A VERIFICATION AND VALIDATION OF THE GEOMETRICALLY EXACT BEAM THEORY WITH LEGENDRE SPECTRAL FINITE ELEMENTS FOR WIND TURBINE BLADE ANALYSIS

by

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ABSTRACT

Composite wind turbine blades continue to get larger and have more complex geometry than ever before. Additionally, they are becoming lighter in proportion to their size. Lighter and larger wind turbine blades result in structures that are highly flexible. It is necessary to have computer aided engineering (CAE) tools that are capable of modeling the nonlinear behavior of composite structures with complex geometry in a robust yet computationally efficient manner. The National Renewable Energy Laboratory (NREL) has developed an aeroelastic CAE tool, FAST, which is used for wind turbine analysis. The current wind turbine blade model in FAST is based on linear Euler-Bernoulli beam theory. A new finite element beam model, BeamDyn, which is based on the geometrically exact beam theory (GEBT) has been proposed to replace the incumbent wind turbine blade model in FAST. In the work reported here, GEBT and its spectral finite element implementation in BeamDyn is presented, and a number of numerical and experimental cases show the efficacy of the proposed model.
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LIST OF ABBREVIATIONS

U.S. Department of Energy .................................................. DOE
Computer aided engineering ............................................... CAE
Original equipment manufacturer ........................................... OEM
Degree of freedom ............................................................. DOF
Geometrically exact beam theory .......................................... GEBT
One-dimensional ............................................................... 1-D
Three-dimensional ............................................................. 3-D
Finite element ................................................................. FE
Variational-asymptotic method ............................................. VAM
Two-dimensional ............................................................... 2-D
Variational Asymptotic Beam Section .................................... VABS
Legendre spectral finite element .......................................... LSFE
Direction cosine matrix ...................................................... DCM
Gauss-Lobatto-Legendre ..................................................... GLL
Gauss-Legendre ............................................................... GL
Quadratic finite elements ................................................... QFE
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CHAPTER 1
INTRODUCTION

Wind energy is an important element in the nation’s overall energy supply. In the past 10 years wind power installations in the U.S. have exceeded 61 GW [1]. The U.S. Department of Energy (DOE) has set a target of 20% of the nation’s electricity to be provided by wind power by the year 2030 [2].

Over recent years the size of wind turbines has increased in the quest for economies of scale. Additionally, wind turbine blades have become lighter in proportion to their size and have more complex geometry. Larger and lighter wind turbine blades result in structures that are highly flexible [3]. To ensure the performance and reliability of wind turbines it is crucial to make use of computer aided engineering (CAE) tools that are capable of analyzing wind turbine blades in an accurate and efficient manner. Modern supercomputers make full 3-D computational analysis an option, but these simulations are computationally expensive and not always the preferred option, thus it is ideal to have an efficient high fidelity alternative.

FAST is an aeroelastic CAE tool developed by the National Renewable Energy Laboratory (NREL) for the purposes of wind turbine analysis using realistic operating conditions for both land-based and offshore wind turbines. FAST currently has over 4,000 users, including original equipment manufacturers (OEMs), students and faculty, and government research labs. The current wind turbine blade model in FAST is not fully capable of analyzing composite or highly flexible wind turbine blades [4]. The limitations of the current blade model in FAST are [5] [6]:

- The model assumes that the blade is straight (i.e., no initial curvature)
- The model assumes that the blade is idealized as an Euler-Bernoulli beam (i.e., transverse shear effects are ignored)
• Warping is not modeled explicitly, though its effect may be included implicitly via torsion stiffness

• For a blade, the effect of chordwise offsets of the center of mass, shear center, and tension center normal to the chord is ignored

• The blade material is assumed to be isotropic

• The model is not capable of having axial or torsional degrees of freedom (DOF)

Modern wind turbine blades are constructed of composite materials with initial curvature. Structural analysis of composite blades is complicated due to the elastic-coupling effects that exist under an applied load. To mitigate these issues and to add more utility to the users, NREL has developed a new software package, called BeamDyn. BeamDyn offers the following advantages to the incumbent blade model in FAST:

• BeamDyn is based on the geometrically exact beam theory (GEBT)

• BeamDyn offers a finite element formulation

• BeamDyn accommodates initial twist and initial curvature of blades

• BeamDyn accommodates transverse shear by using a Timoshenko-like formulation

• BeamDyn accommodates warping in the formulation

• BeamDyn accommodates the effect of chordwise offsets of the center of mass, shear center, and tension center normal to the chord

• BeamDyn accommodates anisotropic material properties

• BeamDyn accommodates six DOF at each node

• BeamDyn accommodates geometric nonlinearities (i.e., large displacements/rotations)
1.1 Beam theory introduction

Beam models are widely used to analyze structures that have one of its dimension much larger than the other two. Many engineering structures are modeled as beams: bridges, joists, and helicopter rotor blades. Beam models are also well suited for analysis of wind turbine blades, towers, and shafts. These models have their beginnings around 1750, starting with Euler and have since seen great improvements in accuracy. The Euler-Bernoulli beam theory is considered to be a linear model and is limited to small deflections. Although limited, it has served as a backbone for beam theory and is a theory which still has many applications in structural engineering today.

Timoshenko’s beam theory improved upon Euler-Bernoulli beam theory by allowing the cross-section to shear with respect to the centerline of the beam with an applied load. GEBT, proposed by Reissner [7], also considers the shearing of the cross-section, and is said to be “Timoshenko-like” in this regard. The term geometrically exact refers to a model that is capable of capturing geometric nonlinearities, initial curvatures and assumes small strains [8]. Reissner defined the one-dimensional (1-D) strains in terms of virtual displacement and virtual rotation quantities; this is known as an intrinsic formulation. This work allowed the formulation to be independent of displacement or rotation variables. However, this treated the beam as a 1-D continuum and fails to consider the three-dimensional (3-D) strains or how the strain are distributed within the cross section [8].

The effort to extend Reissner’s work to include 3-D effects was undertaken by Hodges and Danielson [9], which allowed the 3-D strain field to be expressed in terms of the intrinsic 1-D measures. Simo [10], and Simo and Vu-Quoc [11] extended Reissner’s initial work to include 3-D dynamic problems. Jelenić and Crisfield [12] determined that simply linearly interpolating the rotational field violates the objectivity criterion, which states that rigid-body motion can not contribute to the overall strain field since rigid-body motion, by definition, generates no strain. Jelenić and Crisfield derived a finite-element (FE) method that interpolates the rotation field by splitting the rotations into elastic and rigid-body components.
by preserving the geometric exactness, and non-linearity of this theory. It is noted that Ibrahimbegović and his colleagues implemented GEBT for static [13] and dynamic [14] analysis.

The variational-asymptotic method (VAM) of Berdichevsky [15] was found to split a 3-D geometrically nonlinear elasticity analysis for beam-like structures into a nonlinear 1-D analysis and a linear two-dimensional (2-D) analysis by analyzing the energy of the beam [8]. This work was extremely important and lead to a Variational Asymptotic Beam Section (VABS) analysis tool, which is capable of analyzing complex composite cross-sections, and recovering 3-D stress information based on 1-D beam results. An application of GEBT to wind turbine blades was completed by Luscher et al. where the CX-100, a well characterized wind turbine blade, was analyzed and compared to experimental test data [16].

The spectral element method was introduced by Patera [17] and applied to the incompressible-flow Navier-Stokes equations. Spectral FEs have seen successful implementation in the fields of fluid dynamics [17–19], geophysics [20], elastodynamics [21], and acoustics [22]. There is also an implementation of spectral FEs for a Timoshenko beam [23]. The Legendre polynomial has also seen extensive use as the basis function [18], these are known as Legendre spectral finite elements (LSFEs). Wang et al. implemented spectral FEs into GEBT [24] and [25].

The goal of this thesis is to systematically present the established geometrically exact beam theory which was developed by Hodges [8] and Bauchau [26], present its spectral finite-element implementation in BeamDyn which was completed by NREL, and to detail verification and validation of BeamDyn which is this author’s contribution. In this context, verification refers to comparison of BeamDyn to similar numerical tools and analytical solutions where they are available. Validation is the comparison of BeamDyn results to experimental data.

This thesis is organized as follows. First, a theoretical background section will be presented to equip the reader with the necessary physical and mathematical concepts to un-
derstand the theoretical foundation of GEBT. Next, the geometrically exact beam theory will be presented in two parts, first a dimensional reduction, then the 1-D beam theory and spectral FE implementation. Finally, verification and validation of BeamDyn will illustrate its efficacy to wind turbine blade modeling.

1.2 Notation

Throughout this thesis, vector and tensor analysis will be presented. As such, it is helpful to introduce the notation that will be used from this point forward. Scalars are denoted by an italic letter, e.g., \(a\). Vectors are denoted by an underline, e.g., \(\underline{a}\). Unit vectors are denoted \(\hat{e}\). Tensors are denoted by a double underline, e.g., \(\underline{A}\). Summation index convention applies throughout the text unless otherwise noted. Latin indices (e.g., \(i, j, k\)) range over \(1, 2, 3\). Greek indices (e.g., \(\alpha, \beta\)) range from \(2, 3\). The superscript \(\cdot^T\) is the transpose, an overdot denotes the time derivative, and \(\cdot’\) is the spatial derivative of the given quantity with respect to \(x_1\) (the spatial axis along which the span of the beam is defined) unless otherwise noted. Vector and tensor operators for dot and cross products are denoted with \(\cdot\) and \(\times\), respectively. The tilde operator, \(\tilde{\cdot}\), always denotes a cross product, and can only be applied to a vector\(^1\). When used in an equation \(I\) will refer to the identity matrix unless otherwise noted. When a vector is resolved in the material (moving) basis it is denoted \(\cdot^*\).

\(^1\)Please see Appendix A for more information on Mathematical Preliminaries
CHAPTER 2
BACKGROUND

In order to properly present the geometrically exact beam theory (GEBT) and its finite-element implementation, it is necessary to equip the reader with background information on
the subjects of 3-D rotations, beam kinematics, Hamilton’s principle, variational calculus,
and asymptotic methods. This chapter is divided into sections that cover these subjects
as they relate to GEBT. The information presented in this chapter is necessary to the
understanding of GEBT and has been presented by Bauchau [26], Hodges [8], and Yu [27].

2.1 3-D Rotations

Since the study of wind turbine blades in operation inherently involves rotation it is
necessary to understand how rotations affect the beam formulation. This subsection covers
the direction cosine matrix, rotation tensors, and the composition of rotations.

2.1.1 The Direction Cosine Matrix

For two orthonormal bases $I = (\hat{i}_1, \hat{i}_2, \hat{i}_3)$ and $E = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$, the operation that brings
basis $I$ into $E$ is called a finite rotation. This operation is depicted in Figure 2.1. Equation
2.1 shows that vector $e_1$ can be described in the basis $I$

$$\tilde{e}_1 = D_{11}\hat{i}_1 + D_{21}\hat{i}_2 + D_{31}\hat{i}_3$$ (2.1)

The coefficients of Equation 2.1 are the components of the direction cosine matrix (DCM).
The DCM can be found using [26]

$$D_{kl} = \tilde{e}_k^T \tilde{e}_l$$ (2.2)

Since $\hat{i}_k$ and $\hat{e}_l$ are unit vectors it can be seen that Equation 2.2 can also be expressed as the cosine of the angle between basis $I$ and $E$ as

$$D_{kl} = \cos(\hat{i}_k, \hat{e}_l)$$ (2.3)
Figure 2.1: Finite rotation that brings basis $\mathcal{I}$ into $\mathcal{E}$

For planar rotation, one of the unit vector directions remain unchanged. For example

$$\begin{align*}
\bar{e}_1 &= \cos \phi \bar{e}_1 - \sin \phi \bar{e}_3 \\
\bar{e}_2 &= \bar{e}_2 \\
\bar{e}_3 &= \sin \phi \bar{e}_1 + \cos \phi \bar{e}_3
\end{align*}$$

The direction cosine matrix for rotation of $\mathcal{E}$ about $\bar{e}_2$ is then given by Equation 2.2 as

$$D_2(\phi) = \begin{bmatrix}
\cos \phi & 0 & \sin(\phi) \\
0 & 1 & 0 \\
-\sin(\phi) & 0 & \cos(\phi)
\end{bmatrix}$$

In a similar fashion one can find $D_1(\phi)$ and $D_3(\phi)$.

### 2.1.2 The Rotation Tensor

Euler’s theorem on finite rotations states any displacement of a rigid body can be described by a single rotation of magnitude $\phi$ about a unit vector $\bar{n}$ if the point which the unit vector passes through remains fixed. Figure 2.2 shows rotation of $\mathcal{E}$ about $\bar{n}$ by $\phi$, where $O$ is the origin and $\bar{n}$ is the unit vector about which the rotation occurs.

Euler’s theorem can be used to define the rotation tensor as [26]

$$R = I + \sin \phi \bar{n} + (1 - \cos \phi) \bar{n} \bar{n}$$  \hspace{1cm} (2.4)
Figure 2.2: Finite rotation about a unit normal vector

where \(^2\),

\[
\tilde{n} = \begin{bmatrix}
0 & -n_3 & n_2 \\
n_3 & 0 & -n_1 \\
-n_2 & n_1 & 0
\end{bmatrix}
\]  

(2.5)

The rotation tensor and the DCM are related by considering a rotation that brings basis \(\mathcal{I}\) into \(\mathcal{E}\). First, the vectors are resolved in the basis \(\mathcal{I}\), which implies that \(\check{\mathcal{I}}_{1}^{[\mathcal{I}]} = [1, 0, 0]\), \(\check{\mathcal{I}}_{2}^{[\mathcal{I}]} = [0, 1, 0]\), and \(\check{\mathcal{I}}_{3}^{[\mathcal{I}]} = [0, 0, 1]\). Then basis \(\mathcal{I}\) is brought to \(\mathcal{E}\) using the DCM, where \(D\) is the direction cosine matrix.

\[
\check{e}_1^{[\mathcal{I}]} = D\check{\mathcal{I}}_1^{[\mathcal{I}]}
\]  

(2.6)

If the basis \(\mathcal{I}\) is brought into basis \(\mathcal{E}\) by using the rotation tensor the expression is given by

\[
\check{e}_1^{[\mathcal{I}]} = R^{[\mathcal{I}]}\check{e}_1^{[\mathcal{I}]}
\]  

(2.7)

\(^2\)Please see Appendix A for more information on the tilde operator
It can therefore be seen that the relationship is simply that the direction cosine matrix is the rotation tensor resolved in $\mathcal{I}$

$$D = R^{[\mathcal{I}]}$$ (2.8)

An important property of the rotation tensor is that it is orthogonal, i.e.,

$$R \, R^T = R^T \, R$$ (2.9)

### 2.1.3 Composition of Rotations

In order to calculate rotation across more than two bases, the rotation from the first basis to the final basis must be composed rather than summed. For example, consider three bases $\mathcal{I} = (\hat{i}_1, \hat{i}_2, \hat{i}_3)$, $\mathcal{E} = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$, and $\mathcal{B} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ as shown in Figure 2.3.

![Figure 2.3: Rotation from basis $\mathcal{I}$ to $\mathcal{E}$, and basis $\mathcal{E}$ to $\mathcal{B}$](image)

The rotation from basis $\mathcal{I}$ to basis $\mathcal{B}$ is $R$. From Equation 2.7, $\hat{e}_1 = R_{\mathcal{I}} \hat{i}_1$, and $\hat{b}_1 = R_{\mathcal{E}} \hat{e}_1$. Therefore, $\hat{b}_1 = R_{\mathcal{E}} R_{\mathcal{I}} \hat{i}_1 = R_{\mathcal{B}} \hat{i}_1$, where

$$R = R_{\mathcal{E}} R_{\mathcal{I}}$$ (2.10)
However it is easier to express the second rotation, $R_2$, in terms of basis $\mathcal{E}$. Using the transformation law for second order tensors we have

$$R_2^{[\tau]} = R_1^{[\tau]} R_2^{[\epsilon]} R_1^{[\epsilon]^T} \quad (2.11)$$

Where $R_2^{[\tau]} R_1^{[\tau]} = R_1^{[\tau]} R_2^{[\epsilon]}$ is the same as $R_2^{[\tau]} = R_1^{[\tau]} R_2^{[\epsilon]} R_1^{[\epsilon]^T}$. Therefore, using Equation 2.10

$$R^{[\tau]} = R_1^{[\tau]} R_2^{[\epsilon]} \quad (2.12)$$

### 2.2 Rotation Parameterization

While finite rotations can be fully described using the DCM, this method requires the use of nine components to define finite rotation. Parameterization of rotation allows us to define rotations in different ways, and there are two classes or parameterization: vectorial and non-vectorial. The DCM is an example of a non-vectorial parameterization. Vectorial parameterization allows for the rotations to be expressed as a vector which contains only three components, and thereby reduces the computational expense for operations containing rotations. There are a number of rotation parameters to choose from, and for the purpose of BeamDyn the Wiener-Milenković rotation parameterization has been selected, which is a vectorial rotation parameterization.

