A CLASS OF EFFICIENT ALGORITHMS
FOR STOCHASTIC SEISMIC GROUND MOTIONS

by

Sarah A. Verros
A thesis submitted to the Faculty and the Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science (Applied Mathematics and Statistics).

Golden, Colorado
Date ____________________

Signed: ____________________
Sarah A. Verros

Signed: ____________________
Dr. Mahadevan Ganesh
Thesis Advisor

Signed: ____________________
Dr. David J. Wald
Thesis Co-Advisor

Golden, Colorado
Date ____________________

Signed: ____________________
Dr. Willy Hereman
Professor and Head
Department of Applied Mathematics and Statistics
ABSTRACT

Modeling the spatial correlation of ground motion residuals, caused by coherent contributions from source, path, and site, can provide valuable loss and hazard information, as well as a more realistic picture of ground motion intensities. The USGS computer model, ShakeMap, utilizes a deterministic approach to simulate median ground motions based on observed seismic data. ShakeMap based simulations are used to estimate fatalities and economic losses after a seismic event. Incorporating the spatial correlation of ground motion residuals has been shown to improve seismic loss estimation. The method of Park et al. (2007) has been investigated for computing spatially correlated random fields of residuals. However, for large scale ShakeMap models, computational requirements of the method by Park et al. (2007) are prohibitive. In this thesis, for our application-specific seismic ground motion problem, we develop and implement three new computationally efficient methods to model spatially correlated random field of residuals, in conjunction with ShakeMap.

First, we develop a memory efficient algorithm to improve the approach proposed by Park et al. (2007). This new, multilevel parallel algorithm is based on decay properties of an associated ground motion correlation function. The first approach is dependent on input grids and the stochastic dimension is induced by the grid size. In the second method, we seek to reduce the dimensionality associated with the computation through global Karhunen Loève (KL) expansions for random fields on the sphere. In the third method, we use a localized version of the KL representation using needlet approximations. We demonstrate the three approaches using extensive simulations.
# TABLE OF CONTENTS

ABSTRACT ................................................................................................................................................... iii

LIST OF FIGURES........................................................................................................................................ vii

LIST OF TABLES............................................................................................................................................ xi

LIST OF ABBREVIATIONS .......................................................................................................................... xii

ACKNOWLEDGMENTS .................................................................................................................................. xiii

CHAPTER 1 INTRODUCTION...................................................................................................................... 1

CHAPTER 2 SPATIAL VARIABILITY IN EARTHQUAKE GROUND MOTIONS ...... 5

2.1 Ground Motion Estimation.................................................................................................................. 5

2.1.1 A Brief Introduction to ShakeMap .............................................................................................. 7

2.2 Impact of Spatial Variability on Seismic Hazard Analysis ............................................................... 9

2.3 Initial Investigations into Spatial Variability .................................................................................... 12

2.4 Modeling Spatial Correlation of Residuals....................................................................................... 14

2.4.1 Spatial Correlation of Intraevent residuals ................................................................................ 15

2.4.2 Comparing correlation models ..................................................................................................... 17

2.5 Inconsistencies in Modeling Variability .......................................................................................... 24

2.6 Additional Aspects of Spatial Variability ......................................................................................... 27

2.6.1 Spatial Cross-Correlation ........................................................................................................... 27

2.7 Spatial Correlation of Other Intensity Measures .............................................................................. 36

2.8 Furthering the Findings ...................................................................................................................... 37

2.8.1 Topographic Site Amplification ................................................................................................. 38
4.4.2 KL Expansion for Various Correlation Ranges ........................................ 98
4.4.3 A Modified Angular Power Spectrum ................................................. 107
4.4.4 KL Expansion for Modified Angular Power Spectrum ......................... 108
4.4.5 Computing Losses with the KL Expansion ......................................... 115

4.5 Dimension Reduction Results .................................................................. 117

CHAPTER 5 NEEDLET APPROXIMATIONS FOR SPHERICAL RANDOM
FIELDS ........................................................................................................... 119

5.1 Mathematical Background ....................................................................... 119
  5.1.1 Preliminary Background ................................................................. 119
  5.1.2 Spherical Needlets ......................................................................... 122
  5.1.3 Fully Discrete Needlet Approximation ............................................. 125
  5.1.4 Needlet Decomposition of Random Fields ....................................... 126

5.2 Numerical Examples ............................................................................... 131
  5.2.1 Needlet Approximation of a Localized Deterministic Function .......... 135
  5.2.2 Needlet Approximation of Isotropic Random Fields ......................... 138

CHAPTER 6 CONCLUSION ............................................................................. 142

REFERENCES CITED .................................................................................... 145
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>ShakeMap estimated peak ground acceleration values for the Northridge earthquake (1994), simulated unconditionally (left) and conditionally (right) upon station data. .................................................................2</td>
</tr>
<tr>
<td>2.1</td>
<td>Hazard curves for PGA generated with Boore et al. Figure after Bommer &amp; Abrahamson. ...............................................................10</td>
</tr>
<tr>
<td>3.1</td>
<td>One instance of the generated correlation matrix for the Loma Prieta earthquake (a,c,e) (1989) with 0 seismic stations and the Northridge earthquake (b,d,f) (1994) conditioned on 185 stations. These results were generated using the SCS approach with Jayaram &amp; Baker with radius 45km. ........................................................................50</td>
</tr>
<tr>
<td>3.2</td>
<td>Point by point differences in the correlation matrices of the conditional simulation method versus the SCS method with various radii. The correlation matrices were computed for the Northridge event with no included observation data. ........................................................................52</td>
</tr>
<tr>
<td>3.3</td>
<td>CPU times and speedup ...............................................................................................................................56</td>
</tr>
<tr>
<td>3.4</td>
<td>Efficiency and Karp-Flatt Metric ..............................................................................................................57</td>
</tr>
<tr>
<td>3.5</td>
<td>CPU time for full method versus SCS method for $P = 1, 2, 4$ cores and for various number of realizations. Shakemap used was Northridge with 33,366 grid points. ........................................................................................................59</td>
</tr>
<tr>
<td>3.6</td>
<td>Convergence of the sum of spatially correlated fields to zero for increasing realizations. ..............................................................................................................60</td>
</tr>
<tr>
<td>3.7</td>
<td>Economic loss distributions (in billions) for Northridge without (left) and with (right) station data. .............................................................................................................................63</td>
</tr>
<tr>
<td>3.8</td>
<td>ShakeMaps generated for the San Diego scenario event with directivity included in sub-figures (c-f). The epicenter is shown on the map with a black diamond. ........................................................................................................64</td>
</tr>
<tr>
<td>3.9</td>
<td>ShakeMaps generated for the San Diego scenario event with directivity included in subfigures (c-f). The epicenter is shown on the map with a black diamond. ........................................................................................................65</td>
</tr>
</tbody>
</table>
Figure 3.10  ShakeMaps generated for the Loma Prieta event using a fixed area and variable grid spacing. ...................................................... 67

Figure 3.11  Estimated economic losses corresponding to 1000 realizations of spatially variable ShakeMaps with variable grid spacing. ....................... 68

Figure 3.12  The steps to computing $n$ realizations of spatially variable ShakeMaps using $P$ cores. ................................................................. 69

Figure 4.1  An illustration of the decay of the angular power spectrum specified in Section 4.3, given by $A_l = (l + 1)^{-\alpha}$. ............................................. 88

Figure 4.2  A sample GRF for each angular power spectrum specified in Section 4.3 with $\alpha = 3, 5,$ and $9$. Each is computed using a $\kappa = 100$ term truncation. .89

Figure 4.3  Average absolute error of a $\kappa$-term KL approximation to a $2^7$-term reference solution using various angular power spectra with 100 realizations. ............................................................. 91

Figure 4.4  Absolute error of a $\kappa$-term KL approximation to a $2^7$-term reference solution using various angular power spectra for a single realization........... 92

Figure 4.5  Angular power spectrum derived from a correlation function for various correlation lengths. ........................................................................... 97

Figure 4.6  GRFs computed with the KL expansion with covariance-derived angular power spectrum using various $b$ values with a $\kappa$-term truncation .. 100

Figure 4.7  GRFs computed with the KL expansion with covariance-derived angular power spectrum using various $b$ values with a $\kappa$-term truncation .. 101

Figure 4.8  Relative truncation errors computed a $\kappa$-term KL expansion for various $b$. ......................................................................................... 103

Figure 4.9  Relative truncation error in $\kappa_{n-1}$-term approximation to a $\kappa_n$-term approximation for various $b$. ......................................................... 104

Figure 4.10  A comparison of the truncation errors between the KL expansions obtained with the angular power spectra derived using the algebraic decay with $\alpha = 3$ and those derived using the covariance function with various $b$. ................................................................. 106
Figure 4.11 A comparison of the random fields generated with the method introduced in Chapter 3 and the KL expansion with the angular power spectrum derived using the covariance function. Both figures cover the same area in the LA basin and the same color scheme was used in both figures (ranging from approximately -3.14 to 3.14). The random field was generated for the ShakeMap of the Northridge (1994) earthquake. ..... 107

Figure 4.12 Modified angular power spectrum plotted for $b = 8.5$ and $b = 128$ with various $\alpha$. ................................................................. 108

Figure 4.13 GRFs generated using $b = 8.5$ and $b = 128$ with the modified angular power spectrum with $\alpha = 2.0, 2.5$ and 3.0 with a $\kappa$-term truncation .......... 110

Figure 4.14 Various truncation errors computed for the KL expansion computed with the modified angular power spectrum. .................................................... 112

Figure 4.15 Relative truncation error in KL expansions using angular power spectrum $B_l$ for $b = 8.5, 128$ and $\alpha = 2, 2.5, 3$ and for angular power spectrum $A_l$ with $\alpha = 3$. ................................................ 113

Figure 4.16 A comparison of the random fields generated with the method introduced in Chapter 3 and the KL expansion with the modified angular power spectrum derived using the covariance function. Both figures cover the same area in the LA basin and the same color scheme was used in both figures (ranging from approximately -3.14 to 3.14). The random field was generated for the ShakeMap of the Northridge (1994) earthquake. ................................................................. 114

Figure 4.17 Estimated economic losses for the Northridge event with 1000 realizations of a spatially variable random field. ................................. 116

Figure 4.18 A comparison of the spatially variable ShakeMaps generated using the KL expansion from the angular power spectrum derived from the covariance function (a), and with the modified angular power spectrum (b). The random fields were generated for and added to the ShakeMap of the Northridge (1994) earthquake. ................................................................. 117

Figure 5.1 In (a) and (b), we plot the amplitude of the needlet against the points on the HEALPix grid. Altering the ‘center’ of the needlet, $x_{jk}$, between (a) and (b) results in a translation of the needlet, with the maximum amplitude achieved at $x_{jk}$. We plot the Hammer projection of two needlets in (c) and (d) with $j = 3$ and $j = 5$ respectively. We observe that larger $j$ cause the needlet to become more localized. .............................. 133
Figure 5.2  Illustrative results for the needlet approximation of the Wendland radial basis function. (a) Displays the values of the function at the sample locations. These values are used to compute the spherical needlet approximation (b). The absolute error in the needlet approximation to the actual values of the Wendland radial basis function is shown in (c)....137

Figure 5.3  Illustrative results for the isotropic random field. In (a) we have the KL expansion of the isotropic random field sampled at sparse locations on the sphere. This field is generated using the algebraically decaying angular power spectrum with $\alpha = 9$. In (b) we have the corresponding needlet approximation, and in (c) we plot their absolute difference. ........140
TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Mean and standard deviation of fatality loss distributions for Northridge with 0 and 185 stations and Loma Prieta with 0 and 185 stations using the conditional method and various radii.</td>
<td>52</td>
</tr>
<tr>
<td>3.2</td>
<td>Distribution properties for estimated fatalities using R realizations for Northridge.</td>
<td>61</td>
</tr>
<tr>
<td>3.3</td>
<td>Distribution properties for estimated economic losses using R realizations for Northridge. All values in billions.</td>
<td>62</td>
</tr>
<tr>
<td>4.1</td>
<td>Comparison of values of $A_0$ for various correlation lengths.</td>
<td>96</td>
</tr>
<tr>
<td>4.2</td>
<td>Maximum and minimum values obtained for angular power spectrum with various $b$.</td>
<td>98</td>
</tr>
<tr>
<td>4.3</td>
<td>Values of the modified angular power spectrum for $l = 0$ and $l = 1000$ with $\alpha = 2.0, 2.5, \text{ and } 3.0$.</td>
<td>109</td>
</tr>
</tbody>
</table>
LIST OF ABBREVIATIONS

Gaussian Random Field........................................................................................................GRF
Ground Motion Prediction Equation.......................................................................................GMPE
Ground Motion to Intensity Conversion Equation.................................................................GMICE
Intensity Measure ................................................................................................................IM
Karhunen Loève................................................................................................................MKL
Modified Mercalli Intensity....................................................................................................MMI
Pacific Earthquake Engineering Research Center..............................................................PEER
Peak Ground Acceleration..................................................................................................PGA
Peak Ground Velocity..........................................................................................................PGV
Prompt Assessment of Global Earthquakes for Response................................................PAGER
Pseudo Spectral Acceleration ..............................................................................................PSA
Successive Conditional Simulation.....................................................................................SCS
United States Geological Survey........................................................................................USGS
ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Mahadevan Ganesh, who has guided me through this project and who first inspired me to do research. I am grateful for the USGS, whose support made this work possible. In particular, this thesis would not be completed as written without David Wald, who introduced me to this subject and who offered constant guidance and encouragement. I would also like to thank Bruce Worden, whose assistance helped shape this work, and who was able to find even the smallest bugs in my code. The support of the Colorado Golden Energy Computing Organization (GECO) is gratefully acknowledged. Finally, I extend a big thanks to my family and friends, whose support has meant the world to me.
CHAPTER 1
INTRODUCTION

Earthquakes are widely studied phenomenon which have the potential to impact thousands of people every year. Although no method is available to predict earthquakes, capturing the ground motion displacement, velocity, and acceleration during an earthquake has helped seismologists understand the mechanics of seismic events, which can be used to better prepare for future hazards. Several equations, known as ground motion prediction equations (GMPEs), have been derived using earthquake data and can be used to estimate a ground motion intensity measure, such as peak ground acceleration or peak ground velocity, at a particular location. The United States Geological Survey (USGS) has developed a software for earthquake ground motions known as ShakeMap, which uses data from seismic events and GMPEs to form a latitude-longitude grid of estimated ground motion intensity measures (Wald et al., 2005). See Figure 1.1 for an illustrative example. The resultant grid of predicted data can then be run though the USGS Prompt Assessment of Global Earthquakes for Response (PAGER) system, which computes an estimated range of fatalities and economic costs, providing invaluable information to earthquake response teams (Jaiswal et al., 2011).

Current estimation models are adept at capturing the underlying physics of ground motions during an earthquake; however spatial variability, the existence of which has been well documented, is not explained by these models (Bolt, 1982; Loh, 1985). This variability can be seen today in seismic data from highly monitored areas, and has been seen historically in structural damage reports following an earthquake. The installation of dense networks of seismic arrays in the 1980’s enabled the study of spatial variability of earthquake ground motions which continues today.

Spatial variability present in ground motions has been thought to be well-modeled by an isotropic, spatially correlated, random field, which is defined by a correlation function
dependent on the period of the ground motions and the correlation length. Several correlation models have been developed for different regions of the earth (Baker & Cornell, 2006; Boore, 2003; Goda & Atkinson, 2010; Goda & Hong, 2008; Kawakami & Mogi, 2003). This study will implement the correlation model described by Jayaram & Baker (2009). Using this correlation function, an isotropic, spatially correlated, random field may be generated conditionally upon seismic data. This random field may then be added to a predicted ground motion grid to create a single realization of spatially variable ground motions. Although this method produces an exact representation of the random field, the computational requirements, particularly memory requirements, are extensive. To alleviate this, a new iterative algorithm is introduced in Chapter 3 which computes an approximation to the random field, thereby substantially reducing memory requirements. Further, the algorithm is parallelized to reduce CPU times and to make the method viable for real-time computations.

The developed algorithm, referred to as the successive conditional simulation method, is highly dependent on the ShakeMap grid of estimated ground motions. For example, the number of grid points, the distances between each point, and the latitude and longitude ranges determine the the random field. Thus, for a ShakeMap grid with 30,000 points, the dimension of the spatially correlated random field is itself 30,000. In order to reduce the
stochastic dimension and make the method more portable, Karhunen Loève expansions are explored. Using this method, the random field on the sphere is represented as an infinite linear combination of orthogonal functions, namely the spherical harmonic functions, with coefficients in the form of the angular power spectrum (Marinucci & Peccati, 2011). The coefficients can be computed using the correlation function described by Jayaram and Baker (2009), thus using the same kernel properties in this method as for the successive conditional simulation (Magneville & Pansart, 2007). Although this method will reduce the dimension, we will investigate spherical needlets as a method to approximate a highly localized random field.

By definition, spherical needlets are highly localized radial polynomials on the sphere. A recent work showed that spherical needlets can be used to capture fine details on the sphere which are highly localized in space (Wang et al., 2015). Future work could involve the development of a fast and memory efficient algorithm to compute the needlet approximation of a localized, isotropic random field.

The structure of the thesis is as follows:

In Chapter 2, the background and development of spatial variability in the context of earthquake ground motions is described. An overview of the ShakeMap modeling process is provided, as this software is vital to this project. As well as giving a brief overview of the importance of spatial variability in seismic modeling and probabilistic seismic hazard analysis, the most widely used correlation models will be discussed and their weaknesses will be analyzed. The correlation model that is used for the remainder of the project, namely the correlation model described by Jayaram & Baker (2009), is described in detail. Further, additional factors related to spatial variability are discussed, such as spatial cross-correlation models and regional dependence, which present the next steps toward accurately modeling earthquake ground motions.

Building off of the correlation model and method described by Jayaram & Baker (2009) and Park et al. (2007), Chapter 3 provides an in-depth explanation of the new and effi-
cient method for generating the spatially correlated random fields of residuals, the method of successive conditional simulations. This chapter includes a discussion of the statistical approach, as well as techniques to increase efficiency, including parallelization and memory reduction. Further, the method is used to investigate the effects of spatial variability of residuals on estimated fatality and economic loss distributions with other factors such as station influence and directivity.

In Chapter 4, we discuss the method of Karhunen Loève expansions with application to the spatial variability of earthquake ground motions. The angular power spectrum is derived using the Jayaram & Baker (2009) correlation model such that the covariance structure of the random field is completely captured within the set of coefficients. Additional examples of isotropic, Gaussian random fields are computed using simple functional forms of the angular power spectrum. Truncation results are provided and the generated random field is added to the earthquake ground motions to compute loss results.

The spherical needlet decomposition of an isotropic Gaussian random field is described and implemented in Chapter 5. This method is based on the work of Wang et al. (2015), which describes the fully discrete needlet approximation for a function on the sphere.
CHAPTER 2
SPATIAL VARIABILITY IN EARTHQUAKE GROUND MOTIONS

In order to give a comprehensive review of spatial variability in the context of earthquake ground motions, we will first give relevant details and background information about modeling seismic ground motions. In particular, we are interested in the modeling of ground motion intensity measures (IMs) with ShakeMap, the USGS software for generating estimated ground motion maps in near real-time after significant earthquakes (Worden & Wald, 2016). The ShakeMap software takes into account many variables which capture the effect of the source, path, and site components of an earthquake. This review will introduce the units used to measure ground motions, ground motion prediction equations (GMPEs), and the algorithm used to compute the estimated ground motions.

2.1 Ground Motion Estimation

The most commonly used IMs include peak ground acceleration (PGA), peak ground velocity (PGV), and pseudo spectral accelerations (PSA), which relate ground shaking to the fundamental period of a structure. These data are collected from seismometers which are located all over the world. The Modified Mercalli Intensity scale (MMI) is yet another IM which is a qualitative scale based on damage and felt reports during shaking (Wald et al., 2011). For example, an intensity IV is described as being felt by many people indoors and a few people outdoors, with some disturbance to windows, dishes, and doors, while an intensity VIII would include considerable damage to substantial buildings and partial collapse. Each ground motion IM can be used to predict a grid of earthquake ground motions using ShakeMap.

Conversions between these intensity measures, known as Ground Motion to Intensity Conversion Equations (GMICEs), allow multiple types of data to be used in the generation
of a ShakeMap grid. The GMICE used within ShakeMap is the Worden et al. (2012) conversion equation, which linearly relates IMs with dependence on the source, site conditions, and distance from each point to the rupture. The source term is captured using the earthquake magnitude, while the distance term can be computed in one of several ways (Joyner and Boore distance, hypocentral distance, epicentral distance, etc.). The geologic site conditions may greatly affect the ground motions. Sites located on soft rock and soil may experience ground motion amplification that neighboring sites located on rock will not. The site conditions are often captured as Vs30 measurements, which by definition is the average shear wave velocity down to 30 m. Unfortunately, site conditions are often unknown and therefore must be estimated using geologic conditions or topographic information (Allen & Wald, 2009; Wills & Clahan, 2006).

Using known seismic data and source information, ground motions can be estimated at any location near the rupture using GMPEs. Although GMPEs have many forms, here we will focus on a general GMPE given by Abrahamson & Youngs (1992) and Joyner & Boore (1993),

$$\ln Y_{ij} = f(M_i, R_{ij}, \lambda_{ij}, T_n) + \eta(T_n) + \epsilon_{ij}(T_n).$$

In this equation, $\ln Y_{ij}$ is the logarithmic ground motion at site $j$ for earthquake $i$ which is dependent on the period, $T_n$. $f$ is a function which incorporates the magnitude $M_i$, the distance from site $j$ to the rupture, $R_{ij}$, and a single term to represent other source, path and site effects, $\lambda_{ij}$, for simplicity. The last two terms in this equation capture the uncertainty present in ground motions. $\eta$ is known as the interevent uncertainty and is independent of the site, or is a constant for all sites $j$ for a given earthquake $i$. This term represents the variability in ground motions seen between different earthquakes. The last term $\epsilon_{ij}$ is known as the intraevent uncertainty, and captures the variability observed between sites during the same earthquake. Both uncertainty terms are often modeled as centered, normally distributed random variables with standard deviations $\tau_i$ and $\sigma_{ij}$, for the interevent and intraevent terms respectively. When fitting the coefficients for a GMPE, the data from several
earthquakes is often used, allowing the modeler to compute the intraevent and interevent standard deviations, which will then be reported along with the GMPE equation. Therefore, when modeled as random variables, the uncertainty terms add randomness to the estimated ground motions. In particular, the interevent variability will either increase or decrease the entire field of ground motions, while the intraevent term adds variability from site to site.

In the next section we will present a brief description of the ShakeMap software and how the GMICEs and GMPEs are utilized.

2.1.1 A Brief Introduction to ShakeMap

In order to estimate the ground motions at a given latitude and longitude coordinate, the ShakeMap software uses observational data and GMPEs along with source and site information. We begin with a grid point \((x, y)\) where we are interested in estimating the ground motions from a nearby earthquake. The estimate at this point will be determined by any nearby stations, and corrected for relative source to site distances with the GMPE. For example, if there is an observation point near the grid point, denoted \(Y_{obs}\), the predicted amplitude at \((x, y)\) is given as

\[
Y_{obs,xy} = Y_{obs} \times \frac{Y_{GMPE,xy}}{Y_{GMPE,obs}} \times C_{site},
\]

where \(Y_{GMPE,xy}\) is the GMPE predicted ground motions at site \((x, y)\), \(Y_{GMPE,obs}\) is the GMPE predicted ground motions at the observation, and \(C_{site}\) corrects for relative site amplification of the observation and point \((x, y)\). By convention, the IM computed using this equation is called the native IM. If two types of data (i.e., PGA and MMI) are used to predict the ground motions, the non-native IM will be converted to the native IM through a GMICE. This conversion adds uncertainty to the estimation, which is discussed below.

The uncertainty of the GMPE, denoted \(\sigma_{GMPE}\), can be provided as either a total uncertainty term or separated into the intraevent and interevent components. If enough data are available, ShakeMap will compute the interevent bias between the observational data and
estimated value at the site. If this estimate is made, only the intraevent GMPE uncertainty is used, otherwise both terms are used. The uncertainty from estimating the ground motion at a point given a nearby observation is determined by the distance between the grid point \((x, y)\) and the observation, denoted \(r_\Delta\). In ShakeMap, an observation point can influence a grid point only if it lies within a radius of \(r_{max}\) of the observation; however, the observation has the greatest influence if the grid point is within a radius \(r_{ROI}\) where \(r_{ROI} < r_{max}\). The uncertainty of the station influence is modified by \(r_\Delta\) as follows.

\[
\sigma_{obs,xy} = \begin{cases} 
\sigma_{GMPE}(1 - \exp(-\sqrt{0.6r_\Delta})) & : r_\Delta \leq r_{ROI}, \\
\frac{r_{max} - r_{ROI}}{r_{max} - r_\Delta} & : r_{ROI} < r_\Delta < r_{max}, \\
\infty & : r_\Delta \geq r_{max}
\end{cases}
\]

where \(r_{ROI}\) is the empirically determined distance at which the observation’s standard deviation is equal to the GMPE’s (i.e., \(\sigma_{r=ROI}\)). To account for the uncertainty in the GMICE conversions, we can define

\[
\sigma_{conv,obs,xy} = \sqrt{\sigma_{obs,xy}^2 + \sigma_{conv}^2},
\]

where \(\sigma_{conv}\) is the uncertainty introduced through the GMICE conversion.

The full equation for the prediction of median ground motion at point \((x, y)\), defined to be

\[
\bar{Y}_{xy} = \frac{Y_{GMPE,xy}}{\sigma_{GMPE}^2} + \frac{\sum_{i=1}^{n} Y_{obs,i} \times Y_{GMPE,obs,i}}{\sum_{i=1}^{n} 1 / \sigma_{obs,xy,i}^2 + \sum_{j=1}^{m} 1 / \sigma_{conv,xy,j}^2} + \frac{\sum_{j=1}^{m} Y_{conv,j} \times Y_{GMPE,conv,j}}{\sum_{j=1}^{m} 1 / \sigma_{conv,xy,j}^2},
\]

where \(n\) is the total number of native observations, and \(m\) is the total number of non-native observations. Therefore, the total uncertainty is given as

\[
\sigma_{\bar{Y}_{xy}}^2 = \frac{1}{\sigma_{GMPE}^2} + \frac{1}{\sum_{i=1}^{n} 1 / \sigma_{obs,xy,i}^2 + \sum_{j=1}^{m} 1 / \sigma_{conv,xy,j}^2}.
\]

An additional factor to consider is the amplification of the median ground motions based on the site conditions. This is accomplished by first de-amplifying all observational data to
rock site conditions. The GMPE computations for both grid points and observations are computed as if they were located on rock, and subsequently these results are used to compute the median ground motion, $\bar{Y}_{xy}$, on rock site conditions. Finally, this median estimation is amplified according to the site conditions at that point.

2.2 Impact of Spatial Variability on Seismic Hazard Analysis

As previously stated, the uncertainty in earthquake ground motions in the form of interevent and intraevent uncertainty are often modeled as centered random variates with standard deviations given by the GMPE. Variability between earthquakes and within a single earthquake has been well documented in historical observations. In particular, intraevent variability is observed in damage reports, where two similarly constructed structures located closely in space experience significantly different levels of damage. Further, after the installation of dense networks of seismic arrays, variability in ground motions was observed which was not well characterized by current ground motion models. Another factor to consider is the likelihood that such variability is correlated. For example, two points which are located closely in space, have a high probability of exhibiting similar ground shaking, regardless of whether the level of shaking is at, above or below the median predicted value. To remedy this, we consider the computation of a spatially correlated random field of intraevent residuals which can then be added to the median ground motions. This is the general method for capturing spatial variability as discussed in this chapter, and also the inspiration for Chapters 3-5. Another important consideration, is the effect of adding this variability on seismic hazard analysis. We consider the effect in this section.

The main motivation for adding variability to the ShakeMap estimated ground motions is the impact on hazard and loss estimates. Although the existence of spatial variability has been documented for a few decades, it was not until recently that the impact of including variability on hazard and loss estimation was fully realized. This struggle was thoroughly documented by Bommer & Abrahamson (2006), who analyzed the use of the uncertainty in hazard analysis. In the 1970’s and 1980’s, the uncertainty, which we will denote as $\sigma$,
was often neglected entirely in the formulation of GMPEs. Although the inclusion of $\sigma$ became standard in the 1980’s, it was still often ignored. One reason for this, suggested by Bommer & Abrahamson (2006), is that the standard reference for seismic hazard analysis, namely Cornell (1968), did not include the uncertainty. Even though this was corrected a few years later in Cornell (1971) and Merz & Cornell (1973), the original reference was the most commonly used.

The impact of including the uncertainty on the hazard curve is captured well by Figure 2.1, where the PGA level is plotted against the annual frequency of exceedance. Although at low levels of ground shaking the hazard curve is seemingly unaffected by the change in $\sigma$, the annual frequency of exceedance is increased drastically for higher levels of ground motion shaking. Thus, the design of structures such as power plants and nuclear waste storage facilities, which are built to last thousands of years, could be greatly affected by including $\sigma$. In fact, as a result of including $\sigma$ in the past decade, modern seismic hazard analysis reports increased hazard estimates for the long period. Even though the inclusion of uncertainty is unlikely to impact risk assessment for small return periods, under prediction of risk at the long period could expose our society to avoidable dangers.

