MIXED FIVE-POINT NINE-POINT FORMULATION
OF MULTIPHASE FLOW IN PETROLEUM
RESERVOIRS

by
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ABSTRACT

Grid orientation effects are problems observed in numerical simulation of pattern flood performance in petroleum reservoirs. The cause of these problems is the finite-difference approximation of the equations describing multiphase flow in porous media, where the fluid tends to flow along the grid lines of the grid network used. Prior work introduced the nine-point, implicit pressure and nine-point, explicit saturation formulation which significantly reduces the grid orientation effects.

A new formulation is proposed in this study that has a five-point, implicit pressure, a four-point, explicit pressure, and a nine-point, explicit saturation finite-difference solution describing fluid flow diagonal to the network grid lines. This formulation is fairly easy to implement in existing five-point, finite-difference schemes, as compared to the nine-point scheme introduced in previous work.

The proposed model, called the split operator scheme, is compared to the five-point scheme and to the nine-point, implicit pressure and explicit saturation scheme and is
tested for end-point mobility ratios in the range from 1.0 to 50.0, uniform and nonuniform grid systems, as well as homogeneous and heterogeneous permeability distributions.

The split operator scheme is shown to give results identical to the nine-point scheme. It is also shown that grid orientation effects are dependent on the difference in numerical dispersion induced by the networks used in the various grid orientations.
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GLOSSARY

Five-spot - refers to geometric pattern of injection and production wells.

Five-point - refers to number of numerical grid blocks connected through the finite-difference formulation of the pressure equation.

Nine-point - same definition as five-point.

Split operator - the nine-point, finite-difference approximation for the Laplacian operator is split between five points at the present time level and four points at the old time level.

Numerical dispersion - the smearing of saturation fronts that results from discretization of partial differential equations.

Pore volume - defined as the pore volume associated with movable oil in the evaluation of results. In the finite-difference formulations, the pore volume is defined as the product of bulk-volume and porosity.
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Chapter 1
INTRODUCTION

1.1 Subject

Grid orientation effects are encountered in numerical simulation of pattern flood performance for unfavorable mobility ratio displacement. Diverging results in recovery curves, saturation profiles, and breakthrough times are obtained when grid lines are parallel or at 45° angle (diagonal) to the line joining an injector-producer pair in a five-spot flood pattern. That is, in the solution of the equations describing pattern flooding, fluid tends to flow more along the grid lines.

1.2 Objective

This study has been undertaken to investigate if a combination of five-point, implicit, finite-difference formulation for the pressure and nine-point, explicit, finite-difference formulation for the saturation will reduce the grid orientation effects in immiscible pattern flooding. If such an improvement can be documented, especially for the coarse grid network used in practice, it will be fairly easy to implement this mixed five-point nine-point scheme into
existing IMPES computer simulators which use five-point formulation. IMPES stands for IMplicit Pressure, Explicit Saturation formulation of the fluid flow in porous media.

Appendix E contains illustrations of how the numerical grid cells are connected in the various finite-difference schemes studied in this project.

1.3 Literature Review

The grid orientation dependency in areal displacement simulations has been the focus of intensive work for the last 20 years. The following is a brief summary of what has been published in the petroleum literature to date on this topic.

In 1972, Todd et al. investigated the sensitivity to grid orientation effects and proposed a two-point, upstream weighting of fluid mobility scheme to reduce these effects in areal displacement. They observed some stability problems with the two-point upstream, weighting scheme and included an automatic time-step selector to handle this phenomena. The improvement in recovery curves and breakthrough times was considerable for the set of data used.
Others, like Vinsome\textsuperscript{2} and Coats\textsuperscript{3} et.al., have shown that the two-point upstream, weighting scheme is insufficient to alleviate grid orientation effects in steamflooding where highly unfavorably mobility ratios may occur.

Holloway\textsuperscript{4} et.al. published their work in 1975 where they proposed a modification to the interblock transmissibilities which took into account diagonal flow between the nodes. Combining the diagonal flow modification with the two-point, upstream weighting scheme reduced the grid orientation effects only marginally over the two-point scheme alone.

In 1976, Langsrud\textsuperscript{5} proposed a finite-element method which is less prone to grid orientation effects. However, the complexity of the computer code and the computational work involved makes his model less advantageous than the finite-difference formulation in existing simulators.

A breakthrough in the work with grid orientation effects came in 1976 when Yanosik and McCracken\textsuperscript{6} presented their nine-point, finite-difference reservoir simulator. In their revised paper of 1979 they include the derivation of the nine-point formulation by simply combining the five-point, finite-difference approximations for diagonal and parallel
grid orientation. The eight surrounding grid nodes are weighted differently as the corner nodes have a weight of one and the parallel blocks have a weight of four. Yanosik and McCracken evaluated a number of scenarios, including immiscible as well as piston-type displacement, high end-point mobility ratios (M=100 for the immiscible displacement simulations), and nonsquare-grid network problems. They demonstrated the nine-point models ability to overcome grid orientation effects with respect to recovery curves and saturation contours, however, some problems were realized when a "bullet-shaped" saturation contour appeared for the piston-type displacement simulations at M=50. Yanosik and McCracken used the single-point, upstream weighting of fluid mobility in their formulation, and even though this may be less accurate than the two-point, upstream weighting scheme, it is also less sensitive to changes in saturation, thus, giving less instability problems. Most of the work on grid orientation effects following Yanosik and McCracken's uses the nine-point, finite-difference as a reference base because of its promising features, and also to point out the disadvantages of the scheme.

Robertson and Woo7 noted, quite correctly, that the
nine-point, finite-difference formulation would require extensive revision and additional calculations in existing computer code if it should be implemented into a regular five-point formulation. They proposed a curvilinear grid network that would require few changes to existing code. The results were good for the recovery curves except for the fact that they seemed to fall outside the range predicted by diagonal and parallel grid systems. The saturation contours were also off compared to previous work.

An interesting paper dealing with grid orientation effects using a weighted nine-point, finite-difference scheme was published by Ko and Au\textsuperscript{8} in 1979. They realized that several options exist for combinations of nine-point and five-point solutions in an IMPES (implicit pressure and explicit saturation) simulator and they showed that considerable overshoot occurred for the following three models:

1. Implicit usual five-point pressure and explicit weighted nine-point saturations.

2. Implicit weighted nine-point pressure and explicit usual five-point saturation.
3. Implicit weighted nine-point pressure and explicit weighted nine-point saturation (weighted nine-point pressures are used in the explicit saturation equation).

Ko and Au stated that all of the above models exhibit strong overshoot and/or undershoot characteristics and found them unsuitable.

Without being able to confirm the overshoot characteristics, model 1 was run as a part of this thesis project and figure 1 shows that recovery curves for this combination of the five-point and nine-point schemes falls outside the range predicted by the five-point scheme.

Finally, the model Ko and Au recommended:

4. Weighted nine-point IMPES (individual five-point pressures are used to compute the corresponding five-point saturations before the linear combination is performed at the end of a time-step).

It should be noted that the computational work increases with the increasing model number. The point in Ko and Au's work
is that the weighting factors for the corner blocks relative to the parallel blocks should have some physical basis, not purely mathematical as Yanosik and McCracken arbitrarily chose to use. Ko and Au found that the weighting factors were functions of the shock, or end-point mobility ratio, and independent of block size and interblock mobility. In their study, however, they conclude that the most important factors to improve Yanosik and McCracken's results include proper allocation of injection and production to eliminate the unusual "bullet-shaped" displacement fronts, and a refined areal grid network to predict accurately the areal sweep for the flood pattern, which Yanosik and McCracken already had done.

Vinsome and Au\textsuperscript{9} presented in 1979, and published in 1981 a short paper on the inclusion of harmonic fluid mobility in a five-point, finite-difference formulation with two-point, upstream weighting. The results indicate an improvement over the standard two-point, upstream weighting scheme but the recovery curves are not as close as the nine-point operator.

In January 1982, Abou-Kassem and Aziz\textsuperscript{10} published a comparison of some of the methods referred to above,
including the nine-point scheme, the two-point, upstream weighting scheme, and the total harmonic fluid mobility. They were more or less only confirming what had been published in earlier studies.

Yoshiaki\textsuperscript{11} proposed a mid-point weighting scheme in February 1982 which appeared to be somewhat better than the single-point and the two-point, upstream weighting schemes with respect to accuracy, and comparable to the latter in truncation error. The sensitivity of unfavorable mobility ratios on this scheme were not clear from the paper.

The more sophisticated approaches to the grid orientation problems includes the methods of characteristics and the variational methods. Larson\textsuperscript{12} proposed the variably timed flux updating, which is a method of characteristics, in June 1982. The method produces exact solutions for the data used, but the complex computer code makes it less advantageous than the simpler types of finite-difference schemes.

Coats and Ramesh presented a paper\textsuperscript{13} in September 1982, which they published in JPT, May 1986\textsuperscript{21}. In these publications they disclose that the nine-point scheme gives
large grid orientation error when applied to nonsquare (2:1 aspect ratio) grid blocks, while it was still better than the five-point scheme. No alternative methods were proposed in these papers.

In 1983 Potempa\textsuperscript{14} published a method which is a combination of finite-difference and finite-elements that has a relatively low sensitivity to grid orientation but would be fairly complicated to implement into existing computer code.

Pruess and Bodvarsson\textsuperscript{15}, also in 1983, presented a new method called the seven-point, finite-difference scheme. For the data used this scheme gave the same recovery curves as the nine-point scheme for a five-spot pattern steamflooding simulation. The model was reported to be intermediate between five- and nine-point methods with regard to computational effort and that is probably the reason why it has not replaced the five-point scheme in the industry.

In November 1983, Coats and Modine\textsuperscript{16} presented, what they called, a consistent method for calculating transmissibilities in nine-point, finite-difference equations. They revealed that the formulation as given by Yanosik and McCracken could result in negative
transmissibilities when simulating a heterogeneous reservoir. Coats and Modine indicates that their method can be applied to nonuniform grid networks as well as heterogeneous reservoirs, without showing actual proof for this.

Bertiger and Padmanabhan\textsuperscript{17} presented, in November 1983, a modification to the nine-point scheme that included integration of fluid fluxes across cell faces to improve the schemes handling of nonuniform grid networks. They used a length ratio between the grid blocks as 1:2:4 in both x- and y-direction. The velocity interpolation method they proposed is shown to have less grid orientation sensitivity than the five-point scheme and about the same as the nine-point scheme. However, the results of the proposed scheme do not coincide with the nine-point scheme results even though both are within the range given by the diagonal and parallel five-point simulations.

Also, in November 1983, Shah\textsuperscript{18} recognized the nine-point scheme's weakness in handling nonuniform grid systems and heterogeneities (negative transmissibilities). They proposed a modified nine-point formulation that was founded on physical considerations and which should have the ability to model nonuniform grids as well as reservoir
heterogeneities. It is difficult to say how well this scheme works because only a very limited number of simulations are included in their paper.

Chavent et.al.\textsuperscript{19} published a mixed finite-element method in 1984 that was claimed to give sharp displacement fronts and to be computationally economic. However, as with finite-element methods in general this method is also difficult to introduce in the petroleum industry due to the complexity of the coding.

In 1985, Frauenthal et.al.\textsuperscript{20} proposed a generalized upstream weighting scheme which they had found in the convective-heat transfer literature. The method is first order accurate as is the single-point, upstream weighting but it performs with little grid orientation effects for the data set studied. The method was not tested extensively in the paper, thus, it is hard to tell how it will behave for unfavorably mobility ratios.

The last paper included in this literature review is the one by Taggart and Pinczewski\textsuperscript{22} (August 1987). These authors studied the nine-point, finite-difference scheme as well as higher-order formulations. By using a standard
five-point (second order) scheme to solve for pressure and then solving for saturation by using a proposed third-order scheme they documented that grid orientation effects were still significant and concluded that truncation errors in the solution of the saturation equation are not the only factors responsible for grid orientation sensitivity. Then, they proposed a modified nine-point scheme that provides additional linking to the standard nine-point, finite-difference scheme given by Yanosik and McCracken but without providing the derivation or the physical justification for the modification. The results obtained show that grid orientation effects were still present. A conclusion of major importance in this study is that it is necessary to apply the higher-order schemes not only to the saturation equation but also to the pressure equation, because accurate velocity approximations are essential to simulate sharp, moving fronts accurately.
Chapter 2
A PROPOSED SCHEME

2.1 Introduction

The finite-difference scheme to be proposed and explained in this chapter will be referred to as the split operator scheme. This scheme will be compared to the standard five-point, finite-difference scheme used in most existing implicit pressure and explicit saturation (IMPES) simulators. The split operator scheme will also be compared to Yanosik and McCracken's nine-point scheme. Chapter 2 will therefore cover the difference in the pressure and the saturation equations between these schemes. The proposed split operator scheme will be shown to be fairly easy to implement in existing simulators.

The equations will be written in finite-difference form and they will be based on the assumptions that the IMPES formulation is to be used, that a horizontal reservoir is to be described in two-dimensional form, and that nonuniform grid networks as well as heterogeneities are to be included.

The derivation of the five-point, finite-difference scheme will be given. The derivation procedure for the
nine-point, finite-difference scheme is given in Yanosik and McCracken's paper and is therefore not included here. The split operator scheme is identical to the derivation of the nine-point scheme except for a few simple changes that will be explained in section 2.4. The assumptions made in the derivation of the five-point formulation will apply to the other IMPES schemes as well.

All nomenclature, subscripts, and superscripts are explained in the Nomenclature List.