The vectorial parameterization of rotation is given by

$$p = p(\phi) \bar{n} \quad (2.13)$$

Where $p(\phi)$ is the generating function. There are many choices for the generating function. As stated above, the Wiener-Milenković rotation parameter was selected for BeamDyn, for which the generating function is given by

$$p(\phi) = 4 \tan \frac{\phi}{4} \quad (2.14)$$
An expression of the rotation tensor is given as a function of the rotation parameter by substituting Equation 2.13 into Equation 2.4

\[ R = I + \frac{\sin \phi}{p(\phi)} \vec{p} + \frac{(1 - \cos \phi)}{p(\phi)^2} \vec{p} \vec{p} \]  

Variables \( R_1 \) and \( R_2 \) are introduced as in Bauchau [28] where,

\[ R_1 = \nu \cos \frac{\phi}{2}; \quad R_2 = \nu^2 \]  

and,

\[ \nu = \frac{2 \sin \frac{\phi}{2}}{p} \]  
\[ \varepsilon = \frac{2 \tan \frac{\phi}{2}}{p} \]  

Using Equation 2.16, Equation 2.15 becomes

\[ R = I + R_1(\phi) \vec{p} + R_2(\phi) \vec{p} \vec{p} \]  

### 2.2.1 Rotation Rescaling

All operations with Wiener-Milenković rotation parameters are purely algebraic. It can be seen that the expression in Equation 2.14 will yield a singularity at \( \phi = 2\pi \). As such, in order to be of practical use a rescaling operation must be applied before the rotation reaches \( 2\pi \) in order to avoid the singularity.

Another important factor to consider is composition of rotation parameters. As discussed in Section 2.1.3, rotations must be composed rather than summed. Jelenić and Crisfield showed that standard interpolation methods of rigid body motion contribute to the strain field [29]. Per its definition, rigid body motion cannot contribute to the overall strain. Therefore to maintain the objectivity criterion we adopt the interpolation method proposed by Jelenić and Crisfield. It is helpful to demonstrate this by way of example. If \( \vec{p} \) and \( \vec{q} \) are rotation parameters, and \( \vec{r} \) is the composition of \( \vec{p} \) and \( \vec{q} \), with rotation angles \( \phi_p \), \( \phi_q \), and \( \phi_r \) respectively, then we can consider the composition of rotations \( \vec{p} \) and \( \vec{q} \) as,
\( \mathbf{R}(r) = \mathbf{R}(p)\mathbf{R}(q) \). From Bauchau [28], the composition formulæ is

\[
\cos \frac{\phi}{2} = v_pv_q \left( \frac{1}{\varepsilon_p \varepsilon_q} - \frac{1}{4} \frac{p^T q}{4} \right) \quad (2.19a)
\]
\[
v_r \mathbf{L} = v_pv_q \left( \frac{p}{\varepsilon_q} + \frac{q}{\varepsilon_p} + \frac{\tilde{p}q}{2} \right) \quad (2.19b)
\]

where \( r \) is found by first solving Equation 2.19 to compute \( \phi \). When using the Wiener-Milenković rotation parameters the composition formulæ is

\[
r = 4 \left( \frac{q_0 p + p_0q + \tilde{p}q}{\Delta_1 + \Delta_2} \right) \quad (2.20)
\]

where,

\[
p_0 = 2 - \frac{1}{8} p^T p \quad (2.21a)
\]
\[
q_0 = 2 - \frac{1}{8} q^T q \quad (2.21b)
\]
\[
\Delta_1 = (4 - p_0)(4 - q_0) \quad (2.21c)
\]
\[
\Delta_2 = p_0q_0 - p^T q \quad (2.21d)
\]

For the static case it is clear that if the rotation within an element exceeds \( 2\pi \) the rescaling operation will be required. To err on the side of caution BeamDyn rescales when the rotation reaches \( \pi \). The composition and rescaling operations may be combined into a single operation when using the Wiener-Milenković rotation parameters as follows [28]

\[
r = \begin{cases} 
4 \left( \frac{q_0 p + p_0q + \tilde{p}q}{\Delta_1 + \Delta_2} \right), & \text{if } \Delta_2 \geq 0 \\
-4 \left( \frac{q_0 p + p_0q + \tilde{p}q}{\Delta_1 - \Delta_2} \right), & \text{if } \Delta_2 < 0 
\end{cases} \quad (2.22)
\]

The following notation will be used throughout the document to denote composition of rotations

\[
\mathbf{R}(r) = \mathbf{R}(p)\mathbf{R}(q) \iff r = p \oplus q \quad (2.23)
\]

And for the commonly encountered transpose operation

\[
\mathbf{R}(r) = \mathbf{R}^T(p)\mathbf{R}(q) \iff r = p^\perp \oplus q \quad (2.24)
\]
where $\rho$ indicates that the sign of rotation parameter should be changed.

### 2.2.2 Angular Velocity

Angular velocity is an important concept for understanding the implementation of Beam-Dyn as we are most concerned with rotating wind turbine blades. Consider the orthonormal bases $I$ and $E$, where $I$ is a stationary basis and $E(t)$ is now a basis that changes position with respect to time. Using Equation 2.7, and from Bauchau [26]

$$\vec{e}_1 = \overline{R}\vec{e}_1$$  \hspace{1cm} (2.25)

Taking the time derivative of Equation 2.25

$$\dot{\vec{e}}_1 = \dot{\overline{R}}\vec{e}_1$$  \hspace{1cm} (2.26)

and employing, $\vec{e}_1 = \overline{R}^T\vec{e}_1(t)$ from Equation 2.25

$$\dot{\overline{R}}\vec{e}_1 = \dot{\overline{R}}\overline{R}^{T}\vec{e}_1$$  \hspace{1cm} (2.27)

Also, from kinematics we have

$$\dot{\vec{e}}_i = \vec{\omega}\vec{e}_i$$  \hspace{1cm} (2.28)

Since $\dot{\overline{R}}\overline{R}^T$ is skew symmetric, combination of Equations 2.26, 2.27, and 2.28 yields

$$\vec{\omega} = \dot{\overline{R}}\overline{R}^T$$  \hspace{1cm} (2.29)

The angular velocity can also be defined as $\omega = \text{axial}(\dot{\overline{R}}\overline{R}^T)$, where the axial operator is defined as

$$\omega = \text{axial}(\overline{B}) = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} B_{32} - B_{23} \\ B_{13} - B_{31} \\ B_{21} - B_{12} \end{pmatrix}$$  \hspace{1cm} (2.30)

where $\overline{B}$ is the composition of $\dot{\overline{R}}, \overline{R}^T$, and $B_{ij}$ are the components of $\overline{B}$. Using the result from Equation 2.29 and the definition of the rotation tensor in Equation 2.4 it can be seen

$$\vec{\omega} = \dot{\phi}\vec{n} + \sin \phi \vec{\phi} + (1 - \cos \phi) (\vec{n}\dot{\vec{n}} - \dot{\vec{n}}\vec{n})$$  \hspace{1cm} (2.31)
The angular velocity is then defined as
\[ \omega = \dot{\phi} \hat{n} + \sin \phi \dot{n} + (1 - \cos \phi) \hat{n} \dot{n} \]  
(2.32)

The angular velocity resolved in the material basis is
\[ \tilde{\omega}^* = R^T \tilde{\omega} R = R^T \dot{R}; \quad \omega^* = R^T \omega \]  
(2.33)

The angular velocity can also be defined as a function of the rotation parameters by
\[ \omega = \left( \frac{1}{p(\phi)'} I + \frac{1 - \cos \phi}{p(\phi)^2} \ddot{p} + \frac{1}{p(\phi)^2} \frac{\sin \phi}{p(\phi)} \ddot{p} \right) \dot{p} \]  
(2.34)

Equation 2.34 can also be expressed in terms of the time derivative of the rotation parameter and the tangent tensor, \( \overline{H}(p) \), as follows
\[ \omega = \overline{H}(p) \dot{p} \]  
(2.35)

where the tangent tensor is given by
\[ \overline{H}(p) = \frac{1}{p(\phi)'} + R_2 \ddot{p} + \frac{1}{p(\phi)^2} \frac{1}{p(\phi)} - R_1 \dddot{p} \]  
(2.36)

### 2.2.3 Angular Acceleration

The angular acceleration is found by taking the time derivative of Equation 2.28
\[ \ddot{e}_1 = \tilde{\omega} \dot{e}_1 + \tilde{\omega} \dot{e}_1 \]  
(2.37)

From equation 2.28 it can be seen that
\[ \ddot{e}_1(t) = (\tilde{\omega} + \tilde{\omega} \ddot{\omega}) \ddot{e}_1 \]  
(2.38)

The angular acceleration vector, \( \alpha \), is \( \alpha = \ddot{\omega} \). Therefore
\[ \ddot{e}_1 = (\alpha + \tilde{\omega} \ddot{\omega}) \ddot{e}_1 \]  
(2.39)

### 2.2.4 Curvature Vector

There is an analogy between the curvature vector and the angular velocity, where the angular velocity is the time derivative of Equation 2.7, the curvature is the spatial derivative
of Equation 2.7. If \( \mathcal{E}(s) \) is a space dependent orthonormal basis

\[
\tilde{e}_1(s) = R(s)\tilde{i}_1 \tag{2.40}
\]

taking the spatial derivative of Equation 2.40

\[
\tilde{e}'_1(s) = R'R_1 = R'R_1^T \tilde{e}_1(s) = \kappa \tilde{e}_1(s) \tag{2.41}
\]

where \( \kappa = R'R_1^T \). In the material basis, \( \kappa^* = R^T \kappa \) and \( \kappa^* = R^T R' \)

### 2.3 Beam Kinematics

This section will cover the basic kinematics for the geometrically exact beam theory. It is assumed by this theory that the strains remain small (i.e., material linearity is maintained), and understanding of the kinematics of the beam are of utmost importance. The geometrically exact beam theory will be derived using the variational method.

#### 2.3.1 Reference Configuration

Following Hodges [8], Figure 2.4 shows a reference beam with a given coordinate system \( b_i \) in the undeformed state. Here, \( b_1 \) is in the direction of the span of the beam. \( b_2 \) and \( b_3 \) define the plane in which the cross-section of the beam lies. The same beam is also shown in its deformed state with coordinates \( B_i \).

Here \( x_1 \) represents the direction down the span of the beam. For each point along the beam \( b_i \) is tangent to \( x_i \). We can then define a position vector

\[
\hat{r}(x_1, x_2, x_3) = r(x_1) + x_\alpha b_\alpha \tag{2.42}
\]

Using the rotation tensor (see Section 2.1.2) we may relate the deformed coordinate system to the undeformed coordinate system by

\[
B_i = R b_i \tag{2.43}
\]

We can then write the position vector for the deformed beam as

\[
\hat{R}(x_1, x_2, x_3) = r(x_1) + u(x_1) + x_\alpha b_\alpha + \bar{w}_i(x_1, x_2, x_3) B_i \tag{2.44}
\]
Here \( \mathbf{u} \) is the displacement vector and \( \bar{\mathbf{w}}(x_1, x_2, x_3) \) is the warping function. We must be able to express Equation 2.44 in terms of a tangent triad to account for the warping in the deformed beam. The triad, \( \mathbf{T}_i \), which is tangent to the deformed beam’s reference line, \( \mathbf{R} \), is therefore created as shown in Figure 2.5. Based on the small strain assumption we can relate the triad \( \mathbf{B}_i \) to \( \mathbf{T}_i \) as given by Yu [27]

\[
\begin{bmatrix}
\mathbf{B}_1 \\
\mathbf{B}_2 \\
\mathbf{B}_3
\end{bmatrix} =
\begin{bmatrix}
1 & -2\gamma_{12} & -2\gamma_{13} \\
2\gamma_{12} & 1 & 0 \\
2\gamma_{13} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{T}_1 \\
\mathbf{T}_2 \\
\mathbf{T}_3
\end{bmatrix}
\] (2.45)

Equation 2.44 then becomes

\[
\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + \mathbf{u}(x_1) + x_\alpha \mathbf{T}_\alpha(x_1) + w_\alpha(x_1, x_2, x_3)\mathbf{T}_i
\] (2.46)

where,

\[
w_1 = \bar{w}_1 + 2\gamma_{1\alpha}(w_\alpha + x_\alpha)
\] (2.47)

\[
w_\alpha = \bar{w}_\alpha - 2\gamma_{1\alpha} w_1
\] (2.48)
We must include constraints on the warping function to define a unique displacement field. The first three constraints are given by
\[ w_i(x_1, 0, 0) = 0 \] (2.49)
where these constraints mean that the displacement is given by the difference between the position vectors of the undeformed and deformed reference line. The last constraint is on the definition of the torsional deformation, which yields
\[ (w_{2,3}(x_1, x_2, x_3) - w_{3,2}(x_1, x_2, x_3)) = 0 \] (2.50)
where, the notation \( \langle \rangle \) means integration over the cross-section and the notation \( w_{i,j} \) means \( \frac{\partial w_i}{\partial x_j} \). We can write the four warping constraints as
\[ \langle \Gamma_e w \rangle = 0 \] (2.51)
where,
\[ \Gamma_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_2} \end{bmatrix} \] (2.52)

### 2.3.2 Strain Field

We denote the one-dimensional strains as

\[ \tilde{\epsilon} = [\tilde{\gamma}_{11} \tilde{\kappa}_1 \tilde{\kappa}_2 \tilde{\kappa}_3]^T \] (2.53)

where, \( \kappa_i \) is based on the change of \( x_i \) in the \( T \) basis.

\[ T'_i = (k_j + \tilde{\kappa}_j)T_j \times T_i \] (2.54)

and, \( k_1 \) and \( k_a \) are the initial twist and initial curvatures in the \( b \) basis. The 3-D stain for small local rotation are given by

\[ \Gamma_{ij} = \frac{1}{2} \left( \chi_{ij} + \chi_{ji} \right) - \delta_{ij} \] (2.55)

where, \( \delta_{ij} \) is the Kronecker delta, and

\[ \chi_{ij} = T_i \cdot G_k g^k \cdot b_j \] (2.56)

where,

\[ g_i = \frac{\partial \hat{r}}{\partial x_i} \] (2.57)

\[ G_i = \frac{\partial \hat{R}}{\partial x_i} \] (2.58)

\[ g^i \cdot g_j = \delta_{ij} \] (2.59)
We can now write the 3-D strain field as a linear function in terms of warping, \( w(x_1, x_2, x_3) \) and 1-D strain measures, \( \vec{\epsilon}(x_1) \) as

\[
\Gamma = \Gamma_a w + \Gamma_\epsilon \vec{\epsilon} + \Gamma_R w + \Gamma_l w'
\]  

(2.60)

Further,

\[
\Gamma_a = \begin{bmatrix}
0 & 0 & 0 \\
\frac{\partial}{\partial x_2} & 0 & 0 \\
\frac{\partial}{\partial x_3} & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & 0 \\
0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_3} \\
0 & 0 & \frac{\partial}{\partial x_3}
\end{bmatrix}
\]  

(2.61)

\[
\Gamma_\epsilon = \frac{1}{\sqrt{g}} \begin{bmatrix}
1 & 0 & x_3 & -x_2 \\
0 & -x_3 & 0 & 0 \\
0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  

(2.62)

\[
\Gamma_R = \frac{1}{\sqrt{g}} \begin{bmatrix}
k^* & -k_3 & k_2 \\
k_3 & k^* & -k_1 \\
-k_2 & k_1 & k^*
\end{bmatrix}
\]  

(2.63)

where, \( k^* = k_1 (x_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}) \), and

\[
\Gamma_l = \frac{1}{\sqrt{g}} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  

(2.64)

where,

\[
\sqrt{g} = g_1 \cdot (g_2 \times g_3)
\]  

(2.65)
2.3.3 Strain Energy

The main assumption for this beam theory is that strain remain small. Therefore the strain can be written using Hooke’s law as

\[ \sigma = D \varepsilon \]

(2.66)

Where \( D \) is the \( 6 \times 6 \) sectional stiffness matrix. Then the strain energy can be written as

\[ U = \int \frac{1}{2} \Gamma^T D \Gamma dV = \int_0^\ell \frac{1}{2} \langle (\Gamma^T D \Gamma) \rangle dx \equiv \int_0^\ell U dx_1 \]

(2.67)

where,

\[ \langle \cdot \rangle = \int_S (\cdot) \sqrt{g} dx_2 dx_3 \]

(2.68)

Therefore, Equation 2.67 becomes

\[ U = \frac{1}{2} \langle (\Gamma^T D \Gamma) \rangle \]

(2.69)

where, \( U \) is the strain energy per unit span. This result for the strain energy will be used in upcoming sections to derive dimensional reduction.

2.4 Variational Calculus and Hamilton’s Principle

The section presents a brief overview of variational calculus and Hamilton’s principle. Both of these concepts will be used in the next chapter for the presentation of dimensional reduction.

2.4.1 Variation of Function

The stationary points of a function, \( F = f(x_1, x_2, \ldots, x_i) \), are defined as the points where \( \frac{\partial F}{\partial x_i} = 0 \). These stationary points can be the maximum, minimum, or saddle points of a function. From the above definition it can also be said that for any arbitrary function, \( w_i \),

\[ \frac{\partial F}{\partial x_1} w_1 + \frac{\partial F}{\partial x_2} w_2 + \ldots + \frac{\partial F}{\partial x_i} w_i = 0. \]

While true, it doesn’t hold a great deal of meaning so we will now refer to the arbitrary functions, \( w_i \), as the variation of \( x_i \), given the notation, \( \delta x_i \). Taking the variation at a stationary point results in a “virtual” change around a stationary
point and do not result in a change in the value of $F$. They are simply an “imaginary” change.

A variation of a function, $F = f(x_1, x_2, ..., x_n)$ is

$$\delta F = \frac{\partial F}{\partial x_1}\delta x_1 + \frac{\partial F}{\partial x_2}\delta x_2 + ... + \frac{\partial F}{\partial x_i}\delta x_i = 0.$$  \hspace{1cm} (2.70)

Where the variation symbol “$\delta$” represents an imaginary, or virtual changes and behaves much like the differential operator. The difference is that the differential operator “$d$” represents an actual, infinitesimal change, whereas the variation operator, “$\delta$” represents a virtual change.

