GLOBAL SENSITIVITY METRICS
FROM ACTIVE SUBSPACES
WITH APPLICATIONS

by
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Predictions from science and engineering models depend on input parameters. Global sensitivity metrics quantify the importance of input parameters, which can lead to model insight and reduced computational cost. Active subspaces are an emerging set of tools for identifying important directions in a model’s input parameter space; these directions can be exploited to reduce the model’s dimension enabling otherwise infeasible parameter studies. We develop global sensitivity metrics called activity scores from the estimated active subspace and analytically compare the active subspace-based metrics to established sensitivity metrics. These commonly used metrics include Sobol’ indices derived from a variance-based decomposition and derivative-based metrics. Additionally, we outline practical computational methods to estimate the activity scores. We then consider three numerical examples with algebraic scalar valued functions from engineering and biological models. In each case, the models admit reduced dimensional active subspaces. For each of the models, a variety of sensitivity metrics are compared to the activity scores.
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LIST OF SYMBOLS

Dimension of input space............................................................................................ $m \in \mathbb{N}$
Dimension of the active subspace ................................................................................ $n \in \mathbb{N}$
Number of Monte Carlo samples ............................................................................. $M \in \mathbb{N}$
Input space ................................................................................................................ $\Omega \subset \mathbb{R}^m$
Shifted and scaled inputs.......................................................................................... $x \in [-1, 1]^m$
Differentiable deterministic model.......................................................................... $f \in L^2(\Omega)$
Partial derivative of $f$ ......................................................................................... $\frac{\partial f}{\partial x_i} \in L^2(\Omega)$
Model gradient function ......................................................................................... $\nabla f \in \mathbb{R}^m$
Probability density function .................................................................................. $\rho(x) \in \mathbb{R}^m$
Outer product matrix ............................................................................................. $C \in \mathbb{R}^{m \times m}$
Orthogonal matrix of eigenvectors of $C$ ............................................................... $W \in \mathbb{R}^{m \times m}$
Diagonal matrix of eigenvalues of $C$ .................................................................... $\Lambda \in \mathbb{R}^{m \times m}$
Monte carlo estimate of $C$ .................................................................................. $\hat{C} \in \mathbb{R}^{m \times m}$
Orthogonal matrix of eigenvectors of $\hat{C}$ ............................................................ $\hat{W} \in \mathbb{R}^{m \times m}$
Diagonal matrix of eigenvalues of $\hat{C}$ ................................................................. $\hat{\Lambda} \in \mathbb{R}^{m \times m}$
Model mean .......................................................................................................... $\mathbb{E}[f]$  
Model variance...................................................................................................... $\text{Var}[f]$  
Sobol’ total sensitivity index .................................................................................. $\tau_i$
Derivative-based metric ......................................................................................... $\nu_i$
Standardized regression coefficient ...................................................................... $\hat{\beta}_i$
Modified Morris metric ............................................................................................................... $\mu_i^*$

First eigenvector component .................................................................................................. $w_i$

Activity score ......................................................................................................................... $\alpha_i$

Standard error ...................................................................................................................... $se$

Bootstrap standard error ...................................................................................................... $\hat{se}$

Relative error ....................................................................................................................... $e$

Piston cycle time (seconds) ................................................................................................... $t$

OTL Circuit mid-point voltage (volts) .................................................................................. $V$

Ebola basic reproductive ratio ............................................................................................. $R_0$
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For those that shall follow after.
CHAPTER 1
INTRODUCTION

This thesis documents the majority of the research we have worked on for the past year and a half as it relates to active subspaces. While most of this information is currently in revision for publication, this document includes elementary effects (see section 2.4), which are not covered in [1]. For more information about the research related to Ebola, much of which was excluded from this document, see [2].

1.1 Setup & notation

We consider a differentiable and square-integrable function \( f = f(x) \) defined on an \( m \)-dimensional hypercube \( x = [x_1, \ldots, x_m]^T \in [-1,1]^m \). Define the function \( \rho(x) = 2^{-m} \) for \( x \in [-1,1]^m \) and zero elsewhere. Then we can write \( f \)'s mean and variance as

\[
E[f] = \int f \rho \, dx, \quad \text{Var}[f] = \int (f - E[f])^2 \rho \, dx. \tag{1.1}
\]

We emphasize that \( f \) is a deterministic function; the mean and variance do not represent any probabilistic information. Denote the gradient of \( f \) by \( \nabla f(x) \in \mathbb{R}^m \) with partial derivatives \( \frac{\partial f}{\partial x_i} \), and assume the partial derivatives are square-integrable. All vector norms are the standard Euclidean norm and matrix norms are the matrix 2-norm (i.e., the spectral norm). Throughout the paper, \( M \) refers to the number of Monte Carlo samples, and \( n < m \) denotes the dimension of the active subspace when appropriate.
Suppose we have a deterministic model, such that given any set of input values in an input parameter space we can determine a quantity of interest. We write our deterministic model as \( f(x) : \mathbb{R}^m \rightarrow \mathbb{R} \), where the input factors are the vector \( x \) and the quantity of interest they define is \( f(x) \). Sensitivity analysis provides a methodology for identifying a rank of importance among the model inputs. From another point of view, sensitivity analysis apportions uncertainty in the model’s output to its inputs. When exact parameter values of a model remain uncertain, sensitivity analysis is useful to understand the behavior of the model. For this reason we say that it is related uncertainty analysis or quantification.

In order to measure the sensitivities of an input parameter we must first define what mathematical tool we will use to take the measurements with. In sensitivity analysis, the tools for this task are referred to as sensitivity metrics. In most literature there is a distinction between local and global sensitivity metrics. To help understand this, consider the \( i \)th partial derivative of \( f \) evaluated at a point \( x^* \) in the domain of the function. The quantity \( \frac{\partial f}{\partial x_i}(x^*) \) represents how the model changes with respect to the \( i \)th input parameter at the point \( x^* \). Because of this, one might think that it could be utilized as a sensitivity metric. This intuition is partially correct but we must be careful to understand what we are measuring. The partial derivative is evaluated at a single point. Hence, this value does not reflect the behavior of the function over the entire input parameter domain. For this reason we would say it is a local sensitivity metric, i.e., it illuminates how the function behaves locally at the point \( x^* \). By contrast, a global sensitivity metric would measure how sensitive the model is to changes in an input parameter over its entire domain of a definition.

We review four global sensitivity metrics used in practice: standardized regression coefficients, elementary effects, variance-based metrics, and derivative-based metrics. In each
case, we (i) define the metric, (ii) discuss a method for high accuracy computation with numerical quadrature, (iii) detail a Monte Carlo-based method for low accuracy estimation, and (iv) mention practical error estimates for the Monte Carlo-based estimates.

2.1 Variance based sensitivity metrics

In this section, we describe the total sensitivity indices (a.k.a., the total Sobol’ indices). The total sensitivity indices were introduced by Sobol’ [3]; several papers describe these metrics, propose extensions [4], and develop efficient computational procedures [5, 6, 7, 8]. The total sensitivity indices are derived from a particular decomposition of the square-integrable $f(x)$, often called the function ANOVA decomposition or the variance-based decomposition; we use the concise notation from Liu and Owen [5]. Let $u$ be a subset of the integers $\{1, \ldots, m\}$ with cardinality $|u|$. Define $x^u \in [-1, 1]^{|u|}$, and let $\rho^u(x^u) = 2^{-|u|}$ for $x^u \in [-1, 1]^{|u|}$ and zero elsewhere. The subset $-u$ is the complement of $u$ in $\{1, \ldots, m\}$. The ANOVA decomposition can be written

$$f(x) = \sum_{u \subseteq \{1, \ldots, m\}} f_u(x),$$

(2.1)

where $f_u$ depends on $x$ only through $x^u$. The function $f_u$ is defined as

$$f_u(x) = \int f(x) \rho^{-u} dx^{-u} - \sum_{v \subset u} f_v(x),$$

(2.2)

and $f_{\emptyset} = \mathbb{E}[f]$. The functions $f_u$ are mutually orthogonal, and the variance of $f$ can be decomposed as

$$\text{Var}[f] = \sum_{u \subseteq \{1, \ldots, m\}} \text{Var}[f_u].$$

(2.3)

Define the set $S_i$ to be the set of subsets of $\{1, \ldots, m\}$ containing the index $i$. The total sensitivity index $\tau_i$ for the $i$th variable $x_i$ can be written as sums of subsets of the variances $\text{Var}[f_u]$ divided by the total variance,

$$\tau_i = \frac{\sum_{u \subseteq S_i} \text{Var}[f_u]}{\text{Var}[f]}.$$

(2.4)
Other sensitivity metrics can be defined from the variances $\text{Var}[f_u]$; we focus exclusively on $\tau_i$ from (2.4) to compare with the metrics derived from active subspaces; see Section 3.1.

### 2.1.1 Computation

When the dimension of $\mathbf{x}$ is sufficiently low, $f(\mathbf{x})$ is sufficiently smooth, and $f(\mathbf{x})$ can be computed quickly, one can accurately estimate $\tau_i$ using a high order numerical quadrature rule. This approach is equivalent to using the Fourier coefficients as in [7], which we use to compute reference values for the numerical examples in Chapter 4.

