A NEW ALGORITHM TO SOLVE LARGE SCALE MINE PRODUCTION SCHEDULING PROBLEMS BY USING THE LAGRANGIAN RELAXATION METHOD

by

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ABSTRACT

In open pit mining, the life of the mine production schedule defines the timing and the magnitude of annual cash flows. For a large scale mining project, optimization of the production schedule that maximizes the Net Present Value (NPV) of the cash flows is a challenge because of the size and the complexity of the operations. To determine optimum mine production schedules, the Mixed Integer and Linear Programming (MILP) approach has been used to model mining operations with complex operational constraints. The full scale MILP model of a mine consists of a significantly large number of variables and constraints, and requires specially crafted algorithms to solve it, in a reasonable amount of time.

This dissertation develops an algorithm and presents a methodology to solve large scale MILP models for complex mining operations using the Lagrangian relaxation method of the Operations Research (OR) technique. The proposed method defines two smaller tractable subproblems by way of the Lagrangian relaxation of a pre-determined set of constraints. The algorithm iteratively solves each of the proposed subproblems, moving toward optimality until a desired level is achieved. The solutions from the two subproblems help to define a tighter feasible region resulting in elimination of variables and constraints from consideration during the construction of the MILP model. In this process, the size of the actual MILP model is efficiently reduced without violating optimality. As such, the proposed method significantly reduces the size of the MILP model, and improves the solution time of the MILP approach for large scale, multi-time period production scheduling problems. This contribution is illustrated in a large scale open pit mine case study.
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CHAPTER 1

OVERVIEW OF MINE PLANNING

1.1 INTRODUCTION

Mining is defined as an ore extraction activity from the earth by excavating the ground. To generate mine planning, there are several important steps that have to be followed. The first step in the mine planning is economic block model generation. After an economic block model is developed based on available geologic information, an ultimate pit limit is determined to define the limits of the deposit that is economically mineable. Then pushbacks or phases are designed to be used as guidance for yearly production scheduling. Since mine planning deals with large amounts of data sets, optimality of each step is critical to the resulting production scheduling. The next section reviews these mine planning steps. The cutoff grade concept and the influence of the cutoff grade policy to the project’s net present value (NPV) are also discussed in this chapter.

Due to limited computing power, finding the optimum yearly production scheduling is quite challenging. A mathematical formulation of a production schedule and a current solution time issue are discussed in the next chapter.

1.2 REVIEW OF THE STEPS OF MINE PLANNING

At the initial stage of the mining project, exploration effort collects massive amounts of geologic information. The geologic data are analyzed by using the
geostatistical approach and modeled as a three dimensional (3-D) block model as shown in Figure 1.1.

Figure 1.1: 3-D block model for an open pit mine (Crawford and Davey, 1979)

Once a 3-D geologic block model is developed, initial assumption of equipment capacities and related costs, and process recoveries, are made and grade information is converted into economic values by using predicted commodity prices. Then the ultimate pit limit is designed by analyzing how deep the deposit can be economically mined while keeping predetermined overall slope angles. Positive valued blocks have to pay off removal costs of all overlaying waste blocks. The optimum ultimate pit limit is the one that yields the maximum profit among many other feasible pit limit designs.
To find pit limits, a heuristic ‘Moving Cone’ method or a graph theory based optimization technique, ‘Lerchs and Grossmann algorithm’ (Lerchs and Grossmann 1964) and its modified algorithm (Zao and Kim 1992), are widely incorporated into the commercial mine planning packages. The Moving Cone method is computationally faster and easier to implement, but it is known to have logical shortcomings in optimizing pit limits, whereas the Lerchs and Grossmann algorithm guarantees optimum pit limits. Pit limits are analyzed on the basis of undiscounted values and there is no consideration of equipment capacity limitations. As such, the ultimate pit limit can be viewed as a single time period production scheduling with unlimited operational capacities. Figure 1.2 shows an example of the ultimate pit limit.

Figure 1.2: Example of the ultimate pit limit (Dagdelen, 2004)
In reality, depletion of the deposit cannot be completed in just one year because of capacity limitations. Therefore, a consideration of the ‘time value of money’ has to be incorporated in the determination of production scheduling. A final pit shape may be smaller than the ultimate pit limit because ore blocks located at a deep portion of the pit can only be mined at later years of the project, and the waste removal usually takes place at earlier years. Therefore, the discounting effect of these positive ore blocks is larger than the overlying negative waste blocks. As a result, an ore block that can be economically mined based on undiscounted value may not be economically mined, when block values are discounted differently, depending on when they are mined.

After the ultimate pit limit is determined, pushbacks or phases are determined as guidance for yearly production scheduling. The procedure starts with the nested pits generations. These nested pits are generated by changing the economic parameters such as commodity prices. This procedure is often referred as a “Pit Parameterization.” If commodity prices are set lower than ones used in the ultimate pit limit model, a new pit limit will become smaller because some of the ore blocks will no longer pay off overlying waste blocks. Then they will be excluded from the pit limit. A series of different economic parameters will generate a series of different sizes of pit limits. By taking the equipment’s working width in each bench and the haul road location into account, these nested pits become pushbacks or phases. The smallest pit contains the highest economic value per ton of the material, and subsequent pits have lower average values. A pushback corresponding to the highest valued pit is referred to as the first pushback, and the other pushbacks will be numbered according to pit values. Although the time value of money is not included in pushback designs, mining from the first pushback to later ones over the project life will improve the project’s NPV. This improvement occurs because cash flows in earlier years are improved compared to slicing the entire ultimate pit limit. This pushback design is used as guidance for annual production scheduling. Figure 1.3 shows an example of a pit cross section showing different pushbacks.
Once pushbacks are designed, the next step is to come up with the yearly production scheduling by considering all the operational constraints such as mining and mill capacities, truck availability, blending requirements, sequencing constraints, and maximum bench advance rates per year for each pushback, depending on the complexity of the project. As mentioned above, economic value calculations at this stage have to be based on discounted values. When all the operational constraints are taken into account, completing the first pushback and moving to the second pushback may not generate the highest NPV. Therefore, multiple pushbacks should be mined simultaneously if the strategic target is to maximize the NPV of the project. In a large scale open pit mine operation, typically 3 or 4 pushbacks are mined in a given year (See Figure 1.4). To analyze all the possible scenarios to determine one scheduling that will maximize the NPV, the model size can easily grow large.
Since the production scheduling has to follow certain sequences, e.g., overlying blocks or benches have to be mined before mining underlying ore blocks, binary variables (0 or 1) are introduced in mathematical formulation. Adding one binary variable doubles the possible number of combinations; a size of the problem and the solution time grow exponentially. Due to limited computing power, large scale open pit production scheduling models are known to be quite challenging.

For a decision as to whether a given block is mined, and where to be sent requires one binary variable for each block and for each time period. If a block model contains one million blocks and if there are 10 time periods, there will be 10 million binary variables. If this kind of problem were solved by a branch and bound algorithm in one of today’s available computers, a solution would not be found even after running several days or weeks. Even a single feasible solution, not an optimum solution, might not be found. When production scheduling is developed as part of the corporation’s strategic
decision making processes, various runs are required to answer series of 'what if' questions. So the software has to provide solutions within a limited amount of time with high optimality levels.

To overcome this issue of long solution time, many of the commercial packages use the heuristic approach to find a solution to production scheduling. This approach is easier to implement and faster to get a series of solutions. But there is no guarantee that each of the results is truly optimum to the objective of scheduling, usually maximizing the NPV.

1.3 DEFINITION OF THE CUTOFF GRADE

A cutoff grade is defined as a grade that discriminates between ore and waste under a given deposit characteristics and given operational and economical parameters. From the grade-tonnage distribution of the deposit, a decision of what portion of the deposit is to be processed and what portion to be wasted needs to be made for a reserve calculation. These calculations are based on the cutoff grade determination. Therefore, resulting recoverable reserves are based on the cutoff grade policy.

A breakeven cutoff grade is defined as a grade at which the value obtained from the mill or process equals the cost of wasting the block. If a block is located within the ultimate pit limit and an assumption is made that all the blocks within the pit limit are mined, then the cost used to calculate the breakeven cutoff grade is the difference between the total cost incurred by sending a block to the mill and the total cost incurred by sending it the waste dump. A simplified example, assuming there is no additional dumping cost, is illustrated in Figure 1.5, followed by a mathematical explanation discussed in Lane (1964) and Dagdelen (1992).
Figure 1.5: Example of the breakeven cutoff grade calculation

A value per ton of a block if a block is sent to the mill \((V_{mill})\) is calculated as:

\[ V_{mill} = (P - S) \times g \times y - m - c \]  \[1.1\]

If there is no additional dumping cost, the value per ton of a block when a block is sent to the waste dump \((V_{damp})\) is simply:

\[ V_{damp} = -m \]  \[1.2\]

Where

\(P\): Metal price

\(S\): Sales and refinery cost
\( g: \) Grade of the block  
\( y: \) Recovery  
\( m: \) Mining cost  
\( c: \) Processing cost

Definition of the breakeven cutoff grade \((COG_{BE})\) is at which \(V_{mili} = V_{dump}\). Therefore, it can be expressed as:

\[
(P - S) \times COG_{BE} \times y - m - c = -m
\]

Mining costs can be dropped from both sides, and then the breakeven cutoff grade is calculated as:

\[
COG_{BE} = \frac{c}{(P - S) \times y}
\]  \[1.3\]

Regardless of the block destination, the mining cost is paid to remove the block if it is mined, so the mining cost is dropped from the breakeven cutoff grade calculation, Equation \[1.3\]. If a deposit is mined by using the breakeven cutoff grade, the total of undiscounted cash flows will be maximized.

To maximize the NPV for long term production scheduling, cutoff grades need to be chosen by considering the time value of money. Cutoff grades should be set to be higher grades than the breakeven cutoff grade during early years of the operation. By increasing cutoff grades in a given year, average ore grades will be increased, therefore the resulting revenue in a given year will become higher. A “Dynamic cutoff grade policy” and a “Declining cutoff grade policy” are suggested to improve the project’s NPV. The concept is to get inflows of money (revenues) as soon as possible and outflows of money (operating costs) as late as possible. But, of course, in order to get revenues by
mining ore materials, certain portion of the waste materials have to be removed. Also, to mine high grade ore materials located in the deep portion of the pit as early as possible, a stripping of the waste materials has to be done aggressively. This may result in uneconomical cash flows for the first few years of the project. But by doing so, the overall NPV of the project can be improved in comparison to delaying the waste stripping during the initial few years. Therefore, a cutoff grade determination has to consider locations of the ore, grade tonnage distributions, and all other operational constraints in the project. Therefore, scheduling has to consider multi-time periods.

When stockpiles are available, the marginal materials between the chosen cutoff grade and the breakeven cutoff grade can be stockpiled, and then rehandled in later years, if these marginal materials cover additional stockpiling and rehandling costs. Thus, the stockpile option makes the problem more complicated.
CHAPTER 2

PRODUCTION SCHEDULING

2.1 INTRODUCTION

Production scheduling has been a challenging part of mining projects because it deals with diverse factors and entails an enormous amount of information. A project’s profitability will largely depend on production scheduling. If a project is subject to inefficient scheduling, this may affect the project’s economic outcomes, and could also result in unsustainable resource development.

Since the ultimate pit limit determination is based on predetermined equipment capacity and cost assumptions, the production scheduling outcome has to match the initial capacity and related cost assumptions. It is observed that there is a circular relationship between each of the stages in mine planning (Pana, 1965, Pana et al., 1966, Johnson, 1968). These relationships are illustrated by Dagdelen (1985), which is shown in Figure 2.1. Optimizing the production scheduling usually requires multiple runs to evaluate many alternative strategic scenarios. In reality, the pushback design may have many alternatives, several different capital expenditure scenarios may need to be evaluated, equipment selections may have many alternatives, additional exploration efforts provides updated block model, and so on.
Figure 2.1: Circular relationships in mine planning (Dagdelen 1985)

As an industry’s decision making tool for actual projects, the solvability and the flexibility, as well as the optimality, of the algorithm are the key elements. This is because the current open pit mine operation is mainly a large scale operation with complex processing flows. The mining engineers need to evaluate various alternative options within a limited amount of time by using currently available tools.
2.2 OPTIMIZATION OF THE PRODUCTION SCHEDULING

In simplest term, the production scheduling involves decision as to

1. If a given block should be mined
2. When it should be mined
3. Once it is mined, how it should be processed

The resulting answers to these questions above define annual cash flows of a given project that will impact the NPV. The production scheduling optimization is to come up with answers to above questions for each block in the block model, such that NPV of future annual cash flows is maximized. The following small example will demonstrate this concept.

Assume that the ultimate pit limit is defined, and four blocks are located within the pit limit. Each block has two possible destinations; mill and dump. Figure 2.2 shows a 2-D block model. For each block, undiscounted economic values based on the breakeven cutoff grade are also presented.

![Figure 2.2: Example with four blocks showing undiscounted economic values](image)

Figure 2.2: Example with four blocks showing undiscounted economic values
Block \((i, j) = (1, 1)\) and \((1, 3)\) have grades less than the breakeven cutoff grade, so the block value shows only mining cost assuming there is no additional dumping cost as shown in Equation [1.2]. Block \((1, 2)\) and \((2, 2)\) have grades greater than breakeven cutoff grade, therefore the values are calculated based on Equation [1.1] discussed in Chapter 1. Assume that the mill capacity is limited to be one block per year, and the mining capacity is unlimited to keep the discussion as simple as possible. Level \(i = 1\) blocks can be mined at any time but level \(i = 2\) block, block \((2, 2)\), can only be mined after level \(i = 1\) blocks are all removed.

If a discount rate is assumed to be 10%, then the scheduling to maximize NPV by using the breakeven cutoff grade is achieved by mining block \((1, 2)\) as ore in year 1, and by mining block \((1, 1)\), \((1, 3)\) as waste, and \((2, 2)\) as ore in year 2 as shown in Figure 2.3. Numbers in the blocks show discounted economic values by using the following equation:

\[
BV' = \frac{BV_o}{(1 + disc)^t}
\]

Where

- \(BV':\) Discounted block value at the end of year \(t\)
- \(BV_o:\) Undiscounted block value
- \(disc:\) Discount rate

Resulting NPV by mining with breakeven the cutoff grade at 10% of the discount rate is “9.17”, calculated as follows:

\[
0.91 - 0.83 - 0.83 + 9.92 = 9.17
\]

\[\framebox{0.91 - 0.83 - 0.83 + 9.92 = 9.17}\]

Year 1 Year 2
If a dynamic cutoff grade policy is used for this example and if a stockpile is available, the schedule should aim to mine block (2, 2) in year 1 rather than in year 2 if it yields higher NPV. To mine block (2, 2) in year 1, block (1, 2) needs to be treated as a stockpile material in year 1, because the mill capacity is limited at one block per year.
Figure 2.4: Dynamic cutoff grade scheduling
Assuming that the block value of block (1, 2), which is “1” in Figure 2.2, is calculated by a revenue of “4”, a term “(P - S) × g × y” in Equation [1.1], a mining cost of “-1”, a term “-m” in Equation [1.1], and a process cost of “-2”, a term “-c” in Equation [1.1]. And a stockpile rehandle cost is assumed to be “0.1”. Then a higher NPV can be obtained by mining all the blocks and process block (2, 2) in year 1, and rehandling block (1, 2) from the stockpile in year 2, as illustrated in Figure 2.4.

Resulting NPV is “9.75”, calculated as follows:

\[-0.91 - 0.91 - 0.91 + 10.91 + 1.57 = 9.75\]

\[\text{Year 1} \quad \text{Year 2}\]

Since potential ore blocks (1, 2) and (2, 2) are both mined and only (2, 2) is processed in year 1, a mill cutoff grade in year 1 is higher than the breakeven cutoff grade. This example shows that the dynamic cutoff grade policy can provide higher NPV compared to the breakeven cutoff grade policy. If this applies to large scale deposits, differences can become large.

To find the optimum cutoff grades, Lane (1964) suggested a heuristic algorithm to maximize the NPV of the project under mining, milling and refinery constraints. In his algorithm, a determination of cutoff grades has to consider additional costs associated with not receiving the future cash flows quicker due to the cutoff grade decision taken now. Since the remaining cash flows are larger in earlier years of the project, declining cutoff grades are obtained. The future cash flows that are used to determine optimum cutoff grades are the function of cutoff grades used to calculate future cash flows. This cyclical relationship requires iterative calculations before converging to the optimum cutoff grades. Iterations are repeated until input future cash flows equal resulting cash flows with a certain tolerance. This approach is widely implemented in commercial mine
planning packages. But they are still in a limited use of optimization because the cutoff grades are optimized under the predetermined mine sequences (Thomas 2001). Also, in a realistic deposit model where grade distributions are not homogeneous and economic factors are time dependent, the best cutoff grade policy is no longer the simple declining cutoff grade policy (King 2001). If enough amounts of high grade ore materials are not accessible in early years of the project, cutoff grades have to be decreased to keep the mill full.

When there are multiple processes, such as mills and leaches, and if stockpiles are available, a decision as to where the material should be sent is based on multiple cutoff grades. If there are multiple material types in the deposit, such as oxide material and sulfide material, cutoff grades are dependent on the material types if process costs and recoveries differ depending on the material types. If the heuristic approach is applied to the complicated operation with various constraints, the number of iterative calculations can grow very large; then the solution may not converge within a reasonable amount of time. Figure 2.5 shows an example of an open pit mine operation with multiple destinations and multiple material types.

Therefore, a selection of the cutoff grade policy has to be done strategically because it is a function of many different parameters such as deposit characteristics, economic parameters, process recoveries, and all the operational constraints. If all of those aspects are not considered simultaneously, it will cause sub-optimum decision of the production scheduling.
Figure 2.5: Open pit mine operation at the Round Mountain Gold Mine (Nevada, USA)
To simplify the problem, a schedule can be incrementally generated by only considering shorter time periods at a time. For instance, one can determine the year 1 schedule first, and then determine the following schedules year by year. This way, model sizes in each run are small, therefore, an overall schedule can be found quickly. But there is a pitfall in this methodology, because it misleads decisions of the mine sequences and the cutoff grades.