![Hazard curves for PGA generated with Boore et al. (2003). Figure after Bommer & Abrahamson (2006).](image)

Figure 2.1: Hazard curves for PGA generated with Boore et al. (2003). Figure after Bommer & Abrahamson (2006).
Another example of the influence of uncertainty in loss estimates is presented by Sokolov & Wenzel (2014) in the context of lifeline, a set of components that are essential to sustain life and growth of a community. This may include chemical and military plants, gas pipelines, hospitals, highways, etc. In particular, the authors consider damage to an electric power system during an earthquake, which could profoundly disrupt power supply and affect many other systems within lifeline. Although the spatial correlation of residuals, including intraevent and interevent uncertainty are understood to affect loss estimates, typically studies of distributed networks are conducted without consideration spatial variability. Thus, this study attempts to understand the impact of adding these residuals on risk assessment.

To understand the impact of an earthquake on the lifeline system, it is important to know which systems may be affected simultaneously (i.e., multiple roads on a highway system) or if one critical element would be damaged. For the purposes of this study, electrical substations were considered to be critical facilities in the portfolio. These substations were considered to be dependent on one another, and thus spatially correlated ground motions must be considered. The results of including both types of variability lead to the following conclusions.

• Both the intraevent and interevent terms are essential to understanding the entire loss picture.

• Interevent variability, which is a constant factor across the entire portfolio, causes the level of ground motions to either increase or decrease everywhere, which consequently will either increase or decrease the losses everywhere.

• Intraevent variability, which is modeled as a spatially correlated, random field, causes the predicted losses to increase in some locations, and decrease in others.

• If the facilities are considered to be correlated at large separation distances (large correlation range), there is a increased probability of joint non-functionality of all substations in the portfolio.
• If the facilities are considered to be correlated only at short separation distances (small correlation range), there is an increased probability of non-functionality at some stations, but not others.

• If the correlated range is large, the maximum and minimum levels of damage are closer than if the correlation range is small.

The conclusions stated here emphasize two factors. First, the addition of variability in seismic hazard analysis has the ability to greatly affect the results. Therefore, the uncertainty term should be considered to investigate all possibilities of loss. Second, there are several components of the interaction of seismic hazard analysis with the uncertainty that are complex. Since the resultant losses are dependent on the range of correlation, much work still remains to determine the proper range for a given portfolio.

In the following section, we discuss in greater detail the analysis of spatial variability in earthquake ground motions and the attempts to model variability as a spatially correlated, random field.

2.3 Initial Investigations into Spatial Variability

Even before the construction of the first dense seismic array networks, it was known that variability existed in ground motions during an earthquake. It was not until the construction of dense seismic arrays, however, that such variability was captured in a way that could be analyzed. Up to that point in time, earthquake engineers mainly focused on the effects of temporal variation in ground motions, which were known to have a great effect on seismic design of structures (Harada, 1982). The construction of the SMART-1 dense instrument array in Taiwan in the 1970’s enabled some of the first real analysis of spatial variability present in ground motions. It was discovered that spatial variability dominated the dynamics of underground structures and could have a profound effect on the seismic response of tall buildings, extended structures, and buildings with wide-spread foundations (Bolt, 1982). In one of the first attempts to capture the spatial variability, T. Harada modeled the ground
motion displacement as a zero-centered, homogenous Gaussian space-time process. Using the
data from the SMART-1 array, a spatial correlation function was constructed which could
then be used to capture the effects of earthquake ground motions on pipelines, maximum
ground strain, and maximum relative displacement between points (Harada, 1982).

The installation of several additional dense instrument arrays a few years later not only
enabled more in-depth analysis of the spatial variability of ground motions, but also investi-
gation into the effects of site conditions, which were considered a likely source of spatial
variability (Abrahamson & Schneider, 1992; Abrahamson et al., 1992; Schneider et al., 1992).
Abrahamson et al. (1992) and Abrahamson & Schneider (1992) began their investigations
into these effects by analyzing the data from the EPRI/LSST dense instrument arrays in
Lotung, Taiwan. This arrays captured data for a wide range of events with varying magni-
tudes, focal mechanisms, and source distances. Using this data, Abrahamson et al. (1992)
developed a model to capture the effects of coherency, and Abrahamson & Schneider (1992)
proposed a model for amplitude variation. For a brief summary of these models, refer to
Schneider et al. (1992) where the equations are repeated and re-used for further analysis with
a set of data from ten dense seismic array networks. The results from Schneider et al. (1992)
led to several conclusions regarding the dependence of spatial variability on local site condi-
tions. For example, it was found that the coherence function developed for the LSST array in
Lotung, Taiwan was appropriate for use at other soil sites; however, the geologic complexity
of site conditions for rock sites could significantly influence the coherence function, making
the developed model inadequate. Further, it was found that the estimated correlation range
varied with site conditions and variability was highest on rock. This suggests that amplitude
variability is sensitive to local site conditions, and one general model is likely inadequate to
capture the effects of spatial variability.

Another study conducted around that time was Beresnev et al. (1994), which investi-
gated the impact of source and site effects on variability in the context of the dominant
frequency of an earthquake. Abrahamson observed that spatial variability of strong ground
motions are inversely proportional to earthquake magnitude through the SMART-1 array data (Abrahamson, 1988). This was later confirmed by Sadigh (1983) and Idriss (1991), who documented the decrease in spatial variability of a particular set of earthquake aftershocks in comparison to the main events. Beresnev et al. (1994) claimed that this relationship was due to the dependence of the spatial variability on the dominant frequency of the wave field, which is itself dependent on the magnitude. The results confirmed that the main factor in the formation of the dominant frequency was the magnitude, and thus the magnitude can strongly influence the level of spatial variability. An additional study, namely Kawakami & Sharma (1999), investigated the effects of magnitude, focal depth, epicentral distance, station separation, and other components on the spatial variability present in data from Chiba, Japan and Lotung, Taiwan. Although no solid conclusions were made on the dependence of spatial variability on focal depth and epicentral distance, it was confirmed that the level of spatial variability was dependent on the frequency of the ground motions as well as the station separation distance.

2.4 Modeling Spatial Correlation of Residuals

In order to conduct seismic hazard analysis, assumptions about the distribution of the ground motion intensity measures are necessary. Based on findings in data, seismic hazard analysis conducted before 2008 was based on the assumption that the logarithmic spectral accelerations follow a normal distribution marginally, although this assumption had not been proven. In other analysis, such as the spatial cross-correlation of residuals, vectors of ground motions are used for seismic hazard analysis where some assumptions must be made as to the statistical distribution of the vector. It was not until 2008, when Jayaram and Baker worked to confirm that vectors of ground motions, which may contain data from different sites or different periods, follow a multivariate normal distribution. This validation process was conducted by testing the normality assumption of the inter and intra event residuals obtained from ground motion models using a quantile-quantile (Q-Q) plot test. Results indicated that the assumption that both residuals follow a univariate normal distribution is
valid. Furthering this, several tests were conducted to test for the multivariate normality of the residuals at different periods. Although the details of their analysis are not repeated here, the results strongly indicate that the assumption of normality for the marginal and joint distributions of logarithmic ground motions is valid.

2.4.1 Spatial Correlation of Intraevent residuals

In 2003, Kawakami and Mogi published a study which focused on the intraevent variability in PGA as a function of separation distance, with a goal of describing the correlation between amplitudes at closely located stations and the impact of station separation distance. This study used data from the Chiba array, SMART-1 array and SIGNAL array. Like in Kawakami & Sharma (1999), the ratios of PGA values (defined as smaller value over larger value) were computed for each station pair and were used as the statistic to measure the correlation between PGA values. Thus, the closer the ratio was to 1, the higher the correlation between the two stations. Further, the PGA ratio was corrected for local site conditions, which may affect the variations in PGA values.

After correcting for local site conditions, the PGA ratios were binned into several groups based on their separation distance and regression analysis was performed. The following conclusions were made:

- The mean value of the ratios gradually decreases with respect to increasing separation distance, while the standard deviation gradually increases
- Some station pairs which were located closely together exhibited low PGA ratio values, suggesting some variability is not caused by local site conditions
- For ratios obtained without the site correction parameter, it was noted that the dispersion in PGA ratios increased for each separation distance group
- The mean, standard deviation, and other statistics were different for the different sets of array data used, suggesting a regional dependence for the spatial correlation of
ground motion data that may not be characterized by a general correlation equation

• There is little dependence on the empirical attenuation relation in terms of the correlation model coefficients

Although these conclusions provide great insight into some aspects of the modeling of spatial correlation, a few criticisms are made. First, the amplification factor used in this experiment was a constant 2.0 across the entire region. Uniformly increasing the ground motion may increase the spatial correlation of the intraevent residuals, resulting in an artificially high correlation range. Further, the method used to fit the correlation model, namely least squares regression, evenly weights each data point in computing the fit of the correlation model. However, the fit of the correlation model should be the best at short separation distances because correlation at large separation distances is typically very small and has little effect on the predicted ground motions.

In another study, Boore et al. (2003) investigates the spatial variability of PGA in the Northridge (California, 1994) dataset. Unlike other models discussed previously, this study uses only data from the Northridge earthquake to exactly compute the spatial variability within the ground motion data for this event. The difference in the logarithmic ground motions was computed for each station pair and the data were binned by separation distance. The binning was completed in a way such that each bin had at least 15 station pairs and excluded pairs where the separation distance was greater than 10 km. The correlation function was fitted such that,

\[ \rho(h) = \exp(-\sqrt{0.6 \times h}), \]

where \( h \) is the separation distance. Unlike Kawakami & Mogi (2003), the data in this study were not corrected for site conditions, which likely influenced the resulting correlation model. Further, since this model was fit using only the Northridge data set, it is unknown whether this model is applicable to any other earthquake, even within a similar region.
A few years later, Wang & Takada (2005) worked to create a model of spatial correlation of ground motion intensities that could be used for earthquake damage predictions and portfolio analysis using data from the K-NET and KiK-NET high density arrays in Japan. To fit a model, the normalized difference in logarithmic standard deviation for each pair of sites was binned by separation distance. Least squares regression was used to fit the data into the form of a decaying exponential,

$$
\rho(h) = \exp(-h/b),
$$

where $b$ is the correlation length. Six earthquakes and two GMPEs, namely Annaka et al. (1997) and Midorikawa & Ohtake (2002), were used to fit the parameter $b$. Although the values obtained from using the Annaka relation are slightly larger than those obtained for the Midorikawa-Ohtake relation, the differences are small enough to conclude the choice of GMPE is insignificant, confirming the results in Kawakami & Mogi (2003). The range for PGA was found to be slightly smaller than that for PGV, suggesting a dependence on the period of ground motions. Overall, it was found that the model fit the logarithmic deviation quite well except at short separation distances, where the model tended to over-predict. This is attributed to the lack of data at short separation distances, although many factors such as source, site effects, and wave propagation could affect the model. Recalling the criticisms made of Kawakami & Mogi (2003), this could be remedied by hand fitting the model or using weighted least squares.

### 2.4.2 Comparing correlation models

In 2008, Goda and Hong published a paper which investigated the inconsistencies in modeling intra-event variability and total variability in ground motions. Although much research had been conducted on the subject, the correlation functions from several papers including Kawakami & Mogi (2003), Boore et al. (2003), and Wang & Takada (2005), reported different rates of correlation decay for PGA with respect to increasing separation distance.
Although some of these differences could be attributed to factors such as site conditions, source and path effects, different modeling techniques used in these studies may also contribute to the differences. In an attempt to relate the coefficients from different studies, an empirical equation is developed to predict the correlation coefficient based on the separation distance between sites and the period of ground motion.

Given the interevent ($\eta_i$) and intraevent ($\epsilon_{ij}$) uncertainty terms with respective standard deviations $\tau_i$ and $\sigma_{ij}$, the total uncertainty is given by

$$\sigma_T(T_n) = \sqrt{\sigma_{ij}(T_n)^2 + \tau_i(T_n)^2}.$$ 

By definition, the correlation coefficient between two site $j$ and $k$ which are separated by a distance $h$ is given by

$$\rho_T(h, T_n) = \left(\frac{\tau_i(T_n)}{\sigma_T(T_n)}\right)^2 + \rho_{\epsilon}(h, T_n)\left(\frac{\sigma_{ij}(T_n)}{\sigma_T(T_n)}\right)^2,$$

where $\rho_{\epsilon}(h, T_n)$ is the correlation coefficient between $\epsilon_{ij}(T_n)$ and $\epsilon_{ik}(T_n)$. This value can be calculated in two ways:

$$\rho_{\epsilon}(h, T_n) = \frac{COV[\epsilon_{ij}(T_n), \epsilon_{ik}(T_n)]}{\sigma_{ij}(T_n)^2},$$
$$\rho_{\epsilon}(h, T_n) = 1 - \frac{\sigma_d(h, T_n)^2}{2\sigma_{ij}(T_n)^2},$$

where $\sigma_d(h, T_n)^2$ is the variance of $\epsilon_{ij}(T_n) - \epsilon_{ik}(T_n)$. These two functions are mathematically identical; however, they are not interchangeable numerically as they may produce different results. For example, equating these equations yields

$$2COV[\epsilon_{ij}(T_n), \epsilon_{ik}(T_n)] = 2\sigma_{ij}(T_n)^2 - \sigma_d(h, T_n)^2.$$
It is clear that $COV[\epsilon_{ij}(T_n), \epsilon_{ik}(T_n)]$ and $\sigma_d(h, T_n)^2$ depend on the separation distance between data points, $h$, while $\sigma_{ij}(T_n)$ does not. Thus, since there are finite data available, different data sets may be used to compute these statistics resulting in slightly different values of $\rho_e(h, T_n)$. It is noted that Wang & Takada (2005) make use of the second equation to compute $\rho_e$, while Boore et al. (2003) and further analysis in Goda & Hong (2008) make use of the first.

To confirm the results of Boore et al. (2003), the authors perform regression analysis under the same assumptions. The form of $\rho_e$ was taken to be

$$
\rho_e(h, T_n) = \exp(-\alpha h^\beta).
$$

A similar assumption was made in Wang & Takada (2005) and Boore et al. (2003), where $\beta$ was assumed to be 1.0 and 0.5 respectively. For this study, records from the Pacific Earthquake Engineering Research Center (PEER) NGA database, specifically a set of California records and the Chi Chi (Taiwan, 1999) event, were used to study the correlation of PGA, PGV, and PSA. Initially, a bin width of 3 km was used and results confirmed the decay of $\rho_e$ with respect to increasing separation distance. Further, it was shown that the coefficients for PSA decrease more rapidly with smaller periods (0.3 s as compared to 1.0 and 3.0 s). After re-performing the experiment with differing bin widths, it was shown that bin width has little effect on the empirical function, provided enough data exist for each bin. Using the California records, the coefficients were fitted to be

$$
\alpha = -0.16 \ln(T_n) + 0.62, \quad \beta = 0.50,
$$

which are compatible with the results from Boore et al. (2003).

This study was taken further by Goda & Atkinson (2009), in which the K-NET and KiK-net databases of ground motion data were used to continue the work of Goda & Hong (2008). These databases provide a unique opportunity to compute the spatial correlation for
a wide range of earthquake magnitudes in a subduction zone environment where numerous ground motion records exist. In addition to providing another empirical function to estimate the correlation of residuals, this study provided an opportunity to compare the differences in correlation functions for different regions (i.e., Japan and California). Using the data from 106 earthquakes in this region, coefficients were derived using least square assuming the form,

\[ \rho_e(h, T_n) = \max[\gamma(T_n) \exp(-\alpha(T_n)h^{\beta(T_n)}) - \gamma(T_n) + 1, 0]. \]

The PGA correlation function for all earthquake types is given by

\[ \rho_e(h, T_n) = \max[2.6 \exp(-0.095h^{0.336}) - 2.6 + 1, 0]. \]

It can be seen that this function has a steeper correlation curve and the function intersects the zero axis at shorter separation distances than the correlation model for Japan. Thus, if the California correlation equation were used for events in Japan, the spatial correlation would have been under-predicted. Comparing the models derived for the different types of earthquakes, it was shown that the outcome of a seismic loss assessment could be changed by up to 50%. Thus, the results from this paper strongly demonstrate the importance of using the correct correlation model for the tectonic environment and region. Although the differences between the empirical relations were clearly shown, exact reasons for the differences were not given as many parameters, likely inter-dependent parameters, may cause the differences shown.

In another comparative study, Jayaram & Baker (2009) use the ground motion records from 7 large earthquakes to develop a spatial correlation model, namely Northridge, Chi Chi, Big Bear (California, 1992), Parkfield (California, 2004), Alum Rock (California, 2007), Anza (California, 2005), and Chino Hills (California, 2008). The model developed is then compared to other, previously discussed correlation models and their differences are explained.
To begin, define the normalized intraevent residual,

\[ \epsilon'_{ij} = \frac{\epsilon_{ij}}{\sigma_{ij}}, \]

and further define,

\[ \tilde{\epsilon}_{ij} = \frac{\epsilon_{ij} + \eta_i}{\sigma_{ij}} = \frac{\ln Y_{ij} - \ln(\bar{Y}_{ij})}{\sigma_{ij}}, \]

where we will assume the ground motions, \( \ln Y_{ij} \) are of the form

\[ \ln Y_{ij} = \ln \bar{Y}_{ij} + \epsilon_{ij} + \eta_i. \]

Then if we are interested in computing the semivariogram, a measure of the average dissimilarity of data, of the normalized intraevent residual \( \epsilon' \) at sites \( a, b \) for a given earthquake \( i \) which are separated by a distance \( h \). By definition, we can write the semivariogram as

\[
\hat{\gamma}(h) = \frac{1}{2N(h)} \sum_{\alpha=1}^{N(h)} (\epsilon'_a - \epsilon'_b)^2,
\]

\[
= \frac{1}{2N(h)} \sum_{\alpha=1}^{N(h)} \left( \frac{\ln(Y_a) - \ln(\bar{Y}_a) - \eta}{\sigma_a} - \frac{\ln(Y_b) - \ln(\bar{Y}_b) - \eta}{\sigma_b} \right)^2,
\]

\[
\approx \frac{1}{2N(h)} \sum_{\alpha=1}^{N(h)} \left( \frac{\ln(Y_a) - \ln(\bar{Y}_a)}{\sigma_a} - \frac{\ln(Y_b) - \ln(\bar{Y}_b)}{\sigma_b} \right)^2,
\]

\[
= \frac{1}{2N(h)} \sum_{\alpha=1}^{N(h)} (\tilde{\epsilon}_a - \tilde{\epsilon}_b)^2.
\]

Thus, an approximation for the semivariogram of \( \epsilon' \) is in terms of \( \tilde{\epsilon} \). The assumption made here is that \( \frac{\eta}{\sigma_a} \approx \frac{\eta}{\sigma_b} \). For the GMPE used in this study (Boore & Atkinson, 2008), the standard deviation of the intraevent residuals are only dependent on the period of the ground motions, making this assumption reasonable. This approximation is important because it shows that the covariance of the intraevent residuals may be estimated without accounting
for the interevent residual.

Using the earthquakes mentioned in the previous paragraph, the spatial correlations computed for each earthquake are used to develop an empirical model for the spatial correlation of the form

\[ \rho(h) = \exp(-3h/b). \]

One difference in this work versus previous papers considered in this review is the method of regression. In this experiment, regression is performed by hand, placing more weight at smaller separation distances. This is because correlation is expected to be low at large separation distances and due to larger correlation values from closer sites, correlation at the large separation distances has little effect on the predicted ground motions.

The records from all earthquakes considered provided generally consistent characteristics for periods longer than 2 seconds; however, at shorter periods significant differences were found. These differences were attributed to the Vs30 ranges for each event. If the Vs30 values are correlated (i.e., if the local site conditions are homogenous), it is likely that the ground motions will be more correlated resulting in a higher correlation range. The opposite is true for heterogeneous site conditions. Another factor to consider is the error in estimating the Vs30 values. For example, errors in Vs30 approximation at sites which are located closely in space are likely correlated, which may artificially increase the range of the semivariogram, particularly at short periods. Taking this into consideration, the correlation model developed is dependent on the period of the ground motions and the expected clustering of the Vs30 values. The correlation range is given below.

\[
b = \begin{cases} 
8.5 + 17.2T & : \text{Vs30 uncorrelated} \\
40.7 - 15.0T & : \text{Vs30 correlated} \\
22.0 + 3.7T & : T \geq 1. 
\end{cases}
\]
Although this paper provides a comprehensive model for each period and $V_s30$ condition, it is not entirely clear what determines the correlation of the $V_s30$ values. Thus, the choice of correlation range is left up to the user, which may end in differing results.

In comparing this model to those previously shown, the work by Boore et al. (2003) and Goda & Hong (2008) showed remarkably similar results to those obtained here. Since Northridge was used to derive this correlation model and the other correlation models, some similarity may be attributed the influence of the Northridge data set on this model. In contrast, the model developed by Wang & Takada (2005) for PGV and the Chi Chi event predicts a much higher correlation range (83.4 km versus 8.5 or 40.7 km). Differences in fitting, including amplification factors and regression method, may account for these differences.

Using data from the European Strong Motion database and the Italian Accelerometric Archive, Esposito & Iervolino (2011) fit a spatial correlation model following the approach outlined in Jayaram & Baker (2009). As described in Jayaram & Baker (2010), the sample variance may be used to estimate the true variance to obtain the standardized residuals, or the GMPE provided variance may be used. After implementing both approaches, it was found that the correlation coefficients differed by less than 10%, and thus the GMPE uncertainty was a fair substitute. The exponential model was used as this model allows for high correlation at small separation distances. For the European Strong Motion Database, the correlation ranges were 13.5 km and 21.5 km for PGA and PGV respectively. For the Italian data, ranges were 11.5 km and 14.5 km for PGA and PGV respectively. These results suggest that ground motion with lower frequency content yields higher correlation, and confirms that the correlation model is dependent on regional differences.

A further study concerning the dependence of spatial correlation on local site conditions was conducted by Sokolov et al. (2012). Following up on a previous study, Sokolov et al. (2010), which showed the dependence of spatial variability on local geology, this study examined how the intraevent correlation of PGA depends on array or site specific grouping.
Further, it was determined whether such a grouping could reduce the standard deviation associated with GMPEs. The arrays used in this study were the ILA, TAP, and CHY in Taiwan. Findings from this study are summarized below.

- Intraevent correlation is dependent on site classes, attributed to the sensitivity of PGA on high frequency soil amplification

- Variations in geographical features, such as softness of surface rocks and thickness of sediments, can cause variations in residuals when paired with high frequency amplitudes in some site classes results. Thus spatial correlation is smallest in regions with large variability in geographic features

- Intraevent correlation is dependent on the ground motion array, specifically it is small for ILA and TAP and large for CHY

Thus, this study provides further evidence that spatial correlation is dependent on regional geology, and in general, it is likely that one correlation function is not applicable to other regions.

### 2.5 Inconsistencies in Modeling Variability

Although the effects of spatial correlation of ground motions have been estimated and documented, spatial correlation is often ignored when developing GMPEs. In fact, it is often assumed that ground motions are uncorrelated in the process of fitting a GMPE, which ironically can then be used to compute the spatial correlation of residuals. This inconsistency was studied by Hong et al. (2009). In particular, the effect of assuming spatial correlation on the GMPE and on the spatial correlation coefficients was investigated. A GMPE was derived assuming spatial variability of the intraevent residuals and the coefficients obtained were compared to those obtained by Goda & Hong (2008), which uses the same data set. The bias was determined by the change in the parameter set $\lambda$ of variables in the GMPE. It was found that no significant bias was introduced in these coefficients; however, it was
found that the variance of the intra and interevent residuals did change, which would likely impact hazard and loss estimation.

An additional examination into the effect of adding spatial correlation into GMPEs was conducted by Jayaram & Baker (2010). Starting with the work from Hong et al. (2009), this paper seeks to explain the changes in the intraevent and interevent standard deviations, and to discuss the impact of these changes on seismic hazard and loss analysis. As a last step, a modified algorithm based on Abrahamson & Youngs (1992) which incorporates spatial correlation is introduced and used to refit the Campbell & Bozorgnia (2008) ground motion model. In comparison to the Campbell & Bozorgnia (2008) fitted without the assumption of spatial correlation, results confirmed that the model coefficients, as well as the total standard deviation, were not significantly altered. However, the intraevent standard deviation increased from 0.578 to 0.654, and the interevent standard deviation decreased from 0.223 to 0.157. The changes in these coefficients imply that the model coefficients for the GMPE may be used to predict ground motions with or without spatial variability, but the intra and interevent standard deviations should be computed under the assumption of spatial correlation.

An exception to this comes from the work of Jayaram & Baker (2009). It was shown that the spatial correlation of intraevent residuals can be estimated from the total residuals. Thus, since the total residuals are unaffected by the assumption of spatial variability, the spatial correlation model can be accurately fitted independent of the GMPE. This method allows different data sets to be used to fit the GMPE and the spatial correlation model, which may better constrain one or both of these models. Further, the correlation model may be fit giving precedence to the smaller separation distances and may depend on site conditions, both of which would be challenging tasks for the GMPE method. Therefore, even though it was shown that fitting the GMPE with the assumption of spatial correlation changed the intra and inter event residuals, there are several advantages to using spatial correlation models and GMPEs fitted separately. The impact of incorrectly estimating the residuals on hazard and loss analysis would likely overestimate the likelihood of jointly observing extreme
Another inconsistency in modeling spatial variability is the treatment of local site and regional conditions, which have been shown to affect spatial correlation of ground motion residuals. More specifically, assumptions made about the Vs30 values and amplification factors could significantly impact our estimation of spatial correlation in a region. One example of this has already been seen in Wang & Takada (2005), where uniform site amplification terms lead to a very high correlation range. In a study conducted by Baker & Miller (2011), Monte Carlo simulations were used to generate PSA values at several pairs of sites for a rupture of the Northern Hayward segment. This data formed a synthetic catalog of earthquakes in the region with various magnitudes, distances, and site conditions. Using this data, it was shown that the spatial correlation of residuals is dependent on the source, rupture distance, and separation distance. Thus, providing more support that spatial correlation will vary from region to region since fault structure and geometry vary regionally. Another factor discussed is the impact of using Vs30 values to capture site conditions. If Vs30 values are known and are strongly variable, it is likely that the GMPE will not be able to explain ground motions, resulting in artificially high correlation of intraevent residuals. In another case, if the Vs30 values are predicted, there is a good chance that the errors in the predictions are spatially correlated and the Vs30 grid will be more homogenous than it is in reality, which will result in falsely high spatial correlation of residuals. Thus, Vs30 values should be estimated carefully as they can influence errors in correlation length.

Though many studies have been conducted on the correlation of intraevent residuals, the outcomes of these studies have produced different correlation models. Some studies have shown the effects of frequency content on correlation models, (Goda & Hong, 2008; Jayaram & Baker, 2009) or regional differences (Goda & Atkinson, 2009; Sokolov et al., 2010) which may explain some of the discrepancies. While some authors analyzed data from a single event (Boore et al., 2003; Jayaram & Baker, 2009; Wang & Takada, 2005), others have investigated using multiple events (Goda & Atkinson, 2009, 2010; Goda & Hong, 2008; Jayaram &
Baker, 2009; Sokolov et al., 2010). While Goda & Atkinson (2009) investigated the effects of earthquake type on spatial correlation and reported little dependency, Sokolov et al. (2010), Kawakami & Mogi (2003), Goda & Atkinson (2009), and others showed the impact of regional differences on the model. Wang & Takada (2005) and Jayaram & Baker (2009) use existing GMPEs to fit their models, but Goda & Hong (2008), Goda & Atkinson (2009), and Sokolov et al. (2010) fit their GMPEs to the data before calculating the correlation model. All of these factors, which may contribute to the different correlation models, can have a significant impact on the resultant correlation model derived.

2.6 Additional Aspects of Spatial Variability

In this section we will consider several aspects of spatial variability which expand upon the spatial correlation of intraevent residuals.