If $f(\mathbf{x})$ does not satisfy these assumptions, one can use a less accurate Monte Carlo method as in [9, Section 4.6]. Next we review the Monte Carlo method and its central limit theorem-based standard error. First draw $2M$ samples $\mathbf{x}_i^T$ independently according to the uniform density $\rho(\mathbf{x})$ on $[-1,1]^m$. We arrange these samples into two $M \times m$ matrices, $\mathbf{A}$ and $\mathbf{B}$, defined as

$$
\mathbf{A} = \begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_M^T \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{x}_{M+1}^T \\ \vdots \\ \mathbf{x}_{2M}^T \end{bmatrix}.
$$

(2.5)

Define the $M \times m$ matrix $\mathbf{C}_i$ to be the matrix $\mathbf{B}$ with the $i$th column replaced by the $i$th column of $\mathbf{A}$. Define the $M$-vector $\mathbf{f}_A$ as

$$
\mathbf{f}_A = \begin{bmatrix} f(\mathbf{x}_1) \\ \vdots \\ f(\mathbf{x}_M) \end{bmatrix}.
$$

(2.6)

In words, the vector $\mathbf{f}_A$ contains evaluations of the function $f$ at each row of the matrix $\mathbf{A}$. Similarly, define the $M$-vectors $\mathbf{f}_B$ and $\mathbf{f}_C_i$ whose elements contain evaluations of $f$ at the rows of their respective subscripted matrices. These calculations use $(m+2)M$ evaluations of $f$. The total sensitivity indices may be estimated as

$$
\tau_i \approx \hat{\tau}_i = 1 - 2 \frac{\mathbf{f}_A^T \mathbf{f}_C_i - \hat{f}_0^2}{\mathbf{f}_A^T \mathbf{f}_A + \mathbf{f}_B^T \mathbf{f}_B - \hat{f}_0^2},
$$

(2.7)

where the sample mean $\hat{f}_0$ is
\[
\hat{f}_0 = \frac{(f_A + f_B)^T e}{M},
\] (2.8)

and \(e\) is an \(M\)-vector of ones.

### 2.1.2 Error estimates

Monte Carlo estimates come equipped with an asymptotically valid estimate of the error derived from the central limit theorem; in statistics, these error estimates are called the standard error. The standard error \(se_{\tau_i}\) for the Monte Carlo estimate \(\hat{\tau}_i\) is

\[
se_{\tau_i} = \left[ \frac{\|f_{A,C_i}\|^2 - (f_{A}^T f_{C_i})^2}{M} \right]^{1/2},
\] (2.9)

where \(f_{A,C_i}\) is the element-wise product of \(f_A\) and \(f_{C_i}\). In Chapter 4, we study the standard error of the estimator by investigating the values of \(se_{\tau_i}\) for different values of \(M\); see Figure 4.3, Figure 4.6, and Figure 4.9.

### 2.2 Derivative-based sensitivity metrics

A natural notion of sensitivity is how a model output responds to changes in its inputs, i.e., a derivative. However, derivatives only provide information at a single parameter value. Morris developed global sensitivity metrics based on sampling and averaging coarse finite difference approximations to derivatives [10]. Kucherenko, et al [11] extended this idea to continuous derivatives and integration, where the averages are estimated with Monte Carlo methods. We focus on one particular derivative-based sensitivity metric studied by Sobol’ and Kucherenko [12]:

\[
\nu_i = \int \left( \frac{\partial f}{\partial x_i} \right)^2 \rho dx, \quad i = 1, \ldots, m.
\] (2.10)

This metric averages local sensitivity information to make it a global metric. We discuss this metric’s connection to active subspaces in Section 3.1.
2.2.1 Computation

If $f$’s derivatives are sufficiently smooth, and if the dimension of $x$ is low, then one can compute accurate estimates of $\nu_i$ using a high order numerical quadrature rule; we use the quadrature approach for reference values in the numerical examples in Chapter 4. The Monte Carlo estimate of $\nu_i$ using $M$ samples is

$$\nu_i \approx \hat{\nu}_i = \frac{1}{M} \sum_{j=1}^{M} \left( \frac{\partial f}{\partial x_i}(x_j) \right)^2,$$

where the $x_j$’s are drawn independently according to $\rho(x)$.

2.2.2 Error estimates

The standard error $se_{\nu_i}$ is

$$se_{\nu_i} = \left[ \frac{1}{M-1} \sum_{j=1}^{M} \left( \left( \frac{\partial f}{\partial x_i}(x_j) \right)^2 - \hat{\nu}_i \right)^2 \right]^{1/2}.$$

In Chapter 4, we study the standard error of the estimator by investigating $se_{\nu_i}$ for different values of $M$; see Figure 4.3, Figure 4.6, and Figure 4.9.

2.3 Linear model coefficients

A simple and easily computable sensitivity metric can be derived from the coefficients of a least-squares fit linear approximation to $f(x)$. Such a metric is valid when $f$ is smooth and monotonic along each component of $x$. Saletelli, et al [9, Section 1.2.5] discuss a similar metric in connection with a linear regression model.

2.3.1 Computation

Since $f(x)$ is deterministic, the statistical interpretation of the regression coefficients is not valid. Nevertheless, we can devise a Monte Carlo-based least-squares procedure to estimate the coefficients of a linear approximation as follows. Draw $x_j$ independently according to $\rho(x)$ for $j = 1, \ldots, M$, and compute $f_j = f(x_j)$. Next, use least-squares to fit the intercept $\hat{b}_0$ and coefficients $\hat{b} = [\hat{b}_1, \ldots, \hat{b}_m]^T$ of a linear approximation,
\[ f_j \approx \hat{b}_0 + \hat{b}^T x_j, \quad j = 1, \ldots, M. \] (2.13)

When the coefficients \( \hat{b}_i \) are properly scaled, they become useful sensitivity metrics. Define \( \hat{\beta}_i \) as

\[ \hat{\beta}_i = \frac{\hat{b}_i}{\sqrt{3} \hat{\sigma}_f}, \] (2.14)

where \( \hat{\sigma}_f \) is the standard deviation of the samples \( \{f_j\} \). The \( 1/\sqrt{3} \) comes from scaling by the variance of the uniform random variable \( x_i \) with support \([-1, 1]\). If \( f(x) \) is monotonic and sufficiently smooth, then \( \hat{\beta}_i \) gives a rough indication of how \( f \) changes in response to a change in \( x_i \) over the space of inputs. In contrast to the total sensitivity index \( \tau_i \) from (2.7) and the derivative-based metric \( \nu_i \) from (2.10), the \( \hat{\beta}_i \)'s are signed so they indicate if \( f \) will increase or decrease given a positive change in \( x_i \).

We can compute reference values for \( \hat{\beta}_i \) by noting the relationship between the \( m \)-dimensional Legendre series truncated to degree 1 (i.e., only a constant and \( m \) linear terms) and the least-squares linear approximation; the Legendre series is often referred to as the generalized polynomial chaos associated with the uniform density [7, 13]. In particular, the degree-1 Legendre approximation is the best linear approximation in the \( L_2 \) norm.

Its coefficients admit a simple integral representation by virtue of the Legendre polynomials’ orthogonality. Thus, we can estimate the coefficients with high order numerical quadrature [14] and scale them to get the coefficients of a linear monomial approximation as in (2.13).

Let \( \eta_i \in \mathbb{N}_1^m \) be a multi-index such that it has the value 1 in the \( i \)th position and is 0 elsewhere. Then, a Legendre polynomial that is linear in \( x_i \) and constant in all other terms may be written \( L_{\eta_i}(x) \), and the standardized regression coefficients are

\[ \hat{\beta}_i = \frac{\hat{f}_{\eta_i}}{\text{Var}[f]}, \] (2.15)

where
\[ \hat{f}_{\eta_i} = \int_{\Omega} f(x)L_{\eta_i}(x)\rho dx. \] (2.16)

In Chapter 4, we use the integral representation in (2.15) to compute reference values of the standardized regression coefficients.

2.3.2 Error estimates

When reference values are too expensive to compute, we can estimate the standard error in \( \hat{\beta}_i \) with a bootstrap standard error, which we denote \( \hat{se}_{\beta_i} \). See [15, Section 9.5] for a detailed description of using the bootstrap for linear regression coefficients. Note that randomness in the estimates is a result of the Monte Carlo method; there is no randomness in the evaluation of \( f(x) \). In Chapter 4, we study the bootstrap standard error by investigating the values of \( \hat{se}_{\beta_i} \) for different values of \( M \); see Figure 4.3, Figure 4.6, and Figure 4.9.

2.4 Elementary effects

The elementary effects method (a.k.a., Morris Method) was first introduced by Morris who developed an effective screening method for sensitivity metrics [10]. Campolongo et al. proposed a revised version of the elementary effects method, which improved the definition of the metric and its sampling strategy [16]. An excellent summary of the method and metrics can be found in [9, Chapter 3]. The metrics are based on the analysis of local variation around a base point in the input parameter domain. The method is well suited when the number of input factors is simply too large to allow the application of more computationally expensive variance-based techniques. First, we discretize the input space \( \Omega \) into \( p \geq 2 \) levels. Define \( x_i' = [x_1, x_2, \ldots, x_{i-1}, x_i + \Delta, \ldots, x_m]^T \in \Omega \), for \( i = 1, \ldots, m \). The point \( x_i' \) can be thought of as a perturbation of the point \( x \), by a distance \( \Delta \) in the \( i \)th parameter direction, where \( \Delta \) is a value in the set \( \left\{ \frac{1}{p-1}, \ldots, 1 - \frac{1}{p-1} \right\} \). The sensitivity metrics are defined in terms of elementary effects, for a given value of \( x \in \Omega \), the elementary effect of the \( i \)th parameter is defined as
To compute the metric we draw $M$ samples of $x_j$ independently according to $\rho(x)$, and for each $j = 1\ldots M$, compute $EE_i(x_j)$. Then, the revised Morris importance metric $\mu_i^*$ in [16], is defined as the average of the absolute values of all samples, i.e.,

$$\mu_i^* = \sum_{j=1}^{M} |EE_i(x_j)|$$

(2.18)

In [12] it was shown that $\mu_i^*$ is an approximation of the average absolute value of the $i$th partial derivative of $f$, which we refer to as $\mu_i$, i.e.,

$$\mu_i^* \approx \mu_i = \int_{\Omega} \left| \frac{\partial f}{\partial x_i} \right| \rho \, dx, \quad i = 1, \ldots, m.$$  

(2.19)

Because of this relation we use the value $\mu_i$ a global sensitivity metric rather than $\mu_i^*$. Other sensitivity metrics can be defined from the elementary effects in (2.17) see [9, Chapter 3].