2.2 The Five-Point Scheme

The partial differential equations for water and oil:

\[
\delta \left\{ \left[ 0.006328 \frac{K_x}{K_w} \frac{\mu_w}{(B_w \mu_w)} \right] \frac{\delta P_w}{\delta x} \right\}/\delta x \\
+ \delta \left\{ \left[ 0.006328 \frac{K_y}{K_w} \frac{\mu_w}{(B_w \mu_w)} \right] \frac{\delta P_w}{\delta y} \right\}/\delta y \\
+ Q_w \\
= \delta (\Phi \frac{S_w}{B_w})/\delta t \\
\] (water equation)

\[
\delta \left\{ \left[ 0.006328 \frac{K_x}{K_o} \frac{\mu_o}{(B_o \mu_o)} \right] \frac{\delta P_o}{\delta x} \right\}/\delta x \\
+ \delta \left\{ \left[ 0.006328 \frac{K_y}{K_o} \frac{\mu_o}{(B_o \mu_o)} \right] \frac{\delta P_o}{\delta y} \right\}/\delta y \\
+ Q_o \\
= \delta (\Phi \frac{S_o}{B_o})/\delta t \\
\] (oil equation)
The discretized and expanded differential equations for water and oil take the form:

\[
(T_{wx_{i+1/2,j}})^n[ (P_{w_{i+1,j}})^{n+1} - (P_{w_{i,j}})^{n+1} ]
-(T_{wx_{i-1/2,j}})^n[ (P_{w_{i,j}})^{n+1} - (P_{w_{i-1,j}})^{n+1} ]
+(T_{wy_{i,j+1/2}})^n[ (P_{w_{i,j+1}})^{n+1} - (P_{w_{i,j}})^{n+1} ]
-(T_{wy_{i,j-1/2}})^n[ (P_{w_{i,j}})^{n+1} - (P_{w_{i,j-1}})^{n+1} ]
+Q_w
=\left[ \frac{V_{r_{i,j}}}{(t(n+1)-t(n))} \right] \left[ ((S_{w_{i,j}})^{n+1} - (S_{w_{i,j}})^n)(\Phi_{i,j}/B_{w_{i,j}})^n \right]
+(S_{w_{i,j}})^n( (\Phi_{i,j}/B_{w_{i,j}})^{n+1} - (\Phi_{i,j}/B_{w_{i,j}})^n ) 
\] (2.01)

\[
(T_{ox_{i+1/2,j}})^n[ (P_{o_{i+1,j}})^{n+1} - (P_{o_{i,j}})^{n+1} ]
-(T_{ox_{i-1/2,j}})^n[ (P_{o_{i,j}})^{n+1} - (P_{o_{i-1,j}})^{n+1} ]
+(T_{oy_{i,j+1/2}})^n[ (P_{o_{i,j+1}})^{n+1} - (P_{o_{i,j}})^{n+1} ]
-(T_{oy_{i,j-1/2}})^n[ (P_{o_{i,j}})^{n+1} - (P_{o_{i,j-1}})^{n+1} ]
+Q_o
=\left[ \frac{V_{r_{i,j}}}{(t(n+1)-t(n))} \right] \left[ ((S_{o_{i,j}})^{n+1} - (S_{o_{i,j}})^n)(\Phi_{i,j}/B_{o_{i,j}})^n \right]
+(S_{o_{i,j}})^n( (\Phi_{i,j}/B_{o_{i,j}})^{n+1} - (\Phi_{i,j}/B_{o_{i,j}})^n ) 
\] (2.02)

where the gravity terms are neglected and the transmissibility terms are given as:
\[
T_{x_{i+1/2,j}} = 0.001127 \frac{(K_{xh} K_{rw}/(\mu w B_{w}))_{i+1/2,j} \, dydz}{((dx_{i+1}+dx_i)/2)}
\]

\[
T_{x_{i-1/2,j}} = 0.001127 \frac{(K_{xh} K_{rw}/(\mu w B_{w}))_{i-1/2,j} \, dydz}{((dx_{i-1}+dx_i)/2)}
\]

\[
T_{y_{i,j+1/2}} = 0.001127 \frac{(K_{yh} K_{rw}/(\mu w B_{w}))_{i,j+1/2} \, dx/dz}{((dy_{j+1}+dy_j)/2)}
\]

\[
T_{y_{i,j-1/2}} = 0.001127 \frac{(K_{yh} K_{rw}/(\mu w B_{w}))_{i,j-1/2} \, dx/dz}{((dy_{j-1}+dy_j)/2)}
\]

\[
T_{o_{x_{i+1/2,j}}} = 0.001127 \frac{(K_{xh} K_{ro}/(\mu o B_{o}))_{i+1/2,j} \, dydz}{((dx_{i+1}+dx_i)/2)}
\]

\[
T_{o_{x_{i-1/2,j}}} = 0.001127 \frac{(K_{xh} K_{ro}/(\mu o B_{o}))_{i-1/2,j} \, dydz}{((dx_{i-1}+dx_i)/2)}
\]

\[
T_{o_{y_{i,j+1/2}}} = 0.001127 \frac{(K_{yh} K_{ro}/(\mu o B_{o}))_{i,j+1/2} \, dx/dz}{((dy_{j+1}+dy_j)/2)}
\]

\[
T_{o_{y_{i,j-1/2}}} = 0.001127 \frac{(K_{yh} K_{ro}/(\mu o B_{o}))_{i,j-1/2} \, dx/dz}{((dy_{j-1}+dy_j)/2)}
\]

in oil field units. The harmonic permeabilities have the following form when the reservoir height, dz, is constant throughout the reservoir:

\[
K_{xh_{i+1/2}} = \frac{(dx_{i+1}+dx_i)/2}{(dx_{i+1}/(2 K_{xh_{i+1,j}})+dx_i/(2 K_{xh_{i,j}}))}
\]
\[ K_{x, h_{1-1/2}} = ((d_{x} + d_{x-1})/2)/(dx/(2K_{x_{i,j}})+d_{x-1}/(2K_{x_{i-1,j}})) \]

\[ K_{y, j_{1+1/2}} = ((d_{y} + d_{y+1})/2)/(dy/(2K_{y_{i,j+1}})+d_{y+1}/(2K_{y_{i,j}})) \]

\[ K_{y, j_{1-1/2}} = ((d_{y} + d_{y-1})/2)/(dy/(2K_{y_{i,j}})+d_{y-1}/(2K_{y_{i,j-1}})) \]

Using the following relationships:

\[ P_{o_{i,j}} = P_{w_{i,j}} + P_{cwo_{i,j}} \quad (2.03) \]

\[ S_{o_{i,j}} = 1 - S_{w_{i,j}} \quad (2.04) \]

\[ \Phi_{i,j} = \Phi_{b_{i,j}} (1 + cf(P_{w_{i,j}} - P_{b})) \quad (2.05) \]

\[ B_{w_{i,j}} = B_{wb} (1 - cw(P_{w_{i,j}} - P_{b})) \quad (2.06) \]

\[ B_{o_{i,j}} = B_{ob} (1 - co(P_{w_{i,j}} - P_{b})) \quad (2.07) \]

and making these assumptions:

\[ B_{w_{i,j}} \approx B_{wb} \]
\[ \text{Bo}_{i,j} = \text{Bob} \]

\[ \Phi_{i,j} = \Phi_b \]

\[ (P_{cwo_{i,j}})^{n+1} = (P_{cwo_{i,j}})^n \]

Then, by substituting (2.03), (2.04), (2.05), (2.06), and (2.07) into (2.01) and (2.02), performing the proper differentiation operations on the right hand side of the water and the oil equations, and adding the new expressions gives the pressure equation:

\[ B_{i,j}(P_{w_{i,j-1}})^{n+1} + D_{i,j}(P_{w_{i-1,j}})^{n+1} + E_{i,j}(P_{w_{i,j}})^{n+1} \]

\[ + H_{i,j}(P_{w_{i,j+1}})^{n+1} + F_{i,j}(P_{w_{i+1,j}})^{n+1} \]

\[ = R_{i,j} \]

(2.08)

where

\[ B_{i,j} = (a_{i,j})^n (T_{wy_{i,j-1/2}})^n + (T_{oy_{i,j-1/2}})^n \]

(2.09)

\[ D_{i,j} = (a_{i,j})^n (T_{wx_{i-1/2,j}})^n + (T_{ox_{i-1/2,j}})^n \]

(2.10)

\[ F_{i,j} = (a_{i,j})^n (T_{wx_{i+1/2,j}})^n + (T_{ox_{i+1/2,j}})^n \]

(2.11)
\[ H_{i,j} = (a_{i,j})^n (Twy_{i,j+1/2})^n + (Toy_{i,j+1/2})^n \]  
(2.12)

\[ E_{i,j} = -B_{i,j} - D_{i,j} - F_{i,j} - H_{i,j} \]
\[-(\Phi_{i,j} Vr_{i,j} / (t^{n+1}-t^n))((a_{i,j})^n(Sw_{i,j})^n(c_f+c_w)/Bw_{i,j} \]
\[ + (1-(Sw_{i,j})^n)(c_f+c_o)/Bo_{i,j}) \]  
(2.13)

\[ R_{i,j} = -((\Phi_{i,j} Vr_{i,j} / (t^{n+1}-t^n))((a_{i,j})^n(Sw_{i,j})^n(c_f+c_w)/Bw_{i,j} \]
\[ + (1-(Sw_{i,j})^n)(c_f+c_o)/Bo_{i,j} (Pw_{i,j})^n \]
\[-((Tox_{i+1/2,j})^n((Pcwo_{i+1,j})^n-(Pcwo_{i,j})^n) \]
\[-(Tox_{i-1/2,j})^n((Pcwo_{i,j})^n-(Pcwo_{i-1,j})^n) \]
\[ + (Toy_{i,j+1/2})^n((Pcwo_{i,j+1})^n-(Pcwo_{i,j})^n) \]
\[-(Toy_{i,j-1/2})^n((Pcwo_{i,j})^n-(Pcwo_{i,j-1})^n) \]
\[ + (a_{i,j})^n(Qw_{i,j})^n + (Qo_{i,j})^n \]  
(2.14)

where \((a_{i,j})^n = (Bw_{i,j}/Bo_{i,j})^n \) \((a_{i,j})^{n+1} = (Bw_{i,j}/Bo_{i,j})^{n+1} \)

Equation (2.08) thus yields a system of "IMAX x JMAX" equations for calculating the unknown pressure of the water phase at the current time level for each of the grid blocks, or nodes. The equation is solved using a banded Gaussian elimination algorithm in this study, however, other methods are available that will be more efficient. Once the updated
pressures are known, the corresponding water saturation is calculated explicitly by rearranging the water equation (2.01), like this:

\[(S_{w_{i,j}})^{n+1} = (S_{w_{i,j}})^n + (t^{n+1} - t^n)(B_{w_{i,j}})^{n+1}((T_{w_{i+1/2,j}})^{n}((P_{w_{i+1,j}})^{n+1} - (P_{w_{i,j}})^{n+1})
- (T_{w_{i-1/2,j}})^{n}((P_{w_{i,j}})^{n+1} - (P_{w_{i-1,j}})^{n+1})
+ (T_{w_{i,j+1/2}})^{n}((P_{w_{i,j+1}})^{n+1} - (P_{w_{i,j}})^{n+1})
- (T_{w_{i,j-1/2}})^{n}((P_{w_{i,j-1}})^{n+1} - (P_{w_{i,j}})^{n+1})
+ Q_w_{i,j}/(V_{r_{i,j}}(\Phi_{i,j})^{n+1})
- (S_{w_{i,j}})^n(cf+cw)((P_{w_{i,j}})^{n+1} - (P_{w_{i,j}})^n)\]  \hspace{1cm} (2.15)

2.3 The Nine-Point Scheme

Yanosik and McCracken derives the following expression for the water phase in block \((i,j)\):
\[ T_{x_{i+1/2}, j} (L_{w_{i+1/2}, j})^n ((P_{w_{i+1}, j})^{n+1} - (P_{w_{i}, j})^{n+1}) \]

\[ -T_{x_{i-1/2}, j} (L_{w_{i-1/2}, j})^n ((P_{w_{i}, j})^{n+1} - (P_{w_{i-1}, j})^{n+1}) \]

\[ +T_{y_{1, j+1/2}} (L_{w_{1, j+1/2}})^n ((P_{w_{1}, j+1})^{n+1} - (P_{w_{1}, j})^{n+1}) \]

\[ -T_{y_{1, j-1/2}} (L_{w_{1, j-1/2}})^n ((P_{w_{1}, j})^{n+1} - (P_{w_{1-1}, j})^{n+1}) \]

\[ +T_{x_{i+1/2}, j-1/2} (L_{w_{i+1/2}, j-1/2})^n ((P_{w_{i+1}, j-1})^{n+1} - (P_{w_{i}, j})^{n+1}) \]

\[ -T_{x_{i-1/2}, j+1/2} (L_{w_{i-1/2}, j+1/2})^n ((P_{w_{i}, j})^{n+1} - (P_{w_{i-1}, j+1})^{n+1}) \]

\[ +T_{y_{1+1/2}, j+1/2} (L_{w_{1+1/2}, j+1/2})^n ((P_{w_{1+1}, j+1})^{n+1} - (P_{w_{1}, j})^{n+1}) \]

\[ -T_{y_{1-1/2}, j-1/2} (L_{w_{1-1/2}, j-1/2})^n ((P_{w_{1}, j})^{n+1} - (P_{w_{1-1}, j-1})^{n+1}) \]

\[ +Q_w \]

\[ = (V_{r_{1}, j}/(t(n+1) - t(n))) \left( (S_{w_{1}, j})^{n+1} - (S_{w_{1}, j})^n \right) \left( \Phi_{i, j}/B_{w_{1}, j} \right)^n \]

\[ + (S_{w_{i}, j})^n \left( \Phi_{i, j}/B_{w_{1}, j} \right)^{n+1} - (\Phi_{i, j}/B_{w_{1}, j})^n \]  

(2.16)
And similarly for the oil phase:

\[
T_{x_{i+1/2,j}} (L_{o_{i+1/2,j}})^n ((P_{o_{i+1,j}})^{n+1} - (P_{o_{i,j}})^{n+1})
\]

\[-T_{x_{i-1/2,j}} (L_{o_{i-1/2,j}})^n ((P_{o_{i,j}})^{n+1} - (P_{o_{i-1,j}})^{n+1})
\]

\[+T_{y_{i,j+1/2}} (L_{o_{i,j+1/2}})^n ((P_{o_{i,j+1}})^{n+1} - (P_{o_{i,j}})^{n+1})
\]

\[-T_{y_{i,j-1/2}} (L_{o_{i,j-1/2}})^n ((P_{o_{i,j}})^{n+1} - (P_{o_{i,j-1}})^{n+1})
\]

\[+T_{xy_{i+1/2,j+1/2}} (L_{o_{i+1/2,j+1/2}})^n ((P_{o_{i+1,j+1}})^{n+1} - (P_{o_{i,j+1}})^{n+1})
\]

\[-T_{xy_{i-1/2,j+1/2}} (L_{o_{i-1/2,j+1/2}})^n ((P_{o_{i,j+1}})^{n+1} - (P_{o_{i-1,j+1}})^{n+1})
\]

\[+T_{xy_{i+1/2,j-1/2}} (L_{o_{i+1/2,j-1/2}})^n ((P_{o_{i+1,j}}}^n - (P_{o_{i,j}})^{n+1})
\]

\[-T_{xy_{i-1/2,j-1/2}} (L_{o_{i-1/2,j-1/2}})^n ((P_{o_{i,j}})^{n+1} - (P_{o_{i-1,j}}}^n)
\]

\[+Q_o
\]

\[= (V_{r_{i,j}}/(t(n+1)-t(n)))(((S_{o_{i,j}})^{n+1} - (S_{o_{i,j}})^n) (\Phi_{i,j}/B_{o_{i,j}})^n
\]

\[+(S_{o_{i,j}})^n ((\Phi_{i,j}/B_{o_{i,j}})^{n+1} - (\Phi_{i,j}/B_{o_{i,j}})^n))
\]