2.6.1 Spatial Cross-Correlation

One aspect of spatial correlation that has not yet been discussed is the correlation between ground motion intensity measures. Baker & Cornell (2006) worked with PSA values at several different periods and orientations, and estimated the correlation between periods, which can then be used to measure the joint distributions of PSA. This information can be incredibly valuable for modeling probabilistic seismic hazard using vector-valued probabilistic seismic hazard analysis (VPSHA), which requires information about the joint distribution of spectral acceleration values. In order to begin this analysis, the assumption is made that the logarithmic PSA values are modeled by a normal distribution, as was shown by Jayaram & Baker (2008). To complete the VPSHA, only the the correlation coefficients between PSA values were needed. The procedure for the analysis of ground motions used 267 (3-component) records from 30 earthquakes from the PEER database which all had the following properties: the earthquake was a shallow crust event, the source to site distance was less than 100 km, the magnitude was greater than 5.5, and the soil was classified as stiff soil. A final event, namely the Chi Chi earthquake, was used to validate the results.
To capture the correlation coefficient, the three components of the ground motion were represented as

\[
\ln Sa_x(T_n) = f_H(M_i, R_{ij}, T_n, \lambda_{ij}) + \sigma_H(M_i, T_n)\epsilon_x(T_n)
\]
\[
\ln Sa_y(T_n) = f_H(M_i, R_{ij}, T_n, \lambda_{ij}) + \sigma_H(M_i, T_n)\epsilon_y(T_n)
\]
\[
\ln Sa_z(T_n) = f_V(M_i, R_{ij}, T_n, \lambda_{ij}) + \sigma_V(M_i, T_n)\epsilon_z(T_n)
\]

where \(f_H\) and \(f_V\) are the mean ground motion predictions for the horizontal and vertical components respectively, which are dependent on the magnitude \(M\), distance to fault measure \(R\), period \(T\) and other parameters \(\lambda\). \(\sigma_H\) and \(\sigma_V\) are the standard deviation of PSA values for a given magnitude and period, and \(\epsilon_i, i = x, y, z\), capture the variability in the observations. The goal of Baker and Cornell (2006) is to find the correlation between \(\epsilon\) at different frequencies or for different components. It is clear to see that the values of \(\epsilon\) are dependent on the model used to calculate the median ground motion. An interesting result found in this experiment was that the correlation of \(\epsilon\) was independent of the model chosen. This result was confirmed using the Abrahamson & Silva (1997), Boore et al. (1997), and Campbell (1997) GMPEs. The following equation is used to compute the correlation coefficient:

\[
\rho_{A,B} = \frac{\sum_{i=1}^{n}(A_i - \hat{A})(B_i - \hat{B})}{\sqrt{\sum_{i=1}^{n}(A_i - \hat{A})^2 \sum_{i=1}^{n}(B_i - \hat{B})^2}}
\]

where \(A\) and \(B\) are the random variables of interest, \(\hat{A}, \hat{B}\) are the sample means, \(A_i\) is the \(i\)th observation of \(A\), and \(n\) is the total number of observations. This equation can be used to capture the correlation coefficient for each pair of orientations (xx, xy, xz, zz) and for each pair of periods of interest. After these coefficients are computed, they are smoothed by an average to remove excess noise, and then a Fisher z transformation \((z = \frac{1}{2} \ln(\frac{1+\rho}{1-\rho}))\) was applied so that the data would have a constant standard error. Least squares was applied to the resultant data, \(z\), and the following equations were derived.
\[
\begin{align*}
\rho_{\varepsilon_x, \varepsilon_y} &= 0.79 - 0.023 \times \ln(T), \\
\rho_{\varepsilon_z, \varepsilon_z} &= 0.63, \\
\rho_{\varepsilon_x, \varepsilon_z} &= 1 - \cos\left(\frac{\pi}{2} - \ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right) (0.359 + 0.163 \times I_{T_{\text{min}} < 0.189 \times \ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right)}\right), \\
\rho_{\varepsilon_z, \varepsilon_z} &= 1 - 0.77 \ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right) + 0.315 (\ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right))^{1.4}, \\
\rho_{\varepsilon_x, \varepsilon_y} &= (0.79 - 0.023 \times \ln\left(\frac{T_{\text{min}}}{T_{\text{max}}}\right) (1 - \cos\left(\frac{\pi}{2} - (0.359 + 0.163 I_{T_{\text{min}} < 0.189 \times \ln\left(\frac{T_{\text{min}}}{T_{\text{max}}}\right)}\right) \ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right))\right), \\
\rho_{\varepsilon_z, \varepsilon_z} &= (0.64 - 0.021 \times \ln\left(\frac{T_{\text{min}}}{T_{\text{max}}}\right) (1 - \cos\left(\frac{\pi}{2} - (0.29 + 0.094 I_{T_{\text{min}} < 0.189 \times \ln\left(\frac{T_{\text{min}}}{T_{\text{max}}}\right)}\right) \ln\left(\frac{T_{\text{max}}}{T_{\text{min}}}\right))\right),
\end{align*}
\]

where

\(\rho_{\varepsilon_x, \varepsilon_y}\): Correlation between orthogonal horizontal components for the same period,

\(\rho_{\varepsilon_x, \varepsilon_z}\): Correlation between horizontal and vertical component as same period,

\(\rho_{\varepsilon_x, \varepsilon}\): Correlation between periods for horizontal component,

\(\rho_{\varepsilon_z, \varepsilon}\): Correlation between periods for vertical component,

\(\rho_{\varepsilon_x, \varepsilon_y}\): Correlation between periods and horizontal components,

\(\rho_{\varepsilon_z, \varepsilon}\): Correlation between periods, horizontal, and vertical components.

Although these equations provided a good fit to the data, a few criticisms present themselves. It is noted that neither \(\rho_{\varepsilon_z, \varepsilon_x}\) nor \(\rho_{\varepsilon_z, \varepsilon_z}\) is dependent on separation distance. This has been demonstrated several times in this review, however no good explanation for this was given. Further, the forms of these equations, particularly of the between-period correlation functions, is very complex. Although these equations were fitted using a very specific dataset, the complexity begs the question of whether they are general enough to use in other regions, even if those specifications are met.

Another study which investigates the spatial cross-correlation of ground motion intensity measures is Wang & Du (2013). It is stressed here that more than one intensity measure is often useful in engineering applications, particularly in analyzing structural portfolios with
various building types. Thus, knowing the cross-correlation of these ground motion IMs can greatly impact hazard and loss estimation. Using data from earthquakes in California, Japan, Mexico, and Taiwan, two vectors of data are created, the first containing PGA, Arias intensity (Ia), and PGV values, and the second containing PSA values at nine periods. In this study, the intraevent residuals are corrected to remove bias as a result of Vs30 predictions and rupture distances. The method used to find the cross-correlation is the linear model of coregionalization described by Journel & Huijbregts (1978). We define the empirical cross-semivariogram as

$$\gamma_{ij}(h) = \frac{1}{2N(h)} \sum_{\alpha=1}^{N(h)} (x_i(u_{\alpha} + h) - x_i(u_{\alpha}))(x_j(u_{\alpha} + h) - x_j(u_{\alpha})),$$

where $i, j$ are ground motion IMs, $x_i(u_{\alpha} + h)$ and $x_i(u_{\alpha})$ are the $\alpha$th data pair separated by a distance $h$ for ground motion component $i$, of which there are $N(h)$. Then we will define the cross-semivariogram matrix as

$$\Gamma(h) = \begin{bmatrix}
\gamma_{11}(h) & \cdots & \gamma_{1n}(h) \\
\vdots & \ddots & \vdots \\
\gamma_{n1}(h) & \cdots & \gamma_{nn}(h)
\end{bmatrix}.$$

In order to ensure that this model is positive definite for all values of $h$, we decompose $\Gamma$ as

$$\Gamma(h) = \sum_{l=1}^{L} B^l g_l(h),$$

where $g_l(h)$ is a single model (i.e., $g_1(h) = \exp(-3h/r_1)$), and $B^l$ is referred to as a coregionalization matrix, whose entries are determined by fitting the empirical semivariogram. In this case, two models are chosen such that

$$\Gamma(h) = B^1(1 - \exp(-3h/10)) + B^2(1 - \exp(-3h/60)).$$
For further details on fitting these matrices, refer to WLS method (Goulard & Voltz, 1992). Results indicate that the coregionalization matrices vary significantly with the spatial correlation of Vs30 values.

Since the main motivation for considering the spatial correlation of ground motions is the impact on hazard and loss estimated, it is important to know the additional impact of spatial cross-correlation on loss results. Weatherill et al. (2015) discuss this impact and suggests that ignoring cross-correlation of intraevent residuals, even if spatial correlation of intraevent residuals is considered, may not be enough to really capture the ground motions and accurately depict loss estimates. Seismic hazard analysis typically works with a portfolio of structures, which may be diverse in terms of structure type, usage, seismic code design and age. Therefore, it is often the case that structures within a portfolio will have various fundamental elastic periods, which becomes an issue since fragility models are often defined in terms of a single intensity measure. Thus, it would be beneficial if multiple intensity measures could be considered when performing hazard analysis. Another consideration is the spatial correlation of structures within a portfolio, which could greatly change risk assessments, particularly if structures are affected by the same earthquake or are located closely in space. The purpose of Weatherill et al. (2015) is to demonstrate the importance of spatial cross-correlation in seismic risk assessment for portfolios consisting of heterogenous building types. To begin, three methods are outlined to compute the spatial cross-correlation.

The first method considered is the conditional hazard approach, which is described by Iervolino et al. (2010). In this method, the cross-correlation of each ground motion IM is simulation conditionally upon a primary IM, denoted $IM_1$. Then for another IM, we can compute

$$\mu_{IM_k|IM_1,M,R} = \mu_{IM_k|M,R} + \rho_{IM_1,IM_k}\sigma_{IM_k}\frac{z - \mu_{IM_1,M,R}}{\sigma_{IM_1}},$$

$$\sigma_{IM_k|IM_1} = \sigma_{IM_k}\sqrt{1 - \rho_{IM_1,IM_k}^2},$$
where $\mu_{IM_k|IM_1,M,R}$ and $\sigma_{IM_k|IM_1}$ are the mean and total standard deviation of the logarithmic ground motion for IM $k$, $\mu_{IM_k|M,R}$ and $\sigma_{IM_k}$ are the unconditional mean and standard deviation, and $z$ is a random variable. This method is an approximation in that the spatial correlation between two intensity measures, neither of which are the primary measure, is not explicitly captured. In most cases, this is a disadvantage of the method because the covariance structure of the IMs other than the primary IM are not accurately reproduced and the user is faced with the selection of the appropriate primary IM.

Another method to compute the spatial cross-correlation is called the full-block cross-correlation. This method is described by Oliver (2003). For $Y_k$ and $Y_l$ two ground motion IMs, let the mean ground motions be denoted $\mu_{IM_k}$ and $\mu_{IM_l}$ respectively. Then using this method, we write

$$
\begin{bmatrix}
Y_k \\
Y_l
\end{bmatrix} =
\begin{bmatrix}
\mu_{IM_k} \\
\mu_{IM_l}
\end{bmatrix} +
\begin{bmatrix}
L_{IM_k} & 0 \\
0 & \rho_{IM_k,IM_l} L_{IM_l} \sqrt{1 - \rho_{IM_k,IM_l}^2 L_{IM_l}}
\end{bmatrix}
\begin{bmatrix}
Y_{IM_k} \\
Y_{IM_l}
\end{bmatrix}
$$

where $L_{IM_i}, i = k, l$ is the Cholesky decomposition of the correlation matrix $C_{IM_i}, i = k, l$ obtained using the method described by Jayaram & Baker (2009), and $Y$ are random variables. $\rho_{IM_k,IM_l}$ is the cross-correlation between IM $k$ and $l$ defined by $\rho_{IM_k,IM_l} = \rho_{IM_k,IM_l}(h = 0)$.

The cross-covariance, given as $LL^T$ may be computed in a general case for $k$ total IMs as

$$
LL^T =
\begin{bmatrix}
L_{IM_1} L_{IM_1}^T & \rho_{IM_1,IM_2} L_{IM_1} L_{IM_2}^T & \cdots & \rho_{IM_1,IM_k} L_{IM_1} L_{IM_k}^T \\
\rho_{IM_2,IM_1} L_{IM_2} L_{IM_1}^T & L_{IM_2} L_{IM_2}^T & \cdots & \rho_{IM_2,IM_k} L_{IM_2} L_{IM_k}^T \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{IM_k,IM_1} L_{IM_k} L_{IM_1}^T & \rho_{IM_k,IM_2} L_{IM_k} L_{IM_2}^T & \cdots & L_{IM_k} L_{IM_k}^T
\end{bmatrix}
$$

A necessary condition for the positive-definiteness of the cross-covariance matrix $LL^T$ is the positive definiteness of the correlation matrix for each IM, which is itself ensured by the method of Jayaram & Baker (2009). Unlike the conditional hazard approach, this method computes the correct covariance structure for each IM and does not require the choice of a
primary IM. A further consideration, is the adoption of multiple correlation models. A benefit of this method is that different correlation models may be easy implemented to compute the correlation matrices $C_{MI,i}, i = 1, \ldots, k$, which gives this method a great amount of flexibility.

A final method proposed by the authors to consider is the linear model of coregionalization as discussed by Loth & Baker (2013) and Wang & Du (2013) presented earlier in this section. Although simple to use, this model fits a single model to capture both the spatial correlation and spatial cross-correlation of the IMs, and thus it may fail to capture regional differences or inter-event differences.

The influence of spatial cross correlation was studied using a synthetic portfolio derived using Italian data. The correlation model used was Jayaram & Baker (2009), the cross-correlation model used was Baker & Cornell (2006), and the spectral correlation of the intra-event residuals is computed using the Goda & Atkinson (2009) model. It is noted that while most models have been developed to capture both the intra and interevent terms into one, this study separates the terms resulting in an inconsistency in which the models were fit to how they were applied.

An exposure model was developed which includes the spatial distribution of each building with its structural replacement cost. Buildings are separated into groups based on primary construction material, age, and number of stories. A limitation to this method is that the ground motion assumed is taken from the centroid of a cell, for which the buildings where distributed evenly based on population density, however, measures were taken to ensure that differences shown in these results are not attributed to this bias. A vulnerability model for each type of building was derived in order to fully understand the impact of spatial correlation and spatial cross-correlation on seismic risk assessment. The method for deriving these models is based on the work of Silva et al. (2013) and each function is dependent on the spectral acceleration for the period of vibration, which was calculated using the period-height relationships proposed by Crowley et al. (2004).

Weatherill et al. (2015) uses seven models to generate losses:
1. No spatial correlation or cross-correlation, inter-event residuals sampled independently for each period

2. Spatial correlation is considered separately for each quantity and inter-event residuals sampled independently for each period

3. Spatial correlation and cross correlation are modeled using the conditional hazard approach with $S_a(0.2s)$ chosen as the primary IM.

4. Spatial correlation and cross correlation are modeled using the conditional hazard approach with $S_a(1.2s)$ chosen as the primary IM.

5. Spatial correlation and cross correlation are modeled using the full-block cross correlation method. Spectral correlation of the inter-event residual is simulated using Goda and Atkinson 2009.

6. Spatial correlation and cross correlation are modeled using the LMCR method. Spectral correlation of the inter-event residual is simulated using Goda and Atkinson 2009.

7. Spatial correlation and cross correlation are modeled using the LMCR method. Uncertainty is represented using only the total $\sigma$ term.

Initially, analysis is computed using a single building type to ensure that the spatial correlation is in fact influencing the loss results. When spatial correlation is included, greater losses are observed at lower annual probabilities of exceedance. Further, the inclusion of spatial correlation has a greater impact on portfolios with a smaller spatial scale. A further observation is that for higher annual loss probabilities, spatial correlation may reduce loss estimates.

For a heterogeneous profile, including both types of correlation increases loss results at lower annual probability of exceedance for both spatial scales, although the trend is clearer for smaller spatial scale. The full-block model and the LMCR provide similar results when the residuals are separated, suggesting that the model for cross-correlation has little impact
on the losses. However, it was found that the predictions for the model without correlation were very similar to the model with correlation but without cross-correlation, suggesting that neglecting the cross-correlation prohibits the correct covariance structure. The conditional hazard method, when conditioned on the lowest period of spectral acceleration, was close to the uncorrelated results at the smaller scale and spatial correlation only at the higher spatial scale. However, when the longer period was used to condition the matrices, the results align more with the full-block and LMCR cross-correlation methods. Specifically, the conditional hazard method produces a sort of median between these two methods with less extreme values. Weatherill et al. (2015) suggest that the results obtained using this method may change depending on the loss portfolio and thus this method may not be viable. However, if it is to be used, conditioning on the longer period IM is suggested.

It is observed in Weatherill et al. (2015) that the full-block cross correlation, LMCR and LMCR using total $\sigma$ resulted in higher losses at lower annual probability of exceedance and lower losses at higher annual probabilities of exceedance, in comparison to models generated when cross-correlation was ignored. It is seen that the LMCR method when using total $\sigma$ produces lower losses than the split intraevent and interevent residual, suggesting that the way the uncertainty is handled may have a more significant influence on the losses than the method of computing cross-correlation.

It is also important to note that the portfolio itself plays a large role in the losses produced. The impact of spatial correlation on a portfolio is highly dependent on the degree of spatial clustering in the portfolio, and even the degree of clustering of the highest valued assets. To ensure that the full aggregation of the portfolio in this experiment was not biasing the results, results were obtained for simpler portfolios of distributed assets. It was found that the overall trends discussed previously still held, but the impact of spatial cross-correlation was smaller, suggesting that the study overestimated the impact but the trends hold true.
2.7 Spatial Correlation of Other Intensity Measures

In 2012, Foulser-Piggott & Stafford (2012) investigated the spatial correlation of Arias intensity. Arias intensity is a scalar measure that has been shown to reflect multiple characteristics of the ground motion and to be applicable for use in various problems in earthquake engineering. Arias intensity is defined as

\[ I_a = \frac{\pi}{2g} \int_0^{t_{tot}} a(t)^2 dt, \]

where \( a(t) \) is the ground acceleration, \( t_{tot} \) is the duration of the ground motion, and \( g \) is gravitational acceleration. Arias intensity has only been used for a few GMPEs, none of which involve spatial correlation. Using the same approach to fitting the semivariogram as Jayaram & Baker (2009), the data from the Chi Chi and Northridge events were used to fit semivariograms for Arias intensity with weighted least squares. In this case, two types of models are considered, the exponential model

\[ \gamma(h) = a(1 - \exp(-3h/b)), \]

and the sigmoid model,

\[ \gamma(h) = \frac{1}{1 + \exp\left(-\frac{\ln(h) + \phi_1}{\phi_2}\right)}, \]

where \( \phi_1, \phi_2 \) are free parameters. It was found that the exponential model overestimated the semi-variance in the Chi Chi event, while the Sigmoid model does better but still does not capture the trend exactly. The results from Northridge further confirmed that neither model completely captured the trends in the data, and further that the differences between the trends in Chi Chi and Northridge suggest one model may not be fit for use in multiple events.

In Du & Wang (2013), the intraevent spatial correlation of Arias intensity, cumulative absolute velocity (CAV), and PSA values were investigated. CAV is found to be well corre-
lated to structural damage and is therefore a useful statistic, especially in the area of loss modeling (Reed & Kassawara, 1988). In this study, more than 1,500 records from earthquakes in Taiwan, Japan, and California are used, and Vs30 information is gathered for each site. In the process of fitting, correction methods are used to fix bias in distance scaling and Vs30 estimation. As previously discussed, Vs30 prediction may artificially increase spatial correlation ranges of Vs30 values. To fix this issue, Vs30 values which are inferred are randomly redistributed about the mean value using Monte Carlo simulations with a log normal distribution. For each realization, the data was fit using the process outlined by Jayaram & Baker (2009), and the mean correlation range was found. Results are outlined below.

- For inhomogeneous Vs30 values, the correlation ranges of CAV, Ia, and PGA are relatively similar.

- For homogeneous Vs30, the range for PGA is the greatest, and the range for Ia is slightly larger than the range for CAV.

- As the period increased, the influence of regional site conditions on the spatial range of Sa values becomes weaker, consistent with Jayaram & Baker (2009).

- The use of different GMPEs to fit correlation coefficients results in very small changes, suggesting correlation coefficients are independent of the GMPE used.

Comparing the outcomes from this study with the results from Jayaram & Baker (2009), we see that the results are similar except at periods greater than 2s, where Jayaram & Baker (2009) predicts a smaller correlation range. This difference is attributed to the use of different databases to fit the model.

2.8 Furthering the Findings

In this section, we consider studies in topographic site amplification, and geographically varying GMPEs. Both of these methods attempt to capture the variability present in earthquake ground motions by relating it to the underlying physical parameters that cause
variability. In contrast the works previously discussed in this overview, these works attempt to remove the uncertainty from the ground motion model instead of statistically capture it. The two works discussed in this section are not comprehensive and serve to give brief examples of future work in this area.

### 2.8.1 Topographic Site Amplification

Numerous studies have related topographic site effects to the amplification of ground motion intensity measures. According to Maufroy et al. (2015), the maximum ground motion amplification is generally found at crests on the opposite side of the upcoming wavefield, while the minimum amplification occurs in deep valleys. This suggests that the topography impacts the spatial variability present in ground motions. In order to model this, Maufroy et al. (2015) introduces a new methodology to predict the topographic site effect based on the Earth’s curvature derived from digital elevation maps. By definition, the Earth’s curvature is the second derivative of the elevation map, which can be calculated using the method proposed by Zevenbergen & Thorne (1987). The curvature may be represented in grid form, where a positive value at a location means there is convex topography and a negative values indicates concave features. A smoothing algorithm in combination with linear regression can then be used to determine the relationship between the curvature and the median amplification factors.

In this experiment, a homogeneous linear elastic, half-space was assumed and a synthetic database was constructed for which focal mechanisms and locations are randomly generated beneath the area of interest, and where valleys, slopes, and hills are present. After several simulations, observations were made about the probability of exceeding amplification factors 2 and 3 with reference to the topology. It was found that the probability of exceedance for deep valleys was 2% and 0% for 2 and 3 respectively. However, for hills and crests, the probability of exceedance ranged from 50% to 80% for amplification factor 2 and 10% for hills and 30% for crests for a factor of 3. Thus it follows that the probability of exceedance is correlated with the curvature of the Earth, more so than the elevation. Further, this study
showed that the variability in PGA was the largest on slopes and large summits, while deep valleys experienced little variability and lower ground motion levels.

2.8.2 Geographically Varying GMPEs

It is often the case that GMPEs are fitted under the assumption that ground motion IMs are independent of the wave propagation path and/or focal mechanism and geometries because enough data is usually not available to fit GMPEs specifically for a specific focal mechanism or geometry. Spatial variability can be introduced into ground motion estimation maps through a method known as kriging. Kriging uses a linear combination of weighted measured values to estimate the ground motion at a point, while co-kriging uses similar techniques except that it incorporates other factors, such as correlations between other variables. However, neither of these techniques are physically informed. Another method, presented by Hong & Liu (2015), is to develop a geographically varying ground motion prediction model (GVGMPM) based on ground motion records and the underlying physics of the ground motion. Beginning with a physics based GMPE, a new GMPE is fit by using geographically weighted linear regression (Fotheringham et al., 2002). The essence of this method is to estimate a coefficient using the data at neighboring sites, weighting these data by the separation distance. The coefficient is the obtained by minimizing the sum of weighted square error.

This method was used on the Chi Chi and Wenchuan earthquakes, and the results were compared against traditional kriging and co-kriging techniques, as well as a regular GMPE. For the Chi Chi event, the GVGMPM outperformed the GMPE and both kriging techniques, although the mean error was slightly smaller for the traditional kriging technique than the GVGMPM. Results were similar for the Wenchuan earthquake, and suggest that this method should be considered for future ground motion predictions. Unfortunately, this method does require a significant amount of data in order to compute the GVGMPM coefficients, making the method difficult to use for smaller, or poorly documented events.
2.9 Conclusion

The consideration of spatial variability in earthquake ground motions can impact hazard and loss assessments. This field of research is still very active and much work remains to fully understand the best methods to capture spatial variability and the impact on seismic hazard analysis. In Chapter 3, we focus on one method to compute a spatially correlated random field of ground motion residuals as suggested in Park et al. (2007). Specifically, we will use the Jayaram & Baker (2009) correlation model to compute several realizations of spatially variable earthquake ground motions, which can then be used to compute hazard and loss estimates.
CHAPTER 3
SPATIALLY CORRELATED RANDOM FIELDS FOR SHAKEMAP

Although variability in seismic ground motions was documented well before the 1980’s, it wasn’t until the installation of dense seismic array networks that enough data was available to enable a more in depth analysis on the spatial variability of ground motions. Variability in ground motions can be present in two forms: waveform variability (coherency) or amplitude variability (correlation). Modeling the spatial variation in amplitude as a result of coherent contributions from source, path, and site can present a more realistic picture of ground motions. As such, many studies (Boore et al., 2003; Goda & Hong, 2008; Kawakami & Mogi, 2003) have investigated spatial correlation present in ground motions. Further, realizations of spatially variable ground motions can provide useful loss distributions and hazard information. In order to capture the spatial correlation, we will investigate the site-to-site intra-event standard normal error term discussed by Park et al. (2007). This term, in combination with the inter-event term, is modeled as a spatially correlated field of random residuals, possibly conditioned upon ground motion data, which can then be added to the ShakeMap intensities to give a single realization. The spatially correlated field may be computed using an isotropic correlation function dependent only on the distance from one site to another and the frequency of the intensity measure (IM). Although several correlation models have been developed which describe the decay in correlation from intensities at one site to another with increasing separation distance, this paper will focus on the model described by Jayaram & Baker (2009).

Computing the spatially correlated field using the conditional simulation method is memory intensive and computationally expensive. Since one aim of this paper is to produce loss results in near real-time, it is necessary to optimize these calculations to produce results for many realizations quickly and with as little memory as possible. The method of successive
conditional simulation (SCS) implemented in this paper is an iterative method which breaks down the computation of the spatially correlated random field into several calculations. Under the assumption of a radius of influence, or a specified separation distance for which points are no longer correlated, this method is capable of computing the field with reduced memory. Further, this method enables parallelization which can greatly decrease computational time and increase the efficiency of the computation. Utilizing the SCS method, several realizations of the field are computed for the Northridge, Loma Prieta, and Chi Chi events and are added to the original ShakeMap to produce loss distributions. Both estimated fatalities and economic losses are computed using the USGS PAGER system, and the resultant distributions are analyzed to see the effects of both spatial variability and the addition of observational data.

A description of the mathematics used in this paper is given in Section 3.1. The SCS method is described in Section 3.2 and optimization techniques are provided in Section 3.3. Efficiency and convergence results, as well as outcomes from directivity and a description of the economic loss and fatality calculations as well as results experiments are given in Sections 3.4-3.6.

3.1 Mathematical Background

We will begin with the standard form of a Ground Motion Prediction Equation (GMPE) as described by Abrahamson & Silva (1997):

\[
\ln Y_{i,j} = \ln Y_{i,j}^{\text{original}} + \tau \eta_i + \sigma \epsilon_{i,j},
\]

where \( Y_{i,j} \) is the ground motion IM at site \( j \) during earthquake \( i \), \( \eta_i \) is the inter-event standard normal error term with standard deviation \( \tau \), and \( \epsilon_{i,j} \) is the intra-event term with standard deviation \( \sigma \). \( \ln Y_{i,j} \) is the original ShakeMap ground motion intensity data. The inter-event term, which is currently present in ground motion prediction equations (GMPEs), captures the correlation of IMs at sites that is a result of being induced by the same earthquake.
Following Park et al. (2007), we will rewrite the equation as

\[ \ln Y_{i,j} = \ln \gamma_{i,j} + \tilde{\sigma} \tilde{\epsilon}_{i,j}, \]  

(3.2)

where \( \tilde{\sigma} = \sqrt{\sigma^2 + \tau^2} \) and \( \tilde{\epsilon}_{i,j} = \frac{\tau \eta_{i,k} + \sigma \epsilon_{i,j}}{\tilde{\sigma}} \) such that \( \tilde{\epsilon}_{i,j} \) captures both the intra and inter event standard normal error terms. Thus, we are interested in calculating \( \tilde{\sigma} \tilde{\epsilon}_{i,j} \) for an earthquake \( i \) at each site \( j \), and for a given number of realizations. For a single realization and earthquake, \( \tilde{\sigma} \tilde{\epsilon}_{i,j} \) is a field of spatially correlated random ground motions, where the correlation between two points depends only on their separation distance and the frequency of the ground motions.

Jayaram & Baker (2009), Goda & Hong (2008), Goda & Atkinson (2009), and several other authors have fit separation distance and frequency dependent correlation functions using semivariograms. Grouping both the intra and inter event terms into one eases this process as a single correlation function may be used for both the intraevent and interevent variability. In this paper we will use the correlation function described by Jayaram & Baker (2009) defined to be

\[ \rho(h, T) = \exp(-3h/b), \]

where \( h \) is the separation distance between two sites, \( T \) is the period of the ground motions, and

\[ b = \begin{cases} 
8.5 + 17.2T & : V_{s30} \text{ clustering} \\
40.7 - 15.0T & : V_{s30} \text{ do not cluster} \\
22.0 + 3.7T & : T \geq 1.
\end{cases} \]

The conditions on the boolean \( V_{s30} \) clustering parameter are not fully described so by default we will assume \( V_{s30} \) clustering for all earthquakes considered. Further, we will narrow our consideration to PGA and PGV values where we will assume \( T = 0, 1 \) respectively. Thus the correlation models used in this paper are given by

\[ \rho(h, 0) = \exp(-3h/8.5), \]
for PGA and
\[ \rho(h, 1) = \exp(-3h/25.7), \]
for PGV.

Consider a ShakeMap grid for a fixed earthquake \( i \) at sites \( j = 1, 2, \cdots, j_N \), where \( j_N \) is the total size of the grid and each site may be either ShakeMap grid points or observational data. Let \( h_{jk} \) denote the geodetic distance between sites \( j \) and \( k \) and define
\[
D = \begin{bmatrix}
    h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,j_N} \\
    h_{1,2} & h_{2,2} & h_{2,3} & \cdots & h_{2,j_N} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    h_{1,j_N} & h_{2,j_N} & h_{3,j_N} & \cdots & h_{j_N,j_N}
\end{bmatrix},
\]
to be the geodetic distance matrix between all sites \( j, k = 1, \cdots, j_N \). Note that under the assumption of isotropy, the distance matrix is symmetric and positive with 0s on the diagonal. The covariance matrix, \( \Sigma \), for all sites \( j, k = 1, \cdots, j_N \) can then be calculated by applying the correlation function to the distance matrix, \( D \). Since the correlation function is a decaying exponential, we have that \( \Sigma \) is a symmetric matrix with 1’s on the diagonal and positive values less than one on the off diagonal, under the assumption that no two grid points are at the same location.

