FINITE DIFFERENCE AND CONTROL-VOLUME
FINITE ELEMENT SIMULATION OF
NATURALLY FRACTURED RESERVOIRS

by

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ABSTRACT

Two models for the simulation of naturally fractured reservoirs were developed. Both models are three-dimensional, two-phasic (water and oil). The first model uses a dual-porosity formulation (DPM) in which matrix and fracture systems are accounted for. The second model uses a single-porosity formulation (SPM) in which all calculations are performed in the fracture system. The single-porosity formulation of the model equations is accomplished via an empirical mathematical matrix/fracture transfer function.

The model equations are discretized using both the finite difference (FD) and the control-volume finite element (CVFE) methods. The FD discretization scheme was the first step toward the CVFE discretization of the model equations. In the CVFE discretization scheme, the pressure is treated in a finite element manner, while the mobility terms are upstream-weighted as in the usual FD method. The use of the perpendicular-bisection (PEBI) method or the Voronoi grid, and the seven-point FD method are only special cases of the CVFE discretization scheme.

A general purpose two-dimensional triangulation code is used to prepare the CVFE grid of the reservoir areal layout. For the three-dimensional case, the third dimension is modeled by vertical projection of the areal two-dimensional grid. The principal advantage of the CVFE method over the FD
method is the flexibility to represent complex reservoir geometries more accurately and to minimize the grid orientation effect (GOE).

Because the CVFE form of the discretized equations is similar to the usual FD method, standard techniques are used to solve the Jacobian. The equations are solved fully implicit for pressure, saturation, and wellbore variables using the iterative Newton-Raphson method of solution. Due to the sensitivity of the selection of timestep size, two options for timestepping were included in this work, manual and automatic. Automatic timestep selection is handled by restricting saturation and/or pressure changes to meet specified tolerance criteria selected by the user.

Detailed description of reservoir heterogeneity can improve the accuracy of simulating petroleum reservoirs. Permeability variations, however, appear to have the strongest influence on reservoir heterogeneity. A Geostatistical approach, Turning Bands method, for creating random fields of reservoir permeability values at arbitrary points in space was implemented in this work.

Tracer injection is a powerful method for characterizing permeability heterogeneities. The chemical transport model (CTM) implemented in this work is the advective-dispersive model which is used to simulate the tracer flow in the fractures. The tracer concentration distribution at the model grid nodes is calculated implicitly at each timestep directly after the solution of the pressure and saturation variables.
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CHAPTER 1
INTRODUCTION

1.1 Naturally Fractured Reservoirs

A reservoir fracture is a naturally occurring macroscopic planar discontinuity in a rock due to deformation or physical diagenesis, see Fig. 1.1. Fractures fall into two different categories: (1) structure-related fractures which show a consistency of orientation and regularity over a given field of observation, and (2) structure-unrelated fractures which include curved or irregular discontinuous breaks that show no consistency of orientation.

A naturally fractured reservoir is defined as a reservoir in which naturally occurring fractures have a significant effect on reservoir fluid flow in the form of increased reservoir permeability and/or porosity or increased permeability anisotropy. Naturally fractured reservoirs are very complicated and more difficult to evaluate than conventional reservoirs. They are probably the most complex of all reservoir systems. Effective evaluation and prediction in these reservoirs requires an early recognition of the role of the fractured system and a systematic approach to the gathering and analysis of pertinent data. They possess many inherent obstacles to proper analysis due to difficulties in prediction, evaluation, and characterization. These obstacles stem from problems such as: (1) a general lack of in-depth quantitative approaches to description and characterization of highly anisotropic reservoirs,
(2) failure of geologists or engineers to recognize the fractures and/or the regularity of their distribution, and (3) over-simplistic approaches for describing fracture distribution and morphologies.

These obstacles are compounded by the improper use or nonuse of the many techniques available to detect natural subsurface fractures. Detecting subsurface fractures or predicting their occurrence is indeed the most basic step in fully evaluating a naturally fractured reservoir.

Fig. 1.1: Idealization of heterogeneous porous media: Actual reservoir (Modified from reference no. 93)
There are three basic types of evaluation in fractured reservoir analysis. They are, in order of increasing complexity, amount of data, and time to completion: (1) early exploration evaluations to predict gross reservoir quality, (2) evaluations of the economic potential (e.g. reserves, flow rates, etc.), and (3) evaluations for recovery planning and detailed reservoir modeling.

The origin of the fracture system is postulated from data on fracture dip, morphology, strike (if available), relative abundance, and the angular relationships between fracture sets. These data can be obtained from a full-diameter core (oriented or conventional), Borehole Televiewer, Formation Microscanner, or other less oriented logging tools, and applied to empirical models of fracture generation. Once the origin of a fracture system has been determined in a reservoir, the physical properties of the rock/fracture system must be addressed. This involves characterization of the fracture system in terms of the physical morphology, distribution, and estimation of reservoir properties (porosity, permeability, etc.) resulting from the fracture system characteristics.

1.2 Idealization of Naturally Fractured Reservoirs

The behavior of naturally fractured reservoirs is complicated and more difficult to predict than conventional reservoirs due to the different mechanisms involved in the production of oil and gas. The complexity arises from the existence of two flow paths (matrix and fracture which have totally
different properties) that communicate with each other. Because of the irregularity of the fracture morphology, distribution, orientation, and extension, some sort of idealization of these reservoirs is needed for simulation purposes.

![Diagram of Matrix and Fractures]

**Fig. 1.2:** Idealization of heterogeneous porous media: Model reservoir (Modified from reference no. 93)

Warren and Root (1963) suggested a physical idealization for fractured reservoirs, see Figs. 1.1 and 1.2. Their idealization was based on the following assumptions: (1) the matrix blocks are homogeneous and isotropic and form
a uniform assemblage of identical, rectangular parallelepipeds, (2) the fractures are contained within an orthogonal system of continuous, uniform, and constant-width paths where the fractures are parallel to the principal axes of permeability, (3) the flow takes place from fracture to fracture, from fracture to matrix, and from matrix to fracture, (4) the matrix blocks have high storage volume and low flow capacity whereas fractures have low storage volume and high flow capacity, and (5) the wellbore intersects the fracture network and hence the fractures provide the main path for fluid flow into the wellbore.

Kazemi et al. (1976) set the foundation for field-scale, dual-porosity, multi-phase simulators. In their idealization, an elemental reservoir volume in a naturally fractured reservoir is represented by a set of grid cells, see Fig. 1.3. Each grid cell may contain one or several matrix blocks. Accordingly, within any grid cell, all matrix blocks will have the same pressures and saturations. Similarly, the fractures within a grid cell will have identical pressures and saturations which differ from the matrix pressures and saturations. A typical grid cell with its pressure distribution is shown in Fig. 1.4. In many practical problems, this idealization is sufficient. In cases where a matrix block contains several grid nodes, the gravity segregation within the matrix block is calculated. This description points to the critical engineering decision in the selection of the number of grid nodes with respect to the number of matrix blocks.
Fig. 1.3: Schematic of an elemental reservoir volume in a naturally fractured reservoir (Modified from reference no. 50)
Fig. 1.4: Idealization of flow and elemental reservoir volumes containing matrix blocks in a naturally fractured reservoir (Modified from reference no. 50)
CHAPTER 2
LITERATURE REVIEW

In this chapter, the main concepts and the most important works related to: recovery mechanisms of naturally fractured reservoirs, grid orientation effect (GOE), and treatment of heterogeneities are briefly described.

2.1 Recovery Mechanisms of Naturally Fractured Reservoirs

The problem of oil recovery from a naturally fractured reservoir is the extraction of oil from the low permeability matrix blocks via the fracture conduits. Gravity drainage and imbibition play the major role in oil recovery from the matrix blocks.

2.1.1 Gravity Drainage

Gravity drainage is sometimes a major oil recovery mechanism in naturally fractured reservoirs. In these reservoirs, the presence of vertical or sub-vertical fractures could allow the gas-oil contact (GOC) or the water-oil contact (WOC) to advance ahead of the corresponding contact in the matrix blocks. It is the difference between the density of the fluids and the elevation difference of the fluid contacts in fracture and matrix that causes the oil in the matrix block to be produced into the fracture network.

Gravity drainage in reservoirs takes place as free fall and forced
drainage. Free fall gravity drainage usually takes place in reservoirs with high permeability and fair formation thickness. It could leave a very low residual oil saturation in the matrix if the rate of oil production is correctly controlled. Forced drainage, on the other hand, takes place in dual permeability reservoirs where gas or water moves in the higher permeability fracture ahead of the oil in the low permeability matrix. The pressure difference between the two fluids in the two permeability zones provides the necessary force for the displacing fluid to enter the low permeability zone and displace the oil. If the height of the low permeability zone is limited by discontinuities, whether they are barriers or fractures, higher residual oil remains in this type of reservoir.

Two essential points in the gravity drainage processes must be mentioned. These points are: (1) when fluid is flowing, under gravity drainage and constant uniform pressure, through a homogeneous rock, the flow is always in the vertical direction. In other words, if we consider a homogeneous porous block, saturated with a fluid which is in equilibrium with the displacing phase, undergoing gravity drainage, the recovery of the displaced fluid is independent of whether the vertical boundaries are open or closed, and (2) forced gravity drainage, whether the GOC or WOC passes a block or stays just in contact with the block, has an insignificant effect on the oil recovery with time.

In general, the basic differential equations describing the fluid flow are
basically the same for the two types of gravity drainage except that their boundary conditions are different.

The fundamental development of the gravity drainage equation, which describes the frontal displacement of two immiscible fluids in a porous medium, was first set by Leverett (1941) and then applied by Buckley and Leverett (1942). Cardwell and Parsons (1948) were the first research workers to describe the concept of free fall gravity drainage theoretically. Birks (1955) used the concept, developed by Muskat (1937), and Dykstra and Parsons (1950), to study the effect of forced gravity drainage on the recovery from limestone blocks separated by fissures for both water and gas drive.

Kyte (1970) described a new method to predict oil recovery from matrix blocks of weakly water-wet or oil-wet reservoirs where gravity segregation can be more important than imbibition. Lefebvre du Prey (1978) studied gravity and capillarity effects on the matrix imbibition in fissured reservoirs. He found that for large matrix blocks, gravity effects were very important and recovery time was proportional to the size of the block.

2.1.2 Imbibition

Imbibition is defined as the spontaneous intake of a fluid into rock under the effect of capillary pressure gradient. This process continues until a balance is created by gravitational gradient, and can be a dominant factor for oil recovery from water-wet fissured rocks. Brownscombe and Dyes (1952)
presented the idea of a water imbibition displacement. The idea occurred when the rock fractures were filled with water, the water tends to soak into the rock (water imbibition) while the oil expels out of it.

![Diagram of water imbibition](image)

**Fig. 2.1:** Oil production by water imbibition (Modified from reference no. 12)

Figure 2.1 illustrates the concept of oil production by water imbibition. In this figure, the water is being injected into the fracture only. Part of the injected water soaks into the rock and expels oil out of it which becomes part of the flow string. In the case where the imbibition rate is greater than the
rate of injection, all the water will imbibe into the matrix and only oil will reach the end of the fracture. With time, the imbibition rate will decrease, water will travel longer distances in the fracture, and eventually it will reach the producing well. At this moment water production will start. The water-oil ratio (WOR) appears then to be a function of both water injection rate and rate of water imbibition.

Graham and Richardson (1959) presented theoretical and laboratory studies of imbibition. They found that oil production was rate-sensitive. For a given fracture width, the lower the injection rate; i.e. the lower the pressure gradients, the more is the oil recovered for a given amount of water injected.

Mattax and Kyte (1962) have applied the scaling laws proposed by Rapoport (1955) to the study of imbibition phenomena. They showed that the time required to recover a given fraction of oil from a matrix block was proportional to the square of the distance between fractures. Iffly et al. (1972) conducted extensive studies in imbibition and concluded that only laboratory studies with fluids and cores of the actual reservoir could provide meaningful insight into the actual performance of a naturally fractured reservoir. They indicated that lithology plays an important role on the imbibition process. Lefebvre du Prey (1978) studied gravity and capillarity effects on the matrix imbibition in fissured reservoirs. He found that for small blocks, the capillarity effects were very important, and recovery time was proportional to the square of the size of the block. Blair (1964) numerically studied the
countercurrent imbibition of water into a porous medium. He found that the increase in capillary pressure will reduce the time required for imbibition and the effect of relative permeability was found to be less, possibly due to the data sets he used.

Aronofsky et al. (1958) suggested an empirical exponential model for recovery from highly fractured reservoirs where the dominant recovery mechanism is imbibition. The model has proved useful in some of the largest naturally fractured reservoirs of the world, thus it was implemented in this study for the formulation of the single-porosity model.

2.2 Grid Orientation Effect (GOE)

Finite difference solutions of the two-dimensional frontal displacement problems can be strongly influenced by the orientation of the finite difference grid. Several investigations\textsuperscript{11} have been reported on the influence of various parameters and discretization schemes on the grid orientation. These include: mobility ratio and grid size, and truncation errors and discretization schemes.

2.2.1 Mobility Ratio and Grid Size

For favorable or unit mobility ratio and homogeneous media, no GOE is observed, whereas in heterogeneous media, averaging should be done so that GOE does not exist.

The GOE becomes more and more pronounced as the mobility ratio
increases. In the solution of adverse mobility ratio displacement (mobility ratio of the displacing fluid to the displaced fluid is greater than one) problems by the five-point finite difference schemes, fluids appear to flow along grid lines. This results in solutions that depend on the orientation and size of the grid. Hence different results are obtained when, for example, the same one quarter five spot problem is solved by grids that are parallel and diagonal to the line joining the injector and the producer. As long as numerical diffusion is greater than physical diffusion, solutions depend on the grid orientation and the block size. As the grid is refined, the GOE may become even more severe.

Todd et al. (1972) described the GOE resulting from using the diagonal and parallel grids. They showed that a two-point upstream weighting relative permeability was found effective for reducing the GOE. Since their method of weighting has less truncation error, compared with single-point weighting, a coarser grid can be used to a certain extent with the same accuracy as a finer grid. The diagonal system was found to give more accurate results in water flooding.

Coats et al. (1974) reported that the use of a parallel system resulted in a more pronounced finger toward the producer, while the diagonal system had a more radial pattern front. They also showed that the two-point upstream weighting is not enough to alleviate the GOE in steam flooding where a highly unfavorable mobility ratio is used. Instead, they proposed the use of a variational method or higher order finite difference schemes.
2.2.2 Truncation Errors and Discretization Schemes

For standard five-point difference schemes, first-order truncation error terms resemble an anisotropic numerical diffusion "tensor". There is less GOE for nine-point schemes for which this diffusion is nearly isotropic. A complete removal of first-order diffusion terms by means of higher-order weightings improves the results in certain cases.

Yanosik and McCracken (1976) were the first research workers who set the foundation for an effective solution to the GOE. They proposed the use of a nine-point finite difference scheme. However, this scheme is designed for point distribution, square grid and isotropic systems. It results in negative transmissibilities for nonuniform grids or anisotropic systems.

Coats and Ramesh (1982) studied the GOE on steamflooding in five-spot, seven-spot, and nine-spot patterns. They concluded that the nine-point scheme holds for all patterns, even off-centered wells. Nonsquare grids resulted in a more GOE for both nine and five-points. However, the nine-point scheme gave smaller GOE for a limited rate. In a vertical plane, the GOE was examined and found to be less pronounced than the areal one.

Pruess and Bodvarsson (1983) came out with a new seven-point finite difference method, where the domain was partitioned by identical and regular hexagons; i.e. each node is connected to six neighbors. The GOE was reduced as effective as the nine-point formulation with less computational time, particularly, in favorable symmetrical spatial domains such as seven-spot.
Shiralkar and Stephenson (1987) published one of the most rigorous, rotationally invariant, nine-point schemes that is capable of reducing the GOE. Shiralkar (1989) accommodated the general anisotropic system with the same conditions of symmetry and positivity of transmissibilities. Wolcott (1991) significantly reduced the GOE for an unfavorable mobility ratio using the formulation suggested by Shiralkar.

Palagi (1992) proposed a new gridding technique known as a Voronoi grid to simulate fluid flow in heterogeneous reservoirs. He mentioned that he could minimize the GOE for unfavorable mobility displacements in a five-spot pattern.

Along the line of finite element method (FEM), Young (1978) proposed a Galerkin based FEM. A two-dimensional, three-component, single-phase model was developed to predict the behavior of tracer performance, ion exchange, and adverse mobility ratio miscible displacement as in the five-spot pattern. The author reported that this technique has reduced the GOE in less computational time than other FEM.

Forsyth (1990) described a control-volume, finite-element technique for coupling coarse grids with local fine meshes. The pressure was treated in a finite-element manner, while the mobility terms were upstream weighted as in the finite difference methods. Fung et al. (1991) implemented the control-volume finite-element method for the simulation of multiphase thermal flow in porous media. They found that this method significantly reduced the GOE
and at the same time the computational cost is much lower than the nine-point finite difference schemes.

2.3 Treatment of Heterogeneities

Reservoir heterogeneity has long been recognized as an influential factor in affecting reservoir performance. Variations in any reservoir property can degrade the efficiency of a recovery process. The variability of permeability, however, appears to be particularly influential. In this study, we restrict our attention to permeability heterogeneity. Detailed descriptions of reservoir heterogeneity can improve the accuracy of simulating petroleum reservoirs. For an efficient simulation process, every node in the model grid must be identified by a set of reservoir property values which include rock properties, fluid properties, relative permeability, and capillary pressure data. However, in practical situations, this may be restricted to some special cases because it requires substantial amount of human effort and time.

Reservoir characterization is the discipline that seeks to define quantitatively the input data, at lesser effort and time, required by the simulator. One of the goals of reservoir characterization is to place more emphasis on the actual geology of the reservoir. However, with such a minute fraction of the reservoir actually sampled by a well, a complete description of the reservoir is rarely possible.

Geostatistical methods address this problem by using statistical
techniques to create an image of the reservoir based on the information observed at the wells. These methods for describing reservoir heterogeneities have many applications in petroleum engineering.

Warren and Price (1961) used log-normal uncorrelated permeability distribution at the small scale. By analyzing the numerical results obtained with fine grids, they concluded that a good estimation of the average permeability of a gridblock is the geometric average of the microblock permeabilities within it. Mejia and Rodriguez-Iturbe (1974), Journel and Huijbregts (1978), and Delhomme (1978) used kriging technique in the hydrosciences to generate random fields. See also Smith and Freeze (1979), and Smith and Schwartz (1981).


Jensen and Lake (1988) studied the effect of sample size on the measures of heterogeneity. They also studied the influence of permeability distribution on heterogeneity. Emanuel et al. (1989) described a method based
on fractal geostatistics for applying to reservoir modeling. They incorporated this method to improve the estimation of vertical sweep efficiency. Ouenes (1992) developed a new kriging approach based on the Simulated Annealing Method (SAM). He checked the validity of this method against three sets of laboratory data and two sets of reservoir data.
CHAPTER 3
MATHEMATICAL MODELING OF
NATURALLY FRACTURED RESERVOIRS

3.1 Theory of Modeling

Reservoir modeling is the process of inferring the behavior of a real reservoir (*the prototype system*) from the performance of a model of that reservoir. The model may be physical, such as a scaled laboratory model, or *mathematical*. A mathematical model of a real reservoir is a set of partial differential equations, together with an appropriate set of boundary conditions, which are believed to adequately describe the significant physical processes taking place in the real reservoir.

The processes taking place in a real reservoir are basically fluid flow and mass transfer. Up to three immiscible phases (water, oil, and gas) may flow simultaneously where gravity, capillary, and viscous forces play an important role in the flow process. Whereas mass transfer may take place between the phases (chiefly between gas and oil phases). The model equations must account for all forces, and should also take into account an arbitrary reservoir description with respect to heterogeneity and geometry. The equations are obtained by combining the equation for conservation of mass (*continuity equation*) with the equation of motion (*Darcy's law*).

To use the mathematical model for predicting the behavior of a real reservoir, it is necessary to solve the model equations subject to the
appropriate boundary conditions. The methods of solution are basically divided into two main methods, *analytical* and *numerical*. Analytical methods are applicable only to the simplest cases involving homogeneous reservoirs and very regular boundaries. Numerical methods, on the other hand, are extremely general in their applicability and have proved to be highly successful for obtaining solutions to very complex reservoir situations.

3.1.1 Modeling Similitude

In modeling petroleum reservoirs two systems are being considered: a *prototype system* (*the petroleum reservoir*) and a *model system* (*the mathematical model*). The conditions that a model of a certain phenomenon reproduces all aspects of the behavior of the prototype system are known as conditions of *similitude*. A *complete similarity* should be satisfied between the two systems. The concepts of *geometric similarity*, *kinematic similarity*, and *dynamic similarity* are well known in different simulation fields.

Geometric similarity implies that the ratios between all corresponding lengths in the two considered systems must be the same. Kinematic similarity is the similarity of the flow net composed of streamlines and equipotentials. It implies that the direction of the velocity remains unchanged and that the ratio between velocities (and accelerations) at all homologous points in the two systems is the same throughout the flow domain. Dynamic similarity requires that forces at homologous points and homologous times acting on homologous
elements of fluid mass must be in the same ratio throughout the two systems.

3.1.2 Modeling Analysis

In general, two approaches to model scaling are usually employed: *dimensional analysis* and *inspectional analysis*. Dimensional analysis states that any equation expressing a physical relationship between quantities must be *dimensionally homogeneous*; i.e. all terms in such an equation are similar in dimensions. In general, the variables involved in a physical phenomenon are known, while the relationships among them are unknown. By a procedure of dimensional analysis the phenomenon considered may be formulated as relationships among a set of *dimensionless groups* of variables.

We assume that all the aspects of the physical prototype system and the macroscopic phenomena occurring in it can be completely described by a known system of mathematical equations, say $P$ (prototype). Let $M$ (model) be another system of equations analogous to $P$, completely defining the dynamic and kinematic behavior of the model system. This means that for every equation in $P$ there is a corresponding equation in $M$ and for each variable and operation in $P$, there is a corresponding variable and operation in $M$. Inspectional analysis is possible once the two sets of homologous equations -for the prototype and for the model- are established. Ratios (scales) of the model values are introduced to corresponding prototype values for all variables and constants involved. These ratios are then inserted into one set of equations,
e.g., the prototype equations, and the resulting set is compared with the unchanged one. By demanding that the two sets of equations become identical, scales are obtained.

3.2 Model Choice

In modeling the history of a naturally fractured reservoir and then predicting the future performance of that reservoir, the following points must be taken into consideration:

(1) Matrix blocks may contain over 90 per cent of the total oil reserves, and the problem of oil recovery from a fractured reservoir is essentially that of extracting oil from these blocks and not necessarily from the fractures. Thus the main aim should be to simulate the behavior of the matrix blocks as accurately as possible, so the fracture pressure, which is used essentially as the boundary condition for matrix blocks, can be known accurately. When the pressure in a fractured reservoir does not behave fairly uniformly throughout the reservoir, such a reservoir may be divided into as many regions as required. With such a division of the reservoir (whether areally or vertically), the pressure in each region will be fairly horizontally constant or its potential will be vertically constant at any given time.

(2) The main producing mechanisms in these types of reservoirs are gravity drainage and imbibition which are very pressure-sensitive. Therefore, pressure and all other related matters which may have a considerable effect
on gravity drainage and imbibition processes, should be well defined and calculated as accurately as possible. In other words, the equations describing the exchange flow between fracture and matrix at their boundaries should be derived as accurately as possible.

(3) The number of parameters that can be varied during history matching fractured reservoirs is far greater than those used in history matching conventional reservoirs. This makes history matching fractured reservoirs subject to different sources of error. To avoid this problem, one has to avoid the use of "lumped" parameters as much as possible, whereas each parameter and its relationship to the flow of fluids should be described as accurately as possible.

3.3 Assumptions of this Model

In this study, the simulator to be discussed represents a single region of a naturally fractured reservoir. In formulating the fluid flow equations for this region, the following assumptions were implicitly made:

(1) Pressure and saturation distribution are fairly areally uniform at all times.

(2) In addition to the conservation of mass, Darcy's law is used to describe fluid flow through fractures.

(3) For a dual-porosity model formulation, Darcy's law is used to describe fluid flow from matrix blocks to fractures. Whereas for single-porosity
model formulation, the empirical mathematical model presented by Aronofsky is used to describe fluid recovery from matrix blocks.

(4) When there is a tracer or chemical injected, the advective-dispersive model is used to describe this flow process.

(5) At the end of a time-step, the reservoir system is in a complete equilibrium.

3.4 Mathematical Operators

Let \( f \) be a scalar function of the spatial variables \( x, y, \) and \( z, \) and the time \( t. \) The following mathematical operators are defined:

3.4.1 The Difference Operator

The difference operator of the function \( f \) in the spatial variable \( x \) is denoted by \( \Delta_x f \) and is defined by:

(1) Forward Difference

\[
\Delta_x f = f(x+\Delta x,y,z,t) - f(x,y,z,t)
\]

(2) Backward Difference

\[
\Delta_x f = f(x,y,z,t) - f(x-\Delta x,y,z,t)
\]
3.4.2 The Differential Operator

The differential operator of the function $f$ in the spatial variable $x$ is denoted by $\frac{\partial f}{\partial x}$ or $\frac{df}{dx}$ or $f_x$ and is defined by:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y,z,t) - f(x,y,z,t)}{\Delta x}$$

The same formulae hold for $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, and $\frac{\partial f}{\partial t}$.

3.4.3 The Gradient Operator

The gradient of the scalar function $f$, written as $\nabla f$, is a vector whose $x$-component is $\frac{\partial f}{\partial x}$, $y$-component is $\frac{\partial f}{\partial y}$, and $z$-component is $\frac{\partial f}{\partial z}$.

3.4.4 The Divergence Operator

Let $\vec{u}$ be a vector whose components are $u_x$, $u_y$, and $u_z$. The divergence of the vector $\vec{u}$, written as $\nabla \cdot \vec{u}$, is a differential operator on $\vec{u}$ and is defined as:
\[ \nabla . \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \]

3.4.5 Combination of Operators

The divergence of the gradient of the function \( f \), written as \( \nabla . (\nabla f) \) and abbreviated as \( \nabla^2 f \), is called the Laplacian of \( f \) and is defined as:

\[ \nabla^2 f = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

More generally, if \( f \) and \( p \) are both scalar functions, then the divergence of the vector \((f \nabla p)\) is:

\[ \nabla . (f \nabla p) = \frac{\partial}{\partial x} \left( f \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( f \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left( f \frac{\partial p}{\partial z} \right) \]

3.5 Conservation of Mass (Continuity Equation)

For single-phase flow, the general form of the continuity equation is:

\[- \nabla . (\delta \rho \mathbf{v}) + \delta \rho \dot{q} = \delta \frac{\partial}{\partial t} (\rho \phi)\]

Where:

\[ \delta \] = geometric factor function defined as follows:

one dimension: \[ \delta(x, y, z) = A(x) \]
two dimension: $\delta(x,y,z) = \Delta z(x,y)$

three dimension: $\delta(x,y,z) = 1$

$\rho$ = fluid density

$\vec{v}$ = velocity vector

$\dot{q}$ = volume flow rate per unit reservoir volume

$\phi$ = porosity

all are in consistent units.

For multi-phase flow, the continuity equation takes the following form:

$$- \nabla \cdot (\delta \rho_a \vec{v}_a) + \delta \rho_a \dot{q}_a = \delta \frac{\partial}{\partial t} (\rho_a \phi S_a)$$ (3.1)

Where $\alpha$ represents the phase, and $S_a$ is the phase saturation.

3.6 Equation of Motion (Darcy's Law)

For single-phase flow, the compact differential form of Darcy's law is:

$$\vec{v} = - \frac{k}{\mu} (\nabla p - \gamma \nabla D)$$

Where:

$\vec{v}$ = velocity vector

$k$ = absolute permeability tensor of the porous medium

$\mu$ = fluid viscosity

$p$ = fluid pressure
\[ \gamma = \text{fluid specific weight} \]

\[ D = \text{depth} \]

all are in consistent units.

For multi-phase flow, Darcy's law takes the following differential form:

\[ \vec{v}_\alpha = - k \left( \frac{k_r}{\mu}_\alpha \right) (\nabla p_\alpha - \gamma \nabla D) \]  \hspace{1cm} (3.2)

Where \( \alpha \) represents the phase and \( k_r \) is the phase relative permeability. The permeability tensor in equation (3.2) must be determined experimentally. In most practical problems it is possible (or necessary) to assume that \( k \) is a diagonal tensor given by:

\[
\begin{bmatrix}
k_{xx} & 0 & 0 \\
0 & k_{yy} & 0 \\
0 & 0 & k_{zz}
\end{bmatrix}
\]

If \( k_{xx} = k_{yy} = k_{zz} \), the medium is called isotropic, otherwise it is anisotropic.

### 3.7 Matrix/Fracture Transfer

In a naturally fractured reservoir, the interaction between matrix blocks and fractures is described by transfer functions. These functions can be generated in the laboratory and supplied to the model as a correlation, or as digitized data. The following drawbacks can be expected:

(1) different reservoir conditions require different sets of transfer functions.
(2) the flow between matrix blocks and fractures may be reversed during the gas or water injection (resaturation). Thus this process can not be handled by these types of transfer functions.

(3) these functions are not general purpose models and their applications are limited.

In this study, two formulations are considered: dual-porosity model formulation and single-porosity model formulation.

### 3.7.1 DPM Matrix/Fracture Transfer

Since Darcy's law applies to the flow in porous media in both matrix and fracture, a reasonable extension for this fact is to consider that the same law is applicable to the flow between the two media, thus:

\[
\dot{\varepsilon}_{amf} = -k_m \left( \frac{k_r}{\mu} \right)_{amf} \sigma \left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right]
\]  

(3.3a)

Where \( \dot{\varepsilon}_{amf} \) is the volumetric dual-porosity model matrix/fracture transfer per unit reservoir volume, and \( \sigma \) is the shape factor.

If in a water-oil system, the water comes in contact with water-wet matrix saturated with oil, the water will imbibe into the matrix, and the oil will expel into the fracture at the same volumetric rate. Accordingly, the transfer from/to the matrix is equal to the loss/gain in the matrix and thus:
3.7.2 SPM Matrix/Fracture Transfer Function

The single-porosity model matrix/fracture transfer function is an analytical and dynamic function that is capable of reducing a dual-porosity model into a single-porosity model. The single-porosity model carries out all calculations in the fractures, which are represented by continuous continuum, with matrix blocks as source/sink terms. This property makes it much faster than the conventional dual-porosity model. Because of the difficulties and uncertainty involved in the evaluation of some of the single-porosity model parameters, these parameters need to be tuned through history matching the dual-porosity model or the field recovery performance.

In naturally fractured reservoirs, gravity drainage and imbibition are the most important mechanisms for oil recovery from matrix blocks. In this study, imbibition is assumed the dominant mechanism for oil recovery from matrix blocks where water is imbibed into matrix blocks while oil is expelled out at the same volumetric rate. Aronofsky et al. presented an empirical mathematical model to describe the recovery of oil from matrix blocks in a decreasing exponential form as:

\[
R(t) = R_a \left( 1 - e^{-\lambda t} \right)
\]

Where:
\( R(t) = \) recovery of a single block at time \( t = \phi_m \) \( (S_{wm} - S_{wcm}) \)

\( R_w = \) maximum recoverable oil from a single block = \( \phi_m \) \( (1 - S_{orm} - S_{wcm}) \)

\( \lambda = \) exponential fitting parameter = reverse time at 63\% recovery or \((1 - 1/e)\).

The above model is valid if the following conditions are satisfied: (1) Viscous and gravitational forces are negligible, (2) The recovery is a continuous monotonic function of time that converges to a finite limit, and (3) None of the properties that determine the ultimate oil recovery changes much during the oil recovery process. The imbibition rate is given by:

\[
\frac{\partial R(t)}{\partial t} = R_w \lambda e^{-\lambda t}
\]

In this approach, the matrix block response to a unit Heaviside step function change of saturation in the fracture is considered. This response is the rate of fluid flow to/from the matrix; i.e.

\[
S_w(t) = H_0(t) = \begin{cases} 
0 & t = 0 \\
1 & t > 0 
\end{cases}
\]

For continuous change of \( S_w(t) \) in the time interval \([0, t]\), the response is given by the accumulation of the differential form of \( S_w(t) \) with time multiplied by a unit step matrix response; i.e. this response is given by the following convolution:

\[
\frac{\partial R(t)}{\partial t} = R_w \lambda \int_0^t e^{-\lambda(t - \tau)} \frac{\partial S_w}{\partial \tau} \, d\tau
\]
From which we get:

\[ \frac{dS_{af}}{dt} = - R_w \lambda \int_0^t e^{-\lambda(t-\tau)} \frac{dS_{af}}{d\tau} \, d\tau \quad (3.4a) \]

For more flexibility in matching reservoir performance, Aronofsky’s model can be expanded in a more general form as follows:

\[ R(t) = R_w - R_1 e^{-\lambda_1 t} - R_2 e^{-\lambda_2 t} - R_3 e^{-\lambda_3 t} - ... \]

Where:

\[ R_1 + R_2 + R_3 + ... = R_w \]

Plugging the second equation into the first one yields:

\[ R(t) = R_1 (1 - e^{-\lambda_1 t}) + R_2 (1 - e^{-\lambda_2 t}) + R_3 (1 - e^{-\lambda_3 t}) + ... \]

Which can be written as:

\[ R(t) = \sum_{N=1}^{NFIT} R_N (1 - e^{-\lambda_N t}) \]

From which we get:

\[ \frac{dS_{af}}{dt} = - \sum_{N=1}^{NFIT} R_N \lambda_N \int_0^t e^{-\lambda_N (t-\tau)} \frac{dS_{af}}{d\tau} \, d\tau \quad (3.4b) \]

Where \( \frac{dS_{af}}{dt} \) is the volumetric single-porosity model matrix/fracture transfer function per unit reservoir volume, and \( NFIT \) is the number of fitting parameters required to fit the oil recovery curve of a single matrix block.
It is clear that the above transfer function has no matrix variables. Thus, the incorporation of this function reduces the dual-porosity model into a single-porosity model.

3.8 Fracture Flow Equation

Combining equations (3.1) and (3.2) and realizing that $\rho_a = \frac{(\rho_a)_{sc}}{B_a}$ yields:

$$\nabla \cdot \left[ \delta k \left( \frac{k_r}{\mu B_a} \right) \left( \nabla p_a - \gamma_a \nabla D \right) \right] + \delta \frac{\dot{q}_a}{B_a} = \delta \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)$$

For fluid flow in the fracture, the above equation takes the form:

$$\nabla \cdot \left[ \delta k_f \left( \frac{k_r}{\mu B_a \phi_f} \right) \left( \nabla p_{af} - \gamma_{af} \nabla D_f \right) \right] + \delta \frac{\dot{q}_{af}}{B_a} = \delta \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)$$

Since the idealization of fractured reservoirs considers the matrix as a source and the fracture as the main path for fluid flow toward the wellbore, the matrix/fracture transfer should be added to the above equation. Thus we get:

$$\nabla \cdot \left[ \delta k_f \left( \frac{k_r}{\mu B_{af} \phi_f} \right) \left( \nabla p_{af} - \gamma_{af} \nabla D_f \right) \right] + \delta \frac{\dot{q}_{af}}{B_a} = \delta \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)$$

In fractured reservoirs, the effective fracture permeability is:

$$k_f = k_f \phi_f = (k \phi_f)$$

Substitution of $k_f$ yields the fracture flow equation:
\[
\n\nabla \cdot \left[ \delta (k \phi) \left( \frac{k_r}{\mu B} \right) \left( \nabla p_{af} - \gamma_{af} \nabla D_f \right) \right] + \delta \frac{\dot{a}_{mf}}{B_a} + \delta \frac{\dot{a}_{af}}{B_a} = \delta \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_{f} \tag{3.5}
\]

### 3.9 Matrix/Fracture Flow Equation

The matrix/fracture flow equation is obtained by combining equations (3.3a) and (3.3b):

\[
k_m \left( \frac{k_r}{\mu B} \right)_{amf} \sigma \left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right] = \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_{m} \tag{3.6}
\]

### 3.10 Wellbore Rate Equation

If the wellbore total flow rate is specified, the following wellbore equation is applied:

\[
Q_T = \sum_{a} \left\{ \sum_{k=1}^{KK} q_{ak} + c_a \frac{\partial p_{bh}}{\partial t} \right\} \tag{3.7}
\]

### 3.11 Initial Conditions

Initially, at time zero, the reservoir should be in a gravity-capillary pressure equilibrium. In other words, the phase velocity in the vertical direction is zero.

For fracture initialization, this condition is expressed mathematically as:

\[
\left[ (k \phi) \left( \frac{k_r}{\mu B} \right) \left( \frac{\partial}{\partial z} p_{af} - \gamma_{af} \frac{\partial}{\partial z} D_f \right) \right]_{z=0} = 0 \tag{3.8a}
\]
For dual-porosity model matrix initialization, there would be no flow between
the fracture and the matrix at equilibrium condition; i.e.

$$
\left[ k_m \left( \frac{k_r}{\mu B} \right) \sigma \left[ \nabla p_{amf} - \nabla (\gamma_a D_{mf}) \right] \right]_{t=0} = 0 \quad (3.8b)
$$

### 3.12 Boundary Conditions

For times greater than zero, boundary conditions must be specified. Four
different types of boundary conditions are encountered in reservoir engineering
applications. These are:

1. **Constant Influx Boundary**

$$
\left[ (k_\xi \phi_f) A \left( \frac{k_r}{\mu B} \right) \left( \frac{\partial}{\partial \xi} p_{af} - \gamma_{af} \frac{\partial}{\partial \xi} D_f \right) \right]_{\xi = b} = C \quad (3.9a)
$$

2. **Constant Pressure Boundary**

$$
P_{af}\vert_{\xi = b} = C \quad (3.9b)
$$

3. **No-Flow Boundary**

$$
\left[ (k_\xi \phi_f) A \left( \frac{k_r}{\mu B} \right) \left( \frac{\partial}{\partial \xi} p_{af} - \gamma_{af} \frac{\partial}{\partial \xi} D_f \right) \right]_{\xi = b} = 0 \quad (3.9c)
$$
(4) Aquifer Boundary

\[
\left[ (k \phi)_f \frac{A}{\mu B} \left( \frac{\partial}{\partial \xi} P_{af} - \gamma_{af} \frac{\partial}{\partial \xi} D_f \right) \right]_b - a = q_{aq} \tag{3.9d}
\]

3.13 Auxiliary Equations

The mathematical model discussed so far is not complete without the necessary auxiliary equations. The dual-porosity model has nine unknowns to be evaluated at each node. These unknowns are: \( p_{wf}, p_{of}, S_{wf}, S_{af}, P_{wm}, P_{om}, S_{om}, \) and \( p_{bh} \) for rate control wellbore boundary condition. The equations available so far are: two fracture equations represented by equation (3.5), two matrix/fracture equations represented by equation (3.6), and one rate-control wellbore equation represented by equation (3.7). The other four auxiliary equations are:

\[
P_{of} = p_{wf} + P_{cf} \tag{3.10a}
\]

\[
S_{of} = 1 - S_{wf} \tag{3.10b}
\]

\[
P_{om} = p_{wm} + P_{cm} \tag{3.10c}
\]

\[
S_{om} = 1 - S_{wm} \tag{3.10d}
\]

The above equations, (3.10a) through (3.10d), eliminate four unknowns. Thus only five unknowns; i.e. \( p_{wf}, S_{wf}, P_{wm}, S_{wm}, \) and \( p_{bh} \) have to be evaluated at each
node.