### 2.4.1 Computation

If $f$’s derivatives are sufficiently smooth, and if the dimension of $x$ is low, then one can compute accurate estimates of $\mu_i$ using a high order numerical quadrature rule; we use the quadrature approach for reference values in the numerical examples in Chapter 4.

The Monte Carlo estimate of $\mu_i$ using $M$ samples is

$$\mu_i \approx \hat{\mu}_i = \frac{1}{M} \sum_{j=1}^{M} \left| \frac{\partial f}{\partial x_i}(x_j) \right|,$$

(2.20)

where the $x_j$’s are drawn independently according to $\rho(x)$.

### 2.4.2 Error estimates

The standard error $se_{\mu_i}$ is

$$se_{\mu_i} = \left[ \frac{1}{M-1} \sum_{j=1}^{M} \left( \left| \frac{\partial f}{\partial x_i}(x_j) \right| - \hat{\mu}_i \right)^2 \right]^{1/2}.$$  

(2.21)
In Chapter 4, we study the standard error of the estimator by investigating \( se_{\mu_i} \), for different values of \( M \); see Figure 4.3, Figure 4.6, and Figure 4.9. The relative rankings produced by this metric are the same as those produced by the derivative based metric outlined in Section 2.2. Because of the similarity between \( \mu_i \) and \( \nu_i \) we have omitted the tabulated reference values of this metric in Table 4.2, Table 4.4, and Table 4.6.
CHAPTER 3
ACTIVE SUBSPACES

Active subspaces [17] are defined by the eigenvectors of the following $m \times m$ symmetric, positive semidefinite matrix,

$$ C = \int \nabla f \nabla f^T \rho \, dx = W \Lambda W^T, \quad (3.1) $$

where $W = [w_1, \ldots, w_m]$ is the orthogonal matrix of eigenvectors, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$ is the diagonal matrix of eigenvalues in decreasing order. The eigenpairs satisfy

$$ \lambda_i = \int (\nabla f w_i)^2 \rho \, dx. \quad (3.2) $$

If $\lambda_i = 0$, then $f$ is constant along the direction $w_i$, and such structure can be exploited to reduce the model’s dimension. An example is the function $f(x_1, x_2) = \exp(7x_1 + 3x_2)$, whose inactive subspace corresponds to the direction $[-3, 7]$. Suppose $\lambda_n > \lambda_{n+1}$ for some $n$ less than $m$. Then we partition

$$ \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \quad (3.3) $$

where $\Lambda_1$ is a diagonal matrix of the first $n$ eigenvalues, and $W_1$ is the $m \times n$ matrix containing the first $n$ eigenvectors. If $\lambda_{n+1}, \ldots, \lambda_m$ are sufficiently small, then a reasonable low-dimensional approximation of $f$ is

$$ f(x) \approx g(W_1^T x), \quad (3.4) $$

where $g : \mathbb{R}^n \to \mathbb{R}$. A particular form for $g$ based on a conditional average is proposed in [18], and its mean-squared error is bounded in terms of the eigenvalues $\lambda_{n+1}, \ldots, \lambda_m$.

To identify an active subspace, we must estimate the eigenpairs $\Lambda$, $W$ and look for a gap in the eigenvalues. If the dimension $m$ is sufficiently small, then one might use high order numerical integration rules to estimate $C$ in (3.1). A Monte Carlo method for estimating these eigenpairs is proposed and analyzed in [19]. For $i = 1, \ldots, M$, draw $x_i$ uniformly at
random from the hypercube $[-1, 1]^m$. Compute $\nabla f_i = \nabla f(x_i)$. Then estimate

$$C \approx \hat{C} = \frac{1}{M} \sum_{i=1}^{M} \nabla f_i \nabla f_i^T = \hat{W} \hat{\Lambda} \hat{W}^T. \quad (3.5)$$

How large $M$ must be so that (i) the estimated eigenvalues $\hat{\Lambda}$ are close to the true eigenvalues $\Lambda$ and (ii) the subspace distance,

$$\text{dist}(\text{ran}(W_1), \text{ran}(\hat{W}_1)) = \|W_1 W_1^T - \hat{W}_1 \hat{W}_1^T\|_2, \quad (3.6)$$

where $\hat{W}_1$ contains the first $n$ columns of $\hat{W}$, is well-behaved, has been studied in [19].

3.1 Sensitivity metrics from active subspaces

We propose two possible metrics derived from the eigenvectors $W$ and eigenvalues $\Lambda$ from (3.1). The first set of metrics uses the components of the first eigenvector (i.e., the eigenvector associated with the largest eigenvalue). The second set of metrics uses linear combinations of the squared eigenvector components weighted by the eigenvalues, which we call activity scores.

3.2 The first eigenvector

In [20, Section 4.4], we used the first eigenvector from the matrix $\hat{C}$ in (3.5) to identify the most important parameters in a single-diode model of a photovoltaic solar cell. For this particular model, the magnitudes of the eigenvector components gave the same ranking as the total sensitivity indices (see Section 4.4). However, these two metrics indicate different characteristics of the function. The total sensitivity indices measure the proportion of the variance attributed to each parameter while the eigenvector identifies an important direction in the parameter space. Like the linear model coefficients in Section 2.3, the eigenvector components are not all positive, and the relative difference in signs (since eigenvectors are only unique up to a sign) provides insight into the relationship between inputs and outputs. For example, if two eigenvector components are equal in magnitude but differ in sign, then we expect $f$ to increase when one parameter is perturbed but decrease when the other
is perturbed. This is similar to the gradient-based metrics described by Morris [10] and Kucherenko [11]. However, the trouble of averaging a gradient that might change signs does not effect the eigenvector. In the numerical examples in Chapter 4, we compare the first eigenvector’s components to other sensitivity metrics, but we do not provide a theoretical comparison. We use the bootstrap to estimate a standard error in the eigenvector components, though one must be careful to keep the signs consistent when computing the bootstrap replicates. Section 7.2 of [15] shows an example of using the bootstrap to estimate standard errors in eigenvector components. When the context is clear, we denote the first eigenvector by \( w = [w_1, \ldots, w_m]^T \) and the bootstrap standard error by \( \hat{se}_w \).

### 3.3 Activity scores

We develop a more comprehensive and justifiable sensitivity metric from the eigenpairs in (3.1). We propose the following metric for global sensitivity analysis. Define \( \alpha \in \mathbb{R}^m \) as

\[
\alpha = \alpha(n) = \sum_{j=1}^{n} \lambda_j w_j^2,
\]

(3.7)

where the exponent on the vector \( w_j \) means squaring each component of \( w_j \). We call \( \alpha(n) \) the *activity scores*, and we use these numbers to rank the importance of a model’s inputs.

The active subspace’s construction gives insight into the interpretation of the activity scores. The eigenvector \( w \) identifies the most important direction in the parameter space in the following sense: perturbing \( x \) along \( w \) changes \( f \) more, on average, than perturbing \( x \) orthogonal to \( w \), see (3.2). The entries of \( w \) measure the relative change in each component of \( x \) along this most important direction, so they impart significance to each component of \( x \). The second most important direction is the eigenvector \( w_2 \), and the relative importance of \( w_2 \) is measured by the difference between the eigenvalues \( \lambda_1 \) and \( \lambda_2 \). For example, if \( \lambda_1 \gg \lambda_2 \), then \( f(x) \) has dominant one-dimensional active subspace, and the importance of \( x \)’s components is captured in \( w \)’s components. Therefore, to construct the global sensitivity metric, it is reasonable to scale each eigenvectors by its corresponding eigenvalue.
Squaring each eigenvector component in (3.7) removes information provided by the signs. But the resulting metric is much easier to compare to existing sensitivity metrics.

### 3.4 Comparison to existing metrics

We compare the activity scores to the derivative-based metrics and the total sensitivity indices reviewed in Section 2.1.

**Theorem 3.1** The activity scores are bounded above by the derivative-based metrics,

\[ \alpha_i(n) \leq \nu_i. \]  

(3.8)

The inequality becomes an equality when \( n = m \).

**Proof** Note that the derivative-based metrics are the diagonal elements of the matrix \( C \) in (3.1). Then

\[ \nu = \text{diag}(C) = \text{diag}(W\Lambda W^T). \]

(3.9)

So

\[ \nu_i = \sum_{j=1}^{m} \lambda_j W_{ij}^2 = \alpha_i(m), \]

(3.10)

where \( W_{ij} \) is the \((i, j)\) element of \( W \), and \( \alpha_i(m) \) is the \( i \)th component of \( \alpha(m) \). This proves the equality statement. To see the inequality, note that \( W_{ij}^2 \geq 0 \) and \( \lambda_j \geq 0 \), so

\[ \alpha_i(n) = \sum_{j=1}^{n} \lambda_j W_{ij}^2 \leq \sum_{j=1}^{m} \lambda_j W_{ij}^2 = \nu_i, \]

(3.11)

as required. \( \square \)

Theorem 3.1 shows that the activity score can be interpreted as a truncation to the derivative-based metric \( \nu_i \) from Section 2.2. In many practical situations, the activity scores and the derivative-based metrics give comparable rankings. However, it is possible to construct cases where the rankings differ. Consider the quadratic function \( f(x_1, x_2) = (x_1^2 + x_2^2)/2 \). For this case, the matrix \( C \) from (3.1) is
\[ C = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}. \]  

(3.12)

The derivative-based metrics are the diagonal elements of \( C \)—both 1/3 implying equal importance. However, the activity scores with \( n = 1 \) are \([1/3, 0]\), which implies that the second variable is not important at all. This case illustrates that the activity scores are appropriate when there is a gap between the eigenvalues of \( C \) (i.e., \( f \) admits an active subspace), and \( n \) is chosen according to the gap. In the case of (3.12), there is no gap between the two eigenvalues and therefore, \( f \) doesn’t admit an active subspace.