Figure 2.6 shows the difference between the variation and differential of a function. It can simply be stated that $\delta f$ is the virtual change that brings $f(x)$ into $\tilde{f}(x)$.

Figure 2.6: Variation of a function compared to the differential of a function

The following relations between differentials and variations can be shown

$$\frac{d}{dx}(\delta f) = \frac{d}{dx}(\tilde{f} - f) = \frac{df}{dx} - \frac{df}{dx} = \delta \left( \frac{df}{dx} \right)$$ \hspace{1cm} (2.71)

Also, for integrals

$$\delta \left( \int_{a}^{b} f \, dx \right) = \int_{a}^{b} \tilde{f} \, dx - \int_{a}^{b} f \, dx = \int_{a}^{b} (\tilde{f} - f) \, dx = \int_{a}^{b} \delta f \, dx$$ \hspace{1cm} (2.72)
2.4.2 Virtual Displacement Vector

The virtual displacement vector is found using Equation 2.70. For a Cartesian coordinate system the variation of, \( x_i = x_i(q_1, q_2, ..., q_n) \), is given by

\[
\delta x_i = \frac{\partial x_i}{\partial q_1} \delta q_1 + \frac{\partial x_i}{\partial q_2} \delta q_2 + ... + \frac{\partial x_i}{\partial q_n} \delta q_n = 0
\]  
(2.73)

and in terms of the position vector

\[
\delta r = \frac{\partial r}{\partial q_1} \delta q_1 + \frac{\partial r}{\partial q_2} \delta q_2 + ... + \frac{\partial r}{\partial q_n} \delta q_n = 0
\]  
(2.74)

2.4.3 Virtual Rotation Vector

The virtual rotation vector, \( \delta \psi \), is defined as

\[
\delta \psi = \text{axial}(\delta R R^T)
\]  
(2.75)

It is important to point out that notation for this vector is \( \overline{\delta \psi} \), indicating that the vector itself includes the “\( \delta \)”. This indicates that there is no vector \( \delta(\psi) \). The relationship between the virtual rotation and the virtual change in the angular velocity is given by

\[
\delta \omega = \overline{\delta \psi} - \overline{\omega \delta \psi}
\]  
(2.76)

The virtual change in the angular velocity can also be defined in the rotating frame as

\[
\delta \omega = \overline{\delta \psi} - \overline{\omega \delta \psi}, \quad \delta \omega = \overline{R \delta \psi}^* \quad \delta \omega^* = \overline{\delta \psi}^* + \overline{\omega^* \delta \psi}^*, \quad \delta \omega^* = \overline{R^T \delta \psi}
\]  
(2.77)

2.4.4 Virtual Work

The virtual work, \( \delta W \), performed by external forces by a virtual distance \( \delta r \) is given by

\[
\delta W = F^T \delta r
\]  
(2.79)

Using Equation 2.74 we find

\[
\delta W = F^T \left( \frac{\partial r}{\partial q_1} \delta q_1 + \frac{\partial r}{\partial q_2} \delta q_2 + ... + \frac{\partial r}{\partial q_n} \delta q_n \right)
\]  
(2.80)
2.4.5 Hamilton’s Principle

For elastodynamic deformation of a structure the extended Hamilton’s principle can be written as [30]

\[
\int_{t_1}^{t_2} \left[ \delta (K - U) + \delta W \right] dt = \delta A
\]  

(2.81)

where, \( t_1 \) and \( t_2 \) are arbitrary fixed times, \( K \) and \( U \) are the kinetic and strain energy per unit length, \( \delta W \) is the virtual work of the applied loads, and \( \delta A \) is the virtual action at the ends of the time interval. The 3-D functionals in Hamilton’s extended principle may also be defined in terms of 1-D functionals as

\[
\int_{t_1}^{t_2} \left[ \delta (K_{1D} + K^* - U) + \delta W_{1D} + \delta W^* \right] dt = \delta A
\]  

(2.82)

where \( K_{1D} \), \( \delta W_{1D} \), and \( \delta A \) are expressed only in terms of 1-D variables. The remaining terms are functions of 3-D variables. Since the cross-section of a beam is much smaller than the length of the beam we can use an asymptotic expansion (covered in detail in the next section) of Equation 2.82 to get the 1-D variational statement

\[
\int_{t_1}^{t_2} \int_{0}^{\ell} \left[ \delta (K - U) + \delta W \right] dx_1 dt = \delta A
\]  

(2.83)

where \( \ell \) is the length of the beam, \( K \) is the 1-D kinetic energy density, \( U \) is the 1-D strain energy density, and \( \delta W \) is the 1-D virtual work density. For the derivation of BeamDyn the virtual actions are ignored.
CHAPTER 3  
DIMENSIONAL REDUCTION

The purpose of this chapter is to present the dimensional reduction of the 3-D beam theory into a 2-D cross-sectional analysis and a 1-D beam theory as shown by Berdichevsky [15]. First, the equation of strain energy will be analyzed using the variational asymptotic method. This will yield 1-D constitutive equations that will be used for the 1-D beam analysis in the next chapter.

3.1 Variational Asymptotic Method

The VAM is employed to minimize a functional depending on a small parameter, where a functional is defined as a function of a function. For example, the strain energy is given by $U = f(\Gamma)$, and $\Gamma = f(x_1, x_2, x_3)$, therefore the strain energy is the functional. In this instance the purpose is to find a function (in the vector space) which minimizes the functional (i.e., $\frac{\partial F}{\partial x} = 0$). The approach is to then find a warping function in Equation 2.60 to minimize the strain energy of the beam given in Equation 2.69. Then, the 1-D strain, $\tilde{\epsilon}$, will be solved through a 1-D global analysis as introduced in Chapter 4. It should be noted that this process cannot be done exactly, there are small errors associated with this method due to the removal of higher order terms [8].

The first step in dimensional reduction is asymptotic analysis, which is covered extensively in Hodges [8] and Yu [27]. This step requires that we consider the order of magnitude of the terms in Equation 2.82. As detailed in Section 2.4.5, $K^*$ and $\delta W^*$ (which contain 3-D variables $w$) are high order terms asymptotically speaking, and are therefore eliminated from the analysis. It should be noted that Equation 2.69 is the 3-D representation of strain energy for the elastic beam problem. The goal is to minimize this equation subject to the warping constraints in Equation 2.51 to find the unknown warping functions as this is the
only remaining term that carries 3-D variables in 2.82. In order to find the warping functions
the finite element method is used to define the warping field as
\[
\mathbf{w}(x_1, x_2, x_3) = S(x_2, x_3) V(x_1)
\]  
(3.1)
where \( V \) is the nodal value of the warping displacement, and \( S(x_2, x_3) \) is the shape function. Substituting into Equation 2.69 we get
\[
2\mathcal{U} = V^T E V + 2V^T (D_{ae} \mathbf{e} + D_{aR} V + D_{al} V') + \mathbf{e}^T D_{ee} \mathbf{e} + V^T D_{RR} V + V^T D_{ll} V' + 2V^T D_{Re} \mathbf{e} + 2V^T D_{lR} V'
\]  
(3.2)
where,
\[
E = \langle \langle [\Gamma_a S]^T D [\Gamma_a S] \rangle \rangle \quad D_{ae} = \langle \langle [\Gamma_a S]^T D [\Gamma_e S] \rangle \rangle \\
D_{aR} = \langle \langle [\Gamma_a S]^T D [\Gamma_R S] \rangle \rangle \quad D_{al} = \langle \langle [\Gamma_a S]^T D [\Gamma_l S] \rangle \rangle \\
D_{ee} = \langle \langle [\Gamma_e S]^T D [\Gamma_e S] \rangle \rangle \quad D_{RR} = \langle \langle [\Gamma_R S]^T D [\Gamma_R S] \rangle \rangle \\
D_{ll} = \langle \langle [\Gamma_l S]^T D [\Gamma_l S] \rangle \rangle \quad D_{Re} = \langle \langle [\Gamma_R S]^T D [\Gamma_e S] \rangle \rangle \\
D_{lR} = \langle \langle [\Gamma_R S]^T D [\Gamma_l S] \rangle \rangle
\]  
(3.3)
and \( D \) is the 6 × 6 elastic material constant matrix. We may also define the kernel, \( \psi \), of matrix \( \Gamma_a \) (Equation 2.61) as
\[
\psi = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -x_3 \\
0 & 0 & 1 & x_2
\end{bmatrix}
\]  
(3.4)
and,
\[
\psi = S \Psi
\]  
(3.5)
where, \( \Psi \) are the nodal values of \( \psi \).

In order minimize the strain energy in Equation 2.69, we first assume that since the structure is a beam (which means that the length is much larger than the height and width of the cross-section) \( a/l \ll 1 \) and \( a/R \ll 1 \), where \( l \) is the characteristic wavelength of deformation along the beam, \( R \) is the characteristic radius of initial curvature/twist of the beam, and \( a \) is the characteristic length of the cross-section. For the purposes of this analysis
it is safe to assume that $l$ and $R$ are of the same order. 

We then assume that the characteristic strain magnitude of the 1-D and 3-D strains, $\hat{\varepsilon}$, is

$$\hat{\varepsilon} = O(\bar{\varepsilon}) = O(\varepsilon) = O(\Gamma)^{\ll 1} \quad (3.6)$$

and,

$$w_i = O(a\bar{\varepsilon}) \quad (3.7)$$

It can be seen that the last term of Equation 2.69 includes a spatial derivative, thereby reducing the magnitude by an order of $a$. Now the magnitude of each of the terms in Equation 3.2 is determined. For example, $V$ is of the order of the warping function, i.e., $O(a\bar{\varepsilon})$. The term $E$ is on the order of the elastic constants and is denoted as $v$. Therefore, the term $V^T EV$ is of the order $O(a^2\bar{\varepsilon}^2v)$. Using this method for each term in Equation 3.2 and getting rid of the higher order terms we get the zeroth-order approximation for the strain energy

$$2\mathcal{U}_0 = V^T EV + 2V^T D_{ae} \bar{\varepsilon} + \bar{\varepsilon}^T D_{ee} \bar{\varepsilon} \quad (3.8)$$

We then set the first variation of Equation 3.8 equal to zero and discretize the warping constrains in Equation 2.51 by using Equation 3.1 to give $V^T(\Gamma_c S) = 0$. We then use a Lagrange multiplier to get

$$EV = -D_{ae} \bar{\varepsilon} \quad (3.9)$$

The compete solution for Equation 3.9 is expressed as

$$V = V^* + \Psi \lambda \quad (3.10)$$

where $V^*$ is the solution of Equation 3.9 which is linearly independent of the null space, and $\lambda$ is determined from the warping constrains $V^T(\Gamma_c S) = 0$. Finally, solving for the warping function, $V$, which minimizes Equation 3.8 we get the expression

$$V = \left[ I - \Psi(\hat{D}_c^T \Psi)^{-1} \hat{D}_c^T \right] \hat{V}_0 \bar{\varepsilon} = V_0$$ \quad (3.11)
where, $D_T = \langle \Gamma_c S \rangle$, $V_0$ is the zeroth-order values of $V$, and $\hat{V}_0$ is the zeroth-order warping influence coefficients. Finally, plugging back into Equation 3.8 we get a 1-D constitutive equation that is asymptotically correct up to $O(v^2)$ without correction for initial curvature and twist

$$2\mathcal{U}_0 = \varepsilon^T (\hat{V}_0^T D_{ae} + D_{ee}) \hat{\varepsilon}$$

in matrix form,

$$2\mathcal{U}_0 = \begin{bmatrix} \ddot{y}_{11} \\ \ddot{k}_1 \\ \ddot{k}_2 \\ \ddot{k}_3 \end{bmatrix}^T \begin{bmatrix} \ddot{S}_{11} & \ddot{S}_{12} & \ddot{S}_{13} & \ddot{S}_{14} \\ \ddot{S}_{21} & \ddot{S}_{22} & \ddot{S}_{23} & \ddot{S}_{24} \\ \ddot{S}_{31} & \ddot{S}_{32} & \ddot{S}_{33} & \ddot{S}_{34} \\ \ddot{S}_{41} & \ddot{S}_{42} & \ddot{S}_{43} & \ddot{S}_{44} \end{bmatrix} \begin{bmatrix} \ddot{y}_{11} \\ \ddot{k}_1 \\ \ddot{k}_2 \\ \ddot{k}_3 \end{bmatrix}$$

which implies the 1-D constitutive law

$$\begin{bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} \ddot{S}_{11} & \ddot{S}_{12} & \ddot{S}_{13} & \ddot{S}_{14} \\ \ddot{S}_{21} & \ddot{S}_{22} & \ddot{S}_{23} & \ddot{S}_{24} \\ \ddot{S}_{31} & \ddot{S}_{32} & \ddot{S}_{33} & \ddot{S}_{34} \\ \ddot{S}_{41} & \ddot{S}_{42} & \ddot{S}_{43} & \ddot{S}_{44} \end{bmatrix} \begin{bmatrix} \ddot{y}_{11} \\ \ddot{k}_1 \\ \ddot{k}_2 \\ \ddot{k}_3 \end{bmatrix}$$

(3.14)

As stated above, the result in Equation 3.14 does not contain a correction for initial curvature or twist. The result also does not account for transverse shear strains. Therefore this derivation represents the generalized Euler-Bernoulli model. While this model does have applications, it is not the model we seek for the geometrically exact beam theory which is a Timoshenko-like beam model. In order arrive at the Timoshenko-like model one must also consider the first-order approximation to minimization of the strain energy. This is done in the same way as was illustrated before but by perturbing the unknown warping function, $V$, as $V = V_0 + V_1$, where $V_0$ was found in 3.11. The result is a full $6 \times 6$ stiffness matrix with coupling terms. The strain energy is given by

$$2\mathcal{U} = \begin{bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix}$$

(3.15)
and 1-D constitutive equation is

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}
\end{bmatrix} \begin{bmatrix}
\gamma_{11} \\
\gamma_{12} \\
\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\] (3.16)

We can see that the strain includes all shear terms and is given by

\[
\varepsilon = \begin{bmatrix}
\gamma_{11} & 2\gamma_{12} & 2\gamma_{13} & \kappa_1 & \kappa_2 & \kappa_3
\end{bmatrix}^T
\] (3.17)

The cross-sectional mass matrix, which is used in dynamic simulations, is given by

\[
\mathcal{M}^* = \begin{bmatrix}
m & 0 & 0 & 0 & m\eta_{x3}^* & -m\eta_{x2}^* \\
0 & m & 0 & -m\eta_{x3}^* & 0 & 0 \\
0 & 0 & m & m\eta_{x2}^* & 0 & 0 \\
0 & -m\eta_{x3}^* & m\eta_{x2}^* & i_{11}^* & 0 & 0 \\
m\eta_{x3}^* & 0 & 0 & 0 & i_{22}^* & i_{23}^* \\
-m\eta_{x2}^* & 0 & 0 & 0 & i_{23}^* & i_{33}^*
\end{bmatrix}
\] (3.18)

which can also be written in compact form as

\[
\mathcal{M}^* = \begin{bmatrix}
\frac{mL}{m} & m\tilde{\eta}^* \\
m\tilde{\eta}^* & \tilde{\varphi}^*
\end{bmatrix}
\] (3.19)

where \(m\) is the mass per unit length, \(\eta^*\) are the components of the position vector of the sectional center of mass with respect to the reference line, and \(\varphi^*\) the sectional moment of inertia tensor per unit length.

### 3.1.1 VABS

For the purposes of BeamDyn we rely on a preprocessor such as the Variational Asymptotic Beam Sectional Analysis (VABS) tool to calculate the \(6 \times 6\) stiffness matrix and mass matrix shown in Equations 3.15 and 3.18, respectively. VABS is based on the formulation presented in this chapter and is asymptotically correct up to the first-order. Figure 3.1 shows how the 3-D beam analysis is broken into a 2-D cross-sectional analysis, and a 1-D nonlinear beam analysis. For the purposes of BeamDyn, VABS completes all actions in the red shaded...
areas, and BeamDyn solves the 1-D beam problem presented in the next chapter. Inputs to the 2-D cross-sectional analysis are the cross-sectional geometry, 3-D elastic constants, density, initial twist and curvature of the beam. It should also be noted here that the analysis for this project stops short of the 3-D recovery analysis which is an additional step shown here to highlight the full capability of this method.

Figure 3.1: Beam Analysis Procedure[8].
CHAPTER 4
1-D BEAM THEORY

The intent of this chapter is to present the 1-D beam theory and also detail its finite element implementation for BeamDyn. This approach is detailed in the work by Bauchau [26] and Hodges [8].

4.1 The Sectional Strain Measures

The strain measures for BeamDyn are solved in the inertial frame, which is not necessarily coincident with the reference frame shown in Figure 2.4. The rotation that brings the inertial frame to the undeformed reference frame is $R_0$. The rotation that brings the undeformed reference frame to the deformed reference frame is $R$. The sectional strain measures for beams with shallow curvature are defined as

$$
\varepsilon = \begin{bmatrix} \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} x_0' + u' - (R R_0)^T \tilde{n}_1 \\ \frac{k}{k} \end{bmatrix}
$$

(4.1)

where $k = \text{axial} \left( (R R_0)'(R R_0)^T \right)$ and is the sectional curvature vector resolved in the inertial basis as shown in Equation 2.41.

The strain components given in Equation 4.1 resolved in the material basis, are denoted $\varepsilon^* = (R R_0)^T \varepsilon$. Likewise, the curvature components are denoted $\kappa^* = (R R_0)^T \kappa$.