The optimization for a shorter time periods than the life of the mine considers only limited areas that can be mined during these time periods. To optimize the mine sequence and the cutoff grades properly, it is important to consider the life of the mine. For instance, if a bottom bench in a given pushback contains high grade ore but they can be mined only after year 3 due to limited mining capacities, then the schedule should consider stripping overlying materials with the full capacities in year 1 and 2. This way, high grade ore can be available to be mined in year 3. With this scheduling, cash flows in year 1 and 2 may not be very high, because a large amount of waste stripping generates large costs, but a high profit in year 3 may yield high NPV. If a model considers only one year at a time, then waste stripping in year 1 and 2 may not be very aggressive because high grade ore is not available in this model. This will cause a delay to reach high grade ore, resulting sub-optimum scheduling.

The Mixed Integer and Linear Programming (MILP) approach that is discussed in the next chapter can model life of the mine scheduling. This mathematical programming approach is very powerful and provides complete flexibility in modeling complex operational environments as we see in many large scale operations.
2.3 PREVIOUS WORK

The previous literatures in the area of open pit mine planning optimization can be viewed by categorizing them into three different topics:

1) Ultimate pit limit determination
2) Pushback design
3) Production scheduling

Previous work is reviewed by these categories.

2.3.1 Ultimate Pit Limit Determination

In 1965, Lerchs and Grossmann proposed a graph-theory based algorithm, known in the industry as the “LG” algorithm, to find an ultimate pit limit that guarantees to yield the maximum profit under a given overall slope angle of the pit (Lerchs and Grossmann 1965). In the LG algorithm, each block is represented as a node, then arcs are generated by connecting two nodes. A tree is defined as a graph in which any two nodes are connected by only one arc. Each arc contains attributes to keep the cumulative values of the arcs within the mass or the closure. The direction of the arcs depicts the waste block supported by the ore blocks under the slope constraints. The trees are transformed until the entire waste block nodes are connected by the arcs. The set of nodes that gives a maximum closure is the blocks that are within the ultimate pit limit.

Zhao and Kim (1992) improved the LG algorithm by utilizing the concept of ore-waste support as well as the geometric constraints. Many of the recent commercial mine planning packages using the LG algorithm were able to implement the optimum ultimate pit limit determinations (Alford and Whittle 1986). These algorithms replaced the conventional, heuristic approach of the ‘Moving Cone’ method, which is quick and easy
to implement but known for its logical problem of finding a joint ore support for the waste blocks. Another approach to find an ultimate pit limit of an open pit mine is the use of a Network Flow algorithm. Johnson (1968) observed that the structure of the ultimate pit limit is a network flow and suggested an ultimate pit limit determination method based on the Maximum Flow (Max-Flow) algorithm to solve the problem.

Underwood and Tolwinski (1996) analyzed the solution steps of the Lerchs and Grossmann algorithm and discussed the relationship with the Dual Simplex algorithm. They observed that some of the pit slope constraints are ignored during each step of the algorithm, but fewer constraints are violated as the steps progress. When the algorithm steps terminate, the solution does not have any violation and it reaches the optimum solution. These steps are related to the steps of Dual Simplex algorithm solution methodology. This is also equivalent to solving the dual of the Max-Flow problem, which can be viewed as solving the Minimum Cost Flow (Min-Cost) problem.

### 2.3.2 Pushback Design

After the ultimate pit limit is determined, incremental pits are generated within the ultimate pit limit. After evaluating these pits, then the pushbacks are designed. To generate many small pits in a pit limit, Matheron (1975a, 1975b) introduced the concept of ‘Pit Parameterization’ for pushback design. It was further developed by Francois-Bongarcon and Marechal (1976), and Dagdelen and Francois-Bongarcon (1982), and implemented in a commercial software by Whittle (1986). By changing one or two economic parameters that are used in block value calculation then different sizes of the pit limits called ‘nested pits’ are generated for each set of parameters. Since each pit limit for a given parameter set is calculated by pit limit optimization method such as the Lerchs and Grossmann algorithm or the Max-Flow algorithm, each of these nested pits
has a highest profit. Commonly used parameters are a commodity price, or a ratio of commodity price to mining cost (Whittle 1988).

Seymour (1995a) extended the LG algorithm so that each of the generated trees (small pits) keeps the cumulative pit information, typically the tonnage and the cumulative pit value. This algorithm generates different sizes of the pits at the end of the steps. This method also guarantees that pits containing maximum metal will be found. The relationship between pit size and pit cumulative value can then be analyzed to evaluate how much profit the deposit can potentially generate.

These nested pits are also used for pushback or phase designs. Wang and Sevim (1993) presented an algorithm to sequence the nested pits to come up with production scheduling. Seymour (1995b) applied a Dynamic Programming (DP) algorithm to find, by exhaustive search, mining sequences and optimum cutoff grades to obtain maximum NPV. DP has the flexibility of adding constraints, such as minimum after tax cash flow rates each year. The objective of the method can be maximizing NPV, or maximizing metal production, or minimizing refractory ore process amounts, depending on the target of the project.

When the production scheduling is developed by sequencing the existing pushbacks or the phases, a problem called 'gap' may occur. A 'gap' is a discrepancy between annual production rates and the incremental size of each nested pit. Since the incremental size cannot be controlled by a parameterization, because a geologic location of the high grade and low grade materials cannot be controlled, target production rates may not be matched by sequencing the nested pits when size increment between two consecutive pits is too large. Due to this gap problem, many of the existing production scheduling algorithms can fail. Wang and Sevim (1995) proposed a methodology to generate nested pits with a controllable size increment to meet the target of yearly mining rates. The method heuristically searches the least metal content blocks by using the inverted cone template, and eliminates these blocks from the block model. The remaining blocks of each of the elimination steps have the highest probability of being the maximum metal
pits of their size. Since the elimination process is done by using a predetermined number of the blocks, the size increment of nested pits can be controlled. This algorithm does not guarantee each of the generated pits to be truly optimum, but the gap problem is eliminated.

2.3.3 Production Scheduling

The ultimate pit limit design is a determination of the final shape of the pit by using undiscounted prices and costs. The nested pit generation is a series of the ultimate pit limits by using a parameterization to generate different sizes of the pits. There are no capacity limitations such as mining, or process capacities, and there are no blending constraints. To develop realistic production scheduling, many factors need to be taken into account when maximizing the project's NPV. Johnson (1968) applied a Linear Programming (LP) approach to optimize the production scheduling, by using 3-D block model information. He solved the multi-time period production scheduling problem by applying the Danzig-Wolfe decomposition technique. His approach is to decompose the scheduling problem into master and subproblems and solve the multi-time period problem by solving a series of ultimate pit limit problems iteratively. Gershon (1982) extended Johnson's LP formulation by adding more operational constraints to generate realistic production scheduling models.

Dagdelen (1985) and Dagdelen and Johnson (1986) extended the pit parameterization concept and applied the Lagrangian relaxation method and the subgradient method to solve the multi-time period production scheduling problem. The multi-time period production scheduling formulation is transformed into the relaxed problem by incorporating some of the constraints into the objective function by introducing Lagrange multipliers. Then the problem is decomposed into series of single time period problems such that the LG algorithm or the Max-Flow algorithm can be
applied. The subgradient method is applied to find a set of Lagrange multipliers in order to meet the production target.

Akaike (1999) extended the Lagrangian relaxation method and introduced the 4D Network Relaxation method so that the LG algorithm or the Max-Flow algorithm can be applied to solve multi-time period production scheduling problems, instead of directly solving them by using the Integer Programming (IP) method. This method was applied to schedule individual blocks to specify when a given block should be mined and where it should be sent so that NPV is maximized. Mine sequencing and process optimization are solved simultaneously under the relaxed constraints. This approach improves the applicability of the optimization approach to large scale production scheduling problems because it takes advantage of the mathematical structure of the formulation. Since the mining capacity constraints are treated as side constraints and relaxed from the original formulation, a gap problem may occur depending on the grade distribution in the deposit.

Another way of solving production scheduling problem is a reblocking approach. Ramazan (2001) proposed the fundamental tree algorithm to reduce the number of integer variables. In his algorithm, blocks are aggregated into new units called ‘fundamental trees’. A fundamental tree is defined as an aggregate of a minimum number of the blocks which have cumulative positive values which can be mined without violating the slope constraints. The reblocking processes are done by the LP method, and the schedule is optimized by arranging these fundamental trees. These two methods do not require the pushback designs prior to production scheduling, but these algorithms have not yet been implemented to solve the large scale open pit problems with the various operational constraints. The possibilities of assigning the complex operational constraints need to be further investigated.

In many deposits, extraction of completing one pushback followed by the next pushback does not provide cash flows that generate the highest NPV for the project. Many of the commercial mine planning packages generate production scheduling by mining from multiple pushbacks in the same time period by an exhaustive search and
evaluating the best mining sequences (Cai 1993). To speed up the computation time, Tolwinski (1998) proposed to combine the blocks and update the attribute of the combined blocks. In most of these cases, the production scheduling starts with finding the mine sequencing by using the constant cutoff grade. A breakeven cutoff grade is normally used, and the cutoff grade is optimized under the determined mine sequences. Lane (1964) proposed a heuristic approach to determine the cutoff grade to maximize the project’s NPV. This approach is further discussed by Lane (1988), Dagdelen (1992), and Whittle and Wharton (1995).

The concept of Lane’s algorithm is to use a cutoff grade higher than the breakeven cutoff grade for production scheduling during early years of the project. This will provide higher cash flows in earlier years and, therefore, the resulting NPV is improved. This algorithm iteratively finds the optimum cutoff grades by taking the future cash flows into account in determining each year’s cutoff grades. Lane also extended his theory to be able to have mining, processing and refining constraints (Lane 1988). Recent sophisticated production scheduling tools deal with various mill throughputs depending on the rock types. The mill capacity is determined as a function of blending the materials instead of using fixed tons (Wooller 2001). Lane’s heuristic approach has been modified to improve the solution time, and has been implemented in commercial packages (Wooller 2004).

This heuristic method works well in relatively simple models, but when the process flow gets more complicated, the solution to determine the optimum extraction sequence and the cutoff grade may not converge in a reasonable amount of time. Also, if the mine sequencing is determined by using a constant cutoff grade prior to optimizing the cutoff, resulting sequences may not be truly optimum. Therefore, cutoff grades determined under sub-optimum sequences yield a sub-optimum economic outcome for the project.

When the mine has a complex interaction of operational constraints in addition to mining, processing and refining constraints, development of the optimum production scheduling becomes more challenging. Dagdelen (1996) proposed a special Mixed
Integer Linear Programming (MILP) formulation, which will be described in detail in Chapter 3, to model and solve large scale open pit production scheduling problems under various operational constraints by using a block aggregation method. Blocks with similar characteristics, such as the same rock type and the same grade interval located in the same sequence, are aggregated. Mine sequencing constraints are controlled by binary variables, and the cutoff grade determinations and the material flow related constraints are controlled by linear variables. The concept of this MILP approach is demonstrated and discussed on a small scale model given in Urbaez and Dagdelen (1999). This approach was applied to production scheduling of complex gold ore mining and processing at Newmont Mining Corporation’s Carlin Trend Operation (Hoerger, Bachmann, Criss and Shortridge 1999).

Unlike heuristic algorithms, this MILP based optimization approach guarantees to optimize mine sequencing, cutoff grades, and process flows simultaneously. Cutoff grade optimization concepts based on the MILP approach have been discussed for open pit mine applications (Dagdelen and Kawahata 2005). Although MILP is a powerful tool, when it is applied to the large scale open pit mine model, long solution time is a serious problem due to the number of the variables and the constraints.

To overcome the solution time issue, Smith (2001) attempted to use the solution obtained by heuristic approaches as a starting point of the MILP calculation. Since a heuristic approach can provide several good scheduling scenarios quickly, this information can be used to reduce the feasible area of the MILP model. Although there is no guarantee that the scheduling is truly optimal. This approach was able to make the MILP problem to be able to solve within reasonable amount of time. In this proposed method, the production rates in the MILP problem are bounded by the best and the worst scheduling results from the heuristic approach.

For underground operation, Topal (2003) and Kuchta et al. (2003, 2004) proposed to assign the earliest and the latest possible start times to machine placements so that the number of the binary variables can be reduced. In his underground mine model, one
binary variable is assigned to each machine placement and each time period. To reduce
the number of the binary variables, the earliest and the latest start times are calculated for
each machine placement. Then unnecessary variables are eliminated from the model.
Earliest start time for a given machine placement is determined by the sequencing and
other physical constraints. The latest start time is determined by the consideration of
machine placement successors to ensure a given machine placement starts early enough
so that it does not affect the ore production in later time periods. Since each machine
placement does not need variables before the earliest start time and after the latest start
time, unnecessary variables can be eliminated.

Whittle (2004) also addressed the issue of an excessive sizes of full scale
optimization models and applied a heuristic search algorithm that coherently works with
the LP optimization approach. He also mentioned the application of the random sampling
approach to provide the capability of handling larger and more complex models.

Incorporation of deposit’s uncertainties is another important issue in mine design.
Dimitrakopoulos et al. (2002) discussed potential uncertainties and risks that may arise in
the mining project. They proposed to quantify these uncertainties in the optimization
proposed a formulation based on consideration of expected block grades, and applied to
the multi element deposits in the mine. They further discussed a methodology to include
the uncertainty of the ore body derived from the variability of in-situ grades, which may
cause a failure to meet the production targets of the project. The objective function is
formulated by including the probability that a given block is mined in a certain time
period derived from the simulation method (Ramazan and Dimitrakopoulos 2004).
Dagdelen and Coskun (1998) discussed a potential risk derived from the misclassification
of the selective mining units (SMU), which would result in an incorrect decision.
Consequently, the materials would be sent to an improper destination. The conditional
simulation method is applied to characterize the cost of block misclassification errors.
To include the uncertainty in the decision making process for a large scale, complex operation, the solvability of the MILP problem within a reasonable amount of time is still the key element for further implementation.

2.4 THESIS OBJECTIVE

When formulating a large scale production scheduling problem, the solution time for the large scale MILP problem is directly dependent on number of linear and binary variables, and constraints. Especially, the number of binary variables that are used to model sequencing requirements between mining areas affects the solution time significantly. In a typical MILP formulation consisting of multiple mines, multiple phases and multiple processes with multiple time periods, the number of binary variables becomes more than a couple of thousand. This results in a solution time to the MILP problem to be in excess of days or sometimes more than weeks. In order to achieve an optimum or near optimum solution in a reasonable amount of time (less than three hours), the number of binary variables used in the MILP formulation needs to be reduced.

This dissertation research will focus on developing an algorithm to overcome solution time issues for large scale, multi-time period production scheduling problems in the MILP model by way of variable reduction technique.

At the first step of the study, general mine planning concepts and cutoff grade discussions are reviewed. This will demonstrate how each of the mine planning steps is interactively related to one another. Then, discussion of the MILP method (Dagdelen 1996) to provide optimum mine sequences and cutoff grades simultaneously, will be presented. The discussion will demonstrate the flexibility of the MILP formulation which can handle various operational constraints in the long term production scheduling model.
To reduce the number of binary variables, the Lagrangian relaxation method proposed by Dagdelen (1985) can be used to solve multi-time period production scheduling problems. The proposed method was applied to the Integer Programming (IP) formulation of the production scheduling (block by block) model to determine when a given block should be mined, and where it should be sent. The application of this method was demonstrated on a small scale model due to the available computing power at that time. The research work in this dissertation will investigate the applicability of this method to solve the MILP model for large scale production scheduling problems with various operational constraints.

The original contribution from this research will be an investigation of the applicability of the Lagrangian relaxation method to provide guidance for the MILP production scheduling model. It will improve the solution time of the MILP model without violating optimality. It is envisioned that the solution time for the large scale production scheduling problems will be drastically reduced. Therefore, realistic, complex multi-time period production scheduling problem will be solved to optimality within a reasonable amount of time.

Chapter 3 presents the MILP formulation for the production scheduling problem as presented in Dagdelen (1996). Then Chapter 4 presents the general discussion of the Lagrangian relaxation method and the subgradient technique. The Chapter 5 discusses the proposed solution algorithm to solve large scale production scheduling MILP problems by applying the Lagrangian relaxation method.

The proposed approach is implemented on a large scale, open pit mine production scheduling model representing a case study coming from a real world project, which is discussed in Chapter 6. Chapter 7 presents another case study to demonstrate the applicability of the MILP approach to the strategic planning by analyzing a series of different run options. The conclusion of the dissertation will be made and the recommendations for the future research will be discussed in the Chapter 8.
CHAPTER 3

MATHEMATICAL FORMULATION OF THE PRODUCTION SCHEDULING PROBLEM

3.1 INTRODUCTION

In this chapter, the mathematical optimization approach is presented to solve multi-time period production scheduling problems. The Mixed Integer Linear Programming (MILP) approach, proposed by Dagdelen (1996), is reviewed to demonstrate how this approach can be utilized in realistic, large scale production scheduling models where complex operational constraints exist.

Because of the flexibility to include various operational constraints, the MILP approach is considered to be a powerful mathematical approach. It guarantees to provide the optimum solution by achieving the best mine sequence and the best cutoff grade policy simultaneously (Dagdelen 1996). However, the number of the variables and constraints can easily increase if a large scale model is formulated and all the complex factors of the operation are included. Because the number of the variables and the constraints are large, the solution time becomes an issue for a large scale production scheduling model.

To overcome this solution time issue, the Lagrangian relaxation method, presented in Chapter 4, is applied to reduce the model size of the MILP problem efficiently, without violating optimality. This proposed solution algorithm is presented in Chapter 5.
3.2 PROBLEM DEFINITIONS

Production scheduling determines when a given material is mined from various sources, and where a given material is sent. Sources are typically pits or underground mines or initial stockpile inventories, and destinations can be mills, leach pads or waste dumps. The objective of the scheduling varies project by project depending on the management philosophy. However, in this dissertation, an objective of the scheduling is maximizing the NPV of the project, which is the most common target that many companies are using.