Sobol’ and Kucherenko connect the total sensitivity indices to the derivative-based metrics in [12]; we use [12, Theorem 2] to relate the total sensitivity indices to the activity scores.

**Theorem 3.2** The total sensitivity indices \( \tau \) are bounded by

\[ \tau_i \leq \frac{1}{4\pi^2 V} \left( \alpha_i(n) + \lambda_{n+1} \right), \]  

(3.13)

where \( V = \text{Var}[f], \lambda_{n+1} \) is the \( n+1 \)st eigenvalue in (3.1).

**Proof** Theorem 2 from [12] shows

\[ \tau_i \leq \frac{1}{4\pi^2 V} \nu_i. \]  

(3.14)

Note that an additional factor of 1/4 appears in this case due to scaling the domain and the partial derivatives to the hypercube \([-1, 1]^m\) with a weight function \((1/2)^m\). Using Theorem 3.1,

\[ \nu_i = \alpha_i(n) + \sum_{j=n+1}^m \lambda_j W_{ij}^2 \leq \alpha_i(n) + \lambda_{n+1} \sum_{j=n+1}^m W_{ij}^2 \leq \alpha_i(n) + \lambda_{n+1}. \]  

(3.15)

The last inequality follows since the rows of \( W \) have norm 1. Combining (3.14) with (3.15) completes the proof. ■

Theorem 3.2 tells us that if \( \lambda_{n+1} \) is small—which often indicates an \( n \)-dimensional active subspace—and a component of \( \alpha(n) \) is small, then the corresponding component of the total
sensitivity index will also be small. However, Sobol’ and Kucherenko show an example of
a function in [12, Section 7] where the total sensitivity indices and the derivative-based
metrics identify different sets of variables as important. The situation is similar for the
activity scores. This should not be surprising since different sensitivity metrics measure
different characteristics of the function. The derivative-based metrics measure the average
response to small perturbations in the inputs; the activity scores are comparable when the
function admits an active subspace. In contrast, the total sensitivity indices measure the
variance attributable to each parameter. In many nicely behaved functions derived from
practical engineering models (such as the examples in Chapter 4), all metrics are consistent.
However, it is possible to construct functions where the metrics induce different rankings.
The appropriate choice of metric depends on the application.

If the dimension of $x$ is sufficiently small and the derivatives of $f$ are sufficiently smooth,
then one may estimate the integrals defining the matrix $C$ in (3.1) using a high order numer-
ical quadrature rule. Then the eigenvalues are computed from the numerically integrated
matrix; we use this approach to compute reference values in Chapter 4.

When these conditions are not satisfied, which is often the case in complex simulation
models, one may get coarser estimates of the activity scores $\alpha$ from the eigenpairs $\hat{\Lambda}$, $\hat{W}$ of
the Monte Carlo estimate $\hat{C}$ in (3.5). Previous work studied the approximation properties of
the Monte Carlo-based estimates of the eigenvalues and the subspaces [19]. Unfortunately,
those results do not directly translate to a priori error measures on the estimated activity
scores. We are currently working to find a lower bound on the number $M$ of samples in
(3.5) such that the activity scores are accurate to within a user-specified tolerance—similar
to [19, Corollary 3.3], which bounds the number of samples needed for accurate eigenvalue
estimation. We conjecture that this number may be smaller than the number of samples
needed for accurate Monte Carlo estimates of the derivative-based metrics $\nu$ from (2.10),
but such analysis is beyond the scope of the thesis.
Since the Monte Carlo-based estimates of the eigenpairs cannot be interpreted as sums of independent random variables, we cannot compute central limit theorem-based standard errors. This limitation extends to the activity scores derived from the eigenpairs. Therefore, we turn to bootstrapping to estimate the standard error in the Monte Carlo-based estimates of the activity scores. The bootstrap algorithm is described in detail by Efron and Tibshirani [15, Algorithm 6.1]. We denote the bootstrap standard error of the activity score $\alpha_i$ by $\hat{se}_{\alpha_i}$. In Chapter 4, we study the bootstrap standard error’s behavior as the number of samples $M$ increases in the Monte Carlo-based estimates.
We perform numerical experiments on two models from Bingham’s *Virtual Library of Simulation Experiments* [21]: (i) an algebraic model of a cylindrical piston (Section 4.1) and (ii) an algebraic model of a transformerless push-pull circuit (Section 4.2). We also perform numerical experiments on the reproductive ratio for a modified SEIR model. The purpose of these experiments is to compare the sensitivity metrics constructed from active subspaces to the conventional sensitivity metrics. In each model we evaluate analytic gradients for computations. The piston and otl circuit model admit a dominant one-dimensional active subspace using the Monte Carlo method represented by (3.5), where the Ebola model admits a two-dimensional active subspace. To analyze the error in the Monte Carlo-based estimates of all metrics, we compute reference values using a tensor product Gauss-Legendre quadrature rule with seven points in each dimension, which results in $7^m$ points for the piston ($m = 7$) and otl circuit ($m = 6$) models. We use nine points per dimension for the Ebola reproductive ratio model because it has a higher dimensional input space ($m = 8$) than the other models, i.e., we use $9^8$ quadrature points. For a summary of the reference values, see Table 4.2, Table 4.4, and Table 4.6.

For each value of $M$ (the number of Monte Carlo samples), we compute the (i) relative error in the Monte Carlo-based estimates (e.g., $|\hat{\nu}_i - \nu_i|/\nu_i$, where $\nu_i$ is the quadrature-based reference value) and (ii) the a posteriori error estimates—either central limit theorem-based standard error or bootstrap standard error. Since the Monte Carlo estimates are random, we repeat the error computations 10 times for each $M$ and take the average. The average relative error plots for the two models are in Figure 4.2 and Figure 4.5. And the average a posteriori error estimates are in Figure 4.3, Figure 4.6, and Figure 4.9.
We ran the experiments in MATLAB R2015b on a dual-core 2011 MacBook Air with 4GB of memory. The complete executable Matlab code for these experiments is available at https://github.com/PaulMDiaz/as_sensitivity_analysis.

4.1 Piston model

The following algebraic model for a cylindrical piston can be found at http://www.sfu.ca/~ssurjano/piston.html, which also contains Matlab code for evaluating the model predictions for specified inputs. It appears in [22, 23] as a test model for statistical screening. The model is defined by the following nonlinear relationships:

\[
\begin{align*}
    t &= 2\pi \sqrt{\frac{M}{k + S^2 \frac{P_0 V_0}{T_0} \frac{T_a}{V^2}}}, \\
    V &= \frac{S}{2k} \left( \sqrt{A^2 + 4k \frac{P_0 V_0}{T_0} T_a - A} \right), \\
    A &= P_0 S + 19.62M - \frac{kV_0}{S}.
\end{align*}
\]

The quantity of interest is \( t \), which is the time in seconds it takes the piston to complete one cycle. The remaining terms in the model are input parameters. The parameters’ descriptions and bounds are in Table 4.1 ordered from top to bottom. For computation, we shift and scale the 7-dimensional parameter space to the hypercube, and we equip it with a uniform probability density function. In the notation in previous sections, the parameters in Table 4.1 are \( \mathbf{x} \), and the time \( t \) is \( f(\mathbf{x}) \).

We apply Algorithm 1.1 from [17, Chapter 1] to test the piston model for an active subspace. The 7 eigenvalues from \( \hat{C} \) in (3.5) are shown in Figure 4.1(a) on a logarithmic scale. The order-of-magnitude gap between the first and second eigenvalues suggests a dominant and computable one-dimensional active subspace. Figure 4.1(b) is a summary plot for the piston model, which plots the active variable (i.e., the linear combination of the 7 parameters with weights from the first eigenvector of \( \hat{C} \)) on the horizontal axis versus the corresponding model output on the vertical axis. The near-one-dimensional relationship between the active variable and the model output confirms the one-dimensional active subspace suggested by
Table 4.1: Input parameters’ descriptions and ranges for piston model (4.1). The parameters are ordered from top to bottom.

<table>
<thead>
<tr>
<th>Input parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$ ∈ [30, 60]</td>
<td>Piston Weight (kg)</td>
</tr>
<tr>
<td>$S$ ∈ [0.005, 0.020]</td>
<td>Piston Surface Area (m$^2$)</td>
</tr>
<tr>
<td>$V_0$ ∈ [0.002, 0.010]</td>
<td>Initial Gas Volume (m$^3$)</td>
</tr>
<tr>
<td>$k$ ∈ [1000, 5000]</td>
<td>Spring Coefficient (N/m)</td>
</tr>
<tr>
<td>$P_0$ ∈ [90000, 110000]</td>
<td>Atmospheric Pressure (N/m$^2$)</td>
</tr>
<tr>
<td>$T_a$ ∈ [290, 296]</td>
<td>Ambient Temperature (K)</td>
</tr>
<tr>
<td>$T_0$ ∈ [340, 360]</td>
<td>Filling Gas Temperature (K)</td>
</tr>
</tbody>
</table>

The eigenvalue gap. See [24, 17] for more information on summary plots. From these two figures, we choose $n = 1$ for the dimension of the active subspace, so we use $n = 1$ when computing the activity scores.

Figure 4.1: Figure 4.1(a) shows the eigenvalues of $\hat{C}$ from (3.5). The order-of-magnitude gap between the first and second eigenvalue suggests a one-dimensional active subspace. This is confirmed by the summary plot in Figure 4.1(b), which shows that the piston model’s quantity of interest can be reasonably well approximated by a univariate function of one active variable.