4.2 Governing Equations for the Static Problem

In order to derive a set of governing equations of motion for the static problem it is necessary to first consider Hamilton’s principle in 2.83. We find that the only terms that remain for the static case are the strain energy and virtual work terms. In the material basis the strain energy is given by

$$
\int_0^\ell \delta U^* \, dx_1 = \int_0^\ell \left[ \delta \varepsilon^* T \left( \frac{\partial U^*}{\partial \varepsilon^*} \right)^T + \delta \kappa^* T \left( \frac{\partial U^*}{\partial \kappa^*} \right)^T \right] \, dx_1
$$

(4.2)
where

\[
\begin{align*}
E^* &= \left( \frac{\partial U^*}{\partial \epsilon^*} \right)^T \\
M^* &= \left( \frac{\partial U^*}{\partial \kappa^*} \right)^T
\end{align*}
\]  

(4.3)

where $E^*$ and $M^*$ are the beam’s sectional forces and moments, respectively. The constitutive law can be written as

\[
\begin{bmatrix} E^* \\ M^* \end{bmatrix} = C^* \begin{bmatrix} \epsilon^* \\ \kappa^* \end{bmatrix}
\]

(4.4)

where $C^*$ is the beam’s $6 \times 6$ sectional stiffness matrix in the material basis, and is given by a preprocessor such as VABS as discussed in Chapter 3.

Next, it is necessary to define the variations in strain components in order to substitute into the expression for virtual work. The variation in strain components are found using Equations 4.1 and 2.70

\[
\begin{align*}
\delta \epsilon^* &= (R R_0)^T \left[ \delta u' + (\bar{x}' + \bar{u}') \delta \psi \right] \\
\delta \kappa^* &= (R R_0)^T \delta \psi'
\end{align*}
\]

(4.5a, 4.5b)

where $\delta \psi = \text{axial}(\delta)(R R_0^T)$ is the virtual rotation vector as shown in Equation 2.75. Equation 4.2 then becomes

\[
\int_0^\ell \left\{ \left[ \delta \bar{u}^T + \delta \psi^T (\bar{x}' + \bar{u}') \right] F + \delta \psi^T M \right\} \, dx_1
\]

(4.6)

where $F = (R R_0) E^*$ and $M = (R R_0) M^*$ are the beam’s internal forces and moments, respectively, resolved in the inertial basis.

The virtual work is expressed as

\[
\delta W = \int_0^\ell \left[ \delta \bar{u}^T f + \delta \psi^T m \right] \, dx_1
\]

31
where \( f \) and \( m \) are the externally applied forces and moments per unit span of the beam, respectively. Equation 2.83 then becomes

\[
\int_0^\ell \left\{ -\left( \delta u' T + \delta \psi' T (\ddot{x}_0 + \ddot{u}) T \right) F - \delta \psi' T M + \delta u' T f + \delta \psi' T m \right\} \, dx_1
\]

Integration by parts yields the governing equations of motion for the static problem

\[
F' = -f, \quad (4.8a)
\]
\[
M + (\ddot{x}_0 + \ddot{u}) F = -m. \quad (4.8b)
\]

### 4.3 Dynamic Problems

For dynamic problems it is necessary to solve for the kinetic energy term in Equation 2.83. The inertial velocity vector, \( \mathbf{v} \), of a material point is found by taking a time derivative of its inertial position vector, and using Equation 2.33, to find

\[
\mathbf{v} = \dot{\mathbf{u}} + (\mathbf{R} \mathbf{R}_0) \mathbf{s}^* = \dot{\mathbf{u}} + (\mathbf{R} \mathbf{R}_0) \mathbf{\omega}^* \mathbf{s}^*
\]

where \( \mathbf{s}^* = \{0, x_2, x_3\} \). Using tilde transformation Equations A.4 and A.2, we get

\[
\mathbf{v} = \dot{\mathbf{u}} + (\mathbf{R} \mathbf{R}_0) \tilde{\mathbf{s}}^* T \mathbf{\omega}^*
\]

The contributions of warping of the cross-section may be ignored for the beam analysis since it has been considered for the 2-D sectional analysis [8]. The velocity vector resolved in the material frame is given by

\[
\mathbf{v}^* = (\mathbf{R} \mathbf{R}_0)^T \mathbf{v} = (\mathbf{R} \mathbf{R}_0)^T \dot{\mathbf{u}} + \tilde{\mathbf{s}}^* T \mathbf{\omega}^*
\]

### 4.3.1 Kinetic Energy

The kinetic energy per unit length of the beam is defined as

\[
\mathcal{K} = \frac{1}{2} \int_0^\ell \int_{\mathcal{A}} \rho \mathbf{v}^* T \mathbf{v}^* \, d\mathcal{A} \, dx_1
\]
where $\rho$ is the mass density of the material per unit volume of the reference configuration.

Substituting Equation 4.11 we get

$$ K = \frac{1}{2} \int_0^\ell \int_A \rho \left[ \dot{u}^T (R R_0) + \omega^T \ddot{s}^* \right] \left[ (R R_0)^T \dot{u} + \ddot{s}^* T \omega^* \right] \, dA \, dx_1 $$

(4.13)

where,

$$ m = \int_A \rho \, dA, \quad \eta^* = \frac{1}{m} \int_A \rho \ddot{s}^* \, dA, \quad \varrho^* = \int_A \rho \ddot{s}^* \dot{s}^* \, dA $$

(4.14)

and are the same variables defined in Equation 3.19.

Integrating over the beam’s cross-section Equation 4.13, becomes

$$ K = \frac{1}{2} \int_0^\ell \left[ m \ddot{u}^T + 2m \ddot{u}^T (R R_0) \ddot{\eta}^* T \omega^* + \omega^* T \ddot{\varrho}^* \omega^* \right] \, dx_1 $$

$$ = \frac{1}{2} \int_0^\ell \left[ \ddot{\gamma}^* T \mathcal{M}^* \ddot{\gamma}^* \right] \, dx_1 $$

(4.15)

(4.16)

where $\mathcal{M}^*$ is the sectional mass matrix, resolved in the material basis, and is given by 3.19. $\mathcal{M}$ resolved in the inertial frame is given by

$$ \mathcal{M} = (R R_0) \mathcal{M}^*(R R_0)^T $$

(4.17)

the location of the center of mass and moment of inertia tensor resolved in the inertial frame, are defined as $\eta = (R R_0) \eta^*$, and $\varrho = (R R_0) \varrho^* (R R_0)^T$ respectively. The term $R R_0$ is a 6 $\times$ 6 rotation matrix given by

$$ (R R_0) = \begin{bmatrix} (R R_0) & 0 \\ 0 & (R R_0) \end{bmatrix} $$

(4.18)

The sectional velocities resolved in the material basis are given by

$$ \gamma^* = \begin{bmatrix} (R R_0)^T \dot{u} \\ \omega^* \end{bmatrix} = \begin{bmatrix} (R R_0)^T & 0 \\ 0 & (R R_0)^T \end{bmatrix} \begin{bmatrix} \dot{u} \\ \omega \end{bmatrix} = (R R_0)^T \gamma $$

(4.19)

it can be seen that

$$ \gamma = \begin{bmatrix} \dot{u} \\ \omega \end{bmatrix} $$

(4.20)
At this time we can also define the sectional linear and angular momenta resolved in the material system, denoted $\mathbf{h}^*$ and $\mathbf{g}^*$ as

$$ P^* = \begin{bmatrix} h^* \\ g^* \end{bmatrix} = \mathbf{M}^* \mathbf{V}^* $$

(4.21)

and the sectional linear and angular momenta resolved in the inertial system is given by

$$ P = \begin{bmatrix} h \\ g \end{bmatrix} = (\mathbf{R} \mathbf{R}_0) P^* = \mathbf{M} \mathbf{V} $$

(4.22)

### 4.3.2 The Governing Equations of Motion for Dynamic Problems

Next, to find the kinetic energy term in Equation 2.83 we take the variation of the kinetic energy

$$ \int_0^\ell \delta K \! dx_1 = \int_0^\ell \left[ \delta \dot{\mathbf{u}}^* T \left( \frac{\partial K}{\partial \ddot{\mathbf{u}}^*} \right)^T + \delta \omega^* T \left( \frac{\partial K}{\partial \omega^*} \right)^T \right] \! dx_1 $$

(4.23)

where,

$$ h^* = \left( \frac{\partial K}{\partial \dot{\mathbf{u}}^*} \right)^T $$

(4.24)

$$ g^* = \left( \frac{\partial K}{\partial \omega^*} \right)^T $$

(4.25)

or in the compact form of Equation 4.16

$$ \int_0^\ell \delta K \! dx_1 = \int_0^L \delta \mathbf{V}^* T \mathbf{M}^* \mathbf{V}^* \! dx_1 $$

(4.26)

by the applying the variation and Equation 2.78 we get

$$ \delta \begin{bmatrix} \dot{\mathbf{u}}^T (\mathbf{R} \mathbf{R}_0) \\ \dot{\omega}^T \end{bmatrix} = (\delta \dot{\mathbf{u}}^T + \delta \dot{\psi} \dot{\mathbf{u}}^T) (\mathbf{R} \mathbf{R}_0) $$

(4.27)

$$ \delta \omega^* T = \dot{\mathbf{V}}^* \mathbf{R} \mathbf{R}_0 $$

(4.28)

Combining this result with Equations 4.21 and 4.23 we get

$$ \int_0^\ell \delta K \! dx_1 = \int_0^\ell (\delta \dot{\mathbf{u}}^T h + \delta \dot{\psi} \dot{\mathbf{u}}^T h + \delta \dot{\mathbf{V}}^T \! g) \! dx_1 $$

(4.29)
Adding this term to the virtual work term in Equation 4.7 gives a complete expression for Hamilton’s principle, and therefore the governing equations of motion for the dynamic problem are

\[
\int_{t_i}^{t_f} \int_{0}^{\ell} \left\{ (\delta \dot{u}^T + \delta \psi^T \dot{u}^T) h + \delta \dot{\psi} g - (\delta \dot{u}^T + \delta \psi^T \dot{E}_1^T) F \right. \\
\left. - \delta \psi^T M + \delta u^T f + \delta \psi^T m \right\} \text{d}x_1 \text{d}t = 0
\]

where \( E_1 = \xi_0' + \xi' \). As before, integration by parts yields the governing equations of motion

\[
\begin{align*}
\dot{\dot{h}} - F' &= f, \quad (4.30a) \\
\dot{\delta} + \dot{h} h - M' - (\xi_0' + \dot{\xi}') F &= m. \quad (4.30b)
\end{align*}
\]

### 4.4 Inertial Forces

The inertial forces acting in the beam are obtained from the governing equations of motion, Equations 4.30.

\[
\mathcal{F}^I = \dot{\mathcal{P}} + \begin{bmatrix} 0 & 0 \\ \dot{u} & 0 \end{bmatrix} \mathcal{P} 
\]

(4.31)

The expression for \( \mathcal{P} \) is given by

\[
\mathcal{P} = \begin{bmatrix} m \ddot{u} & m \ddot{n}^T \\ m \ddot{n} & \ddot{g} \end{bmatrix} \begin{bmatrix} u \\ \omega \end{bmatrix} = \begin{bmatrix} m \ddot{u} + m \ddot{n}^T \omega \\ m \ddot{n} + \ddot{g} \omega \end{bmatrix} 
\]

(4.32)

The next step is to find the time derivative of Equation 4.32. Using the chain rule and the identities: \( m \ddot{n} = \ddot{\omega} m \ddot{g} \), and \( \ddot{g} = \ddot{\omega} + \ddot{\omega}^T \), we have

\[
\dot{\mathcal{P}} = \begin{bmatrix}
m \dddot{u} + (\ddot{\omega} + \ddot{\omega}^T) m \dddot{n} \\
m \dddot{n} + \dddot{\omega} m \dddot{g} + \dddot{\omega} m \dddot{g} + \dddot{\omega}
\end{bmatrix} 
\]

(4.33)

Equation 4.31 can now be written in as

\[
\mathcal{F}^I = \begin{bmatrix} m \dddot{u} + (\ddot{\omega} + \ddot{\omega}^T) m \dddot{n} \\ m \dddot{n} + \dddot{\omega} + \dddot{\omega} m \dddot{g} \end{bmatrix} 
\]

(4.34)
4.5 Elastic Forces

Similar to the inertial forces, the elastic forces acting in the beam element are obtained from the governing equations of motion, Equations 4.30, defined as

$$\mathcal{F}^C = \begin{bmatrix} E \\ M \end{bmatrix}, \quad \text{and} \quad \mathcal{F}^D = \begin{bmatrix} 0 & 0 \\ E_1^T & 0 \end{bmatrix} \begin{bmatrix} E \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ E_1^T E \end{bmatrix}$$

(4.35)

and the strain components are given by

$$e = \begin{bmatrix} E_1 - (R R_0) \hat{n} \end{bmatrix}$$

(4.36)

where $\kappa$ is the curvature vector given in Section 2.2.4 as, $\kappa = \text{axial}(R'R_T)$. The elastic forces are then give by

$$\begin{bmatrix} E \\ M \end{bmatrix} = C e$$

(4.37)

where $C$ is the sectional stiffness matrix.

4.6 Linearization of Forces

Both the inertial and elastic forces are nonlinear and the finite element process will require linearization. From Bauchau [26] the linearization of the inertial forces then yields

$$\Delta \mathcal{F}^I = \kappa^I \begin{bmatrix} \Delta u \\ \Delta \psi \end{bmatrix} + \mathcal{G}^I \begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\psi} \end{bmatrix} + \mathcal{M}^I \begin{bmatrix} \Delta \ddot{u} \\ \Delta \ddot{\psi} \end{bmatrix}$$

(4.38)

where $\kappa^I$, $\mathcal{G}^I$, and $\mathcal{M}^I$ are the stiffness, gyroscopic, and mass matrices associated with the inertial forces, and are given by

$$\kappa^I = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathcal{G}^I = \begin{bmatrix} (\hat{w} + \hat{\omega} \hat{n}) m \hat{n}^T \\ \hat{w} \hat{m} + (\hat{\omega} \hat{\omega} \hat{n} - \hat{\omega} \hat{n} \hat{\omega}) \end{bmatrix}$$

(4.39a)

$$\mathcal{G}^I = \begin{bmatrix} (\hat{\omega} \hat{m})^T + \hat{\omega} m \hat{n}^T \\ 0 \\ \hat{\omega} \hat{q} - \hat{\omega} \hat{\omega} \hat{q} \end{bmatrix}$$

(4.39b)

$$\mathcal{M}^I = \begin{bmatrix} m I & m \hat{n}^T \\ m \hat{n} & q \end{bmatrix}$$

(4.39c)
Substituting these results back into 4.38 we get

\[ \Delta \mathcal{F}' = \mathcal{K}' \Delta \mathbf{q} + \mathcal{G}' \Delta \mathbf{v} + \mathcal{M}' \Delta \mathbf{a} \]  

(4.40)

The linearization for the elastic forces are given below

\[ \Delta \left\{ \frac{F}{M} \right\} = \left\{ \frac{\tilde{F}^T \Delta \psi}{\tilde{M}^T \Delta \psi} \right\} + \mathcal{C} \left\{ \frac{\tilde{E}_1 \Delta \psi + \Delta u'}{\Delta \psi'} \right\} \]

Taking variations of Equation 4.35 yields the following expression for increments in the elastic forces

\[ \Delta \mathcal{F}^C = \mathcal{C} \left\{ \frac{\Delta u'}{\Delta \psi} \right\} + \mathcal{O} \left\{ \frac{\Delta u}{\Delta \psi} \right\}, \quad \Delta \mathcal{F}^D = \mathcal{P} \left\{ \frac{\Delta u'}{\Delta \psi} \right\} + \mathcal{Q} \left\{ \frac{\Delta u}{\Delta \psi} \right\} \]  

(4.41)

where the stiffness matrices are

\[ \mathcal{O} = \begin{bmatrix} 0 & C_{11} \tilde{E}_1 - \tilde{F} \\ 0 & C_{21} \tilde{E}_1 - \tilde{M} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0 & 0 \\ \tilde{F} + (C_{11} \tilde{E}_1)^T & (C_{21} \tilde{E}_1)^T \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{E}_1^T O_{12} \end{bmatrix} \]  

(4.42)

The following were introduced

\[ \mathcal{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & O_{12} \\ 0 & O_{22} \end{bmatrix} \]  

(4.43)

Substituting these results back into 4.41 we get

\[ \Delta \mathcal{F}^C = \mathcal{C} \Delta \mathbf{q}' + \mathcal{O} \Delta \mathbf{q}, \quad \Delta \mathcal{F}^D = \mathcal{P} \Delta \mathbf{q}' + \mathcal{Q} \Delta \mathbf{q} \]  

(4.44)

4.7 Gravity Forces

Gravitational forces are given by

\[ \mathcal{F}^G = \begin{bmatrix} mg \\ m \ddot{z} \end{bmatrix} \]  

(4.45)

where \( g \) is the acceleration due to gravity.

