shown in Figure 2.

Aluminum plate of 0.25" thickness was used for the model mold because aluminum is light, easy to machine, and releases well from the epoxy resin. To obtain a spherical void in the model, a core material was required that could be capable of being compressed slightly due to the shrinkage of the resin upon gelation. It was necessary for the core to be capable of withstanding temperatures in excess of 130° C. Rosswax 160° C. wax was tested using lead shot to increase the density, but the compressive strength of the wax was so high that internal stress was developed in the resin upon gelation of the resin in Small Model #2. Dow Corning "Silastic RTV" silicone rubber was found to be softer and stronger so it was used for the core. 1/

1/ The purpose of attempting to use wax was to allow easy removal of the core in the crusher chamber model. After the resin had gelled and had been removed from the mold, the
model would be heated until the wax melted and flowed from the tunnel.

Suspension of the core during the casting of the model was initially attempted with fine rubber thread, but the temperature affected the tensile strength to such a great extent that the thread broke in the Small Model #1. Monofilament nylon thread of 0.005" diameter was tested and found to be satisfactorily stable at high temperatures and amply elastic as well.

**Casting Procedure**

The casting procedure for the 100/55/15 epoxy resin is as follows:

1. Assemble and seal the mold and place in oven.
2. Suspend the core and set clearances.
3. Preheat mold and core to 100° C.
4. Heat resin to 130° C.
5. Heat hardener and plasticizer to 100° C.
6. Add hardener to resin gradually by stirring.
7. After all the hardener has dissolved, add plasticizer.
8. Measure core clearances regularly by gauge until gelation while maintaining oven temperature at 100° C.

The hardener is normally obtained in the form of solid flakes and should be ground with a mortar and pestle or
electric blender to facilitate dissolving. According to Leven (1956, Fig. 2) the melting point of the phthallic anhydride in a 100/55 concentration is 117° C., but precipitation will occur below 97° C. so the temperature of the mixture must be carefully controlled by gradual addition of the hardener to the resin with stirring. 

2/ Warning -- The phthallic anhydride vapor is extremely toxic, and a respirator and goggles should be worn when handling the hardener at high temperatures. The dibutyl phthalate plasticizer should not be subjected to temperatures in excess of 140° C. because vaporization occurs which causes stresses in the resin.

The mixture begins to gel after about 10 hours, and once begun, gelation will be complete within a few hours; thus the clearances of the core should be carefully measured at least an hour before gelation begins. The clearance gauge should not be inserted once gelation begins.

Loading

The small loading frame is shown in Figure 2. A 1" diameter hydraulic cylinder was used to develop the loading force. Aircraft type high pressure flexible tubing connects the cylinder to the pressure pump outside the oven. Figure 7 shows the pressure pump, the two pressure gauges and a surge tank.
Two pivot blocks, with their axes 90° apart, are placed between the loading platen on the model and the loading bar of the loading frame to assure uniform application of the load to the model.

Small Model #3 was successfully cast and loaded to produce uniform fringe patterns. Some leakage of the cylinder occurred because the rubber seals hardened after having been exposed to high temperatures for prolonged periods of time. To prevent leakage problems in the large loading frame to be used on the crusher chamber model, a hydraulic cylinder with teflon seals was obtained.
CRUSHER CHAMBER MODEL

The exterior dimensions of the actual crusher chamber are 195'x60'x60'. A model scale of 1" = 40' was chosen. To prevent edge effects, the distance between the chamber wall and the outside edge of the model should be at least three times the chamber radius.

Thus exterior model dimensions of 14"x8"x8" were determined as shown in Figure 3.

Model Mold and Core

The crusher chamber model mold in Figure 5 was constructed similar to the small model mold. Aluminum plate of 0.25" thickness was used for the sides and ends. Aluminum angle 1.5"x1.5" was used to reinforce the sides and ends to prevent buckling. The ends are held in place by four 0.375" diameter steel all-thread rods. A base plate of 0.25" aluminum is secured by eight 0.25" diameter bolts.