The MILP approach takes all the possible material flows into account over the life of the mine, and determines the best scheduling and the best cutoff grade policy among many alternatives. An overview of the production scheduling is illustrated in Figure 3.1.

A “source” is defined as the original location of the materials before they are mined. Sources are divided into “sequences,” where predetermined sequencing arrangements have to be followed when they are mined. In order to start mining a given sequence, all of the previous sequences have to be completely mined out. These sequencing arrangements are usually determined by pushback designs. A given sequence is further broken down into another unit called “increments.” A determination of the increments is predefined based on scheduling criteria. The average grade and the tonnage in each increment are assumed to be known. When the materials are mined from certain increments and sent to one destination, economic factors such as commodity prices and related costs, and recoveries, are assumed to be known.
Figure 3.1: Overview of production scheduling
A cutoff grade is defined as a grade that is used to differentiate between ore and waste. If there are multiple processes or dumps, multiple cutoff grades are used to differentiate material's destination. In the MILP model, dynamic cutoff grades are determined as the result of scheduling, because the results show when a given increment should be mined and where it should be sent. A cutoff grade concept is illustrated in Figure 3.2.

In the MILP model, a set of blocks that belongs to the same source, sequence and increment are aggregated. These aggregated blocks are in the same location because they belong to the same source and the same sequence. Also, these blocks have similar rock properties and grades because they belong to the same increment. The block aggregation concept is illustrated in Figure 3.3.

By this block aggregation, there is no need to assign one binary variable to each block and each time period. Instead, a linear variable is assigned to each increment and each time period. Since a given increment can be mined partially in a given time period, an increment can be mined over multiple time periods, or a part of the increment is sent to the mill, and the rest of it is sent to the dump if necessary.

Since mining activities usually take place over multi-time periods, all the activities are associated with discounted economic values. When deciding the best production scheduling, discounted economic values are used in order to consider the time value of money. Various constraints are formulated to reflect realistic operational conditions. In this dissertation, only some of the commonly used constraints are discussed, but the model can be extended to include more constraints.
Figure 3.2: Cutoff grade concept

Figure 3.3: Block aggregation in the MILP model
3.3 NOTATIONS AND DECISION VARIABLES

The notations that are used in this dissertation are defined as follows:

\( i \): Source (\( i = 1, \ldots, I \))

\( j \): Sequence (\( j = 1, \ldots, J \))

\( \Gamma_j \): Set of sequence indices that have to be mined out before starting sequence \( j \)

\( k \): Increment (\( k = 1, \ldots, K \))

\( d \): Destination (\( d = 1, \ldots, D \))

\( t \): Time period an activity is taken (\( t = 1, \ldots, T \))

\( P^t \): Price of main commodity in time period \( t \)

\( S^t \): Sales and refinery costs in time period \( t \)

\( \text{Att} \): Total number of attributes

\( \text{Att}^n P^t \): Price of \( n^{th} \) attribute in time period \( t \)

\( \text{Att}^n S^t \): Sales and refinery costs of \( n^{th} \) attribute in time period \( t \)

\( g_{i,j,k} \): Grade of main commodity located in source \( i \), sequence \( j \), increment \( k \)

\( \text{Att}^n g_{i,j,k} \): Grade of \( n^{th} \) attribute located in source \( i \), sequence \( j \), increment \( k \)

\( y_{i,j,k,d} \): Recovery of commodity mined from source \( i \), sequence \( j \), increment \( k \), and sent to destination \( d \)

\( \text{Att}^n y_{i,j,k,d} \): Recovery of \( n^{th} \) attribute mined from source \( i \), sequence \( j \), increment \( k \), and sent to destination \( d \)

\( m^t_{i,j,k} \): Mining cost of the material mined from source \( i \), sequence \( j \), increment \( k \) in time period \( t \) (To the exit of the source)

\( tp^t_{i,d} \): Transportation cost from exit of source \( i \) to destination \( d \) in time period \( t \)

\( c^t_{i,d} \): Processing cost at destination \( d \) in time period \( t \)
\textit{Res}_{i,j,k}: \quad \text{Reserve at source } i, \text{ sequence } j, \text{ increment } k

disc: \quad \text{Discount rate of the project}

Z: \quad \text{Objective function}

Decision variables are defined as follows:

\begin{align*}
X_{i,j,k,d}^t & : \quad \text{Tons mined from source } i, \text{ sequence } j, \text{ increment } k \text{ and sent to destination } d \text{ in time period } t \\
Y_{i,j}^t & : \quad \begin{cases} 
1 & \text{if source } i, \text{ sequence } j \text{ is completely mined out by the end of time period } t \\
0 & \text{otherwise} 
\end{cases} \quad \text{(Binary variables)}
\end{align*}

3.4 \textbf{MIXED INTEGER AND LINEAR PROGRAMMING (MILP) MODEL}

In mining operations, all the material movements are associated with economic activities. When a given material is sent to the process destination, sales revenues and operating costs are generated. Prior to the mathematical formulation for the optimization model, discounted values of these sales revenues and operating costs are calculated.
Sales revenue per ton of material mined from source $i$, sequence $j$, increment $k$ and sent to destination $d$ in time period $t$ is calculated as follows:

\[
SR'_{i,j,k,d} = \left[ (P' - S') \times g_{i,j,k} \times y_{i,j,k,d} \times \sum_{n=1}^{\text{NR}} \left[ (\text{Att}'' P' - \text{Att}'' S') \times \text{Att}'' g_{i,j,k} \times \text{Att}'' y_{i,j,k,d} \right] \right] \times \frac{1}{(1 + \text{disc})^t}
\]

[3.1]

Attributes in this model can be any values that are associated with each block. It can be byproducts that generate revenues, or chemical values that are used for blending constraints. If a given attribute does not generate revenue, then associated price would be 0.

Destinations include both processing sites and waste dumps. If a given destination is a waste dump, then the associated recovery would be 0.

Operating cost per ton of material mined from source $i$, sequence $j$, increment $k$ and sent to destination $d$ in time period $t$ is calculated as follows:

\[
OC''_{i,j,k,d} = (m'_{i,j,k} + tp'_{i,d} + c'_d) \times \frac{1}{(1 + \text{disc})^t}
\]

[3.2]
Objective function

Objective function is to maximize the NPV by analyzing all the possible material flows throughout the life of the mine.

\[
MaxZ = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{d=1}^{D} \sum_{t=1}^{T} \left( SR'_{i,j,k,d} - OC'_{i,j,k,d} \right) X'_{i,j,k,d}
\]  

[3.3]

Constraints

The objective function is subject to the following constraints:

1. Reserve constraints

The material mined is up to what is available in the geologic reserves \( Res_{i,j,k} \).

\[
\sum_{d=1}^{D} \sum_{t=1}^{T} X'_{i,j,k,d} \leq Res_{i,j,k} \quad \forall i, j, k
\]  

[3.4]
2. Mining capacity constraints

Mining capacity at each source can be limited in each time period. $Mcap'_i$ is the upper limit capacity at source $i$ in time period $t$.

$$\sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{d=1}^{D} X'_{i,j,k,d} \leq Mcap'_i \quad \forall i, t \quad [3.5]$$

3. Global mining capacity constraints

Total mining tons from all the sources can be limited in each time period. $GMcap'$ is the upper limit capacity in time period $t$.

$$\sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{d=1}^{D} X'_{i,j,k,d} \leq GMcap' \quad \forall t \quad [3.6]$$

4. Process capacity constraints

Process capacity at each process destination can be limited in each time period. $Pcap'_d$ is the upper limit capacity at process destination $d$ in time period $t$.

$$\sum_{j=1}^{J} \sum_{k=1}^{K} X'_{i,j,k,d} \leq Pcap'_d \quad \forall d \in \{Process\}, t \quad [3.7]$$
5. Attribute blending constraints

Attribute blending constraints at each process destination can be limited as lower and upper bound in each time period. $Attr^n L_d$ is the lower bound and $Attr^n U_d$ is the upper bound of blending constraints for $n^{th}$ attribute at process destination $d$.

$$\sum_{i=1}^{l} \sum_{j=1}^{j} \sum_{k=1}^{K} (Attr^n g_{i,j,k} - Attr^n L_d) \times X'_{i,j,k,d} \geq 0$$

$$\sum_{i=1}^{l} \sum_{j=1}^{j} \sum_{k=1}^{K} (Attr^n g_{i,j,k} - Attr^n U_d) \times X'_{i,j,k,d} \leq 0 \quad \forall d \in \{Process\}, t$$

[3.8]

6. Attribute cumulative amount constraints

Attribute cumulative amount constraints at each process destination can be limited as lower and upper bound in each time period. $Attr^n CL_d$ is the lower bound and $Attr^n CU_d$ is the upper bound of cumulative amount constraints for $n^{th}$ attribute at process destination $d$.

$$\sum_{i=1}^{l} \sum_{j=1}^{j} \sum_{k=1}^{K} Attr^n g_{i,j,k} \times X'_{i,j,k,d} \geq Attr^n CL_d$$

$$\sum_{i=1}^{l} \sum_{j=1}^{j} \sum_{k=1}^{K} Attr^n g_{i,j,k} \times X'_{i,j,k,d} \leq Attr^n CU_d \quad \forall d \in \{Process\}, t$$

[3.9]
7. Mine proportion constraints

Within sequence \( j \), all the increments are mined proportionally by keeping the original grade tonnage distribution.

\[
\sum_{d=1}^{D} X'_{i,j,k,d} = \left( \sum_{k=1}^{K} \sum_{d=1}^{D} X'_{i,j,k,d} \right) \left( \frac{\sum_{k=1}^{K} Re_{s_{i,j,k}}}{\sum_{k=1}^{K} Re_{s_{i,j,k}}} \right) \quad \forall i, j, k, t
\]  \[3.10\]

8. Sequencing constraints

To control sequencing constraints, binary variables \( Y'_{i,prev} \) are introduced. A sequence \( j \) can only be mined after the set of sequences \( \forall prev \in \Gamma j \) are all mined out. The following set of constraints control these sequencing arrangements.

\[
\sum_{k=1}^{K} \sum_{d=1}^{D} \sum_{\theta=1}^{\theta} X'_{i,\theta,prev,k,d} - Y'_{i,prev} \sum_{k=1}^{K} Re_{s_{i,prev,k}} \geq 0 \quad \forall i, prev \in \Gamma j, t
\]

\[
\sum_{k=1}^{K} \sum_{d=1}^{D} \sum_{\theta=1}^{\theta} X'_{i,\theta,k,d} - Y'_{i,prev} \sum_{k=1}^{K} Re_{s_{i,j,k}} \leq 0 \quad \forall i, j, prev \in \Gamma j, t
\]

[3.11]

9. Non-negativities

All the decision variables have to be non-negative numbers.

\[
X'_{i,j,k,d} \geq 0 \quad \forall i, j, k, d, t
\]

\[
Y'_{i,j} \geq 0 \text{ and binary} \quad \forall i, j, t
\]

[3.12]
These are typical constraints for the open pit mine production scheduling model. Because of the flexibility of the MILP formulation logic, other set of the constraints and the stockpile option can easily be incorporated as discussed in Hoerger et al. (1999).

3.5 MILP MODEL SIZE AND THE SOLUTION TIME

The MILP formulation includes binary variables to control the sequencing arrangement. A binary variable is turned on (=1) when a given sequence is mined out. In the MILP model, one binary variable is associated with each sequence and each time period. Therefore, the number of the binary variables can be a large number. The next example shows this size issue.

If there are three material sources with 50 sequences in each source, if each source is broken down into 30 increments, if there are three destinations, a mill, a leach and a dump, and if the mine life is projected to be 15 years, then the number of the variables can be calculated as follows:

**Number of linear variables:**

\[ i \times j \times k \times d \times t = 3 \times 50 \times 30 \times 3 \times 15 = 202,500 \]

**Number of binary variables**

\[ i \times j \times t = 3 \times 50 \times 15 = 2,250 \]
The number of the constraints depends on the complexity of the model. If there are mining and process capacity constraints, and if increments have to be mined proportionally by keeping the original grade tonnage distribution, then the number of the constraints can be calculated as follows:

**Reserve constraints:**

\[ i \times j \times k = 3 \times 50 \times 30 = 4,500 \]

**Mining capacity constraints:**

\[ i \times t = 3 \times 15 = 45 \]

**Process capacity constraints:**

\[ d \in \{ \text{Process} \} \times t = 2 \times 15 = 30 \]

**Mine proportion constraints:**

\[ i \times j \times k \times t = 3 \times 50 \times 30 \times 15 = 67,500 \]

**Sequencing constraints:**

The number of the sequencing constraints varies depending on how the sequences are related to each other. One example is:

\[ i \times \text{prev} \in \Gamma \times i \times 2 \text{ sets of constraints} = 3 \times 49 \times 15 \times 2 = 4,410 \]
Total number of constraints:

\[4,500 + 45 + 30 + 67,500 + 4,410 = 76,485\]

As can be seen in these calculations, the number of the variables and the constraints can grow large easily. Considering today’s computing power, it is difficult to solve a MILP problem with this size. But in reality, mining operations can be even larger sizes than this example.

For example, Figure 3.4 illustrates a large scale operation in North America. This operation has 14 sources in the region. The largest source has a total of 254 sequences, and each sequence is broken down to 255 increments. To process gold, there are two mills, an oxide mill and a roaster mill, and two leach pads, a run of mine leach and a crushed leach. A given material can be directly sent to the mill, or sent by way of a combination of bio oxidation and flotation circuits to improve recoveries. These options add up to 6 different material flows (destinations). Including one waste dump, there are 7 possible destinations.

If this model is formulated as the MILP problem for next 10 years, then the number of the linear variables will become close to 10 million, and the number of the binary variables will become more than 7 thousand. These model size calculations are shown in Table 3.1.
Figure 3.4: A large scale operation in North America
Table 3.1: The model size of a large scale operation in North America

<table>
<thead>
<tr>
<th>Sources (i)</th>
<th>Sequences (j)</th>
<th>Increments (k)</th>
<th>Destinations (d)</th>
<th>Time periods (t)</th>
<th># of linear variables</th>
<th># of binary variables</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>255</td>
<td>7</td>
<td>10</td>
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<td>2,540</td>
</tr>
<tr>
<td>2</td>
<td>98</td>
<td>255</td>
<td>7</td>
<td>10</td>
<td>1,749,300</td>
<td>980</td>
</tr>
<tr>
<td>3</td>
<td>114</td>
<td>255</td>
<td>7</td>
<td>10</td>
<td>2,034,900</td>
<td>1,140</td>
</tr>
<tr>
<td>4</td>
<td>37</td>
<td>238</td>
<td>7</td>
<td>10</td>
<td>616,420</td>
<td>370</td>
</tr>
<tr>
<td>5</td>
<td>19</td>
<td>255</td>
<td>7</td>
<td>10</td>
<td>339,150</td>
<td>190</td>
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<tr>
<td>6</td>
<td>4</td>
<td>40</td>
<td>7</td>
<td>10</td>
<td>11,200</td>
<td>40</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>42</td>
<td>7</td>
<td>10</td>
<td>20,580</td>
<td>70</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>42</td>
<td>7</td>
<td>10</td>
<td>14,700</td>
<td>50</td>
</tr>
<tr>
<td>9</td>
<td>11</td>
<td>27</td>
<td>7</td>
<td>10</td>
<td>20,790</td>
<td>110</td>
</tr>
<tr>
<td>10</td>
<td>37</td>
<td>56</td>
<td>7</td>
<td>10</td>
<td>145,040</td>
<td>370</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>26</td>
<td>7</td>
<td>10</td>
<td>30,940</td>
<td>170</td>
</tr>
<tr>
<td>12</td>
<td>25</td>
<td>45</td>
<td>7</td>
<td>10</td>
<td>78,750</td>
<td>250</td>
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<tr>
<td>13</td>
<td>20</td>
<td>39</td>
<td>7</td>
<td>10</td>
<td>54,600</td>
<td>200</td>
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<tr>
<td>14</td>
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<td>28</td>
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<td>10</td>
<td>215,600</td>
<td>1,100</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9,865,870</td>
<td>7,580</td>
</tr>
</tbody>
</table>

Other examples are illustrated in Figure 3.5 and Figure 3.6. These figures show the model views of a large scale operation in Africa. This operation has 10 sources in the region. 6 sources are located in a northern region, and 4 sources are located in a southern region. The largest source has 170 sequences. Each sequence is broken down to 20 increments. For strategic decision making purposes, several different cases have to be evaluated. One of the options is to have process facilities only in the southern region. Ore materials from the northern region have to be sent to the southern region, so the capital expenditures of this ore transportation have to be analyzed. Figure 3.5 illustrates this option. Another option is to construct process facilities at both regions. This is illustrated in Figure 3.6.

If this model is formulated for next 15 years by considering one mill and one dump to be possible destinations, then the number of the linear variables will become more than 1 million, and the number of the binary variables will become more than 7 thousand. These model size calculations are shown in Table 3.2.
Figure 3.5: A large scale operation in Africa (Case1)
Figure 3.6: A large scale operation in Africa (Case2)
Table 3.2: The model size of a large scale operation in Africa

<table>
<thead>
<tr>
<th>Sources (i)</th>
<th>Sequences (j)</th>
<th>Increments (k)</th>
<th>Destinations (d)</th>
<th>Time periods (T)</th>
<th># of linear variables</th>
<th># of binary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>82</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>114,800</td>
<td>820</td>
</tr>
<tr>
<td>2</td>
<td>39</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>54,800</td>
<td>390</td>
</tr>
<tr>
<td>3</td>
<td>170</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>238,000</td>
<td>1,700</td>
</tr>
<tr>
<td>4</td>
<td>22</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>30,800</td>
<td>220</td>
</tr>
<tr>
<td>5</td>
<td>51</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>71,400</td>
<td>510</td>
</tr>
<tr>
<td>6</td>
<td>56</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>78,400</td>
<td>560</td>
</tr>
<tr>
<td>7</td>
<td>77</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>107,800</td>
<td>770</td>
</tr>
<tr>
<td>8</td>
<td>141</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>197,400</td>
<td>1,410</td>
</tr>
<tr>
<td>9</td>
<td>94</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>131,600</td>
<td>940</td>
</tr>
<tr>
<td>10</td>
<td>19</td>
<td>20</td>
<td>7</td>
<td>10</td>
<td>26,600</td>
<td>190</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>1,051,400</strong></td>
<td><strong>7,510</strong></td>
</tr>
</tbody>
</table>

These examples show that how large the real world mining operations could be. Mining engineers have to evaluate many different scenarios within a limited time to make a decision. The algorithm’s solvability for large scale models is the key for strategic mine planning.