On the other hand, the single-porosity model has five unknowns to be evaluated at each node. These unknowns are: \( p_{wf}, p_{of}, S_{wf}, S_{of} \) and \( p_{bh} \) for a rate control wellbore boundary condition. The equations available so far are: two fracture equations represented by equation (3.5), and one rate-control wellbore equation represented by equation (3.7). The other two auxiliary equations are given by equations (3.10a) and (3.10b). These equations eliminate two unknowns. Thus only three unknowns; i.e. \( p_{wf}, S_{wf} \), and \( p_{bh} \) have to be evaluated at each node.

3.14 Chemical Transport Model

For cases in which a chemical such as a tracer or polymer is injected, a chemical transport equation is needed to describe the chemical flow in the fracture. The main mathematical approach to depict the tracer dispersion in the fracture is the advective-dispersive model. The general differential equation for the advective-dispersive model in one porosity system is:

\[- \nabla \cdot (\delta \rho \; \vec{v} \; C) + \nabla \cdot (\delta \rho \; \phi \; D \; \nabla C) + \delta \rho \; \dot{\phi} \; C_{dp} = \delta \frac{\partial}{\partial t} (\rho \; \phi \; C)\]

Where \( C \) is the tracer concentration, \( D \) is the dispersion coefficient, and \( C_{dp} \) is the injected concentration (\( C_i \)) for injection wells and produced concentration (\( C_p \)) for production wells. Many tracers might undergo adsorption to the solid
surface of the matrix rock. Including adsorption in the above equation and
simplifying yields:

\[- \nabla \cdot (\delta \rho \ \vec{v} \ C) + \nabla \cdot (\delta \rho \ \phi \ D \ \nabla C) + \delta \rho \ \hat{q} \ C_{i/p} = \delta \frac{\partial}{\partial t} (\rho \ R_d \ C)\]

Where:

\[R_d = \phi + K_d \ \rho_s \ (1 - \phi)\]

\[K_d = \frac{a}{c}\]

In many applications the tracer used might experience radioactive decay.

The concentration change resulting from this decay is:

\[\frac{\partial}{\partial t} (\rho \ R_d \ C) = - \rho \ \lambda \ R_d \ C\]

Where \(\lambda\) is the decay constant and is equal to the reciprocal half life of the
radioactive material. If radioactive decay is included, the chemical transport
equation for the single-porosity system becomes:

\[- \nabla \cdot (\delta \rho \ \vec{v} \ C) + \nabla \cdot (\delta \rho \ \phi \ D \ \nabla C) - \delta \rho \ \lambda \ R_d \ C\]

\[+ \delta \rho \ \hat{q} \ C_{i/p} = \delta \frac{\partial}{\partial t} (\rho \ R_d \ C)\]

For double-porosity systems, two extra terms are added to the left hand-
side of the above equation. The first term is the amount of tracer transferred
by convection between the matrix and the fracture, and the second term is the
mass transfer of tracer diffused between the two continua. Therefore the chemical transport equation for the double-porosity system is:

\[- \nabla \cdot (\delta \rho \; \vec{v} \; C) + \nabla \cdot (\delta \rho \; \phi \; D \; \nabla C) - \delta \rho \; \lambda \; R_d \; C + \delta \rho \; \hat{\alpha}_{mf} \; C\]

\[- (\delta \rho \; \hat{q} \; C_{\|p}) + \delta \rho \; \hat{q} \; (\rho \; R_d \; C) = \delta \frac{\partial}{\partial t} (\rho \; R_d \; C)\]

The use of a water-soluble, non-adsorbable, non-radioactive tracer will require only computation with the water-phase equation. The high injection rate of the tracer minimizes the loss of tracer by matrix imbibition. A proper choice of the grid size can make the numerical dispersion the same as the physical dispersion. With these in mind, dropping the second, third, and fifth terms on the left-hand side as well as dropping the adsorption rate on the right-hand side of the above equation become justifiable. Thus the above equation becomes:

\[- \nabla \cdot (\delta \rho \; \vec{v} \; C) + \delta \rho \; \hat{q} \; C_{\|p} = \delta \frac{\partial}{\partial t} (\rho \; \phi_s \; S_{wf} \; C)\]

Substitution of velocity vector, equation (3.2), into the above equation yields the chemical transport equation:

\[
\nabla \cdot \left[ \delta \; (k \; \phi)_f \left( \frac{k_r}{\mu B_{w}} \right) \left( \nabla P_{wf} - \gamma_{wf} \; \nabla D_f \right) \; C \right] + \delta \frac{\hat{q}_{wf}}{B_w} \; C_{\|p} = \delta \frac{\partial}{\partial t} \left( \frac{\phi_s \; S_{wf}}{B_w} \; C \right)
\]

(3.11)
CHAPTER 4
FD DISCRETIZATION SCHEME

4.1 Grid Geometry

The finite difference discretization scheme is either a five-point difference or a nine-point difference technique. A schematic of the flow directions considered in the five- and nine-point formulations is shown in Fig. 4.1. The five-point formulation only considers flow between a block and the four blocks that are adjacent to its boundaries. The nine-point formulation considers these blocks as well as the flow between the block and the four blocks located at its corner.

Fig. 4.1: Schematic representation of the five-point and nine-point finite difference formulations
4.2 Discretization of the Fracture Flow Equation

The divergence operator on the left-hand side of the fracture flow equation, equation (3.5), can be expanded in any coordinate system. For generality, the three-dimensional expansion and the nine-point formulation will be considered. For the 3D case, equation (3.5) is written as follows:

\[
\nabla \cdot \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \nabla p_{af} - \gamma_{af} \nabla D_f \right) + \frac{\dot{\tau}_{amf}}{B_a} + \frac{\dot{q}_{af}}{B_a} = \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_f
\]

The expansion of the above equation is written as follows:

\[
\frac{\partial}{\partial x} \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \frac{\partial}{\partial x} p_{af} - \gamma_{af} \frac{\partial}{\partial x} D_f \right) + \frac{\partial}{\partial y} \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \frac{\partial}{\partial y} p_{af} - \gamma_{af} \frac{\partial}{\partial y} D_f \right) + \frac{\partial}{\partial z} \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \frac{\partial}{\partial z} p_{af} - \gamma_{af} \frac{\partial}{\partial z} D_f \right)
\]

\[
+ \frac{\partial}{\partial d2} \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \frac{\partial}{\partial d2} p_{af} - \gamma_{af} \frac{\partial}{\partial d2} D_f \right) + \frac{\partial}{\partial d1} \left[ \frac{(k \phi_f r^r)}{B_{af}} \right] \left( \frac{\partial}{\partial d1} p_{af} - \gamma_{af} \frac{\partial}{\partial d1} D_f \right)
\]

\[
+ \frac{\dot{\tau}_{amf}}{B_a} + \frac{\dot{q}_{af}}{B_a} = \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_f
\]

If i represents node numbering in the x-direction, j represents node numbering
in the y-direction, and k represents node numbering in the z-direction, then, for node \(ijk\), the fully implicit central finite-difference approximation of the above equation is written as follows:

\[
\frac{1}{\Delta x_{jk}} \left[ (k_x)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial x} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial x} D_f \right) \right]_{k, \frac{1}{2}}
\]

\[
- \left[ (k_x)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial x} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial x} D_f \right) \right]_{k, \frac{1}{2}}
\]

\[
+ \frac{1}{\Delta y_{jk}} \left[ (k_y)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial y} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial y} D_f \right) \right]_{y, \frac{1}{2}}
\]

\[
- \left[ (k_y)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial y} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial y} D_f \right) \right]_{y, \frac{1}{2}}
\]

\[
+ \frac{1}{\Delta z_{jk}} \left[ (k_z)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial z} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial z} D_f \right) \right]_{z, \frac{1}{2}}
\]

\[
- \left[ (k_z)f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial z} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial z} D_f \right) \right]_{z, \frac{1}{2}}
\]

\[
+ \frac{1}{\Delta d_{2jk}} \left[ (k_{d2})f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial d_{2}} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial d_{2}} D_f \right) \right]_{d_{2}, \frac{1}{2}}
\]

\[
- \left[ (k_{d2})f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial d_{2}} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial d_{2}} D_f \right) \right]_{d_{2}, \frac{1}{2}}
\]

\[
+ \frac{1}{\Delta d_{1jk}} \left[ (k_{d1})f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial d_{1}} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial d_{1}} D_f \right) \right]_{d_{1}, \frac{1}{2}}
\]

\[
- \left[ (k_{d1})f_j \left( \frac{k_r}{\mu B_{af}} \right)^{n+1} \left( \frac{\partial}{\partial d_{1}} p_{af}^{n+1} - \gamma_{af}^{n+1} \frac{\partial}{\partial d_{1}} D_f \right) \right]_{d_{1}, \frac{1}{2}}
\]

\[
* \frac{\dot{q}_{af}^{n+1}}{B_{af}^{\psi k}} - \frac{\dot{a}_{af}^{n+1}}{B_{af}^{\psi k}} = \frac{1}{\Delta t} \left[ \left( \Phi S_{a}^{n+1} \right)_{n+1} - \left( \Phi S_{a}^{n} \right)_{n} \right]_{\psi k}
\]
Expanding potential terms, multiplying all through by \( (V_R)_{\phi} \), including the conversion constant, and rearranging, then the above equation is written as:

\[
0.001127 \left( \frac{A_r}{\Delta x_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
- 0.001127 \left( \frac{A_r}{\Delta x_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
+ 0.001127 \left( \frac{A_r}{\Delta y_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
- 0.001127 \left( \frac{A_r}{\Delta y_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
+ 0.001127 \left( \frac{A_r}{\Delta z_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
- 0.001127 \left( \frac{A_r}{\Delta z_j} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1} - D_{ij} \right) \right] \\
+ 0.001127 \left( \frac{A_r}{\Delta d_{j, \frac{1}{2}, \frac{1}{2}}} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1, \frac{1}{2}} - D_{ij, \frac{1}{2}} \right) \right] \\
- 0.001127 \left( \frac{A_r}{\Delta d_{j, \frac{1}{2}, \frac{1}{2}}} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1, \frac{1}{2}} - D_{ij, \frac{1}{2}} \right) \right] \\
+ 0.001127 \left( \frac{A_r}{\Delta d_{j, \frac{1}{2}, \frac{1}{2}}} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1, \frac{1}{2}} - D_{ij, \frac{1}{2}} \right) \right] \\
+ 0.001127 \left( \frac{A_r}{\Delta d_{j, \frac{1}{2}, \frac{1}{2}}} \right) \left[ k_r \Phi \right]_{\Delta t, \frac{1}{2}} \left( \frac{k_r}{\mu B_{\alpha j}} \right)_{\Delta t, \frac{1}{2}} \left[ \left( \psi_{\alpha ij, \frac{1}{2}}^{n, 1} - \psi_{\alpha ij}^{n, 1} \right) - \gamma_{\alpha ij}^{n, 1} \left( D_{ij, -1, \frac{1}{2}} - D_{ij, \frac{1}{2}} \right) \right]
\]

\[ \frac{\gamma_{\alpha ij}^{n, 1}}{\eta_{\alpha ij}^{n, 1}} = \frac{(V_R)_{\phi}}{5.6146 \Delta t} \left( \Phi_S_{\alpha}^{n, 1} - \Phi_S_{\alpha}^{n, 1} \right) \eta_{\alpha ij} \]
By introducing transmissibility coefficients, the final form of the discretized fracture flow equation is obtained:

\[
\begin{aligned}
(T_a)_{a(f,i)}^{n+\frac{1}{2}} & \left[ p_{af,i+1}^{n+1} - p_{af,i}^{n+1} \right] - \gamma_{af,i}^{n+1} \left( D_{fi,i+1} - D_{fi,i} \right) \\
- (T_d)_{a(f,i)}^{n+\frac{1}{2}} & \left[ p_{af,i}^{n+1} - p_{af,i-1}^{n+1} \right] - \gamma_{af,i}^{n+1} \left( D_{fi,i} - D_{fi,i-1} \right) \\
+ (T_j)_{a(f,j)}^{n+\frac{1}{2}} & \left[ p_{af,j}^{n+1} - p_{af,j}^{n+1} \right] - \gamma_{af,j}^{n+1} \left( D_{fj,j} - D_{fj,j} \right) \\
- (T_j)_{a(f,k)}^{n+\frac{1}{2}} & \left[ p_{af,k}^{n+1} - p_{af,k}^{n+1} \right] - \gamma_{af,k}^{n+1} \left( D_{fk,k} - D_{fk,k} \right) \\
+ (T_d)_{a(f,k)}^{n+\frac{1}{2}} & \left[ p_{af,k}^{n+1} - p_{af,k}^{n+1} \right] - \gamma_{af,k}^{n+1} \left( D_{fk,k} - D_{fk,k} \right) \\
+ (T_d)_{a(f,j)}^{n+\frac{1}{2}} & \left[ p_{af,j}^{n+1} - p_{af,j}^{n+1} \right] - \gamma_{af,j}^{n+1} \left( D_{fj,j} - D_{fj,j} \right) \\
- (T_d)_{a(f,j)}^{n+\frac{1}{2}} & \left[ p_{af,j}^{n+1} - p_{af,j}^{n+1} \right] - \gamma_{af,j}^{n+1} \left( D_{fj,j} - D_{fj,j} \right) \\
+ (T_d)_{a(f,i)}^{n+\frac{1}{2}} & \left[ p_{af,i}^{n+1} - p_{af,i}^{n+1} \right] - \gamma_{af,i}^{n+1} \left( D_{fi,i} - D_{fi,i} \right) \\
- (T_d)_{a(f,i)}^{n+\frac{1}{2}} & \left[ p_{af,i}^{n+1} - p_{af,i}^{n+1} \right] - \gamma_{af,i}^{n+1} \left( D_{fi,i} - D_{fi,i} \right) \\
+ (\tau_{ijk}^{\alpha})_{ijk}^{n+\frac{1}{2}} & \left[ q_{af}^{n+1} - q_{af}^{n+1} \right] - \left( \frac{V_R}_{ijk} \right)^{n+\frac{1}{2}} 5.6146 \Delta t \left[ \left( \frac{\phi S_a}{B_a} \right)^{n+1} - \left( \frac{\phi S_a}{B_a} \right)^{n} \right]_{ijk} = R_{a,ijk}^{n+\frac{1}{2}} (X)
\end{aligned}
\]
4.3 Von Neumann Stability Criteria of the Fully Implicit Formulation

The von Neumann stability criteria is sometimes known as the Fourier stability criteria because it uses Fourier series in the representation of error terms. The Fourier series expansion of a function can be expressed as:

\[ f(x) = \sum_p A_p \cos \frac{p\pi x}{L} = \sum_p A_p e^{i \frac{p\pi x}{L}} \quad i = \sqrt{-1} \]

The von Neumann stability analysis consists in expanding the error in a Fourier series as:

\[ \varepsilon_j = \sum_p A_p e^{i \frac{p\pi j}{L}} = \sum_p A_p e^{i \frac{p\pi \Delta x}{L}} = \sum_p A_p e^{i \beta j \Delta x} \]

Since the errors from all terms are additive, then we can analyze only a single term, thus:

\[ \varepsilon_j = A \ e^{i \beta j \Delta x} \]

To include the growth of error as time increases and in such a way that the error term reduces to the initial value term at time zero, the following form is assumed:

\[ \varepsilon_j^n = e^{an} e^{i \beta j \Delta x} = \gamma^n e^{i \beta j \Delta x} \]

It is obvious that the error term at any point \( i \) in space and \( n \) in time will not increase as long as \( |\gamma| \leq 1 \).
The stability of the fully implicit formulation will be examined by considering the following parabolic one-dimensional equation:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t}$$

The fully implicit central finite-difference approximation of the above equation is written as:

$$\frac{p_{j+1}^{n+1} - 2p_j^{n+1} + p_{j-1}^{n+1}}{\Delta x^2} = \frac{p_j^{n+1} - p_j^n}{\Delta t}$$

Performing a von Neumann stability analysis by the substitution of the error term into the above equation, thus:

$$\frac{\Delta t}{\Delta x^2} \left[ \gamma^{n+1} e^{i(\theta+1)\Delta x} - 2\gamma^{n+1} e^{i\theta\Delta x} + \gamma^{n+1} e^{i(\theta-1)\Delta x} \right] = \gamma^{n+1} e^{i\beta\Delta x} - \gamma^n e^{i\beta\Delta x}$$

Rearranging and solving for the amplification factor $\gamma$, we obtain the final form of error as follows:

$$\frac{\gamma^{n+1}}{\gamma^n} = \gamma = \frac{1}{1 - \frac{\Delta t}{\Delta x^2} \left[ e^{i\beta\Delta x} - 2 + e^{-i\beta\Delta x} \right]}$$

$$= \frac{1}{1 + \frac{\Delta t}{\Delta x^2} \cdot \frac{4}{\sin^2 \left( \frac{\beta \Delta x}{2} \right)} }$$

Therefore, for any choice of $\Delta x$ or $\Delta t$ the fully implicit formulation is unconditionally stable.
4.4 Discretization of the Matrix/Fracture Flow Equation

Realizing that the matrix/fracture potential term appearing on the left-hand side of the matrix/fracture flow equation, equation (3.6), can be expanded as follows:

\[
\left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right]_{ijk} = \left[ (p_a - \gamma_a D)_f - (p_a - \gamma_a D)_m \right]_{ijk}
\]

Therefore the discretization of the matrix/fracture flow equation becomes straightforward and can be written as:

\[
T_{am,ijk}^{n+1} \left[ (p_{af}^{n+1} - \gamma_{af}^{n+1} D_f) - (p_{am}^{n+1} - \gamma_{am}^{n+1} D_m) \right]_{ijk} = \left( \frac{V_R}{5.6146} \right)_{ijk} \left[ \left( \frac{\Phi S_a}{B_a} \right)^{n+1} - \left( \frac{\Phi S_a}{B_a} \right)^n \right]_{m,ijk} = R_{am,ijk}^{n+1} (X)
\]  \hspace{1cm} (4.2)

4.5 Discretization of the Wellbore Rate Equation

Expanding the sums appearing in the wellbore rate equation, equation (3.7), then, for two-phase flow, the discretized wellbore rate equation is written as follows:

\[
Q_T - \sum_a \left\{ \sum_{k=1}^{KK} q_{ak} + c_a \frac{\partial p_{bh}}{\partial t} \right\} = Q_T - \sum_{k=1}^{KK} (q_w + q_o)_k - (c_w + c_o) \frac{\partial p_{bh}}{\partial t} = R_{bh,ij}^{n+1} (X)
\]  \hspace{1cm} (4.3)
4.6 Discretization of the Chemical Transport Equation

The chemical transport equation, equation (3.11), is discretized by following the same discretization procedure shown for the fracture flow equation:

\[
(T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{s,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} - P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} - D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{a,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} + P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} + D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{a,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} + P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} + D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{a,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} + P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} + D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
- (T_{a,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} + P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} + D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ (T_{a,\text{wf}j})^{n+1}_r \left[ (P_{\text{wf}j}^{n+1} + P_{\text{wf}j}^{n}) - \frac{\gamma^{n+1}}{2} (D_{fj}^{n+1} + D_{fj}) \right] C^{n+1}_{\text{wf}j} \]

\[
+ \left( \tau_{\text{wfj}} \right)_j^{r+1} C^{n+1}_{\text{wf}j} + (\mathcal{A}_{\text{wfj}})_{ij}^{r+1} \right) C^{n+1}_{\text{wf}j} \]

\[
= \frac{(V_R)_{jkl}^{r+1}}{5.6146 \Delta t} \left[ \left( \Phi S_w C_w \right)^{n+1}_{jkl} - \left( \Phi S_w C_w \right)^n_{jkl} \right]_{jkl}^{49}
\]
4.7 Fracture Transmissibilities

Fracture transmissibilities have been decomposed into two parts. The first part is the transmissibility constant which has the conversion constant, the interfacial flow area, the characteristic flow length, and the harmonic average permeability. The interfacial flow area depends on the transmissibility formulation considered (whether a five-point or nine-point formulation). The second part is the mobility at the blocks interface and it has the phase mobility of the fracture with upstream weighted relative permeability, formation volume factor and constant viscosity. Thus the transmissibility between nodes \(i\) and \(i+1\); i.e. \(T_{a,f_i}^{n+1} \), is given by:

\[
(T_{a,f_i}^{n+1}) = 0.001127 \left( \frac{A_x}{\Delta x_{i\frac{1}{2}}} \right) \left( k_x \phi \right)_{f_{i\frac{1}{2}}} \left( \frac{k_r}{\mu B} \right)_{a,f_{i\frac{1}{2}}}^{n+1}
\]

Where:

\[
\Delta x_{i\frac{1}{2}} = \frac{(\Delta x_i + \Delta x_{i-1})}{2}
\]

\[
(k_x \phi)_{f_{i\frac{1}{2}}} = \frac{(k_x \phi)_{f_{i\frac{1}{2}}} + (k_x \phi)_{f_{i+1\frac{1}{2}}}}{k_x \phi_{f_{i\frac{1}{2}}} + (k_x \phi)_{f_{i+1\frac{1}{2}}}} = \frac{(\Delta x_i + \Delta x_{i-1})}{(k_x \phi)_{f_{i\frac{1}{2}}} + (k_x \phi)_{f_{i+1\frac{1}{2}}}}\left(\frac{(k_x \phi)_{f_{i\frac{1}{2}}} + (k_x \phi)_{f_{i+1\frac{1}{2}}}}{k_x \phi_{f_{i\frac{1}{2}}} + (k_x \phi)_{f_{i+1\frac{1}{2}}}}\right)
\]

\[
= \frac{(\Delta x_i + \Delta x_{i-1})}{\Delta x_i (k_x \phi)_{f_{i\frac{1}{2}}} + \Delta x_{i+1} (k_x \phi)_{f_{i+1\frac{1}{2}}}}
\]
\[
\left( \frac{k_r}{\mu B} \right)_{af,i+1}^{n+1} = \left[ \omega_{ax,i+1} \left( \frac{k_r}{\mu B} \right)_{af,i}^{n+1} + (1 - \omega_{ax,i+1}) \right] \left( \frac{k_r}{\mu B} \right)_{af,i+1}^{n+1}
\]

The upstream weighting factor \( \omega \) is calculated between \( i \) and \( i+1 \) and stored in \( i+1 \) as \( \omega_{ax} \). \( \omega \) has two values, either 1 or 0, obtained from the node of higher potential; i.e.:

\[
\omega_{ax,i+1} = 1 \quad \text{if} \quad p_{af,i}^{n+1} - \frac{\gamma_{af,i+1}^{n+1}}{2} D_{fi} \geq p_{af,i+1}^{n+1} - \frac{\gamma_{af,i+1}^{n+1}}{2} D_{fi+1}
\]

\[
\omega_{ax,i+1} = 0 \quad \text{otherwise}
\]

\[
\gamma_{af,i+1}^{n+1} = \frac{p_{af,i}^{n+1} + p_{af,i+1}^{n+1}}{288}
\]

The transmissibility is calculated between \( i \) and \( i+1 \) and stored in \( i+1 \) as \( T_x \).

Thus for nodes \( i \) and \( i+1 \), the transmissibility is given by:

\[
(TAX)_{i+1}^{n+1} = (TCONX)_{i+1} \cdot \left[ \omega_{ax,i+1} \left( \frac{k_r}{\mu B} \right)_{af,i}^{n+1} + (1 - \omega_{ax,i+1}) \right] \left( \frac{k_r}{\mu B} \right)_{af,i+1}^{n+1}
\]

Similar treatments are utilized for other directions' transmissibilities.

### 4.7.1 Five-Point Transmissibility Formulation

Considering fluid flow from node \( i \) to node \( i+1 \), then for a five-point transmissibility formulation, the interfacial flow area is given by:

\[
A_x = (\Delta y \Delta z)_{ijk}
\]

A similar manipulation is performed for the rest of directions.
4.7.2 Nine-Point Transmissibility Formulation

To examine the mathematical formulation of the nine-point finite-difference discretization scheme, we will consider the solution of the prototype Laplacian equation; \( \nabla^2 p = 0 \) over the square grid of side length \( h \) shown by Fig. 4.1. The key point is to approximate the solution of \( p_u \) in terms of values of \( p \) at all surrounding nodes. The starting point is the Taylor series expansion of \( p(x+h) \) which is written as:

\[
p(x+h) = p(x) + h \frac{\partial}{\partial x} p(x) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} p(x) + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} p(x) + \frac{h^4}{4!} \frac{\partial^4}{\partial x^4} p(x) + ...
\]

\[
= \left[ 1 + h \frac{\partial}{\partial x} + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} + \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} + \frac{h^4}{4!} \frac{\partial^4}{\partial x^4} + ... \right] p(x)
\]

\[
= \left[ e^{h \frac{\partial}{\partial x}} \right] p(x) = e^h p(x)
\]

Similarly, for \( p(y+h) \):

\[
p(y+h) = p(y) + h \frac{\partial}{\partial y} p(y) + \frac{h^2}{2!} \frac{\partial^2}{\partial y^2} p(y) + \frac{h^3}{3!} \frac{\partial^3}{\partial y^3} p(y) + \frac{h^4}{4!} \frac{\partial^4}{\partial y^4} p(y) + ...
\]

\[
= \left[ 1 + h \frac{\partial}{\partial y} + \frac{h^2}{2!} \frac{\partial^2}{\partial y^2} + \frac{h^3}{3!} \frac{\partial^3}{\partial y^3} + \frac{h^4}{4!} \frac{\partial^4}{\partial y^4} + ... \right] p(y)
\]

\[
= \left[ e^{h \frac{\partial}{\partial y}} \right] p(y) = e^h p(y)
\]

Following the same analogy, we can write:
\[ P_{i-1,j} = e^\xi p_{ij} \quad P_{i+1} = e^\eta p_{ij} \quad P_{i-1,j} = e^{-\xi} p_{ij} \quad P_{i,j-1} = e^{-\eta} p_{ij} \]

\[ P_{i+1,j} = e^{\xi+\eta} p_{ij} \quad P_{i+1,j-1} = e^{-\xi+\eta} p_{ij} \quad P_{i-1,j} = e^{-\xi-\eta} p_{ij} \quad P_{i+1,j-1} = e^{\xi-\eta} p_{ij} \]

Define the following symmetric sums:

\[
S_1 = P_{i-1,j} + P_{i,j-1} + P_{i-1,j} + P_{i,j-1}
\]

\[
S_2 = P_{i+1,j+1} + P_{i-1,j+1} + P_{i+1,j-1} + P_{i-1,j-1}
\]

Substitute all \( p \)'s into the above equations; we obtain:

\[
S_1 = \left[ e^\xi + e^\eta + e^{-\xi} + e^{-\eta} \right] p_{ij}
\]

\[
S_2 = \left[ e^{\xi+\eta} + e^{-\xi+\eta} + e^{-\xi-\eta} + e^{\xi-\eta} \right] p_{ij}
\]

After substitution of all \( e^a \)'s into the above equations and rearranging; we get:

\[
S_1 = 4p_{ij} + h^2 \nabla^2 p_{ij} + \frac{1}{12} h^4 \nabla^4 p_{ij} - \frac{1}{6} h^4 D^4 p_{ij} + \ldots
\]

\[
S_2 = 4p_{ij} + 2h^2 \nabla^2 p_{ij} + \frac{1}{6} h^4 \nabla^4 p_{ij} + \frac{4}{6} h^4 D^4 p_{ij} + \ldots
\]

where \( D^4 = \partial^4/\partial x^2 \partial y^2 \)

Multiplying \( S_1 \) by 4, adding to \( S_2 \), realizing that \( \nabla^4 p_{ij} = 0 \), and solving for \( \nabla^2 p_{ij} \) we obtain:

\[
\nabla^2 p_{ij} = \frac{4 S_1 + S_2 - 20 p_{ij}}{6 h^2} + O(h^4)
\]

Substitution of \( S_1 \) and \( S_2 \) into the above equation yields the nine-point finite-
difference solution of the Laplacian equation:

\[
\nabla^2 p_y = \frac{1}{6h^2} \left[ 4 \left( p_{i-1,j} + p_{i,j+1} + p_{i-1,j+1} + p_{i,j-1} \right) \\
+ \left( p_{i+1,j} + p_{i-1,j-1} + p_{i+1,j-1} + p_{i+1,j+1} \right) - 20 p_y \right]
\]

It is most convenient to exhibit the above nine-point formulation as follows:

\[
\nabla^2 p_y = \frac{1}{6h^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} p_y = 0
\]

The above equation is fourth order correct; i.e. it has lower numerical dispersion and thus reduces the GOE. This formulation gives the weighting factor of 4/6 for orthogonal blocks and 1/6 for diagonal blocks. In other words, fluid flow parallel to the coordinate axes is assumed to occupy 4/6 of the penetrated interface; whereas fluid flow diagonal to the coordinate axes is assumed to occupy 1/6 of each penetrated interface, see Fig. 4.2.

![Fig. 4.2: Interfacial flow area for parallel and diagonal fluid flow](image)

---

**Fig. 4.2**: Interfacial flow area for parallel and diagonal fluid flow
Therefore, for flow from node $i$ to node $i+1$ (parallel flow), the interfacial flow area is given by:

$$A_s = \frac{4}{6} \left[ \frac{(\Delta y \Delta z)_i}{2} + \frac{(\Delta y \Delta z)_{i+1}}{2} \right] = \frac{1}{3} \left[ (\Delta y \Delta z)_i + (\Delta y \Delta z)_{i+1} \right]$$

On the other hand, for flow from node $ij$ to node $i+1,j+1$ (diagonal flow), the interfacial flow area is given by:

$$A_{d2} = \frac{1}{6} \left\{ \left[ \frac{(\Delta x \Delta z)_y}{2} + \frac{(\Delta x \Delta z)_{i+1,j+1}}{2} \right] + \left[ \frac{(\Delta y \Delta z)_y}{2} + \frac{(\Delta y \Delta z)_{i+1,j+1}}{2} \right] \right\}$$

However, the above formulation is valid only for uniform grids and homogeneous systems. For nonuniform grids and heterogeneous systems, the formulation falls down and might lead to negative transmissibilities. To extend this formulation to general anisotropic systems, new methods, that always yield positive transmissibilities, were suggested. Modified Shiralkar, half-weighting, and unit-weighting formulations are such methods. For brevity only half- and unit-weightings are considered in this work. Full treatment of the modified Shiralkar formulation can be found in reference #81.
4.7.2.1 Half-Weighting Nine-Point Transmissibilities

In the half-weighting nine-point transmissibility formulation, the fluid flow parallel to the coordinate axes is assumed to occupy 1/2 of the penetrated interface. Whereas fluid flow diagonal to the coordinate axes is assumed to occupy 1/4 of each penetrated interface, see Fig. 4.3.

![Diagram](image)

**Fig. 4.3:** Interfacial flow area for parallel and diagonal fluid flow (half-weighting formulation)

Therefore, for flow from node \(i\) to node \(i+1\) (parallel flow), the interfacial flow area is given by:

\[
A_i = \frac{1}{2} \left[ \frac{(\Delta y \Delta z)_i + (\Delta y \Delta z)_{i+1}}{2} \right] = \frac{1}{4} \left[ (\Delta y \Delta z)_i + (\Delta y \Delta z)_{i+1} \right]
\]

On the other hand, for flow from node \(ij\) to node \(i+1,j+1\) (diagonal flow), the interfacial flow area is given by:

\[
A_{d2} = \frac{1}{4} \left\{ \left[ \frac{(\Delta x \Delta z)_y + (\Delta x \Delta z)_{y+1}}{2} \right] + \left[ \frac{(\Delta y \Delta z)_y + (\Delta y \Delta z)_{y+1}}{2} \right] \right\}
\]
4.7.2.2 Unit-Weighting Nine-Point Transmissibilities

To satisfy the unit-weighting nine-point transmissibility formulation, the orthogonal interfacial flow area (BC) should equal the diagonal interfacial flow area (AB), see Fig. 4.4; i.e. AB = BC.

![Diagram showing interfacial flow areas for parallel and diagonal fluid flow.](image)

**Fig. 4.4:** Interfacial flow area for parallel and diagonal fluid flow (unit-weighting formulation)

Since AB = \sqrt{2} a, BC = Δy - 2a, solving for a gives \[ a = \frac{Δy}{\sqrt{2} + 2}. \] By equating the parallel and diagonal areas, the following weighting factor is obtained \[ w = \frac{\sqrt{2}}{\sqrt{2} + 2}. \] Therefore, for fluid flow from node \( i \) to node \( i+1 \), the interfacial flow area is calculated as follows:

\[ A_z = w \left[ \frac{(ΔyΔz)_i + (ΔyΔz)_{i+1}}{2} \right]. \]

A similar manipulation is performed for other flow directions.
4.8 DPM Matrix/Fracture Transfer

The dual-porosity model matrix/fracture transfer term is given by equation (3.3a) which is written as follows:

\[
\frac{\dot{\tau}_{amf}}{B_a} = -k_m \left( \frac{k_r}{\mu B} \right)_{amf} \sigma \left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right]
\]

Multiplying by \((V_R)_{ijk}\), including the conversion constant, and rearranging, the above equation is written as:

\[
(\tau_{amf})_{ijk} = -0.001127 \left( k_m \sigma V_R \right)_{ijk} \left( \frac{k_r}{\mu B} \right)_{amf} \left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right]_{ijk}
\]

By introducing the matrix/fracture transmissibility coefficient and realizing that the matrix/fracture potential term is expanded as follows:

\[
\left[ \nabla p_{amf} - \nabla (\gamma_a D)_{mf} \right]_{ijk} = \left[ (p_a - \gamma_a D)_{f} - (p_a - \gamma_a D)_{m} \right]_{ijk}
\]

then, the above equation, for time level \(n+1\), is written as:

\[
(\tau_{amf})^{n+1}_{ijk} = -T_{am,ijk}^{n+1} \left[ (p_a^{n+1} - \gamma_a^{n+1} D)_{f} - (p_a^{n+1} - \gamma_a^{n+1} D)_{m} \right]_{ijk}
\]
4.8.1 Matrix/Fracture Transmissibilities

The matrix/fracture transmissibility is written as:

\[ T_{am,ijk}^{n-1} = 0.001127 \left( k_m \sigma V_R \right)_{ijk} \left( \frac{k_r}{\mu B} \right)_{am,ijk}^{n-1} \]

Where:

\( \sigma \) is the matrix block's shape factor. For single-phase flow applications, Warren and Root have derived the following equation for \( \sigma \):

\[ \sigma = \frac{4N(N+2)}{L^2} \]

Where \( N \) = number of normal sets of fractures: 1, 2, or 3, and \( L \) is the matrix block side length. If matrix blocks have dimensions \( L_x, L_y, \) and \( L_z \), then:

\[
L = \begin{cases} 
L_x & \text{for } N = 1 \\
\frac{2L_xL_y}{L_x + L_y} & \text{for } N = 2 \\
\frac{3L_xL_yL_z}{L_xL_y + L_yL_z + L_xL_z} & \text{for } N = 3 
\end{cases}
\]

In finite difference formulation, Kazemi has shown that for a full three-dimensional case,

\[ \sigma = 4 \left( \frac{1}{L_x^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2} \right) \]
\[
\left( \frac{k_r}{\mu B} \right)_{am,ijk}^{n+1} = \left( 1 - \omega_{am} \right) \left( \frac{k_r}{\mu B} \right)_{af}^{n+1} + \omega_{am} \left( \frac{k_r}{\mu B} \right)_{am,ijk}^{n+1}
\]

The weighting factor \( \omega \) has two values, either 1 or 0, depending on the potential difference between matrix and fracture. Thus

\[
\omega_{am,ijk} = 1 \quad \text{if} \quad \left( P_{am}^{n+1} - \gamma_{am}^{n+1} D_m \right)_{ijk} < \left( P_{af}^{n+1} - \gamma_{af}^{n+1} D_f \right)_{ijk}
\]

\[
\omega_{am,ijk} = 0 \quad \text{otherwise}
\]

\[
\gamma_{am}^{n+1} = \frac{P_{am}^{n+1}}{144}
\]

\[
\gamma_{af}^{n+1} = \frac{P_{af}^{n+1}}{144}
\]

Thus the matrix/fracture transmissibility is given by:

\[
(TAM)^{n+1}_{ijk} = (TCONM)_{ijk} \cdot \left[ \left( 1 - \omega_{am} \right) \left( \frac{k_r}{\mu B} \right)_{af}^{n+1} + \omega_{am} \left( \frac{k_r}{\mu B} \right)_{am,ijk}^{n+1} \right]
\]

The gravity and capillary pressure in the water invaded zone is not treated thoroughly, and the matrix/matrix flow is not considered. Saturation and pressure front tracking in matrix blocks is included by further subdividing the matrix block which in turn improves the gravity segregation.
4.9 SPM Matrix/Fracture Transfer

The single-porosity model matrix/fracture transfer term is given by equation (3.4a) for the simple Aronofsky's model and by equation (3.4b) for the extended Aronofsky's model. For generality, the extended model, equation (3.4b), will be considered which is written as follows:

\[
\frac{\dot{t}_{amf}}{B_a} = - \sum_{N=1}^{NFIT} \frac{R_N \gamma_N}{B_a} \int_0^t e^{-\gamma_N (t-\tau)} \frac{\partial S_{af}}{\partial \tau} d\tau
\]

Multiplying by \((V_{R})_{ljk}\), and rearranging, then, the above equation is written as:

\[
(V_{R})_{ljk} \frac{N_{F}}{5.6146} \sum_{N=1}^{NFIT} \frac{R_N \gamma_N}{B_a} \int_0^t e^{-\gamma_N (t-\tau)} \frac{\partial S_{af}}{\partial \tau} d\tau
\]

4.9.1 Numerical Solution of the Convolution Integral

The convolution integral of the above equation is approximated by:

\[
\int_0^t e^{-\gamma (t-\tau)} \frac{\partial S_{af}}{\partial \tau} d\tau = \sum_{i=0}^{n} \left( S_{af}^{t+1} - S_{af}^t \right) e^{-\gamma (t-\tau)}
\]

Expanding the summation and collecting similar terms, the above equation can be written as:

\[
\int_0^t e^{-\gamma (t-\tau)} \frac{\partial S_{af}}{\partial \tau} d\tau = \sum_{i=0}^{n} \left( S_{af}^{t+1} - S_{af}^t \right) \prod_{m=l}^{n} e^{-\gamma \Delta t_m}
\]
By decomposing the summation of the above equation, we obtain:

\[
\int_0^t e^{-\lambda (t-\tau)} \frac{\partial S_{af}}{\partial \tau} \, d\tau = \sum_{l=0}^{n-1} \left( S_{af}^{l+1} - S_{af}^l \right) \prod_{m=l}^{n-1} e^{-\lambda \Delta t_m} + \left( S_{af}^{n+1} - S_{af}^n \right) e^{-\lambda \Delta t_n}
\]

Where:

\[\Delta t_m = t_{m+1} - t_m\]

The single-porosity matrix/fracture transfer function can be implemented in a code as a progressive formulation accumulated through each time-step as:

\[
(\tau_{am})^{n+1}_{ijk} = \frac{(V_R)_{ijk}}{5.6146} \sum_{N=1}^{N_{FIT}} R_N \lambda N \left\{ SUM^{n+1}_{ijk} \left[ \frac{S_{af}}{B_{af}} \right]^{n+1} - \left( \frac{S_{af}}{B_{af}} \right]^n \right\} e^{-\lambda N \Delta t_n}
\]

Where:

\[SUM^{n+1} = \left\{ SUM^{n+2} + \left[ \left( \frac{S_{af}}{B_{af}} \right)^n - \left( \frac{S_{af}}{B_{af}} \right)^{n+1} \right] \right\} e^{-\lambda N \Delta t_{n-1}}\quad n \geq 2\]

\[SUM^0 = \left[ \left( \frac{S_{af}}{B_{af}} \right)^1 - \left( \frac{S_{af}}{B_{af}} \right)^0 \right] e^{-\lambda N \Delta t_0}\quad n = 1\]
4.10 Wellbore Variables

A thorough understanding of the assumptions implicit in the way a simulator treats wells as "sources and sinks" is essential if one is to avoid serious mistakes in forecasting water-oil ratio (WOR), gas-oil ratio (GOR), well pressures, well requirements and recovery efficiency. In essence, a gridblock containing a well is similar in all aspects to all other gridblocks in the model. There is nothing internal to the grid system that recognizes the existence of a well, and the performance of a simulated well usually can not be derived directly from pressures and fluid saturations in the well block.