The core was made by casting Dow Corning RTV silicone rubber around a lead core in a laminated core-mold composed of five sheets of machined plexiglass. The core and its mold are shown in Figure 6. Three nylon suspension threads were attached to the lead core and placed through three 0.0312"
Figure 3 Crusher chamber model
Figure 4
Plan view of crusher chamber model showing slices analyzed
Figure 5  Crusher chamber model mold and loading frame

Figure 6  Core and core mold
diameter holes drilled in the top sheet of plexiglass. One large conical hole was drilled in the top sheet of plexiglass to allow injection of the liquid rubber. After the rubber was injected and vulcanized, the bolts were removed, and the plexiglass sheets were separated to release the core.

**Casting**

The epoxy resin components are combined and cast in the procedure previously described. After gelling and cooling, the model was removed from the mold and sliced to remove the core. The cutting was accomplished with a Do-All band saw using a precision blade and a speed of 1600 fpm. The area being cut was cooled with a water jet, and was fed very slowly in order to prevent thermal stresses in the model. The cut was made 1.25" from the end of the chamber.

After removing the core the model was glued back together with a mixture of 100/12 "Bakelite 2774" resin and "Araldite 591" hardener.

**Loading**

To load the model under conditions similar to those acting on the actual crusher chamber. \( \sigma_z = 1300 \text{ psi}, \sigma_x = y = 800 \text{ psi} \) A vertical stress \( \sigma_z = 32.5 \text{ psi} \) was imposed by a hydraulic cylinder loading frame as shown in Figure 5. Pivot blocks were used to insure uniform loading.

It was intended that by loading the model in the mold so that the sides of the model were restrained, the Poisson's
ratio \((v = 0.38)\) of the resin would cause an effective \(\sigma_x\) and \(\sigma_y\) equal to 20 psi. The ratio of the actual stress acting on the crusher chamber is:

\[
\frac{\sigma_z}{\sigma_x} = \frac{1300 \text{ psi}}{800 \text{ psi}} = 1.625
\]

and

\[
\sigma_x = \sigma_y = \left(\frac{V}{1-V}\right) z = \left(\frac{0.38}{1-0.38}\right) 32.5 = 20 \text{ psi}
\]

Contraction of the model upon gelation left gaps between the model and the mold on the sides and ends. The model was removed from the mold and the model surfaces were sanded and polished. After placing the model back into the model, the gaps amounted to 0.125" on each side and 0.200" on each end. The gaps were filled with liquid RTV silicone rubber which then vulcanized to a solid.

The model and mold were placed back into the loading frame and the loading cycle was begun. The oven temperature was raised to 100° C. at a rate of 2° C./hour at which time the load (32 psi) was applied. The load was maintained throughout the remainder of the 12 hour cycle at 100° C. and the temperature decrease rate of 2° C./hour.

**Model Failure**

The model was removed from the mold and found to have a large horizontal fracture originating at the bottom of the roof arch of the model and extending to the edges of the model. The fracture, as seen in Figure 8, apparently resulted from high lateral stresses developed by expansion of the resin.
Figure 7  Pressure apparatus

Figure 8  Crusher chamber model #1 after failure

Figure 9  Flatjacks being placed between mold and model
Second Crusher Chamber Model

A second model of the crusher chamber, identical to the first model, was then constructed. To produce uniform controlled loading of the sides and ends of the model, flatjacks were fabricated from 0.010" brass shim stock. Figure 9 shows how the flatjacks were inserted between the model and mold wall. The flatjacks were connected by steel tubing which was in turn connected to flexible tubing passed through the oven top to a helium bottle and regulator, with two precision gauges to control the pressure. (Figure 7)

The vertical loading was produced by the hydraulic cylinder as in the first model. Planes of interest 0.25" thick were sliced from the model, sanded, and polished.
Figure 10 Comparator-polarscope

Figure 11 Detail of polarscope showing slice 20 in analysis position
Figure 12 Isochromatic fringe pattern of Slice 20 showing shear grids

Figure 13 Isochromatic fringe pattern of Slice 20
ANALYSIS

The analysis was performed on a Kodak 10 power optical comparator-polariscope at the photoelastic laboratory of the Rock Mechanics section, Denver Mining Research Center, U. S. Bureau of Mines. (Figures 10, 11, 12)

Shear-Difference Method

The shear-difference method of separating the principal stresses was used as described by Frocht (1965, p. 252-286). Grid lines I through VII were drawn on the slice to be analyzed using 0.05" grid spacing. The analysis was begun on the boundaries of the crusher chamber, and the isoclinic angle and isochromatic fringe order was determined at each corner of the grid squares.