The MILP approach is potentially a powerful mathematical approach to find optimum mine sequences and cutoff grades simultaneously. However, this approach cannot handle a large scale model due to the excessive size of the problem and solution time. Therefore, additional methods need to be developed to overcome these issues. In Appendix B, some of the real world mining projects are presented.

The next chapter discusses general concepts of the Lagrangian relaxation method that is used in the proposed solution algorithm to solve large scale MILP models.
CHAPTER 4

LAGRANGIAN RELAXATION METHOD

4.1 INTRODUCTION

When a given problem is difficult to optimize, related but easier problems can be analyzed to find the optimum, or near optimum solution of the original model. "Relaxations" are an auxiliary optimization models formed by weakening either the constraints or the objective function of the original model (Rardin 1998).

There are several different ways of relaxing the problem. The LP relaxation method is often used in branch and bound searches. This method drops the integrality constraints and solves the problem by the simplex algorithm. In the production scheduling problem, if binary variables are relaxed and are allowed to have any numbers between 0 and 1, this problem will allow scheduling to start the next sequence without competing the previous sequence. Therefore, the results are not acceptable in reality.

A better relaxation approach that avoids starting the next sequence without completing the previous one in the production scheduling is the application of the Lagrangian relaxation method. This method is discussed in Fisher (1981) based on Held and Karp (1970, 1971) and Held et al. (1974), and applied to multi-time period production scheduling problem by Dagdelen (1985). This method relaxes a set of constraints that makes a problem difficult to solve, and it tries to push the solution to the feasible solution by penalizing violation of those relaxed constraints.

In this dissertation, the Lagrangian relaxation method is applied to reduce the size of the large scale production scheduling MILP problem. By this method, the number of the variables in the MILP model is drastically reduced, and by doing so, the optimum or near optimum solution can be found within a reasonable amount of time, without
violating any constraints. The flexibility of the MILP model discussed in Chapter 3 can be fully utilized.

4.2 GENERAL CONCEPT OF THE LAGRANGIAN RELAXATION METHOD

In this section, the general concept of the Lagrangian relaxation method is presented as an introduction. All the notations are presented in a general format, but they will be related to the production scheduling problem in the next chapter.

Suppose an IP formulation is given as follows:

(P)
Max $Z_p(x) = cx$

s.t.
$Ax \leq b$
$Dx \leq e$
$x \geq 0$ and integer

[4.1]

where $x$, $b$, $e$ are defined as column vectors, $c$ is a row vector, and $A$ and $D$ are defined as matrices with conformable dimensions. Assuming that optimizing the problem without having $Dx \leq e$ is relatively easy, a set of the constraints $Dx \leq e$ is then called “side constraints.” If we can relax these side constraints from the original problem, then the solution can be obtained easily.

This problem is called the “original problem” and denoted as (P).
By introducing a vector of non-negative multipliers \( \lambda \), called "Lagrange multipliers," side constraints are relaxed into the objective function as follows:

\[
\text{(LR)} \quad \begin{align*}
\text{Max} \quad Z_{LR}(x) &= cx + \lambda(e - Dx) \\
\text{s.t.} \quad Ax &\leq b \\
x &\geq 0 \text{ and integer}
\end{align*}
\]

This problem is called the "Lagrangian relaxed problem" and denoted as (LR). Note that the side constraints are relaxed but not completely dropped out. They are weighted in an objective function with Lagrange multipliers that work as penalties of violation. Therefore these multipliers will discourage the violations.

4.3 RELATIONSHIP BETWEEN THE ORIGINAL PROBLEM AND THE LAGRANGIAN RELAXED PROBLEM

Fisher (1982) discussed a number of important relationships between the original problem (P) and the Lagrangian relaxed problem (LR), and Dagdelen (1985) summarized these relationships within the context of open pit mine production scheduling optimization problem as follows:

Theorem 4.1

For any \( \lambda \geq 0 \), if the optimal solution for the Lagrangian relaxed problem (LR) is feasible to the original problem (P), then this optimal solution to the Lagrangian relaxed
problem will always be greater than or equal to the optimal solution of the original problem, i.e., \( Z_{LR} \geq Z_P \).

**Proof**

Since the solution for the Lagrangian relaxed problem is also feasible for the original problem, the condition \( \mathbf{Dx} \leq \mathbf{e} \) holds. Hence \( (\mathbf{e} - \mathbf{Dx}) \geq 0 \). Since \( \lambda \geq 0 \), the product \( \lambda (\mathbf{e} - \mathbf{Dx}) \geq 0 \).

Therefore, this gives

\[
\max c\mathbf{x} + \lambda (\mathbf{e} - \mathbf{Dx}) \geq \max c\mathbf{x}
\]

then

\[
Z_{LR} \geq Z_P
\]

From this theorem, it is proved that for any set of non-negative Lagrange multiplies, the solution for the Lagrangian relaxed problem that is feasible for the original problem provides an upper bound on the original problem. The next theorem presents the condition as to when the solution obtained from the Lagrangian relaxed problem optimum for the original problem.

**Theorem 4.2**

For a given \( \lambda^* \geq 0 \), if a vector \( \mathbf{x}^* \) satisfies the following three conditions

(i) \( \mathbf{x}^* \) is optimum in Lagrangian relaxed problem \( Z_{LR} \)

(ii) \( \mathbf{Dx}^* \leq \mathbf{e} \)

(iii) \( \lambda^* (\mathbf{e} - \mathbf{Dx}^*) = 0 \)

then \( \mathbf{x}^* \) is an optimum solution to the original problem (P).
Proof

The solution $x^*$ is clearly feasible in the original problem (P) since $Ax \leq b$ and $ Dx \leq e$ by condition (ii).

By condition (i)

$$
Z_{LR}(x^*) = cx^* + \lambda^*(e - Dx^*)
$$

and by condition (iii), $\lambda^*(e - Dx^*) = 0$, therefore, $Z_{LR}(x^*) = cx^*$ and this completes the proof.

4.4 STRATEGY FOR THE SOLUTION ALGORITHM

Advantages of solving the Lagrangian relaxation problem is that the solution yields an upper bound of the optimal solution for any set of multipliers $\lambda \geq 0$, as discussed in the previous section. Also, by definition, it is easy to be solved. Since the optimum solution to a given Lagrangian relaxed problem yields an upper bound on the optimum solution of the original model, finding the minimum value of the optimum solution of any valid Lagrangian relaxation problems is to find the optimum solution to the original problem. In other words, we are looking for the value of $\lambda \geq 0$, that yield the lowest upper bound.

Define $x(\lambda)$ as the solution variables of the Lagrangian relaxed problem as a function of a set of Lagrange multipliers $\lambda$, and $\varphi(\lambda) = Z_{LR}(x(\lambda))$ as the objective function value of the Lagrangian relaxed problem. To find the optimum solution in the original problem, a problem becomes how to find the best set of Lagrange multipliers so that the function $\varphi(\lambda) = Z_{LR}(x(\lambda))$ is minimized. This discussion yields the following problem:
(D)  
\[
\begin{align*}
\text{Min} \quad & \varphi (\lambda) = Z_{LR}(x(\lambda)) \\
\text{s.t.} \quad & \lambda \geq 0
\end{align*}
\]  \quad [4.3]

Since this problem can be viewed as a dual problem, it is denoted as (D) and often referred as a “Lagrangian dual” problem. Discussions about the duality are presented in the later section in this chapter.

**Corollary 4.3**

If \( x^*(\lambda^*) \) satisfies the optimality conditions presented in the theorem 4.2 for the original problem (P), then \( \lambda^* \) is the optimum for the dual problem (D).

**Proof**

We have \( Z_{LR}(x^*(\lambda^*)) = cx^* + \lambda^*(e - Dx^*) = cx^* = Z_P(x^*) \) by theorem 4.2. Since \( Z_P \leq Z_{LR} \) for all \( \lambda \geq 0 \), and for all feasible \( x \), then by theorem 4.1 \( \varphi (\lambda) \leq Z_{LR}(x(\lambda)) \) for all \( \lambda \geq 0 \).

Thus, the critical problem in solving the original problem becomes finding an optimum vector \( \lambda^* \) where \( \lambda^* \geq 0 \) such that \( \varphi (\lambda) = Z_{LR}(x(\lambda^*)) \) is minimum.

When this Lagrangian dual problem \( \varphi (\lambda) \) is viewed as a function of \( \lambda \), it has properties of the piecewise linear and the convex function (Fisher and Shapiro 1974, Fisher 1981), as conceptually illustrated in Figure 4.1.
In this example, the lowest point of the Lagrangian dual function $Z_{LR}(\mathbf{x}(\lambda))$ is the optimum solution. This piecewise linear and convex function can be iteratively minimized by using the technique called the "subgradient method." The next section discusses this subgradient method in detail.
4.5 SUBGRADIENT OPTIMIZATION

The subgradient method iteratively adjusts the multipliers according to the degree of violation in the previous solution. This method is widely accepted because of its performance to the convergence and applicability to the computer programming, and applied to solve the production scheduling optimization problem in Dagdelen (1985). Theoretical discussions of this method are presented in Held et al. (1974). Here, the basic steps of the subgradient method are presented again in order to solve the Lagrangian dual problem (D).

The Lagrange multipliers after $k^{th}$ iteration are denoted as $\lambda^k$, and the initial Lagrange multipliers are denoted as $\lambda^0$.

Step 1: Choose the initial Lagrange multipliers $\lambda^0 (k = 0)$.
Step 2: Solve the Lagrangian relaxed problem $Z_{LR}(x^k (\lambda^k))$.
Step 3: Calculate the violation of the side constraints $e - Dx^k$.
Step 4: Calculate the next set of the Lagrange multipliers as follows:

$$\lambda^{k+1} = \max \{0, \lambda^k - t_k \times (e - Dx^k)\}$$

Step 5: Set $k = k + 1$ and go to Step 2.

In Step 4, $t_k$ is defined as a step width. To guarantee this algorithm to converge to the optimum solution, the step width $t_k$ needs to satisfy the following conditions:

$$t_k \to 0$$

$$\lim_{k \to \infty} \sum_{i=0}^{k} t_i = \infty$$

The determination of the step size is discussed in Held and Karp (1971) and Held et al. (1974). The suggested method is as follows:
\[ t_k = \gamma_k \times \frac{Z_{LR}^k - Z_P^*}{\|\theta^k\|^2} \]

where

\[ \theta^k = \mathbf{e} - \mathbf{D}x^k \]

\( \gamma^k \): a scalar and \( 0 \leq \gamma^k \leq 2 \)

\( Z_P^* \): a lower bound of \( Z_{LR}^k \), often obtained by applying a heuristics approach to the original problem (P)

### 4.6 DUALITY

In the operations research study, a primal is defined as a given optimization model, which is the original problem formulated for the specific interest. A dual is defined as a subsidiary optimization model defined over the same input parameters as the primal, but characterizing the sensitivity of primal results to changes in inputs (Rardin 1998). In the previous formulations, the original problem (P) is a primal, and the Lagrangian dual (D) is a dual.

There are two important definitions in duality, which are the weak duality and the strong duality. The following definitions are quoted from Rardin (1998).

**Definition 4.4: Weak duality**

The primal objective function evaluated at any feasible solution to a maximized primal is less than or equal to the objective function value of the corresponding dual evaluated at any dual feasible solution.
Definition 4.5: Strong duality

If either a primal or its dual has an optimum solution, both do, and their optimum objective function values are equal.

The definition of the strong duality is the extension of the weak duality to exact equality when both primal and dual reach the optimum solution. As discussed in the previous section, the Lagrangian dual problem (D) yields an upper bound of the optimal solution of the original problem (P) for any multipliers $\lambda \geq 0$. This shows that the Lagrangian dual problem (D) is a weak dual of the original problem (P). Therefore, the function illustrated in Figure 4.1 is always the upper bound of the optimum solution in the original problem. In the production scheduling problem, since the objective function is to maximize the NPV, the Lagrangian dual problem provides an upper bound of the NPV that the project can generate.

4.7 CONDITION OF NON-CONVERGENCE

By using the Lagrangian relaxation method and the subgradient method, many of the difficult problems can be solved. However, this algorithm may not converge to the optimum solution depending on the problem structure. For the condition of the non-convergence, the following discussions are quoted from Dagdelen (1985):

The non-convergence can take place when there is no vector consisting of the Lagrange multipliers which will give a solution feasible to the constraint of the problem.
When it is not possible to find a set of the Lagrange multipliers to satisfy the original problem’s feasibility conditions, then it is said that the condition of “gaps” exists for the problem being solved (Everett 1963, Bazaraa and Shetty 1979).

The mathematical explanation for the existing gaps is that the feasible region of the dual space is not convex and as a result by using linear multipliers one can only determine the solutions which lie on the convex hull of the feasible region. This is graphically illustrated in Figure 4.2.

In Figure 4.2, if the right hand side requirement of the problem being solved is b, then the optimal solution is said to exist in the gap region and cannot be found by adjusting the multipliers. The only solutions which can possibly be found by the modification of the multipliers are the ones corresponding to the right hand sides b_1 and b_2, which lie on the convex hull shown in Figure 4.2.

Although there might be a case where the exact solution cannot be found, it is proved by Everett (1963) that the solutions which can be found by the Lagrange multiplier approach will still be optimum for the set of right hand sides which the existing solution satisfies. The Theorem 4.6 proves this.
Figure 4.2: Condition of the gaps
Theorem 4.6 (Everett)

Given the following two conditions are true:

1. \( \lambda^k, k = \{1, \ldots, n\} \) are non-negative multipliers
2. \( \{x^* \mid Ax^* \leq b\} \) maximizes the function \( Z_P(x) + \lambda^k (e - Dx) \) over all \( \{x \mid Ax \leq b\} \)

then:

3. \( x^* \) maximizes \( Z_P \) over all those \( \{x \mid Ax \leq b\} \) such that \( Dx \leq Dx^* \)

Proof

For the solution \( x^* \) of the Lagrangian relaxed problem to be optimum to the original problem, it must satisfy the optimality condition of theorem 4.2. Since replacing the right hand side of the relaxed constraint \( Dx^* \leq e \) with \( Dx^* = e \) will satisfy the conditions (ii) and (iii), and since \( x^* \) is the optimum solution to (i), \( x^* \) is optimum to the original problem for the case where \( e \) is replaced by \( e = Dx^* \).

In the case of the example of Figure 4.2, Theorem 4.6 says that if one can accept the solution with the right hand side equal to \( b_1 \) for example, instead of \( b \), the solution obtained from the Lagrangian multiplier approach will be optimum for this case. When the results obtained by the Lagrangian multiplier approach do not exactly satisfy the constraint requirements, it is also possible to measure the difference from the true optimum.

The following theorem discussed in Everett (1963) determines the deviations of the solution from the true optimum under the condition of gaps.
Theorem 4.7 (Everett)

If $x^*$ comes within $\varepsilon$ of minimizing the dual problem (D) for all \( \{ x \mid Ax \leq b \} \)

\[ Z_P(x^*) \leq Z_{LR}(x(\lambda)) - \varepsilon \]

then $x^*$ is an $\varepsilon$-optimal solution to the original problem with $\varepsilon = \lambda (e - Dx)$

Proof

From theorem 4.2, $Z_P(x^*) = Z_{LR}(x^*(\lambda^*))$; otherwise from theorem 4.1, $Z_P(x) \leq Z_{LR}(x(\lambda))$, let's say $Z_{LR}(x(\lambda))$ is $\varepsilon$ amount larger than $Z_P(x)$ then

\[ Z_P(x) = Z_{LR}(x(\lambda)) - \varepsilon \]

but $Z_{LR}(x(\lambda)) = Z_P(x) + \lambda (e - Dx)$. Substituting this equation into the equation above,

\[ Z_P(x) = Z_P(x) + \lambda (e - Dx) - \varepsilon \]

which gives $\varepsilon = \lambda (e - Dx)$, completing the proof.

Fisher and Shapiro (1974) observed that generally there exists a duality gap in the IP problem. In production scheduling applications, this results in a violation of the some of the relaxed constraints. Akaike (1999) stated that the violation of the mill constraints and the mining capacity constraints were observed in his case study results.

Based on these mathematical discussions, it is observed that a gap problem cannot be avoided as long as the Lagrangian relaxation method is applied to solve the production scheduling problems. To overcome the gap problem, this dissertation utilizes the
Lagrangian relaxation method to reduce the solution space and the variables of the MILP problem. Therefore, the optimum or near optimum solution can be achieved within a reasonable amount of time without violating any constraints in the MILP model. This proposed algorithm is discussed in the next chapter.
CHAPTER 5

PROPOSED SOLUTION ALGORITHM TO THE MILP MODEL

5.1 INTRODUCTION

In this chapter, a proposed solution algorithm to solve a large scale MILP production scheduling problem is presented. Based on the MILP formulation discussed in Chapter 3, subproblems are formulated as the relaxed IP model, and solved by using the Lagrangian optimization method.

In the MILP model, binary variables control the sequencing arrangements and linear variables control the material flows, such as cutoff grade determination, process flow determination, and the stockpile management. If cutoff grades are predetermined, such as using the breakeven cutoff grades, a destination of each increment is also predetermined. To model this problem, indices for the increments \((k)\) and the destinations \((d)\) can be eliminated; also there is no need to have linear variables to formulate this model. This model is called “IP problem” in this dissertation.