Define \( X \in \mathbb{R}^{j_N \times 1} \) to be the vector of intraevent residuals,
\[
X = [\tilde{\epsilon}_{i,1}, \tilde{\epsilon}_{i,2}, \cdots, \tilde{\epsilon}_{i,j_N}]^T.
\]
In 2010, Jayaram and Baker published a paper which validated the assumption of normality of joint distributions of ground motion data. Therefore, under the assumption that the ground motions \( \ln Y_{i,j} \), \( j = 1, 2, \cdots, j_N \) are jointly normally distributed, \( X \) may be calculated using a multivariate normal distribution with site-to-site covariance matrix, \( \Sigma \). Or
equivalently,

\[ X \sim N_M(0, \Sigma). \]

If \( \Sigma \) is positive definite, we will use the computationally efficient form,

\[ X = R y, \]

where \( R \) is the Cholesky decomposition of the covariance matrix \( \Sigma \) and \( y \) is a standard normal random vector. We notice here that the covariance matrix, \( \Sigma \), and corresponding Cholesky decomposition, \( R \), is independent of the realization. Thus, the variation in each realization is dependent only on the standard normal random vector \( y \).

In the case where ground motion observations are present, it is necessary to condition \( X \) on the standard normal error terms for the empirical observations, which we will denote \( X_{\text{obs}} \). We can calculate \( X \) using a conditional multivariate normal distribution given by

\[
\begin{bmatrix}
X_{\text{obs}} \\
X
\end{bmatrix} \sim N_M \left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} \right),
\]

where \( \Sigma_{11} \) is the covariance matrix between observation points, \( \Sigma_{22} \) is the covariance between grid points \( j = 1, \cdots, j_N \), and \( \Sigma_{12} = \Sigma_{21}^T \) is the covariance between observation and grid points. For \( X_{\text{obs}} = x \), this expression may be rewritten as

\[
[X|X_{\text{obs}} = x] \sim N_M(\Sigma_{21} \Sigma_{11}^{-1} x, [\Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}] ),
\]

where \( \Sigma_{11}^{-1} \) is the generalized inverse of \( \Sigma_{11} \). If \( R \) is defined to be the Cholesky decomposition of the modified covariance matrix, \( \Sigma_{22} = \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \), the equivalent expression,

\[
[X|X_{\text{obs}} = x] = \Sigma_{21} \Sigma_{11}^{-1} x + R y,
\]

may be used where \( y \) is a standard normal random vector. Under the assumption that the uncertainty is 0 at observation sites, we have that \( X_{\text{obs}} = 0 \) and this expression may be
further reduced to

\[ [X|X_{obs} = 0] = Ry. \]

Thus, for a ShakeMap grid with or without observational data, we may compute the vector of spatially correlated random data using the covariance matrix and a standard normal random vector. After \( \tilde{\epsilon}_{i,j} \) is computed for all sites of interest, \( X \) may be added to the median ground motions using Equation 3.2. Although \( X \) captures the inter and intra event variation for every point on a ShakeMap grid, computing the entire vector at once requires a significant amount of memory and computation time. This is largely due to the geodetic distance matrix calculation, which for a \( M \times N \) grid must produce a \( M \times N \times M \times N \) matrix which requires four times the final storage memory for intermediate storage. An additional consideration is the number of seismic stations included in the data. A large number of stations will greatly increase the memory usage and computational time required. A method to reduce the computational requirements, namely the SCS method, is explored.

### 3.2 Successive Conditional Simulation

By using the appropriate geodetic distance matrices and defined correlation function, the vector of \( \tilde{\epsilon}_{i,j} \), denoted \( X \) can be calculated using the expression

\[ X = Ry, \]

where \( y \) is a standard normal random vector and \( R \) is the cholesky decomposition of either \( \Sigma \) or \( \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \) depending on whether observation data is used. To alleviate the computational requirements needed to compute the entire vector \( X \) at once, we will compute the entries one at a time using the SCS method. For each grid point \( j = 1, 2, \ldots, j_N \), \( \tilde{\epsilon}_{i,j} \) is calculated conditionally upon observation data and any previously calculated \( \tilde{\epsilon}_{i,j} \). To facilitate this calculation we will define a new vector, \( X_{obs+prev,j} \), which is the concatenation of \( X_{obs} \) and \( X[1 : j - 1] \). We may compute the value of \( \tilde{\epsilon}_{i,j} = [X]_j \) using the following equation:

\[ ([X]_j|X_{obs+prev,j} = x) = \Sigma_{21,j}\Sigma_{11,j}^{-1}x + R_jy_j, \]
where $y_j$ is a standard normal random variable, $R_j$ is the Cholesky decomposition of $\Sigma_{22,j} - \Sigma_{21,j}^{-1}\Sigma_{21,j}$ and the form of each $\Sigma$ is described below.

In the case where no observation data is included, the value for each $\tilde{\epsilon}_{i,j}$ is computed conditionally on previously calculated $\tilde{\epsilon}_{i,j}$ only. For example, when computing the first entry, $\tilde{\epsilon}_{i,1}$, no data is available and thus

$$\tilde{\epsilon}_{i,1} = y_1.$$ 

The second entry, $\tilde{\epsilon}_{i,2}$ is computed conditioned upon the first entry. In this case,

$$\Sigma_{11,2} = \rho(D_{11}), \quad D_{11} = \begin{bmatrix} h_{1,1} \end{bmatrix},$$

$$\Sigma_{12,2} = \Sigma_{21,2} = \Sigma_{22,2} = \rho(D_{12}), \quad D_{12} = \begin{bmatrix} h_{1,2} \end{bmatrix},$$

$$x_2 = \tilde{\epsilon}_{i,1}.$$ 

In the $n$th case, $\tilde{\epsilon}_{i,1}, \ldots, \tilde{\epsilon}_{1,n-1}$ have been calculated and each $\Sigma$ can be calculated as follows:

$$\Sigma_{11,n} = \rho(D_{11}),$$

$$D_{11} = \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n-1} \\ h_{1,2} & h_{2,2} & \cdots & h_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1,n-1} & h_{2,n-1} & \cdots & h_{n-1,n-1} \end{bmatrix},$$

$$\Sigma_{12,n} = \rho(D_{12}), \quad D_{12} = \begin{bmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{n-1,n} \end{bmatrix},$$

$$\Sigma_{22,n} = \rho(D_{22}), \quad D_{22} = [h_{n,n}],$$

$$x_n = [\tilde{\epsilon}_{i,1}, \ldots, \tilde{\epsilon}_{1,n-1}]^T.$$ 

After computing each $\tilde{\epsilon}_{i,j}$, the spatially correlated field may be added to the ground motion using Equation 3.2.
In the case where observation data is used, the value for each \( \tilde{\epsilon} \) is simulated conditioned upon the observation data in addition to each previously calculated \( \tilde{\epsilon}_{i,j} \). For \( K \) observations, let \( s_{j,k} \) denote the geodetic distance between two observed data points, \( j \) and \( k \), for \( j, k = 1, \cdots, K \). Further, let \( d_{j,k} \) denote the geodetic distance between observation \( j \), \( j = 1, \cdots, K \) and site \( k \), \( k = 1, \cdots N \). Then for the first entry, \( \tilde{\epsilon}_{i,1} \),

\[
\Sigma_{11,1} = \rho(D_{11}),
\]
\[
D_{11} = \begin{bmatrix}
  s_{11} & s_{12} & \cdots & s_{1,K} \\
  s_{1,2} & s_{2,2} & \cdots & s_{2,K} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{1, K} & s_{2, K} & \cdots & s_{K, K}
\end{bmatrix},
\]
\[
\Sigma_{12,1} = \rho(D_{12}), \quad D_{12} = \begin{bmatrix}
  d_{1,1} \\
  d_{2,1} \\
  \vdots \\
  d_{K,1}
\end{bmatrix},
\]
\[
\Sigma_{22,1} = \rho(D_{22}), \quad D_{22} = [h_{1,1}],
\]
\[
x_1 = [0] \in \mathbb{R}^{K+1}.
\]

After calculating the first entry, the second entry, \( \tilde{\epsilon}_{i,2} \), can be simulated conditional upon both the observation data and \( \tilde{\epsilon}_{i,1} \). For this entry,

\[
\Sigma_{11,2} = \rho(D_{11}),
\]
\[
D_{11} = \begin{bmatrix}
  s_{11} & s_{12} & \cdots & s_{1,K} & d_{1,1} \\
  s_{1,2} & s_{2,2} & \cdots & s_{2,K} & d_{2,1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{1, K} & s_{2, K} & \cdots & s_{K, K} & d_{K,1} \\
  d_{11} & d_{2,1} & \cdots & d_{K,1} & h_{1,1}
\end{bmatrix},
\]
\[
\Sigma_{12,2} = \rho(D_{12}), \quad D_{12} = \begin{bmatrix}
  d_{1,2} \\
  d_{2,2} \\
  \vdots \\
  d_{K,2} \\
  h_{1,2}
\end{bmatrix},
\]
\[
\Sigma_{22,2} = \rho(D_{22}), \quad D_{22} = [h_{2,2}],
\]
\[
x_2 = \begin{bmatrix}
  0 \\
  \tilde{\epsilon}_{i,1}
\end{bmatrix} \in \mathbb{R}^{K+1}.
\]
Using the SCS method, the entire vector $X$ can be calculated one entry at a time and the results from using this method are mathematically identical to calculating the entire vector, $X$ at once. A notable difference in this method is that the distance matrix and corresponding covariance matrix must be computed for each point. It can be seen that the computational requirements for computing just the last entry, $\tilde{\epsilon}_{i,jN}$, are nearly the same as those for computing the entire vector $X$ at once. Thus, the benefit of using this method comes from utilizing a radius of influence, where it is assumed that only points within a certain range of the current point will influence the value of each $\tilde{\epsilon}_{i,j}$. This assumption is validated because the correlation function decreases exponentially as a function of increasing distance and eventually asymptotes to a zero correlation. Using a radius of influence will greatly decrease the size of the covariance matrices at each point. The reduced size of these matrices greatly reduces the memory requirements to complete the simulation as well as the computational time required to compute the geodetic distances, invert $\Sigma_{11}$, and compute the Cholesky factorization. Additionally, this method lends itself to parallelization as it can be separated into several independent tasks as described in Section 3.3. For an illustrative example of the correlation matrix, ShakeMap, and spatial variable ShakeMap, see Figure 3.1.

The method may be implemented in a few different ways depending on the sampling path, or the order in which $\tilde{\epsilon}_{i,j}$ is calculated. Goovaerts (1997) suggests that a random sampling path for each realization will prevent the formation of artificial continuity along a given path. However, if the same path is used for each realization, several key components of the covariance matrices may be saved and reused. Following a different, random path for each simulation would require significantly more memory and computational time than following the same path. Thus once a sampling path is chosen, it will be reused for each subsequent realization. For this paper, the path chosen is row by row through the matrix from left to right. This path calculates the $\tilde{\epsilon}_{i,j}$ in the same order as calculating the entire vector $X$ at once, and therefore using this path will allow for direct comparisons between the conditional simulation method and the approximate SCS method when using various radii.
Figure 3.1: One instance of the generated correlation matrix for the Loma Prieta earthquake (a,c,e) (1989) with 0 seismic stations and the Northridge earthquake (b,d,f) (1994) conditioned on 185 stations. These results were generated using the SCS approach with Jayaram & Baker (2009) with radius 45km.
A second path considered, but not described in this paper, is a random path through the grid that will be saved and followed for each subsequent realization. This method calculates $\tilde{\epsilon}$ in a different order than the conditional simulation method and therefore cannot be directly compared as it produces different results. Additionally, since the same random path is used for each realization, the benefits of an initially random path are unknown.

Multiple realizations may be simulated quickly using this method by saving select elements from each iteration. Since each $\tilde{\epsilon}$ is dependent on previously calculated values, the entries of the correlation matrix must be calculated linearly in time following the selected sampling path. However, computationally expensive processes, such as computing the distance matrices and corresponding covariance matrices for each point, may be computed independently of each other and saved for reuse in realizations.

### 3.2.1 Choosing the Optimal Radius of Influence

In order for the $\tilde{\epsilon}$ values derived from the SCS method to provide a good approximation to those derived from the conditional simulation method, an appropriate radius needs to be chosen. Mathematically, if the radius of influence was chosen to be infinite, the conditional and SCS methods would yield the exact same result. However, as larger radius values increase the computational time required to complete the simulation, it is desired to find the minimum radius that will give a close approximation to the $\tilde{\epsilon}$ terms derived from the conditional simulation method. Determining the radius is then a problem of defining what a close enough approximation is.

Using the same set of random vectors, several simulations were performed using the successive method with various radii and the results can be compared to those derived using the conditional simulation approach. The differences in the correlation matrices derived using the conditional method and the SCS method in vector form are plotted in Figure 3.2 for various radii. The errors in the correlation matrix for radius of 15 km are on the order of $\pm 0.15$, while the errors for a radius 65 km are $\pm 4e^{-5}$ and it can be seen that error decreases by a factor of approximately 4 for each 10 km increase in the radius. The magnitude of the
errors is approximately constant across all of the points, and thus it is assumed the errors
decrease evenly across the grid as the radius increases. Since \( X \) is approximately standard
normal with data ranging from \( \pm 3.14 \), the error using a radius of 35 km is small enough to
be ignored. Thus we will use a radius of at least 35 km.

Figure 3.2: Point by point differences in the correlation matrices of the conditional simulation
method versus the SCS method with various radii. The correlation matrices were computed
for the Northridge event with no included observation data.

Table 3.1: Mean and standard deviation of fatality loss distributions for Northridge with 0
and 185 stations and Loma Prieta with 0 and 185 stations using the conditional method and
various radii

<table>
<thead>
<tr>
<th></th>
<th>NR0 mean</th>
<th>std.</th>
<th>NR185 mean</th>
<th>std.</th>
<th>LP0 mean</th>
<th>std.</th>
<th>LP185 mean</th>
<th>std.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional Method</td>
<td>32</td>
<td>25.8</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
<tr>
<td>R = 15</td>
<td>32</td>
<td>25.0</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.0</td>
<td>29</td>
<td>3.6</td>
</tr>
<tr>
<td>R = 25</td>
<td>32</td>
<td>25.6</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
<tr>
<td>R = 35</td>
<td>32</td>
<td>25.8</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
<tr>
<td>R = 45</td>
<td>32</td>
<td>25.7</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
<tr>
<td>R = 55</td>
<td>32</td>
<td>25.8</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
<tr>
<td>R = 65</td>
<td>32</td>
<td>25.8</td>
<td>47</td>
<td>4.5</td>
<td>10</td>
<td>7.1</td>
<td>29</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Since the correlation matrices computed using this method will be used to calculate the
predicted fatalities and economic losses for a given earthquake event or scenario, the radius
that is selected should be able to produce approximately the same loss distribution as the conditional simulation method in a reasonable number of realizations. Loss calculations are completed using the USGS PAGER (Prompt Assessment of Global Earthquakes for Response) system which computes a range of possible fatalities and economic losses. Using the earthquake’s magnitude and location, PAGER estimates the ground shaking and combines these estimates with Oak Ridge National Laboratory’s Landscan global population database to calculate the number of people exposed to various levels of shaking. Estimates of the ranges of potential fatalities and economic losses are then computed based on country-specific loss models. The mean and standard deviation of computed fatality distributions for Northridge with 0 and 185 stations (NR0, NR185) and Loma Prieta with 0 and 112 stations (LP0, LP112) are in Table 3.1.

It can be observed in Table 3.1 that the addition of seismic stations reduces the spread of losses for all radii considered. In fact, for both events with added seismic stations, a radius as small as 25 km produces the same loss distribution as the conditional simulation method. In light of this data, the optimal radius was chosen to be 45 km as each loss distribution converged to the conditional method for this radius and it is still small enough to increase the efficiency of the method. A more in depth discussion of fatality and economic loss results is presented in Section 3.6.

3.3 Approaches to Increase Efficiency

The SCS approach lends itself to optimization in many ways. The distance calculations that must be done to generate the covariance matrices are computationally expensive as well as memory intensive. In using the SCS method with the assumption of a radius of influence, the memory requirements for this computation are already reduced. By using the properties of the distance matrix, the computational requirements can be reduced further. Since both the distance and covariance matrices are symmetric, computing the matrices requires only half of the entries need to be calculated. We may also take advantage of the gridded structure of a ShakeMap to reduce calculations. Using the ShakeMap grid, the
distances between points vertically and horizontally is the same for each row. Thus, we can compute one distance matrix that captures the distances from one point to all possible points within the radius once per row that can be used as a stencil for every point in the row. This stencil matrix may be modified for points on the edge of the grid by simply removing rows and columns for the points which are not included. Additionally, if any stations are included within the radius, the stencil distance matrix may be expanded to include the stations while preserving the original stencil as a submatrix of the new matrix. For those points within the row that contain every point in the stencil and no added station data, the calculated $\Sigma$ can be stored and reused. Taking advantage of these optimization techniques greatly reduces the number of correlation matrices that are necessary to compute and the number of inversions of $\Sigma_{11}$ the code requires.

3.3.1 Parallelization

Although using the SCS method and the further optimization techniques greatly reduced the computational requirements necessary to compute the spatially correlated fields, the desire for near real-time speeds requires the utilization of parallel processes. In order to implement this, the computation must be able to be divided into several independent tasks. While the computation of each $\tilde{\epsilon}_{i,j}$ is dependent on previous values and cannot be parallelized directly, several computationally expensive tasks, such as computing the distance matrix for each point, can be done in any order and saved for computations at a later time. The parallelization of this method begins with splitting up the grid into several ($n$) subgrids, where $n$ is the number of cores to be used. Each core will compute the distance matrices and any other information necessary and store these values for each point with its subgrid. When finished the cores will send the information to the master core which will assemble all the information. The master may then compute the correlation matrix point by point using the computed information and avoiding any computationally expensive calculations. Additionally, the master may send the information to all other cores so that multiple realizations can
be computed simultaneously.\footnote{Parallelization of this method was completed using mpi4py, a Message Passing Interface (MPI) for Python library developed by SciPy.}

The approach of dividing the grid into subgrids was given careful consideration as the method of doing so affects the load balancing and efficiency. Since the distance matrices, and therefore correlation matrices, for each point in a row can be derived using a single, stencil distance matrix, it is efficient to divide up the grid by groups of rows. As points near the top of the grid will have fewer points in their radius, the computational time required to process these rows will be shorter; whereas, points that are located by stations will need more computational time to compute the additional distance vectors within their distance matrix. To achieve load balancing, it is necessary to distribute these rows as evenly across the cores as possible. This was achieved by distributing the rows one at a time to the cores. For example, for 8 rows and 4 cores, core 0 would work with rows 1 and 5, core 1 with rows 2 and 6, and so forth. This method guarantees that a single core will not have all the rows at the top of the grid while also ensuring that the rows with a high concentration of stations are split between the cores.

To measure the effect of parallelization on the code, the speedup, efficiency, and Karp-Flatt metric are calculated. The relative speedup is given by

$$S_P^Q = \frac{T_Q}{T_P},$$

where $S_P$ is the speedup from using $P$ cores and $S_Q$ is the speedup of the first relevant core, $T_Q$ is the time for $Q$ cores to run the code, and $T_P$ is the time for $P$ cores to run the code. If $Q = 1$, then this equation gives the simple speedup equation. A speedup of $P$ is referred to as linear speedup and represents perfectly efficient parallelization. The efficiency of using $P$ cores is given by

$$E_P = \frac{S_P^Q}{P}.$$
Although using additional cores may reduce computational time, the added benefit from using an additional core is likely to decrease as more cores are added and communication time is increased. In a perfectly efficient system with linear speedup, it is noted that $E_P = 1$. An additional measure, the Karp-Flatt metric, is calculated using the following equation:

$$
\hat{f} = \frac{1}{SP} - \frac{1}{P}
$$

This metric will represent the portion of the code run in serial. Ideally, as the $P$ increases, $f$ should decrease; however, as efficiency decreases $f$ may increase as a result.

Figure 3.3: CPU times and speedup
The figures included have two time measurements: the total CPU time and the time for realizations only. Although the entire code was parallelized, the realizations are naturally independent, and thus statistics were gathered on realization time to support this. Figure 3.3 and Figure 3.4 summarize the results in graphical form.

In Figure 3.3(a) and Figure 3.3(b) the total and realization CPU time are plotted against the number of realizations for $P = 1 - 128$ cores. As expected, for $P < 32$ the CPU time grows nearly linearly with the number of realizations for both the total and realization CPU times. For larger $P$ values, the CPU time does not grow with the number of realizations but instead appears to be stagnant. This is likely a result of using many cores and the necessary
communication between the cores.

In Figure 3.3(c), the speedup results are shown for the total CPU time and various realizations. In general, it can be seen that the speedup gradually decreases as the number of cores increases, but decreases at a slower rate for a higher number of realizations. The former is likely caused by the increase in communication time between cores as the number of cores increases. The latter is caused by the efficiency of the parallelization for computing the realizations. In Figure 3.3(d), it can be seen that the speedup for all numbers of cores and all realizations is linear or very close to linear. This is because the computation is embarrassingly parallel. As the number of realizations increases, the amount of CPU time to compute the realizations increases with respect to the total CPU time. Therefore the phenomena in Figure 3.3(c) can be explained by the linear speedup in Figure 3.3(d).

Figure 3.4 shows the resultant efficiency and Karp-Flatt metric for both the total and realization CPU time. The efficiency for the total CPU time is reflective of the observations made in the previous paragraph for the total CPU time. The Karp-Flatt metric, which is an approximate measure of the serial percentage of a code, decreases with the number of cores for most realizations and is in general lower for higher number of realizations. Figure 3.4b and Figure 3.4d are reflective of the linear speedup shown for the realization time. Sharp or random fluctuations in these graphs are likely not representative of any sort of pattern or oddity and in high probability would be smoothed by averaging the results of many simulations.

A flowchart is presented in Figure 3.12 which briefly summarizes the steps of the SCS method.

3.4 Efficiency and Memory Reduction

In order to measure the efficiency of the SCS method as compared to the original method, we gather the CPU time and memory requirements for both methods on the Northridge event with 33,366 grid points (3 km spacing). We use Monte Carlo simulations with 10,000 to 100,000 samples and plot the CPU times for the original method and the SCS method
using $P = 1, 2, 4$ cores in Figure 3.5. The CPU time for the original method increases nearly linearly with increasing realizations, as expected. In comparison, the CPU time for the SCS method run with 1 core increases at a faster rate, or is slower by comparison. When the method is run with 2 cores, the CPU times are approximately equal to those of the full method, and with increasing cores the CPU time continues to decrease. Thus, for this example, the SCS method achieves better computational times when the number of cores used is at least 2. Another factor to consider is the memory requirements for each method. When using the Northridge event with 33,366 grid points, the original method uses approximately 25 Gb memory, while the SCS method needs only 980 Mb when run with 1 core. If the SCS method is instead run with 2 cores, the memory increases to about 1200 Mb, which is still remarkably less than that of the original method. The trade off between memory consumption and CPU time for the SCS method is apparent, but with the slow growth of memory for increasing cores the SCS method may achieve less CPU time and memory consumption than the original method. To further this, both methods were re-run using a Shakemap for Northridge with the same area but with 1/2 grid spacing (1.5 km).
This approximately quadruples the number of grid points. The regular method consumed about 35 Gb of memory while the SCS method needs 1.5 Gb.

### 3.5 Convergence Results

![Log-log Plot of Convergence Errors vs Number of Realizations](image)

Figure 3.6: Convergence of the sum of spatially correlated fields to zero for increasing realizations.

Since the spatially correlated field is constructed using a standard normal random field, we expect that realizations of the spatially correlated random field would have mean zero. To test this assumption, we compute the convergence of the field, $X$, to the zero field using Monte Carlo simulation. The measure of convergence is defined to be:

$$C_N = \frac{1}{N \times (N - 1)} \sum_{n=1}^{N} ||X||_2,$$

where $N$ is the number of realizations. Note that $|| \cdot ||_2$ denotes the Euclidean norm, or the matrix 2-norm. Using the Shakemap for the Northridge event with 33,366 grid points, the convergence was tested for a range of $N$ values from 10 to 100,000. Graphical results are provided in Figure 3.6 in the form of a log-log plot. These results confirm that $C_N$ decreases to zero as the number of realizations increases.
3.6 Loss Results

Adding spatial variability to earthquake ground motions requires the addition of a spatially correlated random field to the ShakeMap estimated values. Thus, a single realization, although easy to visualize, cannot completely characterize the variability in ground motions. In contrast, when hundreds or thousands of realizations are computed, visualization becomes an issue but the variability is likely characterized by the entire set of realizations. To capture the results, the PAGER system is used to compute estimated fatalities and economic losses for each realization which are combined to form a loss distribution. A few other considerations are tested, such as the convergence of the loss distribution with the number of realizations and the impact of stations on the loss distribution, which are described below. We also present results from the combined effects of directivity and spatial variability.

Using the successive conditional simulation method, we compute sets of 10-10,000 realizations and compute the losses associated with each for Northridge with and without stations. By comparing the mean, median, standard deviation, and max and min values, we determine how many realizations are needed to capture the distribution of the ground motions. These values are shown in Table 3.2 and Table 3.3.

Table 3.2: Distribution properties for estimated fatalities using R realizations for Northridge

<table>
<thead>
<tr>
<th>R</th>
<th>No Station Data</th>
<th></th>
<th></th>
<th></th>
<th>185 Stations</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>µ</td>
<td>Median</td>
<td>σ</td>
<td>Max</td>
<td>Min</td>
<td>µ</td>
<td>Median</td>
<td>σ</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>11.09</td>
<td>6.40</td>
<td>34</td>
<td>13</td>
<td>35.1</td>
<td>35.5</td>
<td>1.92</td>
</tr>
<tr>
<td>25</td>
<td>19.76</td>
<td>19</td>
<td>7.69</td>
<td>34</td>
<td>9</td>
<td>34.84</td>
<td>35</td>
<td>3.06</td>
</tr>
<tr>
<td>50</td>
<td>21</td>
<td>20.5</td>
<td>9.05</td>
<td>44</td>
<td>7</td>
<td>34.96</td>
<td>35</td>
<td>3.52</td>
</tr>
<tr>
<td>100</td>
<td>22</td>
<td>21</td>
<td>8.61</td>
<td>44</td>
<td>8</td>
<td>35</td>
<td>35</td>
<td>3.20</td>
</tr>
<tr>
<td>500</td>
<td>21</td>
<td>20</td>
<td>8.38</td>
<td>49</td>
<td>5</td>
<td>35</td>
<td>35</td>
<td>3.16</td>
</tr>
<tr>
<td>1,000</td>
<td>21</td>
<td>20</td>
<td>8.55</td>
<td>59</td>
<td>5</td>
<td>35</td>
<td>35</td>
<td>3.21</td>
</tr>
<tr>
<td>5,000</td>
<td>21</td>
<td>20</td>
<td>8.74</td>
<td>60</td>
<td>3</td>
<td>35</td>
<td>35</td>
<td>3.34</td>
</tr>
<tr>
<td>10,000</td>
<td>21</td>
<td>20</td>
<td>8.67</td>
<td>60</td>
<td>3</td>
<td>35</td>
<td>35</td>
<td>3.34</td>
</tr>
</tbody>
</table>

The differences shown in these tables confirm that the loss distribution is well represented by using 1,000 realizations, and in fact, most of the properties are captured using only 50
Table 3.3: Distribution properties for estimated economic losses using R realizations for Northridge. All values in billions.

<table>
<thead>
<tr>
<th>R</th>
<th>No Station Data</th>
<th>185 Stations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>μ</td>
<td>Median</td>
</tr>
<tr>
<td>10</td>
<td>44.3</td>
<td>40.6</td>
</tr>
<tr>
<td>25</td>
<td>32.5</td>
<td>28.1</td>
</tr>
<tr>
<td>50</td>
<td>35.2</td>
<td>30.5</td>
</tr>
<tr>
<td>100</td>
<td>38.5</td>
<td>34.5</td>
</tr>
<tr>
<td>500</td>
<td>37.5</td>
<td>32.8</td>
</tr>
<tr>
<td>1,000</td>
<td>38.1</td>
<td>33.4</td>
</tr>
<tr>
<td>5,000</td>
<td>37.5</td>
<td>32.5</td>
</tr>
<tr>
<td>10,000</td>
<td>37.0</td>
<td>31.7</td>
</tr>
</tbody>
</table>

realizations. Further, this suggests that the median loss value is not achieved until at least 25 realizations are used, and thus this should be the minimum considered. Hereafter, we will use 1,000 realizations to ensure we have accurate results while still saving CPU time.

In considering the differences between the distributions with and without station data, we can see noticeable changes between nearly every statistic measured. In particular, the mean and median values increased when stations were added for both distributions, suggesting that the lack of data in the model can result in an underestimation of loss values. Further, the standard deviation decreased and the maximum and minimum values become smaller and larger respectively, resulting in a narrower distribution. This result is expected, as conditioning on the data will result in lower overall variability in the ground motion grid. These results are illustrated in Figure 3.7, where the economic loss distributions for 1000 realizations are shown with and without station data. As can be seen from this figure, adding variability to ground motions without station data will increase the median expected losses, while adding variability when stations are present will likely have no effect on the median.