The model in this study allows for two different wellbore constraints: the rate-control well boundary condition and the pressure-control well boundary condition. For a rate-control well boundary condition and fully-implicit formulation, the wellbore equation is given by:

\[
(Q_T)_y = \sum_a \left( \sum_{k=1}^{KK} q_{ak} + c_a \frac{\partial p_{bh}}{\partial t} \right)
\]

Where:

\( (Q_T)_y \) = wellbore total flow rate, STB/DAY

\( KK \) = number of perforated layers

\( (q_a)_{ijk} \) = node ijk phase flow rate, STB/DAY; i.e.

\( (q_w)_{ijk} = WWP_{ijk} [p_{bh} - p_{node_{ijk}}] \)

\( (q_o)_{ijk} = WOP_{ijk} [p_{bh} - p_{node_{ijk}}] \)
\[(q_t)_{ijk} = (q_w)_{ijk} + (q_o)_{ijk}\]

\[WWP_{ijk} = \left\{ WI \left( \frac{k_{rw}}{\mu_v} + \beta \frac{k_{ro}}{\mu_o} \right) \frac{1}{B_{wfr}} \right\}_{ijk}\]

\[WOP_{ijk} = \left\{ WI (1 - \beta) \left( \frac{k_{ro}}{\mu_o B_o f} \right) \right\}_{ijk}\]

\[WI_{ijk} = \left\{ \frac{0.001127 (2 \pi) \Delta z \left( \bar{k} \phi \right)_f}{\ln \left( \frac{r_1}{r_w} \right) + s} \right\}_{ijk}\]

\[\bar{k}_{ijk} = \left\{ \sqrt{k_x \bar{k}_y} \right\}_{ijk}\]

\[(r_1)_{ijk} = \left\{ \frac{e^{-\gamma}}{2} \left[ \left( \frac{k_y}{k_x} \right)^{\frac{1}{2}} \Delta x^2 + \left( \frac{k_x}{k_y} \right)^{\frac{1}{2}} \Delta y^2 \right]^{\frac{1}{2}} \right\}_{ijk}\]

\[\beta_{ijk} = \left\{ \begin{array}{ll} 1 & \text{for injection} \\ 0 & \text{for production} \end{array} \right\}\]

\(\gamma\) is the Euler constant = 0.57721566..., and

\[c_a \frac{\partial p_{bh}}{\partial t}\] is the wellbore storage
For a rate-control well boundary condition and explicit solution of the wellbore variables, the wellbore equation is given by:

\[
(q_t)_{ijk} = (Q_T)_{ij} \left[ \frac{WP_k}{\sum_{k=1}^{KK} WP_k} \right]_{ij}
\]

Where:

\[
WP_{ijk} = WWP_{ijk} + WOP_{ijk}
\]

Consequently water and oil rates are calculated as follows:

\[
(q_w)_{ijk} = (q_t)_{ijk} \left( \frac{k_{rw}}{\mu_w B_w} \right)_{ijk}
(q_o)_{ijk} = (q_t)_{ijk} \left( \frac{k_{ro}}{\mu_o B_o} \right)_{ijk}
\]

For a pressure-control well boundary condition, when bottom hole pressure, \( p_{bh} \), is specified as a producing-well boundary condition, an appropriate gravity gradient is used to relate this pressure to the individual well block pressures. If \( p_{node} < p_{bh} \), the well is shut in, whereas if \( p_{node} > p_{bh} \), water and oil flow rates are calculated from Darcy's law as:

\[
(q_w)_{ijk} = WWP_{ijk} \left[ p_{bh} - p_{node} \right]_{ijk}
(q_o)_{ijk} = WOP_{ijk} \left[ p_{bh} - p_{node} \right]_{ijk}
\]
CHAPTER 5
CVFE DISCRETIZATION SCHEME

5.1 Introduction

A flexible discretization method to define complex reservoir geometries and to enhance the resolution near wells is extremely useful. The use of FD methods has created difficulties and complexities for simulating irregular reservoir boundaries and grid refinements. It is desirable to adopt the grid flexibility of the finite element method while retaining the phase upstream weighting of the usual FD methods. Phase upstream weighting is used for mobility terms since it converges to the physically correct solution. It is difficult, however, to combine the upstream weighting with the usual finite element method for multiphase multidimensional flow problems. Such methods are not mass-conservative in the local sense. Additionally, local mass conservation is a specific requirement of the control volume methods. The ability of the control volume methods to retain the easy physical interpretation can be a distinct advantage in programming and testing these simulators.

The CVFE method was proposed in different simulation fields where flexible gridding as well as local mass, momentum, and energy conservation are achieved. The use of the perpendicular-bisection (PEBI) method or the Voronoi grid, and the seven-point FD method are only special cases of this discretization scheme. In this research, the CVFE method, using linear
triangular elements, is applied to the simulation of naturally fractured reservoirs. The pressure is treated in a finite element manner, while the mobility terms are upstream weighted in the usual finite difference way. The control-volume surrounding node $i$ is constructed by connecting the set of barycenters and connection mid-points of all elements connected to node $i$, see Fig. 5.2. The CVFE scheme for the solution of the model equations uses the integral formulation of these equations. It results in a set of nonlinear equations similar to those obtained in the FD method. Therefore the method of solution using the Jacobian is the same for both methods.

Spatial variables do not appear either in the matrix/fracture flow equation or in the wellbore rate equation. Thus the FD and the CVFE discretization processes of these equations are exactly the same. To exhibit the complexity of the usual finite element method for multiphase multidimensional flow equations, the derivation of this method, for two-dimensional, two-phase flow is shown in APPENDIX C.

For uniform regions, the CVFE triangular mesh is easy to generate by simple algorithms. On the other hand, very general regions are not. In general, one simple algorithm is carried out by subdividing the region into subregions. These subregions are triangulated separately and, then, assembled together to form the complete triangular domain (see APPENDIX B for details). The subregions are chosen in such a way that there are no obtuse angles to assure positive transmissibility constraints.
5.2 Grid Geometry

The first step in developing a finite element model is the domain triangulation process. A typical triangulation using straight-sided (linear) triangular elements is shown in Fig. 5.1. Nodes, elements, and interelement boundaries are indicated in the figure.

Fig. 5.1: Typical mesh for a two-dimensional domain using linear interpolated triangular elements
5.3 Discretization of the Fracture Flow Equation

For the two-dimensional case, the fracture flow equation, equation (3.5), is written as follows:

\[
\nabla \cdot \left[ \Delta z \left( k \phi \right)_f \left( \frac{k_r}{\mu B}_a \right) \left( \nabla P_{af} - \gamma_{af} \nabla D_f \right) \right] + \Delta z \frac{\dot{t}_{amf}}{B_a} + \Delta z \frac{\dot{q}_{af}}{B_a} = \Delta z \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_f
\]

Expanding the divergence operator of the above equation, we obtain:

\[
\frac{\partial}{\partial x} \left[ \Delta z \left( k \phi \right)_f \left( \frac{k_r}{\mu B}_a \right) \frac{\partial}{\partial x} \left( \Phi_{af} \right) \right] + \frac{\partial}{\partial y} \left[ \Delta z \left( k \phi \right)_f \left( \frac{k_r}{\mu B}_a \right) \frac{\partial}{\partial y} \left( \Phi_{af} \right) \right]  \\
+ \Delta z \frac{\dot{t}_{amf}}{B_a} + \Delta z \frac{\dot{q}_{af}}{B_a} = \Delta z \frac{\partial}{\partial t} \left( \frac{\phi S_a}{B_a} \right)_f
\]

The CVFE discretization scheme starts with multiplying the above equation by a test function, \( v(x,y) \), and integrating over the edges, \( e_e \), (see Fig. 5.2):

![Figure 5.2: Schematic of a control-volume surrounding node i](image)

1. connection mid-point
2. element barycenter
3. node i control-volume
4. 2 3 4
\[ \int_{\Omega^e} \left\{ \frac{\partial}{\partial x} \left[ \Delta z \left( k_x \phi \right)_f \left( \frac{k_r}{\mu B}_{af} \right) \frac{\partial \Phi_{af}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \Delta z \left( k_y \phi \right)_f \left( \frac{k_r}{\mu B}_{af} \right) \frac{\partial \Phi_{af}}{\partial y} \right] \right\} v d \Omega^e \]

\[ + \left( \frac{\tau_{amf}}{B_a} \right)_i + \left( \frac{q_{af}}{B_a} \right)_i = \left( V_R \right)_i \frac{\partial}{\partial t} \left( \frac{\Phi_{Sa}}{B_a} \right)_f \]

where \( (V_R)_i \) is the volume of the box surrounding node \( i \). Thus we write:

\[ \int_{\Omega^e} \left\{ \frac{\partial}{\partial x} \left[ \Delta z \left( k_x \phi \right)_f \left( \frac{k_r}{\mu B}_{af} \right) \frac{\partial \Phi_{af}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \Delta z \left( k_y \phi \right)_f \left( \frac{k_r}{\mu B}_{af} \right) \frac{\partial \Phi_{af}}{\partial y} \right] \right\} v d \Omega^e \]

\[ + \left( \tau_{amf} \right)_i + \left( q_{af} \right)_i = \left( V_R \right)_i \frac{\partial}{\partial t} \left( \frac{\Phi_{Sa}}{B_a} \right)_f \]

The discretization process of the first term on the left-hand side is carried out by transferring differentiation from the original function to the test function by performing integration by parts (divergence theorem):

\[ - \int_{\Omega^e} \left\{ \Delta z \left[ \left( k_x \phi \right)_f \frac{\partial \Phi_{af}}{\partial x} + \left( k_y \phi \right)_f \frac{\partial \Phi_{af}}{\partial y} \right] \left( \frac{k_r}{\mu B}_{af} \right) \right\} d \Omega^e \]

\[ + \int_{\Gamma^e} \left\{ \Delta z \left[ \left( k_x \phi \right)_f \frac{\partial \Phi_{af}}{\partial x} n_x + \left( k_y \phi \right)_f \frac{\partial \Phi_{af}}{\partial y} n_y \right] \left( \frac{k_r}{\mu B}_{af} \right) \right\} d \Gamma^e \]

The boundary integral of the above equation will disappear upon the assembly of the global stiffness matrix as the fluxes across element boundaries cancel one another\(^\text{78}\). The variational approximation to the above variational form is performed by selecting:
\[ \Phi(x, y, t) = \sum_j \Phi_j(t) \psi_j(x, y) \quad \quad \nu(x, y) = \psi_i(x, y) \]

By referring to Fig. 5.2, it is clear that the connection between node \(i\) and node \(j\) will include contributions from the two elements sharing the common edge.

Thus, for control-volume \(i\), the above integral can be written as follows:

\[
\sum_{j \in \eta_i} \left\{ -\sum_{e=1}^2 \int_{\Omega_e^i} \left[ \langle k_e \Phi \rangle_{fj} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \langle k_e \Phi \rangle_{fj} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] d\Omega^i \right\} \Delta z_y \left( \frac{k_r}{\mu B_{afj}} \right)^{n+1} \Phi_{ij}^{n+1} \]

Where:

\[ \Phi_{ij} = \Phi_j - \Phi_i = (p_j - p_i) - \gamma_{ij}(D_j - D_i) \]

Including the conversion constant, introducing transmissibilities, and rearranging, we obtain the final form as follows:

\[
\sum_{j \in \eta_i} \left( T_{afj}^{n+1} \left[ \left( p_{afj}^{n+1} - p_{afj}^{n-1} \right) - \gamma_{afj}^{n+1} D_{fj} - D_{fi} \right] \right) \]

Since the third dimension is obtained by vertical projection of the two-dimensional case, then we simply add terms for the third dimension. Thus the global equation for each control volume can be written as:

\[
\sum_{j \in \eta_i} \left( T_{afj}^{n+1} \left[ \left( p_{afj}^{n+1} - p_{afj}^{n-1} \right) - \gamma_{afj}^{n+1} D_{fj} - D_{fi} \right] \right)_{ik} + \left( T_{afj}^{n+1} \left[ \left( p_{afj}^{n+1} - p_{afj}^{n-1} \right) - \gamma_{afj}^{n+1} D_{fj} - D_{fi} \right] \right)_{ik} \]

\[
- \left( T_{afj}^{n+1} \left[ \left( p_{afj}^{n+1} - p_{afj}^{n-1} \right) - \gamma_{afj}^{n+1} D_{fj} - D_{fi} \right] \right)_{ik} \]

\[
+ \left( \tau_{amf}^{n+1} \right)_{ik} + \left( a_{afj} \right)_{ik}^{n+1} \frac{V_R}{5.6146 \Delta t} \left[ \left( \frac{\phi_s}{B_a} \right)^{n+1} - \left( \frac{\phi_{s}}{B_a} \right)_{ik}^{n-1} \right] = R_{afj}^{n+1} (X) \]
5.4 Discretization Error and Convergence Rate

The FE approximation to the potential was given by:

$$\Phi(x, y, t) = \sum_j \Phi_j(t) \psi_j(x, y)$$

The exact solution can always be expanded in the vicinity of any node $i$ as a polynomial:

$$p = p_i + \left( \frac{\partial p}{\partial x} \right)_i (x - x_i) + \left( \frac{\partial p}{\partial y} \right)_i (y - y_i) + ...$$

If within an element of size $h$, a polynomial expansion of degree $n$ is employed, this can fit locally the Taylor expansion up to that degree and the error in $p$ will be of the order $O(h^{n+1})$. Thus, when linear triangular elements are used; i.e. $n = 1$, the convergence rate is of order $O(h^2)$. Though the above argument is a trifle heuristic from a mathematical point of view, it is true and it gives correctly the expected order of convergence.

In many well-posed problems, the order of convergence often suffices to extrapolate the solution to the correct result. Therefore if the rate of convergence is $O(h^2)$ and there are two approximate solutions $p^1$ for mesh size $h$ and $p^2$ for mesh size $h/2$, then we can write:

$$\frac{p^1 - p}{p^2 - p} = \frac{O(h^2)}{O\left(\frac{h}{2}\right)^2} = 4$$

From the above, an almost exact solution $p$ can be predicted. This type of
extrapolation is used if convergence is monotonic and nearly asymptotic.
Automatic mesh refinement processes are used for estimating discretization
errors so that specified accuracy can be achieved\(^{38}\).

5.5 Discretization of the Matrix/Fracture Flow Equation

\[
T_{am,ik}^{n+1} \left[ (p_{af}^{n-1} - \gamma_{af}^{n+1} D_f) - (p_{am}^{n+1} - \gamma_{am}^{n+1} D_m) \right]_{ik}
- \frac{(V_{R})_{ik}}{5.6146 \Delta t} \left[ \left( \frac{\phi S_a}{B_a} \right)^{n+1} - \left( \frac{\phi S_a}{B_a} \right)^n \right]_{m,ik} = R_{am,ik}^{n+1} (\tilde{X})
\]  

(5.2)

5.6 Discretization of the Wellbore Rate Equation

\[
Q_T - \sum_{a} \left\{ \sum_{k=1}^{kk} q_{ak} + c_a \frac{\partial p_{bh}}{\partial t} \right\} = Q_T - \sum_{k=1}^{kk} (q_w + q_o)_{k} - (c_w + c_o) \frac{\partial p_{bh}}{\partial t}
= Q_T - \sum_{k=1}^{kk} (q_w + q_o)_{k} - \frac{(c_w + c_o)}{\Delta t} (p_{bh}^{n+1} - p_{bh}^n) = R_{bh,ik}^{n+1} (\tilde{X})
\]  

(5.3)

5.7 Discretization of the Chemical Transport Equation

\[
\sum_{j \in \eta_l} (T_{wxf,ij}^{n+1} \left[ (p_{wzf}^{n+1} - p_{wfs}^{n+1}) - \gamma_{wxf}^{n+1}(D_{fj} - D_{fj}) \right] C_{wxf,ij}^{n+1} \right]_{k}
+ (T_{wxf,ik}^{n+1} \left[ (p_{wfk+1/2}^{n+1} - p_{wfk-1/2}^{n+1}) - \gamma_{wxf}^{n+1}(D_{fjk+1/2} - D_{fjk-1/2}) \right] C_{wxf,ik}^{n+1} \right)_{j}
- (T_{wxf,ik}^{n+1} \left[ (p_{wfk-1/2}^{n-1} - p_{wfk+1/2}^{n-1}) - \gamma_{wxf}^{n-1}(D_{fjk-1/2} - D_{fjk+1/2}) \right] C_{wxf,ik}^{n-1} \right)_{j}
+ (\tau_{wz,ik}^{n+1} C_{wz,ik}^{n+1} + (q_{wz,ik}^{n+1} C_{wz,ik}^{n-1} \right)_{ik} \left[ (V_{R})_{ik} \right] \left[ \left( \frac{\phi S_w C_w}{B_w} \right)^{n+1} - \left( \frac{\phi S_w C_w}{B_w} \right)^n \right]_{f,ik}
\]  

(5.4)
5.8 Lagrange Family of Interpolation Functions

The derivation of the Lagrange family of interpolation functions for the higher-order triangular elements is simplified by the use of what is called the area coordinates $L_i$.

5.8.1 Area Coordinates

Let $P$ be a point inside a triangular element 123 as shown by Fig. 5.3. A convenient set of coordinates, $L_1$, $L_2$, and $L_3$ can be defined in such a way that:

![Diagram of area coordinates](image)

**Fig. 5.3:** Definition of area coordinates $L_1$, $L_2$, and $L_3$
\[
L_1 = \frac{\text{area } P23}{\text{area } 123}
\]
\[
L_2 = \frac{\text{area } P31}{\text{area } 123}
\]
\[
L_3 = \frac{\text{area } P12}{\text{area } 123}
\]

Hence the name area coordinates. Knowing the area coordinates of the point \( P \), then the \((x,y)\)-coordinates are given by the following transformation:

\[
x = L_1 x_1 + L_2 x_2 + L_3 x_3 = \sum_{i=1}^{3} L_i x_i
\]
\[
y = L_1 y_1 + L_2 y_2 + L_3 y_3 = \sum_{i=1}^{3} L_i y_i
\]
\[
1 = L_1 + L_2 + L_3 = \sum_{i=1}^{3} L_i
\]  \(\text{(5.5)}\)

Solving the set of equations in (5.5) for \( L_1, L_2, \) and \( L_3 \) and rearranging yields:

\[
L_i = \frac{1}{2} \frac{1}{A_{123}} \left( \alpha_i + \beta_i x + \gamma_i y \right) \quad i = 1, 2, \text{ or } 3
\]

Where:

\[
\begin{align*}
\alpha_i &= x_j y_k - x_k y_j \\
\beta_i &= y_j - y_k \\
\gamma_i &= x_k - x_j \\
A_{123} &= \frac{1}{2} \sum_{i=1}^{3} \alpha_i
\end{align*}
\]

\[i \neq j \neq k \text{ are permuted in a counter clockwise direction} \]
5.8.2 Construction of the Linear Interpolation Functions

Consider the position and the node numbers of the triangular element as shown in Fig. 5.4:

![Diagram of a triangular element with nodes and edges labeled with L1, L2, and L3.]

**Fig. 5.4:** Position and node numbers for the derivation of the Lagrange linear interpolation functions

The interpolation function for node 1 should take on a value of one at node 1 and a value of zero elsewhere. Equivalently, \( \psi_1 \) should vanish on the side defined by \( L_1 = 0 \), see Fig. 5.4. Therefore, \( \psi_1 \) is of the form:

\[
\psi_1 = c L_1
\]

The constant \( c \) is determined so as to yield \( \psi_1 = 1 \) at \( L_1 = 1 \). We obtain \( c = 1 \), and, therefore, we get:
\[ \psi_1 = l_1 \]

\( \psi_2 \) vanishes along the side \( L_2 = 0 \). Therefore, \( \psi_2 \) is of the form

\[ \psi_2 = c L_2 \]

When \( \psi_2 = 1 \) at \( L_2 = 1 \), then the constant \( c = 1 \), and, therefore, we obtain:

\[ \psi_2 = L_2 \]

Following exactly the same procedure for the third node, we obtain:

\[ \psi_3 = L_3 \]

5.8.3 Construction of the Quadratic Interpolation Functions

Consider the position and the node numbers of the triangular element as shown by Fig. 5.5:

![Diagram](image)

**Fig. 5.5:** Position and node numbers for the derivation of the Lagrange quadratic interpolation functions
The interpolation function for node 1 should take on a value of one at node 1 and a value of zero elsewhere. Equivalently, \( \psi_1 \) should vanish on the sides defined by \( L_1 = 1/2 \), and \( L_1 = 0 \), see Fig. 5.5. Therefore, \( \psi_1 \) is of the form:

\[
\psi_1 = c L_1 (L_1 - 1/2)
\]

The constant \( c \) is determined so as to yield \( \psi_1 = 1 \) at \( L_1 = 1 \). We obtain \( c = 2 \), and therefore we get:

\[
\psi_1 = L_1 (2L_1 - 1)
\]

\( \psi_2 \) vanishes along the sides: \( L_2 = 1/2 \), and \( L_2 = 0 \). Therefore, \( \psi_2 \) is of the form

\[
\psi_2 = c L_2 (L_2 - 1/2)
\]

Similarly \( \psi_2 = 1 \) at \( L_2 = 1 \). Thus the constant \( c = 2 \), and hence we get:

\[
\psi_2 = L_2 (2L_3 - 1)
\]

Similarly, for \( \psi_3 \), we obtain:

\[
\psi_3 = L_3 (2L_3 - 1)
\]

For the mid-side node 4, \( \psi_4 \) vanishes along the sides \( L_1 = 0 \) and \( L_2 = 0 \), thus \( \psi_4 \) is of the form:

\[
\psi_4 = c L_1 L_2
\]

The constant \( c \) should be determined so as to yield \( \psi_4 = 1 \) at \( L_1 = 1/2 \) and \( \psi_4 = 1 \) at \( L_2 = 1/2 \). We obtain \( c = 4 \), and therefore, we get:

\[
\psi_4 = 4 L_1 L_2
\]

Following exactly the same procedure for nodes 5 and 6, we get:

\[
\psi_5 = 4 L_2 L_3 \quad \psi_6 = 4 L_1 L_3
\]
5.9 Fracture Transmissibilities

Consider the evaluation of transmissibility between node $i$ and node $j$; i.e. $(T)_{ai}^{n+1}$, where at least one of the nodes $i$ or $j$ is not on the external boundary. The transmissibility has been decomposed into two parts. The first part is the transmissibility constant which has the conversion constant, the geometric factor between the two nodes $i$ and $j$, the interfacial flow area, and the characteristic flow length. The second part is the mobility term between the two nodes $i$ and $j$ and it has the phase mobility of the fracture with upstream-weighted relative permeability, formation volume factor and constant viscosity. Mathematically, the transmissibility can be written as follows:

$$(T)_{ai}^{n+1} = 0.001127 \gamma_j \left( \frac{A}{r} \right)_j \left( \frac{k_r}{\mu B} \right)_{ai}^{n+1}$$

Where:

$$\gamma_j = -\sum_{e=1}^{2} \int_{\Omega^e} \left[ \frac{(k_x \phi)_{fji}}{A} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{(k_y \phi)_{fji}}{A} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] d\Omega^e$$

$$= -\sum_{e=1}^{2} \left[ \frac{(k_x \phi)_{fji}}{A} \beta_i \beta_j + \frac{(k_y \phi)_{fji}}{A} \gamma_i \gamma_j \right] i \neq j$$

$$(k_x \phi)_{fji} = \frac{2}{(k_x \phi)_{fji} (k_x \phi)_{fji}} (k_x \phi)_{fji} + (k_x \phi)_{fji}$$
\[
\frac{\left(k_y \phi\right)_{f_{ij}}}{\left(k_y \phi\right)_{f_{ii}} + \left(k_y \phi\right)_{f_{jj}}}
\]

\[
\psi_i = \frac{1}{2A_e} \left(\alpha_i + \beta_i x + \gamma_i y\right)
\]

\[
\alpha_i = x_j y_k - x_k y_j
\]

\[
\beta_i = y_j - y_k
\]

\[
\gamma_i = x_k - x_j
\]

\[
A_e = \frac{1}{2} \sum_{i=1}^{3} \alpha_i
\]

\[
x_{\text{barycenter}} = \frac{\sum_{i=1}^{3} x_i}{3}
\]

\[
y_{\text{barycenter}} = \frac{\sum_{i=1}^{3} y_i}{3}
\]

\[
A_{ij} = w_{ij} \Delta z_{ij}
\]

\[
w_{ij} \text{ is the width between nodes } i \text{ and } j
\]

\[
\Delta z_{ij} = \frac{(\Delta z_i + \Delta z_j)}{2}
\]

\[
r_{ij} \text{ is the distance between nodes } i \text{ and } j
\]

\[
\left(\frac{k_r}{\mu B}\right)_{af_{ij}}^{n+1} = \omega_{af_i} \left(\frac{k_r}{\mu B}\right)_{af_i}^{n+1} + (1 - \omega_{af_i}) \left(\frac{k_r}{\mu B}\right)_{af_{ij}}^{n+1}
\]

The upstream weighting factor \(\omega\) is calculated between nodes \(i\) and \(j\) and
stored in $i$ as $\omega_{af,i}$. $\omega$ has two values, either 1 or 0, obtained from the node of higher potential. Thus:

$$\omega_{af,i} = 1 \quad \text{if} \quad p_{af,i}^{n+1} - \gamma_{af,ij}^{n+1} D_{f,i} \geq p_{af,j}^{n+1} - \gamma_{af,ij}^{n+1} D_{f,j}$$

$$\omega_{a,i} = 0 \quad \text{otherwise}$$

$$\gamma_{af,ij}^{n+1} = \frac{\rho_{af,i}^{n+1} + \rho_{af,j}^{n+1}}{288}$$

The transmissibility is calculated between nodes $i$ and $j$ and stored in $i$ as $T_{af,i}$.

Thus for nodes $i$ and $j$, the transmissibility is given by:

$$(T)_{af,ij}^{n+1} = TCON_i \cdot \left[ \omega_{af,i} \left( \frac{k_r}{\mu B}_{af,i} \right)^{n+1} + (1 - \omega_{af,i}) \left( \frac{k_r}{\mu B}_{af,j} \right)^{n+1} \right]$$

The transmissibility in the $z$-direction is given by:

$$(T_z)_{af,k-\frac{1}{2}}^{n+1} = 0.001127 \cdot \left( \bar{k}_z \phi \right)_{f,k-\frac{1}{2}} \cdot \left( \frac{A}{\Delta z} \right)_{k-\frac{1}{2}} \cdot \left( \frac{k_r}{\mu B}_{af,k-\frac{1}{2}} \right)^{n+1}$$

Where:

$$\left( \bar{k}_z \phi \right)_{f,k-\frac{1}{2}} = \frac{(\Delta z_{k-1} + \Delta z_k)}{(k_z \phi)_{f,k-1} + (k_z \phi)_{f,k}} = \frac{(\Delta z_{k-1} + \Delta z_k)}{\Delta z_{k-1} (k_z \phi)_{f,k} + \Delta z_k (k_z \phi)_{f,k-1}}$$

$$= \frac{(\Delta z_{k-1} + \Delta z_k) (k_z \phi)_{f,k-1}}{\Delta z_{k-1} (k_z \phi)_{f,k} + \Delta z_k (k_z \phi)_{f,k-1}}$$
\[ A_{k, \frac{1}{2}} = A_{CV_i} = \text{Area surrounding node } i = \sum_{j \in u_i} A_{e_j} / 3 \]

\[ \Delta z_{k, \frac{1}{2}} = \frac{(\Delta z_{k-1} + \Delta z_k)}{2} \]

\[ \left( \frac{k_r}{\mu B} \right)_{afk, \frac{1}{2}}^{n+1} = \omega_{az,k} \left( \left( \frac{k_r}{\mu B} \right)_{afk-1}^{n+1} + (1 - \omega_{az,k}) \left( \frac{k_r}{\mu B} \right)_{afk}^{n+1} \right) \]

The upstream weighting factor \( \omega \) is calculated between \( k-1 \) and \( k \) and stored in \( k \) as \( \omega_{az} \). \( \omega \) has two values, either 1 or 0, obtained from the node of higher potential as follows:

\[ \omega_{az,k} = 1 \quad \text{if} \quad p_{afk-1}^{n+1} - \gamma_{afk-1}^{n+1} D_{f,k-1} \geq p_{afk}^{n+1} - \gamma_{afk}^{n+1} D_{f,k} \]

\[ \omega_{az,k} = 0 \quad \text{otherwise} \]

\[ \gamma_{afk-\frac{1}{2}}^{n+1} = \frac{p_{afk-1}^{n+1} + p_{afk}^{n+1}}{288} \]

The transmissibility is calculated between \( k-1 \) and \( k \) and stored in \( k \) as \( T_z \).

Thus for nodes \( k-1 \) and \( k \), the transmissibility is given by:

\[ (TAZ)_k^{n+1} = TCONZ_k \cdot \omega_{az,k} \left( \frac{k_r}{\mu B} \right)_{afk-1}^{n+1} + (1 - \omega_{az,k}) \left( \frac{k_r}{\mu B} \right)_{afk}^{n+1} \]
5.10 Well's Equivalent Radius

For isotropic systems, the node i control-volume can be approximated as a circular area around node i and hence the well's equivalent radius is calculated as follows:

\[
\langle r_1 \rangle_{ik} = \sqrt{\frac{A_{CV,i}}{\pi}}
\]

However, for anisotropic media, Peaceman\textsuperscript{78} has shown that for the finite difference discretization scheme, the well equivalent radius is given by:

\[
\langle r_1 \rangle_{yk} = \left\{ \frac{e^{-\gamma}}{2} \cdot \frac{1}{\left(\frac{k_y}{k_x}\right)^{\frac{1}{4}} + \left(\frac{k_x}{k_y}\right)^{\frac{1}{4}}} \cdot \left[ \left(\frac{k_y}{k_x}\right)^{\frac{1}{4}} \Delta x^2 + \left(\frac{k_x}{k_y}\right)^{\frac{1}{4}} \Delta y^2 \right]^{\frac{1}{2}} \right\}_{yk}
\]

By referring to Fig. 5.6, on the assumption that the well is located at the center of the control volume, then the well control-volume (shaded area on figure) can be embedded in a rectangle of side lengths:

\[dx = \max x_j - \min x_j\]

\[dy = \max y_j - \min y_j\]

Where \(j\) is the set of barycenters and connection mid-points of all elements connected to node \(i\) (see Fig. 5.6).
Defining the aspect ratio as $\alpha = dy/dx$, and requiring that $dx \, dy = A_{CV}$, then, the well equivalent radius is given by:

$$
(r_{i})_{uk} = \left\{ \frac{e^{-\gamma}}{2} \cdot \frac{\left(\frac{k_y}{k_z}\right)^{\frac{1}{2}} \frac{A_{CV}}{\alpha} + \left(\frac{k_x}{k_y}\right)^{\frac{1}{2}} a A_{CV}}{\left(\frac{k_y}{k_z}\right)^{\frac{1}{4}} + \left(\frac{k_x}{k_y}\right)^{\frac{1}{4}}} \right\}_{uk}
$$
CHAPTER 6

SOLUTION OF THE MODEL EQUATIONS

6.1 Method of Solution

The dual-porosity and the single-porosity models generate a system of coupled nonlinear algebraic equations in the form \( A(\bar{x}) \bar{x} = \bar{b} \). Iterative methods are used to solve such a system. The basic concept of these methods is summarized as follows:

Let \( \bar{x}^{n} \) be the solution after \( n \) iterations, therefore we write:

\[
A(\bar{x}^{n}) \bar{x}^{n} - \bar{b} = R(\bar{x}^{n})
\]

Where \( R(\bar{x}^{n}) \) is the residual after \( n \) iterations.