From the IP problem, two different subproblems called “Lagrangian subproblem 1” and “Lagrangian subproblem 2” are generated by using the Lagrangian relaxation method. These two Lagrangian dual problems provides the most aggressive mine sequencing case and the most conservative mine sequencing case. Therefore, solutions from the Lagrangian dual problems provide the bounds of the solution space for the MILP model where the optimum or near optimum solution exists. With the reduced size MILP model, an optimum or near optimum solution of the production scheduling problem can be found within a reasonable amount of time.
The influence of the "gap problem" in the Lagrangian optimization is minimized because solutions of the Lagrangian dual problems are only used to find the proper bounds to be assigned in the MILP model.

5.2 SEQUENCE VALUE CALCULATIONS

To formulate the IP problem, which is discussed in details in the following sections, it is necessary to calculate the "sequence values" for all the sequences.

This procedure starts by finding the "best destination" for each increment. This determination is done based on the undiscouned economic values. The undiscounted value that a given increment generates is referred as an "increment value." Then the sequence values are calculated by adding all the increment values for a given sequence.

Increment value ($/\text{ton}$) $IV_{i,j,k}$ for source $i$, sequence $j$, increment $k$ is calculated by utilizing Equation [3.1] and [3.2]:

$$IV_{i,j,k} = \max \left[ \left( P^1 - S^1 \right) \times g_{i,j,k} \times y_{i,j,k,d} + \sum_{n=1}^{\infty} \left( \left( \text{Att}^n P^1 - \text{Att}^n S^1 \right) \times \text{Att}^n g_{i,j,k} \times \text{Att}^n y_{i,j,k,d} \right) \right]$$

$$- \left( m^1_{i,j,k} + tp^1_{i,d} + c^1_d \right), \quad d = 1, \ldots, D \right] \tag{5.1}$$

where

$P^1$: Price of the main commodity in time period 1
$S^1$: Sales and refinery costs in time period 1
$Att$: Total number of attributes

$Att^1 P^1$: Price of $n^{th}$ attribute in time period 1

$Att^1 S^1$: Sales and refinery costs of $n^{th}$ attribute in time period 1

$g_{i,j,k}$: Grade of the main commodity located in source $i$, sequence $j$, increment $k$

$Att^1 g_{i,j,k}$: Grade of the $n^{th}$ attribute located in source $i$, sequence $j$, increment $k$

$y_{i,j,k,d}$: Recovery of the main commodity mined from source $i$, sequence $j$, increment $k$, and sent to destination $d$

$Att^1 y_{i,j,k,d}$: Recovery of $n^{th}$ attribute mined from source $i$, sequence $j$, increment $k$, and sent to destination $d$

$m_{i,j,k}^1$: Mining cost of the material mined from source $i$, sequence $j$, increment $k$ in time period 1 (To the exit of the source)

$tp_{i,d}^1$: Transportation cost from the exit of source $i$ to destination $d$ in time period 1

$c_{d}^1$: Processing cost at destination $d$ in time period 1

Note that the time period index $t$ in Chapter 3 is replaced by $t = 1$. Since economic value calculations are based on the undiscounted values, $t = 1$ values are appropriate to be used for those economic parameters that vary by time. If $t = 0$ values are available, they may also be used.

The sequence value $SV_{i,j}$ ($$/sequence$) for source $i$, sequence $j$ is calculated by adding all the increment values for a given sequence. $Itons_{i,j,k}$ is the tons of increment $k$, in source $i$, sequence $j$:

\[
SV_{i,j} = \sum_{k=1}^{K} IV_{i,j,k} \times Itons_{i,j,k}
\]  

[5.2]
An example of sequence value calculation steps are presented in Appendix B.

5.3 FORMULATION OF THE IP PROBLEM

By using the sequence values, the IP problem is formulated. Since each increment’s destination is predetermined, decision variables are only binary variables to decide when a given sequence should be mined. IP problem is formulated as follows:

\[
I'_{i,j} = \begin{cases} 
1 & \text{if source } i, \text{ sequence } j \text{ is mined in time period } t \\
0 & \text{otherwise} 
\end{cases} \quad \text{(Binary variables)}
\]

**Objective function**

Objective function is to maximize the NPV:

\[
Max Z = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{t=1}^{T} C'_{i,j} \times I'_{i,j} \quad \text{[5.3]}
\]

where

\[
C'_{i,j} = SV_{i,j} \times \frac{1}{(1 + disc)^t}
\]
Constraints

1. Mining capacity constraints

\[ \sum_{j=1}^{J} S_{i,j} \times I'_{i,j} \leq M_{cap}'_i \quad \forall i, t \]  \hspace{1cm} [5.4]

2. Process capacity constraints

\[ \sum_{j=1}^{J} D_{i,j,d} \times I'_{i,j} \leq P_{cap}'_d \quad \forall d \in \{\text{Process}\}, t \]  \hspace{1cm} [5.5]

3. Reserve constraints

\[ \sum_{r=1}^{T} I'_{i,j} \leq 1 \quad \forall i, j \]  \hspace{1cm} [5.6]
4. Sequencing constraints

Sequence $j$ can only be mined after a set of sequences $\forall \text{prev} \in \Gamma j$ are mined out.

\[
\sum_{\beta=1}^{i} l'_{i,\text{prev}} - \sum_{\beta=1}^{i} l'_{i,j} \geq 0 \quad \forall i, j, \text{prev} \in \Gamma j, t \quad [5.7]
\]

5. Binary conditions

All the decision variables are the binary variables.

\[
l'_{i,j} \geq 0 \text{ and binary} \quad \forall i, j, t \quad [5.8]
\]

In order to reduce the solution time to the IP problem, the Lagrangian relaxation method will be used to solve this problem. To do this, two subproblems are generated from the IP problem by using the Lagrangian relaxation method, and they will be iteratively solved to the desired optimality level. The solutions to these two subproblems provide the bounds for the MILP problem. Therefore, the original MILP model size can be reduced to a much smaller size such that it can be solved efficiently.
5.4 INCORPORATION OF THE LAGRANGIAN RELAXATION METHOD TO REDUCE THE PROBLEM SIZE AND THE SOLUTION SPACE

Based on the discussion in Chapter 4, a difficult original IP problem can be solved by relaxing some of the constraints from the original problem. The IP problem can be solved easily by the iterative method if the existing constraints are only the reserve constraints, Equation [5.6], and the sequencing constraints, Equation [5.7], with binary conditions, Equation [5.8] (Dagdelen, 1985 and Akaike, 1999). By modifying suggested methodologies, two different Lagrangian relaxed problems, the Lagrangian subproblem 1 and the Lagrangian subproblem 2, are generated.

Lagrangian subproblem 1

The Lagrangian subproblem 1 is modeled as the most aggressive mine sequence case. Mining capacity constraints, Equation [5.4], are kept, and process capacity constraints, Equation [5.5], are relaxed by assigning Lagrange multipliers.

From the original IP problem from Equation [5.1] to Equation [5.8], the following Lagrangian relaxed problem, denoted as $Z_{LR,SI}$, is derived by introducing a set of Lagrange multipliers $\lambda'_{d}$.
(LR)

\[ \text{Max} Z_{LR,S1} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{t=1}^{T} C_{i,j}^t \times I_{i,j}^t + \sum_{d=1}^{\text{De[process]}} \sum_{t=1}^{T} \lambda_d^t \left( P_{\text{cap}}^d - \sum_{i=1}^{I} \sum_{j=1}^{J} D_{\text{tons}_{i,j,d}} \times I_{i,j}^t \right) \]

s.t.

\[ \sum_{j=1}^{J} B_{\text{tons}_{i,j}} \times I_{i,j}^t \leq M_{\text{cap}}^i \quad \forall i, t \quad \text{(mining capacity)} \]

\[ \sum_{t=1}^{T} I_{i,j}^t \leq 1 \quad \forall i, j \quad \text{(reserve)} \]

\[ \sum_{\theta = 1}^{\text{prev}} I_{i,j,\theta}^t - \sum_{\theta = 1}^{\text{prev}} I_{i,j,\theta}^t \geq 0 \quad \forall i, j, \theta \in \Gamma_j, t \quad \text{(sequencing)} \]

\[ I_{i,j}^t \geq 0 \text{ and binary} \quad \forall i, j, t \quad \text{(binary)} \]

[5.9]

Objective function can be rearranged. Then the problem becomes as follows:

(LR)

\[ \text{Max} Z_{LR,S1} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{t=1}^{T} \left( C_{i,j}^t - \sum_{d=1}^{\text{De[process]}} D_{\text{tons}_{i,j,d}} \times \lambda_d^t \right) \times I_{i,j}^t + \sum_{d=1}^{\text{De[process]}} \sum_{t=1}^{T} \lambda_d^t \times P_{\text{cap}}^d \]

s.t.

\[ \sum_{j=1}^{J} B_{\text{tons}_{i,j}} \times I_{i,j}^t \leq M_{\text{cap}}^i \quad \forall i, t \quad \text{(mining capacity)} \]

\[ \sum_{t=1}^{T} I_{i,j}^t \leq 1 \quad \forall i, j \quad \text{(reserve)} \]

\[ \sum_{\theta = 1}^{\text{prev}} I_{i,j,\theta}^t - \sum_{\theta = 1}^{\text{prev}} I_{i,j,\theta}^t \geq 0 \quad \forall i, j, \theta \in \Gamma_j, t \quad \text{(sequencing)} \]
\[ I'_{i,j} \geq 0 \text{ and binary } \forall i, j, t \] (binary)

[5.10]

This Lagrangian relaxed problem is equivalent to the Equation [4.2] in Chapter 4. As discussed in Chapter 4, if this problem is viewed as a function of Lagrange multipliers $\lambda'_d$, then the problem becomes how to find the best set of Lagrange multipliers so that the objective function value $Z_{LR,s1}$ is minimized. This is the Lagrangian dual problem discussed in Chapter 4.

For each set of the Lagrange multipliers $\lambda'_d$, the Lagrangian relaxed problem, Equation [5.10], is solved. Then multipliers are adjusted according to the degree of the violation, which is a term

\[ P_{cap'_d} - \sum_{i=1}^{I} \sum_{j=1}^{J} D_{tons_{i,j,d}} \times I'_{i,j} \]

and the problem is solved with a new set of the multipliers. This is repeated until the objective function value $Z_{LR,s1}$ is minimized. The multipliers $\lambda'_d$ can be viewed as penalties of process capacity violations. These penalties discourage the violation and push the solution towards the feasible solution.

The adjustment of the multipliers can be done by using the following subgradient method discussed in Chapter 4:

\[ \lambda'_d^{(k+1)} = \max \left\{ 0, \lambda'_d^{(k)} - t_s \times \left( p_{cap'_d} - \sum_{i=1}^{I} \sum_{j=1}^{J} D_{tons_{i,j,d}} \times I'_{i,j}^{(k)} \right) \right\} \]

[5.11]

The step width can be determined by the following equation:
\[ I_k = \gamma_k \times \left( \frac{Z(t_k)}{\sum_{d=1}^{\text{De[process]}} \sum_{r=1}^{T} (p_{c,a}^t - \sum_{j=1}^{I} Dions_{s,j,d} \times I_{i,j}^{(t)})} - Z^* \right)^2 \]  

As discussed in Chapter 4, assignments of \( \gamma_k \) need to be evaluated for each project. When the Lagrangian dual is minimized, a solution is an upper bound of the objective function value of the original IP problem, because of the weak duality condition.

Since a duality gap usually exists in IP problem, a solution from the Lagrangian dual may not be feasible with respect to the relaxed constraints, i.e., process capacity constraints. But all the other existing constraints are strictly honored.

If this Lagrangian subproblem 1 is viewed from the mining engineering point of view, coefficients in the modified objective function in Equation [5.10], a term

\[ \left( C_{i,j} - \sum_{d=1}^{\text{De[process]}} Dions_{s,i,d} \times \lambda_d \right) \]

can be related to the pit parameterization concept, because original sequence values \( C_{i,j} \) are discounted (Dagdelen 1985). If all the Lagrange multipliers are set to be zero, the problem becomes a production scheduling with unlimited process capacities. If this model is solved, it will process as much of ore regardless of the process capacity limitations under the given reserve, mining capacity, and sequencing constraints. By modifying the Lagrange multipliers, some of the sequences that were initially profitable become unprofitable because of the discounting effect when \( C_{i,j} \leq \sum_{d=1}^{\text{De[process]}} Dions_{s,i,d} \times \lambda_d \). This will discourage the violation of the process capacities, and push the solution towards the feasible solution. When the Lagrangian dual is minimized, results will show much less violations of the process capacity constraints, but there will still be some violations due to the gap problem.
Observations of the results from this Lagrangian subproblem 1 are summarized as follows:

- Since this model looks at the life of the mine, it tries to reach highly valued sequences as early as possible under the existing constraints. The time value of money is included in this decision.

- The process capacity constraints are relaxed by assigning Lagrange multipliers, which work as penalties for tonnage violations. If the improvement of the NPV exceeds the costs of penalties due to violations, the solution will show that process capacity constraints should be violated.

- Resulting mine sequences may not be feasible due to possible process capacity violations. In reality, to avoid violations, a mine will operate either by raising the cutoff grades to process fewer amounts of ore, or slowing down mine sequence progresses.

The actual best mine sequences slow down the progress of sequences compared to results from this Lagrangian subproblem 1. Therefore, the resulting mine sequences from this model can be viewed as the most aggressive mine sequence case.

The next step is to model the most conservative mine sequence case by introducing another Lagrangian relaxation model.
Lagrangian subproblem 2

The second subproblem, the Lagrangian subproblem 2, is modeled as the most conservative mine sequence model. This is formulated by keeping the process capacity constraints, and relaxing the mining capacity constraints.

The following Lagrangian relaxed problem, denoted as $Z_{LR,S2}$, is formulated by introducing a set of Lagrange multipliers $\lambda_i^r$.

\[
\text{(LR)}
\]

\[
\max Z_{LR,S2} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{R} C_{i,j}^r X_{i,j}^r + \sum_{i=1}^{I} \sum_{j=1}^{J} \lambda_i^r \left( M \text{cap}^r_i - \sum_{j=1}^{J} \text{Stons}_{i,j} X_{i,j}^r \right)
\]

s.t.

\[
\sum_{i=1}^{I} \sum_{j=1}^{J} D_{\text{tons},i,j,d} X_{i,j}^r \leq P \text{cap}^r_d \quad \forall d \in \{ \text{Process} \}, t
\]

(process capacity)

\[
\sum_{i=1}^{I} I_{i,j}^r \leq 1 \quad \forall i, j
\]

(reserve)

\[
\sum_{\theta \in \mathcal{P}} I_{i,j,\theta} - \sum_{\theta \in \mathcal{P}} I_{i,j,\theta} \geq 0 \quad \forall i, j, \theta \in \mathcal{P}, t
\]

(sequencing)

\[
I_{i,j}^r \geq 0 \text{ and binary} \quad \forall i, j, t
\]

(binary)

\[5.13\]

Similar to the Lagrangian subproblem 1, objective function can be rearranged. Then the problem becomes as follows:
\[
\text{MaxZ}_{LR, S2} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{R} \left( C_{i,j} - S\text{tons}_{i,j} \times \lambda_{i}^{r} \right) \times I_{i,j}^{r} + \sum_{i=1}^{I} \sum_{r=1}^{R} \lambda_{i}^{r} \times S\text{tons}_{i,j} \\
\text{s.t.} \quad \sum_{i=1}^{I} \sum_{j=1}^{J} D\text{tons}_{i,j,d} \times I_{i,j}^{r} \leq P\text{cap}_{d}^{r} \quad \forall d \in \{\text{Process}\}, t \\
\sum_{i=1}^{I} I_{i,j}^{r} \leq 1 \quad \forall i, j \\
\sum_{i=1}^{I} I_{i,prev}^{r} - \sum_{i=1}^{I} I_{i,j}^{r} \geq 0 \quad \forall i, j, prev \in \Gamma_{j}, t \\
I_{i,j}^{r} \geq 0 \text{ and binary} \quad \forall i, j, t \\
[5.14]
\]

The subgradient method can be used to find the best set of Lagrange parameters \( \lambda_{i}^{r} \) so that the Lagrangian relaxed problem Equation [5.14] is minimized. These steps are similar to the ones in the Lagrangian subproblem 1 presented in Equation [5.11] and [5.12].

Observations of the results from this Lagrangian subproblem 2 are summarized as follows:

- This model also looks at the life of the mine, and tries to reach highly valued sequences as early as possible.

- However, since the process capacity constraints cannot be violated, all the materials above the breakeven cutoff grades have to be processed. In reality,
cutoff grades should be dynamically adjusted to reach highly valued sequences as early as possible, rather than processing low grade ore during early years.

Therefore, the actual mine sequences should progress faster than the results from the Lagrangian subproblem 2.

Since the results from the Lagrangian subproblem 1 provides the most aggressive mine sequencing case, and the results from the Lagrangian subproblem 2 provides the most conservative mine sequence case, then the optimum or near optimum mine sequences should lie between these two cases. Therefore, these results provide good bounds of the mine sequences. Solutions from these Lagrangian relaxation subproblems help to eliminate unnecessary variables from the main MILP model. The remaining solution space is significantly reduced from the original, full scale model. Full production scheduling problems can then be optimized with reduced number of variables by the MILP approach, and the solution can be obtained within a reasonable amount of time. All the possible material flows including cutoff grade determinations, and all the constraints can be considered within the MILP model.

This proposed solution algorithm is illustrated in Figure 5.1.
Figure 5.1: Steps of the MILP production scheduling solution algorithm
This proposed solution algorithm is applied to a large scale open pit mine production scheduling model. The next chapter presents this case study to demonstrate how the MILP production scheduling problem is solved by utilizing the Lagrangian optimization technique.
CHAPTER 6

APPLICATION OF THE STEPS OF THE PROPOSED ALGORITHM
TO SOLVE THE LARGE SCALE MILP MODEL

6.1 INTRODUCTION

The proposed method was applied to multi-time period production scheduling on a large scale, multi-metal deposit model coming from a real world project. This deposit contains gold, copper and silver, and is projected to be mined by an open pit mining method.