The basic formulas used are:

\[
(\sigma_x)_i = \sigma_{x_0} \pm \sum \left( \frac{\Delta T_{xy}}{\Delta x} \right) \Delta y
\]

\[
\sigma_y = \sigma_x \pm [(P-Q)^2 - 4 \tau_{xy}^2]^{1/2}
\]

where \( \tau_{xy} = \left( \frac{P-Q}{2} \right) \sin 2\theta \)

and \( \theta = \) isoclinic angle

The data from shear grids I and VII are given in Appendix IV.
Error Analysis

An error analysis was conducted to determine the errors introduced by the stress-freezing procedure and by analysis.

The results from Appendix III are given below.

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<th>Applied Vertical Load</th>
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<tr>
<td>Max. $f_{\sigma_h}$ = 4.18%</td>
<td>Max. $f_{\sigma_v}$ = 1.08%</td>
</tr>
<tr>
<td>Most prob. $f_{\sigma_h}$ = 3.33%</td>
<td>Most prob. $f_{\sigma_v}$ = 0.077%</td>
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$\frac{f_{\sigma_v}}{\sigma}$

Max. $f_H = 5.26%$
Most prob. $f_H = 3.33%$

$f = \text{fractional error}$

Random points were chosen on two grid lines to determine what fractional errors were present in the shear-difference analysis.

Example 1 Shear grid IV, point 10
Max. $f_{\sigma_x} = 28.2%$
Most prob. $f_{\sigma_x} = 16.3%$

Example 2 Shear grid II, point 22
Max. $f_{\sigma_x} = f_{\sigma_y} = 13.5%$
Most prob. $f_{\sigma_x} = f_{\sigma_y} = 10.4%$
CONCLUSIONS

Four slices were cut from the crusher chamber model as shown in Figure 4. The results of the analysis of the slices follows.

Slice 20

A complete shear-difference analysis was conducted on Slice 20 and the results are shown in Figures 17 and 18 which are $\sigma_x$ and $\sigma_y$ contours in psi.

The results obtained by Climax are shown in Figures 15 and 16. The maximum stresses (4000 psi) obtained by the finite element method occur at the two outside corners. The stress concentration factor is 3.08 times the applied vertical stress of 1300 psi.

The maximum stress obtained by photoelastic analysis is 96 psi which is 3.00 times the applied vertical stress of 32 psi. The location of these maximum stresses is the same as found by Climax for the upper corner and the other maximum stress is at the apex of the bottom corner.

Slice 24

Slice 24 was taken across the chamber 0.25" from the end to determine end effects. The stress state in this location
Figure 15: Horizontal stress contours obtained by finite-element analysis.

Figure 16: Vertical stress contours obtained by finite-element analysis.
Figure 17 Horizontal stress contours obtained by photoelastic analysis
Figure 18 Vertical stress contours obtained by photoelastic analysis
cannot be analyzed by the two-dimensional finite element method. The stress distribution was found to be nearly identical to that of Slice 20, but the maximum stresses were 72 psi which is 25% lower.

**Slice 16**

This slice was cut through the center of the drift to obtain an idea of how the stress state surrounding the chamber is altered by such an opening. Apparently the drift has little effect on the overall stress state. This result is probably due to the fact that the volume of the drift is so much smaller than the volume of the crusher chamber in the area affected.

**Slice E10**

This slice, through the end of the crusher chamber along the axis, was analyzed for maximum stress at the crusher chamber boundary. The maximum stress was 128 psi at the bottom corner and the other high stress was 120 psi at the upper corner. Thus the stress (128 psi) at the lower corner is 25% greater than the maximum stress (96 psi) in Slice 20. Consequently further investigation should be carried out with the finite-element method to analyze the stress along the long axis of openings.