The reference values for the sensitivity metrics, computed with high order numerical quadrature, are shown in Table 4.2. We use these values to compute the relative error in the Monte Carlo estimates for $M \in \{10, 50, 100, 500, 1000, 5000, 10000\}$ samples. Figure 4.2
Table 4.2: Reference values for piston model’s inputs’ sensitivity metrics computed with
tensor product Gauss-Legendre quadrature on $7^7 = 823543$ points. Lighter colors represent
more significant parameters. All sensitivity metrics rank the variables similarly in this model,
with one exception in the ordering of $M$ and $k$ with $\tau_i$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\tau_i$</th>
<th>$\nu_i$</th>
<th>$\beta_i$</th>
<th>$w_i$</th>
<th>$\alpha_i(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M(\text{kg})$</td>
<td>0.0509</td>
<td>0.0032</td>
<td>0.1963</td>
<td>0.1604</td>
<td>0.0018</td>
</tr>
<tr>
<td>$S(\text{m}^2)$</td>
<td>0.5994</td>
<td>0.0449</td>
<td>-0.7345</td>
<td>-0.7936</td>
<td>0.0437</td>
</tr>
<tr>
<td>$V_0(\text{m}^3)$</td>
<td>0.3528</td>
<td>0.0265</td>
<td>0.5567</td>
<td>0.5768</td>
<td>0.0231</td>
</tr>
<tr>
<td>$k(\frac{\text{N}}{\text{m}})$</td>
<td>0.0669</td>
<td>0.0040</td>
<td>-0.1424</td>
<td>-0.1035</td>
<td>0.0007</td>
</tr>
<tr>
<td>$P_0(\frac{\text{N}}{\text{m}^2})$</td>
<td>0.0013</td>
<td>0.0001</td>
<td>-0.0352</td>
<td>-0.0305</td>
<td>0.0001</td>
</tr>
<tr>
<td>$T_a(\text{K})$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0018</td>
<td>0.0015</td>
<td>0.0000</td>
</tr>
<tr>
<td>$T_0(\text{K})$</td>
<td>0.0001</td>
<td>0.0000</td>
<td>-0.0051</td>
<td>-0.0042</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

shows average relative error over 10 independent trials. Figure 4.3 shows the central limit
theorem-based standard error or bootstrap standard error, as appropriate, for the Monte
Carlo estimates.

4.2 Circuit model

The transformerless push-pull circuit model can be found at http://www.sfu.ca/~ssurjano/
otlcircuit.html, along with Matlab code for evaluating the model’s predictions given input
parameters. This model appears in [22, 23] as a test function for statistical screening. The
model is defined as

$$
V = \left\{ \frac{(12R_{b2}/(R_{b1} + R_{b2}) + 0.74)\beta(R_{c2} + 9)}{\beta(R_{c2} + 9) + R_f} + \frac{11.95R_f}{\beta(R_{c2} + 9) + R_f} \right\} \frac{0.74R_f\beta(R_{c2} + 9)}{(\beta(R_{c2} + 9) + R_f)R_{c1}}.
$$

The quantity of interest is $V$, the midpoint voltage. The remaining terms in the model are
input parameters; their descriptions and ranges are in Table 4.3 ordered from top to bottom.
We shift and scale the parameter space to the hypercube $[-1, 1]^6$ and equip it with a uniform
probability density. In the notation in previous sections, the parameters in Table 4.3 are $x$,
and the midpoint voltage $V$ is $f(x)$.
Figure 4.2: The relative error in the Monte Carlo estimates of the sensitivity metrics for the piston model (4.1), computed using the reference values in Table 4.2, for the number $M$ of samples in $\{10, 50, 100, 500, 1000, 5000, 10000\}$. The metrics compared are the total sensitivity index ($\tau_i$, Figure 4.2(a)), the derivative-based metric ($\nu_i$, Figure 4.2(b)), the linear model coefficients ($\beta_i$, Figure 4.2(c)), the elementary effects ($\mu_i$, Figure 4.2(d)), the first eigenvector components ($w_i$, Figure 4.2(e)), and the activity scores ($\alpha_i$, Figure 4.2(f)).

As with the piston model, we apply Algorithm 1.1 from [17, Chapter 1] to test the circuit model for an active subspace. The 6 eigenvalues from $\hat{C}$ in (3.5) are shown in Figure 4.4(a) on a logarithmic scale. As with the piston model, the order-of-magnitude gap between the first and second eigenvalues identifies a dominant and computable one-dimensional active subspace. Figure 4.4(b) is the summary plot. Again, the near-one-dimensional relationship between the active variable and the model output confirms the
Figure 4.3: The central limit theorem-based standard errors \(se\) or the bootstrap standard errors \(\hat{se}\) for the sensitivity metrics associated with the piston model (4.1) as a function of the number \(M\) of Monte Carlo samples. The metrics compared are the total sensitivity index (\(\tau_i\), Figure 4.3(a)), the derivative-based metric (\(\nu_i\), Figure 4.3(b)), the linear model coefficients (\(\hat{\beta}_i\), Figure 4.3(c)) the elementary effects (\(\mu_i\), Figure 4.3(d)), the first eigenvector components (\(w_i\), Figure 4.3(e)), and the activity scores (\(\alpha_i\), Figure 4.3(f)). The bootstrap standard errors were computed using 100 bootstrap replicates.

Table 4.3: Input parameters’ descriptions and ranges for the circuit model (4.2). The parameters are ordered from top to bottom.

<table>
<thead>
<tr>
<th>Input parameters</th>
<th>Description</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_{b1})</td>
<td>Resistance b1 (K-Ohms)</td>
<td>([50, 150])</td>
</tr>
<tr>
<td>(R_{b2})</td>
<td>Resistance b2 (K-Ohms)</td>
<td>([25, 75])</td>
</tr>
<tr>
<td>(R_f)</td>
<td>Resistance f (K-Ohms)</td>
<td>([0.5, 3])</td>
</tr>
<tr>
<td>(R_{c1})</td>
<td>Resistance c1 (K-Ohms)</td>
<td>([0.25, 1.2])</td>
</tr>
<tr>
<td>(R_{c2})</td>
<td>Resistance c2 (K-Ohms)</td>
<td>([90000, 110000])</td>
</tr>
<tr>
<td>(\beta)</td>
<td>Current Gain (Amperes)</td>
<td>([50, 300])</td>
</tr>
</tbody>
</table>
one-dimensional active subspace suggested by the eigenvalue gap. Therefore, we again choose $n = 1$ for the dimension of the active subspace and use $n = 1$ when computing the activity scores.

Figure 4.4: Figure 4.4(a) shows the eigenvalues of $\hat{C}$ from (3.5). The order-of-magnitude gap between the first and second eigenvalue suggests a one-dimensional active subspace. This is confirmed by the summary plot in Figure 4.4(b), which shows that the circuit model’s quantity of interest can be reasonably well approximated by a univariate function of one active variable.

Table 4.4: Reference values for circuit model’s inputs’ sensitivity metrics computed with tensor product Gauss-Legendre quadrature on $7^6 = 117649$ points. Lighter colors represent more significant parameters. All sensitivity metrics rank the variables consistently in this model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\tau_i$</th>
<th>$\nu_i$</th>
<th>$\beta_i$</th>
<th>$w_i$</th>
<th>$\alpha_i(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{b1}$</td>
<td>0.5001</td>
<td>2.4555</td>
<td>-0.6925</td>
<td>-0.7407</td>
<td>2.3778</td>
</tr>
<tr>
<td>$R_{b2}$</td>
<td>0.4117</td>
<td>1.7039</td>
<td>0.6358</td>
<td>0.6112</td>
<td>1.6190</td>
</tr>
<tr>
<td>$R_f$</td>
<td>0.0740</td>
<td>0.2902</td>
<td>0.2662</td>
<td>0.2458</td>
<td>0.2617</td>
</tr>
<tr>
<td>$R_{c1}$</td>
<td>0.0218</td>
<td>0.1090</td>
<td>-0.1341</td>
<td>-0.1318</td>
<td>0.0752</td>
</tr>
<tr>
<td>$R_{c2}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-0.0002</td>
<td>-0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0000</td>
<td>0.0002</td>
<td>-0.0032</td>
<td>-0.0039</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

The reference values for the sensitivity metrics, computed with high order numerical quadrature, are shown in Table 4.4. We run the same Monte Carlo error study as the piston
model in section 4.1. Figure 4.5 shows average relative error over 10 independent trials. Figure 4.6 shows the central limit theorem-based standard error or bootstrap standard error, as appropriate, for the Monte Carlo estimates.

Figure 4.5: The relative error in the Monte Carlo estimates of the sensitivity metrics for the circuit model (4.2), computed using the reference values in Table 4.4, for the number \( M \) of samples in \( \{10, 50, 100, 500, 1000, 5000, 10000\} \). The metrics compared are the total sensitivity index (\( \tau_i \), Figure 4.5(a)), the derivative-based metric (\( \nu_i \), Figure 4.5(b)), the linear model coefficients (\( \hat{\beta}_i \), Figure 4.5(c)), the elementary effect (\( \mu_i \), Figure 4.5(d)), the first eigenvector components (\( w_i \), Figure 4.5(e)), and the activity scores (\( \alpha_i \), Figure 4.5(f)). As in the piston model, the errors in the Monte Carlo estimates of \( \tau_i \) and \( \hat{\beta}_i \) are much larger for the smaller reference values. The errors in the derivative-based metrics, the first eigenvector components, and the activity scores behave similarly across the scores.