The life of the mine production scheduling is performed by using the proposed solution algorithm. The primary purpose of this case study is to investigate how a large scale MILP production scheduling problem can be solved by utilizing the Lagrangian relaxation method.

For a given data set and the operational scenarios, a full scale MILP model is developed. From this model, an IP problem is formulated based on the calculated sequence values. This IP problem is solved to optimality by way of two Lagrangian subproblems, which represent the most aggressive mine sequence case and the most conservative mine sequence case, by the Lagrangian relaxation method. The results provide the bounds of the solution space of the MILP model. The size of the MILP model was efficiently reduced and the optimum production scheduling and the cutoff grade policy were achieved within only an hour (48 minutes).

For comparison purposes, a full scale MILP model was solved without reducing the problem size, but no feasible solution was found after running several hours. Then, a manual approach was applied and the model was solved in 3.5 hours. Since the proposed
method provides a solution in 48 minutes, the solution time reduction is 2 hours 42 minutes compared to the manual approach.

6.2 PROJECT DESCRIPTION AND THE MILP PRODUCTION SCHEDULING MODEL

Based on the ultimate pit limit analysis on a case study reserve model, two separate pits are generated in a region, named Source 1 and Source 2. Tonnages and sizes of block models of these sources are summarized in Table 6.1.

Table 6.1: Tonnages and sizes of block models

<table>
<thead>
<tr>
<th>Source #</th>
<th>Source Names</th>
<th>Number of blocks</th>
<th>Tons (ktons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Source 1</td>
<td>34,006</td>
<td>218,100</td>
</tr>
<tr>
<td>2</td>
<td>Source 2</td>
<td>58,970</td>
<td>360,175</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>92,976</strong></td>
<td><strong>578,275</strong></td>
<td></td>
</tr>
</tbody>
</table>

In the Source 1, there are 75 sequences and each sequence is broken into 20 increments. In the Source 2, there are 82 sequences and each sequence is broken into 80 increments. These are summarized in Table 6.2.

Table 6.2: Model sizes by sources

<table>
<thead>
<tr>
<th>Source #</th>
<th>Source Names</th>
<th># of Sequences</th>
<th># of Increments in each Seq</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Source 1</td>
<td>75</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>Source 2</td>
<td>82</td>
<td>80</td>
</tr>
</tbody>
</table>
These two pits can be mined independently (no sequencing constraints between the sources). Within each pit, a given sequence can only be mined after preceding sequence(s) are mined out. Depending on the location of the sequence, there may be more than one preceding sequences. A given increment is sent to one of two destinations; a mill, or a dump. Each increment has to be mined proportionally by keeping the original grade tonnage distribution within a sequence.

Total mining tons from two pits are constrained by the global mining capacity in each year. Ore productions are limited by the mill capacity in each year.

From these deposit information and the operational constraints, a full scale MILP problem is generated. The size of each index is as follows:

- **i:** Source $(i = 1,2)$
- **j:** Sequence $(j = 1,\ldots,75$ in Source 1 and $j = 1,\ldots,82$ in Source 2$)$
- **k:** Increment $(k = 1,\ldots,20$ in Source 1 and $k = 1,\ldots,80$ in Source 2$)$
- **d:** Destination $(d = 1$ for mill, $d = 2$ for dump$)$
- **t:** Time period an activity is taken $(t = 1,\ldots,15)$

The number of the variables in the full MILP model is calculated as follows:

**Number of linear variables:**

**Source 1**

\[i \times j \times k \times d \times t = 1 \times 75 \times 20 \times 2 \times 15 = 45,000\]

**Source 2**

\[i \times j \times k \times d \times t = 1 \times 82 \times 80 \times 2 \times 15 = 196,800\]
Therefore, total number of linear variables is as follows:
\[ 45,000 + 196,800 = 241,800 \]

**Number of binary variables**

Source 1
\[ i \times j \times t = 1 \times 75 \times 15 = 1,125 \]

Source 2
\[ i \times j \times t = 1 \times 82 \times 15 = 1,230 \]

Therefore, total number of binary variables is as follows:
\[ 1,125 + 1,230 = 2,355 \]

Total numbers of the constraints are calculated as follows:

**Reserve constraints:**

Source 1
\[ i \times j \times k = 1 \times 75 \times 15 = 1,125 \]

Source 2
\[ i \times j \times k = 1 \times 82 \times 15 = 1,230 \]

Total number of reserve constraints
\[ 1,125 + 1,230 = 2,355 \]

**Global mining capacity constraints:**
\[ t = 15 \]
Process capacity constraints:
\[ d \in \{ \text{Process} \} \times t = 1 \times 15 = 15 \]

Mine proportion constraints:
Source 1
\[ i \times j \times k \times t = 1 \times 75 \times 20 \times 15 = 22,500 \]

Source 2
\[ i \times j \times k \times t = 1 \times 82 \times 80 \times 15 = 98,400 \]

Total number of mine proportion constraints
\[ 22,500 + 98,400 = 120,900 \]

Sequencing constraints:
Source 1
The first set of the sequencing constraints:
\[ i \times \text{prev} \in \Gamma j \times t = 1 \times (75-1) \times 15 = 1,110 \]
(All the sequences except for the last one have successors)

The second set of the sequencing constraints:
\[ i \times (\text{number of the combination of } j \text{ and } \text{prev} \in \Gamma j) \times t \]
\[ = 1 \times 117 \times 15 = 1,755 \]
Source 2

Similar to the Sequence 1, numbers of the constraints are calculated as follows:

The first set of the sequencing constraints:
\[ i \times \text{prev} \in \Gamma j \times t = 1 \times (82-1) \times 15 = 1,215 \]
(All the sequences except for the last one have successors)

The second set of the sequencing constraints:
\[ i \times (\text{number of the combination of } j \text{ and } \text{prev} \in \Gamma j) \times t \]
\[ = 1 \times 132 \times 15 = 1,980 \]

Total number of sequencing constraints
\[ 1,110 + 1,755 + 1,215 + 1,980 = 6,060 \]

Therefore, total number of constraints is as follows:
\[ 2,355 + 15 + 15 + 120,900 + 6,060 = 129,345 \]

Solution spaces of the mine sequences in this full scale MILP production scheduling model are illustrated in Figure 6.1 for the Source 1 and in Figure 6.2 for the Source 2.

This large scale MILP production scheduling model is solved by using an optimization package Lindo API 4.0 (Lindo Systems Inc.) on a 3GHz CPU, 4GB RAM, Pentium 4 PC. After running more than several hours, no feasible solution was found.
Figure 6.1: Solution space in Source 1
Figure 6.2: Solution space in Source 2
6.3 GENERATION OF THE SUBPROBLEMS

From the main MILP problem, sequence values are calculated and the IP problem is generated.

Since there are only binary variables in the IP model, and the destination of each increment is predetermined based on the breakeven cutoff grade, the model size is reduced compared to the main MILP model.

The number of the variables in the IP problem is calculated as follows:

**Number of binary variables**

Source 1

\[ i \times j \times t = 1 \times 75 \times 15 = 1,125 \]

Source 2

\[ i \times j \times t = 1 \times 82 \times 15 = 1,230 \]

Therefore, total number of binary variables is as follows:

\[ 1,125 + 1,230 = 2,355 \]

Total numbers of the constraints are calculated as follows:

**Global mining capacity constraints:**

\[ t = 15 \]

**Process capacity constraints:**

\[ d \in \{\text{Process}\} \times t = 1 \times 15 = 15 \]
Reserve constraints:

Source 1

\[ i \times j = 1 \times 75 = 75 \]

Source 2

\[ i \times j = 1 \times 82 = 82 \]

Total number of reserve constraints

\[ 75 + 82 = 157 \]

Sequencing constraints:

Source 1

\[ i \times \text{(number of the combination of } j \text{ and } \text{pre} \in \Gamma j) \times t \]
\[ = 1 \times 117 \times 15 = 1,755 \]

Source 2

\[ i \times \text{(number of the combination of } j \text{ and } \text{pre} \in \Gamma j) \times t \]
\[ = 1 \times 132 \times 15 = 1,980 \]

Total number of sequencing constraints

\[ 1,755 + 1,980 = 3,735 \]

Therefore, total number of constraints is as follows:

\[ 15 + 15 + 157 + 3,735 = 3,922 \]
From this IP problem, the Lagrangian subproblem 1 is generated to model the most aggressive mine sequence case. Process capacity constraints are relaxed by assigning Lagrange multipliers. This problem is solved iteratively by using the subgradient method.

For a given set of the Lagrange multipliers, the problem is solved to optimum or near optimum, and the Lagrange multipliers are adjusted according to the degree of the violation of the process capacity constraints. Then the problem is solved with a new set of the Lagrange multipliers. This step is repeated until the desired level of the minimum objective function value is achieved. There are 7 runs required until the optimum solution is achieved. Each run took only 2 to 5 minutes. These results are summarized in Table 6.3a and Table 6.3b.

Lagrange multipliers are initially set to be 0. Then they are adjusted according to the violation of the process capacity constraints based on Equation [5.11] and [5.12]. Example calculations are shown as follows:

**Run 2, Year 2**

\[
t_i = \gamma_i \times \frac{Z_{iR,S1}^{(1)} - Z_p}{\sum_{d=1}^{\text{det}[\text{process}]} \sum_{r=1}^{T} \left( p\text{cap}_{d,r} \sum_{i=1}^{n} \sum_{j=1}^{J} D_{t,n,j,d} \times I_{i,j}^{(1)} \right)^2}
\]

\(\gamma_i\) was set to 1, which was suggested by Held et al. (1974) to select any value between 0 and 2. \(Z_{iR,S1}^{(1)}\) is obtained by solving the Lagrangian subproblem 1 by setting all the Lagrange multipliers to be 0, i.e., Run 1 results, which is $1079M. \ Z_p^* \) was determined to be $960M, which is obtained by heuristically solving the IP problem, as suggested by Fisher (1981).
The term \[ \sum_{d=1}^{[D_{d][process]}^\text{process}] \sum_{t=1}^{T} \left( p\text{cap}_{d,t}^{c^2} - \sum_{i=1}^{I} \sum_{j=1}^{J} D\text{tons}_{i,j,d} \times I_{i,j}^{t^2} \right) \] represents sum of the absolute values of delta differences for the violation of the process capacity constraints for each time period as given in the third column of individual runs shown in Table 6.3a and Table 6.3b. As such, the value for the above term for the second run can be calculated by adding the delta difference values for each time period as shown in the third column of the Run 1.

This is calculated as follows:

\[
|13.0| + |-20.7| + |-10.0| + |10.3| + |-2.3| + |-11.8| + |-17.7| + |-4.7| + |3.2| + |-7.5| + |-12.9| \\
+ |28.0| + |28.0| + |28.0| + |28.0| = 226
\]

Then the step size is calculated as follows:

\[
t_1 = 1 \times \frac{1079 - 960}{226^2} = 0.002
\]

Thus, the Lagrange multiplier associated with Run 2 in Year 2 is calculated as follows:

\[
\lambda_{t=1}^{\text{2, (2)}} = \max \left\{ 0, \lambda_{t=1}^{\text{2, (1)}} - t_1 \times \left( p\text{cap}_{d,t}^{c^2} - \sum_{i=1}^{I} \sum_{j=1}^{J} D\text{tons}_{i,j,d} \times I_{i,j}^{t^2} \right) \right\}
\]

\[
= \max \{ 0, -0.002 \times (28 - 48.7) \} = \max \{ 0, 0.04 \} = 0.04
\]

Similar to Year 2, the Lagrange multiplier in Year 3 was calculated as follows:
Run 2, Year 3

\[
\lambda_{1,2}^{[1]} = \max \left\{ 0, \lambda_{1,1}^{[1]} - t_i \times \left( pcap_{1,1}^{[1]} - \sum_{j=1}^{J} \sum_{i=1}^{I} Dons_{i,j} \times I_{1,j}^{[1]} \right) \right\} \\
= \max \{0.0 - 0.002 \times (28 - 38)\} = \max \{0,0.02\} = 0.02
\]

Table 6.3a and Table 6.3b show that the total violation of the process capacity constraints becomes smaller because the set of the Lagrange multipliers work as the penalty of the process capacity violation. The NPV of Run 2 and Run 3 does not form a non-increasing sequence. Fisher and Shapiro (1974) discussed this as a gap problem. However, in practice, the problems solved by this subgradient method converge in a very few iterations in most of the cases. In this case study, the optimum solution was achieved at Run 7.
<table>
<thead>
<tr>
<th>Yr</th>
<th>Mill Capacity (Mtons)</th>
<th>Run1</th>
<th>Run2</th>
<th>Run3</th>
<th>Run4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>λ</td>
<td>λ</td>
<td>λ</td>
<td>λ</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Mtons)</td>
<td>ΔDiff</td>
<td>(Mtons)</td>
<td>ΔDiff</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
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<td>4.4</td>
<td>18.4</td>
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<td>-10.4</td>
<td>33.8</td>
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<td>42.9</td>
<td>-12.9</td>
<td>35.1</td>
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<td>45.8</td>
<td>-15.4</td>
<td>38.4</td>
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<td>21.8</td>
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<td>-8.8</td>
<td>24.9</td>
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<td>12.4</td>
<td>0.0</td>
</tr>
<tr>
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<td>28</td>
<td>0.00</td>
<td>15.8</td>
<td>12.3</td>
<td>0.0</td>
</tr>
<tr>
<td>13</td>
<td>28</td>
<td>0.00</td>
<td>10.2</td>
<td>17.8</td>
<td>0.0</td>
</tr>
<tr>
<td>14</td>
<td>28</td>
<td>0.00</td>
<td>9.7</td>
<td>18.3</td>
<td>0.0</td>
</tr>
<tr>
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<td>28</td>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Total violation: -87.6 -78.5 -74.5 -75.0
Lagrangian dual NPV: 1079 1068 1079 1072
Solution time: 2min 3min 2min 2min

Table 6.3a: Iteration of the Lagrangian optimization method (Lagrangian subproblem 1)
<table>
<thead>
<tr>
<th>Yr</th>
<th>Mill Capacity (Mtons)</th>
<th>Run5</th>
<th>Run6</th>
<th>Run7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>λ</td>
<td>Δt</td>
<td>λ</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>0.00</td>
<td>5.7</td>
<td>10.3</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
<td>0.20</td>
<td>48.7</td>
<td>-20.7</td>
</tr>
<tr>
<td>3</td>
<td>28</td>
<td>0.16</td>
<td>38.9</td>
<td>-10.9</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>0.00</td>
<td>17.9</td>
<td>10.1</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
<td>0.00</td>
<td>18.9</td>
<td>9.1</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
<td>0.00</td>
<td>35.6</td>
<td>-7.6</td>
</tr>
<tr>
<td>7</td>
<td>28</td>
<td>0.08</td>
<td>42.0</td>
<td>-14.0</td>
</tr>
<tr>
<td>8</td>
<td>28</td>
<td>0.09</td>
<td>28.0</td>
<td>0.0</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
<td>0.02</td>
<td>23.3</td>
<td>4.7</td>
</tr>
<tr>
<td>10</td>
<td>28</td>
<td>0.00</td>
<td>25.2</td>
<td>2.8</td>
</tr>
<tr>
<td>11</td>
<td>28</td>
<td>0.00</td>
<td>43.1</td>
<td>-15.1</td>
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<tr>
<td>12</td>
<td>28</td>
<td>0.00</td>
<td>30.6</td>
<td>-2.6</td>
</tr>
<tr>
<td>13</td>
<td>28</td>
<td>0.00</td>
<td>0.0</td>
<td>28.0</td>
</tr>
<tr>
<td>14</td>
<td>28</td>
<td>0.11</td>
<td>0.0</td>
<td>28.0</td>
</tr>
<tr>
<td>15</td>
<td>28</td>
<td>0.00</td>
<td>0.0</td>
<td>28.0</td>
</tr>
</tbody>
</table>

Table 6.3b: Iteration of the Lagrangian optimization method (Lagrangian subproblem 1)
Similar to the Lagrangian subproblem 1, the Lagrangian subproblem 2 is generated to model the most conservative mine sequence case. The global mining capacity constraints are relaxed by assigning Lagrange multipliers. Then the problem is solved iteratively by using the subgradient method. There are only 2 runs required until the optimum solution is achieved. This is because by using the breakeven cutoff grade, mine sequences slow down due to the large amount of ore (all the materials above the breakeven cutoff grade) that has to be processed. Therefore, a full mining capacity is not used in most of the years. Each run took only 5 minutes. These results are summarized in Table 6.4.