Let \( \bar{x} = \bar{x}^{n} + \delta \bar{x}^{n} \) be the exact solution, thus we write:

\[
A(\bar{x}^{n} + \delta \bar{x}^{n}) (\bar{x}^{n} + \delta \bar{x}^{n}) - \bar{b} = \bar{0}
\]

Using Taylor series for the first term on the left-hand side of the above equation, we obtain:

\[
A(\bar{x}^{n} + \delta \bar{x}^{n}) = A(\bar{x}^{n}) + A'(\bar{x}^{n}) \delta \bar{x}^{n}
\]

Where \( A'(\bar{x}^{n}) \) represents the derivative of \( A(\bar{x}^{n}) \). Substitution into the above equation yields:

\[
A(\bar{x}^{n}) \bar{x}^{n} + A(\bar{x}^{n}) \delta \bar{x}^{n} + A'(\bar{x}^{n}) \bar{x}^{n} \delta \bar{x}^{n} + A'(\bar{x}^{n}) (\delta \bar{x}^{n})^{2} - \bar{b} = \bar{0}
\]
Neglecting the quadratic term of the above equation and rearranging:

\[ R(\hat{x}^n) + [A(\hat{x}^n) + A'(\hat{x}^n) \delta x^*] \delta x^n = 0 \]

Identifying the term in parentheses as the derivative of the residual, we write:

\[ R(\hat{x}^n) + R'(\hat{x}^n) \delta x^n = 0 \]

Rearranging the above equation, we get:

\[ R'(\hat{x}^n) \delta x^n = - R(\hat{x}^n) \tag{6.1} \]

Equation (6.1) represents a system of linear equations. Any linear system solver can be used to solve it. \( R'(\hat{x}^n) \) is called the Jacobian matrix and is denoted by \( J(\hat{x}^n) \) and is defined as:

\[
J(\hat{x}^n) = \begin{bmatrix}
\frac{\partial R_1(\hat{x}^n)}{\partial x_1} & \frac{\partial R_1(\hat{x}^n)}{\partial x_2} & \cdots & \frac{\partial R_1(\hat{x}^n)}{\partial x_n} \\
\frac{\partial R_2(\hat{x}^n)}{\partial x_1} & \frac{\partial R_2(\hat{x}^n)}{\partial x_2} & \cdots & \frac{\partial R_2(\hat{x}^n)}{\partial x_n} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial R_m(\hat{x}^n)}{\partial x_1} & \frac{\partial R_m(\hat{x}^n)}{\partial x_2} & \cdots & \frac{\partial R_m(\hat{x}^n)}{\partial x_n}
\end{bmatrix}
\]

The method of solution outlined above is known as Newton's method for nonlinear systems of equations. This method is generally expected to give quadratic convergence, provided that a sufficiently accurate starting value is known and the inverse of the Jacobian matrix, \( J^{-1}(\hat{x}^n) \), exists.
### 6.2 Solution of the DPM

For each node connected to a well, the system to be solved is a $5 \times 5$ block matrix in the form:

\[
\begin{bmatrix}
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} & \frac{\partial R_{of}}{\partial s_{wm}} & \frac{\partial R_{of}}{\partial p_{bh}} \\
\frac{\partial p_{of}}{\partial R_{of}} & \frac{\partial s_{of}}{\partial R_{of}} & \frac{\partial s_{wm}}{\partial R_{of}} & \frac{\partial p_{bh}}{\partial R_{of}} \\
\frac{\partial R_{om}}{\partial p_{of}} & \frac{\partial R_{om}}{\partial s_{of}} & \frac{\partial R_{om}}{\partial s_{wm}} & \frac{\partial R_{om}}{\partial p_{bh}} \\
\frac{\partial p_{of}}{\partial R_{om}} & \frac{\partial s_{of}}{\partial R_{om}} & \frac{\partial s_{wm}}{\partial R_{om}} & \frac{\partial p_{bh}}{\partial R_{om}} \\
\frac{\partial R_{bh}}{\partial p_{of}} & \frac{\partial R_{bh}}{\partial s_{of}} & \frac{\partial R_{bh}}{\partial s_{wm}} & \frac{\partial R_{bh}}{\partial p_{bh}} \\
\end{bmatrix}
\begin{bmatrix}
\delta p_{of} \\
\delta s_{of} \\
\delta s_{wm} \\
\delta p_{bh} \\
\delta p_{bh}
\end{bmatrix}
= 
\begin{bmatrix}
R_{of} \\
R_{of} \\
R_{om} \\
R_{om} \\
R_{bh}
\end{bmatrix}
\]

Since the idealization of the dual-porosity system considers the fracture as the main flow path toward the wellbore, the above system can be simplified as:

\[
\begin{bmatrix}
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} & \frac{\partial R_{of}}{\partial s_{wm}} & \frac{\partial R_{of}}{\partial p_{bh}} \\
\frac{\partial p_{of}}{\partial R_{of}} & \frac{\partial s_{of}}{\partial R_{of}} & \frac{\partial s_{wm}}{\partial R_{of}} & \frac{\partial p_{bh}}{\partial R_{of}} \\
\frac{\partial R_{om}}{\partial p_{of}} & \frac{\partial R_{om}}{\partial s_{of}} & \frac{\partial R_{om}}{\partial s_{wm}} & \frac{\partial R_{om}}{\partial p_{bh}} \\
\frac{\partial p_{of}}{\partial R_{om}} & \frac{\partial s_{of}}{\partial R_{om}} & \frac{\partial s_{wm}}{\partial R_{om}} & \frac{\partial p_{bh}}{\partial R_{om}} \\
\frac{\partial R_{bh}}{\partial p_{of}} & \frac{\partial R_{bh}}{\partial s_{of}} & \frac{\partial R_{bh}}{\partial s_{wm}} & \frac{\partial R_{bh}}{\partial p_{bh}} \\
\end{bmatrix}
\begin{bmatrix}
\delta p_{of} \\
\delta s_{of} \\
\delta s_{wm} \\
\delta p_{bh} \\
\delta p_{bh}
\end{bmatrix}
= 
\begin{bmatrix}
R_{of} \\
R_{of} \\
R_{om} \\
R_{om} \\
R_{bh}
\end{bmatrix}
\]
We could think of equation (6.2) as being partitioned as:

\[
\begin{bmatrix}
C_1 & C_2 & F \\
C_3 & C_4 & 0 \\
G & 0 & H
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\delta p_{bh}
\end{bmatrix}
=
\begin{bmatrix}
R_f \\
R_m \\
R_{bh}
\end{bmatrix}
\]

Where:

\[
C_1 = \begin{bmatrix}
\frac{\partial R_{wf}}{\partial p_{wf}} & \frac{\partial R_{wf}}{\partial s_{wf}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}}
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
\frac{\partial R_{wf}}{\partial p_{wm}} & \frac{\partial R_{wf}}{\partial s_{wm}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{om}}{\partial p_{om}} & \frac{\partial R_{om}}{\partial s_{om}}
\end{bmatrix}, \quad C_3 = \begin{bmatrix}
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{om}}{\partial p_{om}} & \frac{\partial R_{om}}{\partial s_{om}}
\end{bmatrix}
\]

\[
C_4 = \begin{bmatrix}
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}}
\end{bmatrix}, \quad F = \begin{bmatrix}
\frac{\partial R_{wf}}{\partial p_{bh}} \\
\frac{\partial R_{of}}{\partial p_{bh}} \\
\frac{\partial R_{wm}}{\partial p_{bh}}
\end{bmatrix}, \quad G = \begin{bmatrix}
\frac{\partial R_{bh}}{\partial p_{wf}} & \frac{\partial R_{bh}}{\partial s_{wf}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}}
\end{bmatrix}, \quad H = \begin{bmatrix}
\frac{\partial R_{bh}}{\partial p_{bh}} \\
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial s_{of}} \\
\frac{\partial R_{wm}}{\partial p_{wm}} & \frac{\partial R_{wm}}{\partial s_{wm}}
\end{bmatrix}
\]

\[
X_1 = \begin{bmatrix}
\delta p_{wf} \\
\delta s_{wf}
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
\delta p_{wm} \\
\delta s_{wm}
\end{bmatrix}, \quad R_f = \begin{bmatrix}
R_{wf} \\
R_{of}
\end{bmatrix}, \quad R_m = \begin{bmatrix}
R_{wm} \\
R_{om}
\end{bmatrix}, \quad R_{bh} = \begin{bmatrix}
R_{bh}
\end{bmatrix}
\]

The partitioned system can be expanded as follows:

\[
C_1 X_1 + C_2 X_2 + F \delta p_{bh} = - R_f \quad (6.2a)
\]

\[
C_3 X_1 + C_4 X_2 = - R_m \quad (6.2b)
\]

\[
G X_1 + H \delta p_{bh} = - R_{bh} \quad (6.2c)
\]

Solving equation (6.2c) for \( \delta p_{bh} \), we obtain:
\[ \delta p_{bh} = - \frac{R_{bh}}{H} - \frac{G}{H} X_1 \tag{6.2d} \]

Plugging equation (6.2d) into equation (6.2a) and rearranging, we get:

\[ \left( C_1 - \frac{F G}{H} \right) X_1 + C_2 X_2 = - R_f + \frac{F}{H} R_{bh} \tag{6.2e} \]

Solving equation (6.2b) for \( X_2 \), we obtain:

\[ X_2 = C_4^{-1} \left( - R_m - C_3 X_1 \right) \tag{6.2f} \]

Plugging equation (6.2f) into equation (6.2e) and rearranging, we get:

\[ \left( C_1 - \frac{F G}{H} - C_2 C_4^{-1} C_3 \right) X_1 = - R_f + \frac{F}{H} R_{bh} + C_2 C_4^{-1} R_m \tag{6.2g} \]

Equation (6.2g) represents a system of linear equations in the form \( A \mathbf{X}_f = \mathbf{B} \).

After evaluating \( \mathbf{X}_f \), \( \delta p_{bh} \) is evaluated using equation (6.2d), and \( X_2 \) is evaluated using equation (6.2f).

### 6.3 Solution of the SPM

For each node connected to a well, the system to be solved is a 3 x 3 block matrix in the form:

\[
\begin{bmatrix}
\frac{\partial R_{of}}{\partial p_{of}} & \frac{\partial R_{of}}{\partial p_{bf}} & \frac{\partial R_{of}}{\partial p_{bh}} \\
\frac{\partial S_{of}}{\partial p_{of}} & \frac{\partial S_{of}}{\partial p_{bf}} & \frac{\partial S_{of}}{\partial p_{bh}} \\
\frac{\partial R_{bf}}{\partial p_{bf}} & \frac{\partial R_{bf}}{\partial p_{bf}} & \frac{\partial R_{bf}}{\partial p_{bh}} \\
\end{bmatrix}
\begin{bmatrix}
\delta p_{of} \\
\delta S_{of} \\
\delta p_{bf} \\
\end{bmatrix} =
\begin{bmatrix}
R_{of} \\
R_{of} \\
R_{bf} \\
\end{bmatrix}
\tag{6.3}
\]

Following the same notation of the dual-porosity model, we could think of
equation (6.3) as being partitioned as:

\[
\begin{bmatrix}
C_1 & F \\
G & H
\end{bmatrix}
\begin{bmatrix}
X_1 \\
\delta p_{bh}
\end{bmatrix}
= \begin{bmatrix}
-R_f \\
-R_{bh}
\end{bmatrix}
\]

Where:

\[
C_1 = \begin{bmatrix}
\frac{\partial R_{wf}}{\partial p_{wf}} & \frac{\partial R_{wf}}{\partial S_{wf}} \\
\frac{\partial R_{of}}{\partial p_{wf}} & \frac{\partial R_{of}}{\partial S_{wf}}
\end{bmatrix},
\quad
F = \begin{bmatrix}
\frac{\partial R_{wf}}{\partial p_{bh}} \\
\frac{\partial R_{of}}{\partial p_{bh}}
\end{bmatrix},
\quad
G = \begin{bmatrix}
\frac{\partial R_{bh}}{\partial p_{wf}} & \frac{\partial R_{bh}}{\partial S_{wf}}
\end{bmatrix},
\quad
H = \begin{bmatrix}
\frac{\partial R_{bh}}{\partial p_{bh}}
\end{bmatrix}
\]

\[
X_1 = \begin{bmatrix}
\delta p_{wf} \\
\delta S_{wf}
\end{bmatrix},
\quad
R_f = \begin{bmatrix}
R_{wf} \\
R_{of}
\end{bmatrix}
\]

The partitioned system can be expanded as follows:

\[
C_1 X_1 + F \delta p_{bh} = -R_f \quad (6.3a)
\]

\[
G X_1 + H \delta p_{bh} = -R_{bh} \quad (6.3b)
\]

Solving equation (6.3b) for \( \delta p_{bh} \), we obtain:

\[
\delta p_{bh} = -\frac{R_{bh}}{H} - \frac{G}{H} X_1 \quad (6.3c)
\]

Plugging equation (6.3c) into equation (6.3a) and rearranging, we get:

\[
\left( C_1 - \frac{F G}{H} \right) X_1 = -R_f + \frac{F}{H} R_{bh} \quad (6.3d)
\]

Equation (6.3d) represents a system of linear equations in the form \( AX = B \).

After evaluating \( X_f \), \( \delta p_{bh} \) is evaluated using equation (6.3c).
6.4 Solution of the CTM for the FD Discretization

The only unknown in the discretized chemical transport equation is the concentration distribution, $C_{wf}$. Thus the equation can be rearranged as follows:

$$B1_{ijk}^{n+1} C_{wfj+1/2}^{n+1} + B2_{ijk}^{n+1} C_{wfj-1/2}^{n+1} + B3_{ijk}^{n+1} C_{wfi+1/2}^{n+1} + B4_{ijk}^{n+1} C_{wfi-1/2}^{n+1} +$$

$$B01_{ijk}^{n+1} C_{wfk+1/2}^{n+1} + B10_{ijk}^{n+1} C_{wfk-1/2}^{n+1} + B6_{ijk}^{n+1} C_{wfi+1/2}^{n+1} + B7_{ijk}^{n+1} C_{wfi-1/2}^{n+1} +$$

$$B8_{ijk}^{n+1} C_{wfj+1/2}^{n+1} + B9_{ijk}^{n+1} C_{wfj-1/2}^{n+1} + B5_{ijk}^{n+1} C_{wfk}^{n+1} = RHS_{ijk}^{n+1}$$

Where:

$$B1_{ijk}^{n+1} = + (T_x)^{n+1}_{wfi,j} \frac{1}{2} \left[ (P_{wfi,j+1}^{n+1} - P_{wfi,j}^{n+1}) - \gamma_{wfi,j}^{n+1} (D_{fi,j+1} - D_{fi,j}) \right]$$

$$B2_{ijk}^{n+1} = - (T_x)^{n+1}_{wfi,j} \frac{1}{2} \left[ (P_{wfi,j}^{n+1} - P_{wfi,j-1}^{n+1}) - \gamma_{wfi,j}^{n+1} (D_{fi,j} - D_{fi,j-1}) \right]$$

$$B3_{ijk}^{n+1} = + (T_x)^{n+1}_{wfi,j} \frac{1}{2} \left[ (P_{wfi,j+1}^{n+1} - P_{wfi,j}^{n+1}) - \gamma_{wfi,j+1/2}^{n+1} (D_{fi,j+1} - D_{fi,j}) \right]$$

$$B4_{ijk}^{n+1} = - (T_x)^{n+1}_{wfi,j} \frac{1}{2} \left[ (P_{wfi,j}^{n+1} - P_{wfi,j-1}^{n+1}) - \gamma_{wfi,j-1/2}^{n+1} (D_{fi,j} - D_{fi,j-1}) \right]$$

$$B01_{ijk}^{n+1} = + (T_z)^{n+1}_{wfk,j} \frac{1}{2} \left[ (P_{wfk,j+1}^{n+1} - P_{wfk,j}^{n+1}) - \gamma_{wfk,j+1}^{n+1} (D_{fj,k+1} - D_{fj,k}) \right]$$
\[ B_{10}^{n+1}_{ijk} = - (T_{c_{\text{wfj}}}^{n+1})_{jk} \left[ (P_{\text{wfj}}^{n+1} - P_{\text{wfj-1}}^{n+1}) - \frac{\gamma_{w_{\text{fj-1/2}}}^{n+1}}{2} (D_{f_{j-1}} - D_{f_{j-1}}) \right] \]

\[ B_{6}^{n+1}_{ijk} = + (T_{d_{\text{wfi-j}}}^{n+1})_{jk} \left[ (P_{\text{wfi-j+1}}^{n+1} - P_{\text{wfi-j}}^{n+1}) - \frac{\gamma_{w_{\text{fi-j-1/2}}}^{n+1}}{2} (D_{f_{j-1}+1} - D_{f_{j-1}}) \right] \]

\[ B_{7}^{n+1}_{ijk} = - (T_{d_{\text{wfi-j}}}^{n+1})_{jk} \left[ (P_{\text{wfi-j-1}}^{n+1} - P_{\text{wfi-j}}^{n+1}) - \frac{\gamma_{w_{\text{fi-j-1/2}}}^{n+1}}{2} (D_{f_{j-1}+1} - D_{f_{j-1}}) \right] \]

\[ B_{8}^{n+1}_{ijk} = + (T_{d_{\text{wfi-j}}}^{n+1})_{jk} \left[ (P_{\text{wfi-j-1}}^{n+1} - P_{\text{wfi-j}}^{n+1}) - \frac{\gamma_{w_{\text{fi-j-1/2}}}^{n+1}}{2} (D_{f_{j-1}+1} - D_{f_{j-1}}) \right] \]

\[ B_{9}^{n+1}_{ijk} = - (T_{d_{\text{wfi-j}}}^{n+1})_{jk} \left[ (P_{\text{wfi-j}}^{n+1} - P_{\text{wfi-j+1}}^{n+1}) - \frac{\gamma_{w_{\text{fi-j-1/2}}}^{n+1}}{2} (D_{f_{j-1}} - D_{f_{j-1}+1}) \right] \]

\[ B_{5}^{n+1}_{ijk} = (\tau_{\text{wmf}_{ij}}^{n+1})_{jk} + (q_{\text{wf}_{ij}}^{n+1})_{jk} - \frac{(V_{R})_{ijk}^{n+1}}{5.6146 \Delta t} \left( \frac{\phi S_{w}}{B_{w}} \right)_{f_{ij}} \]

\[ \text{RHS}^{n+1}_{ijk} = - \frac{(V_{R})_{ijk}^{n+1}}{5.6146 \Delta t} \left( \frac{\phi S_{w} C_{w}}{B_{w}} \right)_{f_{ij}} - (q_{\text{wf}_{ij}}^{n+1})_{ijk} \left( C_{w} \right)_{ijk}^{n+1} \]

Where \((q_{\text{wf}_{ij}}^{n+1})_{ijk}^{n+1}\) is the water production rate, \((q_{\text{wf}_{ij}}^{n+1})_{ijk}^{n+1}\) is the water injection rate and \((C_{w}^{n+1})_{ijk}^{n+1}\) is the injected concentration of the tracer. For concentration weighting, the upstream single-point weighting criterion is preferred for consistency purposes, although any other weighting criterion is acceptable.

Along this line, the above equation is extended to:
\[
B_{1}^{n+1}_{ijk} \left[ \omega_{wz,k+1} C_{wz,k+1}^{n+1} + (1 - \omega_{wz,k+1}) C_{wz,k+1}^{n+1} \right]
+ B_{2}^{n+1}_{ijk} \omega_{wz,j} C_{wz,j}^{n+1} + (1 - \omega_{wz,j}) C_{wz,j}^{n+1}
+ B_{3}^{n+1}_{ijk} \omega_{wy,j} C_{wy,j}^{n+1} + (1 - \omega_{wy,j}) C_{wy,j}^{n+1}
+ B_{4}^{n+1}_{ijk} \omega_{wy,j} C_{wy,j}^{n+1} + (1 - \omega_{wy,j}) C_{wy,j}^{n+1}
+ B_{6}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} C_{wd1,i+1,j+1}^{n+1} + (1 - \omega_{wd1,i+1,j+1}) C_{wd1,i+1,j+1}^{n+1}
+ B_{7}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} C_{wd2,i+1,j+1}^{n+1} + (1 - \omega_{wd2,i+1,j+1}) C_{wd2,i+1,j+1}^{n+1}
+ B_{8}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} C_{wd2,i+1,j+1}^{n+1} + (1 - \omega_{wd2,i+1,j+1}) C_{wd2,i+1,j+1}^{n+1}
+ B_{9}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} C_{wd2,i+1,j+1}^{n+1} + (1 - \omega_{wd2,i+1,j+1}) C_{wd2,i+1,j+1}^{n+1}
+ B_{10}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} C_{wd1,i+1,j+1}^{n+1} + (1 - \omega_{wd1,i+1,j+1}) C_{wd1,i+1,j+1}^{n+1}
+ B_{5}^{n+1}_{ijk} C_{wf,k}^{n+1} = RHS_{ijk}^{n+1}
\]

The above equation can be written as a linear system of equations as follows:

\[
B_{7}^{n+1}_{ijk} \omega_{wd2,i+1,j-1} C_{wz,i}^{n+1} + B_{2}^{n+1}_{ijk} \omega_{wz,i} C_{wz,i}^{n+1} + B_{9}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} C_{wz,k+1}^{n+1} +
\]

\[
B_{4}^{n+1}_{ijk} \omega_{wy,j} C_{wz,k+1}^{n+1} + B_{10}^{n+1}_{ijk} \omega_{wz,k} C_{wz,k+1}^{n+1} + B_{1}^{n+1}_{ijk} \omega_{wz,i+1} +
\]

\[
B_{2}^{n+1}_{ijk} (1 - \omega_{wz,i}) + B_{3}^{n+1}_{ijk} \omega_{wy,j+1} + B_{4}^{n+1}_{ijk} (1 - \omega_{wy,j}) + B_{6}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} +
\]

\[
B_{7}^{n+1}_{ijk} (1 - \omega_{wd2,i+1,j+1}) + (6.4)
\]

\[
B_{5}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{8}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{10}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} +
\]

\[
B_{1}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} + B_{2}^{n+1}_{ijk} (1 - \omega_{wd1,i+1,j+1}) + B_{3}^{n+1}_{ijk} (1 - \omega_{wy,j+1}) +
\]

\[
B_{4}^{n+1}_{ijk} (1 - \omega_{wy,j+1}) + B_{6}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} + B_{7}^{n+1}_{ijk} (1 - \omega_{wd2,i+1,j+1}) +
\]

\[
B_{5}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{8}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} + B_{9}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} +
\]

\[
B_{10}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} + B_{5}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{6}^{n+1}_{ijk} \omega_{wd1,i+1,j+1} +
\]

\[
B_{7}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{8}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} + B_{9}^{n+1}_{ijk} \omega_{wd2,i+1,j+1} +
\]

Equation (6.4) represents a system of linear equations which can be solved efficiently using any banded-system solver.
6.5 Solution of the CTM for the CVFE Discretization

The only unknown in the discretized chemical transport equation is the concentration distribution, $C_{wf}$. Thus the equation can be rearranged as follows:

$$B_{1j}^{n+1} C_{wf,j}^{n+1} + B_{2ik}^{n+1} C_{wf,k-\frac{1}{2}}^{n+1} + B_{3ik}^{n+1} C_{wf,k+\frac{1}{2}}^{n+1} + B_{4ik}^{n+1} C_{wf,ik}^{n+1} = RHS_{ik}^{n+1}$$

Where:

$$B_{1j}^{n+1} = (T)_{wf,j}^{n+1} \left[ (P_{wf,j}^{n+1} - P_{wf,i}^{n+1}) - y_{wf,ik}^{n+1} (D_{f,j} - D_{f,k}) \right]$$

$$B_{2ik}^{n+1} = (T)_{wf,k}^{n+1} \left[ (P_{wf,k}^{n+1} - P_{wf,k-1}^{n+1}) - y_{wf,ik}^{n+1} (D_{f,k} - D_{f,k-1}) \right]$$

$$B_{3ik}^{n+1} = -(T)_{wf,k}^{n+1} \left[ (P_{wf,k}^{n+1} - P_{wf,k-1}^{n+1}) - y_{wf,ik}^{n+1} (D_{f,k} - D_{f,k-1}) \right]$$

$$B_{4ik}^{n+1} = (\tau_{wmf})_{ik}^{n+1} + (q_{wf,ik,prod})^{n+1} - \frac{(V_R)_{ik}}{5.6146 \Delta t} \left( \frac{\phi S_w}{B_w} \right)_{f,ik}^{n+1}$$

$$RHS_{ik}^{n+1} = -\frac{(V_R)_{ik}}{5.6146 \Delta t} \left( \frac{\phi S_w C_w}{B_w} \right)_{f,ik}^{n+1} - (q_{wf,ik,inj})^{n+1} (C_{wf,ik})_{inj}$$

Where $(q_{wf,ik,prod})$ is the water production rate, $(q_{wf,ik,inj})$ is the water injection rate and $(C_{wf,ik,inj})$ is the injected concentration of the tracer. For concentration weighting, the upstream single-point weighting criterion is preferred for
consistency purposes, although any other weighting criterion is acceptable.

Along this line, the above equation is extended to:

\[ B_1^{n+1} \omega_{w,i} C_{w,i}^{n+1} + (1 - \omega_{w,i}) C_{w,f}^{n+1} \]
\[ + B_2^{n+1} \omega_{w,z,k+1} C_{w,f,k}^{n+1} + (1 - \omega_{w,z,k+1}) C_{w,f,k+1}^{n+1} \]
\[ + B_3^{n+1} \omega_{w,z,k} C_{w,f,k-1}^{n+1} + (1 - \omega_{w,z,k}) C_{w,f,k}^{n+1} + B_4^{n+1} C_{w,f,ik}^{n+1} = RHS_{ik}^{n+1} \]

Which is expanded as follows:

\[ B_1^{n+1} \omega_{w,i} C_{w,i}^{n+1} + B_1^{n+1} (1 - \omega_{w,i}) C_{w,f}^{n+1} + B_2^{n+1} \omega_{w,z,k+1} C_{w,f,k}^{n+1} \]
\[ + B_2^{n+1} (1 - \omega_{w,z,k+1}) C_{w,f,k+1}^{n+1} + B_3^{n+1} \omega_{w,z,k} C_{w,f,k-1}^{n+1} \]
\[ + B_3^{n+1} (1 - \omega_{w,z,k}) C_{w,f,k}^{n+1} + B_4^{n+1} C_{w,f,ik}^{n+1} = RHS_{ik}^{n+1} \]

Rearranging the above equation, we obtain:

\[ B_1^{n+1} (1 - \omega_{w,i}) C_{w,f}^{n+1} + B_2^{n+1} (1 - \omega_{w,z,k+1}) C_{w,f,k+1}^{n+1} \]
\[ + B_3^{n+1} \omega_{w,z,k} C_{w,f,k-1}^{n+1} + \left[ B_1^{n+1} \omega_{w,i} + B_2^{n+1} \omega_{w,z,k+1} \right] \]
\[ + B_3^{n+1} (1 - \omega_{w,z,k}) + B_4^{n+1} \right] C_{w,f,ik}^{n+1} = RHS_{ik}^{n+1} \quad (6.5) \]

Equation (6.5) represents a system of linear equations which can be solved using any linear solver.
6.6 Automatic Timestepping Technique

Due to the sensitivity of the selection of the timestep size, an automatic
timestepping technique is included in this research. The timestep selection is
set by calculating saturation and/or pressure changes to meet specified
tolerance criteria selected by the user.

Let: $DTMIN = \text{minimum timestep}$, $DTMAX = \text{maximum timestep}$,
$DSLIM = \text{maximum saturation change over a timestep}$, $DPLIM = \text{maximum}
pressure change over a timestep$, be the tolerance criteria selected by the user.
The following calculations are performed:

$$DSMAX = \max_i \{S_{w}^{n+1} - S_{w}^n\} \quad DPAX = \max_i \{P_{w}^{n+1} - P_{w}^n\}$$

$$DTS = DT \frac{DSLIM}{DSMAX} \quad DTP = DT \frac{DPLIM}{DPAX}$$

The new timestep is calculated as:

$$DT^* = \min\{DTS, DTP\}$$

and is required to satisfy:

$$DTMIN \leq DT^* \leq DTMAX$$

If $DSMAX$ is less than or equal $DSLIM$ and $DPAX$ is less than or
equal $DPLIM$ then proceed to the next timestep. Else if $DSMAX$ is greater
than $DSLIM$ or $DPAX$ is greater than $DPLIM$ then do the calculations of the
current timestep all over again using the new calculated timestep.
CHAPTER 7
MODELS VERIFICATION AND TESTING

Initially, this work's dual-porosity model, using both FD and CVFE discretization schemes, was verified and tested with published results. Two field scale examples, that were first presented by Kazemi et al., are used to verify both discretization schemes of the dual-porosity model. These include a quadrant of a five-spot pattern, and a five-well fractured reservoir having dip and natural water influx.

The relative permeability and capillary pressure curves, for the two examples, are shown in Fig. 7.1. It should be noted that the relative permeability to oil and water in the fracture covers the full spectrum of saturations from 0 to 1. The relative permeability in the matrix is restricted to the mobil saturation range from $S_w = 0.25$ to $S_w = 0.7$. The matrix capillary pressure is generally much greater than the fracture capillary pressure. The fracture capillary pressure declines rapidly with increased water saturation. This sets up a pressure differential from the matrix to the fracture causing oil to flow from matrix to fracture and water to imbibe into the matrix blocks. The capillary pressure endpoints, however, in both the fracture and the matrix are the same, so the transition zones are identical in the fracture and the matrix. This is necessary, otherwise a static equilibrium for pressure and saturation distribution is not possible.
Fig. 7.1: Relative permeability and capillary pressure data
7.1 A Quadrant of a Five-Spot Pattern

In this model problem, water is injected into one quarter of a developed conceptual five-spot pattern at a rate of 200 STB/D. Total liquid production rate is set at 210 STB/D. The reservoir is assumed to be fractured uniformly and was modeled using a 2D grid. Both FD and CVFE grids are shown in Fig. 7.2. Reservoir properties, for this model problem, are given in table 7.1. If reservoir parameters, however, are different from those assumed in the table, then recoveries will be substantially different from the results reported here.

Timesteps were calculated automatically using a maximum fracture saturation change of 0.01. Timestep size was restricted to a minimum of 0.01 day and a maximum of 10 days.

A comparison of water/oil ratio from this work with that calculated by Kazemi et al. and Dutra and Aziz is shown in Fig. 7.3. Both FD and CVFE models show WOR values intermediate between Kazemi and Dutra models at early times. After almost three years, both models tend to follow the Dutra model.

Additional runs using maximum timestep sizes of 20, 30, 40, 50, and 60 days were made to test the stability of the models. Both FD and CVFE discretization schemes gave essentially identical results as shown by Fig. 7.4 for WOR's and by Fig. 7.5 for fracture pressure at the production well. As can be seen from Fig. 7.5, the fracture pressure at the production well has declined to 2400 psi after almost 3 years of continuous production.
(a) The C8x8 FD grid (600x600 ft, $\Delta x = \Delta y = 75$ ft)

(b) The V8x8 CVFE mesh (600x600 ft, 136 elements)

Fig. 7.2: The C8x8 FD grid and the V8x8 CVFE mesh
Table 7.1: Reservoir Properties For the Five-Spot Pattern

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir dip angle, $\theta$, $^\circ$</td>
<td>0</td>
</tr>
<tr>
<td>Drainage area, $A$, acres</td>
<td>8.265</td>
</tr>
<tr>
<td>Reservoir grid</td>
<td>8x8x1</td>
</tr>
<tr>
<td>Grid size, ft</td>
<td>$\Delta x = \Delta y = 75$</td>
</tr>
<tr>
<td>Reservoir thickness, $h$, ft.</td>
<td>30</td>
</tr>
<tr>
<td>Matrix block size, ft.</td>
<td>10x10x30</td>
</tr>
<tr>
<td>Matrix blocks shape factors, $c$, ft$^2$</td>
<td>0.08</td>
</tr>
<tr>
<td>Fracture porosity, $\phi_f$</td>
<td>0.01</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td>0.19</td>
</tr>
<tr>
<td>Fracture compressibility, $c_p$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Matrix compressibility, $c_m$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Fracture permeability, $k_p$, md</td>
<td>10000</td>
</tr>
<tr>
<td>Matrix permeability, $k_m$, md</td>
<td>1</td>
</tr>
<tr>
<td>Fracture connate water saturation, $S_{wcf}$</td>
<td>0</td>
</tr>
<tr>
<td>Fracture residual oil saturation, $S_{orf}$</td>
<td>0</td>
</tr>
<tr>
<td>Matrix connate water saturation, $S_{wcm}$</td>
<td>0.25</td>
</tr>
<tr>
<td>Matrix residual oil saturation, $S_{orm}$</td>
<td>0.30</td>
</tr>
<tr>
<td>Pressure at water/oil contact, $P_{woc}$, psi</td>
<td>3959.89</td>
</tr>
<tr>
<td>Water formation volume factor, $B_w$, RB/STB</td>
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</tr>
<tr>
<td>Oil formation volume factor, $B_o$, RB/STB</td>
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</tr>
<tr>
<td>Water density, $\rho_w$, lb/ft$^3$</td>
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<td>Oil density, $\rho_o$, lb/ft$^3$</td>
<td>52</td>
</tr>
<tr>
<td>Water compressibility, $c_w$, psi$^{-1}$</td>
<td>$3.03 \times 10^{-6}$</td>
</tr>
<tr>
<td>Oil compressibility, $c_o$, psi$^{-1}$</td>
<td>$1.50 \times 10^{-5}$</td>
</tr>
<tr>
<td>Water viscosity, $u_w$, cp</td>
<td>0.5</td>
</tr>
<tr>
<td>Oil viscosity, $u_o$, cp</td>
<td>2</td>
</tr>
<tr>
<td>Water injection rate, $q_{w, inj}$, STB/D</td>
<td>200</td>
</tr>
<tr>
<td>Total production rate, $q_{l, pro}$, STB/D</td>
<td>210</td>
</tr>
</tbody>
</table>
Fig. 7.3: Comparison between calculated water/oil ratios
Fig. 7.4: Comparison of water/oil ratios for different timestep sizes
Fig. 7.5: Fracture pressure at the production well vs. time
7.2 A Five-Well Fractured Reservoir Model

A conceptual five-well fractured reservoir model which is tilted along the x-direction at an angle of 5° is presented. The model has 13 nodes in the x-direction, 5-nodes in the y-direction, and 1-node in the z-direction. Fig. 7.6 shows the FD and the CVFE grids that were used for this model. The reservoir nodes are 500 ft on each side and the reservoir thickness is 90 ft. Each reservoir node is further subdivided into 25 matrix blocks- 100 ft on each side in the x and y directions. The reservoir properties for this example are given in table 7.2.

A total influx of 2,800 STB/D of water was assigned at the five nodes along the lowest part of the reservoir as shown in Fig. 7.6. Five production wells are arranged along the center-line in the x-direction as depicted by Fig. 7.6. Well 1 was initially producing at a total liquid rate of 1000 STB/D. It was shut in after 190 days when it went to a high WOR and well 4 was opened at the same rate. Well 2 was never produced because of its high water saturation. Wells 3 and 5 were producing at a total liquid rate of 1000 STB/D.

Timesteps were calculated automatically using a maximum fracture saturation change of 0.01. Timestep size was restricted to a minimum of 0.01 day and a maximum of 10 days. A comparison of water/oil ratio from this work with that calculated by Kazemi et al., for the four producing wells, is shown in Fig. 7.7. Both FD and CVFE models give identical results and they agree with the results reported by Kazemi et al.
(a) The C13x5 FD grid (6500x2500 ft, $\Delta x = \Delta y = 500$ ft)

(b) The V13x6 CVFE mesh (6500x2500 ft, 162 elements)

Fig. 7.6: The C13x5 FD grid and the V13x6 CVFE mesh
Table 7.2: Reservoir Properties For the Five-Well Reservoir

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir dip angle, $\theta$, $^\circ$</td>
<td>5</td>
</tr>
<tr>
<td>Drainage area, $A$, acres</td>
<td>373.049</td>
</tr>
<tr>
<td>Reservoir grid</td>
<td>13x5x1</td>
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<tr>
<td>Grid size, ft.</td>
<td>$\Delta x = \Delta y = 500$</td>
</tr>
<tr>
<td>Reservoir thickness, $h$, ft.</td>
<td>90</td>
</tr>
<tr>
<td>Matrix block size, ft.</td>
<td>100x100x90</td>
</tr>
<tr>
<td>Matrix blocks shape factors, $\sigma$, ft$^{-2}$</td>
<td>0.0008</td>
</tr>
<tr>
<td>Fracture porosity, $\phi_f$</td>
<td>0.01</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td>0.20</td>
</tr>
<tr>
<td>Fracture compressibility, $c_f$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Matrix compressibility, $c_m$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Fracture permeability, $k_f$, md</td>
<td>21000</td>
</tr>
<tr>
<td>Matrix permeability, $k_m$, md</td>
<td>1</td>
</tr>
<tr>
<td>Fracture connate water saturation, $S_{wcf}$</td>
<td>0</td>
</tr>
<tr>
<td>Fracture residual oil saturation, $S_{ocrf}$</td>
<td>0</td>
</tr>
<tr>
<td>Matrix connate water saturation, $S_{wcm}$</td>
<td>0.25</td>
</tr>
<tr>
<td>Matrix residual oil saturation, $S_{orm}$</td>
<td>0.30</td>
</tr>
<tr>
<td>Pressure at water/oil contact, $P_{woc}$, psi</td>
<td>3983.75</td>
</tr>
<tr>
<td>Water formation volume factor, $B_w$, RB/STB</td>
<td>1</td>
</tr>
<tr>
<td>Oil formation volume factor, $B_o$, RB/STB</td>
<td>1</td>
</tr>
<tr>
<td>Water density, $\rho_w$, lb/ft$^3$</td>
<td>63.14</td>
</tr>
<tr>
<td>Oil density, $\rho_o$, lb/ft$^3$</td>
<td>52</td>
</tr>
<tr>
<td>Water compressibility, $c_w$, psi$^{-1}$</td>
<td>$3.03 \times 10^{-6}$</td>
</tr>
<tr>
<td>Oil compressibility, $c_o$, psi$^{-1}$</td>
<td>$1.50 \times 10^{-5}$</td>
</tr>
<tr>
<td>Water viscosity, $\mu_w$, cp</td>
<td>0.5</td>
</tr>
<tr>
<td>Oil viscosity, $\mu_o$, cp</td>
<td>2</td>
</tr>
</tbody>
</table>
Fig. 7.7: Comparison between calculated water/oil ratios
7.3 Verification of the Single-Porosity Model

The reservoir properties for the five-spot example, Fig. 7.1 and table 7.1, are used to test the validity of the single-porosity model. This model requires a prior knowledge of the fitting variables appearing in Aronofsky’s matrix/fracture transfer function. The fitting variables include: the maximum recoverable oil from a single matrix block, \( R_m \), and the exponential fitting parameter, \( \lambda \).

The expanded form of Aronofsky’s model could have been used instead, but it has been found inefficient for the following reasons:

1. Aronofsky’s expanded model involves a system of 2xNFit nonlinear equations. These equations need more elaborate strategies for their solution. One group of methods, called homotopy methods, is described in a survey article by Watson (1987), can be used for the solution of such a system.

2. Fitting a single block (SBS) oil recovery curve is not enough for matching the DPM performance. Thus these parameters need adjustment with time; i.e. they need to be adjusted via history matching the DPM.

3. The process of history matching is one of the more time-consuming aspects of petroleum reservoir simulation. Therefore, when using the expanded model, there will be more parameters to vary with time rather than only one parameter in the simple Aronofsky’s model which makes history matching much simpler and straightforward.

The initial estimates of the fitting variables are obtained by running a single block simulator (SBS). The SBS is performed by taking one matrix
block having the dimensions of, say, 10x10x10 ft. This block is fine gridded into 10x10x1 cells and is surrounded by a fracture of 0.01 ft width to make a system of 12x12x1 cells as shown in Fig. 7.8. The whole system is, then, treated as a conventional reservoir by assigning the reservoir matrix properties to the matrix blocks and the reservoir fracture properties to the fracture surrounding the blocks. The equivalent injection rates for the SBS are calculated as follows:

\[ q_{SBS} = \frac{q_T}{NMB} \frac{V_{SBS}}{V_{MB}} \]

Where:

- \( q_{SBS} \) is the SBS injection rate, STB/D
- \( q_T \) is the total injection rate, STB/D
- \( NMB \) is the number of matrix blocks used in the model grid
- \( V_{SBS} \) is the volume of the matrix block used in the SBS, ft\(^3\)
- \( V_{MB} \) is the volume of the matrix block used in the model problem, ft\(^3\)

On the other hand, the production well is set at an appropriate pressure control well boundary condition; i.e. if the initial pressure distribution is set at 30 psi, then the bottom-hole pressure at the production well is set at 15-20 psi. Timesteps were calculated manually using a minimum timestep size of 0.01 day and a maximum timestep size of 50 days. Fig. 7.9 shows the oil recovery behavior of a single matrix block. Selecting two points on the SBS curve, plugging into Aronofsky's model, and solving for \( R_w \), and \( \lambda \) to give
Fig. 7.8: A schematic representation of a single matrix block that has been fine gridded for use in an SBS
Fig. 7.9: Oil recovery behavior for a single matrix block (SBS)
1.00 for $R_w$ and 0.0015083074886 for $\lambda$. As a quality check these values were used to regenerate the SBS oil recovery curve at different times and plotted on the same figure as can be seen by the fitted model curve. The DPM, using these values as a first guess, was used to generate $\lambda$ values at different times, as shown by Fig. 7.10. The SPM is then run using the same $R_w$ and the $\lambda$ values that were generated by the DPM. Finite difference approximation to the Jacobian guarantees an excellent match with the DPM. Cubic spline interpolation can be used to get the value of $\lambda$ as a function of time.