(2.17)

Note that the transmissibility terms are now composed of two individual terms, the "conductivity" term T and the mobility term L. The definition of these terms will be given after the complete pressure equation is given. The pressure equation will take on four additional terms for each block (i,j) which also have to be solved implicitly in this scheme:
\[ B_{i,j} (Pw_{i,j-1})^{n+1} + D_{i,j} (Pw_{i-1,j})^{n+1} + E_{i,j} (Pw_{i,j})^{n+1} + H_{i,j} (Pw_{i,j+1})^{n+1} + F_{i,j} (Pw_{i+1,j})^{n+1} + NE_{i,j} (Pw_{i+1,j+1})^{n+1} + WS_{i,j} (Pw_{i-1,j-1})^{n+1} + SE_{i,j} (Pw_{i+1,j+1})^{n+1} + NW_{i,j} (Pw_{i-1,j+1})^{n+1} = R_{i,j} \] (2.18)

\[ B_{i,j} = Ty_{1,j-1/2} ((a_{i,j})^{n} (Lw_{1,j-1/2})^{n} + (Lo_{i,j-1/2})^{n}) \] (2.19)

\[ D_{i,j} = Tx_{1-1/2,j} ((a_{i,j})^{n} (Lw_{1-1/2,j})^{n} + (Lo_{i-1/2,j})^{n}) \] (2.20)

\[ F_{i,j} = Tx_{1+1/2,j} ((a_{i,j})^{n} (Lw_{1+1/2,j})^{n} + (Lo_{i+1/2,j})^{n}) \] (2.21)

\[ H_{i,j} = Ty_{1,j+1/2} ((a_{i,j})^{n} (Lw_{1,j+1/2})^{n} + (Lo_{i,j+1/2})^{n}) \] (2.22)

\[ NE_{i,j} = Txy_{1+1/2,j+1/2} ((a_{i,j})^{n} (Lw_{1+1/2,j+1/2})^{n} + (Lo_{i+1/2,j+1/2})^{n}) \] (2.23)

\[ WS_{i,j} = Txy_{1-1/2,j-1/2} ((a_{i,j})^{n} (Lw_{1-1/2,j-1/2})^{n} + (Lo_{i-1/2,j-1/2})^{n}) \] (2.24)

\[ SE_{i,j} = Txy_{1+1/2,j-1/2} ((a_{i,j})^{n} (Lw_{1+1/2,j-1/2})^{n} + (Lo_{i+1/2,j-1/2})^{n}) \] (2.25)

\[ NW_{i,j} = Txy_{1-1/2,j+1/2} ((a_{i,j})^{n} (Lw_{1-1/2,j+1/2})^{n} + (Lo_{i-1/2,j+1/2})^{n}) \] (2.26)
\[ E_{i,j} = -B_{1,j} - D_{1,j} - F_{1,j} - H_{1,j} - N_{E_{1,j}} - N_{S_{1,j}} - N_{E_{1,j}} - N_{W_{1,j}} \\
- (\Phi_{i,j} \cdot V_{R_{i,j}})/(t^{n+1} - t^n) \cdot ((a_{i,j})^n(S_{W_{i,j}})^n(c_f+c_w)/B_{W_{i,j}}) \\
+ (1 - (S_{W_{i,j}})^n)(c_f+c_o)/B_{O_{i,j}} \]
\[(2.27)\]

\[ R_{i,j} = -((\Phi_{i,j} \cdot V_{R_{i,j}})/(t^{n+1} - t^n)) \cdot ((a_{i,j})^n(S_{W_{i,j}})^n(c_f+c_w)/B_{W_{i,j}}) \\
+ (1 - (S_{W_{i,j}})^n)(c_f+c_o)/B_{O_{i,j}} \cdot (P_{W_{i,j}})^n \\
- (T_{x_{i+1/2,j}}(L_{o_{i+1/2,j}})^n(\cdot (P_{cwo_{i+1,j}})^n-(P_{cwo_{i,j}})^n) \\
- T_{x_{i-1/2,j}}(L_{o_{i-1/2,j}})^n(\cdot (P_{cwo_{i,j}})^n-(P_{cwo_{i-1,j}})^n) \\
+ T_{y_{i,j+1/2}}(L_{o_{i,j+1/2}})^n(\cdot (P_{cwo_{i,j}})^n-(P_{cwo_{i,j+1}})^n) \\
- T_{y_{i,j-1/2}}(L_{o_{i,j-1/2}})^n(\cdot (P_{cwo_{i,j}})^n-(P_{cwo_{i,j-1}})^n) \\
+ T_{x_{y_{i+1/2,j+1/2}}}(L_{o_{i+1/2,j+1/2}})^n(\cdot (P_{cwo_{i+1,j+1}})^n-(P_{cwo_{i,j}})^n) \\
- T_{x_{y_{i-1/2,j-1/2}}}(L_{o_{i-1/2,j-1/2}})^n(\cdot (P_{cwo_{i,j}})^n-(P_{cwo_{i-1,j-1}})^n) \\
+ T_{x_{y_{i+1/2,j-1/2}}}(L_{o_{i+1/2,j-1/2}})^n(\cdot (P_{cwo_{i+1,j-1}})^n-(P_{cwo_{i,j}})^n) \\
- T_{x_{y_{i-1/2,j+1/2}}}(L_{o_{i-1/2,j+1/2}})^n(\cdot (P_{cwo_{i,j}})^n-(P_{cwo_{i-1,j+1}})^n) \\
+ (a_{i,j})^n(Q_{W_{i,j}})^n + (Q_{O_{i,j}})^n \]
\[(2.28)\]

The harmonic permeabilities used in the x- and y-directions, that is the interblock absolute permeability between block (i,j) and its parallel border blocks are the same as for the five-point scheme (assuming constant reservoir thickness):
\[ K_{xh_{i+1/2}} = \frac{(dx_{i+1} + dx_1)}{2} / (dx_{i+1} / (2 K_{x{i+1,j}}) + dx_i / (2 K_{x{i,j}})) \] (2.29)

\[ K_{xh_{i-1/2}} = \frac{(dx_i + dx_{i-1})}{2} / (dx_i / (2 K_{x{i,j}}) + dx_{i-1} / (2 K_{x{i-1,j}})) \] (2.30)

\[ Ky_{j+1/2} = \frac{(dy_{j+1} + dy_j)}{2} / (dy_{j+1} / (2 Ky_{j+1,j}) + dy_j / (2 Ky_{j,j})) \] (2.31)

\[ Ky_{j-1/2} = \frac{(dy_j + dy_{j-1})}{2} / (dy_j / (2 Ky_{j,j}) + dy_{j-1} / (2 Ky_{j-1,j})) \] (2.32)

In the diagonal directions, that is between the block \((i,j)\) and its corner neighbors the harmonic permeabilities take the form:

\[ K_{xyh_{i+1/2,j+i/2}} = \frac{(dx_{i+1} + dx_1)}{2} + ((dy_{j+1} + dy_j)/2) K_{xh_{i+1/2,j}} K_{yh_{j+1/2}} \]

\[ / (K_{xh_{i+1/2,j}} (dy_{j+1} + dy_j)/2 + Ky_{j+1/2} (dx_{i+1} + dx_1)/2) \] \hspace{1cm} (2.33)

\[ K_{xyh_{i-1/2,j-1/2}} = \frac{(dx_{i-1} + dx_1)}{2} + ((dy_{j-1} + dy_j)/2) K_{xh_{i-1/2,j}} K_{yh_{j-1/2}} \]

\[ / (K_{xh_{i-1/2,j}} (dy_{j-1} + dy_j)/2 + Ky_{j-1/2} (dx_{i-1} + dx_1)/2) \] \hspace{1cm} (2.34)
\[ K_{x,y,h_{i+1/2,j-1/2}} = \left( \frac{(dx_{i+1} + dx_i)}{2} \right) + \left( \frac{(dy_{j-1} + dy_j)}{2} \right) K_{x,h_{i+1/2,j}y_{h,i,j-1/2}} \]
\[
\left/ \left( K_{x,h_{i+1/2,j}}, (dx_{i+1} + dx_i) + 2 \cdot Ky_{h,i,j-1/2} \right) \right) \]  
\[ (2.35) \]

\[ K_{x,y,h_{i-1/2,j+1/2}} = \left( \frac{(dx_{i-1} + dx_i)}{2} \right) + \left( \frac{(dy_{j+1} + dy_j)}{2} \right) K_{x,h_{i-1/2,j}y_{h,i,j+1/2}} \]
\[
\left/ \left( K_{x,h_{i-1/2,j}}, (dy_{j+1} + dy_j) + 2 \cdot Ky_{h,i,j+1/2} \right) \right) \]  
\[ (2.36) \]

The mobility terms are calculated using the single-point upstream values of relative permeability, viscosity, and formation volume factor:

\[ L_{w_{i+1/2,j}} = \left( \frac{K_{rw}}{(\mu w_{Bw})} \right)_{i+1/2,j} \]  
\[ (2.37) \]

\[ L_{w_{i-1/2,j}} = \left( \frac{K_{rw}}{(\mu w_{Bw})} \right)_{i-1/2,j} \]  
\[ (2.38) \]

\[ L_{w_{i,j-1/2}} = \left( \frac{K_{rw}}{(\mu w_{Bw})} \right)_{i,j-1/2} \]  
\[ (2.39) \]

\[ L_{w_{i,j+1/2}} = \left( \frac{K_{rw}}{(\mu w_{Bw})} \right)_{i,j+1/2} \]  
\[ (2.40) \]

\[ L_{w_{i+1/2,j+1/2}} = \left( \frac{K_{rw}}{(\mu w_{Bw})} \right)_{i+1/2,j+1/2} \]  
\[ (2.41) \]
\[ L_{w_{1}} = (K_{w}/(\mu w B_{w})) \]

The "conductivity", or transmissibility terms are given as:

\[ \text{T}_{xy_{1}} = K_{xy} h_{1}/2, j_{1} d_{z} \left[ (dx_{1} + dx_{j})/2 \right]/\left[ 3 \left( (dx_{1} + dx_{j})/2 \right)^{2} + (dy_{j} + dy_{j})/2 \right] \]

(2.45)

\[ \text{T}_{xy_{1}} = K_{xy} h_{1}/2, j_{1} d_{z} \left[ (dx_{1} + dx_{j})/2 \right]/\left[ 3 \left( (dx_{1} + dx_{j})/2 \right)^{2} + (dy_{j} + dy_{j})/2 \right] \]

(2.46)

\[ \text{T}_{xy_{1}} = K_{xy} h_{1}/2, j_{1} d_{z} \left[ (dx_{1} + dx_{j})/2 \right]/\left[ 3 \left( (dx_{1} + dx_{j})/2 \right)^{2} + (dy_{j} + dy_{j})/2 \right] \]

(2.47)

\[ \text{T}_{xy_{1}} = K_{xy} h_{1}/2, j_{1} d_{z} \left[ (dx_{1} + dx_{j})/2 \right]/\left[ 3 \left( (dx_{1} + dx_{j})/2 \right)^{2} + (dy_{j} + dy_{j})/2 \right] \]

(2.48)

\[ \text{T}_{xy_{1}} = K_{xy} h_{1}/2, j_{1} d_{z} \left[ (dx_{1} + dx_{j})/2 \right]/\left[ 3 \left( (dx_{1} + dx_{j})/2 \right)^{2} + (dy_{j} + dy_{j})/2 \right] \]

(2.49)
\[ T_{x_{i-1/2},j} = K x_{i-1/2},j dy_j dz / \left[ (dx_{i-1}+dx_i)/2 \right] \]
\[ -T_{xy_{i-1/2},j+1/2} - T_{xy_{i-1/2},j-1/2} \]  
(2.50)

\[ T_{y_{i,j-1/2}} = K y_{i,j-1/2} dx_i dz / \left[ (dy_{j-1}+dy_j)/2 \right] \]
\[ -T_{xy_{i+1/2},j-1/2} - T_{xy_{i-1/2},j-1/2} \]  
(2.51)

\[ T_{y_{i,j+1/2}} = K y_{i,j+1/2} dx_i dz / \left[ (dy_{j+1}+dy_j)/2 \right] \]
\[ -T_{xy_{i-1/2},j+1/2} - T_{xy_{i+1/2},j+1/2} \]  
(2.52)

The water equation is solved for water saturation to yield the nine-point saturation equation:
\[(Sw_{i,j})^{n+1} = (Sw_{i,j})^n + (t^{n+1} - t^n)(Bw_{i,j})^{n+1}(T_{x_{i+1/2,j}}(L_{w_{i+1/2,j}})^n((Pw_{i+1,j})^n - (Pw_{i,j})^n)
-Tx_{i-1/2,j}(Lw_{i-1/2,j})^n((Pw_{i,j})^n - (Pw_{i-1,j})^n)
+Ty_{i,j+1/2}(Lw_{i,j+1/2})^n((Pw_{i,j+1})^n - (Pw_{i,j})^n)
-Ty_{i,j-1/2}(Lw_{i,j-1/2})^n((Pw_{i,j})^n - (Pw_{i,j-1})^n)
+T_{xy_{i+1/2,j+1/2}}(Lw_{i+1/2,j+1/2})^n((Pw_{i+1,j+1})^n - (Pw_{i,j})^n)
-T_{xy_{i-1/2,j-1/2}}(Lw_{i-1/2,j-1/2})^n((Pw_{i,j})^n - (Pw_{i-1,j-1})^n)
+T_{xy_{i+1/2,j-1/2}}(Lw_{i+1/2,j-1/2})^n((Pw_{i+1,j-1})^n - (Pw_{i,j})^n)
-T_{xy_{i-1/2,j+1/2}}(Lw_{i-1/2,j+1/2})^n((Pw_{i,j})^n - (Pw_{i-1,j+1})^n)
+Q_{w_{i,j}}/(V_{r_{i,j}}(\Phi_{i,j})^{n+1})\)

\[(Sw_{i,j})^n(c_f+c_w)((Pw_{i,j})^{n+1} - (Pw_{i,j})^n)\quad (2.53)\]

2.4 The Split Operator Scheme

The split operator scheme is proposed, in this study, as an alternative to the Yanosik and McCracken nine-point scheme. The idea behind this scheme is to keep the relationship between the \((i,j)\) block and its eight surrounding blocks in both the pressure equation and the saturation equation in order to avoid the stability problems observed by Ko and Au. It is also an important feature with the proposed split operator scheme that the \((i,j)\) block is
dependent on only four blocks where the pressures are at the present time-step level while the remaining four are at the old time-step level. The name "split operator" is a consequence of this arrangement of pressures split between the present and the old time-step levels. As a result, the split operator scheme becomes a combination of five-point, implicit pressure and nine-point, explicit saturation formulation.