Since the results from the Lagrangian subproblem 1 provides the most aggressive mine sequences and the results from the Lagrangian subproblem 2 provides the most conservative mine sequences, the critical area to investigate is between these two bounds. Therefore, the solution space in the MILP model is significantly reduced by eliminating unnecessary variables. These are illustrated in Figure 6.3 for the Source 1 and in Figure 6.4 for the Source 2.
<table>
<thead>
<tr>
<th>Yr</th>
<th>Mining Capacity (Mtons)</th>
<th>Run1</th>
<th>Delta Diff (e-Dx)</th>
<th>Run2</th>
<th>Delta Diff (e-Dx)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>λ</td>
<td>Mine tons (Mtons)</td>
<td>λ</td>
<td>Mine tons (Mtons)</td>
</tr>
<tr>
<td>1</td>
<td>32</td>
<td>0</td>
<td>26.4</td>
<td>0.00</td>
<td>26.2</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
<td>0</td>
<td>59.4</td>
<td>0.00</td>
<td>50.0</td>
</tr>
<tr>
<td>3</td>
<td>56</td>
<td>0</td>
<td>66.0</td>
<td>0.01</td>
<td>36.6</td>
</tr>
<tr>
<td>4</td>
<td>56</td>
<td>0</td>
<td>36.9</td>
<td>0.00</td>
<td>30.2</td>
</tr>
<tr>
<td>5</td>
<td>56</td>
<td>0</td>
<td>37.3</td>
<td>0.00</td>
<td>55.6</td>
</tr>
<tr>
<td>6</td>
<td>56</td>
<td>0</td>
<td>29.5</td>
<td>0.00</td>
<td>49.6</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>0</td>
<td>32.1</td>
<td>0.00</td>
<td>38.4</td>
</tr>
<tr>
<td>8</td>
<td>56</td>
<td>0</td>
<td>36.0</td>
<td>0.00</td>
<td>33.2</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>0</td>
<td>30.4</td>
<td>0.00</td>
<td>30.5</td>
</tr>
<tr>
<td>10</td>
<td>56</td>
<td>0</td>
<td>41.1</td>
<td>0.00</td>
<td>51.0</td>
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<td>11</td>
<td>56</td>
<td>0</td>
<td>51.2</td>
<td>0.00</td>
<td>52.1</td>
</tr>
<tr>
<td>12</td>
<td>56</td>
<td>0</td>
<td>42.1</td>
<td>0.00</td>
<td>42.2</td>
</tr>
<tr>
<td>13</td>
<td>56</td>
<td>0</td>
<td>21.1</td>
<td>0.00</td>
<td>34.5</td>
</tr>
<tr>
<td>14</td>
<td>56</td>
<td>0</td>
<td>34.1</td>
<td>0.00</td>
<td>26.7</td>
</tr>
<tr>
<td>15</td>
<td>56</td>
<td>0</td>
<td>23.8</td>
<td>0.00</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>Lagrangian dual NPV</td>
<td>994</td>
<td></td>
<td>983</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Solution time</td>
<td>5min</td>
<td></td>
<td>5min</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.4: Iteration of the Lagrangian optimization method (Lagrangian subproblem 2)
Figure 6.3: Solution space in Source 1 after the Lagrangian optimization

54 binary variables in Source 1
Figure 6.4: Solution space in Source 2 after the Lagrangian optimization
With this reduced solution space, the main MILP problem is regenerated by eliminating unnecessary variables. The reduced size MILP model has 17,774 linear variables (vs. 241,800), 207 binary variables (vs. 2,355), and 16,278 constraints (vs. 128,900). This is a significant reduction of the problem size. These are summarized in Table 6.5.

Table 6.5: Model size comparisons

<table>
<thead>
<tr>
<th></th>
<th>Number of linear variables</th>
<th>Number of binary variables</th>
<th>Number of Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>MILP full scale model</td>
<td>241,800</td>
<td>2,355</td>
<td>128,900</td>
</tr>
<tr>
<td>MILP reduced size model</td>
<td>17,774</td>
<td>207</td>
<td>16,278</td>
</tr>
<tr>
<td>Size reduction</td>
<td>92.6%</td>
<td>91.2%</td>
<td>87.4%</td>
</tr>
</tbody>
</table>

For this large scale open pit mine case study, the optimum solution is found in less than one hour (48 minutes), which is a significant reduction of the solution time (reduction is 2 hours 42 minutes compared to the manual approach). The solution time comparisons are summarized in Table 6.6.

Table 6.6: Solution time comparisons

<table>
<thead>
<tr>
<th></th>
<th>Proposed approach (min)</th>
<th>Manual approach (min)</th>
<th>Full scale MILP (min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagrangian subproblem 1</td>
<td>23</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>Lagrangian subproblem 2</td>
<td>10</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>MILP optimum solution</td>
<td>15</td>
<td>210</td>
<td>inf</td>
</tr>
<tr>
<td>Total</td>
<td>48</td>
<td>210</td>
<td>inf</td>
</tr>
</tbody>
</table>

The proposed solution algorithm is successfully applied to the large scale, multi-time period production scheduling model. The large scale MILP production scheduling problem is solved to optimality within a reasonable amount of time. Both the mine sequences and the cutoff grades are optimized. Scheduling details are presented in Appendix C.
CHAPTER 7

PRODUCTION SCHEDULING AND CUTOFF GRADE OPTIMIZATION
CASE STUDY

7.1 INTRODUCTION

By using the mathematical optimization approach, different run options of the project can easily be analyzed. For each of the run options, cutoff grade strategies can be evaluated for complex mine models. By the solution from the MILP problem, yearly cash flows and the resulting NPV are calculated to analyze the deposit's profitability.

In this section, an application of the MILP approach is presented on a gold deposit at the McLaughlin Mine in California, USA. This deposit was developed by open pit mining method, operated mainly in 1980's. Complete discussions are presented by Dagdelen and Kawahata (2005).

7.2 CASE STUDIES

Table 7.1 shows a grade tonnage distribution within the ultimate pit limit. Table 7.2 shows economic and operational parameters.
Table 7.1: A grade tonnage distribution within the ultimate pit limit

<table>
<thead>
<tr>
<th>Interval</th>
<th>From (oz/ton)</th>
<th>To (oz/ton)</th>
<th>Midpoint (oz/ton)</th>
<th>Ktons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000</td>
<td>0.020</td>
<td>0.010</td>
<td>70,000</td>
</tr>
<tr>
<td>2</td>
<td>0.020</td>
<td>0.025</td>
<td>0.023</td>
<td>7,257</td>
</tr>
<tr>
<td>3</td>
<td>0.025</td>
<td>0.030</td>
<td>0.028</td>
<td>6,319</td>
</tr>
<tr>
<td>4</td>
<td>0.030</td>
<td>0.035</td>
<td>0.033</td>
<td>5,591</td>
</tr>
<tr>
<td>5</td>
<td>0.035</td>
<td>0.040</td>
<td>0.038</td>
<td>4,598</td>
</tr>
<tr>
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<td>0.040</td>
<td>0.045</td>
<td>0.043</td>
<td>4,277</td>
</tr>
<tr>
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<td>0.045</td>
<td>0.050</td>
<td>0.048</td>
<td>3,465</td>
</tr>
<tr>
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<td>0.050</td>
<td>0.055</td>
<td>0.053</td>
<td>2,428</td>
</tr>
<tr>
<td>9</td>
<td>0.055</td>
<td>0.060</td>
<td>0.058</td>
<td>2,307</td>
</tr>
<tr>
<td>10</td>
<td>0.060</td>
<td>0.065</td>
<td>0.063</td>
<td>1,747</td>
</tr>
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<td>0.065</td>
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<td>0.068</td>
<td>1,640</td>
</tr>
<tr>
<td>12</td>
<td>0.070</td>
<td>0.075</td>
<td>0.073</td>
<td>1,485</td>
</tr>
<tr>
<td>13</td>
<td>0.075</td>
<td>0.080</td>
<td>0.078</td>
<td>1,227</td>
</tr>
<tr>
<td>14</td>
<td>0.080</td>
<td>0.090</td>
<td>0.085</td>
<td>1,799</td>
</tr>
<tr>
<td>15</td>
<td>0.090</td>
<td>0.100</td>
<td>0.095</td>
<td>1,799</td>
</tr>
<tr>
<td>16</td>
<td>0.100</td>
<td>0.110</td>
<td>0.105</td>
<td>371</td>
</tr>
<tr>
<td>17</td>
<td>0.110</td>
<td>0.120</td>
<td>0.115</td>
<td>371</td>
</tr>
<tr>
<td>18</td>
<td>0.120</td>
<td>0.130</td>
<td>0.125</td>
<td>371</td>
</tr>
<tr>
<td>19</td>
<td>0.130</td>
<td>0.140</td>
<td>0.135</td>
<td>371</td>
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<tr>
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<td>0.140</td>
<td>0.150</td>
<td>0.145</td>
<td>371</td>
</tr>
<tr>
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<td>0.150</td>
<td>0.160</td>
<td>0.155</td>
<td>371</td>
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<tr>
<td>22</td>
<td>0.160</td>
<td>0.170</td>
<td>0.165</td>
<td>371</td>
</tr>
<tr>
<td>23</td>
<td>0.170</td>
<td>0.180</td>
<td>0.175</td>
<td>371</td>
</tr>
<tr>
<td>24</td>
<td>0.180</td>
<td>0.190</td>
<td>0.185</td>
<td>371</td>
</tr>
<tr>
<td>25</td>
<td>0.190</td>
<td>0.200</td>
<td>0.195</td>
<td>371</td>
</tr>
<tr>
<td>26</td>
<td>0.2</td>
<td>0.358</td>
<td>0.279</td>
<td>5864</td>
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</table>

Total 125,513

Table 7.2: Economic and operational parameters

<table>
<thead>
<tr>
<th></th>
<th>(P)</th>
<th>600</th>
<th>$/oz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>(P)</td>
<td>600</td>
<td>$/oz</td>
</tr>
<tr>
<td>Sales Cost</td>
<td>(s)</td>
<td>5</td>
<td>$/oz</td>
</tr>
<tr>
<td>Processing Cost</td>
<td>(c)</td>
<td>19</td>
<td>$/ton ore</td>
</tr>
<tr>
<td>Recovery</td>
<td>(y)</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>Mining Cost</td>
<td>(m)</td>
<td>1.2</td>
<td>$/ton</td>
</tr>
<tr>
<td>Fixed Cost</td>
<td>(fa)</td>
<td>8.35M</td>
<td>$/year</td>
</tr>
<tr>
<td>Mining Capacity</td>
<td>(Mcap)</td>
<td>Unlimited</td>
<td></td>
</tr>
<tr>
<td>Processing Capacity</td>
<td>(Pcap)</td>
<td>1.05M</td>
<td>tons</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>(disc)</td>
<td>15</td>
<td>%</td>
</tr>
</tbody>
</table>
Base Case: The breakeven cutoff grade schedule

To generate a base case scenario, the breakeven cutoff grade is used throughout the mine life. If there is only one process, e.g., autoclave mill, ore is sent to a crusher and processed at the autoclave mill. Waste is sent to a waste dump. Base case material flows are shown in Figure 7.1.

![Figure 7.1: Material flows for Base Case](image)

To simplify the discussion, the grade tonnage distribution presented in Table 7.1 is assumed to be homogeneous within the ultimate pit limit. The breakeven cutoff grade is then calculated by using the Equation [1.3] in Chapter 1.

\[
COG_{BE} = \frac{\$19}{(\$600/\text{oz} - \$5/\text{oz}) \times 0.9} = 0.035\text{oz/ton}
\]
By using this breakeven cutoff grade, all the materials above a grade of 0.035oz/ton are sent to the process, and the other materials are sent to the waste dump. Therefore, total ore in this deposit is the total tons from the interval 5 to the interval 26, which is calculated as 36,346 ktons. The average grade of the ore is calculated by taking a weighted average of the ore materials, from the interval 5 to the interval 26, which is calculated as 0.102oz/ton. Therefore, yearly cash flows and the NPV can be calculated. Yearly schedules are summarized in Table 7.3.

Case 2: Optimum cutoff grade schedule for base case conditions

For the base case discussed above, the MILP approach is applied to find the optimum cutoff grades to maximize the NPV as Case 2, instead of using the constant breakeven cutoff grade. This MILP formulation contains 1,820 variables, and 971 constraints. The results are shown in Table 7.4.

As can be seen from Table 7.3 and Table 7.4, the optimum cutoff grade policy brings 90% higher NPV than using the breakeven cutoff grade. Scheduling results for Case 2 show a declining cutoff grades. This declining cutoff grade policy improves average grades in earlier years. Therefore cash flows in these years are improved, which result in better NPV of the project. Because of aggressive waste stripping, the mine life becomes shorter, from 35 years in Case 1 to 10 years in Case 2.

Following Case 3 to Case 5 demonstrates the complete flexibility of the MILP approach.
<table>
<thead>
<tr>
<th>Year (i)</th>
<th>Cog</th>
<th>Avg Ore Grade</th>
<th>Mining (Mtons)</th>
<th>Processing (Mtons)</th>
<th>Refining (koz)</th>
<th>Profits ($M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>2</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>3</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>4</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>5</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
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<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
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<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>8</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
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<td>9</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>10</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>11 to 34</td>
<td>0.035</td>
<td>0.102</td>
<td>3.6</td>
<td>1.05</td>
<td>96.3</td>
<td>33.0</td>
</tr>
<tr>
<td>35</td>
<td>0.035</td>
<td>0.102</td>
<td>3.4</td>
<td>1.00</td>
<td>91.7</td>
<td>31.4</td>
</tr>
</tbody>
</table>

|          |        |              | 125.8          | 36.7              | 3,365.9       | 1,154.2      |

Table 7.3: Yearly schedules by using the breakeven cutoff grade

(NPV@15%) $218.5
<table>
<thead>
<tr>
<th>Year (i)</th>
<th>Cog</th>
<th>Avg Ore Grade</th>
<th>Mining (Mtons)</th>
<th>Processing (Mtons)</th>
<th>Refining (koz)</th>
<th>Profits ($M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.160</td>
<td>0.261</td>
<td>18.4</td>
<td>1.05</td>
<td>246.6</td>
<td>96.5</td>
</tr>
<tr>
<td>2</td>
<td>0.150</td>
<td>0.253</td>
<td>16.9</td>
<td>1.05</td>
<td>239.1</td>
<td>93.9</td>
</tr>
<tr>
<td>3</td>
<td>0.140</td>
<td>0.248</td>
<td>16.1</td>
<td>1.05</td>
<td>234.4</td>
<td>92.0</td>
</tr>
<tr>
<td>4</td>
<td>0.120</td>
<td>0.238</td>
<td>14.7</td>
<td>1.05</td>
<td>224.9</td>
<td>87.9</td>
</tr>
<tr>
<td>5</td>
<td>0.110</td>
<td>0.233</td>
<td>14.1</td>
<td>1.05</td>
<td>220.2</td>
<td>85.8</td>
</tr>
<tr>
<td>6</td>
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<td>13.6</td>
<td>1.05</td>
<td>215.5</td>
<td>83.6</td>
</tr>
<tr>
<td>7</td>
<td>0.094</td>
<td>0.202</td>
<td>11.0</td>
<td>1.05</td>
<td>190.9</td>
<td>72.1</td>
</tr>
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<td>0.152</td>
<td>6.8</td>
<td>1.05</td>
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<td>49.3</td>
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<tr>
<td>10</td>
<td>0.050</td>
<td>0.133</td>
<td>5.5</td>
<td>1.05</td>
<td>125.7</td>
<td>39.7</td>
</tr>
<tr>
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<td></td>
<td>125.3</td>
<td>10.5</td>
<td>2,002.5</td>
<td>758.8</td>
</tr>
</tbody>
</table>

Table 7.4: Yearly schedules by optimum cutoff grade policy (Case 2)
Case 3: Optimum cutoff grade schedule with stockpile option

If a stockpile is available for Case 2, then the material flow changes. This is called Case 3 and the material flow is illustrated in Figure 7.2.

![Diagram of material flows for stockpile option (Case 3)](image)

Figure 7.2: Material flows for the stockpile option (Case 3)

The MILP approach can incorporate stockpile option easily. Table 7.5 shows the resulting optimum cutoff grades and the yearly schedules of the stockpile option.

In this option, materials in the ultimate pit limit are all mined out by the end of year 6. Then the stockpiled materials are rehandled after year 7. This stockpile option further improves the NPV of the project, because marginal materials between the optimum cutoff grade and the breakeven cutoff grade that are not processed in early years can be processed in later years.
<table>
<thead>
<tr>
<th>Year (i)</th>
<th>Mill Cog</th>
<th>Stk Cog</th>
<th>Avg Ore Grade Mined</th>
<th>Avg Ore Grade Milled</th>
<th>Mining (Mtons)</th>
<th>Processing (Mtons)</th>
<th>Refining (koz)</th>
<th>Profits ($M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.200</td>
<td>0.055</td>
<td>0.143</td>
<td>0.279</td>
<td>22.5</td>
<td>1.05</td>
<td>263.7</td>
<td>101.3</td>
</tr>
<tr>
<td>2</td>
<td>0.200</td>
<td>0.055</td>
<td>0.143</td>
<td>0.279</td>
<td>22.5</td>
<td>1.05</td>
<td>263.7</td>
<td>101.3</td>
</tr>
<tr>
<td>3</td>
<td>0.200</td>
<td>0.055</td>
<td>0.143</td>
<td>0.279</td>
<td>22.5</td>
<td>1.05</td>
<td>263.7</td>
<td>101.3</td>
</tr>
<tr>
<td>4</td>
<td>0.200</td>
<td>0.050</td>
<td>0.143</td>
<td>0.279</td>
<td>22.5</td>
<td>1.05</td>
<td>263.7</td>
<td>101.3</td>
</tr>
<tr>
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<td>0.200</td>
<td>0.050</td>
<td>0.133</td>
<td>0.279</td>
<td>22.5</td>
<td>1.05</td>
<td>263.7</td>
<td>101.3</td>
</tr>
<tr>
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<td>0.050</td>
<td>0.133</td>
<td>0.244</td>
<td>13.2</td>
<td>1.05</td>
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<td>164.8</td>
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<td>-</td>
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<td>137.6</td>
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<td>-</td>
<td>-</td>
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<td>1.05</td>
<td>59.5</td>
<td>7.0</td>
</tr>
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<td>-</td>
<td>-</td>
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<td>0</td>
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<td>56.5</td>
<td>5.2</td>
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<td>52.7</td>
<td>2.9</td>
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<td>-</td>
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<td>0</td>
<td>0.65</td>
<td>31.1</td>
<td>0.9</td>
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<td>22.7</td>
<td>2,052.8</td>
<td></td>
<td>913.2</td>
<td></td>
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</tbody>
</table>

Table 7.5: Yearly schedules by optimum cutoff grades for the stockpile option (Case 3)
Case 4: Optimum cutoff grade schedule with multiple processes

Many of large scale open pit operations have multiple processes to maximize metal recoveries. Case 4 demonstrates the flexibility of the MILP approach to handle multiple processes. Figure 7.3 shows all the processes used in Case 4. Capacities for each process are also presented in the figure.