4.8 Finite Element Implementation

We are now able to implement the finite element method. The equations of motion of curved beams can be written as, \( \mathcal{F}' - \mathcal{F}' + \mathcal{F}^D = \mathcal{F}^G + \mathcal{F}^ext \), where \( \mathcal{F}^ext \) are the external
forces applied to the beam. Next a weighted residual formulation is used to give

\[ \int_{0}^{\ell} N^T \left( F^I - F^{C'} + F^D - F^G - F^{ext} \right) \, dx_1 = 0 \]

where \( N \) a matrix storing the spectral basis functions. Through integration by parts we have

\[ \int_{0}^{\ell} \left( N^T F^I + N'^T F^C + N^T F^D \right) \, dx_1 = \int_{0}^{\ell} N^T (F^G + F^{ext}) \, dx_1 \]

As discussed before, the values for \( F^I, F^C, \) and \( F^D \) are nonlinear. As such, the linearized force expressions (Equations 4.40, and 4.44) may be substituted to give

\[ \int_{0}^{\ell} \left[ N^T \left( F^I + K^I \Delta q + G^I \Delta v + M^I \Delta a + F^D + P \Delta q' + Q \Delta q \right) \right. \]
\[ + N'^T \left( F^C + C \Delta q' + O \Delta q \right) \left. \right] \, dx_1 = \int_{0}^{\ell} N^T (F^G + F^{ext}) \, dx_1 \]

The elemental displacement, velocity, and acceleration fields may be expressed in terms of their nodal values using the shape functions,

\[ \hat{q}(x_1) = N \hat{\dot{q}}, \]
\[ \hat{q}'(x_1) = N' \hat{\ddot{q}}, \]
\[ \hat{v}(x_1) = N \hat{\dot{v}}, \]
\[ \hat{a}(x_1) = N \hat{\ddot{a}}, \]

where \( \hat{q}, \hat{\dot{v}}, \) and \( \hat{\ddot{a}} \) are the nodal values of the displacements, velocities, and accelerations, respectively. The governing finite element expression is

\[ \hat{M} \Delta \hat{a} + \hat{G} \Delta \hat{\dot{v}} + \hat{K} \Delta \hat{\ddot{q}} = \hat{F}^G + \hat{F}^{ext} - \hat{F} \]  \hspace{1cm} (4.46)

where,

\[ \hat{M} = \int_{0}^{\ell} N^T M^I N \, dx_1 \]  \hspace{1cm} (4.47)
\[ \hat{G} = \int_{0}^{\ell} N^T G^I N \, dx_1 \]  \hspace{1cm} (4.48)
\[ \hat{K} = \int_0^\ell \left[ N^T \left( \kappa' + Q \right) N + N^T \mathcal{P} N' + N^T \mathcal{C} N' + N^T \mathcal{O} N \right] \, dx_1 \] (4.49)

and the forces are given by

\[ \hat{F} = \int_0^\ell \left( N^T \mathcal{F}^I + N^T \mathcal{F}^D + N^T \mathcal{F}^C \right) \, dx_1 \] (4.50a)

\[ \hat{F}^G = \int_0^\ell N^T \mathcal{F}^G \, dx_1 \] (4.50b)

\[ \hat{F}^{ext} = \int_0^\ell N^T \mathcal{F}^{ext} \, dx_1 \] (4.50c)

### 4.9 Interpolation Strategy

The displacement, velocity, and acceleration fields are obtained through interpolation as [25]

\[ u(\xi) = \sum_{k=1}^{p+1} h^k \hat{u}^k \] (4.51)

\[ u'(\xi) = \sum_{k=1}^{p+1} h'{}^k \hat{u}^k \] (4.52)

where \( h^k(\xi) \) are the components of the shape function matrix \( N \). \( h^k(\xi) \) is a \( p^{th} \)-order polynomial Lagrangian-interpolant shape function at node \( k \), \( k = \{1, 2, ..., p + 1\} \). \( \hat{u}^k \) are the nodal values at the \( k^{th} \) node, and \( \xi \in [-1, 1] \) are the natural coordinates.

As discussed in Sections 2.1.3 and 2.2.1 we must use specialized composition rules for rotation parameters. Therefore we adopt the interpolation approach proposed by Jelenić and Crisfield [29]. This is done by first removing the reference rotation, \( \hat{C}^1 \), from the calculation of the finite rotation at each node and then calculating the relative rotation at each node, \( \hat{r}^k \)

\[ \hat{r}^k = (\hat{C}^{1-}) \oplus \hat{c}^k \] (4.53)
where the minus on \((\hat{c}^{1-})\) means that the relative rotation is calculated by removing the reference rotation from each node. Next, we must interpolate the rotation field as

\[
\begin{align*}
\bar{r}(\xi) &= \sum_{k=1}^{p+1} h^k \hat{c}^k \\
\bar{r}'(\xi) &= \sum_{k=1}^{p+1} h'^k \hat{c}^k
\end{align*}
\tag{4.54}
\]

The curvature field \(\kappa(\xi)\) is given by

\[
\kappa(\xi) = R(\hat{c}^{1}) H(\bar{r}) \bar{r}'
\tag{4.55}
\]

where \(H\) is the tangent tensor as given in Equation 2.36. We can also define the curvature vector in terms of the rotation vector as

\[
k = H p'
\tag{4.57}
\]

Finally the rigid-body rotation is restored to give

\[
c(\xi) = \hat{c}^{1} \oplus \bar{r}(\xi)
\tag{4.58}
\]

### 4.10 Spectral Finite Elements

As stated above we now make use of spectral finite-element shape functions. Spectral FEs display many advantages over low-order, \(h\)-type, elements. For example, spectral FEs have a higher accuracy for a set number of DOFs, and better computational efficiency [23], provided that the underlying solution is sufficiently smooth.

For BeamDyn we use \(p^{th}\)-order Lagrangian interpolants with nodes located at the \(p + 1\) Gauss-Lobatto-Legendre (GLL) points with \(p\)-point Gauss-Legendre (GL) quadrature. The location of the GLL points are given by the zeros of

\[
(1 - \xi^2) \frac{dL_p}{d\xi} = 0
\tag{4.59}
\]

where \(L_p\) is the Legendre Polynomial of order \(p\).
We can then write the Lagrangian interpolant, our shape functions, for the \( k \)th node as

\[
h_k(\xi) = \begin{cases} 
\frac{(1 - \xi^2)}{p(p + 1) L_p(\xi_k)(\xi - \xi_k)} \frac{dL_p}{d\xi}, & \xi \neq \xi_k \\
1, & \xi = \xi_k
\end{cases}
\]  

(4.60)

Figure 4.1 shows the Lagrangian-interpolant shape functions for \( p = 4 \), and \( p = 8 \).

\[\begin{array}{c}
\text{(a) } p = 4 \\
\text{(b) } p = 8
\end{array}\]

Figure 4.1: Lagrangian-interpolant shape functions in the element natural coordinates for (a) fourth-order and (b) eight-order LSFEs [25]

The nodes within an element are located at GLL points, and the weak-form integrals are evaluated with reduced \( p \)-point Gauss quadrature at the GL points whose locations are illustrated for \( p = 3 \) in Figure 4.2.

\[\begin{array}{c}
\text{GL} \\
\times \\
\times \\
\bullet \\
\bullet \\
\times \\
\times \\
\times
\end{array}\]

Figure 4.2: Third-order LSFE with nodes at \( p + 1 \) GLL points and \( p \) GL points.

For example, the integral in Equation 4.47 is evaluated as

\[
\hat{M} = \int_0^\ell \mathbf{N}^T \mathbf{M}^T \mathbf{N} \, dx \approx \sum_{j=1}^{n^{GL}} w_j N_j(\xi_j^{GL}) M(\xi_j^{GL}) N(\xi_j^{GL}) J(\xi_j^{GL})
\]  

(4.61)
where, $n^{GL}$ is the number of GL points, and $J$ is the Jacobian of the mapping.
CHAPTER 5
VALIDATION AND VERIFICATION

For the purposes of this thesis, the term “verification” refers to code-to-code and code-to-analytical results comparison. “Validation” refers to code-to-experiment comparison. The purpose of verification is to prove that the code in question is capable of accurately modeling a physical response as compared to the results of an established code. The purpose of validation is to prove that the code in question is capable of replicating real world results. The verification and validation process is a crucial step to ensuring that the code performs as expected for a rigorous set of test cases. For BeamDyn, the test cases were designed with the intent of proving it is a suitable tool for wind turbine blade analysis.

This chapter will consider a mixture of analytical and numerical verification test cases. A validation case for a wind turbine blade will also be examined. The purpose of these test cases is to ensure that BeamDyn is capable of performing as expected for beams with anisotropic material properties, complex geometry, and nonlinear displacements. The efficacy of the spectral element method as formulated with GEBT will also be examined. This section represents the author’s contributions.

5.1 Test Case 1 - Static bending of cantilever beam

The first test case is a static analysis of an isotropic cantilever beam with a point moment applied at the tip of the beam as shown in Figure 5.1. This is a common benchmark problem that was first proposed by Simo [10] and was also demonstrated in Wang et al. [25]. This problem demonstrates the ability of BeamDyn to analyze an isotropic beam with no initial curvature, and with highly nonlinear deflections. In fact, the displacement is so large that it the beam bends into a circular shape. The beam is discretized by two 5\textsuperscript{th}-order elements (i.e., the spectral order, \( p = 5 \)) with a beam total length of 10 inches along \( x_1 \). The cross-sectional stiffness matrix, \( C^* \), is shown below.
Figure 5.1: Configuration of cantilevered 10 inch isotropic beam with a point moment, $M_{x2}$, applied at the tip

\[ C^* = 10^3 \times \begin{bmatrix} 1770 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1770 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1770 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 86.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 215 \end{bmatrix} \]

which has units of $C^*_{ij}$ (lb), $C^*_{i,j+3}$ (lb.in), and $C^*_{i+3,j+3}$ (lb.in²) for $i = 1, 2, 3$. It is noted that the asterisk implies that sectional properties are resolved in the material basis, and the stiffness matrix in the inertial frame is given by $C = (R R_0) C^* (R R_0)^T$.

The analytical displacement in $x_1$ and $x_3$ ($u_1$ and $u_3$ respectively), are given by [31]

\[ u_1 = \rho \sin \left( \frac{x_1}{\rho} \right) - x_1; \quad u_3 = \rho \left( 1 - \cos \left( \frac{x_1}{\rho} \right) \right) \]  

(5.1)

where,

\[ \rho = \frac{EI_2}{M_{x2}} \]

(5.2)

The moment about $x_2$, $M_{x2}$, is given by

\[ M_{x2} = \lambda \ddot{M}_{x2} \]

(5.3)
where,
\[ \tilde{M}_{x2} = \pi \frac{E I_2}{L} \]  
(5.4)

and, \( \lambda \) is the load scaling factor. The load scaling factor varies as, \( \lambda = 0.0, 0.4, 0.8, 1.2, 1.6, \) and \( 2.0. \)

Additionally, \( E I_2 = 86.9 \text{ lb.in}^2 \) and is given in the 5, 5 element of \( C^* \), and \( L = 10 \text{ inches} \).

The moment about \( x_2 \) is given by Equation 5.3 and is found by substituting values for \( \lambda, L \), and \( E I_2 \) into the equation. The values of \( M_{x2} \) corresponding to the varying values of \( \lambda \) are

\[
M_{x2} = \begin{pmatrix}
0 \\
10,920 \\
21,840 \\
32,761 \\
43,681 \\
54,601
\end{pmatrix}
\]

Substituting the vector \( M_{x2} \), and the values of \( x_1 \) (from 0 to 10 inches in increments of one inch), \( \lambda \), and \( E I_2 \) into Equations 5.1 and 5.2, \( u_1 \) and \( u_3 \) can be calculated along the beam. The results at the tip of the beam are compared to the results from the BeamDyn simulation for the same beam. The results from the BeamDyn solution are shown in Figure 5.2, where it can be seen that the beam bends into a complete circle when the load scaling factor is equal to two.

Table 5.1 compares the analytical and BeamDyn solutions for the tip displacements as defined in Wang et al. [25].

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Analytical (( u_1 ))</th>
<th>BeamDyn (( u_1 ))</th>
<th>Analytical (( u_3 ))</th>
<th>BeamDyn (( u_3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>-2.4317</td>
<td>-2.4317</td>
<td>5.4987</td>
<td>5.4987</td>
</tr>
<tr>
<td>0.8</td>
<td>-7.6613</td>
<td>-7.6613</td>
<td>7.1978</td>
<td>7.1978</td>
</tr>
<tr>
<td>1.2</td>
<td>-11.5591</td>
<td>-11.5591</td>
<td>4.7986</td>
<td>4.7986</td>
</tr>
<tr>
<td>1.6</td>
<td>-11.8921</td>
<td>-11.8921</td>
<td>1.3747</td>
<td>1.3747</td>
</tr>
<tr>
<td>2.0</td>
<td>-10.0000</td>
<td>-10.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison of analytical and BeamDyn calculated tip displacements \( u_1 \) and \( u_3 \) (in inches) of a cantilever beam subjected to a constant moment; the BeamDyn model was composed of two \( 5^{th} \)-order LSFEs.
Fig. 5.2: Static deflection of a cantilever beam under six constant bending moments as calculated with two 5th-order Legendre spectral finite elements in BeamDyn [25]

From inspection of Fig. 5.2 we can see that when \( \lambda = 2 \), the rotation about the \( x_2 \) axis exceeds \( \pi \) within one element. As discussed in Section 2.2.1 rescaling of the nodal rotation, \( p_2 \), according to Equation 2.22 must be applied. Fig. 5.3(a) shows how the nodal rotation rescales when the rotation reaches \( \pi \). Fig. 5.3(b) shows the relative rotation, \( r_2 \), within each element, as there were two elements in this analysis. It can be seen that the relative rotation within one element does not rescale.

Next, a convergence study of the BeamDyn LSFEs is conducted. The convergence rate is compared to Dymore [32], which is a well established open-source, flexible multibody dynamics code that is formulated on the same GEBT theory as BeamDyn. Dymore has a range of options for the order of the interpolating function; for this study we have selected quadratic finite elements (QFE). Fig. 5.4 shows the normalized error \( \varepsilon(u) \), where \( u \) is the calculated tip displacement (at \( x = L \)), as a function of the number of model nodes for the calculation with Dymore QFEs and a single LSFE, where

\[
\varepsilon(u) = \left| \frac{u - u^a}{u^a} \right| \quad (5.5)
\]
Figure 5.3: Rescaling of rotation parameter and relative rotations two elements [25]

and where, $u_a$ is the analytical solution. The load scaling factor is set to $\lambda = 1$ for this case. It can be seen from Figure 5.4 that the convergence of the LSFEs are far superior to the QFEs and that the error of the LSFEs reaches machine precision in an exponential manner. For a given model size the LSFE is orders of magnitude more accurate than the QFE.

Figure 5.4: Normalized error of the (a) $u_1$ and (b) $u_3$ tip displacements for $\lambda = 1$ [25]
5.2 Test Case 2 - Static analysis of a composite beam

The second test case is a static analysis of a composite beam with bend-bend, and bend-twist coupling. This is an important case for BeamDyn to be able to model since many wind turbine blades are constructed with these types of coupling effects. The configuration of this beam is shown in Figure 5.5.

The stiffness matrix is given by

\[ C^* = 10^3 \times \begin{bmatrix} 
1368.17 & 0 & 0 & 0 & 0 & 0 \\
0 & 88.56 & 0 & 0 & 0 & 0 \\
0 & 0 & 38.78 & 0 & 0 & 0 \\
0 & 0 & 0 & 16.96 & 17.61 & -0.351 \\
0 & 0 & 0 & 17.61 & 59.12 & -0.370 \\
0 & 0 & 0 & -0.351 & -0.370 & 141.47 
\end{bmatrix} \]

which has units of \( C_{ij}^* \) (lb), \( C_{i+j+3}^* \) (lb.in), and \( C_{i+3,j+3}^* \) (lb.in\(^2\)) for \( i = 1, 2, 3 \). The composite beam is 10 inches long with a boxed cross-section and can be found in Yu et al. [33]. The configuration of the beam is the same as in Figure 5.1 with the exception that \( M_{x2} \) is replaced with a point force, \( P_{x3} = 150 \text{ lbs} \) along the \( x_3 \) direction at the tip.

The simulation is run in Dymore and is discretized by 10 3rd-order elements; BeamDyn uses two 5th-order elements. The results from Dymore and BeamDyn are shown in Table 5.2.
The displacements and rotations are shown in Figure 5.6 as a function of the beam length \((x_1)\).

Table 5.2: Numerically determined tip displacements and rotation parameters of a composite beam

<table>
<thead>
<tr>
<th></th>
<th>(u_1) (inch)</th>
<th>(u_2) (inch)</th>
<th>(u_3) (inch)</th>
<th>(p_1)</th>
<th>(p_2)</th>
<th>(p_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn</td>
<td>-0.09064</td>
<td>-0.06484</td>
<td>1.22998</td>
<td>0.18445</td>
<td>-0.17985</td>
<td>0.00488</td>
</tr>
<tr>
<td>Dymore</td>
<td>-0.09064</td>
<td>-0.06483</td>
<td>1.22999</td>
<td>0.18443</td>
<td>-0.17985</td>
<td>0.00488</td>
</tr>
</tbody>
</table>

Figure 5.6: Displacements and rotation parameters along beam axis for Test Case 2 [25].

It can be seen that the solutions given by BeamDyn and Dymore are in very good agreement. It can also be seen in Figure 5.6(b) that the in-plane force leads to a fairly large twist angle, \(p_1\), due to the bend-twist coupling. A small out of plane deflection can also be observed in Figure 5.6(a), which is also a result of the bend-twist coupling terms. Thus, it has been demonstrated that BeamDyn is capable capturing the effects of coupling terms in a composite material.
5.3 Test Case 3 - Stepped Beam

A beam with a drastic changes in material properties is examined as the next test case. Often times realistic wind turbine blades will have large jumps in cross-sectional material properties as a result of changing geometry or structure as illustrated in Figure 5.7.

![Figure 5.7: General geometry and cross-sectional structure of a wind turbine blade](image)

The root contains many layers of fiberglass and must support the largest moments, and is therefore designed to has the largest bending stiffness, \( EI \), where \( E \) is the modulus of elasticity and \( I \) is the moment of inertia of the cross section. Figure 5.8 shows the normalized bending stiffness along the length of the blade for a typical wind turbine blade (see Section 5.6 for an example of a realistic wind turbine blade). It can be seen from the graph that the bending stiffness jumps to about 10% of its root value in under 10% of the length of the blade.