Keeping the same notation as for the nine-point scheme the pressure equation for the split operator scheme becomes:

\[
B_{i,j}(P_{w_{i,j-1}})^{n+1} + D_{i,j}(P_{w_{i-1,j}})^{n+1} + E_{i,j}(P_{w_{i,j}})^{n+1} \\
+ H_{i,j}(P_{w_{i,j+1}})^{n+1} + F_{i,j}(P_{w_{i+1,j}})^{n+1} \\
= R_{i,j} - NE_{i,j}[(P_{w_{i+1,j+1}})^{n} - (P_{w_{i,j}})^{n}] \\
- WS_{i,j}[(P_{w_{i-1,j-1}})^{n} - (P_{w_{i,j}})^{n}] \\
- SE_{i,j}[(P_{w_{i+1,j-1}})^{n} - (P_{w_{i,j}})^{n}] \\
- NW_{i,j}[(P_{w_{i-1,j+1}})^{n} - (P_{w_{i,j}})^{n}] \\
\]  \hspace{1cm} (2.54)

Coefficient \( E_{i,j} \) is the only term that will change from the nine-point scheme and it becomes like the five-point coefficient \( E \) since the diagonal terms are moved over to the right hand side of the pressure equation:
\[ E_{i,j} = -B_{i,j} - D_{i,j} - F_{i,j} - H_{i,j} \]

\[-(\Phi_{i,j} V_{i,j}/(t^{n+1} - t^n))(a_{i,j})^n(S_{w_{i,j}})^n(cf+cw)/B_{w_{i,j}} \]

\[+(1-(S_{w_{i,j}})^n)(cf+co)/B_{o_{i,j}} \]  \hspace{1cm} (2.55)

From equation (2.54) it is clear that the pressure equation can be solved with the same band-solving or Gaussian elimination routine that is being used for the five-point scheme. Thus, the split operator scheme would not cause any additional computer effort with respect to the implicit calculations. The explicit saturation equation is the same as for the nine-point scheme equation (2.53) and therefore more complex than for the five-point scheme but this is not the most significant work and storage consuming part of the IMPES model. To implement the split operator scheme into an existing five-point, finite-difference IMPES model would require changes in the transmissibility definitions, changes on the right hand side of the pressure equation, and modification of the saturation equation. All these changes are simple to perform relative to changing the implicit part of the model.
Chapter 3

TEST OF THE SPLIT OPERATOR SCHEME

3.1 Introduction

The testing of the split operator mixed five-point nine-point, finite-difference scheme was performed by making parallel simulations with the five-point, finite-difference scheme and the Yanosik and McCracken nine-point, finite-difference scheme. The features tested includes sensitivity to end-point mobility ratios, uniform (square) grid network versus nonuniform, nonsquare grid network, and homogeneous reservoir characteristics versus heterogeneous reservoir characteristics. By comparing recovery curve performances and saturation contours it should be possible to determine whether the split operator scheme is in accordance with the other two formulations or if it is outside the range set by the five-point and nine-point schemes. Furthermore, the simulations should indicate if the model has any particularly weak points that may cause grid orientation problems. It is not thereby implied that the five-point or the nine-point scheme represents the "true" answer when it comes to simulation of reservoir behavior, they are only approximations that have been thoroughly tested and documented in the literature.
The work by Yanosik and McCracken\textsuperscript{6} provides the reservoir and fluid characteristics data chosen for this project, see table 1:

**TABLE 1**

*Input Data*

Porosity = 20%
Viscosity of oil = 16.7 cp
Viscosity of water = 0.05 cp
Flood pattern area = 40 acres
Reservoir thickness = 20 ft
Absolute permeability = 12.5 mD
Initial water saturation = 27.5%
Irreducible oil saturation = 37.5%
Irreducible water saturation = 27.5%
Initial reservoir pressure = 100 psi
Compressibility of water and oil = 10\textsuperscript{-5} psi\textsuperscript{-1}
Formation volume factor for water and oil = 1.0 at 0 psi
Fluid injection and production rate per well = 60 BBL/DAY

Yanosik and McCracken made their simulations with constant porosity, that is, no variation in space or time. They also seem to have assumed the capillary pressure
effects to be negligible. This study will make the same assumptions but the computer code used and listed in appendix A includes the option of porosity variations with pressure (time) and the option for including capillary pressure curves.

The relative permeability curves used are given by the following equations:

\[ K_{rw_{i,j}} = 0.3 \left[ \frac{(Sw_{i,j} - Swr)}{(1-Swr-Sor)} \right]^{3.4} \]
\[ K_{ro_{i,j}} = 1.0 \left[ \frac{(So_{i,j} - Sor)}{(1-Swr-Sor)} \right]^{2.0} \]

and the fractional flow of water is given as:

\[ f_{w_{i,j}} = \frac{(K_{rw}/\mu_{w}/Bw)_{i,j}}{((K_{rw}/\mu_{w}/Bw)_{i,j} + (K_{ro}/\mu_{o}/Bo)_{i,j})} \]

These relative permeability and fractional flow curves match the curves used by Yanosik and McCracken.

3.2 Verification of the Nine-Point Scheme

The nine-point, finite-difference scheme of Yanosik and McCracken was run with the data in table 2 which corresponds to an end-point mobility ratio of \( M=100 \). However, since all
simulations were performed on an IBM PS/2 Model 50Z with a mathcoprocessor, there was not capacity to run the 21x21 diagonal grid and the 29x29 parallel grid that Yanosik and McCracken used in their simulations. The first calculations were therefore made to determine the relationship between grid network size and convergence of recovery curves. Figure 2 shows the results of these simulations, clearly indicating that when the grid blocks get smaller (number of grid blocks increases) the recovery performance curves converge. The explanation is obvious, as the number of grid blocks increases, the size of each block decreases, and the size of the diagonal grid blocks approaches the size of the parallel grid blocks. The reduction in grid cell size leads to a reduction in numerical dispersion\textsuperscript{25} but more important, as the cell sizes for the diagonal and the parallel grid orientations becomes approximately equal (see table 2) the numerical dispersion becomes equal for the two grid orientations, also.
TABLE 2

Grid block sizes for the diagonal
and the parallel grid orientations

<table>
<thead>
<tr>
<th>GRID</th>
<th>SIZE(ft)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Diagonal</td>
<td>Parallel</td>
</tr>
<tr>
<td>3x3</td>
<td>466.7</td>
<td>660.0</td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td>311.1</td>
<td>440.0</td>
<td></td>
</tr>
<tr>
<td>5x5</td>
<td>233.3</td>
<td>330.0</td>
<td></td>
</tr>
<tr>
<td>6x6</td>
<td>186.7</td>
<td>264.0</td>
<td></td>
</tr>
<tr>
<td>7x7</td>
<td>155.6</td>
<td>220.0</td>
<td></td>
</tr>
<tr>
<td>8x8</td>
<td>133.3</td>
<td>188.6</td>
<td></td>
</tr>
<tr>
<td>9x9</td>
<td>116.7</td>
<td>165.0</td>
<td></td>
</tr>
<tr>
<td>21x21</td>
<td>46.7</td>
<td>66.0</td>
<td></td>
</tr>
<tr>
<td>29x29</td>
<td>33.3</td>
<td>47.1</td>
<td></td>
</tr>
</tbody>
</table>

The largest network the IBM PS/2 could handle with reasonable speed was the 9x9 system, thus this was chosen for the parallel grid orientation simulations. From figure 2 it is seen that the closest recovery performance of the diagonal runs was achieved with a diagonal grid system of 5x5. In fact table 3 shows a very close agreement for the diagonal(5x5) and the parallel(9x9), which signifies that the difference in numerical dispersion between the diagonal(5x5) and the parallel(9x9) grid systems are of the same order as
the difference in numerical dispersion between the diagonal (21x21) and the parallel (29x29) grid systems used by Yanosik and McCracken. The diagonal (5x5) and the parallel (9x9) grid network are shown in figures 35 and 36.

**TABLE 3**

*Difference in oil recovery at 1 PV water injected diagonal grid systems versus the parallel (9x9)*

<table>
<thead>
<tr>
<th>Diagonal grid</th>
<th>%difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3x3</td>
<td>-1.73%</td>
</tr>
<tr>
<td>4x4</td>
<td>-0.66%</td>
</tr>
<tr>
<td>5x5</td>
<td>-0.10%</td>
</tr>
<tr>
<td>6x6</td>
<td>0.41%</td>
</tr>
<tr>
<td>7x7</td>
<td>0.47%</td>
</tr>
<tr>
<td>8x8</td>
<td>0.62%</td>
</tr>
<tr>
<td>9x9</td>
<td>0.70%</td>
</tr>
</tbody>
</table>

Figure 3 was obtained using the diagonal (5x5) grid and the parallel (9x9) grid and it shows practically no difference between the diagonal and parallel recovery curves. Figure 3 matches figure 7D in Yanosik and McCrackens' paper and that fact is taken as verification of the nine-point code.
Unless otherwise specified, the following simulations are made for a homogeneous reservoir and uniform square grids.

3.3 Unfavorable Mobility Ratios

The first simulations to be performed after the nine-point scheme was verified were to evaluate the sensitivity of mobility ratios in the range from 1 to 50 with regard to grid orientation.

Figure 4 shows the behavior of the five-point, the nine-point, and the split operator scheme for $M=50$. The five-point scheme shows significant grid orientation effects while the nine-point and the split operator schemes are identical and show no sign of grid orientation effects.

Figure 14 shows the saturation contours for simulations corresponding to figure 4. The 40% water saturation profiles show that the five-point scheme causes a large difference between the diagonal and the parallel front while the nine-point and the split operator schemes give identical contours for the same grid orientation. There is a notable difference in the diagonal and parallel contours for these two schemes indicating grid orientation effects are present.
Figure 5 shows the recovery curves of the split operator scheme only, as the mobility ratios vary between 1 and 50. Notice that no grid orientation effects are detectable.

Figures 15 and 16 show the 40% water saturation contours for the diagonal and the parallel grid systems, respectively, for mobilities in the range from 1 to 50 using the split operator scheme only. The diagonal contours are more advanced than the parallel fronts possibly a result of numerical dispersion and grid orientation effects.

Figures 17, 18, and 19 show the diagonal and the parallel contours on the same plot, for different mobility ratios. It is observed that the greatest divergence between the parallel and the diagonal contours occurs as the mobility ratio increases from 1 to 10. Above M=10 the change in contours happens slower.

3.4 Two-Point Upstream Weighting

The two-point upstream weighting of fluid mobilities is implemented in certain industry five-point, finite-difference simulators and it was therefore with a certain curiosity that the five-point, finite-difference scheme was tested with both upstream weighting formulations for the
unfavorable mobility ratio of $M=50$. The formulation described by Todd et.al.\textsuperscript{1} was in this study used with a constant time-step of one day.

Figure 6 shows the recovery performance curves for the single-point and the two-point upstream weighting of fluid mobility for $M=50$. The recovery curves for the two-point scheme are certainly closer together than the curves for the single-point scheme but the curves are intertwining. Figures 12 and 13 show the 40% water saturation isograms where the two-point contours are closer together than the single-point contours. Not showing on the figures but observed in the parallel grid system calculations was a considerable undershoot in saturations along the diagonal connecting the producing wells. The reason for this would probably be too large time-steps, especially for the early times.

Table 4 represents a comparison of oil recoveries after 1 pore volume movable oil of water has been injected. The breakthrough times are included, as well, and the differences are relative to the parallel(9x9) grid system.
TABLE 4
Comparing single-point to two-point upstream weighting

Mobility Ratio = 50.0
1.0 Pore Volume injected.
Values given as pore volumes.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Single-point</th>
<th>Two-point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagonal(5x5)</td>
<td>0.50</td>
<td>0.30</td>
</tr>
<tr>
<td>Parallel(9x9)</td>
<td>0.46</td>
<td>0.22</td>
</tr>
</tbody>
</table>

| %Difference     | 8.7%          | 36.6%      | 2.1%     | 3.7%    |

The differences are larger than what Todd et al. observed for unit mobility ratio, but then, so is the mobility ratio used in this study (M = 50). The corresponding performance of the split operator scheme which uses the single-point upstream weighting method is illustrated in table 5. It is observed that the split operator scheme is superior to the single-point, five-point, finite-difference scheme both with respect to recovery performance and breakthrough times. The differences are taken relative to the parallel(9x9) grid. Relative to the two-point, five-point, finite-difference scheme the split operator scheme has a better recovery
performance but diverges more on the breakthrough prediction.

TABLE 5

Results for the split operator scheme

Mobility Ratio = 50.0
1.0 Pore Volume injected.

Values given as pore volumes

<table>
<thead>
<tr>
<th>Grid</th>
<th>Recovery</th>
<th>Breakthrough</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagonal(5x5)</td>
<td>0.48</td>
<td>0.25</td>
</tr>
<tr>
<td>Parallel(9x9)</td>
<td>0.48</td>
<td>0.27</td>
</tr>
</tbody>
</table>

%Difference 0.0% -7.4%

3.5 Nonuniform Grid Network

Two different nonuniform grid systems were employed to test the split operator scheme versus the five-point and the nine-point schemes. The first system includes a length distribution of the ratio 1:2:4 which says that the two first and the two last grid cells in each direction have the ratio of 1:2, while the remaining cells have a length of four times the first cell's length. This grid system was first studied by Bertiger and Padmanabhan17 and it is illustrated
in figures 37 and 38.