3.6.1 Directivity

In this section, we investigate the combined effects of directivity and spatial variability. Directivity in ground motions refers to amplitude, frequency, velocity, or acceleration which
Figure 3.7: Economic loss distributions (in billions) for Northridge without (left) and with (right) station data.

varies with direction. In particular, we will investigate the impact of moving the epicenter to different locations on the fault in a scenario event placed in San Diego, CA. Directivity is incorporated into Shakemap using the directivity correction provided by Rowshandel (2010), where it affects both the estimated ground motions and uncertainty. This calculation is dependent on the location of the epicenter in relation to the fault; thus, moving the epicenter will change the resultant Shakemap. For the San Diego event, we will generate a set of Shakemaps by moving the epicenter along the fault from one end to another, which can then be used in the spatial variability calculation to find associated loss distributions.

Illustrations of three different Shakemaps and their spatially variable counterparts are shown in Figure 3.8. One Shakemap is computed without directivity, while the other two are computed with directivity at either end of the fault. These illustrations, particularly Figure 3.8 (e) and (f), show that moving the epicenter to one end of the fault will increase estimated ground motions at the other end of the fault. We can see the combined impacts of variability and directivity in the economic loss distributions shown in Figure 3.9.
Figure 3.8: ShakeMaps generated for the San Diego scenario event with directivity included in sub-figures (c-f). The epicenter is shown on the map with a black diamond.
Figure 3.9: ShakeMaps generated for the San Diego scenario event with directivity included in subfigures (c-f). The epicenter is shown on the map with a black diamond.
We can observe in Figure 3.9 the effects of both adding spatial variability and directivity on the loss distributions. Including spatial variability increases the economic loss estimate for each case (no directivity, directivity on the south and north ends of the fault). The impact of directivity is most noticeable from the case where the epicenter is located at the northern end of the fault. We can see that in this case, the ShakeMap median is higher than the other two cases. This is because moving the epicenter to the Northern end of the fault will increase ground motion estimates at the Southern end, where large populations of people reside.

### 3.6.2 Dimensionality

In the next chapter, we will discuss a method to reduce the dimensionality of the spatially correlated field by using Karhunen Loève expansions. However, it is also of interest to investigate the possibility of dimension reduction through ShakeMap. For example, given a fixed area, if the grid spacing in ShakeMap were to be doubled or tripled, we are interested in how would this effect the resultant estimates of ground motion and losses. To investigate this, we use the records from the Loma Prieta earthquake in 1989 and generate ShakeMaps with grid spacings of 3 km, 6 km, and 9 km. The three ShakeMaps are pictured in Figure 3.10. Although the fine details present in the ShakeMap using 3 km spacing are lost in the ShakeMaps with 6 km and 9 km spacing, the general features are still captured. This is expected because the dominant variables in the ShakeMap, such as the magnitude and fault properties are not changed. Thus the main effect of increasing the grid spacing is a smoother estimated ground motion grid, or a grid with less spatial variability. To investigate the impact on losses, we compute 1000 realizations of spatially variable ShakeMaps using the 3 grid spacings. The distributions of estimated economic losses are shown in Figure 3.11. In general, it can be seen from these figures that increasing the grid spacing alters the estimated economic losses of both the ShakeMap alone, and the distribution of losses from the spatially variable ShakeMaps; however, the magnitudes of the differences are quite small. One significant trend that can be observed here is that a “smoother” ShakeMap grid results
in a smaller loss estimate. This was seen in the previous section using the Northridge data set, where the addition of spatial variability increased the median loss prediction. These results suggest that reducing the dimension of ShakeMap may be counterproductive to the goal of including the spatial variability present in ground motions.

Figure 3.10: ShakeMaps generated for the Loma Prieta event using a fixed area and variable grid spacing.
Figure 3.11: Estimated economic losses corresponding to 1000 realizations of spatially variable ShakeMaps with variable grid spacing.
Figure 3.12: The steps to computing $n$ realizations of spatially variable ShakeMaps using $P$ cores.
CHAPTER 4
KARHUNEN LOÈVE APPROXIMATIONS FOR SPHERICAL RANDOM FIELDS

The goal of using the Karhunen Loéve (KL) transform is to represent a stochastic process as an infinite sum of orthogonal functions. The transform has uses in signal processing, radar and image processing, and seismology as a method to remove redundant information and deliver a result in a more compact form. In the context of this work, the KL transform will be used to give a representation of the spatially correlated random field of ground motion residuals. This Gaussian random field (GRF) will be represented as an infinite sum of spherical harmonic functions and so-called spherical harmonic coefficients, which are determined by the angular power spectrum, a sequence which may be informed on the covariance structure of seismic ground motions residuals.

In Chapter 3, we saw how a single realization of spatially correlated ground motions can be generated using the ShakeMap model. Two methods were described for computing this spatially correlated random field of residuals, both of which involved the construction of covariance matrices. In the first method, a ground motion correlation function was applied to the geodetic distance matrix between points on a ShakeMap grid to obtain the covariance matrix for that particular ShakeMap. Realizations for this method were computed simply by finding the product of the covariance matrix and a standard normal random vector; however, the simplicity of this method came with the price of extensive memory requirements. To make the computation more memory efficient, the method of successive conditional simulations was introduced, which computed the covariance matrix in an iterative fashion. This method enabled parallelization, making the computation memory and temporally efficient. Although both of these methods accomplish the end task of generating a realization of the spatially correlated field, both methods are dependent on the ShakeMap and the particular grid the ShakeMap is layered on. In this sense, moving the ShakeMap to another location would
require the process to be repeated. Further, the stochastic dimensions of the problems are dependent on the size of the grid.

The application of the KL expansion to this project provides another method for computing realizations of the spatially correlated random field of ground motion residuals and alleviates the dependence on the ShakeMap grid. In other words, the KL expansion provides a functional representation of the random field which can be sampled globally, and whose stochastic dimension is independent of the sample size. The expansion requires two key elements, a set of orthogonal basis functions and a set of coefficients. Since we are working on a sphere, the well-documented spherical harmonic functions may be used as the basis functions. Thus, the goal is to compute the coefficients such that they capture the appropriate covariance structure. We begin with an overview of the mathematical background, followed by methods for efficient computation using the KL expansion, and details for the seismic application.

4.1 Mathematical Background

We will begin our investigation into the KL expansion by introducing isotropic Gaussian random fields on the sphere and the properties they possess. From there we can prove the existence of the KL expansion for an isotropic GRF, closely following the work of Lang & Schwab (2013). We will derive the connection between smoothness of covariance kernel of an isotropic GRF on the sphere, $S^2$, and the decay of the angular power spectrum. Further, we will show that the convergence rate of the truncated KL expansion is dependent only on the decay of the angular power spectrum and that it is independent of space and time discretization.

4.1.1 Isotropic Gaussian Random Fields on the Sphere

To begin, we will define several variables used throughout this chapter. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the probability space, where $\Omega$ is a random outcome, $\mathcal{A}$ is the set of outcomes, and $\mathbb{P}$ is the probability function which returns the probability of each event. We will denote the
sphere of radius one in three dimensions by

$$S^2 = \{ x \in \mathbb{R}^3 : ||x|| = 1 \},$$

where $|| \cdot ||$ is the Euclidean norm. Let $(S^2, d)$ be the compact metric space with metric $d(x, y) = \arccos < x, y >_{\mathbb{R}^3}$ $\forall x, y \in S^2$. We recognize this metric as the geodetic distance between points on the surface of the sphere.

Define $\mathcal{B}(S^2)$ to be the Borel $\sigma$-algebra of $S^2$. Then by definition, $\mathcal{B}(S^2)$ is the smallest $\sigma$-algebra containing all open sets of $S^2$. We define a real-valued random field on the unit sphere to be the $\mathcal{A} \otimes \mathcal{B}(S^2)$-measurable mapping $T : \Omega \times S^2 \to \mathbb{R}$. We will define $T$ to be a GRF if for all $k \in \mathbb{N}, x_1, \ldots, x_k \in S^2$, the multivariate random variable $(T(x_1), \ldots, T(x_k))$ is multivariate Gaussian distributed, or $\sum_{i=1}^k a_i T(x_i)$ is a normally distributed random variable for all $a_i \in \mathbb{R}, i = 1, \ldots, k$.

We say $T$ is strongly isotropic if for all $k \in \mathbb{N}, x_1, \ldots, x_k \in S^2$ and $g \in SO(3)$, the group of rotations on $S^2$, $(T(x_1), \ldots, T(x_k))$ and $(T(gx_1), \ldots, T(gx_k))$ have the same finite dimensional distribution. $T$ is called $n$-weakly isotropic for $n \geq 2$ if $\mathbb{E}[|T(x)|^n] < +\infty$ for all $x \in S^2$ and for $1 \leq k \leq n, x_1, \ldots, x_k \in S^2$ and $g \in SO(3)$,

$$\mathbb{E}[T(x_1), \ldots, T(x_k)] = \mathbb{E}[T(gx_1), \ldots, T(gx_k)].$$

Thus, we have that $T$ is strongly isotropic iff $T$ is 2-weakly isotropic (Proposition 5.10(3) Marinucci & Peccati (2011)).

Recall that the Legendre polynomials, denoted $(P_l, l \in \mathbb{N}_0)$, are given by Rodrigue’s formula:

$$P_l(\mu) := 2^{-l} \frac{1}{l!} \frac{\partial^l}{\partial \mu^l} (\mu^2 - 1)^l,$$
for $l \in \mathbb{N}_0$ and $\mu \in [-1, 1]$. The associated Legendre functions, denoted $(P_l, l \in \mathbb{N}_0)$ are defined in terms of the Legendre polynomials,

$$P_{lm}(\mu) := (-1)^m (1 - \mu^2)^{m/2} \frac{\partial^m}{\partial \mu^m} P_l(\mu),$$

for $l \in \mathbb{N}_0, m = 0, \cdots, l, \mu \in [-1, 1]$. Using this definition, we may introduce the surface spherical harmonic functions, $Y := (Y_{lm}, l \in \mathbb{N}_0, m = -l, \cdots, l)$, where

$$Y_{lm} : [0, \pi] \times [0, 2\pi] \to \mathbb{C}$$

and

$$Y_{lm}(\theta, \phi) := \sqrt{2l + 1 \frac{(l - m)!}{(l + m)!}} P_{lm}(\cos(\theta)) e^{im\phi},$$

for $l \in \mathbb{N}_0, m = 0, \cdots, l$, and $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. It is further known that the spherical harmonic functions satisfy

$$Y_{lm} = (-1)^m \overline{Y}_{l-m},$$

for $l \in \mathbb{N}, m = -l, \cdots, -1$.

In order to connect the spherical harmonic functions to the space $L^2(S^2)$, we will use the Peter-Weyl Theorem on the sphere which is shown in Proposition 3.29 in Marinucci & Peccati (2011).

**Theorem 4.1.** (Peter-Weyl Theorem on the sphere) *For all complex-valued functions $f \in L^2(S^2)$, we have*

$$f(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi),$$

*where*

$$a_{lm} = \int_{S^2} f(\theta, \phi) Y_{lm}(\theta, \phi) \sin(\theta) d\theta d\phi = (-1)^m \overline{a}_{l-m}.$$
and convergence holds in the $L^2(S^2)$ sense, i.e.

$$\lim_{L \to \infty} \int_0^\pi \int_0^{2\pi} (f(\theta, \phi) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi))^2 \sin \theta d\theta d\phi = 0.$$ 

By the Peter-Weyl theorem on the sphere, $Y$ is an orthonormal basis of $L^2(S^2; \mathbb{C})$. Then every real-valued function $f$ in $L^2(S^2; \mathbb{C})$ admits a spherical harmonic expansion,

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm},$$

where $f_{lm}$ satisfy $f_{lm} = (-1)^m \overline{f}_{l-m}$. We can expand this sum as follows.

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm},$$

$$= \sum_{l=0}^{\infty} f_{l0} Y_{l0} + \sum_{m=1}^{l} f_{lm} Y_{lm} + \sum_{m=-l}^{l} f_{lm} Y_{lm},$$

$$= \sum_{l=0}^{\infty} f_{l0} Y_{l0} + \sum_{m=1}^{l} f_{lm} Y_{lm} + \sum_{m=1}^{l} (-1)^m \overline{f}_{l-m} (-1)^m Y_{lm},$$

$$= \sum_{l=0}^{\infty} f_{l0} Y_{l0} + \sum_{m=1}^{l} f_{lm} Y_{lm} + \overline{f}_{l0} Y_{l0},$$

$$= \sum_{l=0}^{\infty} f_{l0} Y_{l0} + 2 \sum_{m=1}^{l} (\Re f_{lm} \Re Y_{lm} - \Im f_{lm} \Im Y_{lm}).$$

Now we will show that every 2-weakly isotropic random field admits a convergent Karhunen Loève expansion. The following theorem, which is proven in Theorem 5.13 in Marinucci & Peccati (2011), defines the spectral representation for isotropic fields on $S^2$. To simplify the expression, we will let $y$ be the cartesian vector given by

$$y = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$
Further, we will define the Lebesgue measure on the sphere as

\[ d\sigma(y) = \sin \theta d\theta d\phi. \]

**Theorem 4.2.** Let \( T \) be a 2-weakly isotropic random field on \( S^2 \), then \( T \) satisfies \( P \)-almost surely (with probability 1),

\[ \int_{S^2} T(x)^2 d\sigma(x) < +\infty \]

and \( T \) admits a KarhunenLoève expansion

\[ T = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm} \] (4.1)

where

\[ a_{lm} = \int_{S^2} T(y) Y_{lm}(y) d\sigma(y) \]

for \( l \in \mathbb{N}_0, m \in \{-l \cdots l\} \). The series expansion (4.1) converges in \( L^2(\Omega \times S^2; \mathbb{R}) \), or for fixed \( x \in S^2 \)

\[ \lim_{\kappa \to \infty} \mathbb{E} \left[ \int_{S^2} (T(y) - \sum_{l=0}^{\kappa} \sum_{m=-l}^{l} a_{lm} Y_{lm}(y))^2 d\sigma(y) \right] = 0. \]

The series expansion (4.1) converges in \( L^2(\Omega; \mathbb{R}) \) for all \( x \in S^2 \), or for fixed \( x \in S^2 \)

\[ \lim_{\kappa \to \infty} \mathbb{E} \left[ (T(y) - \sum_{l=0}^{\kappa} \sum_{m=-l}^{l} a_{lm} Y_{lm}(y))^2 \right] = 0. \]

Connecting this result with the Peter-Weyl Theorem on the sphere, this implies that every 2-weakly isotropic random field is an element of \( L^2(\Omega; L^2(S^2)) \). In order to use the KL expansion to compute an isotropic random field, we must define the coefficients \( a_{lm} \). We will define these random coefficients using the angular power spectrum of the random field \( T \). The following lemma described the properties of the angular power spectrum following Lemma 2.4 in Lang & Schwab (2013).
Lemma 4.3. Let $T$ be a strongly isotropic random field on $S^2$ with Karhunen Loève coefficients

$$A := (a_{lm}, l \in \mathbb{N}_0, m = -l, \cdots, l).$$

The elements of the sequence $A$, except for $a_{00}$, are centered random variables, i.e., $E[a_{lm}] = 0$ for all $l \in \mathbb{N}$ and $m = l, \cdots, l$. Furthermore, there exists a sequence $(A_l, l \in \mathbb{N}_0)$ of nonnegative real numbers such that

$$E[a_{l_1m_1} \overline{a}_{l_2m_2}] = A_{l_1} \delta_{l_1l_2} \delta_{m_1m_2}$$

for $l_1, l_2 \in \mathbb{N}$ and $m_i = -l_i, \cdots, l_i, i = 1, 2$ where $\delta_{nm}$ is the Kronecker-Delta function. For the first element we have

$$E[a_{00}\overline{a}_{lm}] = (A_0 + E[a_{00}^2])\delta_{00}\delta_{0m}.$$  

The sequence $(A_l, l \in \mathbb{N}_0)$ is called the angular power spectrum of $T$. By the Peter-Weyl theorem on the sphere, we have that

$$a_{lm} = (-1)^m\overline{a}_{l-m}$$

for $l \in \mathbb{N}_0, m = 1, \cdots, l$.

To obtain further properties of the angular power spectrum, we will utilize several results in Marinucci & Peccati (2011). First, Theorem 6.12 states that the coefficients $(a_{l_0}, a_{l_1}, \cdots, a_{ll})$ are independent if and only if they are Gaussian, provided $T$ is a strongly isotropic random field and $E[|a_l|^2] < +\infty$. Under these same conditions, we state Proposition 6.8

1. For all $m = 1, \cdots, l$,

$$\Re a_{tm} = \Im a_{tm}, \frac{\Re a_{lm}}{\Im a_{lm}} \sim \text{Cauchy}.$$
2. For all \( m = 1, 2, \cdots, l \), \( \Re a_{lm} \) and \( \Im a_{lm} \) are uncorrelated with variance \( \mathbb{E}[\Re a_{lm}]^2 = \mathbb{E}[\Im a_{lm}]^2 = A_l/2 \).

3. The marginal distribution of \( \Re a_{lm}, \Im a_{lm} \) is always symmetric, that is

\[
\Re a_{lm} = -\Re a_{lm}, \Im a_{lm} = -\Im a_{lm}.
\]

Combining these results with Propositions 6.11 and 6.6 in Marinucci & Peccati (2011), a summary of the KL expansion within the context of the angular power spectrum are summarized by Lang & Schwab (2013) in Corollary 2.5.

**Corollary 4.4.** Let \( T \) be a 2-weakly isotropic Gaussian random field on \( S^2 \). Then \( T \) admits a KL expansion

\[
T = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm},
\]

where \( (Y_{lm}, l \in \mathbb{N}_0, m = -l, \cdots, l) \) is the sequence of spherical harmonic functions and the sequence \( \mathbb{A} := (a_{lm}, l \in \mathbb{N}_0, m = -l, \cdots, l) \) is a sequence of complex-valued, centered, Gaussian random variables with the following properties:

1. \( \mathbb{A}_+ := (a_{lm}, l \in \mathbb{N}_0, m = 0, \cdots, l) \) is a sequence of independent, complex-valued, Gaussian random variables

2. The elements of \( \mathbb{A}_+ \) with \( m > 0 \) satisfy \( \Re a_{lm} \) and \( \Im a_{lm} \) are independent and \( N(0, A_l/2) \) distributed

3. The elements of \( \mathbb{A}_+ \) with \( m = 0 \) are real-valued and the elements \( \Re a_{l0} \) are \( N(0, A_l) \) distributed for \( l \in \mathbb{N} \) while \( \Re a_{00} \) is \( N(2\sqrt{\pi} \mathbb{E}[T], A_0) \) distributed

4. The elements of \( \mathbb{A} \) with \( m < 0 \) are deduced from those of \( \mathbb{A}_+ \) by the formula

\[
\Re a_{lm} = (-1)^m \Re a_{l-m}, \quad \Im a_{lm} = (-1)^{m+1} \Im a_{l-m}.
\]
With this corollary, we have a well defined KL expansion provided we have the angular power spectrum, \((A_l, l \in \mathbb{N})\). A few methods exist to derive the spectrum, which will be discussed later in Sections 4.3-4.4. Prior to this, we will discuss the relationship between the decay angular power spectrum with increasing \(l\) and the \(\kappa\)-term truncation of the KL expansion of an isotropic GRF \(T\) on \(S^2\).

### 4.1.2 Decay of the Angular Power Spectrum

In this section, we will show that the decay of the angular power spectrum is associated with the error in a \(\kappa\)-term truncation of the KL expansion. Further, the decay of the angular power spectrum is defined by the behavior of the covariance kernel, which describes the isotropic GRF. Let \(\rho\) denote the kernel of the covariance function of \(T\) with angular power spectrum \((A_l, l \in \mathbb{N}_0)\). Then for \(x, y \in S^2\),

\[
\rho(x, y) = \mathbb{E}(T(x)T(y)) = \sum_{l=0}^{\infty} A_l \sum_{m=-l}^{l} Y_{lm}(x)Y_{lm}(y).
\]

By the addition theorem we have that

\[
P_l(<x, y >_{\mathbb{R}^3}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} (-1)^m Y_{lm}(x)Y_{l-m}(y) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(x)Y_{lm}(y).
\]

Thus we have

\[
\rho(x, y) = \sum_{l=0}^{\infty} A_l \frac{2l+1}{4\pi} P_l(<x, y >_{\mathbb{R}^3}).
\]

Thus the covariance kernel depends solely on the inner-product between points \(x\) and \(y\).

We will show the regularity of \(\rho\) is equivalent to the weighted 2-summability of the angular power spectrum \((A_l, l \in \mathbb{N}_0)\). To connect the regularity of the kernel to the weighted 2-summability of the angular power spectrum, we will make use of weighted Sobolev spaces. First, let \(H^n(-1,1) \subset L^2(-1,1)\) for \(n \in \mathbb{N}_0\) denote the standard Sobolev spaces. We will
denote the closure of $H^n(-1,1)$ as $V^n(-1,1)$ with respect to the weighted norm,

$$||u||^2_{V^n(-1,1)} = \sum_{j=0}^{n} ||u||^2_{V^j(-1,1)},$$

where for $j \in \mathbb{N}_0$, we will define

$$||u||^2_{V^j(-1,1)} = \int_{-1}^{1} \left| \frac{\partial^j u(\mu)}{\partial \mu^j} \right|^2 (1-\mu^2)^j d\mu.$$

Using this definition and following Lang & Schwab (2013), we have that $(V^n(-1,1), n \in \mathbb{N}_0)$ is a decreasing scale of separable Hilbert spaces, or

$$\mathbb{L}^2(-1,1) = V^0(-1,1) \supset V^1(-1,1) \supset \cdots \supset V^n(-1,1) \supset \cdots,$$

and by Ehrling’s lemma we have

$$||u||^2_{V^n(-1,1)} \simeq ||u||^2_{\mathbb{L}^2(-1,1)} + ||u||^2_{V^n(-1,1)},$$

for $u \in V^n(-1,1)$.

We may derive further, equivalent norms of $V^n(-1,1)$ in terms of the summability of the angular power spectrum following the work of Lang & Schwab (2013). For any $u \in \mathbb{L}^2(-1,1)$, we can expand in the $\mathbb{L}^2(-1,1)$ convergent Fourier-Legendre series

$$u = \sum_{l=0}^{\infty} u_l \frac{2l + 1}{2} P_l,$$

where

$$u_l = \int_{-1}^{1} u(x) P_l(x) dx,$$
for \( l \in \mathbb{N}_0 \). If we let \( A_l = 2\pi u_l \), we have that

\[
u = \sum_{l=0}^{\infty} A_l \frac{2l + 1}{4\pi} P_l,
\]

which we recognize as the form of the kernel, \( p \). Thus, we may conclude \( u \) is a valid kernel.

To show the relation between the weighted 2-summability of the angular power spectrum and the regularity of the kernel, we introduce the following theorem from Lang & Schwab (2013).

**Theorem 4.5.** Let \( \rho_I(\mu) := \rho(\arccos(\mu)) \). For \( \rho_I \in V^n(-1,1), n \in \mathbb{N}_0 \), the sequence \( n^{l+1/2}A_l, l \geq n \) is in \( L^2(\mathbb{N}_0) \) if and only if \( (1 - \mu^2)^{n/2} \frac{\partial^{n}}{\partial \mu^n} \rho_I(\mu), \mu \in (-1,1) \) is in \( L^2(-1,1) \), i.e.,

\[
\frac{1}{(4\pi)^2} \sum_{l \geq n} A_l^2 \frac{2l + 1}{2} l^{2n} < \infty
\]

if and only if

\[
\int_{-1}^{1} \left| \frac{\partial^{n}}{\partial \mu^n} \right|^2 (1 - \mu^2)^n d\mu < \infty.
\]

Thus, this theorem shows that a necessary and sufficient condition for the weighted 2-summability of the angular power spectrum is the weighted square integrability of the \( n \)-th weak derivatives of \( \rho_I \) with respect to the weight function \( (1 - \mu^2)^n \).

### 4.2 Efficient Computation of the GRF

In this section we will consider the computation of the GRF for a given angular power spectrum, \( (A_l, l \in \mathbb{N}) \). After we have thoroughly described the methods used to compute the field, we will discuss particular choices of the angular power spectra and the impact on the GRF.

Let \( (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \) and let \( T \) be the isotropic, Gaussian random field on \( S^2 \) having the expansion

\[
T(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).
\]

80
We recall that $Y_{lm}(\theta, \phi)$ is the surface spherical harmonic function which satisfies

$$Y_{lm}(\theta, \phi) = (-1)^m \overline{Y}_{l-m}(\theta, \phi),$$

and $a_{lm}$ are the independent, Gaussian random variables which follow the distributions described in Corollary 4.4. For constant $l \in \mathbb{N}$, we have that

$$\sum_{m=-l}^{l} a_{lm} Y_{lm} = a_{l0} Y_{l0} + \sum_{m=1}^{l} a_{lm} Y_{lm} + a_{l-m} Y_{l-m},$$

$$= a_{l0} Y_{l0} + \sum_{m=1}^{l} a_{lm} Y_{lm} + (-1)^m a_{lm} (-1)^m Y_{lm},$$

$$= a_{l0} Y_{l0} + 2 \sum_{m=1}^{l} (\Re a_{lm} \Re Y_{lm} - \Im a_{lm} \Im Y_{lm}).$$

Following Herrmann (2013), define

$$\tilde{Y}_{lm} = \begin{cases} 
\sqrt{2} \Re Y_{lm} & : m > 0, \\
Y_{l0} & : m = 0, \\
\sqrt{2} \Im Y_{l-m} & : m < 0,
\end{cases}$$

and

$$\tilde{a}_{lm} = \begin{cases} 
\sqrt{2} \Re a_{lm} & : m > 0, \\
a_{l0} & : m = 0, \\
-\sqrt{2} \Im a_{l-m} & : m < 0.
\end{cases}$$

Thus we have that

$$\sum_{m=-l}^{l} a_{lm} Y_{lm} = a_{l0} Y_{l0} + 2 \sum_{m=1}^{l} (\Re a_{lm} \Re Y_{lm} - \Im a_{lm} \Im Y_{lm}).$$
\[ \begin{align*}
&= \tilde{a}_{l0} \tilde{Y}_{l0} + 2 \sum_{m=1}^{l} \left( \frac{1}{\sqrt{2}} \tilde{a}_{lm} \frac{1}{\sqrt{2}} \tilde{Y}_{lm} - \frac{1}{\sqrt{2}} \tilde{a}_{l-m} \frac{-1}{\sqrt{2}} \tilde{Y}_{l-m} \right), \\
&= \tilde{a}_{l0} \tilde{Y}_{l0} + \sum_{m=1}^{l} (\tilde{a}_{lm} \tilde{Y}_{lm} + \tilde{a}_{l-m} \tilde{Y}_{l-m}), \\
&= \sum_{m=-l}^{l} \tilde{a}_{lm} \tilde{Y}_{lm},
\end{align*} \]

or

\[ T = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{lm} Y_{lm} = \sum_{l \geq 0} \sum_{m=-l}^{l} \tilde{a}_{lm} \tilde{Y}_{lm} \]

As previously defined in Corollary 4.4, we have

\[ \Re a_{lm}, \Im a_{lm} \sim N(0, A_l/2), \]

\[ \Re a_{l0} \sim N(0, A_l), \Im a_{l0} = 0, l \in \mathbb{N}. \]

Thus, for \( X_{lm} \) a sequence of standard normal, real-valued, random variables, we may write

\[ \tilde{a}_{l0} = a_{l0} = X_{l0} \sqrt{A_l}, \]

\[ \tilde{a}_{lm} = \sqrt{2} \Re a_{lm} = \sqrt{2} X_{lm} \sqrt{A_l/2} = X_{lm} \sqrt{A_l}, \quad m > 0, \]

and

\[ \tilde{a}_{lm} = -\sqrt{2} \Im a_{l-m} = -\sqrt{2} X_{lm} \sqrt{A_l/2} = X_{lm} \sqrt{A_l}, \quad m < 0. \]

Using this result, we write

\[ T(\theta, \phi) = \sum_{l \geq 0} \sum_{m=-l}^{l} \sqrt{A_l} X_{lm} \tilde{Y}_{lm}(\theta, \phi). \]
We will rewrite \( \tilde{Y}_{lm} \) in terms of the associated Legendre functions so that we may separate the function into its real and imaginary components. Define

\[
L_{lm}(\theta) = \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}} P_{lm}(\cos \theta).
\]

Then by definition,

\[
Y_{lm}(\theta, \phi) = L_{lm}(\theta)e^{im\phi}
\]

and further we have

\[
\sqrt{2} \Re Y_{lm}(\theta, \phi) = \sqrt{2}L_{lm}(\theta) \cos m\phi, \quad \sqrt{2} \Im Y_{lm}(\theta, \phi) = \sqrt{2}L_{lm}(\theta) \sin m\phi.
\]

Thus we may compute

\[
\tilde{Y}_{lm}(\theta, \phi) = \begin{cases} 
\sqrt{2} \cos(m\phi)L_{lm}(\theta) & : m > 0, \\
L_{lm}(\theta) & : m = 0, \\
\sqrt{2} \sin(m\phi)L_{lm}(\theta) & : m < 0
\end{cases}
\]

Combining the results shown above and introducing a \( \kappa \)-level truncation of the KL expansion, we write

\[
T(\theta, \phi) = \sum_{l=0}^{\kappa} \sqrt{A_l}X_{l0}L_{l0}(\theta) + \sqrt{2A_l} \left( \sum_{m=-l}^{-1} X_{lm}^{1} \sin(m\phi)L_{lm}(\theta) + \sum_{m=1}^{l} X_{lm}^{2} \cos(m\phi)L_{lm}(\theta) \right).
\]

The computation of the field was completed using a combination of Python and Matlab. The standard normal random variates were computed using the Python based Numpy library 'random'. In particular, the random variates are precomputed with a single call to the function 'random.randn'. This ensures the distribution of the standard normal random variables follows the said distribution. The angular power spectrum, \((A_l, l \in \mathbb{N})\), which will be discussed in more detail in the next section, is pre-computed in Python and stored for
repeated use. Depending on the sequence used, this computation could be a simple algebraic expression, or it may involve quadrature.