The purpose for this test case is to determine if it is possible to analyze such a beam with one LSFE. The configuration of the beam is shown in Figure 5.9. The beam experiences a sharp jump in material properties at \( L/2 \), where the bending stiffness is reduced by a factor of 2.

A load, \( P \), is applied in the negative \( x_3 \) direction. The beam is modeled as an isotropic beam with a square cross section throughout the beam. The material properties for each section are given in Table 5.3 and Table 5.4.
Figure 5.8: Normalized bending stiffness as a function of normalized blade length

Figure 5.9: Configuration of cantilevered beam with drastic changes material in properties

Table 5.3: Properties for Section AB

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>200 GPa</td>
</tr>
<tr>
<td>G</td>
<td>79.3 GPa</td>
</tr>
<tr>
<td>Height</td>
<td>1.1892 m</td>
</tr>
<tr>
<td>ν</td>
<td>0.26</td>
</tr>
<tr>
<td>L</td>
<td>5 m</td>
</tr>
<tr>
<td>k (shear coefficient)</td>
<td>0.83333</td>
</tr>
</tbody>
</table>
The stiffness matrix for an isotropic material is given by

\[
C = \begin{bmatrix}
EA & 0 & 0 & 0 & 0 & 0 \\
0 & kGA & 0 & 0 & 0 & 0 \\
0 & 0 & kGA & 0 & 0 & 0 \\
0 & 0 & 0 & GJ & 0 & 0 \\
0 & 0 & 0 & 0 & EI_1 & 0 \\
0 & 0 & 0 & 0 & 0 & EI_2 \\
\end{bmatrix}
\]

where \(I_1 = \frac{bh^3}{12}\), \(I_2 = \frac{bh^3}{12}\), and for a square cross-section \(J \approx 2.25\left(\frac{h}{2}\right)^4\). Therefore, the stiffness matrices for sections AB, and BC are given by

\[
C_{AB} = 10^{10} \times \begin{bmatrix}
28.284 & 0 & 0 & 0 & 0 & 0 \\
0 & 9.345 & 0 & 0 & 0 & 0 \\
0 & 0 & 9.345 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.177 & 0 & 0 \\
0 & 0 & 0 & 3.333 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.333 & 0 \\
\end{bmatrix}
\]

and,

\[
C_{BC} = 10^{10} \times \begin{bmatrix}
20.000 & 0 & 0 & 0 & 0 & 0 \\
0 & 6.608 & 0 & 0 & 0 & 0 \\
0 & 0 & 6.608 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.115 & 0 & 0 \\
0 & 0 & 0 & 1.667 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.667 & 0 \\
\end{bmatrix}
\]

which has units of \(C_{ij}^*\) (N), \(C_{i,j+3}^*\) (N.m), and \(C_{i+3,j+3}^*\) (N.m²) for \(i = 1, 2, 3\).
The beam is first analyzed with one LSFE where a single element spans the discontinuity in material properties, as shown in Figure 5.10. The order of the LSFE ranges from 1\textsuperscript{st}-order to 100\textsuperscript{th}-order.

![Figure 5.10: Stepped beam with one LSFE](image)

Next, the beam is analyzed with two elements where the element interface is at the material discontinuity, as shown in Figure 5.11. The order of the elements range from 1\textsuperscript{st}-order to 50\textsuperscript{th}-order to have the same number of nodes as the first case. The results for the tip displacements, for the one-and two-element beams, are given in Table 5.5 and compared to a well refined simulation in Dymore with 200 3\textsuperscript{rd}-order elements.

![Figure 5.11: Stepped beam with two LSFEs](image)
Table 5.5: Numerically determined tip displacements and rotation parameters of a stepped beam discretized by one 100th-order LSFE and two 50th-order LSFEs in BeamDyn compared to Dymore solution discretized by 200 QFEs.

<table>
<thead>
<tr>
<th></th>
<th>$u_1$ (m)</th>
<th>$u_2$ (m)</th>
<th>$u_3$ (m)</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn (one-LSFE)</td>
<td>-0.00725</td>
<td>0.0000</td>
<td>-0.34096</td>
<td>0.0000</td>
<td>0.05620</td>
<td>0.0000</td>
</tr>
<tr>
<td>BeamDyn (two-LSFEs)</td>
<td>-0.00725</td>
<td>0.0000</td>
<td>-0.34096</td>
<td>0.0000</td>
<td>0.05620</td>
<td>0.0000</td>
</tr>
<tr>
<td>Dymore</td>
<td>-0.00725</td>
<td>0.0000</td>
<td>-0.34092</td>
<td>0.0000</td>
<td>0.05619</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Next, the convergence of the one and two-element analyses is examined in order to see if there is any lack of performance using only one LSFE. The convergence plot of the error as a function of nodes is shown in Figure 5.12.

![Figure 5.12: Error ($\varepsilon$) as a function of number of nodes for one and two LSFEs](image)

The error is given by

$$\varepsilon(u) = \left| \frac{u - u^A}{u^A} \right|$$  \hspace{1cm} (5.6)

where, $u^A$ is the solution from the 100th order analysis for each beam. That is, each beam is compared to its most refined solution, thereby making the convergence within each beam apparent. Figure 5.12 shows that the solution reaches machine precision for the case where
LSFEs meet at the discontinuity, whereas, the simulation using only one LSFE is limited to a quadratic convergence rate. To understand why the beam with only one LSFE is limited to quadratic convergence it is instructive to examine the solution at each node along the beam to see if there is any anomalous behavior.

Figure 5.13 shows the rotation parameter, $p_2$, as a function of the beam length. It can be observed that there is a “kink” in the solution at 5 m, which is exactly where the discontinuity in material properties appears. If only one element is used when there is a “kink” in the solution, the convergence rate is reduced quadratic. If however, one knows where the “kink” in the solution is, and interfaces elements at this boundary, the convergence rate is exponential. It can also be said that LSFEs are best suited to smooth solutions.

Figure 5.13: Rotation parameter $p_2$ as a function of beam length for one and two-LSFEs
5.4 Initial Curvature

Next it is necessary to show that BeamDyn is capable of analyzing beams with initial curvature and initial twist. The curvature vector has three components $k_1$, $k_2$, and $k_3$, where $k_1$ is the initial twist and $k_2$, and $k_3$ are the initial curvature as defined in Sections 2.2.4 and 2.3.2.

5.4.1 Twisted Beam

We will first examine the effect of initial twist. A straight beam ($k_2 = k_3 = 0$) with an initial twist ($k_1 \neq 0$) is shown in Figure 5.14. The beam is linearly twisted from $0^\circ$ twist at the root to $90^\circ$ twist at the tip, and the twist is in the positive $\theta$ direction.

![Figure 5.14: Configuration for cantilevered initially twisted beam with a point load, F, at the tip](image)

As discussed in the previous section, the stiffness matrix for an isotropic material is given by

\[
C = \begin{bmatrix}
EA & 0 & 0 & 0 & 0 & 0 \\
0 & kGA & 0 & 0 & 0 & 0 \\
0 & 0 & kGA & 0 & 0 & 0 \\
0 & 0 & 0 & GJ & 0 & 0 \\
0 & 0 & 0 & 0 & EI_1 & 0 \\
0 & 0 & 0 & 0 & 0 & EI_2
\end{bmatrix}
\]

where $I_1 = \frac{bh^3}{12}$, $I_2 = \frac{bh^3}{12}$, and for a rectangular cross-section $J \approx 0.229hb^3$. Table 5.6 shows the material properties for A36 steel, the geometry of the cross-section, and force
applied to the beam. The height and base values reported in the table are the height and base of the rectangular cross-section. The shear stiffness is not rigorously derived value, but it is derived based on the shape of the cross-section, and for rectangular beams it is approximately 5/6. The torsional stiffness coefficient, \( J \), is also an approximate value.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Modulus</td>
<td>200 GPa</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>79.3 GPa</td>
</tr>
<tr>
<td>Height</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Base</td>
<td>0.25 m</td>
</tr>
<tr>
<td>Length</td>
<td>10 m</td>
</tr>
<tr>
<td>Force</td>
<td>4000 kN</td>
</tr>
<tr>
<td>k (shear coefficient)</td>
<td>0.83333</td>
</tr>
</tbody>
</table>

While this approach is not completely accurate due to the approximations in the shear and torsional stiffness coefficients it is helpful for illustrating the physical meaning of the stiffness matrix components. VABS offers a more rigorous way to determine the stiffness matrix. The VABS stiffness matrix for the material properties given in Table 5.6 for the unrotated cross-section is shown below, and is the stiffness matrix used in the subsequent analysis

\[
C = 10^8 \times \begin{bmatrix}
250,000 & 0 & 0 & 0 & 0 & 0 \\
0 & 92.449 & 0 & 0 & 0 & 0 \\
0 & 0 & 83.497 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.498 & 0 & 0 \\
0 & 0 & 0 & 5.208 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.302 & 0
\end{bmatrix}
\]

which has units of \( C_{ij}^* \) (N), \( C_{i,j+3}^* \) (N.m), and \( C_{i+3,j+3}^* \) (N.m\(^2\)) for \( i = 1, 2, 3 \).

The beam discretized using one 7th-order LSFE. The results for the twisted beam are shown in Table 5.7 and compared to the baseline results obtained from a solid-element ANSYS model.
Table 5.7: Numerically determined tip displacements for twisted beam discretized by one 7th-order LSFE in BeamDyn compared with ANSYS results

<table>
<thead>
<tr>
<th></th>
<th>$u_1$ (m)</th>
<th>$u_2$ (m)</th>
<th>$u_3$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn ($k_1 \neq 0$)</td>
<td>-1.132727</td>
<td>-1.715123</td>
<td>-3.578671</td>
</tr>
<tr>
<td>ANSYS</td>
<td>-1.134192</td>
<td>-1.714467</td>
<td>-3.584232</td>
</tr>
<tr>
<td>Percent Error</td>
<td>0.129%</td>
<td>0.038%</td>
<td>0.155%</td>
</tr>
</tbody>
</table>

The percent error between BeamDyn and the result from ANSYS are given by

$$
\varepsilon(u) = \left| \frac{u - u^A}{u^A} \right| \times 100\% \quad (5.7)
$$

It can be seen that the error between the BeamDyn simulation and the ANSYS baseline solution are very close. In fact the error in the tip displacement is below 1%. It can be said that BeamDyn is capable of modeling beams with initial twist.

### 5.4.2 Curved Beams

Next a beam with initial curvature but zero initial twist is examined (i.e., $k_1 = 0, k_2 \neq 0, k_3 \neq 0$). From Section 4.5 it is clear that the initial curvature plays a major role in the distribution of the elastic forces within the beam. As such it is very important to ensure that BeamDyn is capable of modeling this effect properly. A benchmark problem for a curved beam is the case proposed by Bathe [34], and is used here as a verification case. Figure 5.15 shows the configuration of the cantilevered curved beam. The beam is in the $x_1$-$x_2$ plane, and in the positive $x_1$ direction and negative $x_2$ direction. A force of 600 lbs is applied in the positive $x_3$ direction.

The beam is defined by the 45° arc of a 100 inch radius with a center point located at 100 inches on the negative $x_2$ axis. Therefore the coordinates at the tip of the beam in the $x_1$, $x_2$, and $x_3$ coordinate systems are given by (70.7107 in., -29.2893 in., 0.0), respectively. The geometry of the cross section for the curved beam is square, and the material properties are given in Table 5.8.
Figure 5.15: Configuration of cantilevered curved beam with point load at the tip

Table 5.8: Properties for curved beam

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Modulus</td>
<td>$10^7$ psi</td>
</tr>
<tr>
<td>Height of cross-section</td>
<td>1.0 in</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.00</td>
</tr>
<tr>
<td>Radius</td>
<td>100 in</td>
</tr>
<tr>
<td>k (shear coefficient)</td>
<td>0.83333</td>
</tr>
<tr>
<td>Force</td>
<td>600 lbs.</td>
</tr>
</tbody>
</table>
The stiffness matrix from VABS is given by

\[ C = 10^5 \times \begin{bmatrix}
100.000 & 0 & 0 & 0 & 0 & 0 \\
0 & 50.000 & 0 & 0 & 0 & 0 \\
0 & 0 & 42.211 & 0 & 0 & 0 \\
0 & 0 & 0 & 7.686 & 0 & 0 \\
0 & 0 & 0 & 0 & 8.333 & 0 \\
0 & 0 & 0 & 0 & 0 & 8.333 \\
\end{bmatrix} \]

which has units of \( C_{ij} \) (lb), \( C_{i,j+3} \) (lb.in), and \( C_{i+3,j+3} \) (lb.in^2) for \( i = 1, 2, 3 \). The beam is discretized by one 5th-order LSFE. The results for this static analysis are shown in Table 5.9 and compared to the results published in Bathe [34].

Table 5.9: Numerically determined tip displacements for curved beam discretized by one 5th-order LSFE in BeamDyn compared to published results

<table>
<thead>
<tr>
<th></th>
<th>( u_1 ) (inch)</th>
<th>( u_2 ) (inch)</th>
<th>( u_3 ) (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn (one-LSFE)</td>
<td>-23.7</td>
<td>13.5</td>
<td>53.4</td>
</tr>
<tr>
<td>Published</td>
<td>-23.5</td>
<td>13.4</td>
<td>53.4</td>
</tr>
<tr>
<td>Percent Error</td>
<td>0.85%</td>
<td>0.75%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

It can be seen from these results that the simulations from BeamDyn for a initially curved beam match quite well with the published results. It can therefore be said that BeamDyn is capable of modeling beams with initial curvature.

5.5 Equivalent Beams

There are different methods for defining the geometry of the beam (i.e., initial curvature and initial twist). The geometry of the beam has a direct impact on the 1-D beam analysis and 2-D cross-sectional analysis. As shown in Section 3.1.1, the 2-D analysis is an input for the 1-D analysis. The geometry of a beam representing a wind turbine blade may simply be defined as a straight line starting at the root and ending at the tip of the blade. In this case the material properties of each cross-section are defined with respect to this reference axis. Often times, however, OEMs define the geometry of the beam as the line that connects the shear centers of each cross-section as shown in Figure 5.16.
The shear center is defined as the point about which a shear force may be applied without a resulting torsion, and as such, it is convenient to define the beam with respect to this line as it results in many terms in the stiffness matrix being zero. As illustrated in Figure 5.16 the shear center can be located at a different locations for each cross-section along the wind turbine blade depending on the materials, and structural lay out (e.g., shear webs, spar caps, etc) for a particular cross-section. It can be seen that while the stiffness matrix is simplified, choosing the shear center to define the geometry of the beam can produce a beam geometry that is complex with many gradients in the spatial coordinates, which can produce numerical problems for beam solvers due to steep gradients in the curvature term, $\kappa$. It is therefore advantageous to have a method of transforming a beam with complex geometry to a beam with simple geometry.

The purpose of the this section is to determine methods for converting beams that have initial curvature and twist to straight beams with no initial curvature or twist. The goal is to also determine the limits of such methods. A beam with initial twist will be examined first, followed by a beam with initial curvature.
5.5.1 Twisted Beam

First, it is assumed that the beam in Figure 5.14 is straight, i.e., \( k_1 = 0 \). All other dimensions of the beam remain the same for the sake of comparing the results. The beam is discretized using one 7th-order element. In order to develop an equivalent beam model, the stiffness matrix is rotated at each cross-section using the transform

\[
C^{MOD} = R C R^T
\]

where \( C^{MOD} \) is the modified stiffness matrix, and

\[
R = \begin{bmatrix}
R & 0 \\
0 & R
\end{bmatrix}
\]

and,

\[
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\theta) & \sin(\theta) \\
0 & -\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

where, \( \theta \) is the angle between the primed and inertial coordinate systems as shown in Figure 5.17 (where \( \theta \) ranges from 0° to 90°). It is noted that \( \theta \) is shown as positive in the negative z-direction.

Figure 5.17: Rotation of twisted cross-section which brings the primed coordinate system into the inertial coordinate system

The unmodified stiffness matrix is the same as shown in Section 5.4.1, and is given by
\[ C = 10^8 \times \begin{bmatrix}
250.000 & 0 & 0 & 0 & 0 & 0 \\
0 & 92.449 & 0 & 0 & 0 & 0 \\
0 & 0 & 83.497 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.498 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.208 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.302 \\
\end{bmatrix} \]

which has units of \( C_{ij}^* (N) \), \( C_{i,j+3}(N.m) \), and \( C_{i+3,j+3}(N.m^2) \) for \( i = 1, 2, 3 \).

The transformation yields a modified stiffness matrix with bending and shearing coupling terms. An example of the modified stiffness matrix at 2.5 m along the beam axis (i.e., \( \theta = 22.5^\circ \)) is given by

\[ C_{MOD} = 10^8 \times \begin{bmatrix}
250.000 & 0 & 0 & 0 & 0 & 0 \\
0 & 91.138 & 3.165 & 0 & 0 & 0 \\
0 & 3.165 & 84.808 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.498 & 0 & 0 \\
0 & 0 & 0 & 0 & 4.636 & 1.381 \\
0 & 0 & 0 & 0 & 1.381 & 1.874 \\
\end{bmatrix} \]

which has the same units as \( C \). The results for the equivalent beam are shown in Table 5.10 and compared to the results from the twisted beam analysis in Section 5.4.1, and the baseline results obtained from a solid-element ANSYS model.