The second grid distribution system is simpler and more coarse than the first one as the number of grid blocks in the two directions (x and y) are of the ratio 1:4. The IBM PS/2 was able to handle a 4x16 grid system and that was used for the parallel as well as the diagonal simulations. It is obvious that this coarse grid network represents the greatest difference in numerical dispersion between the diagonal and the parallel oriented grids. This grid system is shown 39 and 40.

Only the homogeneous reservoir characteristic cases are discussed in this section, the combined heterogeneous and nonuniform cases will be covered in the next section.

Figure 7 shows the actual recovery curves for the three schemes. It is observed that both the nine-point and the split operator scheme are still overlaying each others performance as the diagonal and the parallel curves slightly diverge. The five-point recovery performance curves diverges, too, but not much more than for the uniform case. This divergence for the nine-point and the split operator schemes shows that these formulations both have a latent grid orientation sensitivity for nonuniform grid systems and that
the split operator scheme is not more sensitive than the nine-point scheme in that respect.

Figures 20, 21, and 22 show the 40% water saturation isograms for the split operator scheme, the five-point scheme, and the nine-point scheme, respectively. The parallel grid front is now ahead of the diagonal front as opposed to the uniform grid system (fig.14) where the diagonal front was ahead. Again, the split operator is overlaying the nine-point contours. The five-point contours are still more divergent than the other schemes.

The second nonuniform grid system, the 4x16 network, has an even more pronounced divergence between the recovery curves for diagonal and parallel grid than the 1:2:4 distribution. Figure 8 shows the recovery curves for the 4x16 system. The curves for the nine-point and the split operator schemes are close to the five-point curves and, as before, the split operator scheme gives identical results to the nine-point scheme. As with the recovery curves, the saturation contours for the five-point scheme are almost identical to the contours of the split operator and the nine-point schemes. This may indicate that for a sufficiently nonuniform, and perhaps coarse, grid network the more sophisticated nine-point and split operator schemes may
not be able to describe the displacement process more accurately than the simpler five-point formulation.

3.6 Heterogeneous Permeability Distribution

The permeabilities have in these simulations been picked by a random number generator using the software @RISK\textsuperscript{23}. Each grid block has a permeability in the x-direction different from the permeability in the y-direction. Appendix B contains a list of the permeabilities generated using a mean permeability of 12.5 mD and a standard deviation of 1, and a lognormal distribution function.

Three different combinations using the random permeability distribution have been simulated for the three finite-difference schemes. The first includes a uniform, square grid network. The second is performed with the 1:2:4 nonuniform grid size system, and the third with the 4x16 network.

Figure 9 shows the recovery curves for the schemes using uniform grid systems. The curves for the split operator and the nine-point schemes are overlaying each other and fall in between the curves for the five-point scheme. No grid
orientation effect is observed for the split operator and the nine-point schemes.

Figures 26 and 28 show good agreement between the diagonal and the parallel saturation contours for both the split operator and the nine-point scheme. The saturation contours for the five-point scheme are again very different for the diagonal versus the parallel grid.

Combining the heterogeneous permeability distribution and the nonuniform 1:2:4 grid size system yields the same results for the recovery curves as the homogeneous reservoir did. The saturation contours shows some differences, however small. Greater variations in permeability would of course give more pronounced differences.

Simulations of the combined heterogeneous reservoir and the 4x16 grid network are also coinciding with the homogeneous results, as observed from figures 11, 32, 33, and 34.

The significance of the simulations with heterogeneous reservoir characteristics and nonuniform grid systems is that the nine-point and the split operator scheme is not very sensitive to heterogeneities in the reservoir, and that the
grid network has more impact on the results obtained. That is, figure 9 shows negligible grid orientation effects as reservoir heterogeneities are introduced while figures 10 and 11 displays grid orientation effects as the nonuniform networks are imposed upon the heterogeneous reservoir.

3.7 Timestep Size Sensitivity

The three finite-difference schemes were run for a timestep of 1 day throughout this study. The injection- and production-rate per well was 60 bbl/day, corresponding to $3.15 \times 10^{-5}$ movable oil pore volume injected per day for the diagonal($5 \times 5$) cases and to $3.89 \times 10^{-5}$ movable oil pore volume injected per day for the parallel($9 \times 9$) cases.

Table 6 shows the effects on the recovery performance of increasing the timestep to 20 days using a uniform grid and homogeneous reservoir properties. The results in table 6 compares to the results in table 5. Note that the percent differences shown in table 6 are relative to the parallel($9 \times 9$) results.
TABLE 6

Effect of increasing timestep to 20 days.

Mobility Ratio = 50.0
1.0 Pore Volume injected

Oil Recovery [Pore Volumes].

<table>
<thead>
<tr>
<th>Grid</th>
<th>Five-point</th>
<th>Nine-point</th>
<th>Split operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagonal(5x5)</td>
<td>0.4961</td>
<td>0.4811</td>
<td>0.4821</td>
</tr>
<tr>
<td>Parallel(9x9)</td>
<td>0.4635</td>
<td>0.4806</td>
<td>0.4825</td>
</tr>
</tbody>
</table>

%difference 7.03% 0.10% -0.08%

Increasing the timestep to 50 days give similar results, although a slightly larger difference between the diagonal and the parallel results is observed. Again, the diagonal recovery performances are shown relative to the parallel results in table 7.
TABLE 7

Effect of increasing timestep to 50 days.

Mobility Ratio = 50.0
1.0 Pore Volume injected

Oil Recovery [Pore Volumes].

<table>
<thead>
<tr>
<th>Grid</th>
<th>Five-point</th>
<th>Nine-point</th>
<th>Split operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagonal(5x5)</td>
<td>0.4971</td>
<td>0.4814</td>
<td>0.4851</td>
</tr>
<tr>
<td>Parallel(9x9)</td>
<td>0.4635</td>
<td>0.4825</td>
<td>0.4856</td>
</tr>
</tbody>
</table>

%difference 7.25% -0.23% -0.10%

3.8 Comparison of Computational Work

The computational work necessary to obtain the solution of the difference equations for one timestep depend upon the grid size, the simulation model (IMPES) and the algorithm used for solving the algebraic equations. The total work per timestep for the IMPES model is not of primary concern in this study but rather the work required by the nine-point, implicit pressure solution routine used by the nine-point scheme versus the five-point, implicit pressure solution routine used by the five-point and the split operator schemes.
Aziz and Settari derived the formulae for calculating work requirements for solving the five-diagonal matrix resulting from two-dimensional finite-difference, equations. Appendix D shows the algorithm used in this study to solve such a matrix and figure 41 shows the standard ordering system used in the algorithm. Figures 42 and 43 show the matrices to be solved using the algorithm for the split operator scheme (and the five-point scheme) and for the nine-point scheme, respectively.

The formulae derived by Aziz and Settari has the form:

\[ \text{Work} = (N-2M+1)[(M+1)^2+M]+M(M+1)(2M+1)/3+M(M-1)+(M+2)^2-M-9 \]

and it gives the number of multiplications/divisions performed per timestep inside the matrix solving algorithm. The results from applying this equation to the split operator and to the nine-point schemes are shown in table 8. Note that M relates to matrix band-width like:

\[ \text{Band-width} = 2M+1 \]

which in this study signify that M=JMAX for the split operator scheme and M=JMAX+1 for the nine-point scheme.
### TABLE 8

**Work per timestep**

<table>
<thead>
<tr>
<th>Grid</th>
<th>Unknowns</th>
<th>Split operator</th>
<th>Nine-point</th>
<th>%difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x5</td>
<td>25</td>
<td>821</td>
<td>1031</td>
<td>25.6%</td>
</tr>
<tr>
<td>9x9</td>
<td>81</td>
<td>7721</td>
<td>9107</td>
<td>18.0%</td>
</tr>
<tr>
<td>21x21</td>
<td>441</td>
<td>209541</td>
<td>227895</td>
<td>8.8%</td>
</tr>
<tr>
<td>29x29</td>
<td>841</td>
<td>747181</td>
<td>795727</td>
<td>6.5%</td>
</tr>
<tr>
<td>100x100</td>
<td>10000</td>
<td>101656996</td>
<td>103656196</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

Note that the percent differences are taken relative to the split operator scheme.
Chapter 4

CONCLUSIONS

Based on the results obtained in this investigation of the split operator scheme the following conclusions are formed:

1. The split operator scheme produces results identical to the nine-point scheme introduced by Yanosik and McCracken in 1976 for the range of mobility ratios evaluated.

2. The split operator scheme shows less sensitivity at coarse grids to grid orientation than does the five-point scheme.

3. The nine-point scheme predicts converging recovery curves as the diagonal and the parallel grid block sizes decreases.

4. The split operator scheme predicts recovery curves that converge to the same value as calculated with the nine-point scheme, for the same grid systems.
5. The five-point scheme will not predict converging recovery curves for the coarse grid systems used in the nine-point scheme.

6. Prediction of saturation contours is more sensitive to grid orientation than is prediction of recovery curves.

7. Both the split operator scheme and the nine-point scheme exhibit grid orientation effects for nonuniform, coarse grid block sizes.

8. Grid orientation effects are less sensitive to reservoir heterogeneities than to nonuniform grid cell sizes, for both the split operator scheme and the nine-point scheme.

9. The split operator scheme requires no modification of the five-point, implicit pressure solution routine, except for the change on the right hand side of the pressure equation in the main code. Thus, the proposed scheme is easier to implement into existing five-point IMPES simulators than the nine-point scheme proposed by Yanosik and McCracken.
10. This study shows that the computational work required to solve the matrix equation for the nine-point scheme range from 18\% (parallel 9x9) to 26\% (diagonal 5x5) above the required work per timestep for the split operator scheme.
Chapter 5
RECOMMENDATIONS

The split operator scheme is shown to behave with the same accuracy and the same insensitivity to grid orientation effects as the nine-point scheme which requires more work. The proposed scheme's features and performance makes it a viable alternative to the five-point scheme.

In order to apply the split operator scheme to nonuniform grid networks more investigation should be focused on how to improve its accuracy and reduce its grid orientation effects when such grid systems are used. Two alternative methods should be compared in such a study. The first of these should look into the numerical dispersion effects, as the difference in such between the diagonal and the parallel grid orientation has been shown to be of importance. The second method to be evaluated is the method for calculating transmissibilities suggested by Coats and Modine\textsuperscript{16} be implemented into the split operator scheme.

The split operator scheme should be implemented in a fully implicit solution method so that stability and accuracy of the split operator scheme in steamflooding, miscible
displacement, and in three-phase flow can be evaluated.
Figure 1 – Effect of various saturation schemes on the recovery performance for an end-point mobility ratio of $M=100.0$ immiscible displacement using an implicit five-point pressure solution.
Figure 2 - Effect of grid size on convergence of diagonal and parallel orientated systems using Yanosik and McCracken's nine-point simulator.
Figure 4 - Recovery performance of the mixed five-point nine-point split operator formulation versus Yanosik and McCracken's nine-point and the standard five-point formulation at M=50.
Figure 5 - Effect of mobility ratio on recovery performance of the mixed five-point nine-point split operator scheme.
Figure 6 - Comparing the single-point upstream mobility weighting with the two-point upstream mobility weighting using the standard five-point scheme for pressure and saturation at M=50.
Figure 7 - Effect of nonuniform grid blocks in a homogeneous reservoir on the five-point, Yanośik & McCrackens nine-point, and on the split operator scheme at M=50 and using a 1:2:4 length ratio in both x- and y-direction.
Figure 8 - Effect of nonuniform grid system using a grid block number of 1:4 in x– versus y–direction.
Figure 9 – Effect of a heterogeneous reservoir using uniform square grid blocks.
Figure 10 – Effect of a heterogeneous reservoir using nonuniform grid blocks (1:2:4).
Figure 11 – Effect of a heterogeneous reservoir using nonuniform grid blocks (1:4).
Figure 12 - Predicted saturation contours for the five-point simulator with single-point upstream weighting of relative permeability and $M=50.0$ (0.32 Movable Oil Pore Volume water injected).
Figure 13 – Predicted saturation contours for the five-point simulator with two-point upstream weighting of relative permeability and M=50.0 (0.32 Movable Oil Pore Volume water injected).
Figure 15 — Saturation contours for diagonal (5x5) grid system using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Figure 16 – Saturation contours for parallel(9x9) grid system using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Homogeneous reservoir.  
Uniform, square grids.  
Single-point upstream weighting.

Figure 17 - Saturation contours for M=1.0 using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Homogeneous reservoir. Uniform, square grids. Single-point upstream weighting.

Figure 18 - Saturation contours for $M=10.0$ using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Figure 19 - Saturation contours for $M=25.0$ using the split operator scheme ($0.32$ Movable Oil Pore Volume water injected).
Figure 20 – Saturation contours for $M=50.0$ using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Figure 21 - Saturation contours for M=50.0 using the five-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 22 – Saturation contours for M=50.0 using the Yanosik & McCracken nine-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 23 - Saturation contours for M=50.0 using the Yanosik & McCracken nine-point scheme (0.32 Movable Oil Pore Volume water injected).
Homogeneous reservoir.
Non-uniform grids (1:4).
Single-point upstream weighting.

Figure 24 - Saturation contours for $M=50.0$ using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Figure 25 - Saturation contours for $M=50.0$ using the five-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 26 - Saturation contours for $M=50.0$ using the split operator scheme ($0.32$ Movable Oil Pore Volume water injected).
Figure 27 - Saturation contours for M=50.0 using the five-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 28 – Saturation contours for M=50.0 using the Yanosik & McCracken nine-point scheme (0.32 Movable Oil Pore Volume water injected).
Heterogeneous reservoir.
Non-uniform grids (1:2:4).
Single-point upstream weighting.

Figure 29 – Saturation contours for M=50.0 using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Heterogeneous reservoir.
Non-uniform grids (1:2:4).
Single-point upstream weighting.

Parallel (9x9)

Diagonal (5x5)

Sw = 0.40

Figure 30 - Saturation contours for M=50.0 using the five-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 31 - Saturation contours for $M=50.0$ using the Yanosik & McCracken nine-point scheme (0.32 Movable Oil Pore Volume water injected).
Heterogeneous reservoir.
Non-uniform grids (1:4).
Single-point upstream weighting.