The computation of the sequence $L_{lm}(\theta)$ is the most computationally expensive part of the simulation. Recall that

$$L_{lm}(\theta) = \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}} P_{lm}(\cos \theta).$$

The associated Legendre functions, $P_{lm}$, may be evaluated in either Python or Matlab; however, the computation of the constant

$$\sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}}$$

can lead to machine precision issues when $l$ is large. The Matlab function ‘legendre’ can be used to compute the entire expression, $L_{lm}$, efficiently and accurately using the option ‘norm’. Using this function ensures accuracy of our computations for very large $l$. Since most of the code used to compute these KL expansions are written in Python, a few options are available for calling the Matlab function ‘legendre’. In the case where Matlab is not installed on the machine used to compute the KL expansion, the values of $L_{lm}(\theta)$ can be pre-computed on another machine and moved so that they may be used during the computation. Such is the case for the computation of the GRF using the Colorado School of Mines AuN machine. This approach can be beneficial as it may reduce the CPU time associated with the computation of the GRF; however, the memory consumed by the files used to store the sequence can be vary large. In particular, when 250 $\theta$ values are used, a total of 40 Gb is used to store the sequence for $l = 0, \cdots, 5,000$. Thus, the memory consumed by these files can limit the number of terms, $l$, that we compute. In addition, we may need to limit the resolution of the GRF. For this reason, the number of terms in the sequence $\kappa$ is limited by 5,000 when the AuN machine is used.
If Matlab and Python are installed on a machine, then a Matlab engine may be used to compute $L_{lm}$ at run time. This approach does not require the storage of large files, nor does it limit the resolution of the grid. Although, the CPU time will be greater for this approach than for the other. To ensure reasonable CPU times, we will again limit the number of terms in the sequence to $\kappa = 5,000$.

We will consider three forms of the angular power spectrum. The first form is described in detail by Lang & Schwab (2013), and consists of an algebraically decaying sequence. Using this form, we reproduce several results found in Lang & Schwab (2013), including convergence and truncation properties, associated with the particular choice of the angular power spectrum. Subsequently, we consider an alternate form of the angular power spectrum which is derived from a representative correlation function of earthquake ground motions (Jayaram & Baker, 2009). These functions describe the correlation in ground motions at points located closely on the earth, and the correlation function decays rapidly with increasing separation distance. This decay is shown to have an effect on the decay of the angular power spectrum, which in turn effects truncation and convergence results. To remedy this, a third form of the angular power spectrum is derived which enforces the decay of the coefficients while still accounting for the covariance structure.

4.3 Angular Power Spectrum: Algebraic Decay

In this section we will consider the angular power spectrum as described by Lang & Schwab (2013). The form of the coefficients is an algebraically decaying sequence, which ensures adequate decay of the sequence, and therefore truncation of the GRF $T$. To motivate the choice of this sequence, we will consider the following proposition (Proposition 5.2 Lang & Schwab (2013)).

**Proposition 4.6.** Let the angular power spectrum $(A_l, l \in \mathbb{N}_0)$ of the centered, isotropic GRF $T$ decay algebraically with order $\alpha > 2$, i.e. there exists constants $C > 0$ and $l_0 \in \mathbb{N}$ such that $A_l \leq C \cdot l^{-\alpha}$ for all $l > l_0$. Then the series of approximate random fields, $(T^\kappa, \kappa \in \mathbb{N})$
converges to the random field $T$ in $L^2(\Omega; L^2(S^2))$ and the truncation error is bounded by

$$||T - T^\kappa||_{L^2(\Omega; L^2(S^2))} \leq \hat{C} \cdot \kappa^{-(\alpha - 2)/2},$$

for $\kappa \geq l_0$ where

$$\hat{C}^2 = C \cdot \left(\frac{2}{\alpha - 2} + \frac{1}{\alpha - 1}\right).$$

Proof. We have that

$$||T - T^\kappa||^2_{L^2(\Omega; L^2(S^2))} = \sum_{l=0}^\infty \sum_{m=-l}^l A_l X_{lm} \tilde{Y}_{lm}(\theta, \phi) - \sum_{l=0}^\kappa \sum_{m=-l}^l A_l X_{lm} \tilde{Y}_{lm}(\theta, \phi) ||^2_{L^2(\Omega; L^2(S^2))},$$

$$= \sum_{l=\kappa+1}^\infty \sum_{m=-l}^l A_l \mathbb{E}[X_{lm}^2] ||\tilde{Y}_{lm}(\theta, \phi)||^2_{L^2(S^2)},$$

$$= \sum_{l=\kappa+1}^\infty A_l ||Y_{l0}(\theta, \phi)||^2_{L^2(S^2)} + 2A_l \left( \sum_{m=1}^{\kappa+1} ||\cos(m\phi)L_{tm}(\theta)||^2_{L^2(S^2)} + \right.$$

$$\left. \sum_{m=-l}^{l-1} ||\sin(m\phi)L_{tm}(\theta)||^2_{L^2(S^2)} \right),$$

$$= \sum_{l=\kappa+1}^\infty A_l ||Y_{l0}(\theta, \phi)||^2_{L^2(S^2)} + 2A_l \sum_{m=1}^{l} ||\Re Y_{lm}(\theta, \phi)||^2_{L^2(S^2)} +$$

$$||\Im Y_{lm}(\theta, \phi)||^2_{L^2(S^2)}.$$

By the properties of the spherical harmonic functions, we have that $||Y_{l0}(\theta, \phi)||^2_{L^2(S^2)} = 1$ and $||\Re Y_{lm}(\theta, \phi)||^2_{L^2(S^2)} + ||\Im Y_{lm}(\theta, \phi)||^2_{L^2(S^2)} = 1$. Therefore,

$$||T - T^\kappa||^2_{L^2(\Omega; L^2(S^2))} = \sum_{l=\kappa+1}^\infty (2l + 1)A_l.$$
We can bound this function using the predefined bound on the angular power spectrum,

\[ \sum_{l=\kappa+1}^{\infty} (2l + 1)A_l \leq \sum_{l=\kappa+1}^{\infty} (2l + 1)(C \cdot l^{-\alpha}) = C \sum_{l=\kappa+1}^{\infty} (2l^{-\alpha+1} + l^{-\alpha}). \]

Reindexing the sum and rewriting with the corresponding integral gives

\[ C \sum_{l=\kappa+1}^{\infty} (2l^{-\alpha+1} + l^{-\alpha}) = C \sum_{l=1}^{\infty} (2(l + \kappa)^{-\alpha+1} + (l + \kappa)^{-\alpha}), \]
\[ \leq C \int_0^{\infty} (2(x + \kappa)^{-\alpha+1} + (x + \kappa)^{-\alpha})dx, \]
\[ = C \left( \frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \kappa^{-1} \right) \kappa^{-\alpha+2}. \]

Since \( \kappa \geq 1, \kappa^{-1} \leq 1 \) and thus we have that

\[ ||T - T^\kappa||^2_{L^2(\Omega; L^2(S^2))} \leq C \left( \frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \kappa^{-1} \right) \kappa^{-\alpha+2}, \]
\[ \leq \hat{C}^2 \kappa^{-(\alpha-2)/2}. \]

Thus, if we choose the angular power spectrum to satisfy \( A_l \leq Cl^{-\alpha} \) for \( l > l_0 \) and \( C \in \mathbb{R} > 0 \), we have a bound on the truncation error.

We will let

\[ A_l = (l + 1)^{-\alpha} \]

for \( \alpha > 2 \). Then \( A_l \) satisfies the condition \( A_l < l^{-\alpha} \) and we expect the results shown in Proposition 4.6. We will consider three choices of \( \alpha \), namely \( \alpha = 3, 5, 9 \). A plot of the decay of the angular power spectrum with each \( \alpha \) is shown in Figure 4.1. We can see the sequences approach zero quickly, with the rate of decay increasing for increasing \( \alpha \). A single realization of a GRF generated with each choice of \( \alpha \) is shown in Figure 4.2. We observe that the level of variability in the GRF decreases with increasing \( \alpha \), as well as the range of values present within the fields. This is due to the greater rate of decay of the angular
Figure 4.1: An illustration of the decay of the angular power spectrum specified in Section 4.3, given by $A_l = (l + 1)^{-\alpha}$.

power spectrum for larger $\alpha$, making fewer terms in the computation of $T$ significant and thus decreasing the level of variability. In the next section, we compute the convergence results for a reference solution $T$ with $\kappa = 2^7$ terms, following the work in Lang & Schwab (2013).

### 4.3.1 Truncation Error Analysis

In a $\kappa$-term truncation of the series expansion, we expect there to be error in our approximation. It has been shown that we may bound the truncation error with respect to assumptions made on the associated angular power spectrum of the isotropic GRF, $T$. To ensure that the calculations from the KL expansion follow this bound, we compute 100 realizations of the GRF for $\kappa = 2, 5, 8, 20, 50, 70$ and compute the maximum error between these fields and the reference solution computed with $\kappa = 2^7$ terms. The maximum error between the GRFs is used to determine the error, as this is a stronger bound than the $L^2$ norm. In
Figure 4.2: A sample GRF for each angular power spectrum specified in Section 4.3 with $\alpha = 3, 5, \text{ and } 9$. Each is computed using a $\kappa = 100$ term truncation.
Figure 4.3 we show the average absolute error from 100 realizations, along with the bound discussed in Proposition 4.6. We also show the absolute error from a single sample of the GRF using various angular power spectrum in Figure 4.4. From these figures, we can see that the truncation follows the bound. Further, the similarity of the results shown here to those shown in Lang & Schwab (2013) confirm that the method of computation for the GRF, which varies slightly from that described in the paper, is accurate.

4.4 Angular Power Spectrum: Seismic Application

In this section, we consider the computation of an isotropic GRF using the KL expansion with the end goal of adding spatial variability to earthquake ground motions. In this case, we have a representative correlation function which estimates the correlation in earthquake ground motion residuals as a function of increasing separation distance. We will continue to use the spherical harmonic functions, and thus the covariance structure of the ground motion correlation model must be captured in the angular power spectrum. In order to derive the angular power spectrum from a correlation function, we follow the work of Magneville & Pansart (2007).

4.4.1 Computing the Angular Power Spectrum from a Covariance Function

We have previously seen the relationship between the coefficients, $a_{lm}$, of the KL expansion and the angular power spectrum ($A_l$, $l \in \mathbb{N}_0$), namely

$$E[a_{l_1m_1} \bar{a}_{l_2m_2}] = A_{l_1} \delta_{l_1l_2} \delta_{m_1m_2}.$$  

In this section, we are interested in finding the angular power spectrum associated with a particular covariance kernel, $\rho$, which is dependent only on the separation distance between two points. Consider the two points $\mathbf{x}$ and $\mathbf{y}$ on the sphere of radius 1 defined by

$$\mathbf{x} = (\sin(\theta_1) \cos(\phi_1), \sin(\theta_1) \sin(\phi_1), \cos(\theta_1)),$$

$$\mathbf{y} = (\sin(\theta_2) \cos(\phi_2), \sin(\theta_2) \sin(\phi_2), \cos(\theta_2)).$$
Figure 4.3: Average absolute error of a $\kappa$-term KL approximation to a $2^7$-term reference solution using various angular power spectra with 100 realizations.
Figure 4.4: Absolute error of a $\kappa$-term KL approximation to a $2^7$-term reference solution using various angular power spectra for a single realization.
\[
\mathbf{y} = (\sin(\theta_2) \cos(\phi_2), \sin(\theta_2) \sin(\phi_2), \cos(\theta_2)).
\]

Let \( \rho(h(\mathbf{x}, \mathbf{y})) \) be the covariance kernel, where \( h(\mathbf{x}, \mathbf{y}) = \arccos \langle \mathbf{x} \cdot \mathbf{y} \rangle \). Then we have that

\[
\rho(h(\mathbf{x}, \mathbf{y})) = \mathbb{E}[T(\mathbf{x}) T(\mathbf{y})],
\]

\[
= \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} \mathbb{E}[a_{l_1 m_1} Y_{l_1 m_1}(\mathbf{x}) \bar{a}_{l_2 m_2} Y_{l_2 m_2}(\mathbf{y})],
\]

\[
= \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} \mathbb{E}[a_{l_1 m_1} \bar{a}_{l_2 m_2} Y_{l_1 m_1}(\mathbf{x}) Y_{l_2 m_2}(\mathbf{y})],
\]

\[
= \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} A_{l_1 l_2} \delta_{m_1 m_2} Y_{l_1 m_1}(\mathbf{x}) Y_{l_2 m_2}(\mathbf{y}),
\]

\[
= \sum_{l=0}^{\infty} A_l \sum_{m=-l}^{l} Y_{l m}(\mathbf{x}) \bar{Y}_{l m}(\mathbf{y}).
\]

By the addition theorem we have

\[
P_l(\cos(h(\mathbf{x}, \mathbf{y}))) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} Y_{l m}(\mathbf{x}) \bar{Y}_{l m}(\mathbf{y}).
\]

Then we may substitute this expression as follows:

\[
\rho(h(\mathbf{x}, \mathbf{y})) = \sum_{l=0}^{\infty} A_l \sum_{m=-l}^{l} Y_{l m}(\mathbf{x}) \bar{Y}_{l m}(\mathbf{y}),
\]

\[
\rho(h(\mathbf{x}, \mathbf{y})) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1) A_l P_l(\cos(h(\mathbf{x}, \mathbf{y}))),
\]

\[
2\pi \rho(h(\mathbf{x}, \mathbf{y})) P_l(\cos h(\mathbf{x}, \mathbf{y})) = \frac{1}{2} P_l(\cos h(\mathbf{x}, \mathbf{y})) \sum_{l=0}^{\infty} (2l + 1) A_l P_l(\cos h(\mathbf{x}, \mathbf{y})),
\]

93
\[ 2\pi \int_{h=0}^{\pi} \rho(h(x, y)) P_l(\cos h(x, y)) d\cos h = \frac{1}{2} \int_{h=0}^{\pi} P_l(\cos h(x, y))^* \left( \sum_{l=0}^{\infty} (2l + 1)A_l P_l(\cos h(x, y)) d\cos h, \right. \]

\[ 2\pi \int_{h=0}^{\pi} \rho(h(x, y)) P_l(\cos h(x, y)) d\cos h = \frac{1}{2} \sum_{l=0}^{\infty} (2l + 1)A_l^* \int_{h=0}^{\pi} P_l(\cos h(x, y)) P_l(\cos h(x, y)) d\cos h. \]

Using the orthogonality between Legendre polynomials, we know that

\[ \int_{-1}^{1} P_l(z) P_l(z) dz = \frac{2\delta_{l_1,l}}{2l_1 + 1}, \]

and thus we may substitute \( z = \cos(h(x, y)) \) such that

\[ \frac{2\delta_{l_1,l}}{2l_1 + 1} = \int_{h=-\pi}^{h=0} P_l(\cos h(x, y)) P_l(\cos h(x, y)) d\cos h. \]

We utilize the previous expression to show

\[ -2\pi \int_{h=0}^{\pi} \rho(h(x, y)) P_{l_1}(\cos h(x, y)) d\cos h = \frac{1}{2} (2l_1 + 1)A_{l_1} \frac{2}{2l_1 + 1}, \]

\[ -2\pi \int_{h=0}^{\pi} \rho(h(x, y)) P_{l_1}(\cos h(x, y)) d\cos h = A_{l_1}. \]

Therefore we have that

\[ A_l = -2\pi \int_{h=0}^{\pi} \rho(h(x, y)) P_l(\cos h(x, y)) d\cos h. \]

We may rewrite this expression so that the integral is in terms of \( x \) as

\[ A_l = 2\pi \int_{x=-1}^{1} \rho(\arccos x) P_l(x) dx. \]
In order to compute this numerically, we require a quadrature scheme which is accurate for high degree polynomials. For this purpose, we use Legendre-Gauss quadrature. Recall that the form of the correlation function given by Jayaram & Baker (2009) is

$$\rho(h) = \exp(-3h/b),$$

where $h$ is the geodetic separation distance between two points, and $b$ is the correlation range. Specifically, for PGA the correlation range was defined to be 8.5 km when site conditions were expected to cluster. Since the spherical harmonic functions (and thus KL expansions) are defined on the sphere of radius one, we must account for the radius of the earth within the correlation function. We define the radius of the earth in kilometers to be $r = 6371$ km and we introduce a modified correlation function that accounts for this radius given by

$$\rho(h) = \exp(-3rh/b).$$

We will consider a range of $b$ values, including $b = 8.5$ km as suggested by Jayaram & Baker (2009). Varying $b$ will control the decay of the angular power spectrum, and thus may influence the truncation results. In order to approximation the angular power spectrum

$$A_l = 2\pi \int_{x=-1}^{1} \rho(\arccos x) P_l(x) dx,$$

we use Legendre-Gauss quadrature. Thus for some points $\{x_1, \cdots, x_N\} \in [-1,1]$ and corresponding weights $\{w_1, \cdots, w_N\}$, we write

$$A_l = 2\pi \int_{x=-1}^{1} \rho(\arccos x) P_l(x) dx = 2\pi \sum_{i=1}^{N} w_i \ast (\rho(\arccos x_i) P_l(x_i)).$$

To ensure that the quadrature method is working, we calculate the first value for each sequence $A_l$ using Mathematica, and compare to our computed sequences. For example, we
can compute

\[ A_0 = 2\pi \int_{x=-1}^{1} \rho(\arccos x) P_0(x) \, dx, \]

\[ = 2\pi \int_{x=-1}^{1} \rho(\arccos x) \, dx, \]

\[ = 2\pi \int_{x=-1}^{1} \exp\left(-\frac{3r}{b} \arccos x\right) \, dx. \]

Empirical results and their corresponding numerical results for various \( b \) values are found in Table 4.1. From these values, we can see that the quadrature method used is accurate.

Table 4.1: Comparison of values of \( A_0 \) for various correlation lengths.

<table>
<thead>
<tr>
<th>( b )</th>
<th>Mathematica</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>1.3537</td>
<td>1.353638292</td>
</tr>
<tr>
<td>7,500</td>
<td>0.83867</td>
<td>0.838670015819</td>
</tr>
<tr>
<td>5,000</td>
<td>0.402454</td>
<td>0.402454174785</td>
</tr>
<tr>
<td>1,000</td>
<td>1.7153E-2</td>
<td>1.71527898915E-2</td>
</tr>
<tr>
<td>500</td>
<td>4.297E-3</td>
<td>4.29698461231E-3</td>
</tr>
<tr>
<td>128</td>
<td>2.81788E-4</td>
<td>2.81259963825E-4</td>
</tr>
<tr>
<td>64</td>
<td>7.0449E-5</td>
<td>7.04156914381e-05</td>
</tr>
<tr>
<td>32</td>
<td>1.76125E-5</td>
<td>1.75462787723e-05</td>
</tr>
<tr>
<td>8.5</td>
<td>1.24268E-6</td>
<td>1.24267757287e-06</td>
</tr>
</tbody>
</table>

The angular power spectrum for various \( b \) values, ranging from 4 to 10,000 is plotted in Figure 4.5. We can see that the rate of decay of the angular power spectrum is proportional to the value of \( b \), or that the angular power spectrum exhibits fast decay for \( b = 10,000 \) km and slow decay for \( b = 4 \) km. In fact, the rate of decay for \( b = 8.5 \) km is such that

\[ A_0 \approx 10E - 06 \text{ and } A_{5000} \approx 10E - 08. \]

In addition, we observed that the initial (maximum) value of the sequence \( A_i \) decreases for decreasing \( b \). The first term of each sequence \( (A_0) \) and the final term in the sequence is included in Table 4.2. For large \( b \), we computed 2,500 terms to ensure decay, while for small
we computed a maximum of 5,000 terms. It is noted here that the choice of \( \kappa = 5000 \) as an upper bound on \( l \) for the sequence \( A_l \) was chosen due to the memory requirements of the KL expansion for \( l > 5000 \), as described in Section 4.2. For some \( b \), the angular power spectrum computed exhibited oscillations near zero for large \( l \). For these sequences, the smallest, consecutive positive value is included as the final term. We observe that for \( b = 5,000, 7,500, \) and 10,000, the angular power spectrum decays to machine precision in 2,500 terms. In contrast, for \( b = 4, 8.5, \) and 16, small decay is observed even up to 5,000 terms.

![Angular Power Spectrum](image)

Figure 4.5: Angular power spectrum derived from a correlation function for various correlation lengths.

Since we know that the rate of decay of the angular power spectrum corresponds to the truncation error, we expect that a \( \kappa \)-term truncation of the KL expansion when \( b > 128 \) km will be a good approximation to the exact solution as these sequences appear to decay
Table 4.2: Maximum and minimum values obtained for angular power spectrum with various $b$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$A_0$</th>
<th>Final Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>1.35366383068</td>
<td>$A_{2500} = 1.538668150460e-12$</td>
</tr>
<tr>
<td>7,500</td>
<td>0.838670015819</td>
<td>$A_{2500} = 2.04645408675e-12$</td>
</tr>
<tr>
<td>5,000</td>
<td>0.402454174785</td>
<td>$A_{2500} = 3.06759609788e-12$</td>
</tr>
<tr>
<td>500</td>
<td>4.29698461231E-3</td>
<td>$A_{2310} = 2.011935464440e-11$</td>
</tr>
<tr>
<td>128</td>
<td>2.81259963825E-4</td>
<td>$A_{3558} = 2.2074126843e-11$</td>
</tr>
<tr>
<td>64</td>
<td>7.04156914381E-5</td>
<td>$A_{3520} = 4.35178417696e-11$</td>
</tr>
<tr>
<td>32</td>
<td>1.75462787723E-5</td>
<td>$A_{3576} = 2.66097303651e-11$</td>
</tr>
<tr>
<td>16</td>
<td>4.39888574097e-6</td>
<td>$A_{5000} = 4.97984356812e-08$</td>
</tr>
<tr>
<td>8.5</td>
<td>1.24267757287E-6</td>
<td>$A_{5000} = 8.57584810748e-08$</td>
</tr>
<tr>
<td>4</td>
<td>2.75005692533e-7</td>
<td>$A_{5000} = 9.05775757623e-08$</td>
</tr>
</tbody>
</table>

quickly. However, for smaller $b$, it is unclear whether a truncation with $\kappa \leq 5000$ will truly
capture the random field. We again emphasize that the upper limit of $\kappa = 5,000$ terms was
set due to computational requirements; thus in the following section we will compute the
KL expansion for various correlation ranges with a maximum of $\kappa = 5,000$ terms.

4.4.2 KL Expansion for Various Correlation Ranges

To determine the effect of the parameter $b$ on the KL expansion, we compute sample
GRFs for each $b$ considered. The sampling grid used is an area of roughly 142 km by 178 km
with

$$\theta \in [0.1946\pi, 0.2169\pi], \phi \in [0.663\pi, 0.691\pi].$$

We use 250 equally spaced grid points for both $\theta$ and $\phi$ for a total of 62,500 grid points.
This area was chosen as it is relatively similar to the size of the Northridge ShakeMap grid
included in Chapter 3. The ShakeMap grid had a total of 33,366 grid points, making the
grid spacing used here finer.

The $\kappa$-term KL expansion is computed for each $b$, with $\kappa$ being the index of the final
term in Table 4.2. A sample GRF for each $b$ is illustrated in Figure 4.6 and Figure 4.7. Each
of the GRFs presented in this section are computed with the same set of random variables,
making the effects of varying $b$ easier to observe. We see that the level of variability within
the field increases with decreasing $b$. In other words, the GRF computed for $b = 4$ km appears
to be stochastic in comparison to the GRF computed with $b = 10,000$ km. Theoretically,
this coincides with our definition of $b$, the correlation length. For $b = 10,000$, points located
far in space are still highly correlated and are expected to exhibit similar behavior. Where
as for small $b$, only points located very close in space are correlated, resulting in a more
spatially variable GRF. Mathematically, this is due to the rate of decay of the angular power
spectrum. For example, for small $b$, the angular power spectrum decays slowly, making terms
at the beginning and end of the sequence similar in magnitude. Therefore, the coefficients
for large $l$, where the spherical harmonic functions are more localized, remain significant. If
we consider large $b$, the fast decay of the angular power spectrum increases the magnitude
difference between terms at the beginning of the sequence and at the end. Thus, for large $b$,
coefficients for large $l$ are less significant.

Another attribute of the computed GRFs that changes with $b$ is the range of values
within the field. We observe that the range of values obtained for $b = 4$ is approximately
$[-2.5, 2]$, while the range for $b = 8.5, 16, 32, 64$, and 128 increases to include $[-3, 3]$. For
larger $b$, the range decreases, with the smallest range $[-0.7, 0.4]$ observed for $b = 10,000$.
This observation can be explained by considering the decay in the angular power spectra
and the magnitude of the terms within the sequence. Since the number of significant terms
is low in the KL expansion for $b = 10,000$, we use fewer terms in the expansion, of which the
magnitude quickly decays. On the other hand, for $b = 8.5$ we use 5,000 terms, all which have
a similar magnitude and contribute to the larger range observed. We note that the range
computed for the KL expansion using $b = 8.5$ is similar to that observed to the spatially
correlated random field observed in Chapter 3, confirming the accuracy of the results in this
section.

We are interested in computing the truncation error associated with a $\kappa$-term KL ex-
pansion. In order to compute these errors, we save the GRFs with $\kappa_n = n \ast 100, n =$
Figure 4.6: GRFs computed with the KL expansion with covariance-derived angular power spectrum using various $b$ values with a $\kappa$-term truncation.
Figure 4.7: GRFs computed with the KL expansion with covariance-derived angular power spectrum using various $b$ values with a $\kappa$-term truncation.
1, 2, 3, \cdots, \frac{\kappa}{100} - 1 \text{ terms. Thus, for a 2,500 term expansion, we save the GRFs with 100, 200, \cdots, 2,400 terms in addition to the final GRF. We will denote the GRF computed with } \kappa \text{-terms as } T_\kappa, \text{ and the GRF computed with } \kappa_n \text{-terms as } T_{\kappa_n}. \text{ We will use two measures of error described below.}

- The relative error, or the maximum error in the } \kappa_{n-1} \text{ term expansion as an approximation to the } \kappa_n \text{ term expansion over the range of values within the field } T_{\kappa_n}, \text{ denoted } r:

\frac{||T_{\kappa_{n-1}} - T_{\kappa_n}||_\infty}{r}

- The final relative error, or the maximum error in the } \kappa_n \text{ term expansion as an approximation to the } \kappa \text{ term expansion over the range of values within the field } T_{\kappa}, \text{ denoted } r:

\frac{||T_{\kappa} - T_{\kappa_n}||_\infty}{r}

The relative error is used because the range of values within each field changes with } b. Using the relative error allows us to compare the errors between fields for different } b. \text{ The maximum error is used as it is a stronger error measure than the } L^2 \text{ norm. We plot the truncation error results in Figure 4.8.}

We can see from Figure 4.8 that the errors for } b < 128 \text{ become smaller quicker than for } b \geq 128. \text{ In fact, in Figure 4.8(b) we see that the errors for smaller } b \text{ are still large in magnitude even for large } \kappa_n. \text{ We replot the data in Figure 4.8(b) in Figure 4.9 in two parts, for } b < 128 \text{ and } b \geq 128. \text{ If we assume that the KL expansion has converged when the relative error is less than 5%, we see that each KL expansion for } b \geq 500 \text{ converges in approximately 1,000 terms. For } b = 128 \text{ and } b = 64, \text{ we see convergence in about 2,500 terms, and for } b = 32 \text{ approximately 3,500 terms. However, we can see that convergence is not observed for } b \leq 16. \text{ This suggests that more than 5,000 terms are needed to see convergence within these GRFs.
(a) Relative $L^\infty$ norm of error in $T_{\kappa_n}$ to reference solution $T_\kappa$.

(b) Relative $L^\infty$ norm of error in $T_{\kappa_{n-1}}$ to $T_{\kappa_n}$.

Figure 4.8: Relative truncation errors computed a $\kappa$-term KL expansion for various $b$. 
Figure 4.9: Relative truncation error in $\kappa_{n-1}$-term approximation to a $\kappa_n$-term approximation for various $b$. 
Finally, we will examine the similarities and differences in the magnitudes of error obtained using the algebraically decaying angular power spectrum versus the angular power spectrum derived using a covariance function. We recall that Lang & Schwab (2013) used a reference solution of $\kappa = 2^7$ terms when computing their truncation error and the KL expansion of 100 terms was considered to converge. We will compare the truncation errors observed for $\alpha = 3$ as this truncation error is the greatest of all three cases tested ($\alpha = 3, 5, 9$). We plot these comparisons in log space in Figure 4.10. We can see that the relative errors between the algebraic decay with $\alpha = 3$ and for the covariance derived angular power spectrum for $b = 500, 1,000, 5,000, 7,500,$ and $10,000$ are visually similar, particularly for $b = 10,000$. This figure emphasizes that the rate of convergence for $b \leq 128$ are slower in comparison to larger $b$ and previously discussed results in Section 4.3.