4.3 Ebola Virus Disease

The following algebraic function represents the basic reproductive ratio of Ebola virus disease for a modified SEIR (Susceptible, Exposed, Infected, and Removed) model presented in
Figure 4.6: The central limit theorem-based standard errors $se$ or the bootstrap standard errors $\hat{se}$ for the sensitivity metrics associated with the circuit model (4.2) as a function of the number $M$ of Monte Carlo samples. The metrics compared are the total sensitivity index ($\tau_i$, Figure 4.6(a)), the derivative-based metric ($\nu_i$, Figure 4.6(b)), the linear model coefficients ($\hat{\beta}_i$, Figure 4.6(c)), the elementary effects ($\mu_i$, Figure 4.6(d)), the first eigenvector components ($w_i$, Figure 4.6(e)), and the activity scores ($\alpha_i$, Figure 4.6(f)). The bootstrap standard errors were computed using 100 bootstrap replicates.

Appendix A.

$$R_0 = \frac{\beta_1 + \frac{\beta_2 \mu_1}{\omega} \gamma_1 + \frac{\beta_3}{\gamma_2} \psi}{\gamma_1 + \psi}. \quad (4.3)$$

The quantity of interest is $R_0$, the basic reproductive ratio (or number) represents the number of new infections one case generates on average over the course of its infectious period, among an uninfected population. If $R_0 < 1$, then eventually the epidemic will end. Conversely, if $R_0 > 1$ the epidemic will continue. The remaining terms in the model are input parameters; their descriptions and ranges are in Table 4.5 ordered from top to bottom. We shift and scale the parameter space to the hypercube $[-1, 1]^8$ and equip it with a uniform probability.
density. In the notation in previous sections, the parameters in Table 4.5 are \( x \), and the basic reproductive ratio \( R_0 \) is \( f(x) \).

Table 4.5: Input parameters’ descriptions and ranges for the basic reproductive ratio of Ebola in (4.3), these values are specific to Liberia. The intervals for the parameters \( \beta_1, \beta_2, \beta_3 \), and \( \omega \) come from the constraints in the barrier method for fitting parameter values in Section A.4. We used data from [25] to construct estimates of the remaining parameter intervals. The parameters are ordered from top to bottom.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 \in [0.1, 0.4] )</td>
<td>transmission rate between infected and susceptible</td>
</tr>
<tr>
<td>( \beta_2 \in [0.1, 0.4] )</td>
<td>transmission rate between removed but still infectious and susceptible</td>
</tr>
<tr>
<td>( \beta_3 \in [0.05, 0.2] )</td>
<td>transmission rate between hospitalized and susceptible</td>
</tr>
<tr>
<td>( \rho_1 \in [0.41, 1] )</td>
<td>proportion of infected who die of the disease and are not hospitalized</td>
</tr>
<tr>
<td>( \gamma_1 \in [0.0276, 0.1702] )</td>
<td>(average time with disease for unhospitalized individuals)(^{-1} )</td>
</tr>
<tr>
<td>( \gamma_2 \in [0.081, 0.210] )</td>
<td>(average time with disease for hospitalized individuals)(^{-1} )</td>
</tr>
<tr>
<td>( \omega \in [0.25, 0.5] )</td>
<td>(average time until a deceased individual is properly buried)(^{-1} )</td>
</tr>
<tr>
<td>( \psi \in [0.0833, 0.7] )</td>
<td>(average time for infected to become hospitalized)(^{-1} )</td>
</tr>
</tbody>
</table>

As with the previous two models, we apply Algorithm 1.1 from [17, Chapter 1] to test the reproductive ratio model for an active subspace. The 8 eigenvalues from \( \hat{C} \) in (3.5) are shown in Figure 4.7(a) on a logarithmic scale. Unlike the previous two models, the order-of-magnitude gap between the second and third eigenvalues identifies a dominant and computable two-dimensional active subspace. Figure 4.7(b) is the summary plot. The near-two-dimensional relationship between the first and second active variables and the model output confirms the two-dimensional active subspace suggested by the eigenvalue gap. Therefore, we choose \( n = 2 \) for the dimension of the active subspace and use \( n = 2 \) when computing the activity scores.

The reference values for the sensitivity metrics, computed with high order numerical quadrature, are shown in Table 4.6. We run the same Monte Carlo error study as the piston model in Section 4.1. Figure 4.8 shows average relative error over 10 independent trials. Figure 4.9 shows the central limit theorem-based standard error or bootstrap standard error, as appropriate, for the Monte Carlo estimates.
Figure 4.7: Figure 4.7(a) shows the eigenvalues of $\hat{C}$ from (3.5). The order-of-magnitude gap between the second and third eigenvalue suggests a two-dimensional active subspace. This is confirmed by the summary plot in Figure 4.7(b), which shows that the reproductive ratio model’s quantity of interest can be reasonably well approximated by a bivariate function of the first and second active variables.

Table 4.6: Reference values for the Ebola reproductive ratio function inputs’ sensitivity metrics computed with tensor product Gauss-Legendre quadrature on $9^8 = 43046721$ points. Lighter colors represent more significant parameters. Not all sensitivity metrics rank the variables consistently in this model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\tau_i$</th>
<th>$\nu_i$</th>
<th>$\beta_i$</th>
<th>$w_i$</th>
<th>$\alpha_i(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>0.2297</td>
<td>0.6628</td>
<td>0.4305</td>
<td>0.3808</td>
<td>0.6076</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.0089</td>
<td>0.0258</td>
<td>0.0790</td>
<td>0.0612</td>
<td>0.0166</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.2796</td>
<td>0.8069</td>
<td>0.5028</td>
<td>0.3365</td>
<td>0.7764</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0.0046</td>
<td>0.0133</td>
<td>0.0551</td>
<td>0.0430</td>
<td>0.0081</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.0852</td>
<td>0.2913</td>
<td>-0.2145</td>
<td>-0.2546</td>
<td>0.2656</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.1887</td>
<td>0.8142</td>
<td>-0.3932</td>
<td>-0.2982</td>
<td>0.7751</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.0033</td>
<td>0.0117</td>
<td>-0.0453</td>
<td>-0.0375</td>
<td>0.0061</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.3112</td>
<td>2.6893</td>
<td>-0.4448</td>
<td>-0.7622</td>
<td>2.6821</td>
</tr>
</tbody>
</table>

4.4 Discussion

In Table 4.2 and Table 4.4, every sensitivity metric provides the same relative ranking among the input parameters; the only exception is the ordering of $k$ and $M$ using $\tau_i$ in the piston
Figure 4.8: The relative error in the Monte Carlo estimates of the sensitivity metrics for the circuit model (4.2), computed using the reference values in Table 4.4, for the number $M$ of samples in $\{1000, 5000, 10000, 25000, 50000, 75000, 100000\}$. The metrics compared are the total sensitivity index ($\tau_i$, Figure 4.8(a)), the derivative-based metric ($\nu_i$, Figure 4.8(b)), the linear model coefficients ($\hat{\beta}_i$, Figure 4.8(c)), the elementary effect ($\mu_i$, Figure 4.8(d)), the first eigenvector components ($w_i$, Figure 4.8(e)), and the activity scores ($\alpha_i$, Figure 4.8(f)). As in the piston model, the errors in the Monte Carlo estimates of $\tau_i$ and $\hat{\beta}_i$ are much larger for the smaller reference values. The errors in the derivative-based metrics, the first eigenvector components, and the activity scores behave similarly across the scores.

Figure 4.8: The relative error in the Monte Carlo estimates of the sensitivity metrics for the circuit model (4.2), computed using the reference values in Table 4.4, for the number $M$ of samples in $\{1000, 5000, 10000, 25000, 50000, 75000, 100000\}$. The metrics compared are the total sensitivity index ($\tau_i$, Figure 4.8(a)), the derivative-based metric ($\nu_i$, Figure 4.8(b)), the linear model coefficients ($\hat{\beta}_i$, Figure 4.8(c)), the elementary effect ($\mu_i$, Figure 4.8(d)), the first eigenvector components ($w_i$, Figure 4.8(e)), and the activity scores ($\alpha_i$, Figure 4.8(f)). As in the piston model, the errors in the Monte Carlo estimates of $\tau_i$ and $\hat{\beta}_i$ are much larger for the smaller reference values. The errors in the derivative-based metrics, the first eigenvector components, and the activity scores behave similarly across the scores.

model. Additionally, all of the sensitivity metrics identify the same sets of non-influential inputs, thought it’s worth noting that each metric measures a different characteristic of the input/output relationship. The signs of the linear model coefficients and the components of the first eigenvector provide information about how the input parameter affects the response, on average. In these two models there is agreement in the sign of the first eigenvector components and the standardized regression coefficients.
Figure 4.9: The central limit theorem-based standard errors $se$ or the bootstrap standard errors $\hat{se}$ for the sensitivity metrics associated with the circuit model (4.2) as a function of the number $M$ of Monte Carlo samples. The metrics compared are the total sensitivity index ($\tau_i$, Figure 4.9(a)), the derivative-based metric ($\nu_i$, Figure 4.9(b)), the linear model coefficients ($\hat{\beta}_i$, Figure 4.9(c)), the elementary effects ($\mu_i$, Figure 4.9(d)), the first eigenvector components ($w_i$, Figure 4.9(e)), and the activity scores ($\alpha_i$, Figure 4.9(f)). The bootstrap standard errors were computed using 100 bootstrap replicates.

There is more variation among metrics in the Ebola reproductive ratio model. As in the other two models, we see consistent sign information from $\hat{\beta}_i$ and $w_i$. In Table 4.6, we see that the standardized regression coefficients identify $\beta_3$ as the most sensitive parameter whereas every other metric identifies $\psi$ as the most sensitive. It is important to note that the standardized regression coefficients are constructed by assuming that $f(x)$ is linear, which is clearly not the case in (4.1), (4.2), or (4.3). This is a big assumption, particularly for (4.3), and Table 4.6 demonstrates how the standardized regression coefficients may lead to inconsistent results for some models. There is however, consistent identification of the same
non-essential sets of parameters ($\beta_2$, $\rho_1$, and $\omega$) among all metrics.