Fig. 7.11 compares water/oil ratio calculated by the SPM with that calculated by Kazemi's model. Also a comparison of the movable oil pore volume water injected versus the movable oil pore volume oil produced for the DPM and the SPM models is shown in Fig. 7.12. The agreement is excellent and the SPM looks a robust formulation for the naturally fractured reservoirs. Its execution time is almost twice faster than the DPM, since it requires only the fracture system, whereas the DPM involves both matrix and fracture systems. The major drawback of this model is the need to history-match it with a pre-existing dual-porosity model.
Fig. 7.10: $\lambda(t)$ values generated by the DPM for use in the SPM
Fig. 7.11: Comparison of the water/oil ratio (WOR) calculated by the SPM model and by Kazemi's model.
Fig. 7.12: Comparison of the movable oil pore volume water injected vs. the movable oil pore volume oil produced for the dual- and the single-porosity models
CHAPTER 8
GRID ORIENTATION EFFECT

In a five-spot pattern, the Cartesian grid lines can be aligned with the line connecting a producer-injector pair (parallel grid) or they can be diagonal to the connection (diagonal grid), see Fig. 8.1.

Fig. 8.1: Examples of (a) parallel and (b) diagonal grid orientations.
When the mobility of the displacing fluid is greater than the mobility of the resident fluid (adverse or unfavorable mobility ratio displacement), a physical instability of the displacement front occurs. There exists a competitive situation between viscous fingers that tend to grow, versus diffusive effects that try to smear out any otherwise sharp fronts. In other words, adverse or unfavorable mobility ratio displacements are unstable physical processes.

Of course the physical processes are independent of the grid geometry, however, the numerical results depend on it. The finite difference solutions of the two-dimensional frontal displacement problems can be strongly influenced by the orientation of the finite difference grid. For example, in the solution of adverse mobility ratio displacement problems by the five-point finite difference schemes, fluids appear to flow along grid lines. This results in solutions that depend on the orientation and size of the grid. Hence there are severe differences in the numerical solutions on the two different grids (parallel and diagonal). These differences do not vanish when finer grids are used.

For grid block sizes typically used in reservoir simulation, numerical diffusion is large enough to completely damp out all viscous fingers. Instead, nonphysical "numerical" fingers develop with a much coarser scale related directly to the grid size. When the grid is refined, the solutions still depend on the size and orientation of the underlying grid, as long as numerical diffusion dominates over physical dispersion and diffusion. This interaction
between anisotropic numerical diffusion and the physical instability of the displacement front is known as the grid orientation effect (GOE).

For simulations of displacements at mobility ratios that are favorable, neutral, or slightly unfavorable, the grid orientation effect can be reduced by refining the grid. On the other hand, for adverse or unfavorable mobility ratio displacements, the grid orientation effect becomes more pronounced. In this case, as the grid is refined, the performance of the parallel and diagonal models diverges. In general, as long as numerical diffusion dominates over physical diffusion, the results will depend on the grid size and will not converge under grid refinement.

The influence of various parameters on the grid orientation effect has been reported. Mobility ratio, grid size, discretization scheme, and front shape are amongst those parameters. On the assumption that numerical diffusion can be controllable, apparently, the only solution to remove the grid orientation effect is the use of extremely fine grids that can capture small scale dispersive forces (physical dispersion) and heterogeneities. However, this may require too many grid blocks which is impractical in many situations.

To reduce the GOE, there are some possibilities to be considered: (1) Making numerical dispersion more isotropic; i.e. the use of the nine-point finite difference scheme or the use of hexagonal grids, (2) In Cartesian grids, there are preferred flow paths following the coordinate axes of symmetry. Thus, in the use of triangular meshes, there will be no preferred direction in the grid
system and so the GOE will be reduced, (3) Reducing the numerical dispersion by reducing the grid size. However, it was observed that GOE may become more severe on finer grids in some situations, and (4) Other ways of reducing numerical dispersion are the application of higher-order schemes such as the finite-element base simulators.

From a practical point of view, however, the focus must be on grid geometries and discretization schemes that at least reduce the GOE without making the blocks very small. Several discretization schemes have been proposed to reduce the GOE. The CVFE method scheme is one such attempt.
8.1 Pertinent Data

The example problem considered in this chapter represents a quadrant of a developed conceptual five-spot pattern. The relative permeability and capillary pressure curves are shown in Figs. 8.2 and 8.3 respectively. It should be noted that the relative permeability to oil and water in the fracture covers the full spectrum of saturations from 0 to 1 (Fig. 8.2), whereas the capillary pressure in the fracture declines rapidly with increased water saturation (Fig. 8.3).

Other pertinent data are given in table 8.1. If reservoir parameters, however, are different from those assumed in the table, then recoveries will be substantially different from the results reported here.

**Fig. 8.2:** Relative perm. curve

**Fig. 8.3:** Cap. pressure curve
<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir dip angle, $\theta$, °</td>
<td>0</td>
</tr>
<tr>
<td>Drainage area, $A$, acres</td>
<td>8.265</td>
</tr>
<tr>
<td>Reservoir grid</td>
<td>8x8x1</td>
</tr>
<tr>
<td>Grid size, ft.</td>
<td>$\Delta x = \Delta y = 75$</td>
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<tr>
<td>Reservoir thickness, $h$, ft.</td>
<td>30</td>
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<tr>
<td>Matrix block size, ft.</td>
<td>10x10x30</td>
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<td>Matrix blocks shape factors, $\sigma$, ft$^2$</td>
<td>0.08</td>
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<td>Fracture porosity, $\phi_f$</td>
<td>0.01</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td>0.19</td>
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<td>Fracture compressibility, $c_p$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Matrix compressibility, $c_m$, psi$^{-1}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Fracture permeability, $k_f$, md</td>
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</tr>
<tr>
<td>Matrix permeability, $k_m$, md</td>
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<tr>
<td>Fracture connate water saturation, $S_{wcf}$</td>
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</tr>
<tr>
<td>Fracture residual oil saturation, $S_{orf}$</td>
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<td>Matrix connate water saturation, $S_{wcm}$</td>
<td>0.25</td>
</tr>
<tr>
<td>Matrix residual oil saturation, $S_{orm}$</td>
<td>0.30</td>
</tr>
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<td>Pressure at water/oil contact, $P_{woc}$, psi</td>
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</tr>
<tr>
<td>Water formation volume factor, $B_w$, RB/STB</td>
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</tr>
<tr>
<td>Oil formation volume factor, $B_o$, RB/STB</td>
<td>1</td>
</tr>
<tr>
<td>Water density, $\rho_w$, lb/ft$^3$</td>
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</tr>
<tr>
<td>Oil density, $\rho_o$, lb/ft$^3$</td>
<td>52</td>
</tr>
<tr>
<td>Water compressibility, $c_w$, psi$^{-1}$</td>
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<td>Water viscosity, $u_w$, cp</td>
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<tr>
<td>Second fitting parameter, $\lambda_2$</td>
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</tr>
<tr>
<td>Minimum time step, $\Delta t_{min}$, day</td>
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<tr>
<td>Maximum time step, $\Delta t_{max}$, day</td>
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8.2 Comparison With Finite Difference Schemes

In simulations with the nine-point finite difference formulation, the performances of diagonal and parallel models tend to converge as the grid is refined. The nine-point finite difference formulation allows flow between a gridblock and all eight surrounding blocks, including those diagonally adjacent. It is possibly the most reliable current solution to the grid orientation problem. However, the increased reliability is obtained for a cost. The CVFE method is an alternative attempt to, hopefully, reduce the grid orientation effect at a lower cost.

Initially, numerical results obtained with the CVFE method were compared with the results obtained with the finite difference scheme. This example problem was solved with the five-point, nine-point, and the CVFE methods for both diagonal and parallel grids. Different adverse mobility ratios of 1, 10, and 50 were used. Finite difference grids of C8x8 for diagonal well and C12x12 for parallel well, see Fig. 8.4, were used. The CVFE meshes, V8x8 for diagonal well and V12x12 for parallel well, see Fig. 8.5, were used for this problem. The predicted recovery performances are shown in Figs. 8.6 and 8.7.

Fig. 8.6 shows that the agreement is excellent between numerical results obtained with the CVFE scheme and with the nine-point finite difference method for all mobility ratios (M) that were investigated for both diagonal and parallel wells. Additionally Fig. 8.6 clearly shows that the 5-point FD predicts considerably higher oil recovery for the diagonal well than for the parallel well
(a) The C8x8 FD grid (600x600 ft, diagonal well)

(b) The C12x12 FD grid (850x850 ft, parallel well)

Fig. 8.4: The C8x8 and C12x12 FD grids
Fig. 8.5: The V8x8 and V12x12 CVFE meshes
for all mobility ratios. Moreover, the parallel well shows less sensitivity to the
grid orientation problem than the diagonal well. For both diagonal and
parallel wells, the predicted recovery tends to converge as the mobility ratio
increases.

Fig. 8.7a depicts the predicted recovery behavior with the 9-point FD
method for both diagonal and parallel wells. As it can be seen from the figure,
for this problem, there is a big difference between the behavior of both grids.
The difference is greater for unit mobility ratio and gets smaller as the
mobility ratio gets bigger. On the other hand, Fig. 8.7b shows the recovery
performance predicted with the CVFE method for both wells. Although the
behavior of the diagonal well is different than the parallel well, the difference
is smaller than that of the FD method.

In conclusion, for this example problem, the CVFE method is as efficient
as the nine-point finite difference scheme to reduce the grid orientation effect
in displacements with unfavorable mobility ratios when approximately the
same number of gridblocks was used. As a matter of fact, for naturally
fractured reservoirs, the CVFE method seems more efficient and realistic than
the 9-point FD method for very high mobility ratios. Still, for naturally
fractured reservoirs, the 9-point FD and the CVFE methods could not
completely eliminate the GOE.
Fig. 8.6: Comparison between results obtained with the CVFE method and with the 5-point and 9-point FD schemes
Fig. 8.7: Comparison between results obtained with the diagonal and parallel grids for both FD and CVFE methods.
8.3 Effect of Grid Size

This section analyzes the effect of grid size on the numerical results obtained with the CVFE method. Two example problems were considered with different triangulation schemes.

Example 8.3.1

In this example problem, three different grid sizes were investigated: (1) a coarse grid with 136 elements, V8x8 (Fig. 8.5a), (2) an intermediate grid with 528 elements, V16x16 (Fig. 8.8), and a fine grid with 2080 elements, V32x32 (Fig. 8.9). Results of the simulation of different mobility ratios of 1, 10, and 50 for the three grid sizes are summarized in Fig. 8.10.

Fig. 8.10 shows that the results, for the mobility ratio of 1, are somewhat sensitive to the size of the grid. In other words, the results do not seem to be converging as the grid is refined; i.e., for favorable mobility ratio displacements, the numerical dispersion is dominating over the physical dispersion. On the other hand, for adverse mobility ratios of 10 and 50, the grid size has negligible effect on the numerical results obtained.
Fig. 8.8: The V16x16 CVFE mesh (600x600 ft, 528 elements)
Fig. 8.9: The V32x32 CVFE mesh (600x600 ft, 2080 elements)
Fig. 8.10: Comparison between the results obtained with the V8x8, V16x16, and V32x32 CVFE meshes
Example 8.3.2

In this example, the effect of grid size as well as the effect of grid refinement near the production well is analyzed using three different grid sizes: (1) a coarse grid with 128 elements, GT1 (Fig. 8.11), (2) an intermediate grid with 512 elements, GT2 (Fig. 8.12), and a fine grid with 2048 elements, GT3 (Fig. 8.13). Results of the simulation of different mobility ratios of 1, 10, and 50 for the three grid sizes are summarized in Fig. 8.14.

The simulation results for this example are identical to the previous one in the way that the results for favorable mobility ratio displacement are somewhat sensitive to the size of the grid. On the other hand, for adverse mobility ratios of 10 and 50, the grid size has almost negligible effect on the numerical results obtained.

In conclusion, the previous two examples show that the model grid, for a specific problem, should be selected properly to minimize the numerical dispersion in a most efficient way. In other words, a stability analysis of the model equations should be performed to predict the most suitable grid size for which the error remains small and negligible.
Fig. 8.11: The GT1 CVFE mesh (600x600 ft, 128 elements)
Fig. 8.12: The GT2 CVFE mesh (600x600 ft, 512 elements)
Fig. 8.13: The GT3 CVFE mesh (600x600 ft, 2048 elements)
Fig. 8.14: Comparison between the results obtained with the GT1, GT2, and GT3 control volume finite element meshes.
8.4 Use of Different Triangulation Schemes

The CVFE mesh is constructed by subdividing the reservoir into subregions or macroblocks (see APPENDIX B for details). These subregions are triangulated separately and then gathered to form the complete triangular domain. The subregions are chosen in such a way that there are no obtuse angles to assure positive transmissibility constraints.

The effect of different triangulation schemes is analyzed in this section. Three examples are considered. In each example, results of two CVFE meshes are compared to results of the base mesh (V8x8 CVFE mesh). The main goal is to check the effect of the triangulation scheme on the simulation results. This is accomplished through varying the orientation of the nodal control volumes of the model grid as shown by figures 8.15, 8.16, 8.18, and 8.19.

Example 8.4.1

Two different meshes are used: (1) The V16x8 CVFE mesh in which the nodal control volumes are oriented vertically (Fig. 8.15). (2) The V8x16 CVFE mesh in which the nodal control volumes are oriented horizontally (Fig. 8.16). The numerical results obtained from these two meshes are compared with the CVFE base mesh (V8x8 CVFE mesh). A comparison of the numerical results obtained from these three meshes is shown in Fig. 8.17. The agreement is excellent which means that the orientation of the nodal control volumes has almost no effect on the simulation results. Yet, the next example considers the diagonal nodal orientation.
Fig. 8.15: The V16x8 CVFE mesh (600x600 ft, 264 elements)
Fig. 8.16: The V8x16 CVFE mesh (600x600 ft, 272 elements)
**Fig. 8.17:** Comparison between the results obtained with the V16x8 and V8x16 control volume finite element meshes
Example 8.4.2

In this example, two additional CVFE meshes of diagonal nodal control volume orientations are used: (1) The R16x16A CVFE mesh in which the nodal control volumes are right-diagonally oriented (Fig. 8.18). (2) The R16x16B CVFE mesh in which the nodal control volumes are left-diagonally oriented (Fig. 8.19). The numerical results obtained from these two meshes are compared with the CVFE base mesh (V8x8 CVFE mesh). A comparison of the numerical results obtained from these three meshes is shown in Fig. 8.20. As before, the agreement is excellent and the orientation of the nodal control volumes seems to have no effect at all.

Example 8.4.3

The last selected CVFE mesh is shown in Fig. 8.21. The numerical results for this mesh, compared with the V8x8 and the GT1 CVFE meshes, are shown in Fig. 8.22. Good agreement is obtained for all three different meshes especially for the unfavorable mobility ratios of 10 and 50.

In conclusion, used this way, the CVFE-based simulators are very powerful tools for simulating very complex reservoir boundaries without having a serious grid orientation effect provided that the reservoir subregions are chosen in such a way that there are no obtuse angles.
Fig. 8.19: The R16x16B CVFE mesh (600x600 ft, 512 elements)
Fig. 8.20: Comparison between the results obtained with the R16x16A and R16x16B control volume finite element meshes.
Fig. 8.21: The GT4 CVFE mesh (600x600 ft, 560 elements)
Fig. 8.22: Comparison between the results obtained with the V8x8, GT1, and GT4 control volume finite element meshes
8.5 Discussion

Results presented in this chapter show that the use of the control volume finite element (CVFE) method (especially the close to equilateral grid) is at least as effective as the nine-point finite difference scheme for the problems that were investigated. The main advantage of the CVFE gridding technique is the ability to represent complex reservoir geometries and to enhance grid resolution near wellbores in a robust, accurate, and easy to use manner. In addition, the reservoir boundaries are treated in a much easier manner than the corresponding FD method.

Nevertheless, the complete elimination of the grid orientation effect is still not possible for gridblocks of usual size. The lack of convergence as the grid is refined represents a very serious problem, which is present in all known discretization schemes. Therefore, this is still an important branch open for research. The use of independent maps to specify physical properties in connection with CVFE discretization scheme may be an important tool to simulate a given description of the reservoir (permeability distribution) with different grid geometries and sizes. Hopefully, it will be possible to identify through future research the conditions (heterogeneity and grid geometry) under which the numerical results converge when the grid is refined.

A linear stability analysis of the model equations can be used to predict the grid size for which the error remains small and limited by the local truncation error.
CHAPTER 9

TREATMENT OF HETEROGENEITIES

Reservoir performance is mainly influenced by reservoir heterogeneity. Permeability variations, however, appear to be the most influential factor in affecting reservoir performance. The accuracy of petroleum reservoir simulation can be substantially improved by assigning a permeability value at each grid node of the reservoir model grid. However, from a practical point of view, we are never lucky enough to have exhaustive sampling of the area of interest. Instead, we have samples of only a tiny fraction of the total area. Our task is to infer permeability values of the whole reservoir from this limited sample information.

Geostatistical methods address this problem by using statistical techniques to create an image of reservoir permeability distribution based on the information observed at some locations (wells). They offer a collection of deterministic and statistical tools aimed at understanding and modeling the spatial variability of a certain phenomenon.

9.1 Review of Geostatistics: Geostatistical Estimation

Geostatistical estimation is concerned with the use of the available sample information to predict values in areas we have not sampled. There are several different estimation methods each of which is applicable to a certain
type of estimation problem. Therefore, it is very important to understand which methods are applicable to which types of problems. All estimation methods, however, involve weighted linear combinations which can be expressed as follows:

\[ \hat{y} = \sum_{i=1}^{n} w_i \cdot v_i \]

\( v_1, \ldots, v_n \) are the \( n \) available data values and \( w_i \) is a weight assigned to the value \( v_i \). These weights are usually standardized so that they sum up to one. Different approaches to assigning the weights to the data values give rise to different methodologies.

A geostatistical estimation requires a model of how the phenomenon of interest behaves at locations where it has not been sampled. Without a model, one has only the sample data and very little inferences can be made about the unknown values at locations that were not sampled. For example, consider having a sample data set of permeability values at some locations. By itself, this sample data set tells us virtually nothing about the permeability distribution over the entire area. All we know from our sample is the permeability values at some particular locations. Thus, estimation of the values at the unknown locations demands that we bring in additional information or make some assumptions.

In certain situations, the process that generated the sample data set might be sufficiently known so that the entire distribution of the phenomenon
can be made from only a few sample values. In such situations a deterministic model is appropriate. In situations where there is uncertainty about how the phenomenon behaves, the random function models (probabilistic models) give us tools for estimating values at the unknown locations once we have made some assumptions about the statistical characteristics of the phenomenon.

9.1.1 Probabilistic Models

In a probabilistic model, the available sample data are viewed as a result of some random process. Therefore we are responsible for defining the random process that might have created the available sample values. Having done so, we could generate many realizations, each one would be a possible reality that is consistent with the available data and with the random function model we have chosen.

9.1.2 Variogram Analysis

The variogram model is our selected interpretation of the spatial correlation of the sample data set. It controls the way that the weights of the estimation method (e.g. kriging or stochastic simulation, etc.) are assigned to the samples during the interpolation process, and consequently controls the quality of the results. In other words, the variogram analysis attempts to quantify this relationship: How well can a measurement be expected to represent another location a specified distance and direction away?
With the limited data we have, we compute the variances for groups of pairs of measurements in class intervals of similar distance and direction. We then plot a graph of the variances versus distance for a particular direction, and fit a model curve (variogram) to the graph; the model is assumed to be an approximation of the true variogram.

**Mean:** The mean, \( m \), is the arithmetic average of some data points; i.e.:

\[
m = \frac{1}{n} \sum_{i=1}^{n} v_i
\]

where \( n \) is the number of data points, and \( v_i \) are the values of data points.

**Variance:** The variance, \( \sigma^2 \), is the average squared difference of the observed values from their mean; i.e.:

\[
\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (v_i - m)^2
\]

**Standard Deviation:** The standard deviation, \( \sigma \), is simply the square root of the variance. It is often used instead of the variance.

**Coefficient of Variation:** The coefficient of variation, \( CV \), is a statistic that is often used to describe the shape of the distribution (skewness) of the sample data points, and is given by:
The Covariance Function: The covariance function, $C(h)$, is calculated from the following:

\[
C(h) = \frac{1}{N(h)} \sum_{i,j|h_i - h_j} \left\{ v_i v_j - m_{-h} m_{+h} \right\}
\]

where the summation is only over the $N(h)$ pairs of data whose locations are separated by $h$. $m_{-h}$ is the mean of all data values whose locations are $-h$ away from some other data locations; i.e.:

\[
m_{-h} = \frac{1}{N(h)} \sum_{i|h_i - h_j} v_i
\]

$m_{+h}$ is the mean of all data values whose locations are $+h$ away from some other data locations; i.e.:

\[
m_{+h} = \frac{1}{N(h)} \sum_{j|h_i - h_j} v_j
\]

The Variogram: The variogram, $\gamma(h)$, is half the average squared difference between the paired data values; i.e.:

\[
\gamma(h) = \frac{1}{2N(h)} \sum_{i,j|h_i - h_j} (v_i - v_j)^2
\]
If we choose to sum over the $N(h)$ pairs whose separation is *approximately* $h$, then the variogram equation is written as:

$$\gamma(h) = \frac{1}{2N(h)} \sum_{(i,j)|h_{ij}=h} (v_i - v_j)^2$$

The most common approach for choosing the tolerance on $h$ for the selection of data pairs is the one shown in Fig. 9.1. Any sample falling within the shaded area would be paired with the sample at $(x,y)$.

**Fig. 9.1:** An illustration of the distance and angular tolerances on $h$ for the selection of data points that would be paired with the sample at $(x,y)$
As the separation distance between pairs increases, the corresponding variogram value will increase. Eventually, however, an increase in the separation distance no longer causes a corresponding increase in the average squared difference between pairs of values and the variogram reaches a plateau. The plateau is called the sill. The distance at which the variogram reaches this plateau is called the range. The nugget effect is the vertical jump from the value of 0 at the origin to the value of the variogram at extremely small separation distances. The ratio of the nugget effect to the sill is called the relative nugget effect.

**Kriging:** Kriging uses the variogram technique to represent the spatial variability of a certain phenomenon along a specific direction \( \alpha \). Statistically, this function is defined as half of the expected value of the difference between a pair of points which are separated in space by a distance \( h \); i.e.:

\[
\gamma(h) = \frac{1}{2} E \left[ (v_i - v_j)^2 \right]
\]

In terms of the variogram, the kriging is given by:

\[
\sum_{j=1}^{n} w_j \tilde{v}_j - \mu = \tilde{v}_{i0} \quad \forall \ i = 1, 2, \ldots, n
\]

With kriging and various methods for generating random fields, conditioning techniques were developed. The most commonly used is the residual conditioning, which combines the residual and the kriged data.
9.1.3 Generation of Random Fields

The concept of a random field is nothing more than a mathematical abstraction which is used to describe a complex natural process, such as a heterogeneous permeability distribution. Random fields are commonly characterized by their statistical properties (e.g. mean, variance, and covariance model). The ability to simulate synthetic random fields of permeability distribution provides a means of assessing the influence and uncertainty of the permeability spatial variations on reservoir performance. The effect of heterogeneity on reservoir performance can be studied via the simulation of different heterogeneous patterns of fracture permeability using different grid sizes. Tracer concentration distribution is a powerful tool for the characterization of petroleum reservoirs. It might be used as a fast and cheap procedure to analyze and identify reservoir heterogeneities.

Public domain software\textsuperscript{28,32,99}, for performing geostatistical analyses are available and can be obtained with their user's manual. In this study, TUBA, a computer code for generating two-dimensional random fields via the Turning Bands method, was used for the generation of random fields of permeability distributions.

9.2 Simulation of a Heterogeneous Five-Spot Pattern

The conceptual five-spot pattern of section 7.1 will be used in this example problem to study the effect of fracture permeability heterogeneity on
reservoir performance. Hypothetical fracture permeability fields were generated with TUBA using three different covariance models: the exponential model, the Gaussian model, and the Bessel model. For all configurations, the fracture permeability mean was held at a value close to the homogeneous fracture permeability; i.e. 10,000 md. Permeability values, however, vary from about 50 md to almost 35,000 md.

Example 1: Data Set #1 (Two-Dimensional Case)

In this example, three fracture permeability fields, as shown by Figs. 9.2, 9.3, and 9.4, were used. The same description of a heterogeneous reservoir was simulated with two CVFE meshes: the V8x8, Fig. 8.5a, and the V16x16, Fig. 8.8. The other reservoir properties remained the same.

Fig. 9.5 compares the bottom-hole pressure for the homogeneous and the three heterogeneous reservoirs for: (a) the V8x8 CVFE mesh, and (b) the V16x16 CVFE mesh. Fig. 9.6 compares the WOR of the homogeneous reservoir with the three heterogeneous reservoirs. Fig. 9.7 compares the oil recovery of the homogeneous with the heterogeneous reservoirs for both meshes. As it can be seen, for all situations, the answers are almost identical; i.e. heterogeneities of the fracture permeability do not affect the reservoir performance (at least for this example). In other words, the results produced by the two meshes are similar, with the grid refinement having only a slight effect on the numerical solutions.
Fig. 9.2: Gray scale map for permeability distribution (exponential covariance model, mean = 10,400 md, variance = 0.1127E+8)
Fig. 9.3: Gray scale map for permeability distribution (Gaussian covariance model, mean = 10,030 md, variance = 0.1246E+8)
Fig. 9.4: Gray scale map for permeability distribution (Bessel covariance model, mean = 9.336 md, variance = 0.1475E+8)
Fig. 9.5: Comparison of the bottom-hole pressure for the homogeneous and heterogeneous reservoirs (example 1)
Fig. 9.6: Comparison of the WOR for the homogeneous and heterogeneous reservoirs (example 1)
Fig. 9.7: Comparison of oil recovery for the homogeneous and heterogeneous reservoirs (example 1)
A slug of tracer was injected after 300 days of continuous production for almost 4 months. The tracer concentration at the effluent is shown in Fig. 9.8 for: (a) the V8x8 CVFE mesh, and (b) the V16x16 CVFE mesh. It is clear that the fracture permeability heterogeneity has some effect on the tracer recovery. The variations, however, are bigger for the finer mesh, the V16x16 mesh, which might imply that a very fine grid is required to simulate this slug of tracer accurately. This is because the gridblocks should be small enough to capture the width of the slug, which changes as the tracer front moves along the pattern.
Fig. 9.8: Tracer concentration at the effluent for the homogeneous and heterogeneous reservoirs (example 1)
Example 2: Data Set #1 (Three-Dimensional Case)

In this case, the reservoir was vertically divided into three layers. Fracture permeability distributions as shown by Figs. 9.2, 9.3, and 9.4, were assigned to the first, second, and third layers respectively. The V8x8 CVFE mesh (Fig. 8.5a) is used for this problem. Due to limitations of the available computer facilities, runs were made only for 600 days.

Fig. 9.9 compares the bottom-hole pressure of the two-dimensional homogeneous with the three-dimensional homogeneous and heterogeneous patterns. From this figure, it is clear that layering enhances reservoir heterogeneity and thus the effect on the reservoir bottom-hole pressure is remarkable. Fig. 9.10 compares the WOR of the two-dimensional homogeneous reservoir with the three-dimensional homogeneous and heterogeneous reservoirs. Fig. 9.11 compares the oil recovery of the two-dimensional homogeneous reservoir with the three-dimensional homogeneous and heterogeneous reservoirs. Variations, however, are minimal; i.e. the WOR or the amount of oil recovery are not the best tools for identifying heterogeneity effects.

The tracer concentration at the effluent, for the homogeneous and the heterogeneous cases is shown in Fig. 9.12. It is obvious that the three-dimensional front is less smeared than the two-dimensional situation. As expected, the vertical subdivision of the heterogeneous reservoir into layers is a better tool for performance predictions.
Fig. 9.9: Comparison of the bottom-hole pressure for the two-dimensional homogeneous and the three-dimensional homogeneous and heterogeneous reservoirs (example 2)
Fig. 9.10: Comparison of WOR for the two-dimensional homogeneous and the three-dimensional homogeneous and heterogeneous reservoirs (example 2)
Fig. 9.11: Comparison of oil recovery for the two-dimensional homogeneous and the three-dimensional homogeneous and heterogeneous reservoirs (example 2)
Fig. 9.12: Comparison of tracer concentration at the effluent for the two-dimensional homogeneous and the three-dimensional homogeneous and heterogeneous reservoirs (example 2)
Example 3: Data Set #2 (Two-Dimensional Case)

In this example, three more heterogeneous fracture permeability fields are considered as shown by Figs. 9.13, 9.14, and 9.15. The V32x32 CVFE mesh (Fig. 8.9.) was used to model this example problem.

Fig. 9.16 compares the bottom-hole pressure of the homogeneous reservoir with the three heterogeneous reservoirs. Fig. 9.17 compares the WOR of the homogeneous reservoir with the three heterogeneous reservoirs. Fig. 9.18 compares the oil recovery performance of the homogeneous and the heterogeneous reservoirs. As it can be seen, for all situations, the results are almost identical; i.e. heterogeneities of the fracture permeability did not affect the reservoir performance even for the very fine grid (at least for this example problem).

The tracer in this example was injected after 300 days of continuous production for only 30 days instead of 4 months. The tracer concentration at the effluent is shown in Fig. 9.19. It appears that the fracture permeability heterogeneity has a stronger effect on the tracer recovery at the effluent for the finer grid.

In general, more studies should be performed to investigate the best tools for studying the effect of fracture permeability heterogeneity on the reservoir performance.
Fig. 9.13: Gray scale map for permeability distribution (exponential covariance model, mean = 9,840 md, variance = 0.8434E+8)
Fig. 9.14: Gray scale map for permeability distribution (Gaussian covariance model, mean = 10,079 md, variance = 0.7834E+8)
Fig. 9.15: Gray scale map for permeability distribution (Bessel covariance model, mean = 10.405 md, variance = 0.9402E+8)
Fig. 9.16: Comparison of the bottom-hole pressure for the homogeneous and heterogeneous reservoirs (example 3)
Fig. 9.17: Comparison of the WOR for the homogeneous and heterogeneous reservoirs (example 3)
Fig. 9.18: Comparison of oil recovery for the homogeneous and heterogeneous reservoirs (example 3)
Fig. 9.19: Tracer concentration at the effluent for the homogeneous and heterogeneous reservoirs (example 3)
CHAPTER 10
CONCLUSIONS AND RECOMMENDATIONS

The numerical results obtained in this work, for both FD and CVFE discretization schemes, were tested and verified with published data. The test problems included a developed conceptual five-spot pattern and a five-well fractured reservoir that were first presented by Kazemi.

The numerical results demonstrate that the CVFE method is at least as accurate as the nine-point finite difference scheme with the added advantage of more flexibility in simulating complex reservoir geometries and enhancing the grid resolution near wellbores.

In this research, however, emphasis was not only on developing algorithms just to solve old problems, but also we investigated other problems previously encountered in simulating naturally fractured reservoirs.

10.1 Conclusions

(1) The CVFE scheme allows the simulation of complex reservoir geometries in the areal plane more accurately, whereas the third dimension is modeled by vertical projection of the two-dimensional grid. Furthermore, a complete three-dimensional finite element formulation does not seem to be necessary.

(2) The single-porosity formulation is more practical and cost-effective
compared to the dual-porosity formulation in the sense that it requires less storage and less computational time. However, Aronofsky's matrix/fracture transfer function for the single-porosity formulation requires more elaborate research to make it a stand-alone formulation instead of the need to history match it with an existing dual-porosity model.

(3) Node numbering of the CVFE mesh has a significant influence on the structure of the resulting Jacobian matrix. Thus, it is important to find a numbering scheme as close to optimum as possible in the sense that it minimizes the band width.

(4) The CVFE's Jacobian matrix is very sparse compared to the banded matrix of the FD method. Therefore, an efficient sparse matrix solver must be used for storage and computational efficiency.

(5) The CVFE method essentially gave identical results to the nine-point FD scheme which is the most reliable solution for reducing the GOE.

(6) Extensive examples were presented to prove the flexibility of the CVFE method for any triangulation scheme provided the triangles are not obtuse.

(7) The CVFE formulation of fluid flow in porous media, where the principal permeability tensor components are in the direction of coordinate axes, is practical. However, in situations where non-tensor permeability channelling does not coincide with the coordinate axes, it can still be accommodated in the CVFE grid.
(8) From the examples we presented for the conceptual fractured reservoirs, which are either homogeneous or homogeneously heterogeneous, mesh refinement beyond a certain point does not affect the numerical results.

(9) Fracture permeability heterogeneity, as presented in our conceptual examples, is not as sensitive as permeability channelling. This was demonstrated in the modeling of a field example\textsuperscript{52}.

(10) A quarter of a five-spot model, as used in this thesis and other publications, is not a proper testing ground for assessing the effect of heterogeneity, because this model inherently assumes symmetry of a full five-spot pattern. A better model would be a single five-spot pattern consisting of an injector and four surrounding producers, and still a better model would be a cluster of repeated five-spot patterns.

(11) The tracer injection is an excellent method for characterizing reservoir channeling as a result of permeability heterogeneity.

10.2 Recommendations

(1) Other forms of the water-oil single-porosity matrix/fracture transfer function should be further studied to better match the dual-porosity model results.

(2) Adaptive remeshing and refinement, where a completely new mesh is created based on the results of the first mesh, is an idea worth exploring.

(3) To improve the representation of wells, a hybrid grid, where a local
cylindrical or elliptical grid is used in the well regions and the CVFE method is applied elsewhere in the remaining reservoir, is worth pursuing.

(4) Quadratic or higher-order triangular elements should be studied in the CVFE method to investigate the benefit of such higher order schemes on the accuracy of results.

(5) The two-phase water-oil formulation should be extended to the three-phase situations.
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<th>Symbol</th>
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<td>A</td>
<td>area, ft$^2$</td>
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<tr>
<td>$D$</td>
<td>depth, ft</td>
</tr>
<tr>
<td>$D$</td>
<td>dispersion coefficient, ft$^2$/day</td>
</tr>
<tr>
<td>DPM</td>
<td>dual-porosity model</td>
</tr>
<tr>
<td>$FD$</td>
<td>finite difference</td>
</tr>
<tr>
<td>FEM</td>
<td>finite element method</td>
</tr>
<tr>
<td>GOE</td>
<td>grid orientation effect</td>
</tr>
<tr>
<td>$h$</td>
<td>thickness, ft</td>
</tr>
<tr>
<td>$H$</td>
<td>Heaviside step function</td>
</tr>
<tr>
<td>$J$</td>
<td>Jacobian matrix</td>
</tr>
<tr>
<td>$k$</td>
<td>absolute permeability, md</td>
</tr>
<tr>
<td>$k_r$</td>
<td>relative permeability, fraction</td>
</tr>
<tr>
<td>$k_{xx}$</td>
<td>xx-component of the permeability tensor, md</td>
</tr>
<tr>
<td>$k_{xy}$</td>
<td>yy-component of the permeability tensor, md</td>
</tr>
<tr>
<td>$k_{xz}$</td>
<td>zz-component of the permeability tensor, md</td>
</tr>
<tr>
<td>$m$</td>
<td>mass, lb</td>
</tr>
<tr>
<td>$M$</td>
<td>mobility ratio</td>
</tr>
<tr>
<td>$nn$</td>
<td>capillary pressure negative power exponent</td>
</tr>
<tr>
<td>$np$</td>
<td>capillary pressure positive power exponent</td>
</tr>
<tr>
<td>$N$</td>
<td>column vector of the shape functions</td>
</tr>
<tr>
<td>NFIT</td>
<td>number of fitting parameters for the SPM</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure, psi</td>
</tr>
<tr>
<td>$P_c$</td>
<td>capillary pressure, psi</td>
</tr>
<tr>
<td>$PV$</td>
<td>pore volume, bbl</td>
</tr>
<tr>
<td>$q$</td>
<td>volume flow rate, bbl/day</td>
</tr>
<tr>
<td>$Q_T$</td>
<td>wellbore total flow rate, bbl/day</td>
</tr>
<tr>
<td>$r$</td>
<td>radial distance, ft</td>
</tr>
<tr>
<td>$r_{ei}$</td>
<td>equivalent wellblock radius, ft</td>
</tr>
<tr>
<td>$r_{ij}$</td>
<td>distance between nodes i and j, ft</td>
</tr>
<tr>
<td>$r_w$</td>
<td>wellbore radius, ft</td>
</tr>
</tbody>
</table>
R = recovery of a single matrix block at a time t, bbl
R = residual of Newton-Raphson scheme
R\infty = maximum recoverable oil from a single matrix block, bbl
s = skin factor
S = saturation, fraction
SPM = single-porosity model
t = time, days
T = transmissibility coefficient, bbl cp/(day psi)
V = volume, ft\(^3\)
V_B = bulk volume, ft\(^3\)
V_G = grain volume, ft\(^3\)
V_P = pore volume, ft\(^3\)
V_R = block volume, ft\(^3\)
w_{ij} = width between nodes i and j, ft
WI = well index, bbl cp/(day psi)
WOC = water/oil contact
WOP = oil-phase well productivity, STB/(day psi)
WOR = water/oil ratio
WP = well productivity = (WWP+WOP), STB/(day psi)
WWP = water-phase well productivity, STB/(day psi)
x = column vector of unknown quantities
x^r = column vector of nodal x-coordinates of some region
y^r = column vector of nodal y-coordinates of some region

GREEK LETTERS

\(\alpha\) = phase (water or oil)
\(\beta\) = well code (0 for production and 1 for injection)
\(\Gamma^n\) = boundary line of an element
\(\gamma\) = fluid phase gradient, psi/ft
\(\gamma_i\) = entry in the global geometric stiffness matrix
\(\delta\) = geometric factor function defined as follows:
\(\delta = \Delta\) for 1D, \(\delta = \Delta z\) for 2D, \(\delta = 1\) for 3D
\(\eta_i\) = set of all surrounding nodes to node i
\(\Delta\) = finite difference operator
\(\Delta_t\) = finite difference operator in time
\(\Delta_t\) = time-step length, day
\(\Delta_x\) = finite difference operator in x-direction
\(\Delta x\) = grid dimension in x-direction, ft
\(\Delta_y\) = finite difference operator in y-direction
\[ \Delta y = \text{grid dimension in y-direction, ft} \]
\[ \Delta z = \text{finite difference operator in z-direction} \]
\[ \Delta \zeta = \text{grid dimension in z-direction, ft} \]
\[ \lambda = \text{exponential fitting parameter for the SPM} \]
\[ u = \text{viscosity, cp} \]
\[ v = \text{velocity, bbl/ft}^2 \]
\[ \xi = \text{direction (x, y, or z)} \]
\[ \rho = \text{density, lb/ft}^3 \]
\[ \sigma = \text{matrix block's shape factor, ft}^2 \]
\[ \tau = \text{volumetric matrix/fracture transfer, bbl/day} \]
\[ \phi = \text{porosity, fraction} \]
\[ \omega = \text{upstream weighting factor} \]
\[ \Omega_s = \text{domain of an element} \]

**MATHEMATICAL SYMBOLS**

\[ \Sigma = \text{sum} \]
\[ \infty = \text{maximum recoverable} \]
\[ \rightarrow = \text{vector quantity} \]
\[ j = \text{integral} \]
\[ \Pi = \text{product} \]
\[ \nabla = \text{gradient operator} \]
\[ \nabla_\cdot = \text{divergence operator} \]
\[ \nabla^2 = \text{Laplacian} \]
\[ \partial = \text{partial derivative} \]
\[ \partial / \partial t = \text{partial derivative with respect to } t \]
\[ \partial / \partial x = \text{partial derivative with respect to } x \]
\[ \partial / \partial y = \text{partial derivative with respect to } y \]
\[ \partial / \partial z = \text{partial derivative with respect to } z \]
\[ \partial / \partial \xi = \text{partial derivative with respect to } \xi \]

**SUBSCRIPTS**

\[ \text{aq} = \text{aquifer} \]
\[ \text{b} = \text{base} \]
\[ \text{b} = \text{boundary} \]
\[ \text{bh} = \text{bottom hole} \]
c = capillary
c = connate
CV = control volume
d1 = diagonal1-direction
d2 = diagonal2-direction
f = fracture
i = reference gridblock in x-direction
ij = connection between nodes i and j
ik = connection between node i and layer k
inj = injected
j = reference gridblock in y-direction
k = reference gridblock in z-direction
m = matrix
max = maximum
min = minimum
o = oil
or = residual oil
orf = fracture residual oil
orm = matrix residual oil
pro = produced
r = relative
r = residual
R = reservoir
sc = standard conditions
t = total
w = water
wcf = fracture connate water
wcm = matrix connate water
x = x-direction
y = y-direction
z = z-direction

SUPERSCRIPTS

e = element
n = old time level
n+1 = new time level
^ = quantity per unit reservoir volume
/ = first derivative
REFERENCES


Orientation Effect in Reservoir Simulation," Paper SPE 21228 was Prepared for Presentation at the 11th SPE Symposium on Reservoir Simulation, Anaheim, CA (Feb. 17-20, 1991).