The results obtained by the method of photoelastic analysis agree well with those obtained by the finite-element
method. The amounts and locations of maximum stress concentrations are nearly identical. It is the author's opinion that the finite-element method in use by Climax can produce accurate, flexible, and rapid results for engineering design.
APPENDIX I

PHOTOELASTIC MATERIAL PROPERTIES
Frocht (1965)

1. The material must be transparent to the type of light used in the polariscope.
2. The material should have high optical sensitivity, indicated by a low fringe value for stress $f_c$.
3. The material should have a high ultimate strength.
4. The material should have linear properties of
   (a) Stress-fringe order
   (b) Strain-fringe order
   (c) Stress-strain
5. The material should be free of residual stress, time edge effects, and optical or mechanical creep.
6. A liquid material must be castable and a solid material must be machineable.
7. The resin should not be highly exothermic upon reaction with the hardener.
8. The material should not be excessively expensive.
APPENDIX II

TESTING AND CALIBRATION

Initial material tests were performed by casting disks 3" in diameter and 1" thick. Aluminum pipe of 3" inside diameter was cut into 1.5" high sections for molds. Bases for the molds were 4"x4" squares of 0.5" thick aluminum plate.

The contact between the mold and base plate was sealed with Pergum #5782 (plastic duct seal) manufactured by Press-tite Division, Interchemical Corp.

Dow Corning #7 Silicone Release Agent was used to coat the mold interior surfaces.

A 3" diameter test disk with 100/55/15 composition was sliced into 0.25" thick disks. The surfaces of the disks were sanded with wet or dry emery paper and then polished with pumice on a buffing wheel.

Two separate calibration tests were performed by diametrically loading the disks at 100° C. in a small oven (12"x12"x14" inside dimensions). Weights were added successively as shown in Figure 19 and the fringe order was determined at each load. Frocht (1965, p. 146) gives the formula for the stress at the center of a diametrically loaded
Figure 19 Small oven and calibration disk

Figure 20 Isochromatic fringe pattern of Slice 16
disk as

\[ p-q = \frac{8P}{tD} \]

Where \( p \) and \( q \) are the maximum and minimum stresses, \( P = \) load in pounds, \( t = \) thickness of the disk and \( D = \) diameter of the disk.

From Obert and Duvall (1967, p. 372) \( p-q = FN \).

The model fringe value \( M \) is equal to the material fringe value \( f \) divided by the thickness \( t \) thus:

\[ p-q = \frac{8P}{\pi tD} = FN = \frac{fN}{t} \]

and

\[ f = \frac{8P}{\pi DN} \]

And for the calibration disk as shown in Figure 19, \( N = 6 \) fringes at center of disk, \( P = 14.1 \) pounds, \( D = 3" \). Thus

\[ f = \frac{8 \times 14.1}{\pi(3)(6)} = 2 \text{ psi/ fringe/in} \]
APPENDIX III

ERROR ANALYSIS

Applied Stress

Horizontal

An error in the applied horizontal stress was introduced due to the fact that the bladder was not in total contact with the sides of the model at the top 0.25" and bottom 0.25". The pressure gauge accuracy was ±0.20 psi.

\[
H = \frac{P_f \cdot h_f \cdot w_f}{h_s \cdot w_s}
\]

\[
f_H = \frac{dH}{H} + \frac{dP_f}{P_f} + \frac{dh_f}{h_f}
\]

\[
\pm \left( \frac{.20 \text{ psi}}{20 \text{ psi}} \right) + \left( \frac{.250''}{7.875''} \right)
\]

Max. \(f_{\sigma_H} = \left| .01 + .0318 \right| = 4.18\%

Most prob. \(f_{\sigma_H} = \left[ (.01)^2 + (.0318)^2 \right]^{1/2} = 3.33\%

\]

\[
p_f = \text{pressure in flatjack}
\]

\[
h_f = \text{height of flatjack in contact}
\]

\[
w_f = \text{width of flatjack in contact}
\]

\[
h_s = \text{height of slice}
\]

\[
w_s = \text{width of slice} = w_f
\]

\[d = \text{error}
\]

\[f = \text{fractional error}\]
Vertical

Error was introduced in the applied vertical stress due to a mistake in calculating pressure. Actual pressure applied to the hydraulic cylinder (R = 1") was 1100 psi. Gauge accuracy = ± 5 psi.

\[ \sigma_v = \frac{(p_A) \text{(Area of cylinder)}}{\text{Area of model}} = \frac{1100 \text{ psi} \ [\pi(1)^2]}{13.755 \times 7.795} = 32.3 \text{ psi} \ (0.2 \text{ psi less than desired}) \]

\[ f_{\sigma_v} = d\sigma_v = (d p_g) + (d p_A) = \pm 5 \text{ psi} + (0.2 \text{ psi}) \]