Figure 7.3: Material flows for multiple processes (Case 4)

In this case study, there are three processing methods; a run of mine leach, a crushed leach, and an autoclave mill. To feed the autoclave mill, there are two different ways to send materials, one is a direct feed to the autoclave mill, and the other is by way of a flotation circuit. If a material goes through the flotation circuit, 20% of tons that came to the flotation go to the autoclave mill, and 80% go to tailings.
Materials which are sent to either the crushed leach or the autoclave mill have to go through a crusher. The capacity of the crusher is assumed to be 5Mtons/yr. Capacities of the flotation circuit, the autoclave mill, and the run of mine leach are assumed to be 2M tons/yr, 1.05Mtons/yr, and 10Mtons/yr, respectively.

To define these processes in the model, material flows from the mine to the run of mine leach is called Process 1, from the mine to the crushed leach through the crusher is called Process 2, from the mine to the autoclave mill through the crusher without going through the flotation circuit is called Process 3, and from the mine to the autoclave mill through the crusher and the flotation circuit is called Process 4. These are illustrated in Figure 7.3. Table 7.6 shows process parameters that are used for Case 4.

Table 7.7 shows resulting optimum cutoff grades and yearly schedules. Table 7.8 shows capacity checks to analyze the bottleneck of the project. As shown in Table 7.8, the crusher, the autoclave mill, and the flotation circuit are operated with full capacities for most of the years.

<table>
<thead>
<tr>
<th>Process</th>
<th>Proc. Cost ($/ton)</th>
<th>Recovery (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1.3</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>90</td>
</tr>
<tr>
<td>4</td>
<td>7.5</td>
<td>80</td>
</tr>
<tr>
<td>Year (i)</td>
<td>ROM Lch Cog</td>
<td>Cr Lch Cog</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
<td>------------</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>0.055</td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
<td>0.047</td>
</tr>
<tr>
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<td>0.025</td>
<td>0.037</td>
</tr>
<tr>
<td>4</td>
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<td>0.031</td>
</tr>
<tr>
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<td>0.020</td>
</tr>
<tr>
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<td>-</td>
<td>0.020</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.7: Yearly schedules by optimum cutoff grades for multiple processes (Case 4)
<table>
<thead>
<tr>
<th>Year</th>
<th>Crusher</th>
<th>ROM Lch</th>
<th>Cr Lch</th>
<th>Autoclave</th>
<th>Flot</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>6,244</td>
<td>2,350</td>
<td>1,050</td>
<td>2,000</td>
</tr>
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<td>2,350</td>
<td>1,050</td>
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<td>1,050</td>
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<td>0</td>
<td>705</td>
<td>2,000</td>
</tr>
</tbody>
</table>

| Capacity | 5,000 | 10,000 | Unlimited | 1,050 | 2,000 |

Table 7.8: Capacity checks (Case 4)
Case 5: Optimum cutoff grade schedule with multiple phases and processes

To demonstrate the multiple phase option, an assumption is made to have two phases in the mine (Phase 1 and Phase 2). Phase 1 has to be completed before Phase 2 starts. Phase 2 mining costs are assumed to be 10% higher than Phase 1 mining costs. Both phases are assumed to have a homogeneous grade tonnage distribution, and Phase 1 contains 40% of the reserves of the mine, and Phase 2 contains 60% of the reserves.

Furthermore, to demonstrate complex constraints, mining capacities are limited to be 12M tons/yr, and refining capacities are limited to be 350k oz/yr. Figure 7.4 shows material flows of Case 5. Table 7.9 shows resulting optimum cutoff grades and yearly schedules for Case 5. Table 7.10 shows capacity checks to analyze the bottleneck of the project.

Figure 7.4: Material flows for multiple phases and processes (Case 5)
<table>
<thead>
<tr>
<th>Year (i)</th>
<th>Stk Cog</th>
<th>Cr Lch Cog</th>
<th>Flot Cog</th>
<th>Autoclave Cog</th>
<th>Avg Ore Grade Mined</th>
<th>Avg Ore Grade Milled</th>
<th>Mining Mtons</th>
<th>Refining koz</th>
<th>Profits ($M)</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>0.022</td>
<td>0.045</td>
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</tr>
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<td>0.165</td>
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<td>0.079</td>
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<td>284.4</td>
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<td>0.165</td>
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<td>0.020</td>
<td>0.045</td>
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<td>0.045</td>
<td>0.170</td>
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<td>12.0</td>
<td>279.8</td>
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<td>-</td>
<td>0.045</td>
<td>0.175</td>
<td>0.077</td>
<td>0.122</td>
<td>12.0</td>
<td>275.9</td>
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<td>0.045</td>
<td>0.175</td>
<td>0.077</td>
<td>0.122</td>
<td>12.0</td>
<td>275.9</td>
<td>120.7</td>
</tr>
<tr>
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<td>0.045</td>
<td>0.175</td>
<td>0.077</td>
<td>0.122</td>
<td>12.0</td>
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<td>120.7</td>
</tr>
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<td>0.122</td>
<td>12.0</td>
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Table 7.9: Yearly schedules by optimum cutoff grades for multiple phases and processes (Case 5)
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Table 7.10: Capacity checks (Case 5)
These case studies demonstrate how the MILP approach can be applied to decide project options and cutoff grade policies. By analyzing different run options, a company can use these results for their strategic decision making processes. With reduced run times by the proposed solution algorithm, the MILP approach can be implemented to analyze a series of ‘what if’ questions in a limited amount of time.
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 CONCLUSIONS

The optimization of the open pit mine production schedules is receiving considerable attention in the mining industry due to their impact on the NPV of the mining projects. It is shown that mathematical programming techniques can be used to model complex mining operations to optimize mine schedules and cutoff grades. Due to limited computing power, optimizing a large scale open pit mine production scheduling problem is quite challenging. The MILP approach has been known to be a powerful mathematical approach, but the solution time has been a big issue due to the size of the models.

To utilize the MILP approach as part of the industry’s decision making processes, this dissertation makes the following major contributions.

1. Development of the solution algorithm to solve large scale MILP production scheduling problems:
   From the full scale MILP problem, two subproblems are developed by using the Lagrangian relaxation method. These subproblems are iteratively solved toward optimality by the subgradient method. The solutions from two subproblems provide the bounds of the solution space in the MILP problem. By this process, the number of the variables and the constraints of the MILP problem are eliminated from the problem, and as a result, the solution time of the large scale production scheduling model is drastically improved.
2. Minimization of heuristic steps in solving large scale production scheduling problems:
Application of the Lagrangian relaxation method replaces many steps that are done heuristically in the industry to solve large scale production scheduling problems. Unlike heuristic approaches, solutions obtained by the Lagrangian relaxation method and the subgradient method are mathematically justified to be optimum.

3. Minimization of the influence of a gap problem:
A gap problem is not avoidable in the Lagrangian relaxation method. But the influence of a gap is minimized because solutions from Lagrangian relaxed problems are used to provide the bounds of the solution space in MILP model, instead of directly solving the scheduling problem.

4. Optimization of mine sequences, material flows, and cutoff grades:
By using the proposed solution algorithm, an optimum or near optimum production scheduling can be found by achieving the best mine sequences and the best material flows. Cutoff grades are dynamically determined as part of the scheduling solution. Stockpiles can be incorporated.

These contributions are illustrated in the case study. Results from the case study show that methodologies discussed in this dissertation can be extended to solve a large scale realistic production scheduling problem. Flexibilities of the MILP model can be fully utilized to handle complex operational constraints. With a significant reduction of the solution time, this approach can be implemented to provide a series of ‘what if’ questions in a reasonable amount of time to evaluate potential financial outcomes as an industry’s decision making tool.
8.2 RECOMMENDATIONS

Ideally, production scheduling problems are solved to optimality without preprocessing input information. Unfortunately, due to an excessive number of variables and constraints in a model, problems cannot be solved without solution guidance. So our research on methodologies to speed up the solution time needs be continued. Based on the conclusions of the research, the following recommendations can be noted for future research.

Since every deposit has its own characteristics, every project has different operational constraints. The MILP approach can handle many of these constraints, and the bounds provided by the Lagrangian relaxation method should still be valid. However, investigation is needed to analyze how narrow the bounds will be for many different combinations of the constraints.

In addition, a sensitivity analysis on the subgradient method may help to improve solution time. Methodologies to find initial set of Lagrange multipliers and step sizes in the subgradient method can be modified by considering the actual penalty for tonnage violations at the mine.

It is important to understand how the mathematical steps in the proposed algorithm relate to operational meanings in an open pit mining. To get actual operational data from mines, it is important to continue cooperation with industries. Understanding what’s happening in mines will help us to understand how mathematical numbers are related to actual costs.

It may be worthwhile to compare results from this proposed method with currently available heuristic approaches. Solution times and resulting optimality levels, as well as flexibilities of algorithms, can be analyzed. Engineers can choose either heuristic approach based tools or optimization approach based tools, depending on the objective of their studies. Also, comparing these results can be used for cross checks of resulting schedules to gain more confidence on outcomes. These comparisons may also be
beneficial to improve algorithms by analyzing the mathematical characteristics of both approaches.
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APPENDIX A

CUTOFF GRADE DECISION IN THE MILP MODEL

A.1 INTRODUCTION

When the MILP problem is solved and the optimality is reached, each increment’s “best” destination and the “best” time period to be mined are determined. Since this is the optimum production scheduling to maximize NPV, there is no other plan that could possibly generate a higher NPV. When each increment’s destinations are determined, cutoff grades are also determined as part of the solution methodology. This determination considers grade-tonnage distribution in all the sources and sequences, all the constraints in the model, and the life of the mine economics of the project; cutoff grades usually vary year by year. An example of the MILP results and the related cutoff grades are shown in FigureA.1.

This hypothetical example shows two sequences that contain 4 increments in each sequence as part of the large scale model. Assuming that economic parameters and the constraints are given to the entire model and solved by the MILP approach, the results of these two sequences are obtained as presented. In year 1, Inc 1 is sent to waste dump, Inc 2 and 3 are sent to the stockpile, and Inc 4 is sent to the mill. From these results, one can interpret the stockpile cutoff grade is the lower bound grade of Inc 2, and the mill cutoff is the lower bound grade of Inc 4. Similar to year 1, year 2 cutoff grades can also be determined. The MILP provides the best destination and the best time period for each of the materials that is defined as the increment located in one of the sequences in one of the sources; this can be viewed as a cutoff grade determination (Dagdelen and Kawahata 2005).
Figure A.1: Cutoff grade decision in the MILP model
Since these processes do not rely on the breakeven cutoff grade concept, resulting cutoff grades can vary year by year. Therefore, a dynamic cutoff grade policy is determined as a result of the MILP solution methodology.
APPENDIX B

SEQUENCE VALUE CALCULATIONS

B.1 INTRODUCTION

This appendix illustrates steps of sequence value calculations by using a small example. A small example of the gold mine model is illustrated in Figure B.1. Assume that there are 4 increments on a given sequence \((j = 1)\) in a given source \((i = 1)\), and there are two possible destinations: mill \((d = 1)\) and dump \((d = 2)\).

\[
\begin{align*}
P^1 & : \quad $550/\text{oz of Au} \\
S^1 & : \quad $5/\text{oz} \\
Att & : \quad \text{There is no attribute} \\
Y_{1,i,k,1} & : \quad \text{Recovery at the mill \((d = 1)\) is 90\% for all increment } k \\
Y_{1,i,k,2} & : \quad \text{Recovery at the dump \((d = 2)\) is 0\% for all increment } k \\
m^1_{1,i,k} & : \quad $1/\text{ton for all increment } k \\
tp^1_{1,d} & : \quad \text{No transportation cost to the mill and the dump ($0/\text{ton}$)} \\
c^1_{1} & : \quad \text{Process cost at the mill destination } d = 1 \text{ is $5/\text{ton}} \\
c^1_{2} & : \quad \text{No dumping cost at the dump destination } d = 2 \text{ ($0/\text{ton}$)}
\end{align*}
\]

Average Au grades and tons for each increment are shown in Figure B.1.
Figure B.1: An example model to illustrate sequence value calculation steps

Increment value calculation, by using Equation [6.1], for the first increment is presented as follows:

Increment 1 \((k = 1)\)

\[
IV_{1,1} = \max \left\{ \left[ (550-5) \times 0.01 \times 0.9 - (1+0+10), \right. \right. \\
\left. \left. (550-5) \times 0.01 \times 0 - (1+0+0) \right] \right\}
\]

\[
d = 1 \text{ (mill)}
\]

\[
= \max \{-6.10, -1.00\} = -1.00 \text{ (\$/ton)} \quad \text{when } d = 2 \text{ (dump)}
\]

Therefore, the best destination for the increment 1 is determined as \(d = 2\) (dump), and its increment value is -1.00 (\$/ton). This procedure is illustrated in Figure B.2.
The same procedure is repeated for all the increments in a given sequence.

**Increment 2 \( (k = 2) \)**

\[
IV_{1,2} = \max \left\{ (550 - 5) \times 0.03 \times 0.9 - (1 + 0 + 10), \ (550 - 5) \times 0.03 \times 0 - (1 + 0 + 0) \right\}
\]

\[
= \max \{3.72, -1.00\} = 3.72 \text{ \textdollar}/\text{ton} \quad \text{when } d = 1 \text{ (mill)}
\]

**Increment 3 \( (k = 3) \)**

\[
IV_{1,3} = \max \left\{ (550 - 5) \times 0.05 \times 0.9 - (1 + 0 + 10), \ (550 - 5) \times 0.05 \times 0 - (1 + 0 + 0) \right\}
\]

\[
= \max \{3.53, -1.00\} = 3.53 \text{ \textdollar}/\text{ton} \quad \text{when } d = 1 \text{ (mill)}
\]

**Increment 4 \( (k = 4) \)**

\[
IV_{1,4} = \max \left\{ (550 - 5) \times 0.07 \times 0.9 - (1 + 0 + 10), \ (550 - 5) \times 0.07 \times 0 - (1 + 0 + 0) \right\}
\]

\[
= \max \{23.34, -1.00\} = 23.34 \text{ \textdollar}/\text{ton} \quad \text{when } d = 1 \text{ (mill)}
\]
These results are illustrated in Figure B.3. Increment values and the best destinations are determined for all the increments.

![Diagram showing increment values and best destinations for four increments: Inc1 (k = 1) with 200 tons and Au Grade 0.01 oz/ton, Inc2 (k = 2) with 300 tons and Au Grade 0.03 oz/ton, Inc3 (k = 3) with 300 tons and Au Grade 0.05 oz/ton, Inc4 (k = 4) with 200 tons and Au Grade 0.07 oz/ton. The sequence value (j=1) is shown with increment values: $-1.00/ton, $3.72/ton, $13.53/ton, $23.34/ton. The best destinations are dump (d = 2), mill (d = 1), mill (d = 1), mill (d = 1).]

Figure B.3: Increment values and the best destinations for all the increments

Then, the sequence value is calculated as follows:

\[ BV_{1,1} = (-1.00) \times 200 + 3.72 \times 300 + 13.53 \times 300 + 23.34 \times 200 \]

\[ = -200.0 + 1114.5 + 4057.5 + 4667.0 \]

\[ = 9639.0 \text{ ($/sequence)} \]
This sequence value can be viewed as an undiscounted value if a given sequence is completely mined and all the increments are sent to their predetermined destinations based on the breakeven cutoff grade. This sequence value concept is illustrated in Figure B.4.

![Figure B.4: A concept of sequence values](image)

By introducing these sequence values, indices $k$ and $d$ that are used in the MILP main problem can be dropped in formulating the IP problem, because destinations are predetermined based on the breakeven cutoff grade. Since the IP problem is formulated to determine when a given sequence should be mined, decision variables in the IP problem are only binary variables, no linear variables exist.
APPENDIX C

OPEN PIT MINE CASE STUDY

C.1 INTRODUCTION

In Chapter 6, a case study on a large scale open pit mine production scheduling is presented. In this appendix, input information and some of the scheduling results are presented.

A 3-D overview of the ultimate pit limits of the Source 1 and the Source 2 are presented in Figure C.1.

Figure C.1: 3-D view of the ultimate pit limits
In a block model, each block contains information of estimated grades and recoveries of gold, silver and copper. Economic parameters that are used for this case study are summarized as follows:

- **Au prices** \( (P) \): $420/oz
- **Au refinery costs** \( (S) \): $5/oz
- **Cu prices** \( (Att^1_P) \): $1.3/lb
- **Cu refinery costs** \( (Att^1_S) \): $0.090/lb
- **Ag prices** \( (Att^2_P) \): $6.85/oz
- **Ag refinery costs** \( (Att^2_S) \): $0.4/oz
- **Mining costs** \( (m_{i,j,k}) \): $1.20/ton at the top bench of each phase.
  - Incremental mining cost per bench is $0.01/ton
- **Process costs** \( (c_{d}) \): $4/ton of ore
- **Discount rate** \( (disc) \): 7%

### C.2 YEARLY PRODUCTION SCHEDULING RESULTS

By applying the proposed solution algorithm, optimum production scheduling is achieved. Based on the results from two Lagrangian subproblems, unnecessary variables are efficiently reduced from the MILP model.

With the reduced number of variables, the main MILP problem is solved to optimality. Figure C.2 shows the mining and milling tons each year without considering stockpiles. Figure C.3 shows resulting cutoff intervals for both pits. Dynamic cutoff grades are determined.
Figure C.2: Mined and milled materials

Figure C.3: Cutoffs for Source 1 and Source 2
Based on these results, yearly pit views are generated. Figure C.4 shows an original topography, and Figure C.5 to Figure C.10 illustrates year end pit shapes.

Figure C.4: 3-D view of the topography
Figure C.5: 3-D view of the year end shape (Year 1)

Figure C.6: 3-D view of the year end shape (Year 2)
Figure C.7: 3-D view of the year end shape (Year 3)

Figure C.8: 3-D view of the year end shape (Year 4)
Figure C.9: 3-D view of the year end shape (Year 10)

Figure C.10: 3-D view of the year end shape (Year 13)