Table 5.10: Numerically determined tip displacements for modified twisted beam \( (k_1 = 0) \), and unmodified twisted \( (k_1 \neq 0) \) beam discretized in BeamDyn by one 7th-order LSFE compared to an unmodified ANSYS analysis

<table>
<thead>
<tr>
<th></th>
<th>( u_1 ) (m)</th>
<th>( u_2 ) (m)</th>
<th>( u_3 ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn ( (k_1 = 0) )</td>
<td>-1.132725</td>
<td>-1.715119</td>
<td>-3.578670</td>
</tr>
<tr>
<td>BeamDyn ( (k_1 \neq 0) )</td>
<td>-1.132727</td>
<td>-1.715123</td>
<td>-3.578671</td>
</tr>
<tr>
<td>ANSYS</td>
<td>-1.134192</td>
<td>-1.714467</td>
<td>-3.584232</td>
</tr>
</tbody>
</table>

The error between the two cases and the result from ANSYS are given by

\[ \epsilon(u) = \left| \frac{u - u^A}{u^A} \right| \] (5.8)
where, \( u_A \) is the solution from ANSYS and the error in tip displacements (\( u_3 \)) is shown in Figure 5.18.

![Figure 5.18: Error in tip displacement (\( u_3 \)) for \( k_1 = 0 \) and \( k_1 \neq 0 \) compared to the baseline solution from ANSYS](image)

It can be seen from Figure 5.18 that the errors are indistinguishable whether the beam was assumed to have initial twist, or assumed to no initial accompanied by a modified stiffness matrix. It can be said that it is reasonable to use an equivalent beam model for the case investigated.

### 5.5.2 Curved Beam

Next, the objective is to determine if it is possible to model the curved beam as a straight beam by transforming the stiffness matrix in a similar manner to what was done with the twisted beam. From 3-D Euler-Bernoulli [35] [36] beam theory, the cross-section may be translated some distance, \( x_{2c} \) and \( x_{3c} \), as shown in Figure 5.19 by decoupling the
constitutive equation

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]

\[ (5.9) \]

Figure 5.19: Cross-section offset in terms of \( x_{2c} \) and \( x_{3c} \)

Equation 5.9 may be decoupled into a axial force-bending problem and a twisting-shear force problem as shown below

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & 0 & 0 & 0 & K_{15} & K_{16} \\
0 & K_{22} & K_{23} & K_{24} & 0 & 0 \\
0 & K_{32} & K_{33} & K_{34} & 0 & 0 \\
K_{41} & K_{42} & K_{43} & K_{44} & 0 & 0 \\
K_{51} & 0 & 0 & 0 & K_{55} & K_{56} \\
K_{61} & 0 & 0 & 0 & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]

\[ (5.10) \]

where the representation for the axial force-bending and twisting-shear force are given as

\[
\begin{bmatrix}
F_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{15} & K_{16} \\
K_{51} & K_{55} & K_{56} \\
K_{61} & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]

\[ (5.11) \]

\[
\begin{bmatrix}
M_1 \\
F_2 \\
F_3
\end{bmatrix} =
\begin{bmatrix}
K_{44} & K_{42} & K_{43} \\
K_{24} & K_{22} & K_{23} \\
K_{34} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
\kappa_1 \\
2\gamma_{12} \\
2\gamma_{13}
\end{bmatrix}
\]

\[ (5.12) \]
For the curved beam case presented in Figure 5.15 (i.e., $x_{3c} = 0$, and $k_1 = 0$) the stiffness matrix in Equation 5.10 becomes:

$$C = \begin{bmatrix}
    EA & 0 & 0 & 0 & x_{2c}EA & 0 \\
    0 & kG_1A & 0 & -x_{2c}kG_1A & 0 & 0 \\
    0 & 0 & kG_2A & 0 & 0 & 0 \\
    0 & -x_{2c}kG_1A & 0 & GJ + x_{2c}^2kG_1A & 0 & 0 \\
    x_{2c}EA & 0 & 0 & 0 & EI_x + x_{2c}^2EA & 0 \\
    0 & 0 & 0 & 0 & 0 & EI_y + x_{2c}^2EA
\end{bmatrix}$$

Table 5.11 shows the distance of translation for the cross-section, $x_{2c}$, at multiple values along the length of the blade in the $x_1$-direction.

Table 5.11: Beam offset, $x_{2c}$, and corresponding values of $x_1$

<table>
<thead>
<tr>
<th>$x_1$ (m)</th>
<th>$x_{2c}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>70.711</td>
<td>29.289</td>
</tr>
<tr>
<td>55.557</td>
<td>16.853</td>
</tr>
<tr>
<td>38.268</td>
<td>7.612</td>
</tr>
<tr>
<td>19.509</td>
<td>1.921</td>
</tr>
<tr>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The length of the equivalent beam is the arc length of the original beam which is 78.54 inches. When the beam is modeled as a straight beam a static force/moment balance must be considered. The sum of the forces remains the same but a moment about $x_1$ and $x_2$ must be added to the analysis. The added moments are $M_{x1} = -17,573$ lb.in, and $M_{x2} = -4,697$ lb.in. The results using the modified stiffness matrix and moments for one 20th-order LSFE are shown in Table 5.12.

Table 5.12: Numerically determined tip displacements for modified curved beam ($k_2 = k_3 = 0$) discretized in BeamDyn by one 20th-order LSFE and compared to published results

<table>
<thead>
<tr>
<th></th>
<th>$u_1$ (inch)</th>
<th>$u_2$ (inch)</th>
<th>$u_3$ (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn (Equivalent beam)</td>
<td>-20.8</td>
<td>6.0</td>
<td>54.2</td>
</tr>
<tr>
<td>Published</td>
<td>-23.5</td>
<td>13.4</td>
<td>53.4</td>
</tr>
</tbody>
</table>

For more information on the 3-D Euler-Bernoulli beam theory please see Appendix B.
It can be seen that the results are mixed for this analysis. While the results for $u_1$ and $u_3$ are relatively close, the result for $u_2$ is very far off. It is obvious that a beam with such drastic initial curvature cannot be approximated in the proposed manner.

For the sake of wind turbine blade engineering it is of interest to examine a beam with a curvature similar to that of an actual wind turbine blade. A beam is defined by the $5^\circ$ arc of a 600 inch radius with a center point located at 600 inches on the negative $x_2$ axis. The coordinates at the tip of the beam in the $x_1$, $x_2$, and $x_3$ directions are given by (52.2934 in., -2.283 in., 0.0). The maximum offset, $x_{2c}$, for this beam is 2.283 inches, and the applied moments from the equivalence static analysis are $M_{x_1} = -1,369$ lb.in, and $M_{x_2} = -40$ lb.in., the same 600 lb. force is applied in the $x_3$ direction as before. The results for a curved and modified beam in BeamDyn are shown in Table 5.13, where the percent error is calculated as before with the “actual” solution being the BeamDyn unmodified solution.

<table>
<thead>
<tr>
<th></th>
<th>$u_1$ (inch)</th>
<th>$u_2$ (inch)</th>
<th>$u_3$ (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BeamDyn ($k_2 \neq 0, k_3 \neq 0$)</td>
<td>-8.26</td>
<td>0.48</td>
<td>25.64</td>
</tr>
<tr>
<td>BeamDyn modified ($k_2 = k_3 = 0$)</td>
<td>-8.22</td>
<td>0.50</td>
<td>26.05</td>
</tr>
<tr>
<td>Percent error</td>
<td>0.48%</td>
<td>4.17%</td>
<td>1.60%</td>
</tr>
</tbody>
</table>

It can be seen that this is a much better approximation of a curved beam, but a relatively large error still remains in $u_2$, which is the direction of curvature for the beam. It is likely that the static approximation of the moments is also a factor in this error.

Therefore, it can be said that for isotropic beams with small curvature it is reasonable to approximate the curved beam with an equivalent beam containing a modified stiffness matrix and applied force/moment. Additionally it can be said that for isotropic beams with a large initial curvature, equivalent beams are not suitable.
It should also be noted that the application for these examples does not represent the most generalized case and there is not a rigorous, Timoshenko-like, method for shifting the material properties. In addition, the methods covered for the curved beam equivalence are based on linear beam theory (i.e., Euler-Bernoulli beam theory), and only applied to isotropic materials. For the most accurate results it is still recommended that users should choose to define the sectional properties with respect to a reference line that does not change. Preprocessors like VABS allow users to define an arbitrary reference line.

5.6 Case Study - CX-100 Wind Turbine Blade

The main utility of BeamDyn will be to analyze anisotropic wind turbine blades, therefore the CX-100 will serve as a validation case. The CX-100 was chosen because it is a well characterized blade with a wealth of publicly available data regarding the construction and material properties of the blade. The CX-100 is a 9 m blade designed by Sandia National Laboratory [37].

The VABS cross-sectional properties for this beam were provided by Dr. D.J. Luscher of Los Alamos National Laboratory. Dr. Luscher conducted a similar study with a finite element code based on GEBT theory, called NLBeam [16]. The cross-sectional properties were provided at 40 points along the beam. A typical stiffness matrix is shown at 2.2 m along the span of the blade, and is given by

\[
C = 10^3 \times \begin{bmatrix}
193,000 & -75.4 & 12.2 & -75.2 & -1970 & -3500 \\
-75.4 & 19,500 & 4,760 & 62.6 & 67.3 & 11.3 \\
12.2 & 4,760 & 7,210 & -450 & 17.0 & 2.68 \\
-75.2 & 62.6 & -450 & 518 & 1.66 & -1.11 \\
-1,970 & 67.3 & 17.0 & 1.66 & 2,280 & -879 \\
-3,500 & 11.6 & 2.68 & -1.11 & -875 & 4,240 \\
\end{bmatrix}
\]

which has units of $C_{ij}^*$ (N), $C_{i,j+3}^*$ (N.m), and $C_{i+3,j+3}^*$ (N.m²) for $i = 1, 2, 3$.

Figure 5.20 [37] shows the different material lay-ups for the CX-100 blade. Each color represents a section with unique material properties. This figure also shows the geometry of the blade. Figure 5.21 [37] shows the cantilevered test configuration for the static test.
performed at the National Wind Technology Center (NWTC) in Boulder, Colorado. The whiffle-tree configuration applies the load at 3.00 m, 5.81 m, and 7.26 m from the root of the blade to achieve a maximum root moment of 128.6 kN m. The loads and positions are given in Table 5.14 below.

Figure 5.20: Material layup and geometry of CX-100 wind turbine blade [37]

Figure 5.21: Configuration for cantilevered static pull test of CX-100 wind turbine blade, conducted at the NWTC [37]
Table 5.14: Positions and applied loads in CX-100 static loads test at NWTC

<table>
<thead>
<tr>
<th>Saddle #</th>
<th>Position (m)</th>
<th>Applied load (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00</td>
<td>16.9</td>
</tr>
<tr>
<td>2</td>
<td>5.81</td>
<td>5.47</td>
</tr>
<tr>
<td>3</td>
<td>7.26</td>
<td>5.59</td>
</tr>
</tbody>
</table>

The displacements, $u_3$, at each of the loading points were tracked for the experiment and are given in Table 5.15. The BeamDyn simulation was completed using four 7th-order LSFEs and the results are also given in Table 5.15, where the percent error is calculated as before with the “actual” solution being the experimental measurements.

Table 5.15: Numerically determined displacements for static pull test of CX-100 wind turbine blade discretized by four 7th-order LSFEs in BeamDyn compared to experimental results

<table>
<thead>
<tr>
<th></th>
<th>$u_3$ at saddle #1 (m)</th>
<th>$u_3$ at saddle #2 (m)</th>
<th>$u_3$ at saddle #3 (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental</td>
<td>0.083530</td>
<td>0.381996</td>
<td>0.632460</td>
</tr>
<tr>
<td>BeamDyn</td>
<td>0.072056</td>
<td>0.381074</td>
<td>0.698850</td>
</tr>
<tr>
<td>Percent error</td>
<td>13.74%</td>
<td>0.24%</td>
<td>10.5%</td>
</tr>
</tbody>
</table>

The displacements are plotted in Figure 5.22. The error in the results are explained in [16] as a difference in the rigidity of the boundary condition when calculating the 2-D sectional properties with VABS. It stands to reason that since we are using the same sectional properties the same errors would be evident, and we experience the same overall effect as the results published in Luscher [16]. Overall the results are in good agreement.

Next a convergence study of the tip displacements is completed for the CX-100 blade in BeamDyn. Figure 5.23 shows the error as a function of the number of nodes. The error is calculated as before, where the “actual” displacement is the experimental tip displacement reported in Table 5.15. It can be seen that the convergence rate is not exponential as desired. This is likely a function of the sharp gradients in the bending stiffness along the length of the blade as the stepped beam example detailed in Section 5.3.
Figure 5.22: Displacement $u_3$ along the length of the blade for experimental data and BeamDyn simulation with four $7^{th}$-order LSFEs

Figure 5.23: Error in $u_3$ compared to experimental data as a function of the number of nodes for four-LSFEs
Figure 5.8 shows the normalized bending stiffness as a function of the normalized length and is representative of the diagonal terms in the cross-sectional stiffness matrix. If the sharp gradients in the material properties are in fact causing the loss of spectral convergence, the stepped beam conclusion inform us not to have a discontinuity within one element. This can be done fairly easily to account for the sharp gradients in the diagonal terms of the cross-sectional stiffness matrix. The beam is discretized by two LSFEs with the elements interfacing at 0.8 along the normalized length of the beam where the normalized bending stiffness is 1.1 as shown in Figure 5.8. Using the same configuration as the initial CX-100 static test case the error plot is generated as shown in Figure 5.24. For this analysis we are simply interested in the convergence of the tip deflection, $u_3$. As such, comparison to the experimental data is no longer useful and the “actual” solution is simply a highly refined solution in BeamDyn for this configuration.

![Graph showing error in $u_3$ compared to a highly refined solution in BeamDyn as a function of the number of nodes for two-LSFEs](image)

Figure 5.24: Error in $u_3$ compared to a highly refined solution in BeamDyn as a function of the number of nodes for two-LSFEs

It can be seen that Figure 5.24 more closely resembles a spectral convergence, but there are still relatively large jumps in the error. It is important to consider that the CX-100 is an anisotropic blade so we must also account for more than the diagonal term with respect to
the sharp gradients in the cross-sectional stiffness matrix. Figure 5.25 shows the normalized extension shear coupling term as a function of normalized length and is representative of the behavior in the off-diagonal terms in the cross-sectional stiffness matrix.

![Normalized extension shear coupling terms as a function of normalized length](image)

Figure 5.25: Normalized extension shear coupling terms as a function of normalized length

The next logical step in determining if the spectral convergence is affected by sharp gradients in material properties within one element is to make the element boundaries coincide with the locations where the sectional properties are defined. It was stated before that the cross-sectional properties for the CX-100 blade are given at 40 locations along the length of the blade. In order to have an element coincide with each sectional property, we must use thirty-nine LSFEs. Just as we have for the previous simulation, with two-LSFEs, the error in this simulation is found by assigning the “actual” solution as a highly refined solution in BeamDyn. Figure 5.26 shows the results of this simulation. Each circle on the plot indicates an additional order of the LSFE, i.e., the maximum LSFE order is six.

It can be seen that we have achieved spectral convergence with this simulation, albeit with many elements. These results have extended the conclusion from the isotropic stepped beam case study, to a more general conclusion for composite beams. It can therefore be stated that for composite beams with sharp gradients in the cross-sectional stiffness matrix
the spectral convergence is compromised if one LSFE spans a discontinuity in any of the $6 \times 6$ cross-sectional stiffness terms. It should be noted here that while the spectral convergence suffers as a result of sharp gradients in the cross-sectional stiffness matrix the simulations still return reasonable results, so the utility of BeamDyn is not compromised in this sense.

![Graph](image)

Figure 5.26: Error in $u_3$ compared to a highly refined solution in BeamDyn as a function of the number of nodes for thirty-nine 1$^{st}$ to 6$^{th}$-order LSFEs coincident with sectional properties

5.7 Dynamic Test Case - CX-100 Wind Turbine Blade

The final test case is to illustrate that BeamDyn is capable of accurately analyzing dynamic movement. Here the CX-100 blade is given a constant rotational velocity and a gravity force load is applied. A boundary condition is specified where the blade is allowed to rotate about the node located at its root. This test case is analyzed in both BeamDyn and Dymore. The beam is discretized by one 8$^{th}$-order element in BeamDyn, and forty 3$^{rd}$-order elements in Dymore. The angular velocity of the blade is $\frac{\pi}{2}$ rad/s, and the mass matrix is given by Equation 3.18. The time integrator for the dynamic case is a Runge-Kutta fourth-order method, and the time step is $5 \times 10^{-5}$ s. The time integrator for Dymore is the generalized-alpha time integrator, with a time step of $1 \times 10^{-3}$ s. The total simulation time
in both BeamDyn and Dymore is 6 s.

Figure 5.27 shows the time history for all tip displacements and rotations given by BeamDyn and Dymore. It can be seen that there are oscillations in the displacement response. This is given by applying a root motion to a stationary beam (i.e., there is no rigid body motion). For the most part the displacements are in good agreement, and the root mean square error for \( u_3 \) is 0.0335 which is given by

\[
\varepsilon_{RMS} = \sqrt{\frac{\sum_{k=0}^{n_{max}} [u_k^3 - u_b(t^k)]^2}{\sum_{k=0}^{n_{max}} [u_b(t^k)]^2}} \quad (5.13)
\]

where, \( u_b(t) \) is the benchmark solution and is given by the highly refined Dymore solution.

It can be see in Figure 5.27(e) that a rescaling occurs halfway through the simulation as the rotation exceeds the \( \pi \) rescaling limit. Since the time step is so small the numerical value of the rotation parameter is close to the singularity for multiple time step, thus triggering multiple rescaling operations.