Max. \[ f_{\sigma_v} = \left| \frac{0.00455 + 0.00620}{0.075} \right| = 0.075\% \]

Most prob. \[ f_{\sigma_v} = \left[ (0.00455)^2 + (0.00620)^2 \right]^{\frac{1}{2}} = 0.077\% \]

Horizontal/Vertical Ratio

Max. \[ f_{\frac{H}{V}} = \left| f_{\sigma_H} + f_{\sigma_V} \right| = \left| 0.0418 + 0.0108 \right| = 0.0526\% \]

Most prob. \[ f_{\frac{H}{V}} = \left[ (f_{\sigma_H})^2 + (f_{\sigma_V})^2 \right]^{\frac{1}{2}} = 3.33\% \]

Shear-Difference Analysis

The general formulas used in solving for \( \sigma_x \) and \( \sigma_y \) are divided into lettered parts for easier error analysis.

\[ \sigma_x = \sigma_{x_0} \pm \sum \left[ \left[ \frac{P-Q}{2} \sin 2\theta \right] - \left[ \frac{P-Q}{2} \sin 2\theta \right] \right] \frac{\Delta X}{\Delta Y} \]

\[ \begin{array}{cccccc}
A & B & C & D & E & F \\
\rightline{H} & G & \end{array} \]

\[ F \]

The values of A and B are obtained from the upper or left shear lines. Values of C and D are obtained from the lower or right shear lines. Values of I, K, and L are obtained from the center shear line. The error in measuring \( \theta \) is \( \pm 1^\circ \), and the error in \((P-Q)\) is 10 fringe order or 0.056 fringes.

The error in drawing the shear grids is \( \pm .01" \).

The fractional error in \( \sigma_x \) is

\[
\frac{f_{\sigma_x}}{\sigma_x} = \frac{df + dG_1 + dG_2 + dG_3 \ldots \ldots dG_N}{F + G_1 + G_2 + G_3 \ldots \ldots G_N}
\]

Thus a formula is obtained which gives the fractional error of \( \sigma_x \) at any point along the shear grids.

**Example 1** Shear grid IV, point 10 (Appendix IV)

1. Error in \( \sigma_x \). Because the grid begins on a vertical free surface and the X axis is perpendicular to the free surface \( \sigma_{x_0} = 0 \). Thus

\[
\frac{f_{\sigma_x}}{\sigma_x} = \frac{df + dG}{F + G_1 + G_2 + G_3 \ldots \ldots G_N} = \frac{df}{F + G_1 + G_2 + G_3 \ldots \ldots G_N} = \frac{f_{G}}{\sum G + 1}
\]
and \( f_G = f_H + f_E \)

where \( f_H = \frac{d(AB) + d(CD)}{AB + CD} \)

\[
d(AB) = AB \left( \frac{dA}{A} + \frac{dB}{B} \right) = (P-Q) \sin 2 \theta \left[ \frac{0.052}{2} + \frac{\sin 2 \theta - \sin 2 (\theta + 1)}{\sin 2 \theta} \right]
\]

\[
= (0.711) \sin 2 (74^\circ) \left[ 0.028 + \frac{\sin 2 (74^\circ) - \sin 2 (74^\circ + 1^\circ)}{\sin 2 (74^\circ)} \right]
\]

\[
= 0.0361
\]

Similarly

\[
d(CD) = dC = dB = (0.906) \sin 2 (71^\circ) \left[ \frac{0.052}{2} - \frac{\sin 2 (71^\circ) - \sin 2 (71^\circ + 1^\circ)}{\sin 2 (71^\circ)} \right]
\]

\[
= 0.0425
\]

Substituting the values obtained into the equation for \( f_H \),

\[
f_H = \frac{d(AB) + d(CD)}{AB + CD} = 0.0361 + 0.0425 = 0.082
\]

Thus \( f_E = \frac{d\Delta x}{\Delta x} + \frac{d\Delta y}{\Delta y} = 0.01 + 0.01 = 0.200 \)

Max. \( \sigma_x = \left| f_G + f_E \right| = \left| 0.082 + 0.200 \right| = 28.2\% \)