Figure 4.2, Figure 4.5, and Figure 4.8 show us that the Monte Carlo estimates of the activity score, first eigenvector, and derivative-based sensitivity metric have smaller errors than estimates of the total sensitivity indices and regression coefficients. Indeed, the relative error in the total sensitivity indices and regression coefficients corresponding to non-essential parameters—in each model—is large. In the case of the total sensitivity index, this demonstrates a known phenomenon of the metric which has been studied in [6], where the authors propose two methods to correct for this effect reduction of the mean value and correlated sampling. We note this expected behavior of the total sensitivity index rather than correcting it.

Figure 4.3, Figure 4.6, and Figure 4.9 show the expected $O(M^{-1/2})$ rate for the central limit theorem-based standard errors, but surprisingly they show a slightly faster rate of decrease for the bootstrap standard error estimates for the active subspace-based metrics. Also, all sample-based a posteriori error estimates under predict the true relative error.
We propose two global sensitivity analysis metrics derived from the eigenvectors and eigenvalues computed while analyzing a general model for an active subspace. The two metrics are (i) the components of the first eigenvector and (ii) the activity scores, which are the first $n$ squared eigenvector components linearly combined with the eigenvalues. We show theoretical connections between the activity scores and the more conventional total sensitivity indices of Sobol’ and the derivative-based metrics of Kucherenko. We discuss high order quadrature methods and Monte Carlo methods to estimate the new sensitivity metrics. And we demonstrate the new metrics on three algebraic engineering and scientific models. The models are simple enough to compute reference values with high order quadrature and study the behavior of the Monte Carlo estimates and their a posteriori error estimates, i.e., the central limit theorem-based standard errors or the bootstrap estimates of the standard errors. For the two of the test models, all metrics consistently identify the same important and unimportant variables. The model for Ebola demonstrates how some metrics, like the standardized regression coefficients, can give inconsistent rankings. However, all of the remaining metrics identified the same sets of important and non-important parameters of the model. While it is possible to construct models where this is not the case, since the metrics measure different characteristics of the model, we expect that this consistency will occur across many models in engineering practice.
REFERENCES CITED


This appendix highlights part of our research on the Ebola outbreak which began in 2013. The goal of our research originally was to determine which of the affected countries (Guinea, Liberia, or Sierra Leone) would benefit most from the allocation of resources. However, the purpose of this appendix is to provide enough background information so that we may investigate the model for possible active subspaces and compute sensitivity metrics for the quantity of interest $R_0$ (the basic reproductive ratio). With this in mind, we limit the focus of the study to the country of Liberia.

A.1 Background

The most recent Ebola outbreak began in December 2013 and resulted in a devastating loss of life in Guinea, Liberia, and Sierra Leone. This outbreak has been the deadliest in the history of the disease claiming more than 10,000 lives to date [26]. The severity of the epidemic has prompted an international response to halt further spread of the disease. In particular, efforts arising within the United States and China have provided additional relief to the region, but within 2014 this had not been enough to end the epidemic.

Some basic facts about Ebola disease pathogenesis are well known. Those exposed to the virus experience an 8-to-10 day incubation period during which they remain noninfectious, though the time between exposure and the onset of symptoms ranges from 2 to 21 days [27]. Once a patient becomes symptomatic, the virus may be transferred to others through direct contact with bodily fluids, such as blood and vomit. One important epidemiological feature of Ebola is that those who are killed by the disease can still transmit the virus to susceptible individuals. Unlike most pathogens, which cannot survive long on a deceased individual, Ebola does remain infectious after a person succumbs to the disease [27]. In fact, the deceased
are even more contagious than living Ebola patients as the virus can force a victim’s body to release infectious fluids including blood, vomit, and fecal matter—especially in later stages of the disease. This fluid release is among the most visually harrowing symptoms of Ebola present in some late-stage patients [27].

The spread of infection from deceased to susceptible individuals is a significant problem in Western Africa where local burial rituals often require washing, touching, or kissing the body of the deceased. Among the traditional practices the WHO cautions against with Ebola victims are family-led body preparation and religious rituals that require direct contact with the corpse. Muslim tradition, for instance, requires that family members of the same gender wash the body themselves before burial. Moreover, the lack of adequate health care within Guinea, Liberia, and Sierra Leone has perpetuated the disease, as improper and unsafe burials place further individuals at risk. Widespread reports from Liberia in late 2014 described Ebola victims laying on the streets for days, drastically increasing the risk of infection [28]. The WHO estimates that contact with deceased individuals has caused at least 20% of all infections [29]. Hence, the bodies of deceased Ebola victims that have not been properly disposed and burial ceremonies, in which mourners have direct contact with the body of the deceased, can play a large role in the transmission of the virus, and these effects cannot be neglected within an informative model.

Another important feature is the effect of hospitalization. Due to their frequent interaction with patients either under investigation or confirmed to have contracted the Ebola virus, healthcare personnel are particularly susceptible to the disease. In particular, they can be exposed to Ebola by coming in contact with a patients body fluids, contaminated medical supplies and equipment, or contaminated environmental surfaces. For these reasons transmission between infectious, hospitalized individuals and medical workers are non-negligible within a descriptive epidemic model. Additionally, hospitalization and subsequent treatment does improve an infected patient’s chances of recovery, and an increase in the use of medical facilities within a population can reduce the damage caused by the epidemic.
While previous studies have focused predominantly on the effects of contact tracing [30, 31], it has recently been determined that other aspects of disease pathogenesis, including hospitalization, case isolation, and further introduction of sanitary funeral processes, must be better addressed in order to fully mitigate further spread of the disease [32]. The goal of the current study is to construct a model for the most recent Ebola epidemic in Western Africa. To this end, a deterministic model for the epidemic is constructed in the next section by accounting for the specific characteristics of the disease, including the incubation period, increased risk of infection from deceased individuals, and the effects of hospitalization. Realistic parameter values are determined by fitting given data to the model, then the basic reproduction number is computed.

A.2 A Mathematical Model for Ebola

Our first objective is to develop a mathematical model for the spread of Ebola in Western Africa by accounting for the specific characteristics of the disease. SEIR models are often implemented when studying the spread of infectious diseases that possess significant incubation periods [33, 30, 31]. To describe the spread of Ebola, the traditional SEIR model is augmented with additional compartments based on the following assumptions. First, as the number of susceptibles is large, we neglect stochastic effects and formulate a deterministic model. Additionally, though the current epidemic has lasted for over a year, we assume that the outbreak is not sustained by the introduction of new susceptible individuals. Hence, the total population is near equilibrium and both births and deaths are neglected amongst the total population. We further assume that infected individuals can move to three different removed compartments: removed and infectious (i.e. not properly buried), removed and properly buried, and removed and recovered. This distinction is introduced to allow for the scenario in which individuals who have died from the disease, but have not been properly buried, may continue to infect susceptible individuals whom they contact.

We assume that those who recover from the disease are no longer susceptible, as survivors of Ebola are thought to be immune to the strain of the virus that infected them [34]. Once
hospitalized, infected individuals can still spread the disease to members of the susceptible population. However, we assume that patients who die within a hospital receive an immediate proper burial, and thus cannot infect others once deceased. Finally, we assume that hospitalized individuals have a greater chance of survival than infected, non-hospitalized individuals [27]. This is consistent with estimates of the general fatality rate around 70% and the hospitalized fatality rate near 64% [25].

A.3 A Modified SEIR Model

With the aforementioned assumptions identified, we formulate a modified SEIR model to account for the dynamics of the disease within a population. The model consists of the following seven states, each a function of time $t$: (i) the susceptible proportion of the population $S(t)$, (ii) the exposed proportion (i.e., infected but asymptomatic) $E(t)$, (iii) the infected proportion $I(t)$, (iv) the infected and hospitalized proportion $H(t)$, (v) the removed but infectious proportion (i.e., those who died from the disease but have not been sanitarily buried) $R_I(t)$, (vi) the removed and buried proportion $R_B(t)$, and (vii) the removed and recovered proportion $R_R(t)$.

There are several ways in which the populations may be altered. First, susceptibles transfer to the exposed population after coming into contact with either an infected individual (including those who are hospitalized) or a body which is improperly buried. After the viral incubation period, all exposed individuals move to the infected population. The infected move to one of three populations: hospitalized, removed and infectious, or removed and recovered. Note that infected individuals cannot move immediately to a removed and buried state since we assume that some time is needed to bury an individual not receiving medical care at death. However, a proportion of removed and infectious individuals transfer to removed and buried. Hospitalized individuals move to either removed and buried or removed and recovered. Finally, all transmission terms are assumed to be of mass-action form. Figure A.1 summarizes these transition pathways between populations, while Table A.1 summarizes model parameters—all of which are nonnegative.
Figure A.1: A graph representing the states \((S, E, I, R_B, R_I, R_R, \text{and } H)\) and transition pathways (arrows) in the Ebola model (A.1). Table A.1 further describes the model parameters included above.

The coupled system of differential equations for \(S(t), E(t), I(t), H(t), \text{and } R_I(t)\) is given by

\[
\begin{align*}
\frac{dS}{dt} &= -\beta_1 SI - \beta_2 SR_I - \beta_3 SH \\
\frac{dE}{dt} &= \beta_1 SI + \beta_2 SR_I + \beta_3 SH - \delta E \\
\frac{dI}{dt} &= \delta E - \gamma_1 I - \psi I \\
\frac{dH}{dt} &= \psi I - \gamma_2 H \\
\frac{dR_I}{dt} &= \rho_1 \gamma_1 I - \omega R_I.
\end{align*}
\]

(A.1)

The \(R_B\) and \(R_R\) proportions decouple from the system above as their values are determined once the others are known. Their respective time evolution is given by
\[
\begin{align*}
\frac{dR_B}{dt} &= \omega R_I + \rho_2 \gamma_2 H \\
\frac{dR_R}{dt} &= (1 - \rho_1) \gamma_1 I + (1 - \rho_2) \gamma_2 H
\end{align*}
\]

(A.2)

With \(R_B\) and \(R_R\) accounted for, the model is conservative, and compartments have been rescaled so that each represents the proportion of the respective populations with respect to the total population. For example, denoting the total population constant by \(N = S + E + I + H + R_I + R_B + R_R\) so that \(\frac{dN}{dt} = 0\), we can express the first unknown function as \(S(t) = \frac{s(t)}{N}\) where \(s(t)\) represents the total number of susceptible individuals within the nation of interest.