Parallel(4x16)
Diagonal(4x16)

Sw = 0.40

Figure 32 - Saturation contours for M=50.0 using the split operator scheme (0.32 Movable Oil Pore Volume water injected).
Figure 33 - Saturation contours for $M=50.0$ using the five-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 34 - Saturation contours for M=50.0 using the Yanosik & McCracken nine-point scheme (0.32 Movable Oil Pore Volume water injected).
Figure 35 – The diagonal 5x5 grid network.
Figure 36 – The parallel 9x9 grid network.
Figure 37 – The diagonal 5x5 nonuniform 1:2:4 grid network.
Figure 38 – The parallel 9x9 nonuniform 1:2:4 grid network.
Figure 39 – The diagonal 4x16 nonuniform grid network.
Figure 40 – The parallel 4x16 nonuniform grid network.
Figure 41 - Standard column ordering of the 5x5 grid network.
Figure 42 – Form of pressure matrix for standard ordering by column for the five-point and the split operator scheme.
Figure 43 - Form of pressure matrix for standard ordering by column for the nine-point scheme.
CITED REFERENCES


SELECTED BIBLIOGRAPHY


NOMENCLATURE

B, D, E, F, H - coefficients in the five-point, finite-difference IMPES pressure equation

NE, NW, SE, WS - additional coefficients to the nine-point, finite-difference IMPES pressure equation

Bo - formation volume factor for oil, res.vol./std.vol.

Bw - formation volume factor for water, res.vol./std.vol.

cf - rock compressibility, psi⁻¹

co - oil compressibility, psi⁻¹

cw - water compressibility, psi⁻¹

dx, dy, dz - dimensions of the individual grid block in x-, y-, and z(vertical)-direction

fw - fractional flow of water

IMAX, JMAX - total number of mesh spaces in x- and y-direction in the IMPES models

Krw, Kro - relative permeability

Kx, Ky - absolute permeability in x- and y-direction for each grid block, mD

Kxh, Kyh, Kxy - harmonic average of permeability, mD

Lw, Lo - fluid mobility
N - product of IMAX and JMAX
M - JMAX
Pw,Po - pressure of the fluid phase, psi
Pcwo - capillary pressure of water-oil, psi
Qw,Qo - source terms in the IMPES models, bbl/d
Sw - water saturation
Swr - irreducible water saturation
Swi - initial water saturation
Sor - residual oil saturation
T - transmissibility in the IMPES formulation

Greek Symbols

α - ratio: Bw/Bo used in IMPES
δ - differential operator
μ - viscosity, cp
Φ - porosity

Subscripts and Superscripts

i,j - subscript denoting the center of the grid block in question
\(i+1,j\) - subscript denoting the center of the grid block to the east of the cell block in question

\(i-1,j\) - subscript denoting the center of the grid block to the west of the cell block in question

\(i,j-1\) - subscript denoting the center of the grid block to the south of the cell block in question

\(i,j+1\) - subscript denoting the center of the grid block to the south of the cell block in question

\(i+1,j+1\) - subscript denoting the center of the grid block to the north-east of the cell block in question

\(i-1,j-1\) - subscript denoting the center of the grid block to the south-west of the cell block in question

\(i+1,j-1\) - subscript denoting the center of the grid block to the south-east of the cell block in question

\(i-1,j+1\) - subscript denoting the center of the grid block to the north-west of the cell block in question

\(n\) - superscript denoting old time level
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>n+1</td>
<td>superscript denoting current time level</td>
</tr>
<tr>
<td>w</td>
<td>subscript denoting the water phase</td>
</tr>
<tr>
<td>o</td>
<td>subscript denoting the oil phase</td>
</tr>
<tr>
<td>b</td>
<td>subscript denoting base conditions (0.0 psi)</td>
</tr>
<tr>
<td>x</td>
<td>subscript denoting variance in parallel to x-direction</td>
</tr>
<tr>
<td>y</td>
<td>subscript denoting variance in parallel to y-direction</td>
</tr>
<tr>
<td>xy</td>
<td>subscript denoting variance in the direction at 45° to the x- and the y-axis</td>
</tr>
</tbody>
</table>
Appendix A

THE SPLIT OPERATOR CODE

C 2-D LINEAR IMPES OIL-WATER MODEL
C
C THIS SIMULATOR MODELS A HORIZONTAL 2D-RESERVOIR
C USING A 9-POINT SPLIT OPERATOR TECHNIQUE FOR
C PRESSURE AND A NINE-POINT SCHEME FOR
C SATURATION.
C
C PARALLEL GRID ORIENTATION.

C*****************************************************************************************
C DEFINING VARIABLES, CONSTANTS AND PARAMETERS
C*****************************************************************************************

INTEGER IMAX, JMAX, MAX, ZX, TIME, SCHEME
INTEGER CONTR, IC, IB, IBMAX, ZY, ST, PERM

REAL UT, SWI, SWR, SOR, QTOT, BI, DEZ, DXUNIT, DYUNIT
REAL PHIB, PWI, PB, BOB, BWB, MR, MYO, MYW, KROST, KRWST

DOUBLE PRECISION TXP(0:5, 0:20), UXR, UXL, UYT, UYB, KU, OWIP
DOUBLE PRECISION QW(5, 20), QO(5, 20), BW(5, 20), BO(5, 20)
DOUBLE PRECISION KA, PCOW(5, 20), DEC(5), DEY(20)
DOUBLE PRECISION TWYP(0:5, 0:20)
DOUBLE PRECISION U5, RESIDU, TRUNC, DIF
DOUBLE PRECISION ALPHA(5, 20), FW(5, 20), RDU(100)
DOUBLE PRECISION CW, CO, CF, PW(0:5, 0:20),
+ PHI(5, 20), KRW(5, 20)
DOUBLE PRECISION KRO(5, 20), PWOLD(5, 20), OP, WI, WP, OX(100)
DOUBLE PRECISION B(5, 20), D(5, 20), F(5, 20),
+ H(5, 20), E(5, 20)
DOUBLE PRECISION R(5, 20), PO(5, 20), A2(5, 20)
DOUBLE PRECISION AGD(5, 20), X(100), U3, CUMBO
DOUBLE PRECISION WX(100), PAVG, PMID(100), MBE, OMB
DOUBLE PRECISION SWOLD(5, 20), SW(5, 20), NE(5, 20), NW(5, 20)
DOUBLE PRECISION VPI, MD, CUMBE, CUWL, CUOL,
+ WIP, OIP, SE(5, 20)
DOUBLE PRECISION PHIOLD(5, 20), VPB, N, LON(0:5, 0:20)
DOUBLE PRECISION BWOLD(5, 20), BOOLD(5, 20), KXHP0:5, 0:20)
DOUBLE PRECISION KYHP(5, 0:20), DELT, LONW(0:5, 0:20)
DOUBLE PRECISION LOSW(0:5, 0:20), LOE(0:5, 0:20)
DOUBLE PRECISION KNW(0:5, 0:20), KSW(0:5, 0:20), WS(5, 20)
DOUBLE PRECISION XX(5, 20), KY(5, 20), TELL(100), PRO
DOUBLE PRECISION LWN(0:5,0:20),LWE(0:5,0:20)
DOUBLE PRECISION LWNW(0:5,0:20),LWSW(0:5,0:20)
DOUBLE PRECISION TXW,TXE,TYN,TYS,
+ TXYNE,TXYSE,TXYNW,TXYSW
DOUBLE PRECISION TSW(5,20),TNW(5,0:20),VR(5,20),VP

OPEN(UNIT=10,FILE='PAR.OUT',STATUS='OLD')

C**************************************************************************
C INITIALIZING DATA
C**************************************************************************

QTOT=336.88000000000000
SWR=0.2750000000000000
SWI=0.2750000000000000
SOR=0.3750000000000000
PHIB=0.2000000000000000
CF=0.0000000000000000
CW=0.0000100000000000
PWI=100.00000000000000
PB=0.000000000
BOB=1.0000000000000000
BWB=1.0000000000000000
MYW=0.0500000000000000
KROST=1.00
KRWST=0.30
OIP=0.0
WIP=0.0
CUWL=0.0
CUOL=0.0
OP=0.0
WI=0.0
WP=0.0
KA=12.5000000000000000
DEZ=20.0000
CO=0.0000100000000000
IB=0
CUMBE=0.0
CUMMBO=0.0
MBE=0.0
OMB=0.0
N=0.0
TXW=0.0
TXE=0.0
TYN=0.0
TYS=0.0
TXYNE=0.0
TXYSE=0.0
TXYSW=0.0
TXYNW=0.0
TEN=0.0
TWS=0.0
TWH=0.0

WRITE(*,*)'Enter the following data in the same order:'
WRITE(*,*)' DELT (real).........................:
WRITE(*,*)' TIME (integer).......................:
WRITE(*,*)' IMA (integer)........................:
WRITE(*,*)' JMA (integer)........................:
WRITE(*,*)' MYO (real)..........................:
WRITE(*,*)' OUTPUT FREQUENCY(real).............:
WRITE(*,*)' SIZE (1=UNIFORM, 2=1:2:4)...........:

READ(*,*)DELT,TIME,IMA,JMA,MYO,UT,PERM

TMA=NINT(TIME/DELT)

IBMA=1+NINT(TIME/UT)

DO 10 I=1,IMA
DO 10 J=1,JMA
   PW(I,J)=PWI
   PWOLD(I,J)=PWI
   SW(I,J)=SWI
   SWOLD(I,J)=SWI
   A2(I,J)=0.0
   A9(I,J)=0.0
   B(I,J)=0.0
   D(I,J)=0.0
   F(I,J)=0.0
   H(I,J)=0.0
   QW(I,J)=0.0
   R(I,J)=0.0
   E(I,J)=0.0
   QQ(I,J)=0.0
   NE(I,J)=0.0
   NW(I,J)=0.0
   SE(I,J)=0.0
   WS(I,J)=0.0
   BW(I,J)=BWB
   BWOLD(I,J)=BWB
   BO(I,J)=BOB
   BOOLD(I,J)=BOB
   PHI(I,J)=PHIB
   PHIOLD(I,J)=PHIB
10    CONTINUE
C***********************************************************************
C GRID BLOCK SIZE DISTRIBUTION
C***********************************************************************

IF(PERM .EQ. 1)THEN
  DXUNIT=1320.0/(IMAX-1)
  DYUNIT=1320.0/(JMAX-1)
  DO 26 I=1,IMAX
  DO 26 J=1,JMAX
    DEX(I)=DXUNIT
    DEY(J)=DYUNIT
  CONTINUE

WRITE(10,*)'GRID BLOCK SIZE IS UNIFORM ALONG EACH AXIS'
ELSE IF(PERM .EQ. 2)THEN

  DXUNIT=1320.0/(5+4*(IMAX-4))
  DYUNIT=1320.0/(5+4*(JMAX-4))

  DO 14 I=1,IMAX
  DO 14 J=1,JMAX
    IF(I .EQ. 1 .OR. I .EQ. IMAX)THEN
      DEX(I)=DXUNIT
    ELSE IF(I .EQ. 2 .OR. I .EQ. (IMAX-1))THEN
      DEX(I)=2*DXUNIT
    ELSE
      DEX(I)=4*DXUNIT
    END IF

    IF(J .EQ. 1 .OR. J .EQ. JMAX)THEN
      DEY(J)=DYUNIT
    ELSE IF(J .EQ. 2 .OR. J .EQ. (JMAX-1))THEN
      DEY(J)=2*DYUNIT
    ELSE
      DEY(J)=4*DYUNIT
    END IF

  WRITE(10,*)'GRID BLOCK SIZE FOLLOWS 1:2:4 DISTRIBUTION'
  CONTINUE
END IF
C**********************************************************************
C   LOOKUP PERMEABILITIES
C**********************************************************************

CALL SPERM(KX,KY)

C**********************************************************************
C   CALCULATING ORIGINAL OIL IN PLACE
C**********************************************************************

VP=0.0

DO 12 I=1,IMAX
DO 12 J=1,JMAX
   VR(I,J)=DEX(I)*DEY(J)*DEZ
   VP=VP+VR(I,J)
12 CONTINUE

MXMY=IMAX*JMAX
VPB=VP*PHIB*(1-SWR-SOR)/5.6146
OOIP=VP*PHIB*(1-SWI)/5.6146/BOB
OWIP=VP*PHIB*SWI/5.6146/BWB
MR=(KRWST/MYW)/(KROST/MYO)
U5=QTOT/5.6146

WRITE(10,70)MXMY,TMAX,OWIP,OOIP,VPB,MR
WRITE(10,*)'Parallel grid solution.'
WRITE(10,*)

WRITE(10,*)'Rate control of production.'
WRITE(10,*)
WRITE(10,*)'Yanosik & McCracken Nine-Point Scheme'
WRITE(10,*)' with Split Operator on pressure.'
WRITE(10,*)
WRITE(10,*)'Production/Injection Rate.:',U5,'
   bbl/d/well'
WRITE(10,*)'Permeability...................:','KA,' mD'
WRITE(10,*)'Unit Block Size, DX...........:','DXUNIT,' ft'
WRITE(10,*)'Unit Block Size, DY...........:','DYUNIT,' ft'
WRITE(10,*)'Water Viscosity............:','MYW,' cp'
WRITE(10,*)'Oil Viscosity................:','MYO,' cp'
WRITE(10,*)'Residual Oil Saturation...:','SOR
WRITE(10,*)'Initial Water Saturation...:','SWI
WRITE(10,*)'Initial Reservoir Pressure:','PWI,' psi'
WRITE(10,*)
U5=0.0
C*************************************************************
C HARMONIC AVERAGING PERMEABILITIES
C*************************************************************

DO 11 I=1,IMAX-1
DO 11 J=1,JMAX

   KXHP(I,J)=(DEX(I)+DEX(I+1))/2/
   (DEX(I+1)/(2*KX(I+1,J))+DEX(I)/(2*KX(I,J)))

11    CONTINUE

DO 27 I=1,IMAX
DO 27 J=1,JMAX-1

   KYHP(I,J)=(DEY(J)+DEY(J+1))/2/
   (DEY(J+1)/(2*KY(I,J+1))+
   +DEY(J)/(2*KY(I,J)))

27    CONTINUE

DO 13 I=2,IMAX
DO 13 J=2,JMAX

   KSW(I,J)=(((DEX(I)+DEX(I-1))/2)
   +((DEY(J)+DEY(J-1))/2))*
   +KXHP(I-1,J)*KYHP(I,J)/
   +KXHP(I-1,J)*((DEY(J)+DEY(J-1))/2)
   +KYHP(I,J)*((DEX(I))+DEX(I-1))/2)

13    CONTINUE

DO 15 I=2,IMAX
DO 15 J=1,JMAX-1

   KNW(I,J)=(((DEX(I)+DEX(I-1))/2)
   +((DEY(J)+DEY(J+1))/2))*
   +KXHP(I-1,J)*KYHP(I,J)/
   +(KXHP(I-1,J)*((DEY(J)+DEY(J+1))/2)
   +KYHP(I,J)*((DEX(I))+DEX(I-1))/2)