Most prob. \( \sigma_x = f_G = \left( f_H \right)^2 + (f_E)^2 \left[ (0.082)^2 + (0.20)^2 \right]^{1/2} = 16.3\% \)
2. Error in $\sigma_y$

\[ f_{\sigma_y} = \frac{d\sigma_x + dM}{\sigma_x + M} \]

where \( dM = \frac{1}{2} \left[ \frac{dI}{I} + d(KL) \right] \)

\[ f_I = 2 \left[ \frac{d(P-Q)}{P-Q} \right] = 2 \left[ \frac{0.056}{1.811} \right] = 0.0618 \]

\[ f_{KL} = 2 d \left[ \frac{P-Q}{2} + \frac{d \sin 2\theta}{\sin 2\theta} \right] = 2 \left[ \frac{0.028}{0.906} \right] \]

\[ + \frac{\sin 2 (70^\circ) - \sin 2 (70^\circ + 1^\circ)}{\sin 2 (70^\circ)} = 0.146 \]

Thus

\[ dM = \frac{1}{2} \left[ \frac{0.0618 (1.811) + 0.146 (0.583)}{1.811 + 0.583} \right] = 0.0308 \]

Max. $f_{\sigma_y} = (\sigma_x \cdot f_{\sigma_x}) + (M \cdot f_M) = \frac{1.823 (0.282) + 1.386 (0.0308)}{1.823 + 1.386} = 17.4\%$

Most prob. $f_{\sigma_y} = (\sigma_x \cdot \text{Most prob. } f_{\sigma_x}) + (M \cdot \text{Most prob. } f_M) = \frac{1.823 (0.163) + 1.386 (0.0294)}{1.823 + 1.386} = 10.5\%$

Example 2 Shear grid II, point 22 (Appendix IV)

The shear grid has its Y axis parallel with the free surface which is 54° measured counter clockwise from vertical. Therefore after finding $\sigma_x$ and $\sigma_y$ along the shear grid
coordinate system, the values of $\sigma_x$ and $\sigma_y$ must be converted to $\sigma_x^1$ (horizontal) and $\sigma_y^1$ (vertical).

$$\sigma_x^1 = (\cos 54^\circ) \sigma_x \pm (\cos 36^\circ) \sigma_y$$

$$\sigma_y^1 = (\sin 54^\circ) \sigma_x \pm (\sin 36^\circ) \sigma_y$$

and

$$f\sigma_x^1 = f\sigma_y^1 = \frac{d\sigma_x + d\sigma_y}{\sigma_x + \sigma_y}$$

1. Error in $\sigma_x$

$\sigma_{x_0} = 0$ because the x axis is perpendicular to the free surface, hence

$$f\sigma_x = f_G = f_H + f_E$$

$$f_{AB} = AB \left[ \frac{dA + dB}{A} \right] = \frac{P-Q}{2} \sin 2 \left[ \frac{0.028}{2} + \sin 2 - \sin 2 ( +1) \right]$$

$$= 1.486 \sin 2 (20^\circ) \left[ \frac{0.028 \sin 2 (20^\circ) - \sin 2 (20^\circ + 1^\circ)}{1.486} \right]$$

$$= 0.0568$$

$$f_{CD} = CD \left[ \frac{dC + dD}{C} \right] = 1.512 \sin 2 (20^\circ)$$

$$\left[ \frac{0.028 + \sin 2 (20^\circ) - \sin 2 (20^\circ + 1^\circ)}{1.512} \right] = 0.0574$$

therefore

$$f_H = 0.0568 + 0.0574 = 0.0593$$

$$\frac{0.955 + 0.922}{0.15}$$

$$f_E = \frac{d\Delta X + d\Delta Y}{\Delta X} \frac{.01}{.2} \frac{.01}{.1} = 0.05 + 0.1 = 0.15$$
Max.  \( f_{\sigma x} = f_{G} = \left| f_{H} + f_{E} \right| = 0.0593 + 0.15 = 20.93\% \)

Most prob.  \( f_{\sigma x} = \left( (0.0595)^2 + (0.15)^2 \right)^{\frac{1}{2}} = 16.1\% \)