[54] Kazemi, H.: "Notes for Reservoir Simulation II," PE-614,


APPENDICES
APPENDIX A

RESERVOIR DATA

In making a reservoir simulation study, the input data must be handled to a greater extent of care and accuracy. The quality of the output is no better than the quality of the input.

The data required to make a simulation study come from several sources and are accessible to a greater or lesser degree. Good judgement should be exercised in differentiating and selecting the best data available for a particular case. The data are usually in a form not directly applicable to a computer solution, and some preprocessing must be undertaken to produce the data in a usable form.

The groups of reservoir data generally required in making a simulation run are as follows: rock data, fluid data, relative permeability data, capillary pressure data, and wells' production/injection history.

A.1 Rock Data

Knowledge of the physical characteristics of the reservoir rock is of vital importance in understanding the nature of a given reservoir. The reservoir rock properties include: porosity, formation compressibility, permeability, and fluid saturations. These properties constitute a set of parameters by means of which the reservoir rock can be quantitatively described.
A.1.1 Porosity

Porosity is a measure of the storage capacity of a rock. It is defined as the ratio of pore volume to bulk volume of the rock; i.e.

\[ \phi = \frac{V_p}{V_B} = \frac{V_p}{V_p + V_G} \]

A.1.2 Formation Compressibility

Formation compressibility or pore volume compressibility is defined as the change in pore volume as a function of pressure per unit pore volume; i.e.

\[ c_f = \left( \frac{1}{V_p} \frac{\partial V_p}{\partial p} \right) \]

In terms of porosity, the above equation can be written as:

\[ c_f = \left( \frac{1}{\phi} \frac{\partial \phi}{\partial p} \right) \]

Rearranging and integrating the above expression yields:

\[ \frac{\partial \phi}{\phi} = c_f \left( \frac{\partial p}{p_b} \right) \Rightarrow \int_{\phi_b}^{\phi} \frac{\partial \phi}{\phi} = \int_{p_b}^{p} c_f \left( \frac{\partial p}{p_b} \right) \]

Evaluating \( c_f \) at the average pressure between \( p_b \) and \( p \) gives:

\[ \ln \left( \frac{\phi}{\phi_b} \right) = c_f (p - p_b) \Rightarrow \phi = \phi_b \ e^{c_f (p - p_b)} \]

Therefore, the following expressions are written for porosity:
\[ \phi(p) = \phi_b e^{c_f(p - p_b)} \]  \hspace{1cm} (A.1)

\[ \frac{\partial \phi(p)}{\partial p} = c_f \phi(p) \]  \hspace{1cm} (A.2)

Porosity values as a function of pressure can be input to a simulator either as an analytical expression or as a table look-up.

(1) **Analytical Representation:** Equations (A.1) and (A.2) are used to generate \( \phi(p) \) and \( \frac{\partial \phi(p)}{\partial p} \) as a function of pressure \( p \).

(2) **Table Look-up:** In some cases, however, porosity data are not easily represented by analytical expressions. The solution to this problem is to use a "look-up table" as shown below:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \phi(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>0.26</td>
</tr>
<tr>
<td>2050</td>
<td>0.27</td>
</tr>
<tr>
<td>2100</td>
<td>0.28</td>
</tr>
<tr>
<td>2150</td>
<td>0.29</td>
</tr>
<tr>
<td>2200</td>
<td>0.30</td>
</tr>
<tr>
<td>2250</td>
<td>0.31</td>
</tr>
<tr>
<td>2300</td>
<td>0.32</td>
</tr>
<tr>
<td>2350</td>
<td>0.33</td>
</tr>
</tbody>
</table>
The dependent variables \( \phi(p) \) and \( \frac{\partial \phi(p)}{\partial p} \) can be evaluated at any pressure within the supplied range using cubic spline interpolation routines.

A.1.3 Permeability

Absolute permeability of a rock is a measure of the ease with which fluid can flow through it. The permeability data form the most voluminous portion of the input data required by the simulator. Every node in the model grid must be identified by a given set of permeability values.

An easy way to get an idea of the spatial variations of permeability from a fragmentary sampling is to draw a contour map by hand. As a limited amount of information is available, the data will have to undergo a certain amount of subjective interpretation in order to produce a reliable map. Because the degree of interpretation can vary within a single map, the precision of such a map cannot be specified.

Alternatively, geostatistical methods for describing reservoir heterogeneities can be used. These methods are focused on the frequency distribution approach where the probability behavior of the quantity variations is determined from a histogram of the available measured values. The ability to generate random fields for a highly variable quantity is important because it is very difficult to both measure and characterize the in situ distribution of these quantities. On the other hand, it may be of interest to observe the
response of a production well history to reservoir heterogeneity (as reflected by the variance of the permeability).

A.1.4 Fluid Saturations

Fluid saturation is defined as the ratio of the fluid volume to the pore volume of the rock. Well logs can be used to obtain fluid saturations. There are, in general, two ways of measuring original fluid saturations: the direct approach and the indirect approach. The direct approach involves either the extraction of the reservoir fluids or the leaching of the fluids from a sample of the reservoir rock. The indirect approach relies on a measurement of some other property, such as capillary pressure, and the derivation of a mathematical relationship between the measured property and saturation.

A.2 Fluid Data

An accurate description of the physical properties of petroleum reservoir fluids is of considerable importance in simulating petroleum reservoirs. Fluid properties of primary interest include: fluid density, fluid formation volume factor, fluid compressibility, and fluid viscosity.

Data on most of these fluid properties is usually determined by laboratory experiments performed on samples of the actual reservoir fluids. In the absence of the experimentally measured properties, it is necessary to determine the properties from empirically derived correlations.
A.2.1 Fluid Density

Fluid density is defined as the mass of a unit fluid volume at a specified pressure and temperature; i.e.

$$\rho_a = \frac{m_a}{V_a} \quad \Rightarrow \quad V_a = \frac{m_a}{\rho_a} \quad (A.3)$$

A.2.2 Fluid Formation Volume Factor

Fluid formation volume factor is defined as the ratio of fluid volume at reservoir conditions to the fluid volume at standard conditions; i.e.

$$B_a = \frac{V_a}{(V_a)_{sc}} \quad \Rightarrow \quad V_a = B_a \cdot (V_a)_{sc} \quad (A.4)$$

A.2.3 Fluid Compressibility

Fluid compressibility is defined as the rate of change in fluid volume as a function of pressure per unit fluid volume; i.e.

$$c_a = -\frac{1}{V_a} \left( \frac{\partial V_a}{\partial p} \right)_T \quad (A.5)$$

Substitution of equation (A.3) into equation (A.5) yields:

$$c_a = -\frac{\rho_a}{m_a} \frac{\partial}{\partial p} \left( \frac{m_a}{\rho_a} \right) = -\frac{\rho_a}{m_a} \frac{\partial}{\partial p} \left( \frac{1}{\rho_a} \right) = -\rho_a \left( -\frac{\partial \rho_a/\partial p}{\rho_a^2} \right) = \frac{1}{\rho_a} \frac{\partial \rho_a}{\partial p}$$
Rearranging and integrating the above expression yields:

\[ \frac{\partial \rho_a}{\rho_a} = c_a \partial p \Rightarrow \int_{\rho_a}^{\rho} \frac{\partial \rho_a}{\rho_a} = \int_{\rho_b}^{\rho} c_a \partial p \]

Evaluating \( c_a \) at the average pressure between \( p_b \) and \( p \) gives:

\[ \rho_a(p) = \rho_{ab} e^{c_a (p - p_b)} \]

\[ \frac{\partial \rho_a(p)}{\partial p} = c_a \rho_a(p) \]

Substitution of equation (A.4) into equation (A.5) yields:

\[ c_a = -\frac{1}{B_a} \frac{\partial B_a}{\partial p} \]

Rearranging and integrating the above expression yields

\[ \frac{\partial B_a}{B_a} = -c_a \partial p \Rightarrow \int_{B_{ab}}^{B_a} \frac{\partial B_a}{B_a} = -\int_{p_b}^{p} c_a \partial p \]

Evaluating \( c_a \) at the average pressure between \( p_b \) and \( p \) gives

\[ B_a(p) = B_{ab} e^{-c_a (p - p_b)} \quad \text{(A.6)} \]

\[ \frac{\partial B_a(p)}{\partial p} = -c_a B_a(p) \quad \text{(A.7)} \]

The fluid formation volume factor can be input to the simulator either as an analytical representation or a table look-up.
(1) Analytical Representation: Equations (A.6) and (A.7) are used to generate $B_\alpha(p)$ and $\frac{\partial B_\alpha(p)}{\partial p}$ as a function of pressure $p$.

(2) Table Look-up: Some PVT data are not easily represented by analytical expressions. The solution to this problem is to use a "look-up table" as shown below:

<table>
<thead>
<tr>
<th></th>
<th>$B_\alpha(p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>1.26</td>
</tr>
<tr>
<td>2050</td>
<td>1.27</td>
</tr>
<tr>
<td>2100</td>
<td>1.28</td>
</tr>
<tr>
<td>2150</td>
<td>1.29</td>
</tr>
<tr>
<td>2200</td>
<td>1.30</td>
</tr>
<tr>
<td>2250</td>
<td>1.31</td>
</tr>
<tr>
<td>2300</td>
<td>1.32</td>
</tr>
<tr>
<td>2350</td>
<td>1.33</td>
</tr>
</tbody>
</table>

The dependent variables $B_\alpha(p)$ and $\frac{\partial B_\alpha(p)}{\partial p}$ can be evaluated at any pressure within the supplied range using cubic spline interpolation routines.
A.2.4 Fluid Viscosity

An external shear stress applied to a portion of a fluid will introduce a movement of the molecules of the affected part of the fluid in the direction of the applied shear stress. The moving molecules will interact with the neighboring molecules. These will start moving too, but with a lower velocity than that of the molecules exposed to the stress. The dynamic viscosity of a Newtonian fluid is defined as the following ratio for the flowing fluid:

\[ \eta = - \frac{\tau_{xy}}{\left( \partial \nu_x / \partial y \right)} \]

Several correlations have been suggested for estimating the viscosity of hydrocarbon mixtures. Constant viscosity, however, is assumed in this work.

A.3 Relative Permeability Data

Relative permeability is often one of the more difficult pieces of data to evaluate or obtain. The relative permeability can be input to the simulator either as an analytical expression or as a table look-up.

1. Analytical Representation: The analytical correlation to be used for evaluating relative permeability to water is the one reported by Honarpour for sandstone and conglomerate/water-wet which is written as:
\[ k_{rw}(S_w) = 0.035388 \frac{(S_w - S_{w1})}{(1 - S_{wi} - S_{or})} - 0.010874 \left( \frac{S_w - S_{or}}{1 - S_{wi} - S_{or}} \right)^{2.9} + 0.56556 (S_w)^{3.6} (S_w - S_{wi}) \]

From which we write:

\[
\frac{\partial k_{rw}(S_w)}{\partial S_w} = \frac{0.035388}{(1 - S_{w1} - S_{or})} \left[ 1 - 0.89110998 \left( \frac{S_w - S_{or}}{1 - S_{wi} - S_{or}} \right)^{1.9} \right] + 0.56556 (S_w)^{2.6} \left[ S_w + 3.6 (S_w - S_{wi}) \right]
\]

For relative permeability to oil, we will use the one for sandstone and conglomerate/any wettability which is written as:

\[ k_{ro}(S_w) = 0.76067 \left( \frac{(1 - S_{w1})}{1 - S_{or}} - S_{or} \right)^{1.8} \left( \frac{1 - S_w - S_{or}}{1 - S_{wi} - S_{or}} \right)^{2.0} + 2.6318 \phi (1 - S_{or}) (1 - S_w - S_{or}) \]

From which we write:
\[
\frac{\partial k_{ro}(S_w)}{\partial S_w} = -1.52134 \left[ \frac{\left(1 - S_w\right) - S_{or}}{1 - S_{or}} \right]^{1.8} \left[ \frac{1 - S_w - S_{or}}{1 - S_{wi} - S_{or}} \right] \left[ \frac{1}{1 - S_{wi} - S_{or}} \right] \\
- 1.369206 \left[ \frac{1 - S_w - S_{or}}{1 - S_{wi} - S_{or}} \right]^{2.0} \left[ \frac{\left(1 - S_w\right) - S_{or}}{1 - S_{or}} \right]^{0.8} \left[ \frac{1}{1 - S_{wi}} \right] \\
- 2.6318 \phi (1 - S_{or})
\]

(2) Table Look-up: Given a table saturation and relative perm. data. The dependent variables \( k_{nw}(S_w) \), \( \frac{\partial k_{nw}(S_w)}{\partial S_w} \), \( k_{ro}(S_w) \), and \( \frac{\partial k_{ro}(S_w)}{\partial S_w} \), can be evaluated at any saturation within the supplied range using cubic spline interpolation routines.

<table>
<thead>
<tr>
<th>( S_w )</th>
<th>( k_{nw}(S_w) )</th>
<th>( k_{ro}(S_w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.10</td>
<td>0.050</td>
<td>0.770</td>
</tr>
<tr>
<td>0.20</td>
<td>0.110</td>
<td>0.587</td>
</tr>
<tr>
<td>0.30</td>
<td>0.180</td>
<td>0.450</td>
</tr>
<tr>
<td>0.50</td>
<td>0.355</td>
<td>0.240</td>
</tr>
<tr>
<td>0.60</td>
<td>0.475</td>
<td>0.173</td>
</tr>
<tr>
<td>0.70</td>
<td>0.585</td>
<td>0.102</td>
</tr>
<tr>
<td>0.80</td>
<td>0.715</td>
<td>0.057</td>
</tr>
<tr>
<td>0.90</td>
<td>0.850</td>
<td>0.021</td>
</tr>
<tr>
<td>1.00</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>
A.4 Capillary Pressure Data

Capillary pressure data are needed to evaluate pressures of other phases. The capillary pressure data can be input to the simulator either as an analytical expression or as a table look-up.

(1) Analytical Representation: The imbibition capillary pressure is expressed by a power equation and has a range extended between positive and negative values as follows:

\[
P_c(S_w) = P_{c_{\text{max}}} \left( \frac{S_{w_{/Pc0}} - S_w}{S_{w_{/Pc0}} - S_{w_{/i}}} \right)^{np} \quad S_w \leq S_{w_{/Pc0}}
\]

\[
P_c(S_w) = P_{c_{\text{min}}} \left( \frac{S_w - S_{w_{/Pc0}}}{1 - S_{or} - S_{w_{/Pc0}}} \right)^{nn} \quad S_w > S_{w_{/Pc0}}
\]

From which we obtain:

\[
\frac{\partial P_c}{\partial S_w} = -(np) \left( P_{c_{\text{max}}} \right) \left( \frac{S_{w_{/Pc0}} - S_w}{S_{w_{/Pc0}} - S_{w_{/i}}} \right)^{np-1} \left( \frac{1}{S_{w_{/Pc0}} - S_{w_{/i}}} \right) \quad S_w \leq S_{w_{/Pc0}}
\]

\[
\frac{\partial P_c}{\partial S_w} = (nn) \left( P_{c_{\text{min}}} \right) \left( \frac{S_w - S_{w_{/Pc0}}}{1 - S_{or} - S_{w_{/Pc0}}} \right)^{nn-1} \left( \frac{1}{1 - S_{or} - S_{w_{/Pc0}}} \right) \quad S_w > S_{w_{/Pc0}}
\]
Where:

\( P_{\text{emax}} \) is the value of capillary pressure at \( S_{\text{wi}} \), positive.

\( P_{\text{emin}} \) is the value of capillary pressure at \( 1 - S_{\text{or}} \), negative.

\( S_{\text{wi}/P_{\text{eo}}} \) is the value of water saturation at zero capillary pressure.

\( np \) is the positive power exponent.

\( nn \) is the negative power exponent.

(2) Table Look-up: Given a table of saturation and capillary pressure data.

The dependent variables \( P_c(S_w) \), and \( \frac{\partial P_c(S_w)}{\partial S_w} \) can be evaluated at any saturation within the supplied range using cubic spline interpolation routines.

<table>
<thead>
<tr>
<th>( S_w )</th>
<th>( P_c(S_w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>4.000</td>
</tr>
<tr>
<td>0.10</td>
<td>1.850</td>
</tr>
<tr>
<td>0.20</td>
<td>0.900</td>
</tr>
<tr>
<td>0.30</td>
<td>0.550</td>
</tr>
<tr>
<td>0.50</td>
<td>0.290</td>
</tr>
<tr>
<td>0.60</td>
<td>0.200</td>
</tr>
<tr>
<td>0.70</td>
<td>0.160</td>
</tr>
<tr>
<td>0.80</td>
<td>0.110</td>
</tr>
<tr>
<td>0.90</td>
<td>0.050</td>
</tr>
<tr>
<td>1.00</td>
<td>0.000</td>
</tr>
</tbody>
</table>
APPENDIX B

TWO-DIMENSIONAL AUTOMATIC TRANGULATION ALGORITHM

B.1 Introduction

The finite element analysis involves three stages of activity: preprocessing, processing, and postprocessing. Preprocessing involves the preparation of data, such as nodal coordinates, element connectivity, boundary conditions, and domain information. The processing stage involves stiffness generation, stiffness modification, and solution of equations, resulting in the evaluation of nodal variables. Other derived quantities may be evaluated at this stage. The postprocessing stage deals with the presentation of results.

The preparation of data and postprocessing require considerable effort if all data are to be handled manually. The tedium of handling the data and the possibility of errors creeping in as the number of elements increases are discouraging factors for the finite element analyst. In practice, one would like to be able to lay out the idealization of a complex problem, and then generate, edit, and plot the mesh, and execute the finite element program within a few brief sessions at a terminal. Therefore, for the preprocessing stage, a computer-aided triangulation program of two dimensional regions is developed. The program generates nodal coordinates, element connectivity, and boundary information of the idealized region. The objective of this section is to present and illustrate the methodology utilized in the triangulation process.
B.2 Region and Block Representation

While uniform meshes are easy to generate by simple algorithms, very general meshes are not. However, it is most convenient to think of the shape of an element in the xy-coordinate system as being defined by a transformation or mapping of a parent (or standard) element in another coordinate system, say the st-coordinate system, see Figs B.1 and B.2. Thus a general quadrilateral region in the xy-coordinate system is defined by a mapping of a standard square in the st-coordinate system as depicted by Fig. B.1. Equivalently, a general triangular region in the xy-coordinate system is defined by a mapping of a standard triangle in the st-coordinate system as shown by Fig. B.2. The transformation or mapping is given by:

\[ x(s, t) = \sum_{i=1}^{n} x_i N_i(s, t) = x^T N \]  
\[ y(s, t) = \sum_{i=1}^{n} y_i N_i(s, t) = y^T N \]  
(B.1)

Where:
\[ x^T = [x_1, x_2, x_3, \ldots, x_n] \] is the given nodal x-coordinates of the region,
\[ y^T = [y_1, y_2, y_3, \ldots, y_n] \] is the given nodal y-coordinates of the region,
\[ N \] is a column vector that contains the shape or interpolation functions of the standard element in the st-coordinate system, and
\[ n \] is the number of nodes that are given for the region.
Fig. B.1: (a) Standard or parent square, (b) Quadrilateral region

Fig. B.2: (a) Standard or parent triangle, (b) Triangular region
The degree of approximation, however, is determined based on the value of \( n \). For linear approximation, two nodes per side are required (i.e. four nodes for quadrilateral regions and three nodes for triangular regions), for quadratic approximation, three nodes per side are required (i.e. eight nodes for quadrilateral regions and six nodes for triangular regions), for cubic approximation, four nodes per side are needed (i.e. twelve nodes for quadrilateral regions and nine nodes for triangular regions), and so forth.

Quadratic and higher approximations are used to map curved boundaries. Note that all nodes are placed on the boundary of the region; i.e. no interior nodes are required to describe the shape of the region. Thus increasing the degree of approximation by one requires adding an additional node on each side of the boundary of the region. This element family, where nodes are placed on the boundary of the region, is called a serendipity family. In conclusion, if the nodal x- and y-coordinates of a given quadrilateral or triangular region and the shape functions of the corresponding standard element are known, the shape of the region can be accurately described. Moreover dividing the region into a number of triangles is easily accomplished through dividing the corresponding standard element.

**B.3 Interpolation Functions**

In general, the interpolation functions are used either to describe the coordinate transformation (hence they are called shape functions) or to
interpolate the unknown dependent variable. In other words, two independent sets of nodes can coexist for a region:

(1) The first set of nodes, which describes the shape of an element, is used for coordinate transformation:

\[ x(s, t) = \sum_{i=1}^{r} x_i N_i(s, t) = x^T N \]  \hspace{1cm} (B.2)

\[ y(s, t) = \sum_{i=1}^{r} y_i N_i(s, t) = y^T N \]

(2) The second set of nodes is used for the interpolation of the unknown dependent variable:

\[ u(x, y) = \sum_{i=1}^{s} u_i N_i(x, y) = u^T N = N^T u \]  \hspace{1cm} (B.3)

Depending on the relationship between the degree of approximation, r, used for the coordinate transformation and the degree of approximation, s, used for the dependent variable interpolation, elements are classified into 3 categories: (1) Subparametric elements \( (r < s) \), (2) Isoparametric elements \( (r = s) \), and (3) Superparametric elements \( (r > s) \). Isoparametric elements are more commonly used due to the ease and efficiency of calculation in the finite
element implementation.

The construction of the interpolation functions does not depend on the specific differential equation being solved. The construction procedure depends only on the geometry, the number and position of the nodes, and the number of (primary) dependent unknowns identified at the nodes of the element. Thus one can develop a library of finite elements which can be used whenever they are applicable. The selection of an appropriate element can be made depending on the admissibility conditions implied by the finite element formulation, the geometry of the domain being modeled, and the degree of accuracy required.

The finite element formulation does not require a priori knowledge of the type of the elements being used. When derived, the shape or interpolation functions are required to satisfy the following:

\[
N_i \left( s_p, t_j \right) = \delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j \\
i, j = 1, 2, ..., n 
\end{cases} 
\]  
(1)

\[
\sum_{i=1}^{n} N_i(s, t) = 1 
\]  
(2)

The interpolation functions are constructed for the standard elements in the st-coordinate system with the help of what is called Pascal's triangle (Fig. B.3).
Fig. B.3: Pascal's triangle

In general, there are two element families: The first family is the serendipity family which is used for the description of the shape of an element; i.e. no interior nodes are required. For the construction of this family, Pascal's triangle is used as shown in Fig. B.4 for quadrilateral regions and as shown in Fig. B.5 for triangular regions. Note that Fig. B.5 can not be used for cubic and higher degrees approximations since for cubic approximation nine nodes are required whereas Pascal's triangle gives ten nodes instead of nine. Alternatives are out the scope of this section and will not be discussed. The second family is the Lagrange family which is used for the interpolation of
**Fig. B.4:** Serendipity quadrilaterals

**Fig. B.5:** Complete polynomial triangles
the unknown dependent variable. In this family, interior nodes may be used. For the construction of this family, Pascal's triangle is used as shown in Fig. B.6 for quadrilateral regions and as shown in Fig. B.5 for triangular regions without limitations.

Fig. B.6 Lagrange quadrilaterals
B.4 The Family of Serendipity Elements

The serendipity elements are those elements which have no interior nodes; i.e. all the node points are on the boundary of the element. The quadratic approximations in the standard square and the standard triangle will be considered here.

B.4.1 Quadratic Approximation in the Square

Consider the position and the node numbers of the standard square as shown in Fig. B.7:

![Diagram of the standard square with node numbers and equations]

**Fig. B.7:** Position and node numbering of the standard square
Here we show how to construct the interpolation functions for the eight-node (quadratic) quadrilateral element. The interpolation function for node 1 should take on a value of one at node 1 and a value of zero elsewhere. Equivalently, \( N_1 \) should vanish on the sides defined by \( 1 - s = 0, \ 1 - t = 0, \) and \( 2s + 2t - 1 = 0 \), see Fig. B.7. Therefore, \( N_1 \) is of the form

\[
N_1(s, t) = c(1 - s)(1 - t)(2s + 2t - 1)
\]

where \( c \) is a constant which should be determined so as to yield \( N_1(0, 0) = 1 \).

We obtain \( c = -1 \), and therefore,

\[
N_1(s, t) = (1 - s)(1 - t)(1 - 2s - 2t)
\]

For \( N_2(s, t) \), we have the sides: \( s = 0, \ (1 - s) = 0, \) and \( (1 - t) = 0 \). Therefore, \( N_2 \) is of the form

\[
N_2(s, t) = cs(1 - s)(1 - t)
\]

the constant \( c \) should be determined so as to yield \( N_2(1/2, 0) = 1 \).

We obtain \( c = 4 \), and therefore,

\[
N_2(s, t) = 4s(1 - s)(1 - t)
\]

Following exactly the same procedure for the remaining nodes, we obtain:

\[
N_3(s, t) = -s(1 - t)(1 - 2s + 2t)
\]

\[
N_4(s, t) = 4st(1 - t)
\]

\[
N_5(s, t) = -st(3 - 2s - 2t)
\]

\[
N_6(s, t) = 4st(1 - s)
\]

\[
N_7(s, t) = -t(1 - s)(1 + 2s - 2t)
\]

\[
N_8(s, t) = 4t(1 - s)(1 - t)
\]
B.4.2 Quadratic Approximation in the Triangle

Consider the position and the node numbers of the standard triangle as shown in Fig. B.8:

![Diagram of a triangle with nodes labeled 1 to 6 and lines indicating coordinates](image)

**Fig. B.8:** Position and node numbering of the standard triangle

Here we show how to construct the interpolation functions for the six-node (quadratic) triangular element. The interpolation function for node 1 should take on a value of one at node 1 and a value of zero elsewhere. Equivalently, \( N_1 \) should vanish on the sides defined by \((s + t - 1) = 0\), and \(2s + 2t - 1 = 0\), see Fig. B.8. Therefor, \( N_1 \) is of the form:

\[
N_1(s, t) = c(s + t - 1)(2s + 2t - 1)
\]
The constant $c$ is determined so as to yield $N_1(0, 0) = 1$. We obtain $c = 1$, and therefore,

$$N_1(s, t) = (s + t - 1)(2s + 2t - 1)$$

For $N_2(s, t)$, we have the sides: $s = 0$, and $(s + t - 1) = 0$. Therefore, $N_2$ is of the form

$$N_2(s, t) = cs(s + t - 1)$$

The constant $c$ should be determined so as to yield $N_2(1/2, 0) = 1$.

We obtain $c = -4$, and therefore,

$$N_2(s, t) = -4s(s + t - 1)$$

Following exactly the same procedure for the remaining nodes, we obtain:

$$N_3(s, t) = s(2s - 1)$$

$$N_4(s, t) = 4st$$

$$N_5(s, t) = t(2t - 1)$$

$$N_6(s, t) = -4t(s + t - 1)$$
B.5 Illustrative Examples

B.5.1 A Quadrilateral Region

Consider a quadrilateral region as indicated by Fig. B.9a. The eight nodes are arranged in a counter-clockwise direction and the $x$- and $y$-nodal coordinates are given. Using these nodal values, the eight interpolation functions of the standard square, and the transformation equations, we can locate any point in the quadrilateral region by knowing only the $s$- and $t$-coordinates of the point. More realistically, the standard square can be divided into a number of divisions in the $s$-direction and a number of divisions in the $t$-direction. The $x$ and $y$ coordinates of the intersection nodal points can be found using the transformation equations.

Fig. B.9a: A quadrilateral region in the $xy$-coordinate system
Fig. B.9b is the same quadrilateral region after performing twenty divisions in the s-direction and twenty divisions in the t-direction and connecting the shorter diagonals.

**Fig. B.9b:** The triangulated quadrilateral region (8x6 ft, 800 elements)
B.5.2 A Triangular Region

Consider a triangular region as shown by Fig. B.10a. The six nodes are arranged in a counter-clock wise direction and the x- and y-nodal coordinates are given. Using these nodal values, the six interpolation functions of the standard triangle, and the transformation equations, we can locate any point in the triangular region by knowing only the s- and t-coordinates of the point. More generally, the standard triangle can be divided into a number of divisions in the s-direction and the same number of divisions in the t-direction. The x and y coordinates of the intersection nodal points can be found using the previous transformation equations.

Fig. B.10a: A triangular region in the xy-coordinate system
Fig. B.10b is the same triangular region after performing twenty divisions in the s-direction and twenty divisions in the t-direction and connecting the shorter diagonals.

Fig. B.10b: The triangulated region (7x6 ft, 400 elements)
B.5.3 A General Two-Dimensional Region

A general two-dimensional domain can be divided into quadrilateral and/or triangular regions, see Fig. B.11a. Each of these regions is called a macroblock or subregion for reference purposes. Each of these macroblocks is separately triangulated, the repeated nodal coordinates at macroblocks’ interfaces are eliminated, and the nodal points are renumbered. The boundary information is handled separately and it is simply a matter of programming skills.

Fig. B.11a: A general two dimensional domain
Fig. B.11b is the same region after performing the triangulation scheme on every quadrilateral or triangular subregion separately. The created meshes are then combined together to form the final mesh.

Fig. B.11b The triangulated domain (450x347 ft, 1152 elements)
APPENDIX C

DERIVATION OF THE FEM FOR THE TWO-DIMENSIONAL TWO-PHASE FLOW

C.1 The Model Equations

1. The Water Equation

\[
\frac{\partial}{\partial x} \left[ h k_x \left( \frac{k_r}{\mu B} \right)_w \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ h k_y \left( \frac{k_r}{\mu B} \right)_w \frac{\partial p_w}{\partial y} \right] + \frac{q_w}{B_w A_{CV}} = h \frac{\partial}{\partial t} \left( \phi \frac{S_w}{B_w} \right)
\]

The expansion of the right hand-side of the above equation yields:

\[
\frac{\partial}{\partial t} \left( \frac{\phi}{B_w} S_w \right) = S_w \frac{\partial}{\partial p_w} \left( \frac{\phi}{B_w} \right) \frac{\partial p_w}{\partial t} + \left( \frac{\phi}{B_w} \right) \frac{\partial S_w}{\partial t}
\]

Since

\[
\frac{\partial}{\partial p_w} \left( \frac{\phi}{B_w} \right) = \frac{B_w \frac{\partial \phi}{\partial p_w} - \phi \frac{\partial B_w}{\partial p_w}}{B_w^2} = \frac{B_w \phi_b c_\phi + \phi B_{wb} c_w}{B_w^2}
\]

Substituting into the above equation yields:

\[
\frac{\partial}{\partial x} \left[ h k_x \left( \frac{k_r}{\mu B} \right)_w \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ h k_y \left( \frac{k_r}{\mu B} \right)_w \frac{\partial p_w}{\partial y} \right] + \frac{q_w}{B_w A_{CV}}
\]

\[
= h \frac{B_w \phi_b c_\phi + \phi B_{wb} c_w}{B_w^2} S_w \frac{\partial p_w}{\partial t} + h \left( \frac{\phi}{B_w} \right) \frac{\partial S_w}{\partial t}
\]

Thus the water equation can be written as:
\[
\frac{\partial}{\partial x} \left[ f_{11} \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_{12} \frac{\partial p_w}{\partial y} \right] + g_1 = h_{11} \frac{\partial p_w}{\partial t} + h_{12} \frac{\partial s_w}{\partial t} \quad \text{(C.1a)}
\]

2. The Oil Equation

\[
\frac{\partial}{\partial x} \left[ h k_x \left( \frac{k_r}{\mu B} \right) \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ h k_y \left( \frac{k_r}{\mu B} \right) \frac{\partial p_w}{\partial y} \right] + \frac{q_o}{B_o A_{CV}} = h \frac{\partial}{\partial t} \left( \phi \left( \frac{1 - S_w}{B_o} \right) \right)
\]

The expansion of the right hand-side of the above equation yields:

\[
\frac{\partial}{\partial t} \left( \phi \left( \frac{1 - S_w}{B_o} \right) \right) = \left( 1 - S_w \right) \frac{\partial}{\partial p_w} \left( \frac{\phi}{B_o} \right) \frac{\partial p_w}{\partial t} - \left( \frac{\phi}{B_o} \right) \frac{\partial s_w}{\partial t}
\]

Since

\[
\frac{\partial}{\partial p_w} \left( \frac{\phi}{B_o} \right) = \frac{B_o \frac{\partial \phi}{\partial p_w} - \phi \frac{\partial B_o}{\partial p_w}}{B_o^2} = \frac{B_o \phi_b c_\phi + \phi B_{ob} c_o}{B_o^2}
\]

Substituting into the above equation yields:

\[
\frac{\partial}{\partial x} \left[ h k_x \left( \frac{k_r}{\mu B} \right) \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ h k_y \left( \frac{k_r}{\mu B} \right) \frac{\partial p_w}{\partial y} \right] + \frac{q_o}{B_o A_{CV}}
\]

\[
= h \frac{B_o \phi_b c_\phi + \phi B_{ob} c_o}{B_o^2} \left( 1 - S_w \right) \frac{\partial p_w}{\partial t} - h \left( \frac{\phi}{B_o} \right) \frac{\partial s_w}{\partial t}
\]

Thus the oil equation can be written as:
\[
\frac{\partial}{\partial x} \left[ f_{11} \frac{\partial p_w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_{22} \frac{\partial p_w}{\partial y} \right] + g_2 = h_{21} \frac{\partial p_w}{\partial t} + h_{22} \frac{\partial S_w}{\partial t}
\] (C.1b)

C.2 Variational Formulation of the Model Equations

1. The Water Equation

\[
0 = \int_{\Omega^*} \left\{ h_{11} \frac{\partial p_w}{\partial t} + h_{12} \frac{\partial S_w}{\partial t} - \frac{\partial}{\partial x} \left[ f_{11} \frac{\partial p_w}{\partial x} \right] - \frac{\partial}{\partial y} \left[ f_{12} \frac{\partial p_w}{\partial y} \right] - g_1 \right\} w_1 \, dx \, dy
\]

\[
= \int_{\Omega^*} \left\{ h_{11} w_1 \frac{\partial p_w}{\partial t} + h_{12} w_1 \frac{\partial S_w}{\partial t} + f_{11} \frac{\partial w_1}{\partial x} \frac{\partial p_w}{\partial x} + f_{12} \frac{\partial w_1}{\partial y} \frac{\partial p_w}{\partial y} \right. \]

\[
- \left. w_1 g_1 \right\} d \Omega - \int_{\Gamma^*} \left\{ f_{11} \frac{\partial p_w}{\partial x} n_x + f_{12} \frac{\partial p_w}{\partial y} n_y \right\} w_1 \, d \Gamma
\]

The boundary integral vanishes, thus we have:

\[
0 = \int_{\Omega^*} \left\{ h_{11} w_1 \frac{\partial p_w}{\partial t} + h_{12} w_1 \frac{\partial S_w}{\partial t} + f_{11} \frac{\partial w_1}{\partial x} \frac{\partial p_w}{\partial x} + f_{12} \frac{\partial w_1}{\partial y} \frac{\partial p_w}{\partial y} - w_1 g_1 \right\} d \Omega
\]

2. The Oil Equation

Similar treatment of the oil equation yields:
\[ 0 = \int_{\Omega} \left\{ h_{21} w_2 \frac{\partial p_w}{\partial t} + h_{22} w_2 \frac{\partial S_w}{\partial t} + f_{21} \frac{\partial w_2}{\partial x} \frac{\partial p_w}{\partial x} + f_{22} \frac{\partial w_2}{\partial y} \frac{\partial p_w}{\partial y} - w_2 g_2 \right\} \, dx \, dy \]

Where \( w_1 \) (variation in \( p_w \)), and \( w_2 \) (variation in \( S_w \)) are appropriate test functions.