15    CONTINUE
C ************************************************************
C  INITIALIZING BOUNDARY TRANSMISSIBILITIES
C ************************************************************

DO 16 I=1,I MAX
DO 16 J=1, J MAX
   TWXP(I MAX, J) = 0.0
   TWXP(0, J) = 0.0
   TWXP(I, 0) = 0.0
   TWXP(I, J MAX) = 0.0
   TWYP(0, J) = 0.0
   TWYP(I, 0) = 0.0
   TSW(1, J) = 0.0
   TNW(1, J) = 0.0
   TNW(I, J MAX) = 0.0
   TNW(I, 0) = 0.0
   TSW(I, 1) = 0.0
   TSW(I MAX+1, J) = 0.0
   TSW(I, J MAX+1) = 0.0
   KXHP(I MAX, J) = 0.0
   KYHP(I, J MAX) = 0.0
   KXHP(0, J) = 0.0
   KYHP(I, 0) = 0.0
   KSW(0, J) = 0.0
   KSW(I, 0) = 0.0
   KSW(I, 1) = 0.0
   KSW(1, J) = 0.0
   KSW(I MAX+1, J) = 0.0
   KSW(I, J MAX+1) = 0.0
   KNW(I, 0) = 0.0
   KNW(0, J) = 0.0
   KNW(I, 1) = 0.0
   KNW(I, J MAX) = 0.0
   LWE(I MAX, J) = 0.0
   LWE(0, J) = 0.0
   LWN(1, 0) = 0.0
   LWN(I, J MAX) = 0.0
   LWNW(I, J MAX) = 0.0
   LWNW(0, J) = 0.0
   LWNW(I, 0) = 0.0
   LWNW(I, 1) = 0.0
   LWSW(I, J) = 0.0
   LWSW(I, 1) = 0.0
   LWSW(0, J) = 0.0
   LWSW(I, 0) = 0.0
   LOE(I MAX, J) = 0.0
   LOE(0, J) = 0.0
   LON(I, 0) = 0.0
   LON(I, J MAX) = 0.0
LONW(I,JMAX)=0.0
LONW(0,J)=0.0
LONW(I,0)=0.0
LONW(1,J)=0.0
LOSW(1,J)=0.0
LOSW(I,1)=0.0
LOSW(0,J)=0.0
LOSW(I,0)=0.0
PW(0,J)=0.0
PW(I,0)=0.0
16 CONTINUE

C******************************
C TIMELOOP BEGINS
C******************************

DO 17 K=1,TMAX
PRINT *, K

CALL FWC(IMAX,JMAX,MYO,SW,FW,KRW,KRO,PCOW)

C***************************************************************************
C UPDATING VARIABLES FOR PRESENT Timestep
C***************************************************************************

DO 18 I=1,IMAX
DO 18 J=1,JMAX
PWOLD(I,J)=PW(I,J)
SWOLD(I,J)=SW(I,J)
PHIOLD(I,J)=PHI(I,J)
BOOLD(I,J)=BO(I,J)
BWOLD(I,J)=BW(I,J)
BO(I,J)=BOB*(1-CO*PW(I,J))
BW(I,J)=BWB*(1-CW*PW(I,J))
PHI(I,J)=PHIB
ALPHA(I,J)=BW(I,J)/BO(I,J)
18 CONTINUE

C***************************************************************************
C DETERMINING UPSTREAM NODES AND
C CALCULATING TRANSMISSIBILITIES
C***************************************************************************

ZX=0
ZY=0

DO 24 I=2,IMAX
DO 45 J=2,JMAX
IF(PW(I-1, J-1) .GT. PW(I, J)) THEN
    ZX = I - 1
    ZY = J - 1
ELSE
    ZX = I
    ZY = J
END IF

LWSW(I, J) = 0.006328*KRW(ZX, ZY)/(MYW*BW(ZX, ZY))
LOSW(I, J) = 0.006328*KRO(ZX, ZY)/(MYO*BO(ZX, ZY))

TSW(I, J) = KSW(I, J)*DEZ*((DEX(I) + DEX(I-1))/2)*
            ((DEY(J) + DEY(J-1))/2)/
            + (3*(((DEX(I) + DEX(I-1))/2)**2
            + (((DEY(J) + DEY(J-1))/2)**2))

ZX = 0
ZY = 0
45    CONTINUE

DO 23 J = 1, JMAX - 1

IF(PW(I-1, J+1) .GT. PW(I, J)) THEN
    ZX = I - 1
    ZY = J + 1
ELSE
    ZX = I
    ZY = J
END IF

LWNW(I, J) = 0.006328*KRW(ZX, ZY)/(MYW*BW(ZX, ZY))
LONW(I, J) = 0.006328*KRO(ZX, ZY)/(MYO*BO(ZX, ZY))

TNW(I, J) = KNW(I, J)*DEZ*((DEX(I) + DEX(I-1))/2)*
            ((DEY(J) + DEY(J+1))/2)/
            + (3*(((DEX(I) + DEX(I-1))/2)*((DEX(I) + DEX(I-1))/2)
            + (((DEY(J) + DEY(J+1))/2)*((DEY(J) + DEY(J+1))/2)))

ZX = 0
ZY = 0
23    CONTINUE
24    CONTINUE

DO 19 I = 1, IMAX - 1
DO 19 J = 1, JMAX

IF(PW(I+1, J) .GT. PW(I, J)) THEN


Zx = I + 1
ELSE
   Zx = I
END IF

Twxp(I, J) = Kxhp(I, J) * dz * Dey(J) / ((Dex(I) +
   Dex(I + 1)) / 2)
   - Tsw(I + 1, J + 1) - Tnw(I + 1, J - 1)

IF(Twxp(I, J) .LT. 0.0) THEN
   Twxp(I, J) = 0.0
   Tsw(I + 1, J + 1) = 0.0
   Tnw(I + 1, J - 1) = 0.0
END IF

Lwe(I, J) = 0.006328 * Krw(Zx, J) / (Myw * bw(Zx, J))
Loe(I, J) = 0.006328 * Kro(Zx, J) / (Myo * bo(Zx, J))
Zx = 0

19 CONTINUE

DO 20 I = 1,imax
DO 20 J = 1, jmax - 1

IF(Pw(I, J + 1) .GT. Pw(I, J)) THEN
   Zx = J + 1
ELSE
   Zx = J
END IF

Twyp(I, J) = Kyhp(I, J) * dz * dex(I) / ((dey(J) + dey(J + 1)) / 2)
   - Tsw(I + 1, J + 1) - Tnw(I, J)

IF(Twyp(I, J) .LT. 0.0) THEN
   Twyp(I, J) = 0.0
   Tsw(I + 1, J + 1) = 0.0
   Tnw(I, J) = 0.0
END IF

Lwn(I, J) = 0.006328 * Krw(I, Zx) / (Myw * bw(I, Zx))
Lon(I, J) = 0.006328 * Kro(I, Zx) / (Myo * bo(I, Zx))
Zx = 0

20 CONTINUE

Qo(1, 1) = 0.0
Qw(1, 1) = qtot
Qo(imax, jmax) = 0.0
QW(IMAX,JMAX)=QTOT
QW(1,JMAX)=FW(1,JMAX)*QTOT
QO(1,JMAX)=-(1.0-FW(1,JMAX))*QTOT
QW(IMAX,1)=FW(IMAX,1)*QTOT
QO(IMAX,1)=-(1.0-FW(IMAX,1))*QTOT

A9(1,1)=ALPHA(1,1)*QW(1,1)+QO(1,1)
   A9(IMAX,JMAX)=ALPHA(IMAX,JMAX)
   *QW(IMAX,JMAX)+QO(IMAX,JMAX)
A9(1,JMAX)=ALPHA(1,JMAX)*QW(1,JMAX)+QO(1,JMAX)
A9(IMAX,1)=ALPHA(IMAX,1)*QW(IMAX,1)+QO(IMAX,1)

DO 28 I=1,IMAX
DO 28 J=1,JMAX

B(I,J)=TWYP(I,J-1)*(ALPHA(I,J)*LWN(I,J-1)+
   LON(I,J-1))

D(I,J)=TWXP(I-1,J)*(ALPHA(I,J)*LWE(I-1,J)+
   LOE(I-1,J))

F(I,J)=TWXP(I,J)*(ALPHA(I,J)*LWN(I,J)+
   LOE(I,J))

H(I,J)=TWYP(I,J)*(ALPHA(I,J)*LWN(I,J)+
   LON(I,J))

NE(I,J)=TSW(I+1,J+1)*(ALPHA(I,J)*LWSW(I+1,J+1)+
   LOSW(I+1,J+1))*(PW(I+1,J+1)-PW(I,J))

SE(I,J)=TNW(I+1,J-1)*(ALPHA(I,J)*LWNW(I+1,J-1)+
   LONW(I+1,J-1))*(PW(I+1,J-1)-PW(I,J))

NW(I,J)=TNW(I,J)*(ALPHA(I,J)*LWNW(I,J)+LONW(I,J)+
   PW(I,J)-PW(I-1,J+1))

WS(I,J)=TSW(I,J)*(ALPHA(I,J)*LWSW(I,J)+LOSW(I,J)+
   PW(I,J)-PW(I-1,J-1))

E(I,J)=-B(I,J)-D(I,J)-F(I,J)-H(I,J)
   +VR(I,J)*PHI(I,J)*
   +ALPHA(I,J)*SW(I,J)*(CW+CF)/BW(I,J)
   +(1-SW(I,J))*(CW+CF)/BW(I,J)
   +CB(I,J)*SW(I,J)*BW(I,J)
   +PW(I,J)*SW(I,J)
   +CSW(I,J)*SW(I,J)
   +SW(I,J)*Sw(I,J)

R(I,J)=-VR(I,J)*PHI(I,J)
   +(ALPHA(I,J)*SW(I,J)*(CW+CF)/BW(I,J)
   +(1-SW(I,J))*(CW+CF)/BW(I,J))
   +PW(I,J)*DELTA-A9(I,J)
   +SE(I,J)*SW(I,J)-NW(I,J)
   +(TWXP(I,J)*LOE(I,J)*(PCOW(I+1,J)-PCOW(I,J)))-
CALL GAUSS(IMAX,JMAX,B,D,F,H,E,R,X)

IC=0
PAVG=0.0

DO 29 I=1,IMAX
DO 29 J=1,JMAX
  IC=IC+1
  PW(I,J)=X(IC)
  PAVG=PAVG+PW(I,J)
  BO(I,J)=BOB*(1-CO*PW(I,J))
  BW(I,J)=BWB*(1-CW*PW(I,J))
  PHI(I,J)=PHIB*(1+CF*(PW(I,J)-PB))
  ALPHA(I,J)=BW(I,J)/BO(I,J)
  CONTINUE

29  CONTINUE

PAVG=PAVG/MXMY
VPI=0.0
SUM=0.0
SSO=0.0

DO 31 I=1,IMAX
DO 31 J=1,JMAX
  SW(I,J)=SW(I,J)*(1-(CW+CF)*(PW(I,J)-PWOLD(I,J)))+
          DELT*BW(I,J)*(QW(I,J))
  +TWXP(I,J)*LWE(I,J)*(PW(I+1,J)-PW(I,J))
  +TWXP(I-1,J)*LWE(I-1,J)*(PW(I-1,J)-PW(I,J))
  +TWYP(I,J)*LWN(I,J)*(PW(I+1,J)-PW(I,J))
  +TWYP(I,J-1)*LWN(I,J-1)*(PW(I,J-1)-PW(I,J))
  +TNW(I+1,J-1)*LWNW(I+1,J-1)*(PW(I+1,J-1)-PW(I,J))
  +TSW(I,J)*LWSW(I,J)*(PW(I-1,J-1)-PW(I,J))
  +TSW(I+1,J)*LWSW(I+1,J)*(PW(I+1,J)-PW(I,J))
  +TNW(I,J)*LWNW(I,J)*(PW(I-1,J+1)-PW(I,J))
  +/(VR(I,J)*PHI(I,J))
  CONTINUE
WIP=0.0
OIP=0.0

DO 37 I=1,IMAX
DO 37 J=1,JMAX

   WIP=WIP+VR(I,J)*PHI(I,J)*SW(I,J)/BW(I,J)/5.6146
   OIP=OIP+VR(I,J)*(1-SW(I,J))*PHI(I,J)/BO(I,J)/5.6146
   VPI=VPI+PHI(I,J)*VR(I,J)

37 CONTINUE

VPI=VPI/5.6146
OP=OP+(QO(1,JMAX)+QO(IMAX,1))*DELT/5.6146
WP=WP+(QW(1,JMAX)+QW(IMAX,1))*DELT/5.6146
WI=WI+(QW(1,1)+QW(IMAX,JMAX))*DELT/5.6146
CUWL=CUMBE
CUMBE=1.0+(OWIP-WIP)/(WI+WP)
MBE=CUMBE-CUWL
CUOL=CUMMBO
CUMMBO=1.0+(OQIP-OIP)/OP
OMB=CUMMBO-CUOL
N=K*DELT

MD=DMOD(N,UT)
IF(MD .EQ. 0.0) THEN
   U3=N
END IF

IF(N .EQ. DELT) THEN
   IB=IB+1
   TELL(IB)=N
   OX(IB)=-OP/VPB
   WX(IB)=WI/VPB
   RDU(R)IB)=QO(1,JMAX)+QO(IMAX,1)/5.6146
   PMID(IB)=PAVG
ELSE IF(N .EQ. U3) THEN
   IB=IB+1
   TELL(IB)=N
   OX(IB)=-OP/VPB
   WX(IB)=WI/VPB
   RDU(R)IB)=QO(1,JMAX)+QO(IMAX,1)/5.6146
   PMID(IB)=PAVG
END IF
IF(N .EQ. UT .OR. N .EQ. 1650.0 .OR. 
+ N .EQ. 9400.0) THEN

WRITE(10,50) N
WRITE(10,54) MBE, CUMBE, OMB, CUMMBO, WIP, OIP
WRITE(10,80) WI, WP, OP, VPI
WRITE(10,52)

DO 40 J=JMAX,1,-1
WRITE(10,60)(PW(I,J), I=1, IMAX)
40      CONTINUE

WRITE(10,53)
DO 41 J=JMAX,1,-1
WRITE(10,61)(SW(I,J), I=1, IMAX)
41      CONTINUE

END IF
N=0.0
17      CONTINUE
WRITE (10,49)