2. Error in \( \sigma_y \)

\[
f_{\sigma x} = \frac{d\sigma_x + dM}{\sigma_x + M}
\]

\[
dM = \frac{1}{2} \left[ dI + \frac{d(KL)}{KL} \right] = \frac{1}{2} \left[ f_{I} \cdot I + f_{KL} \cdot KL \right]
\]

\[
f_{I} = 2 \frac{d}{P-Q} \left[ \frac{0.056}{3.000} \right] = 0.0373
\]

\[
f_{KL} = 2 \left[ \frac{d}{P-Q} \left( \frac{0.028 + \sin 2(20^0)}{\sin 2(20^0)} - \sin 2(20^0 + 10^0) \right) \right]
\]

\[
= 0.0363
\]

therefore

Max.  \( f_{\sigma y} = \left( \sigma_{x} \cdot f_{\sigma x} \right) + (fM \cdot M) = \frac{1.527 (0.2093) + 2.297 (0.0363) = 10.5\%}{1.527 + 2.297} \)

Most prob.  \( f_{\sigma y} = \left( \sigma_{x} \cdot Most \ probation \ f_{\sigma x} \right) + (M \cdot Most \ probation \ f_{M}) = \frac{1.527 (0.161) + 2.297 (0.028) = 8.12\%}{1.527 + 2.297} \)
Referring back to the equation for $f_{\sigma x}^1$ and $f_{\sigma y}^1$

$$f_{\sigma x}^1 = f_{\sigma y}^1 = \frac{(f_{\sigma x} \cdot \sigma_x) + (f_{\sigma y} \cdot \sigma_y)}{\sigma_x + \sigma_y}$$

Max. $f_{\sigma x}^1 = f_{\sigma y}^1 = \frac{0.2093 \times 1.527 + 0.105 \times 3.824}{1.527 + 3.824} = 13.5\%$

Most prob. $f_{\sigma x}^1 = f_{\sigma y}^1 = \frac{0.161 \times 1.527 + 0.0812 \times 3.824}{1.527 + 3.824} = 10.4\%$
### APPENDIX IV

#### Examples of Shear Difference Analysis Data

<table>
<thead>
<tr>
<th>Grid</th>
<th>Point</th>
<th>θ</th>
<th>P-Q</th>
<th>P-Q/2</th>
<th>sin 2θ</th>
<th>τ_{xy}</th>
<th>θ</th>
<th>P-Q</th>
<th>P-Q/2</th>
<th>sin 2θ</th>
<th>τ_{xy}</th>
<th>Δx/Δy</th>
<th>τ_{xy} Δx/Δy</th>
<th>σ_x</th>
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</thead>
<tbody>
<tr>
<td>II</td>
<td>22</td>
<td>20</td>
<td>2.972</td>
<td>1.486</td>
<td>0.643</td>
<td>0.956</td>
<td>20</td>
<td>3.023</td>
<td>1.512</td>
<td>0.643</td>
<td>0.972</td>
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<td>0.032</td>
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<td>1.422</td>
<td>0.711</td>
<td>0.530</td>
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<td>0.684</td>
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#### Center Shear Line

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<th>Grid</th>
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<th>P-Q</th>
<th>(P-Q)^2</th>
<th>P-Q/2</th>
<th>sin 2θ</th>
<th>τ_{xy}</th>
<th>4τ_{xy}^2</th>
<th>P-Q^2-4τ_{xy}^2</th>
<th>((P-Q)^2-4τ_{xy}^2)^{1/2}</th>
<th>σ_y</th>
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<tbody>
<tr>
<td>II</td>
<td>22</td>
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<td>9.000</td>
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<td>0.965</td>
<td>3.725</td>
<td>5.275</td>
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<td>-3.824</td>
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<tr>
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<td>1.811</td>
<td>3.230</td>
<td>0.906</td>
<td>0.643</td>
<td>0.583</td>
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</table>

#### Conversion of Coordinate System

<table>
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<th>φ</th>
<th>90-φ</th>
<th>(sin φ)σ_x</th>
<th>(cos φ)σ_x</th>
<th>sin(90-φ)σ_y</th>
<th>cos(90-φ)σ_y</th>
<th>σ_x^{1/2}</th>
<th>σ_y^{1/2}</th>
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</thead>
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<tr>
<td>II</td>
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<td>-2.196</td>
<td>-3.484</td>
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BIBLIOGRAPHY


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