C.3 Variational Approximation of the Model Equations

From the governing equations, it is clear that \( S_w \) is two orders less differentiable than \( p_w \). Hence, the interpolation used for \( S_w \) should be two degrees less than that used for \( p_w \) (for consistency of approximation). In other words, one can use cubic interpolation for \( p_w \) and linear interpolation for \( S_w \). Alternatively, a quadratic interpolation for \( p_w \) and a constant for \( S_w \).

Over a typical element \( \Omega_e \), \( p_w \) and \( S_w \) are approximated by interpolation functions of the form:

\[ p_w (x, y, t) = \sum_{j=1}^{r} p_{w_j} (t) \psi_j (x, y) \quad w_1 (x, y) = \psi_1 (x, y) \]

\[ S_w (x, y, t) = \sum_{j=1}^{s} S_{w_j} (t) \phi_j (x, y) \quad w_2 (x, y) = \phi_1 (x, y) \]

Where \( \psi \) and \( \phi \) are interpolation functions of degrees \( r \) and \( s \) \((r > s)\), respectively. Substituting both approximations into the variational forms of the equations, we obtain:
1. The Water Equation

\[ 0 = \int_{\Omega^*} \left\{ h_{11} \psi_i \left[ \sum_{j=1}^{r} \frac{\partial p_{w_j}}{\partial t} \psi_j \right] + h_{12} \psi_i \left[ \sum_{j=1}^{s} \frac{\partial s_{w_j}}{\partial t} \phi_j \right] \\
+ f_{11} \frac{\partial \psi_i}{\partial x} \left[ \sum_{j=1}^{r} p_{w_j} \frac{\partial \psi_j}{\partial x} \right] + f_{12} \frac{\partial \psi_i}{\partial y} \left[ \sum_{j=1}^{r} p_{w_j} \frac{\partial \psi_j}{\partial y} \right] - \psi_i \, g_1 \right\} \, dx \, dy \]

Rearranging, we obtain:

\[ 0 = \sum_{j=1}^{r} \left\{ \left( \int_{\Omega^*} h_{11} \, \psi_i \, \psi_j \, dx \, dy \right) \frac{\partial p_{w_j}}{\partial t} + \left( \int_{\Omega^*} h_{12} \, \psi_i \, \phi_j \, dx \, dy \right) \frac{\partial s_{w_j}}{\partial t} \\
+ \left[ \int_{\Omega^*} \left( f_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + f_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \, dx \, dy \right] p_{w_j} - \int_{\Omega^*} \psi_i \, g_1 \, dx \, dy \right\} \]

2. The Oil Equation

Similar treatment of the oil equation, yields:

\[ 0 = \sum_{j=1}^{r} \left\{ \left( \int_{\Omega^*} h_{21} \, \phi_i \, \psi_j \, dx \, dy \right) \frac{\partial p_{w_j}}{\partial t} + \left( \int_{\Omega^*} h_{22} \, \phi_i \, \phi_j \, dx \, dy \right) \frac{\partial s_{w_j}}{\partial t} \\
+ \left[ \int_{\Omega^*} \left( f_{21} \frac{\partial \phi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + f_{22} \frac{\partial \phi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \, dx \, dy \right] p_{w_j} - \int_{\Omega^*} \phi_i \, g_2 \, dx \, dy \right\} \]

In a matrix form, the above two equations are written as:
\[
\begin{bmatrix}
[M_1^{11}] & [M_1^{12}] \\
[M_2^{21}] & [M_2^{22}]
\end{bmatrix}
\cdot
\begin{bmatrix}
[p_1] \\
{s_1]
\end{bmatrix}
+ \begin{bmatrix}
[K_1^{11}] & [K_1^{12}] \\
[K_2^{21}] & [K_2^{22}]
\end{bmatrix}
\cdot
\begin{bmatrix}
[p_2] \\
{s_2]
\end{bmatrix}
= \begin{bmatrix}
{F_1} \\
{F_2}
\end{bmatrix}
\] (C.3)

Where:

\[
M_{ij}^{11} = \int_{\Omega^*} h_{11} \psi_i \psi_j \, dx dy
\]
\[
M_{ij}^{12} = \int_{\Omega^*} h_{12} \psi_i \phi_j \, dx dy
\]
\[
M_{ij}^{21} = \int_{\Omega^*} h_{21} \phi_i \psi_j \, dx dy
\]
\[
M_{ij}^{22} = \int_{\Omega^*} h_{22} \phi_i \phi_j \, dx dy
\]
\[
K_{ij}^{11} = \int_{\Omega^*} \left( f_{11} \frac{\partial \psi_i}{\partial x} + f_{12} \frac{\partial \psi_i}{\partial y} \right) \, dx dy
\]
\[
K_{ij}^{12} = \int_{\Omega^*} \left( f_{11} \frac{\partial \psi_i}{\partial x} + f_{12} \frac{\partial \psi_i}{\partial y} \right) \, dx dy
\]
\[
K_{ij}^{21} = \int_{\Omega^*} \left( f_{21} \frac{\partial \phi_i}{\partial x} + f_{22} \frac{\partial \phi_i}{\partial y} \right) \, dx dy
\]
\[
K_{ij}^{22} = \int_{\Omega^*} \left( f_{21} \frac{\partial \phi_i}{\partial x} + f_{22} \frac{\partial \phi_i}{\partial y} \right) \, dx dy
\]
\[
F_i^1 = \int_{\Omega^*} \psi_i \, g_1 \, dx dy
\]
\[
F_i^2 = \int_{\Omega^*} \phi_i \, g_2 \, dx dy
\]

Where \( i, j = 1, 2, ..., r \) \( I, J = 1, 2, ..., s \) \( (r > s) \)

The above finite element model is non-positive definite because of the zeros appearing on the main diagonal. Therefore, equation solvers that employ "pivoting" must be used in this case. Also, the fact that saturation does not appear as a degree of freedom at every node makes the computer implementation of the assembly more complicated.
C.4 Time Discretization Schemes

Equation (C.3) can be written in the general form

\[
[M] \{\dot{\Delta}\} + [K] \{\Delta\} = \{F\} \quad \{\Delta\} = \begin{bmatrix} \{p_w\} \\ \{S_w\} \end{bmatrix}
\]

By introducing a number of well-known difference schemes, we write the above equation as follows:

\[
\{ [M] + \theta \Delta t [K] \} \{\Delta\}_{n+1} = \{ [M] - (1 - \theta)\Delta t [K] \} \{\Delta\}_n + \Delta t \{F\}
\]

Which is written as a linear system of equations as:

\[
[A] \{\Delta\}_{n+1} = \{B\}
\]

Where:

\[
[A] = [m] + \theta \Delta t [K]
\]

\[
\{B\} = \{ [M] - (1 - \theta)\Delta t [K] \} \{\Delta\}_n + \Delta t \{F\}
\]

\[
\theta = \begin{cases} 
0 & \text{Forward Difference Scheme} \\
\frac{1}{2} & \text{Crank–Nicolson Scheme} \\
\frac{2}{3} & \text{Galerkin Method} \\
1 & \text{Backward Difference Scheme}
\end{cases}
\]
APPENDIX D

DPM AND SPM JACOBIAN MATRIX ENTRIES
FOR THE FD DISCRETIZATION SCHEME

D.1 Fracture Water Equation

(1) \( p_{wi} \) Derivatives

\[
\frac{\partial R_{wfi}}{\partial p_{wfi}} = \left[ (T_x)^{n+1} \right]_{wfi} \left[ 1 - \frac{\partial}{\partial p_{wfi}} \left( \frac{\partial}{\partial p_{wfi}} \right) \left( D_{fi} - D_{fd} \right) \right]
\]

\[
+ \frac{\partial}{\partial p_{wfi}} \left[ (T_x)^{n+1} \right]_{wfi} \left[ \left( p_{wfi}^{n+1} - p_{wfi}^{n+1} \right) - \left( D_{fi} - D_{fd} \right) \right]
\]

Where:

\[
\frac{\partial}{\partial p_{wfi}} \left[ (T_x)^{n+1} \right]_{wfi} = TCONX_{i+1} \left( 1 - \omega_{wi,i+1} \right) c_w \left( \frac{k_r}{\mu B} \right)_{wfi}
\]

\[
\frac{\partial}{\partial p_{wfi}} \left( T_x \right)_{wfi} = \frac{c_w \rho_{wfi}^{n+1}}{288}
\]

Substituting both terms into the above equation yields:

\[
\frac{\partial R_{wfi}}{\partial p_{wfi}} = \left[ (T_x)^{n+1} \right]_{wfi} \left[ 1 - \frac{c_w \rho_{wfi}^{n+1}}{288} \left( D_{fi} - D_{fd} \right) \right] + TCONX_{i+1} \left( 1 - \omega_{wi,i+1} \right)
\]

\[
- c_w \left( \frac{k_r}{\mu B} \right)_{wfi} \left[ \left( p_{wfi}^{n+1} - p_{wfi}^{n+1} \right) - \left( D_{fi} - D_{fd} \right) \right]
\]

\[
- TWX_{i+1} \left( 1 - GW \right) + TCONX_{i+1} \left( 1 - \omega_{wi,i+1} \right) TBW_{i+1} PTW_{i+1}
\]

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{wijk}}{\partial p_{wfi-1}} = TWX_i (1 + GW) - TCONX_i \omega_{wx,i} TBW_{i-1} PTW_{i-1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfi+j-1}} = TWY_{j+1} (1 - GW) + TCONY_{j+1} \omega_{wy,j+1} ) TBW_{j+1} PTW_{j-1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfi-1}} = TWY_j (1 + GW) - TCONY_j \omega_{wy,j} TBW_{j-1} PTW_{j-1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfk+1}} = TWZ_{k+1} (1 - GW) + TCONZ_{k+1} \omega_{wz,k+1} ) TBW_{k+1} PTW_{k+1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfk-1}} = TWZ_k (1 + GW) - TCONZ_k \omega_{wz,k} TBW_{k-1} PTW_{k-1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfi+j+1}} = TWD_{i+1,j+1} (1-GW) + TCOND_{i+1,j+1} (1-\omega_{wd,ij+1}) TBW_{i+1,j+1} PTW_{i+1,j+1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfi-1,j-1}} = TWD_{i-1,j-1} (1 + GW) - TCOND_{i-1,j} \omega_{wd,ij} TBW_{i-1,j-1} PTW_{i-1,j-1}
\]

\[
\frac{\partial R_{wijk}}{\partial p_{wfi+1,j-1}} = TWD_{i+1,j-1} (1 - GW) + TCOND_{i+1,j} \omega_{wd,ij} TBW_{i+1,j-1} PTW_{i+1,j-1}
\]
\[ \frac{\partial R_{w_{jik}}}{\partial p_{w_{j-1,k+1}}} = TWD_{1_{y,1}} (1 + GW) - TCOND_{1_{y}} \omega_{w_{d_{1,i}}} TBW_{i-1,j+1} PTW_{i-1,j+1} \]

\[ \frac{\partial R_{w_{jik}}}{\partial p_{w_{j+1,k}}} = - TWD_{1_{y,1}} (1 + GW) + TCOND_{1_{y}} \omega_{w_{z_{11}}} TBW_{y_{k}} PTW_{i+1} \]
\[ - TWD_{2_{y,1-j+1}} (1 + GW) + TCOND_{2_{y,1-j+1}} \omega_{w_{d_{2,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j+1} \]
\[ - TWD_{2_{y,1-j}} (1 - GW) - TCOND_{2_{y,1-j}} \omega_{w_{d_{2,i}}} TBW_{y_{k}} PTW_{i+1,j-1} \]
\[ - TWD_{1_{y,1-j}} (1 + GW) + TCOND_{1_{y,1-j}} \omega_{w_{d_{1,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j-1} \]
\[ - TWD_{1_{y,1-j}} (1 - GW) - TCOND_{1_{y,1-j}} \omega_{w_{d_{1,i}}} TBW_{y_{k}} PTW_{i+1,j-1} \]
\[ - TWD_{2_{y,1-j}} (1 - GW) - TCOND_{2_{y,1-j}} \omega_{w_{d_{2,i}}} TBW_{y_{k}} PTW_{i+1,j-1} \]
\[ - TWD_{2_{y,1-j}} (1 - GW) + TCOND_{2_{y,1-j}} \omega_{w_{d_{2,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j+1} \]
\[ - TWD_{1_{y,1-j}} (1 - GW) - TCOND_{1_{y,1-j}} \omega_{w_{d_{1,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j+1} \]
\[ - TWD_{2_{y,1-j}} (1 - GW) + TCOND_{2_{y,1-j}} \omega_{w_{d_{2,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j-1} \]
\[ - TWD_{2_{y,1-j}} (1 - GW) + TCOND_{2_{y,1-j}} \omega_{w_{d_{2,i-j+1}}} TBW_{y_{k}} PTW_{i+1,j+1} \]
\[ + WWP_{y_{k}} c_{w} \left( p_{w_{n+1}}^{n+1} - P_{\text{node}}^{n+1} \right) - 1 \]
(2) $S_{w_t}$ Derivatives

$$\frac{\partial R_{wflk}}{\partial S_{wfl}^{n+1}} = \left[ \left( p_{wfl}^{n+1} - p_{wfl}^{n+1} \right) - \gamma_{wfl}^{n+1} \left( D_{fl}^{n+1} - D_{fl} \right) \right] \frac{\partial}{\partial S_{wfl}^{n+1}} \left( T_s \right)_{wfl}^{n+1} \cdot \frac{1}{2}$$

Where:

$$\frac{\partial}{\partial S_{wfl}^{n+1}} \left( T_s \right)_{wfl}^{n+1} = TCONX_{i+1} \left( 1 - \omega_{wxi}^{n+1} \right) \left( \frac{1}{\mu B} \right)_{wfi}^{n+1} \cdot \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{fl}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wflk}}{\partial S_{wfl}^{n+1}} = TCONX_{i+1} \left( 1 - \omega_{wxi}^{n+1} \right) \left( \frac{1}{\mu B} \right)_{wfi}^{n+1} \cdot \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{fl}^{n+1}$$

$$\cdot \left[ \left( p_{wfl}^{n+1} - p_{wfl}^{n+1} \right) - \gamma_{wfl}^{n+1} \left( D_{fl}^{n+1} - D_{fl} \right) \right]$$

$$= TCONX_{i+1} \left( 1 - \omega_{wxi}^{n+1} \right) TKW_{t}^{n+1} PTW_t^{n+1}$$

Following the same procedure outlined above, we obtain:

$$\frac{\partial R_{wflk}}{\partial S_{wfl}^{n+1}} = -TCONX_i \omega_{wxi} TKW_{t}^{n+1} PTW_t^{n+1}$$

$$\frac{\partial R_{wflk}}{\partial S_{wfl}^{n+1}} = TCONY_{j+1} \left( 1 - \omega_{wyj}^{n+1} \right) TKW_{j}^{n+1} PTW_j^{n+1}$$

$$\frac{\partial R_{wflk}}{\partial S_{wfl}^{n+1}} = -TCONY_j \omega_{wyj} TKW_{j}^{n+1} PTW_j^{n+1}$$
\[ \frac{\partial R_{wijk}}{\partial s_{wjk+1}} = TCONZ_{k+1} \left( 1 - \omega_{wz,k+1} \right) TKW_{k+1} PTW_{k+1} \]

\[ \frac{\partial R_{wijk}}{\partial s_{wjk+1}} = - TCONZ_k \omega_{wz,k} TKW_{k+1} PTW_{k+1} \]

\[ \frac{\partial R_{wijk}}{\partial s_{wij+1,j+1}} = TCOND2_{i+1,j+1} \left( 1 - \omega_{wd2,i+1,j+1} \right) TKW_{i+1,j+1} PTW_{i+1,j+1} \]

\[ \frac{\partial R_{wijk}}{\partial s_{wij+1,j+1}} = - TCOND2_{ij} \omega_{wd2,ij} TKW_{i+1,j+1} PTW_{i+1,j+1} \]

\[ \frac{\partial R_{wijk}}{\partial s_{wij+1,j+1}} = TCOND1_{i+1,j} \left( 1 - \omega_{wd1,i+1,j+1} \right) TKW_{i+1,j} PTW_{i+1,j} \]

\[ \frac{\partial R_{wijk}}{\partial s_{wij+1,j+1}} = - TCOND1_{ij} \omega_{wd1,ij} TKW_{i+1,j} PTW_{i+1,j} \]
\[
\frac{\partial R_{wf,ijk}}{\partial S_{wf,ijk}^{n+1}} = TCONX_{i+1} \omega_{wx,i+1} TKW_{ijk} PTW_{i+1}
- TCONX_{i} (1 - \omega_{wx,i}) TKW_{ijk} PTW_{i-1}
+ TCONY_{j+1} \omega_{wy,j+1} TKW_{ijk} PTW_{j+1}
- TCONY_{j} (1 - \omega_{wy,j}) TKW_{ijk} PTW_{j-1}
+ TCONZ_{k+1} \omega_{wz,k+1} TKW_{ijk} PTW_{k+1}
- TCONZ_{k} (1 - \omega_{wz,k}) TKW_{ijk} PTW_{k-1}
+ TCOND2_{i+1,j+1} \omega_{wd2,i+1,j+1} TKW_{ijk} PTW_{i+1,j+1}
- TCOND2_{i,j} (1 - \omega_{wd2,i,j}) TKW_{ijk} PTW_{i-1,j-1}
+ TCOND1_{i+1,j-1} \omega_{wd1,i+1,j-1} TKW_{ijk} PTW_{i+1,j-1}
- TCOND1_{i,j} (1 - \omega_{wd1,i,j}) TKW_{ijk} PTW_{i-1,j+1}
- TCONM_{ijk} (1 - \omega_{wm,ijk}) TKWM_{ijk} PTWM_{ijk}
\]

\[
+ \left[ W(T) \left( \frac{1}{\mu_w} \frac{\partial k_r}{\partial S_w} + \frac{1}{\mu_o} \frac{\partial k_o}{\partial S_w} \right) \frac{n^*}{f} \frac{1}{B_{wf}^{n+1}} \left( p_{bh}^{n+1} - p_{node}^{n+1} \right) \right]_{ijk}
- \frac{(V_R)_{ijk}}{5.6146 \Delta t} \left( \frac{\Phi}{B_w} \right)_{f,ijk}^{n+1}
\]

For the SPM, the matrix/fracture transfer term is replaced by:

\[
- \frac{(V_R)_{ijk}}{5.6146} \left( \frac{1}{B_{wf,ijk}} \sum_{N=1}^{N_{FIT}} R_N \lambda_N e^{-\lambda_N \Delta t_a} \right)
\]
(3) \( p_{wm} \) Derivative

\[
\frac{\partial R_{wf,ijk}}{\partial p_{wm,ijk}^{n+1}} = (T)_{wm,ijk}^{n+1} \left[ 1 - \frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{wm,ijk}^{n+1} D_{m,ijk} \right] \\
\quad - \frac{\partial}{\partial p_{wm,ijk}^{n+1}} (T)_{wm,ijk}^{n+1} \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}
\]

Where:

\[
\frac{\partial}{\partial p_{wm,ijk}^{n+1}} (T)_{wm,ijk}^{n+1} = TCONM_{ijk} \omega_{wm,ijk} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ijk}
\]

\[
\frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{wm,ijk}^{n+1} = \frac{c_w}{144} \rho_{wm,ijk}^{n+1}
\]

Substituting both terms into the above equation and rearranging yields:

\[
\frac{\partial R_{wf,ijk}}{\partial p_{wm,ijk}^{n+1}} = (T)_{wm,ijk}^{n+1} \left[ 1 - \frac{c_w}{144} \rho_{wm,ijk}^{n+1} D_{m,ijk} \right] \\
\quad - TCONM_{ijk} \omega_{wm,ijk} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ijk} \\
\quad \cdot \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}
\]

\[= TWM_{ijk} (1 - GWM) - TCONM_{ijk} \omega_{wm,ijk} TBWM_{ijk} PTWM_{ijk}\]
(4) $S_{wn}$ Derivative

$$\frac{\partial R_{wf,ijk}}{\partial S_{wm,ijk}^{n+1}} = - \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk} \frac{\partial}{\partial S_{wm,ijk}^{n+1}} (T)^{n+1}_{wm,ijk}$$

Where:

$$\frac{\partial}{\partial S_{wm,ijk}^{n+1}} (T)^{n+1}_{wm,ijk} = TCONM_{ijk} \omega_{wm,ijk} \left( \frac{1}{\mu B} \right)_{wm,ijk}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ijk}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wf,ijk}}{\partial S_{wm,ijk}^{n+1}} = - TCONM_{ijk} \omega_{wm,ijk} \left( \frac{1}{\mu B} \right)_{wm,ijk}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ijk}^{n+1} \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}$$

$$= - TCONM_{ijk} \omega_{wm,ijk} TKWM_{ijk} PTWM_{ijk}$$

(5) $p_{bh}$ Derivative

$$\frac{\partial R_{wf,ijk}}{\partial p_{bh,ijk}^{n+1}} = \sum_{k=1}^{KK} \left[ WI \left( \frac{k_{rw}}{\mu_w} + \beta \frac{k_{ro}}{\mu_o} \right)_{f}^{n+1} \frac{1}{B_{wf}^{n+1}} \right]_{ijk} = \sum_{k=1}^{KK} WWP_{ijk}^{n+1}$$
D.2 Fracture Oil Equation

(1) $p_{wf}$ Derivatives

\[
\frac{\partial R_{of,jk}}{\partial p_{wf,i+1}^{n+1}} = \left( T_x \right)^{n+1}_{of,i+1,1/2} \left[ 1 - \frac{\partial}{\partial p_{wf,i+1}^{n+1}} \gamma_{of,i+1/2}^{n+1} (D_{fi+1} - D_{f,i}) \right] \\
+ \frac{\partial}{\partial p_{wf,i+1}^{n+1}} \left( T_x \right)^{n+1}_{of,i+1,1/2} \left[ \left( p_{of,i-1}^{n+1} - p_{of,i}^{n+1} \right) - \gamma_{of,i+1/2}^{n+1} (D_{fi+1} - D_{f,i}) \right]
\]

Where:

\[
\frac{\partial}{\partial p_{wf,i+1}^{n+1}} \left( T_x \right)^{n+1}_{of,i+1,1/2} = TCONX_{i+1} \left( 1 - \omega_{of,i+1} \right) c_o \left( \frac{k_r}{\mu B} \right)^{n+1}_{of,i+1} \\
\frac{\partial}{\partial p_{wf,i+1}^{n+1}} \gamma_{of,i+1/2}^{n+1} = \frac{c_o p_{of,i-1}^{n+1}}{288}
\]

Substituting both terms into the above equation yields:

\[
\frac{\partial R_{of,jk}}{\partial p_{wf,i+1}^{n+1}} = \left( T_x \right)^{n+1}_{of,i+1,1/2} \left[ 1 - \frac{c_o p_{of,i+1}^{n+1}}{288} (D_{fi+1} - D_{f,i}) \right] + TCONX_{i+1} \left( 1 - \omega_{of,i+1} \right) \\
\left( k_r \right)^{n+1}_{of,i+1} \left[ (p_{of,i-1}^{n+1} - p_{of,i}^{n+1}) - \gamma_{of,i+1/2}^{n+1} (D_{fi+1} - D_{f,i}) \right] \\
= TOX_{i+1} (1 - GO) + TCONX_{i+1} \left( 1 - \omega_{of,i+1} \right) TBO_{i+1} PTO_{i+1}
\]

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{ofijk}}{\partial p_{wfj-1}^{n+1}} = TOX_i \left(1 + GO\right) - TCONX_i \omega_{oxj} TBO_{i-1} PTO_{i-1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOY_{j+1} \left(1 - GO\right) + TCONY_{j-1} \left(1 - \omega_{oyj+1}\right) TBO_{j+1} PTO_{j+1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOY_j \left(1 + GO\right) - TCONY_j \omega_{oyj} TBO_{j-1} PTO_{j-1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOZ_{k+1} \left(1 - GO\right) + TCONZ_{k+1} \left(1 - \omega_{ozk+1}\right) TBO_{k+1} PTO_{k+1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOZ_k \left(1 + GO\right) - TCONZ_k \omega_{ozk} TBO_{k-1} PTO_{k-1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOD_{i-1,j+1} \left(1 - GO\right) + TCOND_{i-1,j+1} \left(1 - \omega_{odi,j+1}\right) TBO_{i-1,j+1} PTO_{i-1,j+1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOD_{i-1,j+1} \left(1 + GO\right) - TCOND_{i-1,j+1} \omega_{odi,j} TBO_{i-1,j-1} PTO_{i-1,j-1}
\]

\[
\frac{\partial R_{ofijk}}{\partial p_{wfj+1}^{n+1}} = TOD_{i-1,j+1} \left(1 - GO\right) + TCOND_{i-1,j+1} \left(1 - \omega_{odi,j+1}\right) TBO_{i-1,j+1} PTO_{i-1,j+1}
\]
\[
\frac{\partial R_{oifjk}}{\partial p_{oif-1,j+1}^{n+1}} = T O D I_{ij} (1 + GO) - T CON D I_{ij} \omega_{odi,ij} T B O_{i-1,j-1} P T O_{i-1,j+1} \\
\]

\[
\frac{\partial R_{oifjk}}{\partial p_{oif,jk}^{n+1}} = -T O X_{i+1} (1 + GO) + T CON X_{i+1} \omega_{ox,i} T B O_{yk} P T O_{t+1} \\
- T O X_{i} (1 - GO) - T CON X_{i} (1 - \omega_{ox,i}) T B O_{yk} P T O_{t+1} \\
- T O Y_{j+1} (1 + GO) + T CON Y_{j+1} \omega_{oy,j+1} T B O_{yk} P T O_{j+1} \\
- T O Y_{j} (1 - GO) - T CON Y_{j} (1 - \omega_{oy,j}) T B O_{yk} P T O_{j+1} \\
- T O Z_{k+1} (1 + GO) + T CON Z_{k+1} \omega_{oz,k+1} T B O_{yk} P T O_{k+1} \\
- T O Z_{k} (1 - GO) - T CON Z_{k} (1 - \omega_{oz,k}) T B O_{yk} P T O_{k+1} \\
- T O D 2_{i+1,j+1} (1 + GO) + T CON D 2_{i+1,j+1} \omega_{odi+1,j+1} T B O_{yk} P T O_{t+1,j+1} \\
- T O D 2_{i,j+1} (1 + GO) - T CON D 2_{i,j+1} (1 - \omega_{odi,j+1}) T B O_{yk} P T O_{t+1,j-1} \\
- T O D 1_{i,j+1} (1 + GO) + T CON D 1_{i,j+1} \omega_{odi+1,j+1} T B O_{yk} P T O_{t+1,j+1} \\
- T O D 1_{i,j} (1 - GO) - T CON D 1_{i,j} (1 - \omega_{odi,j}) T B O_{yk} P T O_{t+1,j-1} \\
- T O M_{yk} (1 - GOM) - T CON M_{yk} (1 - \omega_{om,yk}) T B O M_{yk} P T O_{yk} \\
+ W O P_{yk} \left[ c_{o} \left( p_{ne}^{n+1} - p_{node}^{n+1} \right) - 1 \right]_{yk} \\
- \frac{(V R)_{yk}}{5.6146 \Delta t} \left[ \left( \frac{S_{o}}{B_{o}} \right)_{j,yk}^{n+1} (c_{f} + c_{o}) \right] \\
\]

For the SPM, the matrix/fracture transfer term is replaced by:

\[
- \frac{(V R)_{yk}}{5.6146} c_{o} \left( \frac{S_{o}}{B_{of,yk}} \right)^{N_{FIT}} \sum_{N=1}^{N_{FIT}} R_{N} \lambda_{N} e^{-\lambda_{N} \Delta t_{n}} 
\]
(2) $S_{w,i}$ Derivatives

$$\frac{\partial R_{of,i,k}}{\partial S_{wf,i+1}^{n+1}} = (T_x)_{of,i+1}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{f,i+1}^{n+1} \left( 1 - \frac{c_o}{288} \frac{P_{of,i+1}^{n+1}}{D_{f,i+1} - D_{f,i}} \right) \right]$$

$$+ \left[ (p_{of,i+1}^{n+1} - p_{of,i}^{n+1}) - \gamma_{of,i+1}^{n+1} (D_{f,i+1} - D_{f,i}) \right] \frac{\partial}{\partial S_{wf,i+1}^{n+1}} (T_x)_{of,i+1}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{f,i+1}^{n+1}$$

Where:

$$\frac{\partial}{\partial S_{wf,i+1}^{n+1}} (T_x)_{of,i+1}^{n+1} = TCONX_{i+1} (1 - \omega_{ox,i+1}) \left( \frac{1}{\mu B} \right)_{of,i+1}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{f,i+1}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{of,i,k}}{\partial S_{wf,i+1}^{n+1}} = TCONX_{i+1} (1 - \omega_{ox,i+1}) \left( \frac{1}{\mu B} \right)_{of,i+1}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{f,i+1}^{n+1}$$

$$\cdot \left[ (p_{of,i+1}^{n+1} - p_{of,i}^{n+1}) - \gamma_{of,i+1}^{n+1} (D_{f,i+1} - D_{f,i}) \right]$$

$$+ (T_x)_{of,i+1}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{f,i+1}^{n+1} \left( 1 - \frac{c_o}{288} \frac{P_{of,i+1}^{n+1}}{D_{f,i+1} - D_{f,i}} \right) \right]$$

$$= TCONX_{i+1} (1 - \omega_{ox,i+1}) TKO_{i+1} PTO_{i-1}$$

$$+ TOX_{i+1} \left[ DPCF_{i+1} (1 - GO_{i+1}) \right]$$

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{of,ijk}}{\partial S_{wfi}^{n+1}} = - TCONX_i \omega_{ox,i} TKO_{i-1} PTO_{i-1} \\
+ TOX_i \left[ DPCF_{i-1} (1 + GO_{i-1}) \right]
\]

\[
\frac{\partial R_{of,ijk}}{\partial S_{w fj}^{n+1}} = TCONY_{j-1} (1 - \omega_{oy,j-1}) TKO_{j-1} PTO_{j-1} \\
+ TOY_{j-1} \left[ DPCF_{j-1} (1 - GO_{j-1}) \right]
\]

\[
\frac{\partial R_{of,ijk}}{\partial S_{w fj}^{n+1}} = - TCONY_j \omega_{oy,j} TKO_{j-1} PTO_{j-1} \\
+ TOY_j \left[ DPCF_{j-1} (1 + GO_{j-1}) \right]
\]

\[
\frac{\partial R_{of,ijk}}{\partial S_{wfk}^{n+1}} = TCONZ_{k+1} (1 - \omega_{oz,k+1}) TKO_{k-1} PTO_{k-1} \\
+ TOZ_{k+1} \left[ DPCF_{k-1} (1 - GO_{k-1}) \right]
\]

\[
\frac{\partial R_{of,ijk}}{\partial S_{wfk}^{n+1}} = - TCONZ_k \omega_{oz,k} TKO_{k-1} PTO_{k-1} \\
+ TOZ_k \left[ DPCF_{k-1} (1 + GO_{k-1}) \right]
\]

\[
\frac{\partial R_{of,ijk}}{\partial S_{w fij}^{n+1}} = TCOND2_{i,1j} (1 - \omega_{od2,i-1j+1}) TKO_{i,j+1} PTO_{i,j+1} \\
+ TOD2_{i,1j} \left[ DPCF_{i,j+1} (1 - GO_{i,j+1}) \right]
\]
\[
\frac{\partial R_{ofijk}}{\partial S_{wfi-1j-1}^{n+1}} = - TCOND_{i-1j-1} \omega_{odl,ij} TKO_{i-1j-1} PTO_{i-1j-1}
+ TOD_{i-1j-1} [ DPCF_{i-1j-1} (1 + GO_{i-1j-1}) ]
\]

\[
\frac{\partial R_{ofijk}}{\partial S_{wfi+1j-1}^{n+1}} = TCOND_{i+1j-1} (1 - \omega_{odl,i+1j-1}) TKO_{i+1j-1} PTO_{i+1j-1}
+ TOD_{i+1j-1} [ DPCF_{i+1j-1} (1 - GO_{i+1j-1}) ]
\]

\[
\frac{\partial R_{ofijk}}{\partial S_{wfi-1j+1}^{n+1}} = - TCOND_{i-1j+1} \omega_{odl,ij} TKO_{i-1j+1} PTO_{i-1j+1}
+ TOD_{i-1j+1} [ DPCF_{i-1j+1} (1 + GO_{i-1j+1}) ]
\]
\[
\frac{\partial R_{yk}}{\partial s_{yk}} = + TCONX_{i,1} \omega_{yk} \frac{TKO_{yk} PTO_{i,1} - TOX_{i,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 + GO_{yk} \right) \right]
\]
\[- TCONY_{1} \left( 1 - \omega_{yk} \right) \frac{TKO_{yk} PTO_{1,1} - TOY_{1,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ TCONY_{j,1} \omega_{yk} \frac{TKO_{yk} PTO_{j,1} - TOY_{j,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 + GO_{yk} \right) \right]
\]
\[- TCONY_{1} \left( 1 - \omega_{yk} \right) \frac{TKO_{yk} PTO_{1,1} - TOY_{1,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ TCONZ_{k,1} \omega_{yk} \frac{TKO_{yk} PTO_{k,1} - TOZ_{k,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 + GO_{yk} \right) \right]
\]
\[- TCONZ_{k} \left( 1 - \omega_{yk} \right) \frac{TKO_{yk} PTO_{k,1} - TOZ_{k}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ TCOND_{l,1} \omega_{yk} \frac{TKO_{yk} PTO_{l,1} - TOD_{l,1}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 + GO_{yk} \right) \right]
\]
\[- TCOND_{l} \left( 1 - \omega_{yk} \right) \frac{TKO_{yk} PTO_{l,1} - TOD_{l}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ TCOND_{l} \omega_{yk} \frac{TKO_{yk} PTO_{l,1} - TOD_{l}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 + GO_{yk} \right) \right]
\]
\[- TCOND_{l} \left( 1 - \omega_{yk} \right) \frac{TKO_{yk} PTO_{l,1} - TOD_{l}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ TCONM_{yk} \omega_{yk} \frac{TKO_{yk} PTO_{m,1} - TOM_{yk}}{\partial s_{yk}} \left[ DPCF_{yk} \left( 1 - GO_{yk} \right) \right]
\]
\[+ \left[ W_{j} \left( 1 - \beta \right) \frac{1}{\mu_{w}} \frac{\partial k_{wo}}{\partial s_{w}} \right]_{j} \frac{1}{B_{of}^{n-1}} \left( P_{bh}^{n-1} - P_{node}^{n-1} \right)
\]
\[+ \frac{\left( V_{R} \right)_{yk}}{5.6146 \Delta t} \left( \frac{\phi_{f}}{B_{of}^{n-1}} \right)_{jk}
\]

For the SPM, the matrix/fracture transfer term is replaced by:

\[+ \frac{\left( V_{R} \right)_{yk}}{5.6146} \left( \frac{1}{B_{of}^{n-1}} \right)_{jk} \sum_{N=1}^{NFTT} R_{N} \lambda_{N} e^{-\lambda_{N} T_{n}}\]
(3) $p_{wm}$ Derivative

$$\frac{\partial R_{of,ijk}}{\partial p_{wm,ijk}^{n+1}} = T_{om,ijk}^{n+1} \left[ 1 - \frac{\partial}{\partial p_{wm,ijk}^{n-1}} \gamma_{om,ijk}^{n+1} D_{m,ijk} \right]$$

$$- \frac{\partial}{\partial p_{wm,ijk}^{n+1}} T_{om,ijk}^{n+1} \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk}$$

Where:

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} T_{om,ijk}^{n+1} = TCONM_{ijk} \omega_{om,ijk} c_o \left( \frac{k_r}{\mu B} \right)_{om,ijk}$$

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{om,ijk}^{n+1} = \frac{c_o \rho_{om,ijk}}{144}$$

Substituting both terms into the above equation and rearranging yields:

$$\frac{\partial R_{of,ijk}}{\partial p_{wm,ijk}^{n+1}} = T_{om,ijk}^{n+1} \left[ 1 - \frac{c_o \rho_{om,ijk}^{n+1}}{144} D_{m,ijk} \right]$$

$$- TCONM_{ijk} \omega_{om,ijk} c_o \left( \frac{k_r}{\mu B} \right)_{om,ijk}$$

$$\cdot \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk}$$

$$= TOM_{ijk} (1 - GOM) - TCONM_{ijk} \omega_{om,ijk} TBO_{ijk} PTOM_{ijk}$$
(4) $S_wm$ Derivative

$$\frac{\partial R_{of,ijk}}{\partial S_{wm,ijk}^{n+1}} = (T)_{om,ijk}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{m,ijk}^{n+1} \left( 1 - \frac{c_o \rho_{om,ijk}^{n+1}}{144} D_{m,ijk} \right) \right]$$

$$- \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk} \frac{\partial}{\partial S_{wm,ijk}^{n+1}} (T)_{om,ijk}^{n+1}$$

Where:

$$\frac{\partial}{\partial S_{wm,ijk}^{n+1}} T_{om,ijk}^{n+1} = TCONM_{ijk} \omega_{om,ijk} \left( \frac{1}{\mu B} \right)_{om,ijk}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{m,ijk}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{of,ijk}}{\partial S_{wm,ijk}^{n+1}} = - TCONM_{ijk} \omega_{om,ijk} \left( \frac{1}{\mu B} \right)_{om,ijk}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{m,ijk}^{n+1}$$

$$\cdot \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk}$$

$$+ T_{om,ijk}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{m,ijk}^{n+1} \left( 1 - \frac{c_o \rho_{om,ijk}^{n+1}}{144} D_m \right) \right]$$

$$= - TCONM_{ijk} \omega_{om,ijk} TKO_{yk} PTO_{yk} + TOM_{ijk} [DPCM_{ijk} (1 - GOM)]$$