Figure 5.28 shows the root force for the no gravity load applied and gravity load applied cases. It can be seen that the root forces are higher for the case where the gravity force is applied, as expected.
Figure 5.27: Time history of all tip displacements and rotations for BeamDyn and Dymore
Figure 5.28: Root moment for BeamDyn with gravity load and without gravity load
CHAPTER 6
CONCLUSIONS AND FUTURE WORK

The objectives of this thesis were to systematically present the geometrically exact beam theory, present its finite element implementation in BeamDyn, and to detail verification and validation of BeamDyn. In the first chapter a review of prior work was completed to give an understanding of the current state of GEBT within the framework of the problem statement. Chapter two was devoted to mathematical and kinematic fundamentals that were necessary to the formulation of GEBT. In chapter three the VAM was presented which explained how the 3-D beam problem was split into a 2-D cross-sectional analysis and a 1-D beam analysis. Chapter four presented GEBT and the spectral finite element method as implemented in BeamDyn. Chapter five explored a number of benchmark, numerical, and experimental examples that demonstrated the ability of BeamDyn to accurately analyze beam problems with: complex geometry, including initial curvature and initial twist; highly non-linear displacements; isotropic and anisotropic material properties; sharp gradients in material properties and its effect on spectral convergence; realistic wind turbine blades; and dynamic cases.

It was demonstrated in this thesis that BeamDyn is a rigorous tool for users wishing to analyze complex composite structures that may be approximated as beams. The application of spectral finite elements has further improved the performance of BeamDyn with respect to other tools based on GEBT for certain cases. It is unclear if the spectral FE approach improves the accuracy of realistic wind turbine blades with sharp gradients in the material properties.

Future work in this area includes: further validation against utility scale dynamic wind turbine analysis; modal analysis option for BeamDyn; 3-D stress recovery; fully characterizing the effects of sharp gradients in geometry on spectral convergence; and fully coupling into
the aeroelastic code FAST. Once BeamDyn is coupled into FAST it will offer wind turbine designers a highly accurate alternative to full 3-D FEA packages such as ANSYS which have a high computational cost associated.
REFERENCES CITED


This Appendix contains mathematical and tensor fundamentals to support the analysis in the text. A set of vectors is a basis for a space if every vector in the space can be expressed as a unique linear combination of the base vectors [38]. A set of orthogonal axes, $x_1$, $x_2$, and $x_3$ are shown in Figure A.1, where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the unit vectors, which form the basis, $\mathcal{E}$.

Figure A.1: Rectangular, Cartesian coordinate system

Following are some useful vector operations identities:

**The dot product of two vectors ($\vec{u}$ and $\vec{v}$) in basis $\mathcal{E}$**

$$\vec{u} \cdot \vec{v} = uv \cos(\theta),$$

where $\theta$ is the angle between the vectors. More generally,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases}$$

**The cross product of two vectors in basis $\mathcal{E}$**

$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$

More generally: $\vec{e}_i \times \vec{e}_j = \epsilon_{ijk}\vec{e}_k$
Where:

\[ \varepsilon_{ijk} = \begin{cases} 
0, & \text{if any two indices are equal} \\
+1, & \text{if } (ijk) \text{ is an even permutation of } (123) \text{ i.e., 123, 312, 231} \\
-1, & \text{if } (ijk) \text{ is a odd permutation of } (123) \text{ i.e., 213, 321, 132} 
\end{cases} \]

Cross product is anti-commutative

\[ \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \]

Triple vector product

\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \]

The cross product may also be expressed in “tilde notation”

\[ \mathbf{u} \times \mathbf{v} = \tilde{\mathbf{u}} \cdot \mathbf{v} \]

\[ \hat{e}_i \hat{e}_j = \varepsilon_{ijk} \hat{e}_k \]

It can be seen that \( \tilde{\mathbf{u}} \) is a second order skew symmetric tensor that is often referred to as a “cross product operator”, whose components in the \( \mathcal{E} \) basis are given by

\[ \tilde{\mathbf{u}} = \begin{bmatrix} 0 & -u_{e_3} & u_{e_2} \\ u_{e_3} & 0 & -u_{e_1} \\ -u_{e_2} & u_{e_1} & 0 \end{bmatrix} \] \hspace{1cm} (A.1)

Tilde identities

For vectors, \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) [26][8]
\[ \tilde{u}^T = -\tilde{u} \]  
(A.2)

\[ \tilde{u} \tilde{u} = 0 \]  
(A.3)

\[ \tilde{v} \tilde{u} = -\tilde{u} \tilde{v} \]  
(A.4)

\[ v^T \tilde{u} = -\tilde{u}^T v \]  
(A.5)

\[ \tilde{v} \tilde{u} = u v^T - I v^T u \]  
(A.6)

\[ \tilde{v} \tilde{u} = \tilde{u} \tilde{v} + \tilde{u} \tilde{u} \]  
(A.7)

\[ (\tilde{u} v) = \tilde{u} \tilde{v} - \tilde{v} \tilde{u} \]  
(A.8)

\[ \tilde{u} \tilde{v} - \tilde{v} \tilde{u} = v u^T - u v^T \]  
(A.9)

\[ \tilde{u} v u = (u^T w)v - (u v^T)u \]  
(A.10)

\[ \tilde{u} v u = (u^T w)v - (u^T v)w \]  
(A.11)

\[ u v^T w = (v^T w)u \]  
(A.12)

\[ u^T \tilde{v} v = v^T \tilde{w} u = w^T \tilde{u} v \]  
(A.13)

\[ F^{-1} \cdot F = I \]  
(A.14)

### A.1 Tensor operations

A tensor is a linear, homogeneous, entity that when acting on a vector produces a vector as a result[38].

**Definition of a tensor**

\[ F \cdot v = u \]

**Dot product of vector and tensor not commutative**

\[ F \cdot u = u \cdot F^T \]

**Inverse of tensor**

\[ u = F^{-1} \cdot v \]

**Product of inverse tensor dot tensor**

\[ F^{-1} \cdot F = I \]

**Transpose of tensor product**

\[ (F \cdot G)^T = G^T \cdot F^T \]
Transpose of tensor product

$$(F \cdot G)^T = G^T \cdot F^T$$

Inverse of tensor product

$$(F \cdot G)^{-1} = G^{-1} \cdot F^{-1}$$

Tensor product in index notation

$$F \cdot G = (F_{ij} e_i e_j) \cdot (G_{kl} e_k e_l)$$

Tensor scalar product

$$F \cdot G = F_{lk} G_{kl}$$

Tensor scalar product

$$F : G = F_{lk} G_{lk}$$
From the 3-D Euler beam theory and Equation 3.16 and following Bauchau and Yu [35] [36] we have

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{36} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]  

(B.1)

This may be decoupled into a axial force-bending problem and a twisting-shear force problem as shown below

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & 0 & 0 & 0 & K_{15} & K_{16} \\
0 & K_{22} & K_{23} & K_{24} & 0 & 0 \\
0 & K_{32} & K_{33} & K_{34} & 0 & 0 \\
0 & K_{42} & K_{43} & K_{44} & 0 & 0 \\
K_{51} & 0 & 0 & 0 & K_{55} & K_{56} \\
K_{61} & 0 & 0 & 0 & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]  

(B.2)

where the representation for the axial force-bending and twisting-shear force are given as

\[
\begin{bmatrix}
F_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{15} & K_{16} \\
K_{51} & K_{55} & K_{56} \\
K_{61} & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{bmatrix}
\]  

(B.3)

\[
\begin{bmatrix}
M_1 \\
F_2 \\
F_3
\end{bmatrix} =
\begin{bmatrix}
K_{44} & K_{42} & K_{43} \\
K_{24} & K_{22} & K_{23} \\
K_{34} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
\kappa_1 \\
2\gamma_{12} \\
2\gamma_{13}
\end{bmatrix}
\]  

(B.4)

Euler-Bernoulli theory states that for the beams shown in Figure B.1 the 3-D displacement field can be expressed as a function of \(x_1\).

\[
u_1(x_1, x_2, x_3) = \tilde{u}_1 + x_3\Phi_2(x_1) - x_2\Phi_3(x_1)
\]  

(B.5)

\[
u_2(x_1, x_2, x_3) = \tilde{u}_2
\]  

(B.6)

\[
u_1(x_1, x_2, x_3) = \tilde{u}_3
\]  

(B.7)

where \(\Phi_2(x_1)\) and \(\Phi_3(x_1)\) are rigid body rotations shown in Figure B.1(a) and Figure B.1(b) respectively.
Figure B.1: Euler-Bernoulli bending of beam

\begin{align*}
\Phi_2 &= -\ddot{u}_3'(x_1) \\
\Phi_3 &= \dddot{u}_2'(x_1)
\end{align*}

(B.8)  
(B.9)

The curvature \( \kappa \) is given by

\begin{align*}
\kappa_2 &= -\dddot{u}_3'(x_1) \\
\kappa_3 &= \dddot{u}_2'(x_1)
\end{align*}

(B.10)  
(B.11)

The strain is given by

\[ \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \]

(B.12)

then,

\begin{align*}
\varepsilon_{11}(x_1, x_2, x_3) &= u_{i,1} \\
&= \dddot{u}_1' - x_3 \dddot{u}_3' - x_2 \dddot{u}_2' \\
&= \dddot{u}_1' - x_3 \kappa_2 - x_2 \kappa_3
\end{align*}

(B.13)  
(B.14)  
(B.15)

where \( u_{i,1} \) is \( \frac{\partial u_i}{\partial x_1} \).

For the extension bending we first analyze the cross-section shown below in Figure 5.19. It can be seen that the center of the cross section has been offset from the origin by \( x_{2e} \) and
\( x_{3c} \).

\[
\varepsilon^c_{11}(x_1, x_2, x_3) = \ddot{u}_{1}^r - x_{3c}\kappa_2 - x_{2c}\kappa_3 \tag{B.16}
\]

and,

\[
\kappa_{2c} = \kappa_2 \tag{B.17}
\]

\[
\kappa_{3c} = \kappa_3 \tag{B.18}
\]

additionally,

\[
\varepsilon^c_{11}(x_1, 0, 0) = \ddot{u}_{1}^r \tag{B.19}
\]

This can be written in matrix form as

\[
\begin{pmatrix}
\varepsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{pmatrix}
= 
\begin{bmatrix}
1 & -x_{3c} & x_{2c} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
\varepsilon^c_{11} \\
\kappa^c_2 \\
\kappa^c_3
\end{pmatrix} \tag{B.20}
\]

The total strain energy is given by

\[
U = \begin{pmatrix}
\varepsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{pmatrix}^T
\begin{bmatrix}
c_{11} & 0 & 0 \\
0 & c_{22} & c_{23} \\
0 & c_{23} & c_{33}
\end{bmatrix}
\begin{pmatrix}
\varepsilon^c_{11} \\
\kappa^c_2 \\
\kappa^c_3
\end{pmatrix} \tag{B.21}
\]

Substituting Equation B.20 into B.21 we get

\[
U = \begin{pmatrix}
\varepsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{pmatrix}^T
\begin{bmatrix}
c_{11}^c & x_{3c}c_{11}^c & -x_{2c}c_{11}^c \\
x_{3c}c_{11}^c & c_{22}^c + x_{3c}^2c_{11}^2 & c_{23}^c - x_{2c}x_{3c}c_{11}^c \\
-x_{2c}c_{11}^c & c_{23}^c - x_{2c}x_{3c}c_{11}^c & c_{33}^c + x_{3c}c_{11}^c
\end{bmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\kappa_2 \\
\kappa_3
\end{pmatrix} \tag{B.22}
\]

where

\[
c_{11} = S = \int_A E dA \tag{B.23}
\]

\[
c_{22} = H_{22} = \int_A x_3^2 E dA \tag{B.24}
\]

\[
c_{33} = H_{33} = \int_A x_2^2 E dA \tag{B.25}
\]

\[
c_{23} = -H_{23} = -\int_A x_2x_3 E dA \tag{B.26}
\]

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The principal axis of bending is the axis at which the shear stress are equal to zero, and is used to define $H_{22}$, $H_{33}$, and $H_{23}$. The principal axis of bending is shown in Figure B.2 as $x^*_a$.

![Figure B.2: Principle axis of bending](image)

The following relationships occur as a result of rotating to the principal axis of bending:

$$\sin(2\alpha^*) = \frac{H_{23}^c}{\Delta}$$

$$\cos(2\alpha^*) = \frac{H_{33}^c - H_{22}^c}{2\Delta}$$

and,

$$\Delta = \sqrt{\left(\frac{H_{33}^c - H_{22}^c}{2}\right)^2 + (H_{23}^c)^2}$$

and,

$$H_{22}^c = \frac{H_{33}^c + H_{22}^c}{2} - \Delta$$

$$H_{33}^c = \frac{H_{33}^c + H_{22}^c}{2} + \Delta$$
For the twist shear part of the decoupling we use Saint Venant warping for a circular cross-section. The displacement field is given by

\[ u_2(x_1, x_2, x_3) = \bar{u}_2(x_1) - x_3 \Phi_1(x_1)u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) + x_2 \Phi_1(x_1) \quad (B.27) \]

and the curvature and shear terms are given by

\[ \kappa_1(x_1) = \Phi'_1(x_1) \quad (B.28) \]
\[ 2\gamma_{12} = -x_3 \kappa_1 \quad (B.29) \]
\[ 2\gamma_{13} = x_2 \kappa_1 \quad (B.30) \]

Similar to Figure 5.19, it can be seen that the center of the cross section has been offset from the origin by \(x_{2k}\) and \(x_{3k}\) as shown in Figure B.3.

![Figure B.3: Cross-sectional offset](image)

At the shear center

\[ 2\gamma_{12}(x_1, 0, 0) = 0 \quad (B.31) \]
\[ 2\gamma_{13}(x_1, 0, 0) = 0 \quad (B.32) \]

and,
\[ 2\gamma_{12}^k(x_1, x_{2k}, x_{3k}) = -x_{3k}\kappa_1 \]  
\[ 2\gamma_{13}^k(x_1, x_{2k}, x_{3k}) = x_{2k}\kappa_1 \]  

We then have the following relationship

\[
\begin{bmatrix} \kappa_1 \\ 2\gamma_{12} \\ 2\gamma_{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_{3k} & 1 & 0 \\ -x_{2k} & 0 & 1 \end{bmatrix} \begin{bmatrix} \kappa_1^k \\ 2\gamma_{12}^k \\ 2\gamma_{13}^k \end{bmatrix}
\]  

(B.35)

\[
\mathcal{U} = \begin{bmatrix} \kappa_1^k \\ 2\gamma_{12}^k \\ 2\gamma_{13}^k \end{bmatrix}^T \begin{bmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & s_{23} \\ 0 & s_{23} & s_{33} \end{bmatrix} \begin{bmatrix} \kappa_1^k \\ 2\gamma_{12}^k \\ 2\gamma_{13}^k \end{bmatrix}
\]  

(B.36)

combining Equations B.35 and B.36 yields

\[
\mathcal{U} = \begin{bmatrix} \kappa_1^k \\ 2\gamma_{12}^k \\ 2\gamma_{13}^k \end{bmatrix}^T \begin{bmatrix} s_{11} + x_{2k}^2s_{33}^k + x_{3k}^2s_{22}^k - 2x_{2k}x_{3k}s_{23}^k & x_{2k}x_{3k}s_{23}^k - x_{3k}s_{22}^k & x_{2k}s_{23}^k - x_{3k}s_{22}^k \\ x_{2k}s_{23}^k - x_{3k}s_{22}^k & x_{2k}^2s_{33}^k - x_{3k}s_{22}^k & x_{2k}s_{33}^k - x_{3k}s_{22}^k \\ x_{2k}x_{3k}s_{23}^k - x_{3k}s_{22}^k & x_{2k}x_{3k}s_{33}^k - x_{3k}s_{22}^k & x_{2k}s_{33}^k - x_{3k}s_{22}^k \end{bmatrix} \begin{bmatrix} \kappa_1 \\ 2\gamma_{12} \\ 2\gamma_{13} \end{bmatrix}
\]  

(B.37)

where,

\[
s_{11} = S = \int_A EdA
\]  

(B.38)

\[
s_{22} = S_{22} = \int_A x_3^2 EdA
\]  

(B.39)

\[
s_{33} = S_{33} = \int_A x_2^2 EdA
\]  

(B.40)

\[
s_{23} = -S_{23} = -\int_A x_2x_3 EdA
\]  

(B.41)

In an identical manner to the extension bending problem the following relationships occur as a result of rotating to the principal axis of bending

\[
\sin(2\alpha^*) = \frac{S_{23}^k}{\Delta}
\]

\[
\cos(2\alpha^*) = \frac{S_{33}^k - S_{22}^k}{2\Delta}
\]

and,

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\[ \Delta = \sqrt{\left( \frac{S_{33}^k - S_{22}^k}{2} \right)^2 + (S_{23}^k)^2} \]

and,

\[ S_{22}^{k*} = \frac{S_{33}^k + S_{22}^k}{2} - \Delta \]
\[ S_{33}^{k*} = \frac{S_{33}^k + S_{22}^k}{2} + \Delta \]

We can see that if \(x_{2c} = s_{23} = c_{23} = 0\) and therefore the stiffness matrix becomes

\[
C = \begin{bmatrix}
EA & 0 & 0 & 0 & x_{3c}EA & 0 \\
0 & kGA & 0 & -x_{3c}kGA & 0 & 0 \\
0 & 0 & kGA & 0 & 0 & 0 \\
0 & -x_{3c}kGA & 0 & GJ + x_{3c}^2kGA & 0 & 0 \\
x_{3c}EA & 0 & 0 & 0 & EI_x + x_{3c}^2EA & 0 \\
0 & 0 & 0 & 0 & EI_y + x_{3c}^2EA & 0
\end{bmatrix}
\]