Table A.1: Parameters in the Ebola dynamics model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_1)</td>
<td>transmission rate between infected and susceptible</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>transmission rate between removed but still infectious and susceptible</td>
</tr>
<tr>
<td>(\beta_3)</td>
<td>transmission rate between hospitalized and susceptible</td>
</tr>
<tr>
<td>(\delta)</td>
<td>(incubation period)(^{-1})</td>
</tr>
<tr>
<td>(\gamma_1)</td>
<td>(average time with disease for unhospitalized individuals)(^{-1})</td>
</tr>
<tr>
<td>(\gamma_2)</td>
<td>(average time with disease for hospitalized individuals)(^{-1})</td>
</tr>
<tr>
<td>(\psi)</td>
<td>(average time for infected to become hospitalized)(^{-1})</td>
</tr>
<tr>
<td>(\rho_1)</td>
<td>proportion of infected who die of the disease and are not hospitalized</td>
</tr>
<tr>
<td>(\rho_2)</td>
<td>proportion of infected who die of the disease and are hospitalized</td>
</tr>
<tr>
<td>(\omega)</td>
<td>(average time until a deceased individual is properly buried)(^{-1})</td>
</tr>
</tbody>
</table>

A.4 Parameter Fitting

We begin our analysis of the model by identifying values of model parameters that generate predictions which match available data for Liberia. We obtained time-series data from WHO Situation reports [35] and the Network Dynamics and Simulation Science Laboratory at Virginia Tech [36]. The data set contains cumulative values of infections and deaths from each nation. However, the data set is incomplete and/or irregularly reported; we remove outliers and time periods without sufficient reporting. The remaining number of data points is 36.
Since the data represent cumulative quantities, but the model describes instantaneous proportions of active infections, and differing compartments are employed in our model for deaths, a direct fit is not immediately possible. Instead, to generate cumulative quantities, the time-integrated infected population was fit to the cumulative infected data. Also, our model includes three deceased (or removed) states, while the data do not differentiate among differing death compartments. Thus, we assume reported deaths in the data correspond to the properly buried population \( R_B \), and not the deceased but infectious population \( R_I \), as the latter are likely unknown to data collectors.

To fit the model parameters, an unconstrained nonlinear optimization was performed using MATLAB’s fminsearch function, which utilizes a Nelder-Mead direct search method. Within this solver nonlinear parameter constraints were enforced by using a barrier function to ensure positive, realistic parameter values. The objective function, \( D(p) \), where \( p \) represents the vector of parameters defined in Table A.1, is defined as

\[
D(p) := \sum_{t \in T} \left[ R_{\text{data}}(t) - N \cdot R_B(t; p) \right]^2 + \left[ C_{\text{data}}(t) - N \cdot \int_0^t I(s; p) \, ds \right]^2
\]  

(A.3)

where \( T \) is the discrete set of times at which the data is available, \( N \) is the total population for a given nation, \( R_{\text{data}}(t) \) is the recorded number of cumulative deceased individuals, and \( C_{\text{data}}(t) \) is the recorded number of cumulative infections. The initial conditions are the proportions as of March 22, 2014. For example, the total estimated population of Liberia is \( N = 4.29 \) million, while the number of reported infections on this date was 9, and hence \( I(0) = 9/(4.29 \times 10^6) = 2.1 \times 10^{-6} \).

Table A.2 displays the fitted parameter values for Liberia. Estimates of the incubation period range between 8 and 10 days, so the daily probability of transition from the exposed state to the infected state was assumed to be \( \delta = \frac{1}{9} \). Figure A.2 displays model trajectories with fitted parameters compared to the available data for Liberia.
Table A.2: Fitted parameter values for Liberia. The parameter $\delta$ was not fit, but taken directly from [27], while initial guesses for $\rho_1$ and $\rho_2$ were motivated by this paper.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Liberia</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>0.376</td>
<td>Fit</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.135</td>
<td>Fit</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.163</td>
<td>Fit</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\frac{1}{9}$</td>
<td>[27]</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.0542</td>
<td>Fit</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.174</td>
<td>Fit</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.500</td>
<td>Fit</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0.98</td>
<td>Fit</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.88</td>
<td>Fit</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.325</td>
<td>Fit</td>
</tr>
</tbody>
</table>

A.5 Steady States and Basic Reproduction Number

To determine any non-zero steady state solutions, the rates of change for the $S, E, I, H,$ and $R_I$ populations were set to zero. As previously described, the equations for the removed-buried and removed-recovered populations decouple from the model and thus were omitted. With this, we find

\[
\begin{align*}
\frac{dS}{dt} &= -S(\beta_1 I + \beta_2 R_I + \beta_3 H) = 0 \\
\frac{dE}{dt} &= \beta_1 SI + \beta_2 SR_I + \beta_3 SH - \delta E = 0 \\
\frac{dI}{dt} &= \delta E - \gamma_1 I - \psi I = 0 \\
\frac{dH}{dt} &= \psi I - \gamma_2 H = 0 \\
\frac{dR_I}{dt} &= \rho_1 \gamma_1 I - \omega R_I = 0.
\end{align*}
\]

(A.4)

and

(A.5)
The three equations within (A.5) then imply

\[ E = \frac{\gamma_1 + \psi}{\delta} I, \quad H = \frac{\psi}{\gamma_2} I, \quad R_I = \frac{\rho_1 \gamma_1}{\omega} I, \]

so that substitution of these expressions into (A.4) yields

\[
\begin{align*}
SI \left( \beta_1 + \frac{\beta_2 \rho_1 \gamma_1}{\omega} + \frac{\beta_3 \psi}{\gamma_2} \right) & = 0 \\
I \left[ -(\gamma_1 + \psi) + S \left( \beta_1 + \frac{\beta_2 \rho_1 \gamma_1}{\omega} + \frac{\beta_3 \psi}{\gamma_2} \right) \right] & = 0.
\end{align*}
\]

Notice that within (A.6), \( S = 0 \) is one possible solution, which implies \( I = 0 \) and that all populations are zero. As we assume throughout that the total population satisfies \( N > 0 \), this steady state is not possible. Instead, imposing \( I = 0 \) within (A.6), we see that \( S \) is
arbitrary. However, to satisfy the total population constraint, we must have $S = 1$. Hence, we find the unique infection-free steady state $(\bar{S}, \bar{E}, \bar{I}, \bar{H}, \bar{R}) = (1, 0, 0, 0, 0)$.

To compute the basic reproduction number of the system with respect to this steady state, we employ the Next Generation Matrix [37]. In particular, computing the gains and losses matrices associated with (A.1), we find

$$F = \begin{pmatrix}
0 & \beta_1 \bar{S} & \beta_3 \bar{S} & \beta_2 \bar{S} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}$$

and

$$V = \begin{pmatrix}
\delta & 0 & 0 & 0 \\
-\delta & \gamma_1 + \psi & 0 & 0 \\
0 & -\psi & \gamma_2 & 0 \\
0 & -\rho_1 \gamma_1 & 0 & \omega \\
\end{pmatrix}.$$  

Here, $F$ has been evaluated at the steady state determined above. Inverting $V$ yields

$$V^{-1} = \begin{pmatrix}
\frac{1}{\delta} & 0 & 0 & 0 \\
\frac{1}{\gamma_1 + \psi} & \frac{1}{\gamma_1 + \psi} & 0 & 0 \\
\frac{\psi}{\gamma_2 (\gamma_1 + \psi)} & \frac{1}{\gamma_2} & 0 & 0 \\
\frac{-\rho_1 \gamma_1}{\omega (\gamma_1 + \psi)} & \frac{\rho_1 \gamma_1}{\omega (\gamma_1 + \psi)} & 0 & \frac{1}{\omega} \\
\end{pmatrix}$$

and, using $S = 1$, the resulting next generation matrix is

$$FV^{-1} = \begin{pmatrix}
\frac{\beta_1 + \beta_2 \rho_1 \gamma_1 + \beta_3 \psi}{\gamma_1 + \psi} & \frac{\beta_1 + \beta_2 \rho_1 \gamma_1 + \beta_3 \psi}{\gamma_1 + \psi} & \beta_3 \gamma_2 & \beta_2 \omega \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.$$
Finally, it's easy to see that the spectral radius of this matrix is exactly \((FV^{-1})_{1,1}\) as the remaining eigenvalues must be identically zero. Thus, we arrive at an explicit formula for the basic reproduction number, namely

\[
\mathcal{R}_0 := \rho\left(FV^{-1}\right) = \frac{\beta_1 + \frac{\beta_2 \gamma_1}{\omega} + \frac{\beta_3}{\gamma_2} \psi}{\gamma_1 + \psi}
\]  

(A.7)

With parameter values established, the basic reproduction number can be computed. We note that this quantity is independent of the parameters \(\delta\) and \(\rho_2\) since neither the length of the incubation period nor the death rate of hospitalized patients affects the generation of secondary infected cases. Utilizing fitted parameter values in Table A.2 the basic reproduction number for Liberia is \(\mathcal{R}_0 = 1.563\). Without further intervention and new allocation of resources to fight the disease, our model predicts that the epidemic would continue to spread as of December 2014, generating 736 new cases each day in Liberia.