Although we are satisfied with the level of convergence achieved for $b > 128$, many terms are required to observe convergence for $b \leq 128$ and particularly for $b \leq 16$, poor rates of convergence were observed. We also consider the appearance of the GRFs computed for small $b$. For $b = 8.5$, we expect to observe approximately the same level of spatial variability in the GRF produced by the KL expansion as the spatially correlated random field computed in Chapter 3. We will test this assumption using the Northridge ShakeMap. To compare directly, we compute one realization of a GRF with the KL expansion using $b = 8.5$ km and $\kappa = 5,000$ on exactly the same grid points as the Northridge ShakeMap. We plot these two random fields side by side in Figure 4.11. We observe that the degree of variability exhibited by the field generated using the KL expansion is significantly higher than that of the method of successive conditional simulations.

Thus, we have that the KL expansion with $b = 8.5$ neither converges in a reasonable number of terms, nor does it exhibit the degree of variability we expect. Both of these issues are related to the decay of the angular power spectrum. From our analysis, we know that if the angular power spectrum were to decrease at a faster rate, the convergence rate would increase and the degree of variability in the random field would decrease. Therefore, in the
Figure 4.10: A comparison of the truncation errors between the KL expansions obtained with the angular power spectra derived using the algebraic decay with $\alpha = 3$ and those derived using the covariance function with various $b$. 

(a) Relative $L^\infty$ norm of error in $T_{K_{n-1}}$ to $T_{K_n}$

(b) Relative $L^\infty$ norm of error in $T_{K_n}$ to reference solution $T_\kappa$
Figure 4.11: A comparison of the random fields generated with the method introduced in Chapter 3 and the KL expansion with the angular power spectrum derived using the covariance function. Both figures cover the same area in the LA basin and the same color scheme was used in both figures (ranging from approximately -3.14 to 3.14). The random field was generated for the ShakeMap of the Northridge (1994) earthquake.

next section, we explore a modified angular power spectrum which will decay at a faster rate than the angular power spectrum derived using the covariance function.

4.4.3 A Modified Angular Power Spectrum

To ensure the decay of the angular power spectrum for small $b$, we develop a hybrid-approach for its computation. Specifically, we use the angular power spectrum derived in the previous section, which accounts for the covariance function, in combination with the algebraically decaying sequence found in Section 4.3. We will denote this new, modified sequence $(B_l, l \in \mathbb{N})$. Thus, for

$$A_l = 2\pi \int_{x=-1}^{1} \rho(\arccos x) P_l(x) dx,$$

we write

$$B_l = (l + 1)^{-\alpha} A_l,$$
where we let $\alpha \geq 2$ to be consistent with Lang & Schwab (2013). By multiplying the angular power spectrum derived using a covariance function with a decay factor, we ensure decay of the sequence while still capturing kernel properties. We plot the modified angular power spectrum in Figure 4.12. By comparing these figures to Figure 4.5, we can see that the addition of the $(l+1)^{-\alpha}$ causes the function to decay quickly in comparison. We can also see that $\alpha = 3$ causes a higher rate of decay that $\alpha = 2$, as expected. A new table of maximum and minimum values is given in Table 4.3. Note that due to the definition of $B_l$, $A_0 = B_0$ for all $\alpha$ considered. We can see from this table that every angular power spectrum $B_l$ converges to machine precision in 1,000 terms.

![Angular Power Spectrum](image1)

(a) Angular Power Spectrum, $b = 8.5$

![Angular Power Spectrum](image2)

(b) Angular Power Spectrum, $b = 128$

Figure 4.12: Modified angular power spectrum plotted for $b = 8.5$ and $b = 128$ with various $\alpha$.

In the next section, we plot the GRFs computed using the modified angular power spectrum with various $\alpha$. Further, we compute the truncation results associated with this new sequence of coefficients.

### 4.4.4 KL Expansion for Modified Angular Power Spectrum

Using observations from previous sections, we expect the GRFs generated using the KL expansion with the modified angular power spectrum to change in a number of ways.
Table 4.3: Values of the modified angular power spectrum for \( l = 0 \) and \( l = 1000 \) with \( \alpha = 2.0, 2.5, \) and 3.0.

<table>
<thead>
<tr>
<th>( b )</th>
<th>( B_0 )</th>
<th>( B_{1000}, \alpha = 2.0 )</th>
<th>( B_{1000}, \alpha = 2.5 )</th>
<th>( B_{1000}, \alpha = 3.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.75005692533e-07</td>
<td>2.57336189105e-13</td>
<td>8.13361902626e-15</td>
<td>2.5627786335e-16</td>
</tr>
<tr>
<td>8.5</td>
<td>1.24267757287e-06</td>
<td>9.45839164961e-13</td>
<td>2.9895116324e-14</td>
<td>9.449427069e-16</td>
</tr>
<tr>
<td>16</td>
<td>4.39888574097e-06</td>
<td>1.97566352232e-12</td>
<td>6.2447516324e-14</td>
<td>1.9736893249e-15</td>
</tr>
<tr>
<td>32</td>
<td>7.5983581085e-05</td>
<td>2.35274418423e-12</td>
<td>7.4363131463e-14</td>
<td>2.3503937904e-15</td>
</tr>
<tr>
<td>64</td>
<td>7.04319798453e-05</td>
<td>1.6264111043e-12</td>
<td>5.1413208265e-14</td>
<td>1.6250160943e-15</td>
</tr>
<tr>
<td>128</td>
<td>0.00028177934598</td>
<td>8.95250165312e-13</td>
<td>2.82961514398e-14</td>
<td>8.94355809503e-16</td>
</tr>
</tbody>
</table>

Since the modified angular power spectrum decays at a higher rate than the angular power spectrum derived from the covariance function, we expect that the degree of variability within each GRF will decrease, and that the range of values within the field will also decrease. Further, because increasing \( \alpha \) causes the modified angular power spectrum to decrease at a higher rate, we expect that the degree of variability will decrease with increasing \( \alpha \), as well as the range of values within the field. We plot the GRFs obtained for \( b = 8.5 \) and \( b = 128 \) for various \( \alpha \) in Figure 4.13. We observe from these figures that the trends from previous KL expansions apply here.

Next, we consider the truncation error in the GRFs using the modified angular power spectrum. We compute the KL expansion up to \( \kappa \)-terms, where \( \kappa \) is determined by the final term in Table 4.2. We will use the following statistics to measure the truncation error, where we denote the GRF compute with \( \kappa \) and \( \kappa_n \) terms as \( T_\kappa \) and \( T_{\kappa_n} \) respectively.

- The relative error, or the maximum error in the \( \kappa_{n-1} \) term expansion as an approximation to the \( \kappa_n \) term expansion over the range of values within the field \( T_{\kappa_n} \), denoted \( r \):

\[
\frac{||T_{\kappa_{n-1}} - T_{\kappa_n}||_\infty}{r}
\]

- The final relative error, or the maximum error in the \( \kappa_n \) term expansion as an approximation to the \( \kappa \) term expansion over the range of values within the field \( T_\kappa \), which we
Figure 4.13: GRFs generated using $b = 8.5$, $b = 128$ with the modified angular power spectrum with $\alpha = 2.0, 2.5$ and 3.0 with a $\kappa$-term truncation.
\[ \frac{||T_{\kappa} - T_{\kappa_n}||_{\infty}}{r} \]

We plot the truncation error in Figure 4.14 for \( b = 8.5 \) and \( b = 128 \). We can see that the KL expansions are converging at a much higher rate than the KL expansions associated with the unmodified angular power spectrum. Further, we observe that the relative error increases with increasing \( \alpha \). This is because the range of values within the GRF decreases for increasing alpha, while the maximum error decreases. We can see from these figures that modifying the angular power spectrum has increased the rate of convergence.

To better understand how these truncation errors compare with those obtained using the algebraically decaying angular power spectrum, we plot them against one another in Figure 4.15. We include the relative error statistics for the \( b = 8.5 \) and \( b = 128 \) data sets along side the relative error for the algebraically decaying angular power spectrum with \( \alpha = 3 \). We can see that the error observed for \( A_l \) when \( \alpha = 3 \) is relatively close to the error observed for \( B_l \) when \( \alpha = 2.0 \). As \( \alpha \) increases, the errors for \( B_l \) increase, meaning that they converge at a slower rate. Moving forward in this section, we will use \( \alpha = 2.0 \), as it results in the smallest relative truncation error.

Although the GRFs generated here for \( b = 8.5 \) appear to be similar to those obtained by the successive conditional simulation method in Chapter 3, we see that the range of values within the GRF is much smaller. For example, the values in the random fields generated in Chapter 3 were usually contained within the interval \([-3.14, 314]\); however the values obtained using the modified angular power spectrum with \( \alpha = 2 \) range from about \([-3.14e-3, 3e-3]\). Thus, modifying the angular power spectrum may result in better convergence results and be result in a more similar level of variability within the field, but it will decrease the range of values. In order to use the GRFs computed in this section, we scale the fields to reflect the range of values we need. For example, if we would like the range of values to be
Figure 4.14: Various truncation errors computed for the KL expansion computed with the modified angular power spectrum.
Figure 4.15: Relative truncation error in KL expansions using angular power spectrum $B_l$ for $b = 8.5, 128$ and $\alpha = 2, 2.5, 3$ and for angular power spectrum $A_l$ with $\alpha = 3$. 
in the range $[-3.14, 3.14]$, we scale the GRF, $T$, such that

$$T^* = T \ast \frac{3.14}{\max T}.$$ 

provided $T$ has mean zero. Otherwise, we center the random field by subtracting the mean. We plot the scaled KL expansion using $B_l, b = 8.5, \alpha = 2$ with the random field generated using the method of successive conditional simulations in Chapter 3 in Figure 4.16. From this figure we can see that modifying the angular power spectrum, even with the smallest decay parameter $\alpha$, results in a GRF which is less variable than that produced using the methods of Chapter 3 for the Northridge event. This may suggest that other $\alpha$ values should be explored. In particular, a GRF generated with $1 < \alpha < 2$ may exhibit a similar correlation structure to that of the successive conditional simulations in Chapter 3.

Figure 4.16: A comparison of the random fields generated with the method introduced in Chapter 3 and the KL expansion with the modified angular power spectrum derived using the covariance function. Both figures cover the same area in the LA basin and the same color scheme was used in both figures (ranging from approximately -3.14 to 3.14). The random field was generated for the ShakeMap of the Northridge (1994) earthquake.
In the next section, we will compute the GRFs using the KL expansion with both \((A_l, b = 8.5, \kappa = 5,000)\) and \((B_l, b = 8.5, \alpha = 2.0, \kappa = 5,000)\) exactly on the grid points specified by the ShakeMap for the Northridge earthquake (pictured in Figure 4.11 and Figure 4.16). This will enable us to compute a single realization for each of the GRFs of a spatially variable ShakeMap, which can then be used to compute estimated losses.

### 4.4.5 Computing Losses with the KL Expansion

An important motivation for adding spatially correlated random fields to ShakeMap is the impact on estimated hazards and losses (see Chapter 3 Section 3.6). We use the USGS PAGER software to compute estimated fatalities and economic losses using the ShakeMap for the Northridge event with added spatial variability. To begin, we recall the loss results from the Northridge event with added spatial variability and no conditioning data. In particular, we focus on economic losses as these have more precision. The economic losses for the ShakeMap are shown in Figure 4.17. From this figure, we can see that the ShakeMap alone, with no added variability estimated $1.52 Billion in losses, while the mean of the loss distribution for 1,000 realizations of the spatially variable ShakeMap was $3.81 Billion.

Now we will compute the estimated economic losses by adding a GRF generated using the KL expansion to the ShakeMap. In order to do this, we evaluate the GRF exactly on the ShakeMap grid points so that we may add the random field to the ShakeMap data. First, we compute the KL expansion using the angular power spectrum derived from the covariance function with \(b = 8.5\) and \(\kappa = 5,000\). This random field was plotted and compared to a single realization of the spatially correlated random field generated using the method introduced in Chapter 3 in Figure 4.11. Next, we use the modified angular power spectrum with parameters \(b = 8.5, \alpha = 2.0, \text{ and } \kappa = 5,000\) to compute the KL expansion. Further, we scale the random field such that the values range from approximately \([-3.14, 3.14]\). This random field is plotted against the same random field computed with the method from Chapter 3 in Figure 4.16. These random fields are added to the ShakeMap via \((3.2)\) using the uncertainty provided by the ShakeMap. Plots of the ShakeMaps with the added random fields are in Figure 4.18.
Figure 4.17: Estimated economic losses for the Northridge event with 1000 realizations of a spatially variable random field.
We compute the economic losses for each ShakeMap presented in Figure 4.18 as follows: For the spatially variable ShakeMap computed with the covariance derived angular power spectrum, we compute $17.7$ Billion in economic losses, and for the modified angular power spectrum we compute $8.9$ Billion in economic losses. Since we are only using a single realization for each type of GRF, we cannot draw strong conclusions on these results; however, we note that the losses for the spatially variable ShakeMap with the covariance derived angular power spectrum estimates losses much higher than the loss distribution in Figure 4.17, while the spatially variable ShakeMap with the modified angular power spectrum estimates losses which are within the range. This is likely due to the high variability present in the GRF when the covariance derived angular power spectrum is used.

### 4.5 Dimension Reduction Results

The initial motivation for using the KL expansion came from the possibility of reducing the dimensionality of the spatially variable ShakeMap computation, and the independence
from the ShakeMap grid. Recall that for the Northridge event, the stochastic dimension was 33,366. When the angular power spectrum derived from an algebraically decaying sequence was used, both of these goals were seemingly accomplished, although these GRFs have no ShakeMap application. For example, the KL expansion was assumed to decay within 128 terms ($\kappa = 128$), which means that the entirety of the covariance structure was captured in the sequence $\{A_l, l = 0, \cdots , 128\}$, independent of the number of sampling locations. Unfortunately, dimension reduction of this magnitude was not observed in the seismic application. As previously stated, the truncation error of a $\kappa$-term KL expansion is related to the rate of decay of the angular power spectrum. In terms of a global scale, the correlation functions used to estimate the correlation in seismic ground motions have a very small correlation range. As such, the correlation function plotted on a global scale asymptotically decreases from 1 to 0, resulting an angular power spectrum which begins at small values and exhibits hardly any decay. Due to computational limitations, the KL expansion was taken up to a maximum of $\kappa = 5,000$ terms, which still showed high truncation error. Further, the resultant random fields exhibited more variability than expected and estimated higher losses than their counterparts computed using the successive conditional simulation method in Chapter 3 for the Northridge event. To fix these issues, we introduced a hybrid method, namely the modified angular power spectrum, which contained the covariance properties of the correlation functions by also decayed algebraically. For the case where the modified angular power spectrum, $(B_l, b = 8.5, \alpha = 2.0)$, was used, the sequence was shown to decay before the 5,000 term maximum. Further, the loss results computed using this expansion are within range of those computed in Chapter 3. Thus, in this case dimension reduction was achieved.
Spherical wavelets are useful in many mathematical problems, as they can project complicated functions onto different levels of the $L_2$ function space of the sphere. A recent work, Narcowich et al. (2006), showed that spherical wavelets may be further decomposed into highly localized polynomials, called a spherical needlets, which can approximate complex functions through a spherical needlet decomposition. The highly localized nature of spherical needlets allow them to capture fine details of functions on the sphere, making them applicable to a wide variety of uses. A further work, Wang et al. (2015), made the implementation of spherical needlets possible by introducing the fully discrete spherical needlet approximation. Their method utilizes spherical quadrature rules for the numerical computation of integrals which form the coefficients of the needlet approximation. We provide the necessary mathematical background and implementation details below followed by two examples. First, we approximate a localized, deterministic function on the sphere utilizing the Wendland radial basis function. Second, we use the spherical needlet approximation to compute an isotropic random field on the sphere.

5.1 Mathematical Background

To begin, we will give the preliminary mathematical background necessary to derive the functional form of spherical needlets. Next, we will give the details for the approximation of a function on the sphere with the semidiscrete and fully discrete spherical needlet approximations. Numerical examples will follow in the next section.

5.1.1 Preliminary Background

For $d \geq 2$, let $\mathbb{R}^{d+1}$ be the real $d + 1$-dimensional Euclidean space. For $x, y \in \mathbb{R}^{d+1}$, let $x \cdot y$ denote the inner product and let $|x| = \sqrt{x \cdot x}$ denote the Euclidean norm. We will
denote the unit sphere in $\mathbb{R}^{d+1}$ as

$$S^d = \{ x \in \mathbb{R}^{d+1} : |x| = 1 \}. $$

Then $(S^d, h(x, y))$ forms a compact metric space with metric

$$h(x, y) = \arccos(x \cdot y),$$

which we recognize in three dimensions as the geodetic distance between points on the sphere.

For $1 \leq p \leq \infty$, let $L_p(S^d)$ denote the real $L_p$ function space on the sphere $S^d$ endowed with norm $\| \cdot \|_{L_p(S^d)}$, where

$$\|f\|_{L_p(S)} = \left( \int_{S^d} |f(x)|^p d\sigma_d(x) \right)^{1/p}, \quad f \in L_p(S^d).$$

In this definition, $\sigma_d$ denotes the normalized Riemann surface measure. In particular, for $p = 2$, $L_2(S^d)$ forms a Hilbert space with inner product

$$(f, g)_{L_2(S^d)} = \int_{S^d} f(x)g(x)d\sigma_d(x).$$

We will define a spherical harmonic of degree $l$ on $S^d$ to be the restriction of a homogenous and harmonic polynomial of degree $l$ defined on $\mathbb{R}^{d+1}$ to $S^d$ (Wang et al., 2015). If we let $H_l(S^d)$ be the set of all spherical harmonics of exact degree $l$ on $S^d$, then the dimension of the space is given by

$$Z(d, l) := (2l + d - 1) \frac{\Gamma(l + d - 1)}{\Gamma(d)\Gamma(l + 1)}. $$

In particular, if $d = 2$, we have

$$Z(2, l) := (2l + 1) \frac{\Gamma(l + 1)}{\Gamma(2)\Gamma(l + 1)} = (2l + 1). $$
From Chapter 4, we recall the definition of the Legendre polynomial, associated Legendre functions, and surface spherical harmonic functions. For \( l \in \mathbb{N}_0 \), we define the Legendre polynomial using Rodrigue’s formula:

\[
P_l(\mu) = 2^{-l} \frac{1}{l!} \frac{\partial^l}{\partial \mu^l} (\mu^2 - 1)^l, \mu \in [-1, 1].
\]

In the case where \( d \neq 2 \), we will use the normalized Legendre polynomial given by

\[
P_l^{(d+1)}(t) = P_l^{(\frac{d-2}{2}, \frac{d-2}{2})}(t)/P_l^{(\frac{d-2}{2}, \frac{d-2}{2})}(1),
\]

where \( P_l^{(\alpha,\beta)}(t) \) for \(-1 \leq t \leq 1\) are the Jacobi polynomials of degree \( l \).

The Jacobi polynomials form an orthogonal system with respect to the weight \( \omega_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta, -1 \leq t \leq 1 \) (Wang et al., 2015). For \( 1 \leq p \leq \infty \), define the space \( \mathbb{L}_p(w_{\alpha,\beta}) \) to be the \( \mathbb{L}_p \) space with measure \( \omega_{\alpha,\beta}(t)dt \). Then \( \mathbb{L}_p(\omega_{\alpha,\beta}) \) forms a Hilbert space with inner product

\[
(f,g)_{L_2(\omega_{\alpha,\beta})} = \int_{-1}^{1} f(t)g(t)\omega_{\alpha,\beta}(t)dt,
\]

for \( f, g \in \mathbb{L}_2(\omega_{\alpha,\beta}) \).

By definition, a zonal function is a function \( K : S^d \times S^d \to \mathbb{R} \) which depends only on the inner product of the arguments. Thus, \( P_l^{(d+1)}(x \cdot y) \) is a zonal function.

The associated Legendre functions defined in terms of the Legendre polynomial are given as

\[
P_{lm}(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{\partial^m}{\partial \mu^m} P_l^{(d+1)}(\mu).
\]

Lastly, we denote the surface spherical harmonic functions, \( \mathcal{Y} = (Y_{lm}, l \in \mathbb{N}_0, m = -l, \cdots, l) \), where

\[
Y_{lm}(\theta, \phi) : [0, \pi] \times [0, 2\pi] \to \mathbb{C},
\]
and
\[ Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_{lm}(\cos \theta) e^{im\phi}. \]

The set of spherical harmonic functions \( \{ Y_{lm} : l \geq 0, m = 1, \cdots, Z(d, l) \} \) form an orthonormal basis for the space \( \mathbb{L}_2(S^d) \), and satisfy the addition theorem:
\[ \sum_{m=1}^{Z(d,l)} Y_{lm}(x)Y_{lm}(y) = \frac{Z(d,l)}{4\pi} P_l^{(d+1)}(x \cdot y). \]

For later use, we will define the generalized Sobolev space following Wang et al. (2015). Let \( s \in \mathbb{R}_+ \), and define \( b_l^{(s)} = (1 + \lambda_l)^{s/2} \), where \( \lambda_l = l(l + d - 1) \) is an eigenvalue of the negative Laplace-Beltrami operator \( -\Delta^* \) on \( S^d \). The Fourier coefficients for \( f \in \mathbb{L}_1(S^d) \) are given by
\[ \hat{f}_{lm} = \int_{S^d} f(x)Y_{lm}(x)d\sigma_d(x) \]
for \( l \geq 0, m = 1, \cdots, Z(d, l) \). Then we define the generalized Sobolev space \( \mathbb{W}_p^s(S^d) \) to be the set of all functions \( f \in \mathbb{L}_p(S^d) \) which satisfy
\[ \sum_{l=0}^{\infty} b_l^{(s)} \sum_{m=1}^{Z(d,l)} \hat{f}_{lm} Y_{lm} \in \mathbb{L}_p(S^d). \]

The Sobolev space \( \mathbb{W}_p^s(S^d) \) forms a Banach space with norm
\[ \| f \|_{\mathbb{W}_p^s(S^d)} = \| \sum_{l=0}^{\infty} b_l^{(s)} \sum_{m=1}^{Z(d,l)} \hat{f}_{lm} Y_{lm} \|_{\mathbb{L}_p(S^d)}. \]

### 5.1.2 Spherical Needlets

Now that we have the preliminary mathematical background, we will introduce spherical needlets. This section largely follows the work of Wang et al. (2015). We will begin with several key definitions.
Definition 5.7. Positive Quadrature Rule Given $N \geq 1$, for $k = 1, \cdots, N$, let $x_k$ be $N$ nodes on $S^d$ and let $w_k > 0$ be corresponding weights. The set $\{(w_k, x_k) : k = 1, \cdots, N\}$ is a positive quadrature rule exact for polynomials of degree up to $\eta \geq 0$ if

$$\int_{S^d} p(x) d\sigma_d(x) = \sum_{k=1}^{N} w_k p(x_k).$$

Definition 5.8. Filter A continuously compactly supported function, $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a filter.

We can define a filtered kernel on $S^d$ with the associated filter $g$ as

$$v_{T,g}(x \cdot y) = \begin{cases} 1, & 0 \leq T < 1, \\ \sum_{l=0}^{\infty} g\left(\frac{1}{T}\right)Z(d,l)P_l^{(d+1)}(x \cdot y), & T \geq 1, \end{cases}$$

where $T \in \mathbb{R}_+$.

We will define a spherical needlet to be a localized polynomial on the sphere with an associated quadrature rule and a filter. Thus, with the definitions above, we now have the tools to construct a needlet. We will follow Wang *et al.* (2015) in our choice of needlet filter and quadrature rule. For a smoothness parameter $\kappa \geq 1$, define $h$ to be the filter which satisfies

$$h \in C^{\kappa}(\mathbb{R}_+), \quad \text{supp } h = [1/2, 1], \quad (5.1)$$

$$h(t)^2 + h(2t)^2 = 1, \text{ if } t \in [1/2, 1]. \quad (5.2)$$

Define the spherical needlet quadrature to be

$$\{(w_{jk}, x_{jk}) : k = 1, \cdots, N_j\} \quad (5.3)$$

such that $w_{jk} > 0, k = 1, \cdots, N_j$ and the quadrature is exact for polynomials of up to degree $2^{j+1} - 1$. 

123
Definition 5.9. Spherical Needlet

A spherical needlet, \( \psi_{jk}, k = 1, \cdots, N_j \) of order \( j \) with needlet filter \( h \) and needlet quadrature (5.3) is given as

\[
\psi_{jk}(x) = \sqrt{w_{jk}} v_{2j-1,h}(x \cdot x_{jk})
\]

or equivalently,

\[
\psi_{0k}(x) = \sqrt{w_{0k}}, \quad \psi_{jk}(x) = \sqrt{w_{jk}} \sum_{l=0}^{\infty} h\left( \frac{l}{2j-1} \right) Z(d,l) P_{l}^{(d+1)}(x \cdot x_{jk}), \quad j \geq 1.
\]

We now have the definition of a spherical needlet. We can see that \( \psi_{jk} \) is a polynomial of degree \( 2^j - 1 \). Further, we can see that the equation for \( \psi_{jk}, j \geq 1 \) involves an infinite sum over \( l \); however, since our needlet filter has support on \([1/2, 2]\), we can see that the limits on the sum for \( \psi_{jk}(x), j \geq 1 \) can be reduced. For example, if \( d = 2 \) and \( j = 1 \), we can write

\[
\psi_{1k}(x) = \sqrt{w_{1k}} \sum_{l=0}^{\infty} h(l) Z(2,l) P_{l}^{(2)}(x \cdot x_{1k}),
\]

\[
= \sqrt{w_{1k}} \sum_{l=1}^{2} h(l) Z(2,l) P_{l}^{(2)}(x \cdot x_{1k}),
\]

since \( h(l) = 0, l = 0, l = 3, 4, \cdots, \infty \). If \( j = 2 \), we have

\[
\psi_{2k}(x) = \sqrt{w_{2k}} \sum_{l=0}^{\infty} h\left( \frac{l}{2} \right) Z(2,l) P_{l}^{(4)}(x \cdot x_{2k}),
\]

\[
= \sqrt{w_{2k}} \sum_{l=1}^{4} h\left( \frac{l}{2} \right) Z(2,l) P_{l}^{(4)}(x \cdot x_{2k}).
\]

Let \( f \in L_1(S^d) \). We will define a filtered approximation \( V_{T,g} \) on \( L_1(S^d), T \geq 0 \) as an integral operator with filtered kernel \( v_{T,g}(x \cdot y) \). Thus,

\[
V_{T,g}(f, x) = (f, v_{T,g}(x \cdot \cdot))_{L_2(S^d)} = \int_{S^d} f(y) v_{T,g}(x \cdot y) d\sigma_d(y)
\]
We can define the semidiscrete spherical needlet approximation with filter $h$ and needlet quadrature (5.3) as

$$V_{need}^{J}(f ; x) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{L_2(\mathbb{S}^d)} \psi_{jk}(x), x \in \mathbb{S}^d.$$ 

Now that we have the form of the semidiscrete spherical needlet approximation for a function, $f$, we derive the fully discrete form so that the approximation can be computed numerically.

### 5.1.3 Fully Discrete Needlet Approximation

To summarize the results of the previous section, we have that the semidiscrete spherical needlet approximation is given as

$$V_{need}^{J}(f, x) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{L_2(\mathbb{S}^d)} \psi_{jk}(x), x \in \mathbb{S}^d,$$

where

$$\psi_{0k}(x) = \sqrt{w_{0k}}, \quad \psi_{jk}(x) = \sqrt{w_{jk}} \sum_{l=0}^{\infty} h \left( \frac{l}{2j-1} \right) Z(d, l) P_l(x \cdot x_{jk}), j \geq 1,$$

and

$$(f, \psi_{jk})_{L_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(y) \psi_{jk}(y) d\sigma_d(y).$$

In order to make the semidiscrete formulation a fully discrete needlet approximation, we need a quadrature rule for the inner product $(f, \psi_{jk})_{L_2(\mathbb{S}^d)}$ between the function, $f$, and needlet $\psi_{jk}$. We will define the discretization quadrature rule to be

$$Q_N = Q(N, l) = \{(W_i, y_i) : i = 1, \cdots, N\}$$

exact for polynomials of degree up to some $l$. Applying the quadrature rule to the needlet coefficient, we have a discrete representation of the inner product:
\[(f, \psi_{jk})_{Q_N} = \int_{\mathbb{S}^d} f(y) \psi_{jk}(y) d\sigma_d(y),\]
\[= \sum_{i=1}^{N} W_i f(y_i) \psi_{jk}(y_i).\]

Using this quadrature rule, we write the fully discrete needlet approximation:

\[V_{J,N}^{\text{need}}(f; x) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{Q_N} \psi_{jk}(x), x \in \mathbb{S}^d.\]

Thus, with this expression, we have a fully defined, fully discrete needlet approximation to the function \(f\). Our main goal in this chapter is to compute an approximation to an isotropic random field on the sphere. Thus, in the remaining parts of the mathematical background, we extend the spherical needlet approximation to random fields on the sphere.

5.1.4 Needlet Decomposition of Random Fields

In this section, we seek to describe the spherical needlet approximation of isotropic random fields on the sphere. We will use many of the same concepts discussed in Chapter 4, particularly the KL expansion representation of a random field.