DO 42 I=1, IBMAX
WRITE(10,51) TELL(I), OX(I), WX(I), RDU(I), PMID(I)
42      CONTINUE

50      FORMAT(//,5X,'TIME=',F7.2,' DAYS',//)
49      FORMAT(//,5X,'; Days ; PVOP ; PVWI ;
+ STBOPD ; PAVG(psi) ;',//)
51      FORMAT(5X;'---------|---------|--------|--------
+ |---------|--------|--------|--------|',//,
+ 10(5X;'|',F6.0,'|',F5.3,'+
+ ;|',F5.3,';|',F7.2,
+ ;|',F9.2,';',//),//)
52      FORMAT(/,5X,'-------- PRESSURE DISTRIBUTION
+ --------',//)
53      FORMAT(/,5X,'-------- WATER SATURATION PROFILE
+ ---',//)
60      FORMAT(/,1X,10(F8.1),//)
61      FORMAT(/,1X,10(F7.4,1X),//)
54 FORMAT(/,5X,'Instant Water Material Balance
  + Error:','F9.5,/
  + 5X,'Cumulative Water M.B.
  + Error..........:','F9.5,/
  + 5X,'Instant Oil Material Balance
  + Error.........:','F9.5,/
  + 5X,'Cumulative Oil M.B.
  + Error..........:','F9.5,/
  + 5X,'Water in Place,
  + STB...............:','F12.2,/
  + 5X,'Oil in Place,
  + STB...............:','F12.2,/
)

70 FORMAT(/,5X,'----- IMPES Oil - Water Simulation
  + Model ------:','/
  + 5X,'Number of Grid Blocks :','I5,/
  + 5X,'Number of Timesteps :'','I5,/
  + 5X,'Original Water in Place :'','F12.2,'/
  + STB',/
  + 5X,'Original Oil in Place :'','F12.2,'/
  + STB',/
  + 5X,'Movable Pore Volume :'','F12.2,'/
  + BBLS',/
  + 5X,'End-Point Mobility Ratio:','F12.2,/
)

80 FORMAT(/,5X,'----- Cumulative Volumes, STB
  + -----:','/
  + 5X,'----- WI -----:----- WP -----:----- OP -----:
  + -Pore Vol.,BBL-',/
  + 5X,';','F9.1,1X,';','F9.1,1X,';',
  + F9.1,1X,';','F11.1,4X,';',/
  + 5X,';------------------------:------------------------:
  + ;------------------------:','/

CLOSE(UNIT=10)
STOP
END
Appendix B

HETEROGENEOUS PERMEABILITY DISTRIBUTION

SUBROUTINE Sperm(KX,KY)

DOUBLE PRECISION KX(5,20),KY(5,20)

KX(1,1)=12.5  
KX(2,1)=13.4  
KX(3,1)=12.9  
KX(4,1)=12.9  
KX(5,1)=11.1  
KX(1,2)=12.3  
KX(2,2)=13.0  
KX(3,2)=11.7  
KX(4,2)=13.3  
KX(5,2)=11.3  
KX(1,3)=11.7  
KX(2,3)=11.9  
KX(3,3)=13.0  
KX(4,3)=12.1  
KX(5,3)=13.4  
KX(1,4)=13.0  
KX(2,4)=13.4  
KX(3,4)=12.1  
KX(4,4)=11.5  
KX(5,4)=14.1  
KX(1,5)=12.8  
KX(2,5)=14.1  
KX(3,5)=10.9  
KX(4,5)=11.8  
KX(5,5)=13.1  
KX(1,6)=12.8  
KX(2,6)=13.2  
KX(3,6)=14.2  
KX(4,6)=11.8  
KX(5,6)=12.3  
KX(1,7)=13.0  
KX(2,7)=12.2  
KX(3,7)=10.8  
KX(4,7)=12.0  
KX(5,7)=12.6  
KX(1,8)=12.0  
KX(2,8)=11.3  
KX(3,8)=13.4  
KX(4,8)=11.3  
KX(5,8)=12.1
<table>
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<tr>
<th>(i, j)</th>
<th>Value</th>
</tr>
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</tr>
<tr>
<td>(2, 9)</td>
<td>12.2</td>
</tr>
<tr>
<td>(3, 9)</td>
<td>12.4</td>
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<tr>
<td>(4, 9)</td>
<td>14.0</td>
</tr>
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<td>(5, 9)</td>
<td>12.8</td>
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<tr>
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<td>13.9</td>
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<td>(2, 10)</td>
<td>12.2</td>
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<td>11.7</td>
</tr>
<tr>
<td>(4, 10)</td>
<td>11.3</td>
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<td>(5, 10)</td>
<td>12.5</td>
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<td>12.9</td>
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<tr>
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<td>12.4</td>
</tr>
<tr>
<td>(3, 11)</td>
<td>13.7</td>
</tr>
<tr>
<td>(4, 11)</td>
<td>11.7</td>
</tr>
<tr>
<td>(5, 11)</td>
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<td>(4, 12)</td>
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<td>12.5</td>
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<tr>
<td>(1, 13)</td>
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<tr>
<td>(2, 13)</td>
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</tr>
<tr>
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<tr>
<td>(2, 17)</td>
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</tr>
<tr>
<td>(1, 18)</td>
<td>13.2</td>
</tr>
<tr>
<td>(2, 18)</td>
<td>14.5</td>
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<tr>
<td>(3, 18)</td>
<td>12.5</td>
</tr>
<tr>
<td>(4, 18)</td>
<td>12.7</td>
</tr>
</tbody>
</table>
\begin{align*}
KX(5,18) &= 12.5 \\
KX(1,19) &= 13.6 \\
KX(2,19) &= 11.8 \\
KX(3,19) &= 13.0 \\
KX(4,19) &= 11.7 \\
KX(5,19) &= 12.5 \\
KX(1,20) &= 12.9 \\
KX(2,20) &= 12.4 \\
KX(3,20) &= 13.7 \\
KX(4,20) &= 11.7 \\
KX(5,20) &= 14.0 \\
KY(1,1) &= 12.9 \\
KY(2,1) &= 12.4 \\
KY(3,1) &= 13.7 \\
KY(4,1) &= 11.7 \\
KY(5,1) &= 14.0 \\
KY(1,2) &= 11.8 \\
KY(2,2) &= 11.1 \\
KY(3,2) &= 12.7 \\
KY(4,2) &= 12.4 \\
KY(5,2) &= 13.0 \\
KY(1,3) &= 11.0 \\
KY(2,3) &= 14.1 \\
KY(3,3) &= 11.9 \\
KY(4,3) &= 12.7 \\
KY(5,3) &= 12.2 \\
KY(1,4) &= 14.0 \\
KY(2,4) &= 11.2 \\
KY(3,4) &= 12.0 \\
KY(4,4) &= 13.1 \\
KY(5,4) &= 11.5 \\
KY(1,5) &= 11.1 \\
KY(2,5) &= 13.1 \\
KY(3,5) &= 14.0 \\
KY(4,5) &= 12.0 \\
KY(5,5) &= 13.6 \\
KY(1,6) &= 13.1 \\
KY(2,6) &= 11.3 \\
KY(3,6) &= 12.3 \\
KY(4,6) &= 13.9 \\
KY(5,6) &= 12.1 \\
KY(1,7) &= 13.4 \\
KY(2,7) &= 12.5 \\
KY(3,7) &= 11.3 \\
KY(4,7) &= 13.2 \\
KY(5,7) &= 11.1 \\
KY(1,8) &= 12.4 \\
KY(2,8) &= 12.2 \\
KY(3,8) &= 12.9
\end{align*}
KY(4,8) = 13.2
KY(5,8) = 11.6
KY(1,9) = 12.0
KY(2,9) = 11.1
KY(3,9) = 13.7
KY(4,9) = 11.7
KY(5,9) = 12.2
KY(1,10) = 11.8
KY(2,10) = 11.2
KY(3,10) = 12.1
KY(4,10) = 13.1
KY(5,10) = 12.5
KY(1,11) = 12.0
KY(2,11) = 13.4
KY(3,11) = 14.6
KY(4,11) = 11.4
KY(5,11) = 12.5
KY(1,12) = 12.9
KY(2,12) = 13.2
KY(3,12) = 11.6
KY(4,12) = 12.0
KY(5,12) = 11.1
KY(5,12) = 12.5
KY(1,13) = 13.1
KY(2,13) = 12.4
KY(3,13) = 11.7
KY(4,13) = 13.3
KY(5,13) = 12.5
KY(1,14) = 12.2
KY(2,14) = 12.4
KY(3,14) = 13.4
KY(4,14) = 12.8
KY(5,14) = 12.5
KY(1,15) = 12.5
KY(2,15) = 12.8
KY(3,15) = 11.5
KY(4,15) = 12.2
KY(5,15) = 12.5
KY(1,16) = 11.6
KY(2,16) = 11.8
KY(3,16) = 12.8
KY(4,16) = 13.7
KY(5,16) = 12.5
KY(1,17) = 11.7
KY(2,17) = 13.9
KY(3,17) = 12.8
KY(4,17) = 12.1
KY(5,17) = 12.5
KY(1,18) = 11.7
KY(2,18)=14.5
KY(3,18)=13.4
KY(4,18)=11.2
KY(5,18)=12.5
KY(1,19)=12.5
KY(2,19)=15.7
KY(3,19)=12.9
KY(4,19)=13.0
KY(5,19)=12.5
KY(1,20)=14.0
KY(2,20)=11.2
KY(3,20)=12.0
KY(4,20)=13.1
KY(5,20)=11.5

RETURN

STOP

END
Appendix C

FLUID CHARACTERISTICS

SUBROUTINE FWC(IMAX, JMAX, MYO, SW, FW, KRW, KRO, PCOW)
INTEGER IMAX, JMAX
REAL SWI, SWR, SOR, KRWST, KROST, NW, NO, MYW, MYO
DOUBLE PRECISION FW(5, 20), SW(5, 20), PCOW(5, 20)
DOUBLE PRECISION KRW(5, 20), KRO(5, 20), X1

SWI=0.2750000000000000
SWR=0.2750000000000000
SOR=0.3750000000000000
KRWST=0.3000000000000000
KROST=1.0000000000000000
NW=3.4000000000000000
NO=2.0000000000000000
MYW=0.0500000000000000

DO 99 I=1, IMAX
DO 99 J=1, JMAX

    KRW(I, J)=KRWST*((ABS((SW(I, J)-SWR)
+                        /(1-SWR-SOR)))*NW)

    KRO(I, J)=KROST*((ABS((1-SW(I, J)-SOR)
+                        /(1-SWR-SOR)))*NO)

    FW(I, J)=(KRW(I, J)/MYW)
+                        /(KRW(I, J)/MYW+KRO(I, J)/MYO)

PCOW(I,J)=2*(1-(SW(I,J)-SWR+0.1)/(1-SWR-SOR))
PCOW(I,J)=0.0

IF(SW(I,J) .GE. (1-SOR))THEN
    KRO(I,J)=0.0
    FW(I,J)=1.0
ELSE IF(SW(I,J) .LE. SWR)THEN
    KRW(I,J)=0.0
    FW(I,J)=0.0
END IF

99      CONTINUE
RETURN
END
Appendix D

BANDED GAUSS ELIMINATION

SUBROUTINE GAUSS(IMAX,JMAX,B,D,F,H,E,R,X)

C
A BANDED GAUSS ELIMINATION ALGORITHM WITH NO PIVOTING

DOUBLE PRECISION A(100,41),G(100),X(100)
INTEGER N,IM,IMN,IMAX,JMAX,IC
DOUBLE PRECISION MULT,SUM
DOUBLE PRECISION B(5,20),D(5,20),F(5,20),H(5,20)
DOUBLE PRECISION E(5,20),R(5,20)

N=IMAX*JMAX
IC=0
DO 60 I=1,IMAX
   DO 70 J=1,JMAX
      IC=IC+1
      G(IC)=R(I,J)
      70   CONTINUE
   60 CONTINUE

DO 75 K=1,N
   DO 65 J=1,N
      A(K,J)=0.0
   65 CONTINUE
75 CONTINUE

IC=0
DO 90 I=1,IMAX
   DO 80 J=1,JMAX
      IC=IC+1
      A(IC,IC)=E(I,J)
      IF((IC+1) .LE. N)THEN
         A(IC,IC+1)=H(I,J)
      END IF
      IF((IC+JMAX) .LE. N)THEN
         A(IC,IC+JMAX)=F(I,J)
      END IF
      IF(IC .GT. JMAX)THEN
         A(IC,IC-JMAX)=D(I,J)
      END IF
      IF(IC .GT. 1)THEN
         A(IC,IC-1)=B(I,J)
      END IF
90 CONTINUE

C
END IF

80     CONTINUE
90     CONTINUE

IM=JMAX
IMN=0

DO 10 K=1,N-1
IMN=MIN0(K+IM,N)
    DO 20 I=K+1,IMN
        MULT=A(I,K)/A(K,K)
        G(I)=G(I)-MULT*G(K)
            DO 30 J=K,IMN
                A(I,J)=A(I,J)-MULT*A(K,J)
        30     CONTINUE
20     CONTINUE
10     CONTINUE

IMN=0
X(N)=G(N)/A(N,N)

DO 40 K=N-1,1,-1
SUM=0.0
IMN=MIN0(K+IM,N)
    DO 50 J=K+1,IMN
        SUM=SUM+A(K,J)*X(J)
        X(K)=(G(K)-SUM)/A(K,K)
    50     CONTINUE
40     CONTINUE
RETURN
END
Appendix E

GRID CELL CONNECTIONS

The purpose of this appendix is to illustrate how the grid cells are connected in the three different finite-difference schemes studied in this project. Figure E-1 shows that the center grid cell, for which pressure is to be found, is connected to the four grid blocks at its parallel faces, marked with solid arrows.

<table>
<thead>
<tr>
<th></th>
<th>i, j+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>i-1, j</td>
<td>i, j</td>
</tr>
<tr>
<td></td>
<td>i, j-1</td>
</tr>
</tbody>
</table>

Figure E-1: The five-point grid cell connections.

The grid cell connections of the nine-point scheme includes the corner blocks, as shown in figure E-2.

The split operator scheme is similar to the nine-point scheme but the important difference is that the corner blocks are connected explicitly to the center block in the split operator scheme, illustrated with dashed arrows.
Figure E-2: The nine-point grid cell connections.

Figure E-3: The split operator’s grid cell connections.