(5) $p_{bh}$ Derivative

$$\frac{\partial R_{of,ijk}}{\partial p_{bh,ijk}^{n+1}} = \sum_{k=1}^{KK} \left[ WI (1 - \beta) \left( \frac{k_r}{\mu B}_{of} \right)_{ijk}^{n+1} \right] = \sum_{k=1}^{KK} WOP_{ijk}^{n+1}$$
D.3 Matrix/Fracture Water Equation

(1) $p_{wf}$ Derivative

\[
\frac{\partial R_{wm,ijk}}{\partial p_{wf,ijk}^{n+1}} = \frac{\partial}{\partial p_{wf,i,j,k}^{n+1}} T_{wm,ijk}^{n+1} \left[ p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f \right] - \left( p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m \right)_{ijk}
\]

\[
+ T_{wm,ijk}^{n+1} \left[ 1 - D_{f,i,j,k} \frac{\partial}{\partial p_{wf,i,j,k}^{n+1}} \gamma_{wf,i,j,k}^{n+1} \right]
\]

Where:

\[
\frac{\partial}{\partial p_{wf,i,j,k}^{n+1}} T_{wm,ijk}^{n+1} = TCONM_{ijk} (1 - \omega_{wm,ijk}) c_w \left( \frac{k_r}{\mu B} \right)_{wf,i,j,k}^{n+1}
\]

\[
\frac{\partial}{\partial p_{wf,i,j,k}^{n+1}} \gamma_{wf,i,j,k}^{n+1} = \frac{c_w p_{wf,i,j,k}^{n+1}}{144}
\]

Substituting into the above equation and rearranging yields:

\[
\frac{\partial R_{wm,ijk}}{\partial p_{wf,i,j,k}^{n+1}} = TCONM_{ijk} (1 - \omega_{wm,ijk}) c_w \left( \frac{k_r}{\mu B} \right)_{wf,i,j,k}^{n+1}
\]

\[
\cdot \left[ p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f \right] - \left( p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m \right)_{ijk}
\]

\[
+ T_{wm,ijk}^{n+1} \left[ 1 - \frac{c_w p_{wf,i,j,k}^{n+1}}{144} D_{f,i,j,k} \right]
\]
(2) $S_{wf}$ Derivative

$$\frac{\partial R_{wm,ijk}}{\partial S_{wf,ijk}^{n+1}} = \frac{\partial}{\partial S_{wf,ijk}^{n+1}} T_{wm,ijk}^{n+1} \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}$$

Where:

$$\frac{\partial}{\partial S_{wf,ijk}^{n+1}} T_{wm,ijk}^{n+1} = TCONM_{ijk} (1 - \omega_{wm,ijk}) \left( \frac{1}{\mu B}_{wf,ijk}^{n+1} \right) \left( \frac{\partial k_{rw}}{\partial S_w}_{f,ijk}^{n+1} \right)$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wm,ijk}}{\partial S_{wf,ijk}^{n+1}} = TCONM_{ijk} (1 - \omega_{wm,ijk}) \left( \frac{1}{\mu B}_{wf,ijk}^{n+1} \right) \left( \frac{\partial k_{rw}}{\partial S_w}_{f,ijk}^{n+1} \right)$$

$$\cdot \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}$$
(3) $p_{wm}$ Derivative

$$\frac{\partial R_{wm,ijk}}{\partial p_{wm,ijk}^{n+1}} = \frac{\partial}{\partial p_{wm,ijk}^{n+1}} T_{wm,ijk}^{n+1} \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}$$

$$- T_{wm,ijk}^{n+1} \left[ 1 - D_{m,ijk} \frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{wm,ijk}^{n+1} \right]$$

$$- \frac{(V_R)_{ijk}}{5.6146 \Delta t} \left[ \left( \phi \frac{S_w}{B_w} \right)^{n+1}_{m,ijk} (c_m + c_w) \right]$$

Where:

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} T_{wm,ijk}^{n+1} = TCONM_{ijk} \omega_{wm,ijk} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ijk}$$

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{wm,ijk}^{n+1} = \frac{c_w p_{wm,ijk}^{n+1}}{144}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wm,ijk}}{\partial p_{wm,ijk}^{n+1}} = TCONM_{ijk} \omega_{wm,ijk} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ijk}$$

$$\cdot \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ijk}$$

$$- T_{wm,ijk}^{n+1} \left[ 1 - \frac{c_w p_{wm,ijk}^{n+1}}{144} D_{m,ijk} \right] - \frac{(V_R)_{ijk}}{5.6146 \Delta t} \left[ \left( \phi \frac{S_w}{B_w} \right)^{n+1}_{m,ijk} (c_m + c_w) \right]$$
(4) $S_{wm}$ Derivative

$$\frac{\partial R_{wm,ijk}}{\partial S_{wm,ijk}^{n+1}} = \frac{\partial}{\partial S_{wm,ijk}^{n+1}} T_{wm,ijk}^{n+1} \left[ \left( p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f \right) - \left( p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m \right) \right]_{ijk}$$

$$- \frac{\left( V_R \right)_{ijk}}{5.6146 \Delta t} \left( \frac{\phi}{B_w} \right)_{m,ijk}^{n+1}$$

Where:

$$\frac{\partial}{\partial S_{wm,ijk}^{n+1}} T_{wm,ijk}^{n+1} = TCONM_{ijk} \omega_{wm,ijk} \left( \frac{1}{\mu B} \right)_{wm}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ijk}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wm,ijk}}{\partial S_{wm,ijk}^{n+1}} = TCONM_{ijk} \omega_{wm,ijk} \left( \frac{1}{\mu B} \right)_{wm}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ijk}^{n+1}$$

$$\cdot \left[ \left( p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f \right) - \left( p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m \right) \right]_{ijk}$$

$$- \frac{\left( V_R \right)_{ijk}}{5.6146 \Delta t} \left( \frac{\phi}{B_w} \right)_{m,ijk}^{n+1}$$
D.4 Matrix/Fracture Oil Equation

(1) $p_{wt}$ Derivative

$$
\frac{\partial R_{om,ijk}}{\partial p_{n+1_{wf,ijk}}} = \frac{\partial}{\partial p_{n+1_{wf,ijk}}} T_{om,ijk}^{n+1} \left[ \left( p_{of}^{n+1} - \gamma_{of}^{n+1} \right) D_f \right] - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right)_{ijk}
$$

$$
+ T_{om,ijk}^{n+1} \left[ 1 - D_{f,ijk} \frac{\partial}{\partial p_{n+1_{wf,ijk}}} \gamma_{of,ijk}^{n+1} \right]
$$

Where:

$$
\frac{\partial}{\partial p_{n+1_{wf,ijk}}} T_{om,ijk}^{n+1} = TCONM_{ijk} \left( 1 - \omega_{om,ijk} \right) c_o \left( \frac{k_r}{\mu B_{of,ijk}} \right)^{n+1}
$$

$$
\frac{\partial}{\partial p_{n+1_{wf,ijk}}} \gamma_{of,ijk}^{n+1} = \frac{c_o \rho_{of,ijk}^{n+1}}{144}
$$

Substituting into the above equation and rearranging yields:

$$
\frac{\partial R_{om,ijk}}{\partial p_{n+1_{wf,ijk}}} = TCONM_{ijk} \left( 1 - \omega_{om,ijk} \right) c_o \left( \frac{k_r}{\mu B_{of,ijk}} \right)^{n+1}
$$

$$
\times \left[ \left( p_{of}^{n+1} - \gamma_{of}^{n+1} \right) D_f \right] - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right)_{ijk}
$$

$$
+ T_{om,ijk}^{n+1} \left[ 1 - \frac{c_o \rho_{of,ijk}^{n+1}}{144} D_{f,ijk} \right]
$$
(2) \( S_{w_f} \) Derivative

\[
\frac{\partial R_{om,ijk}}{\partial S_{w_f,ijk}^{n+1}} = \frac{\partial}{\partial S_{w_f,ijk}^{n+1}} T_{om,ijk}^{n+1} \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_j) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk}
\]

\[
+ T_{om,ijk}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{f,ijk}^{n+1} \left( 1 - \frac{c_o}{144} p_{of,ijk}^{n+1} D_{f,ijk} \right) \right]
\]

Where:

\[
\frac{\partial}{\partial S_{w_f,ijk}^{n+1}} T_{om,ijk}^{n+1} = TCONM_{ijk} (1 - \omega_{om,ijk}) \left( \frac{1}{\mu B} \right)^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{f,ijk}^{n+1}
\]

Substituting into the above equation and rearranging yields:

\[
\frac{\partial R_{om,ijk}}{\partial S_{w_f,ijk}^{n+1}} = TCONM_{ijk} (1 - \omega_{om,ijk}) \left( \frac{1}{\mu B} \right)^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{f,ijk}^{n+1}
\]

\[
\cdot \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_j) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ijk}
\]

\[
+ T_{om,ijk}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{f,ijk}^{n+1} \left( 1 - \frac{c_o}{144} p_{of,ijk}^{n+1} D_{f,ijk} \right) \right]
\]
(3) $p_{wm}$ Derivative

$$\frac{\partial R_{om,ijk}}{\partial p_{wm,ijk}^{n+1}} = \frac{\partial}{\partial p_{wm,ijk}^{n+1}} \left[ T_{om,ijk}^{n+1} \left[ \left( p_{of}^{n+1} - \gamma_{of}^{n+1} D_f \right) - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right) \right]_{ijk} \right]$$

$$- T_{om,ijk}^{n+1} \left[ 1 - D_{m,ijk} \frac{\partial}{\partial p_{om,ijk}^{n+1}} \gamma_{om,ijk}^{n+1} \right]$$

$$- \frac{(V_R)_{ijk}}{5.6146 \Delta t} \left[ \left( \frac{\phi S_o}{B_o} \right)_{m,ijk}^{n+1} \left( c_m + c_o \right) \right]$$

Where:

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} T_{om,ijk}^{n+1} = TCONM_{ijk} \omega_{om,ijk} c_o \left( \frac{k_r}{\mu B} \right)_{om,ijk}^{n+1}$$

$$\frac{\partial}{\partial p_{wm,ijk}^{n+1}} \gamma_{om,ijk}^{n+1} = \frac{c_o \rho_{om,ijk}^{n+1}}{144}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{om,ijk}}{\partial p_{wm,ijk}^{n+1}} = TCONM_{ijk} \omega_{om,ijk} c_o \left( \frac{k_r}{\mu B} \right)_{om,ijk}^{n+1}$$

$$\left[ \left( p_{of}^{n+1} - \gamma_{of}^{n+1} D_f \right) - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right) \right]_{ijk}$$

$$- T_{om,ijk}^{n+1} \left[ 1 - \frac{c_o \rho_{om,ijk}^{n+1}}{144} D_{m,ijk} \right]$$

$$- \frac{(V_R)_{ijk}}{5.6146 \Delta t} \left[ \left( \frac{\phi S_o}{B_o} \right)_{m,ijk}^{n+1} \left( c_m + c_o \right) \right]$$
(4) $S_{wm}$ Derivative

$$\frac{\partial R_{om,ijk}}{\partial S_{wm,ijk}^{n+1}} = \frac{\partial}{\partial S_{wm,ijk}^{n+1}} \left[ \frac{T_{om,ijk}^{n+1} \left( p_{of}^{n+1} - \gamma_{of}^{n+1} D_f \right) - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right)_{ijk}}{1 - \frac{c_o}{144} \rho_{om,ijk}^{n+1} D_{m,ijk}} \right]$$

$$- \left( \frac{\partial \rho_{cwo}}{\partial s_{w}^{n+1}} \right)_{m,ijk} \left( 1 - \frac{c_o}{144} \rho_{om,ijk}^{n+1} D_{m,ijk} \right)$$

$$+ \frac{(V_R)_{ijk}}{5.6146\Delta t} \left( \frac{\phi}{B_o} \right)_{m,ijk}^{n+1}$$

Where:

$$\frac{\partial}{\partial S_{wm,ijk}^{n+1}} T_{om,ijk}^{n+1} = TCONM_{ijk} \omega_{om,ijk}^{n+1} \left( \frac{1}{\mu B} \right)_{om,ijk}^{n+1} \left( \frac{\partial k_{ro}}{\partial s_{w}^{n+1}} \right)_{m,ijk}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{om,ijk}}{\partial S_{wm,ijk}^{n+1}} = TCONM_{ijk} \omega_{om,ijk}^{n+1} \left( \frac{1}{\mu B} \right)_{om,ijk}^{n+1} \left( \frac{\partial k_{ro}}{\partial s_{w}^{n+1}} \right)_{m,ijk}$$

$$\cdot \left[ p_{of}^{n+1} - \gamma_{of}^{n+1} D_f \right] - \left( p_{om}^{n+1} - \gamma_{om}^{n+1} D_m \right)_{ijk}$$

$$- T_{om,ijk}^{n+1} \left( \frac{\partial p_{cwo}}{\partial s_{w}^{n+1}} \right)_{m,ijk}^{n+1} \left( 1 - \frac{c_o}{144} \rho_{om,ijk}^{n+1} D_{m,ijk} \right)$$

$$+ \frac{(V_R)_{ijk}}{5.6146\Delta t} \left( \frac{\phi}{B_o} \right)_{m,ijk}^{n+1}$$
D.5 Wellbore Equation

(1) $p_{\text{wt}}$ Derivative

\[
\frac{\partial R_{bh,ij}}{\partial p_{\text{wt},ijk}^{n+1}} = WP_{ijk}^{n+1} - \frac{\partial WP_{ijk}^{n+1}}{\partial p_{\text{wt},ijk}^{n+1}} \left( p_{bh}^{n+1} - P_{\text{node},ijk}^{n+1} \right)
\]

(2) $S_{\text{wt}}$ Derivative

\[
\frac{\partial R_{bh,ij}}{\partial S_{\text{wt},ijk}^{n+1}} = - \frac{\partial WP_{ijk}^{n+1}}{\partial S_{\text{wt},ijk}^{n+1}} \left( p_{bh}^{n+1} - P_{\text{node},ijk}^{n+1} \right)
\]

(3) $p_{bh}$ Derivative

\[
\frac{\partial R_{bh,ij}}{\partial p_{\text{bh},ijk}^{n+1}} = - \sum_{k=1}^{KK} WP_{ijk}^{n+1}
\]

Where:

\[
WP_{ijk}^{n+1} = WI \left\{ \left( \frac{k_{rw}}{\mu_w} + \beta \frac{k_{ro}}{\mu_o} \right)^{n-1} \frac{1}{B_{\text{w},ijk}^{n+1}} + (1 - \beta) \left( \frac{k_{ro}}{\mu_o B_o} \right)_{f}^{n+1} \right\}_{ijk}
\]

\[= WWP_{ijk}^{n+1} + WOP_{ijk}^{n+1} \]
\[
\frac{\partial WP_{ij}^{n+1}}{\partial \rho_{w,i,jk}^{n+1}} = WI \left\{ \left( \frac{k_{rw}}{\mu_w} + \beta \frac{k_{ro}}{\mu_o} \right)_f^{n+1} \left( \frac{c_w}{B_{wf}} \right)^{n+1} + (1 - \beta) c_o \left( \frac{k_{ro}}{\mu_o B_o^2} \right)_f^{n+1} \right\}_{ijk}
\]

\[
= WW_{ij}^{n+1} c_w + WOP_{ij}^{n+1} c_o
\]

\[
\frac{\partial WP_{ij}^{n+1}}{\partial S_{w,i,jk}^{n+1}} = WI \left\{ \left( \frac{\partial k_{rw}}{\partial S_w} \frac{1}{\mu_w} + \beta \frac{\partial k_{ro}}{\partial S_w} \frac{1}{\mu_o} \right)_f^{n+1} \frac{1}{B_{wf}^{n+1}} \right\}_{ijk}
\]

\[
+ (1 - \beta) \left( \frac{\partial k_{ro}}{\partial S_w} \frac{1}{\mu_o B_o} \right)_f^{n+1} \right\}_{ijk}
\]
APPENDIX E
DPM AND SPM JACOBIAN MATRIX ENTRIES
FOR THE CVFE DISCRETIZATION SCHEME

E.1 Fracture Water Equation

(1) \( p_{wf} \) Derivatives

\[
\frac{\partial R_{wf,ik}}{\partial p_{wf,j}^{n+1}} = (T)_{wf,ij}^{n+1} \left[ 1 - \frac{\partial}{\partial p_{wf,j}^{n+1}} \gamma_{wf,ij}^{n+1} (D_{fj} - D_{fi}) \right] + \frac{\partial}{\partial p_{wf,j}^{n+1}} (T)_{wf,ij}^{n+1} \left[ p_{wf,j}^{n+1} - p_{wf,i}^{n+1} \right] - \gamma_{wf,ij}^{n+1} (D_{fj} - D_{fi})
\]

Where:

\[
\frac{\partial}{\partial p_{wf,j}^{n+1}} (T)_{wf,ij}^{n+1} = TCON_i (1 - \omega_{wf,i}) c_w \left( \frac{k_r}{\mu B} \right)_{wf,j}^{n+1}
\]

\[
\frac{\partial}{\partial p_{wf,j}^{n+1}} \gamma_{wf,ij}^{n+1} = \frac{c_w p_{wf,j}^{n+1}}{288}
\]

Substituting both terms into the above equation yields:

\[
\frac{\partial R_{wf,ik}}{\partial p_{wf,j}^{n+1}} = (T)_{wf,ij}^{n+1} \left[ 1 - \frac{c_w p_{wf,j}^{n+1}}{288} (D_{fj} - D_{fi}) \right] + TCON_i (1 - \omega_{wf,i}) c_w \left( \frac{k_r}{\mu B} \right)_{wf,j}^{n+1} \left[ p_{wf,j}^{n+1} - p_{wf,i}^{n+1} \right] - \gamma_{wf,ij}^{n+1} (D_{fj} - D_{fi})
\]

\[
= TW_i (1 - GW_j) + TCON_i (1 - \omega_{wf,i}) TBW_j PTW_{ij}
\]

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{w,i,k}}{\partial p_{w,f,i,k+1}^{n+1}} = TWZ_{k+1} (1 - GW_{k+1}) + TCONZ_{k+1} (1 - \omega_{w,f,k+1}) TBW_{k+1} PTW_{k,k+1}
\]
\[
\frac{\partial R_{w,i,k}}{\partial p_{w,f,i,k-1}^{n+1}} = TWZ_k (1 + GW_{k-1}) - TCONZ_k \omega_{w,f,k} TBW_{k-1} PTW_{k,k-1}
\]
\[
\frac{\partial R_{w,i,k}}{\partial p_{w,f,i,k}^{n+1}} = \sum_{j \in \eta_i} - TW_i (1 + GW_i) + TCON_i \omega_{w,f,i} TBW_i PTW_{ij}
\]
\[\quad - TWZ_{k-1} (1 + GW_k) + TCONZ_{k-1} \omega_{w,f,k+1} TBW_k PTW_{k,k+1}\]
\[\quad - TWZ_k (1 - GW_k) - TCONZ_k (1 - \omega_{w,f,k}) TBW_k PTW_{k,k-1}\]
\[\quad - TWM_{ik} (1-GWM_{ik}) - TCONM_{ik} (1-\omega_{w,m,ik}) TBWM_{ik} PTWM_{ik}\]
\[\quad + WWP_{ik} \left[ c_w \left( p_{bh}^{n+1} - p_{node,ik}^{n+1} \right) - 1 \right] \]
\[\quad - \frac{(V_R)_{ik}}{5.6146 \Delta t} \left( \phi \frac{S_w}{B_w} \right)^{n+1} \left( c_f + c_w \right) \]

For the SPM, the matrix/fracture transfer term is replaced by:

\[
- \frac{(V_R)_{ik}}{5.6146} c_w \left( \frac{S_{w,f}}{B_w} \right)^{n+1} \sum_{N=1}^{NPTT} R_N \lambda_N e^{-\lambda_N \Delta t_N}
\]
(2) $S_{w,t}$ Derivatives

\[
\frac{\partial R_{w,f,j,k}}{\partial S_{w,f,j}^{n+1}} = \left[ \left( p_{w,f,j}^{n+1} - p_{w,f,i}^{n+1} \right) - \gamma_{w,f,j}^{n+1} \left( D_{f,j} - D_{f,i} \right) \right] \frac{\partial}{\partial S_{w,f,j}^{n+1}} (T)_{w,f,i}^{n+1}
\]

Where:

\[
\frac{\partial}{\partial S_{w,f,j}^{n+1}} (T)_{w,f,i}^{n+1} = TCON_i \left( 1 - \omega_{w,f,i} \right) \left( \frac{1}{\mu B} \right)_{w,f,j}^{n+1} \cdot \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{f,i}^{n+1}
\]

Substituting into the above equation and rearranging yields:

\[
\frac{\partial R_{w,f,j,k}}{\partial S_{w,f,j}^{n+1}} = TCON_i \left( 1 - \omega_{w,f,i} \right) \left( \frac{1}{\mu B} \right)_{w,f,j}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{f,i}^{n+1}
\]

\[
\cdot \left[ \left( p_{w,f,j}^{n+1} - p_{w,f,i}^{n+1} \right) - \gamma_{w,f,j}^{n+1} \left( D_{f,j} - D_{f,i} \right) \right]
\]

\[= TCON_i \left( 1 - \omega_{w,f,i} \right) TKW_{j} PTW_{j}
\]

Following the same procedure outlined above, we obtain:

\[
\frac{\partial R_{w,f,j,k}}{\partial S_{w,f,k-1}^{n+1}} = TCONZ_{k+1} \left( 1 - \omega_{w,f,k+1} \right) TKW_{k} PTW_{k,k+1}
\]

\[
\frac{\partial R_{w,f,k}}{\partial S_{w,f,k-1}^{n+1}} = - TCONZ_{k} \omega_{w,f,k} TKW_{k-1} PTW_{k,k-1}
\]
\[
\frac{\partial R_{wf,ik}}{\partial S_{wf,ik}^{n+1}} = \sum_{j \in \eta_i} TCON_i \omega_{wf,i} TKW_i PTW_{ij} + TCONZ_{k+1} \omega_{wf,k+1} TKW_k PTW_{k,k+1}
- TCONZ_k (1 - \omega_{wf,k}) TKW_k PTW_{k,k-1}
- TCONM_{ik} (1 - \omega_{wm,ik}) TKWM_{ik} PTWM_{ik}
+ \left[ W1 \left( \frac{1}{\mu_w} \frac{\partial k_{rw}}{\partial S_w} + \frac{1}{\mu_o} \frac{\partial k_{ro}}{\partial S_w} \right) \frac{n-1}{B_{wf}^{n+1}} \left( p_{bh}^{n+1} - p_{node}^{n+1} \right) \right]_ik
- \frac{(V_{R})_{ik}}{5.6146\Delta t} \left( \frac{\phi}{B_{w/f,ik}} \right)^{n+1}
\]

For the SPM, the matrix/fracture transfer term is replaced by:

\[
- \frac{(V_{R})_{ik}}{5.6146} \left( \frac{1}{B_{wf}^{n+1}} \right)_{ik} \sum_{N=1}^{N_{FIT}} R_N \lambda_N e^{-\lambda_N \Delta t_n}
\]
(3) $p_{wm}$ Derivative

$$\frac{\partial R_{wf,ik}}{\partial p_{wm,ik}^{n+1}} = (T)_{wm,ik}^{n+1} \left[ 1 - \frac{\partial}{\partial p_{wm,ik}^{n+1}} \gamma_{wm,ik}^{n+1} D_{m,ik} \right]$$

$$- \frac{\partial}{\partial p_{wm,ik}^{n+1}} (T)_{wm,ik}^{n+1} \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ik}$$

Where:

$$\frac{\partial}{\partial p_{wm,ik}^{n+1}} (T)_{wm,ik}^{n+1} = TCONM_{ik} \omega_{wm,ik} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ik}$$

$$\frac{\partial}{\partial p_{wm,ik}^{n+1}} \gamma_{wm,ik}^{n+1} = \frac{c_w p_{wm,ik}^{n+1}}{144}$$

Substituting both terms into the above equation and rearranging yields:

$$\frac{\partial R_{wf,ik}}{\partial p_{wm,ik}^{n+1}} = (T)_{wm,ik}^{n+1} \left[ 1 - \frac{c_w p_{wm,ik}^{n+1}}{144} D_{m,ik} \right]$$

$$- TCONM_{ik} \omega_{wm,ik} c_w \left( \frac{k_r}{\mu B} \right)^{n+1}_{wm,ik}$$

$$\cdot \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ik}$$

$$= TWM_{ik} (1 - GWM) - TCONM_{ik} \omega_{wm,ik} TBWM_{ik} PTWM_{ik}$$
(4) $S_{wm}$ Derivative

$$\frac{\partial R_{wf,ik}}{\partial S_{wm,ik}^{n+1}} = - \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right] \frac{\partial}{\partial S_{wm,ik}^{n+1}} (T_{wm,ik}^{n+1})$$

Where:

$$\frac{\partial}{\partial S_{wm,ik}^{n+1}} (T_{wm,ik}^{n+1}) = TCONM_{ik} \omega_{wm,ik} \left( \frac{1}{\mu B} \right)_{wm,ik}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ik}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{wf,ik}}{\partial S_{wm,ik}^{n+1}} = - TCONM_{ik} \omega_{wm,ik} \left( \frac{1}{\mu B} \right)_{wm,ik}^{n+1} \left( \frac{\partial k_{rw}}{\partial S_w} \right)_{m,ik}^{n+1}$$

$$\cdot \left[ (p_{wf}^{n+1} - \gamma_{wf}^{n+1} D_f) - (p_{wm}^{n+1} - \gamma_{wm}^{n+1} D_m) \right]_{ik}$$

$$= - TCONM_{ik} \omega_{wm,ik} TKW_{ik} PTW_{ik}$$

(5) $p_{bh}$ Derivative

$$\frac{\partial R_{wf,ik}}{\partial P_{bh,i}^{n+1}} = \sum_{k=1}^{KK} \left[ WI \left( \frac{k_{rw}}{\mu_w} + \beta \frac{k_{ra}}{\mu_o} \right)_{ik}^{n+1} \frac{1}{B_{wf}^{n+1}_{ik}} \right] = \sum_{k=1}^{KK} WW_{ik}^{n+1}$$
E.2 Fracture Oil Equation

(1) \( p_{w_f} \) Derivatives

\[
\frac{\partial R_{o_{f_{ijk}}}^{n+1}}{\partial p_{wf_j}^{n+1}} = \left(T_{o_{f_{ij}}}^{n+1}\right) 
\left[ 1 - \frac{\partial}{\partial p_{wf_j}^{n+1}} \gamma_{o_{f_{i}}}^{n+1} (D_{f_{j}} - D_{f_{i}}) \right] 
+ \frac{\partial}{\partial p_{wf_j}^{n+1}} \left( T_{o_{f_{ij}}}^{n+1} \left[ p_{o_{f_{j}}}^{n+1} - p_{o_{f_{i}}}^{n+1} \right] - \gamma_{o_{f_{i}}}^{n+1} (D_{f_{j}} - D_{f_{i}}) \right)
\]

Where:

\[
\frac{\partial}{\partial p_{wf_j}^{n+1}} (T_{o_{f_{ij}}}^{n+1}) = TCON_i \left(1 - \omega_{o_{f_{j}}}\right) c_o \left( \frac{k_r}{\mu B} \right)_{o_{f_{j}}}^{n+1}
\]

\[
\frac{\partial}{\partial p_{wf_j}^{n+1}} \gamma_{o_{f_{i}}}^{n+1} = \frac{c_o \ p_{o_{f_{j}}}^{n+1}}{288}
\]

Substituting both terms into the above equation yields:

\[
\frac{\partial R_{o_{f_{ijk}}}^{n+1}}{\partial p_{wf_j}^{n+1}} = \left(T_{o_{f_{ij}}}^{n+1}\right) 
\left[ 1 - \frac{c_o \ p_{o_{f_{j}}}^{n+1}}{288} (D_{f_{j}} - D_{f_{i}}) \right] + TCON_i \left(1 - \omega_{o_{f_{j}}}\right)
\]

\[
c_o \left( \frac{k_r}{\mu B} \right)_{o_{f_{j}}}^{n+1} \left[ p_{o_{f_{j}}}^{n+1} - p_{o_{f_{i}}}^{n+1} \right] - \gamma_{o_{f_{i}}}^{n+1} (D_{f_{j}} - D_{f_{i}}) \]

\[
= T0_i \left(1 - GO_j \right) + TCON_i \left(1 - \omega_{o_{f_{j}}}\right) TBO_j \ PTO_{ij}
\]

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{ofik}}{\partial p_{of,ik}^{n+1}} = TOZ_{k+1} (1 - GO_{k+1}) + TCONZ_{k+1} (1 - \omega_{of,z,k+1}) TBO_{k+1} PTO_{k+1}
\]

\[
\frac{\partial R_{ofik}}{\partial p_{of,ik}^{n+1}} = TOZ_k (1 + GO_{k-1}) - TCONZ_k \omega_{of,k} TBO_{k-1} PTO_{k,k-1}
\]

\[
\frac{\partial R_{ofik}}{\partial p_{of,ik}^{n+1}} = \sum_{j \in \eta_i} - TO_i (1 + GO_i) + TCON_i \omega_{of,i} TBO_i PTO_{ij}
\]

\[
- TOZ_{k+1} (1 + GO_k) + TCONZ_{k+1} \omega_{of,z,k} TBO_{k} PTO_{k,k+1}
\]

\[
- TOZ_k (1 - GO_k) - TCONZ_k (1 - \omega_{of,k}) TBO_{k} PTO_{k,k-1}
\]

\[
- TOM_{ik} (1 - GOM_{ik}) - TCOM_{ik} (1 - \omega_{om,ik}) TBOM_{ik} PTOM_{ik}
\]

\[
+ WOP_{ik} \left[ c_o \left( p_{bh}^{n+1} - p_{node}^{n+1} \right)_{ik} - 1 \right]
\]

\[
- \frac{(V_R)_{ik}}{5.6146 \Delta t} \left[ \phi \frac{S_o}{B_o} \frac{n+1}{j_{ik}} \left( c_f + c_o \right) \right]
\]

For the SPM, the matrix/fracture transfer term is replaced by:

\[
- \frac{(V_R)_{ik}}{5.6146} c_o \left( \frac{S_o}{B_o} \right)_{ik}^{n+1} \sum_{N=1}^{NFIT} R_N \lambda_N e^{-\lambda_N \Delta t_n}
\]
(2) $S_w$ Derivatives

$$\frac{\partial R_{of,jk}}{\partial S_{w_{ij}}^{n+1}} = (T_{of,j})^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{fj} \right. ^{n+1} \left( 1 - \frac{c_0 P_{of,j}^{n+1}}{288} (D_{fj} - D_{fi}) \right) \right]$$

$$+ \left[ P_{of,j}^{n+1} - P_{of,i}^{n+1} \right] - \gamma_{of,j}^{n+1} (D_{fj} - D_{fi}) \left( \frac{\partial k_{ro}}{\partial \omega_{fj}} \right)_{of,j}^{n+1} \frac{\partial}{\partial S_{w_{ij}}^{n+1}} (T_{of,j})^{n+1}$$

Where:

$$\frac{\partial}{\partial S_{w_{ij}}^{n+1}} (T_{of,j})^{n+1} = TCON_i (1 - \omega_{of,i}) \left( \frac{1}{\mu B} \right)_{of,i}^{n+1} \left( \frac{\partial k_{ro}}{\partial B} \right)_{of,i}^{n+1}$$

Substituting into the above equation and rearranging yields:

$$\frac{\partial R_{of,jk}}{\partial S_{w_{ij}}^{n+1}} = TCON_i (1 - \omega_{of,i}) \left( \frac{1}{\mu B} \right)_{of,i}^{n+1} \left( \frac{\partial k_{ro}}{\partial B} \right)_{of,i}^{n+1}$$

$$\cdot \left[ P_{of,j}^{n+1} - P_{of,i}^{n+1} \right] - \gamma_{of,j}^{n+1} (D_{fj} - D_{fi})$$

$$+ (T_{of,j})^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)_{fj} \right. ^{n+1} \left( 1 - \frac{c_0 P_{of,j}^{n+1}}{288} (D_{fj} - D_{fi}) \right) \right]$$

$$= TCON_i (1 - \omega_{of,i}) TK_{of,j} PTO_{ij} + TO_i \left[ DPCF_{ij} (1 - GO_j) \right]$$

Following the same procedure outlined above, we obtain:
\[
\frac{\partial R_{of,ik}}{\partial S_{w,f,k+1}^{n+1}} = TCONZ_{k+1} (1 - \omega_{of,k-1}) TKO_{k+1} PTO_{k,k-1} + TOZ_{k+1} \left[ DPCF_{k-1} (1 - GO_{k-1}) \right]
\]

\[
\frac{\partial R_{of,ik}}{\partial S_{w,f,k-1}^{n+1}} = -TCONZ_k \omega_{of,k} TKO_{k-1} PTO_{k,k-1} + TOZ_k \left[ DPCF_{k-1} (1 + GO_{k-1}) \right]
\]

\[
\frac{\partial R_{of,ik}}{\partial S_{w,f,ik}^{n+1}} = \sum_{j \in \eta_i} TCON_j \omega_{of,i} TKO_{i} PTO_{ij} - TO_i \left[ DPCF_i (1 + GO_i) \right]
\]

\[
+ TCONZ_{k+1} \omega_{of,k+1} TKO_k PTO_{k,k+1} - TOZ_{k+1} \left[ DPCF_k (1 + GO_k) \right]
\]

\[
- TCONZ_k (1 - \omega_{of,k}) TKO_k PTO_{k,k-1} - TOZ_k \left[ DPCF_k (1 - GO_k) \right]
\]

\[
- TCONM_{ik} (1 - \omega_{om,ik}) TKO_{ik} PTOM_{ik} - TOM_{ik} \left[ DPCF_{ik} (1 - GO_{ik}) \right]
\]

\[
+ \left[ WI (1 - \beta) \left( \frac{1}{\mu_o B_o} \right)_f \left( \frac{\partial k_{ro}}{\partial S_w} \right)_f \left( p_{bh}^{n+1} - p_{node}^{n+1} \right) \right]_{ik}
\]

\[
+ \frac{(V_R)_{ik}}{5.6146 \Delta t} \left( \frac{\Phi}{B_o} \right)_{f,ik}^{n+1}
\]

For the SPM, the matrix/fracture transfer function is replaced by:

\[
+ \frac{(V_R)_{ik}}{5.6146} \left( \frac{1}{B_{of}^{n+1}} \right)_{ik}^{N_{FIT}} R_N \lambda_N e^{-\lambda_N \Delta t_n}
\]
(3) $p_{wm}$ Derivative

$$\frac{\partial R_{of,i,k}}{\partial p_{wm,i,k}^{n+1}} = (T_{om,i,k})^{n+1} \left[ 1 - \frac{\partial}{\partial p_{wm,i,k}^{n+1}} \gamma_{om,i,k}^{n+1} D_{m,i,k} \right]$$

$$- \frac{\partial}{\partial p_{wm,i,k}^{n+1}} (T_{om,i,k}^{n+1} \left[ p_{of}^{n+1} - \gamma_{of}^{n+1} D_f \right] - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m))_{i,k}$$

Where:

$$\frac{\partial}{\partial p_{wm,i,k}^{n+1}} (T_{om,i,k}^{n+1} = TCONM_{ik} \omega_{om,i,k} c_o \left( \frac{k_r}{\mu B} \right)^{n+1})$$

$$\frac{\partial}{\partial p_{wm,i,k}^{n+1}} \gamma_{om,i,k}^{n+1} = \frac{c_o p_{om,i,k}^{n+1}}{144}$$

Substituting both terms into the above equation and rearranging yields:

$$\frac{\partial R_{of,i,k}}{\partial p_{wm,i,k}^{n+1}} = (T_{om,i,k})^{n+1} \left[ 1 - \frac{c_o p_{om,i,k}^{n+1}}{144} D_{m,i,k} \right]$$

$$- TCONM_{ik} \omega_{om,i,k} c_o \left( \frac{k_r}{\mu B} \right)^{n+1}$$

$$\cdot \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{i,k}$$

$$= TOM_{ik} (1 - GOM) - TCONM_{ik} \omega_{om,i,k} TBOM_{ik} PTOM_{ik}$$
(4) \( S_{wm} \) Derivative

\[
\frac{\partial R_{of,ik}}{\partial S_{wm,ik}^{n+1}} = (T)_{om,ik}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)^{n+1}_{m,ik} \left( 1 - \frac{c_o \rho_{om,ik}^{n+1}}{144} D_{m,ik} \right) \right] \\
- \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ik} \frac{\partial}{\partial S_{wm,ik}^{n+1}} (T)_{om,ik}^{n+1}
\]

Where:

\[
\frac{\partial}{\partial S_{wm,ik}^{n+1}} T_{om,ik}^{n+1} = TCONM_{ik} \omega_{om,ik} \left( \frac{1}{\mu B} \right)_{om,ik}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{m,ik}^{n+1}
\]

Substituting into the above equation and rearranging yields:

\[
\frac{\partial R_{of,ik}}{\partial S_{wm,ik}^{n+1}} = - TCONM_{ik} \omega_{om,ik} \left( \frac{1}{\mu B} \right)_{om,ik}^{n+1} \left( \frac{\partial k_{ro}}{\partial S_w} \right)_{m,ik}^{n+1} \\
\cdot \left[ (p_{of}^{n+1} - \gamma_{of}^{n+1} D_f) - (p_{om}^{n+1} - \gamma_{om}^{n+1} D_m) \right]_{ik} \\
+ T_{om,ik}^{n+1} \left[ \left( \frac{\partial P_{cwo}}{\partial S_w} \right)^{n+1}_{m,ik} \left( 1 - \frac{c_o \rho_{om,ik}^{n+1}}{144} D_m \right) \right]_{ik}
\]

\[
= - TCONM_{ik} \omega_{om,ik} TKO_{ik} PTO_{ik} + TOM_{ik} [DPCM_{ik} (1 - GOM)]
\]

(5) \( p_{bh} \) Derivative

\[
\frac{\partial R_{of,ik}}{\partial p_{bh,i}^{n+1}} = \sum_{k=1}^{KK} \left[ WI (1 - \beta) \right]_{ik}^{n+1} \left( \frac{k_r}{\mu B} \right)_{of,ik}^{n+1} = \sum_{k=1}^{KK} WOP_{ik}^{n+1}
\]