Let \(T\) be a 2-weakly isotropic random field on \(\mathbb{S}^d\), and let

\[\{Y_{lm} : m = 1, \ldots, Z(d,l), l = 0,1, \ldots \}\]

be an orthonormal spherical harmonic basis for \(L_2(\mathbb{S}^d)\). Note here that the meaning of the index \(m\) here is slightly altered from Chapter 4, where \(m = -l, \ldots, l\). Recall that by Marinucci & Peccati (2011) (Theorem 5.1), \(T\) admits a Karhunen-Loève expansion such that \(T\) satisfies \(P\)-almost surely

\[\int_{\mathbb{S}^d} |T(x)|^2 d\sigma_d(x) < \infty\]
\[
T \sim \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} (T, Y_{lm})_{L^2(S^d)} Y_{lm}
\]
in the \(L_2(S^d)\) sense.

Let \(d \geq 2\) and \(T\) be an isotropic random field on \(S^d\). Further, let \(\psi_{jk}\) be a spherical needlet as described in the previous section with needlet filter \(h \in C^\kappa(\mathbb{R}_+), \kappa \geq 1\). According to Wang et al. (2015), for some \(J \in \mathbb{N}_0\), we may write the semidiscrete needlet approximation of order \(J\) for \(T\) as

\[
V^\text{need}_J(T; \omega, x) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{L^2(S^d)} \psi_{jk}(x), \omega \in \Omega, x \in S^d.
\]

We recall from Chapter 4 that the random field depends highly on the angular power spectrum. In particular, we showed that the convergence of a KL expansion depends on the rate of decay of the angular power spectrum. We will show a parallel here for approximation errors for needlet approximations. Namely, for a 2-weakly isotropic random field \(T\) on \(S^d\) the order of convergence of the approximation error depends on the rate of decay of the angular power spectrum. Following Wang et al. (2015), we will discuss the approximation errors for smooth random fields in terms of the angular power spectrum.

For \(T\) a 2-weakly isotropic random field on \(S^d\), the centered random field corresponding to \(T\) is given by

\[
T^c(\omega, x) = T(\omega, x) - \mathbb{E}[T(x)].
\]

Recalling the definition of a zonal function, we write

\[
G(x \cdot y) = \mathbb{E}[T^c(x)T^c(y)]
\]
where $G$ is a zonal kernel on $\mathbb{S}^d$ and is also a covariance function of $T$. Given $d \geq 2$, let $
abla = (d - 2)/2$. When $G(\cdot) \in L_2(\omega_{\alpha,\alpha})$, it has a convergent Fourier expansion,

$$G \sim \sum_{l=0}^{\infty} A_l^{(d)} Z(d, l) P_{l+1}^{(d+1)},$$

where the convergence is in the $L_2(\omega_{\alpha,\alpha})$ sense (Wang et al., 2016). Using the orthogonality of the Legendre polynomials, we solve for the coefficients $A_l$, to find

$$A_l = \int_{\mathbb{S}^d} G(x \cdot y) P_{l+1}^{(d+1)}(x \cdot y) d\sigma_d(x),$$

which we refer to as the angular power spectrum. Using the addition theorem, we may write

$$G(x \cdot y) = \sum_{l=0}^{\infty} A_l \sum_{m=1}^{Z(d,l)} Y_{lm}(x) Y_{lm}(y).$$

We recall the exercise in Section 4.4.1 where we show this result for the specific case of $d = 3$,

$$G(x \cdot y) = \sum_{l=0}^{\infty} A_l \sum_{m=1}^{2l+1} Y_{lm}(x) Y_{lm}(y).$$

Using the orthogonality of $Y_{lm}$, we have

$$(G(\cdot \cdot y), Y_{lm}(\cdot))_{L_2(\mathbb{S}^d)} = A_l Y_{lm}(y).$$

Following Wang et al. (2015), we define the Fourier coefficients of the random field $T$ by

$$\hat{T}_{lm} = (T, Y_{lm})_{L_2(\mathbb{S}^d)}, \ l \geq 0, m = 1, \cdots, Z(d, l).$$

Then we can define the relationship between the centered Fourier coefficients and the angular power spectrum with the following lemma (Lemma 4.1 Wang et al. (2015)).
Lemma 5.10. For $d \geq 2$, let $\alpha = (d - 2)/2$. Let $T$ be a $2$-weakly isotropic random field on $\mathbb{S}^d$ and let $G(x \cdot y)$ be the covariance function for $T$ satisfying $G(\cdot) \in \mathbb{L}_1(\omega_{\alpha,\alpha})$. Then for $l, l' \geq 0$ and $m, m' = 0, \cdots, Z(d, l)$, $E[\hat{T}_{lm} \hat{T}_{l'm'}] = A_l \delta_{ll'} \delta_{mm'}$.

Proof. Consider

$$E[\hat{T}_{lm} \hat{T}_{l'm'}] = E\left[ \int_{\mathbb{S}^d} T(x) Y_{lm}(x) d\sigma_d(x) \int_{\mathbb{S}^d} T(x') Y_{l'm'}(x') d\sigma_d(x') \right]$$

$$= \int_{\mathbb{S}^d} \left( \int_{\mathbb{S}^d} E[T(x)T(x')] Y_{lm}(x) d\sigma_d(x) \right) Y_{l'm'}(x') d\sigma_d(x')$$

by the Fubini Theorem. Given that

$$G(x \cdot x') = E[T(x)T(x')]$$

and

$$(G(\cdot \cdot x'), Y_{lm}(\cdot))_{L_2(\mathbb{S}^d)} = A_l Y_{lm}(x'),$$

we can conclude

$$E[\hat{T}_{lm} \hat{T}_{l'm'}] = \int_{\mathbb{S}^d} A_l Y_{lm}(x') Y_{l'm'}(x') d\sigma_d(x')$$

$$= A_l \delta_{ll'} \delta_{mm'}.$$

□

Before we describe the numerical examples of this chapter, we will give a few important results from Wang et al. (2015). The first result described the mean $L_p$-error for discrete needlets (Theorem 4.9, Wang et al. (2015)).

Theorem 5.11. Let $d \geq 2, 1 \leq p < \infty, s > d/p, J \in \mathbb{N}_0$. Let $T$ be a $[p]$-weakly isotropic random field on $\mathbb{S}^d$ satisfying $T \in \mathbb{W}^s_p(\mathbb{S}^d)$ $P$-almost surely. Let $Q_N$ be a discretization
quadrature exact for degree \(3 \cdot 2^{J-1} - 1\) and let \(h\) be a needlet filter given by (5.1) and satisfying \(h \in C^\kappa(\mathbb{R}_+)\) with \(\kappa \geq \left\lceil \frac{d+3}{2} \right\rceil\). Then

\[
\mathbb{E}[\|T - V_{J,N}^{\text{need}}(T)\|_{L_p(S^d)}^p] \leq c 2^{-pJ_s} \mathbb{E}[\|T^c\|_{W_p^s(S^d)}^p],
\]

where \(T^c\) is the centered random field, and the constant \(c\) depends only on \(d,p,s\), the filter \(h\) and \(\kappa\).

Next, we consider the case \(p = 2\) to find the mean \(L_2\)-error for discrete needlets (Theorem 4.10, Wang et al. (2015)).

**Theorem 5.12.** Let \(d \geq 2, s > d/2, J \in \mathbb{N}_0\). Let \(T\) be a 2-weakly isotropic random field on \(S^d\) with angular power spectrum \(A_l\) satisfying \(\sum_{l=0}^\infty A_l l^{2s+d-1} < \infty\). Let \(Q_N\) be a discretization quadrature exact for degree \(3 \cdot 2^{J-1} - 1\) and let \(h\) be a needlet filter given by (5.1) and satisfying \(h \in C^\kappa(\mathbb{R}_+)\) with \(\kappa \geq \left\lceil \frac{d+3}{2} \right\rceil\). Then

\[
\mathbb{E}[\|T - V_{J,N}^{\text{need}}(T)\|_{L_2(S^d)}^2] \leq c 2^{-2J_s} \mathbb{E}[\|T^c\|_{W_2^s(S^d)}^2],
\]

where \(T^c\) is the centered random field, and the constant \(c\) depends only on \(d,s\), the filter \(h\) and \(\kappa\).

Finally, we will consider the pointwise error for discrete needlets (Theorem 4.11, Wang et al. (2015)).

**Theorem 5.13.** Let \(d \geq 2, s > d/2, J \in \mathbb{N}_0\). Let \(T\) be a 2-weakly isotropic random field on \(S^d\) with angular power spectrum \(A_l\) satisfying \(\sum_{l=0}^\infty A_l l^{2s+d-1} < \infty\). Let \(Q_N\) be a discretization quadrature exact for degree \(3 \cdot 2^{J-1} - 1\) and let \(h\) be a needlet filter given by (5.1) and satisfying \(h \in C^\kappa(\mathbb{R}_+)\) with \(\kappa \geq \left\lceil \frac{d+3}{2} \right\rceil\). Then, \(P\)-almost surely,

\[
\|T - V_{J,N}^{\text{need}}(T)\|_{L_2(S^d)} \leq c 2^{-J_s} \|T^c\|_{W_2^s(S^d)},
\]
where $T^c$ is the centered random field, and the constant $c$ depends only on $d, s$, the filter $h$ and $\kappa$.

Thus, with these theorems, we have an error bound on the discrete needlet approximation of a smooth, 2-weakly isotropic random field in terms of the angular power spectrum.

5.2 Numerical Examples

We include several numerical examples in this section, including the approximation of a localized, deterministic function on the sphere, and the approximation of an isotropic random field using the angular power spectrum. Before we give details relevant to these examples, we will illustrate the properties of spherical needlets, including their highly localized behavior. We take advantage of the NeedMat package, a Matlab package for spherical needlets (Fan, 2015) which computes spherical needlets and needlet approximations efficiently. This package works in adherence with the HEALPix grid, can divide a spherical surface into pixels of equal area (Górski et al., 2005). To maintain consistency with the NeedMat package, we will use the following definition of the spherical needlet,

$$
\psi_{jk}(x) = \sqrt{w_{jk}} \sum_{l=0}^{\infty} b \left( \frac{l}{2j} \right) \frac{2l + 1}{4\pi} P_l(x \cdot x_{jk}),
$$

which is used in both Fan (2015) and Marinucci & Peccati (2011). This definition assumes we are working in $\mathbb{R}^3$, and thus we have $d = 2$. Further, we will assume the following about the needlet filter and needlet quadrature. To define the needlet filter, we follow Marinucci & Peccati (2011). To construct the function $b \in C^\infty$, we define functions $\phi_1(t)$, $\phi_2(u)$, and $\phi_3(t)$:

$$
\phi_1(t) = \begin{cases} 
\exp(-\frac{1}{1-t^2}), & -1 \leq t \leq 1, \\
0, & \text{otherwise,}
\end{cases}
$$

$$
\phi_2(u) = \frac{\int_{-1}^{u} f(t) dt}{\int_{-1}^{1} f(t) dt},
$$
\[
\phi_3(t) = \begin{cases} 
1, & 0 \leq t \leq \frac{1}{B}, \\
\phi_2(1 - \frac{2B}{B-1}(t - \frac{1}{B})), & \frac{1}{B} < t \leq 1, \\
0, & t > 1.
\end{cases}
\]

Finally, to construct \( b \), we let

\[
b(x) = \sqrt{\phi_3\left(\frac{x}{2}\right) - \phi_3(x)}, \quad -\infty < x < \infty.
\]

We see that \( \phi_1(t) \in C^\infty \) and has compact support \((-1, 1)\). Similarly, \( \phi_2(u) \in C^\infty \) and is such that \( \phi_2(-1) = 0 \) and \( \phi_2(1) = 1 \). It is clear that \( \phi_3(t) \) uses the function \( \phi_2(u) \) such that \( \phi_3(t) \) is constant in the interval \((0, \frac{1}{B})\), and monotonically decreases to 0 in the interval \((\frac{1}{B}, 1)\). Thus, we have \( b(\cdot) \in C^\infty \). We also have that the needlet filter satisfies two additional properties, namely that \( b(\cdot) > 0 \) and has compact support in \([1/2, 2]\), and \( \sum_{j=1}^{\infty} b^2(\frac{l}{2^j}) = 1 \) (Marinucci & Peccati, 2011), thus making \( b \) a valid filter.

To compute the spherical needlets, we must also define the needlet quadrature. Following Fan (2015), we use equally weighted cubature weights, \( w_{jk} \), and let the cubature points, \( x_{jk} \), be defined as the center of the pixels in a HEALPix grid (Fan, 2015). The HEALPix grid provides a discretization of the sphere in to \( N_{\text{pix}} \) pixels of equal area. We can specify the number of pixels as \( N_{\text{pix}} = 12N_{\text{side}}^2 \), where \( N_{\text{side}} = 2^n, n \in \mathbb{N}_0 \). We let \( w_{jk} = \frac{4\pi}{N_{\text{pix}}} \), which is the area of each pixel in this implementation since we are working on the unit sphere.

We will look at the behavior of these needlets when \( d = 2 \), and when \( j = 3 \) and 5. We include several plots in Figure 5.1 to demonstrate the effect of varying \( j \) and \( x_{jk} \) on the needlet. We observe that varying the center of the needlet, \( x_{jk} \), causes a translation of the needlet such that the largest amplitude of the needlet is located at the point \( x_{jk} \). If we instead change the value of \( j \), we see that the needlet becomes more localized in space. Thus, varying the values of \( x_{jk} \) and \( j \) have the effect of translation and dilation of the needlet respectively.
Figure 5.1: In (a) and (b), we plot the amplitude of the needlet against the points on the HEALPix grid. Altering the ‘center’ of the needlet, $x_{jk}$, between (a) and (b) results in a translation of the needlet, with the maximum amplitude achieved at $x_{jk}$. We plot the Hammer projection of two needlets in (c) and (d) with $j = 3$ and $j = 5$ respectively. We observe that larger $j$ cause the needlet to become more localized.
Consider the equation for the needlet approximation to \( f \), where \( f \in L^2(S^2) \). We have that
\[
V_{j}^{\text{need}}(f; \mathbf{x}) = \sum_{j=0}^{J} \sum_{k=1}^{N_{j}} (f, \psi_{jk})_{L^2(S^2)} \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in S^2.
\]
From previous analysis in both Chapter 4 and Chapter 5, we can write
\[
f(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\mathbf{x}),
\]
where \( Y_{lm} \) are the spherical harmonic functions defined on the unit sphere \( S^2 \), and \( a_{lm} \) are coefficients. Using this representation, we can define an expression for \( (f, \psi_{jk})_{L^2(S^2)} \), which we will denote \( \beta_{jk} \), or
\[
\beta_{jk} = (f, \psi_{jk})_{L^2(S^2)}.
\]
We observe that
\[
\beta_{jk} = \int_{S^2} f(\mathbf{x}) \psi_{jk} d\mathbf{x}, \quad (5.4)
\]
\[
= \int_{S^2} \left( \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\mathbf{x}) \right) \left( \sqrt{w_{jk}} \sum_{l=0}^{\infty} b \left( \frac{l}{2^j} \right) \frac{2l+1}{4\pi} P_{l}(\mathbf{x} \cdot \mathbf{x}_{jk}) \right) d\mathbf{x}, \quad (5.5)
\]
\[
= \sqrt{w_{jk}} \sum_{l=0}^{\infty} b \left( \frac{l}{2^j} \right) \sum_{m=-l}^{l} a_{lm} Y_{lm}(\mathbf{x}_{jk}). \quad (5.6)
\]
Thus we have an expression for the needlet coefficients, \( \beta_{jk} \) in terms of the spherical harmonic coefficients, \( a_{jk} \), and spherical harmonic function evaluated at the center of the needlet, \( \mathbf{x}_{jk} \).

We note that using this notation, we can write the spherical needlet approximation to \( f \) in terms of the spherical needlet coefficients,
\[
V_{j}^{\text{need}}(f; \mathbf{x}) = \sum_{j=0}^{J} \sum_{k=1}^{N_{j}} (f, \psi_{jk})_{L^2(S^2)} \psi_{jk}(\mathbf{x}), \quad (5.7)
\]
\[
= \sum_{j=0}^{J} \sum_{k=1}^{N_{j}} \beta_{jk} \psi_{jk}(\mathbf{x}). \quad (5.8)
\]
Therefore, if we can compute the coefficients $\beta_{jk}$, we have a method to compute the needlet approximation to the function $f$. The NeedMat package provides a method to estimate the spherical harmonic coefficients from a sample function which utilizes the orthogonality of the spherical harmonic functions. In the event that the coefficients are unknown, they can be estimated as

$$\hat{a}_{lm} = \sum_{i=1}^{N} w_i f(x_i) Y_{lm}(x_i).$$

(5.9)

where $x_i, i = 1, \cdots, N$ are the sampling locations of the function $f$, and $w_i$ is the curvature weight associated with the surface area of $x_i$ (Fan, 2015).

We will consider two example functions to demonstrate the needlet approximation. In the first example, we compute the spherical needlet approximation to a deterministic function on the sphere, which is defined using the Wendland radial basis function. In this example, we must approximate the spherical harmonic coefficients using a small sample in order to compute the coefficients $\beta_{jk}$. In the second example, we compute the needlet approximation to an isotropic random field on the sphere which is generated using a KL expansion. Thus, for the second example, the coefficients $a_{lm}$ are known. We note the relationship used in this implementation between $j_{max}$ and $l_{max}$ is that $j_{max}$ is the largest integer such that $2j_{max} \leq l_{max}$.

5.2.1 Needlet Approximation of a Localized Deterministic Function

In this section, we seek to approximate a deterministic function on the sphere. We will define the function in terms of the Wendland radial basis function, following Fan (2015). This function provides an excellent example to test the needlet approximation, as the function itself is highly localized. To begin, we define the function as

$$\psi(x) = \psi_0 \left( \frac{\arccos(\langle x_0, x \rangle)}{\eta} \right),$$
where \( x_0 \) is the center of the function, \( \eta \) is a spatial scale parameter, and \( \psi_0(\cdot) \) is the Wendland radial basis function in \( C^4 \) given by,

\[
\psi_0(d) = \begin{cases} 
(1 - d)^6(35d^2 + 18d + 3)/3, & 0 \leq d \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

We can see from this definition that the function is non-zero only when \( 0 \leq \frac{\arccos(<x_0,x>)}{\eta} \leq 1 \), making the function highly localized with a radius determined by the parameter \( \eta \).

We choose \( x_0 = (\pi/2, \pi) \) and we let \( \eta = \pi/4 \), which means that \( \psi(\cdot) \) will be centered at \( x_0 \) with a compact support in a radius of \( \pi/4 \). The steps used to complete this example are outlined below.

- To begin, we sample the function at a set of points, \((\theta, \phi)\), on the sphere, which are determined by the HEALPix grid with \( N_{\text{side}} = 8 \) for a total of 786 points. These values are shown in Figure 5.2(a).
- Using the NeedMat package and the data shown in Figure 5.2(a), we approximate the spherical harmonic coefficients \( a_{lm} \) via (5.9) for \( l = 0, \cdots, l_{\text{max}} \). In this example, we use \( l_{\text{max}} = 16 \) and \( l_{\text{max}} = 32 \).
- We compute the needlet coefficients \( \beta_{jk} \) for \( j = 0, \cdots, J_{\text{max}} \) via (5.4), where \( J_{\text{max}} = 3 \) and \( J_{\text{max}} = 5 \) for \( l_{\text{max}} = 16 \) and \( l_{\text{max}} = 32 \) respectively.
- The needlet approximation is evaluated via (5.7) on a finer resolution HEALPix grid. The approximation is interpolated for plotting purposes, and is shown in Figure 5.2(b).
- The absolute error in the needlet approximation is computed point-by-point by evaluating the Wendland radial basis function at the finer resolution and taking the absolute difference. The error is shown in Figure 5.2(c).

We observe that the spherical needlets are adept at capturing the highly localized behavior of this deterministic function. When computed with \( l_{\text{max}} = 16 \) terms, the maximum
Figure 5.2: Illustrative results for the needlet approximation of the Wendland radial basis function. (a) Displays the values of the function at the sample locations. These values are used to compute the spherical needlet approximation (b). The absolute error in the needlet approximation to the actual values of the Wendland radial basis function is shown in (c).
error observed is $1.107498 \times 10^{-2}$. However, if we increase the number of terms used to compute the needlet approximation to 32, we decrease the maximum error in the approximation to $5.377561 \times 10^{-3}$. Thus by doubling the number of terms used to compute the spherical harmonic coefficients, $l_{\text{max}}$, we decrease the maximum error by approximately a factor of two.

### 5.2.2 Needlet Approximation of Isotropic Random Fields

In this example, we seek to compute the spherical needlet approximation to an isotropic random field, $T$.

We will work with the algebraically decaying angular power spectrum described by Lang & Schwab (2013). This sequence decays quickly, and although not very localized, it provides an opportunity to test the accuracy of the computational methods. We recall the KL expansion for isotropic random field $T$ is given by

$$T(\theta, \phi) = \sum_{l=0}^{l_{\text{max}}} \sqrt{A_l} X_{l0} L_{l0}(\theta) + \sqrt{2A_l} \left( \sum_{m=-l}^{-1} X_{lm}^1 \sin(m\phi) L_{lm}(\theta) + \sum_{m=1}^{l} X_{lm}^2 \cos(m\phi) L_{lm}(\theta) \right),$$

where

$$L_{lm}(\theta) = \sqrt{(2l+1)(l-m)!} \frac{P_{lm}(\cos \theta)}{4\pi (l+m)!}.$$

In this sequence, we are computing a $l_{\text{max}}$-term truncation of the KL expansion at locations specified by $(\theta, \phi)$. We let

$$A_l = (l + 1)^{-9}.$$

To connect the angular power spectrum with the spherical harmonic coefficients, we recall their relationship:

$$a_{l0} = X_{l0} \sqrt{A_l},$$

$$\Re a_{lm} = X_{lm} \sqrt{A_l}/2, \quad m > 0,$$

and

$$\Im a_{l-m} = X_{lm} \sqrt{A_l}/2, \quad m < 0.$$
Now we use the angular power spectrum to compute the isotropic random field, $T$, at locations on the HEALPix grid. We use $N_{\text{side}} = 4$, or 192 points. Using the relationship between the angular power spectrum and the spherical harmonic coefficients, we store the coefficients during the computation of $T$. Note that unlike the angular power spectrum which is deterministic, the spherical harmonic coefficients are random and unique to a particular realization of $T$.

The steps used to complete this example are outlined below.

- Compute a realization of an isotropic random field using the KL expansion with $l_{\text{max}} = 100$ on the data points generated by the HEALPix grid with $N_{\text{side}} = 8$. These data are plotted in Figure 5.3(a) This plot serves to show the reader the random field we seek to estimate.

- Using the NeedMat package and known harmonic coefficients $a_{lm}$, compute the corresponding needlet coefficients $\beta_{jk}$ for $j = 0, \cdots, 5$.

- The needlet approximation is computed via (5.7) using the coefficients $\beta_{jk}$ at the same points as the KL expansion. The approximation is plotted in Figure 5.3(b).

- The absolute error in the needlet approximation is computed point-by-point by taking the absolute difference between the KL expansion and the needlet approximation. We plot the error in Figure 5.3(c).

We observe that the needlet approximation is capturing the behavior of the isotropic random field pictured in Figure 5.3(a). The error plot shows that the maximum error is found on the top of the sphere pictured, where the KL expansion becomes negative. The maximum error computed for this approximation is 0.017476; however, we can see from the rest of the error plot that this maximum error is localized. To create a better approximation, we could increase the number of terms, $j_{\text{max}}$, used to compute the approximation. Unfortunately, this comes with heavy computational requirements.
Figure 5.3: Illustrative results for the isotropic random field. In (a) we have the KL expansion of the isotropic random field sampled at sparse locations on the sphere. This field is generated using the algebraically decaying angular power spectrum with $\alpha = 9$. In (b) we have the corresponding needlet approximation, and in (c) we plot their absolute difference.
This approach for computing the needlet approximation requires large computation for each \( j \) to compute \( \beta_{jk} \). In fact, for \( j_{\text{max}} \geq 7 \) we find that the CPU time required to complete computations and the memory consumed are restrictive of this approach. For the seismic application, preliminary investigations suggest that \( l_{\text{max}} \) should be on the order of thousands in order to achieve good results. This would require a substantial increase in the \( j_{\text{max}} \) value which is computationally prohibitive. Thus, we conclude that for the seismic application, the current implementation by Fan (2015) is not suitable.

Future work would require the development of a computationally efficient approach to compute the needlet approximation in a fast, and memory efficient manner. After the development of a new algorithm and the incorporation of the KL expansions detailed in Chapter 4, the application of spherical needlets to the seismic problem could be explored.
 CHAPTER 6
CONCLUSION

In this thesis, we investigate spatial variability present in seismic ground motions, methods for modeling variability and including it in seismic ground motion models, and its impact on estimated hazards and losses. Spatial variability in seismic ground motions has been well documented over the last century. Historically, the variability in structural damage following an earthquake provided evidence of spatial variability in seismic ground motions, and the installation of dense networks of seismic arrays in the 1980's enabled the study of spatial variability which continues today. Variability is attributed to coherent contributions from source, path, and site, and adding spatial variability to ground motion models can provide valuable hazard and loss information, as well as a more realistic picture of ground motion intensities. Our primary focus in this thesis is to capture the spatial variability of ground motions by computing a set of spatially correlated random fields, which capture the covariance properties of seismic ground motions.

We begin our investigation with relevant background on modeling seismic ground motions, and previous research conducted on spatial variability in earthquake ground motions. Throughout this project, we utilize the USGS ShakeMap model, which is adept at simulating median ground motions based on observed seismic data. This model is influential in its field, and is used in combination with the USGS software PAGER, which estimates fatalities and economic losses after seismic events. In Chapter 2, we provide the accepted method for adding spatial variability to seismic ground motions, such as those computed with ShakeMap. Spatial variability present in ground motions has been thought to be well-modeled by an isotropic, spatially correlated, random field, which is defined by a correlation function dependent on the period of the ground motions and the correlation length. We investigate several ground motion correlation functions as well as their possible dependen-
cies such as site and regional characteristics. We use the correlation function described by Jayaram & Baker (2009), as it is one of the most well-known functions; although, evidence is provided within this chapter for the regional dependency of ground motion correlation function. Further, additional factors related to spatial variability are discussed, such as spatial cross-correlation models, which present the next steps toward accurately modeling earthquake ground motions.

In Chapters 3-5, we consider three methods for computing a field of spatially correlated ground motion residuals. In Chapter 3, we discuss the method of successive conditional simulations (SCS) suggested by Park et al. (2007). This method provides an approximation to the conditional simulation method described by the same authors. Both methods provide means to condition the random field on station data, where the random field is expected to converge to zero. While the conditional simulation method is effective for computing spatially correlated random fields, computational limitations prevent this method from being employed by the ShakeMap software. The SCS method was suggested as a way to alleviate the computational requirements associated with the conditional simulation method. It is a multilevel approximation of the conditional simulation method, with various techniques applied to improve efficiency and memory consumption. Under the assumption of a radius of influence, this new, parallel implementation was shown to significantly decrease the memory usage and CPU time in comparison to the conditional simulation method. We test the impact of spatial variability on losses by adding several (1000) realizations of a spatially correlation random field to a ShakeMap estimate. Results indicate that spatial variability may greatly affect the loss estimates, particularly when scenario ShakeMaps are used or when station data is sparse.

The first approach has the capability to efficiently incorporate spatial variability into the ShakeMap software. However, it is highly dependent on size of the input grids, meaning that the algorithm would have to be rerun for every new ShakeMap. Further, the stochastic dimension is induced by the grid size. In Chapter 4, we explore an additional method to
compute the spatially correlated random fields in a global sense, namely the method of Karhunen Loève (KL) expansions, as a way to reduce the dimensionality associated with the computation. The KL expansion represents an isotropic Gaussian random field as an infinite sum of surface spherical harmonic functions with coefficients determined by the angular power spectrum. We find that the truncation error associated with a \( \kappa \)-term truncation of the sum, is related to the rate of decay of the angular power spectrum. For the seismic application of the KL expansion, the slow rate of decay of the angular power spectrum associated with the Jayaram & Baker (2009) correlation function results in poor convergence rates (and therefore dimension reduction). However, the introduction of an algebraically decaying factor into the angular power spectrum results in an isotropic, Gaussian random field with a significantly reduced stochastic dimension. Additionally, we compute the losses associated with a single realization of the random field using the modified angular power spectrum computed for the Northridge ShakeMap, and find that the estimated losses are in agreement with the SCS method.

The final approach used here is a spherical needlet approximation to an isotropic random field. Spherical needlets are highly localized polynomials on the sphere. The fully discrete needlet approximation described by Wang et al. (2015), provides a method to approximate complex functions on the sphere with spherical needlets. In Chapter 5, we explore the properties of spherical needlets and provide two relevant examples. In the first, we approximate a deterministic function on the sphere which is defined with the Wendland radial basis function Fan (2015). In the second, we use a needlet approximation for an isotropic random field generated by the KL expansion. We use the algebraically decaying angular power spectrum to compute the random field and find the needlet approximation is quick and accurate. Future work could involve the development of a new approach to compute the needlet approximation for a seismic application which could compute a localized isotropic random field in a fast, memory efficient way.
REFERENCES CITED


Beresnev, I.A., Wen, K., & Yeh, Y. 1994. Source, path and site effects on dominant frequency and spatial variation of strong ground motion recorded by SMART1 and SMART2 arrays in Taiwan. *Earthquake engineering and structural dynamics*, 23(6), 583–597.


Campbell, KW, & Bozorgnia, Y. 2008. NGA ground motion model for the geometric mean horizontal component of PGA, PGV, PGD and 5% damped linear elastic response spectra for periods ranging from 0.01 to 10 s. *Earthquake Spectra, 24*(1), 139–171.


149


