A thesis submitted to the Faculty and the Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Doctorate of Philosophy (Operations Research with Engineering).

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ABSTRACT

We introduce a new integer programming formulation to solve the underground project scheduling problem, which we define as determining the time period in which to complete each mining activity so as to maximize the value. We introduce a new formulation for the underground project scheduling problem that utilizes a coarse time fidelity and relies on a set of constraints that forces specific pairs of mining activities to be completed in different time periods. By pairing this novel formulation with recently developed linear programming algorithms and heuristics, we show a dramatic decrease in solution time. We use our formulation in combination with an ad hoc branching strategy, i.e., enumeration, to fix the binary variables that determine if material is to be classified as ore or waste in an underground mine. Finally, in conjunction with our underground model, we use an open pit formulation to determine the timing and location at which to transition from open pit to underground mining. We present multiple constraint reformulations that transform non-precedence constraints into precedence constraints to create a desirable math structure for this combined open-pit-to-underground-transition model. Our research allows mining companies to make more informed decisions regarding design aspects of a mine that can affect the net present value of a project by a significant amount.
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CHAPTER 1
GENERAL INTRODUCTION

Mining is a multi-trillion-dollar-per-year industry upon which we are reliant for all materials that cannot be grown, or extracted as a liquid or gas. Bricks, cellphones, cars, and even paper exist only because we have extracted the necessary elements from the earth. Many of these extraction operations are large, and efficiency is paramount to profitability. One of the major avenues in which mining companies improve their profitability is through long-term project scheduling, which consists of determining the start dates for a given set of activities so as to maximize the value of a project, while adhering to operational and resource-availability constraints.

In this thesis, we show that integer and mixed-integer programming models can be used to increase the net present value of mining operations by millions of dollars. Traditionally, optimization models have focused on surface mining, but here we examine underground mine scheduling. The purpose of our work is: (i) to develop an integer program whose formulation can be used to solve the Underground Mine Project Scheduling Problem (UG-PSP), and (ii) to apply our formulation for this problem to multiple strategic mine planning decisions. We define the UG-PSP as that of scheduling a set of mining activities in such a way as to maximize the net present value of the project, while adhering to precedence and resource-availability constraints. We use this formulation to make decisions, such as the selection of the proper cutoff grade for an underground mine and the determination of the time and location at which to transition from open pit to underground mining. We focus on framing the UG-PSP as a resource-constrained project scheduling problem (RCPSP) to exploit the corresponding mathematical structure and produce faster-solving model instances. By framing the problem as an RCPSP, the formulations and methodologies outlined in this thesis are applicable to scheduling problems that have similar features.
This thesis contains three chapters that are formatted as journal-style papers. Barry King is the primary researcher and author for all of the chapters, which are designed to exist independently except for two appendices that apply to Chapters 3 and 4. Chapters 2, 3, and 4 are journal articles constructed for different audiences. In Chapter 2, we focus on the mining industry. Chapter 3 is written for technical practitioners. Chapter 4 is written for an audience that is knowledgeable about operations research, but has little mining knowledge. To help the reader understand the link between chapters, each contains a short introduction and “Chapter Conclusion.” We assume a basic knowledge of linear and integer programming, which Rardin (1998) outlines. Although each chapter focuses on mining applications, no knowledge of mining is assumed, and an adequate description of each problem is provided.

Chapter 2 examines two different integer programming formulations tailored to solve the UG-PSP problem on two distinct time horizons. The UG-PSP tactical formulation is designed to solve problems with a short time horizon, e.g., fewer than three years, and requires that the time intervals are sufficiently short to accurately model the duration of underground mining activities. Alternatively, the UG-PSP strategic formulation is tailored to schedule mining activities over long time horizons, i.e., up to the life expectancy of the mine, and relies on forcing specific pairs of activities to be completed in different time periods. The remaining chapters of thesis utilize the UG-PSP strategic formulation to which we add extensions. We highlight the strengths and weaknesses of each formulation with a computational study using multiple synthetic underground datasets.

Chapter 3 provides an outline for using the UG-PSP strategic formulation to select the optimal cutoff grade, i.e., the minimum degree of mineralization within the ore to be processed into a salable product for an underground mine that is under construction. In addition, we observe that the mathematical structure of the UG-PSP strategic formulation is such that we can leverage new linear programming solution techniques. The underground mine is separated into distinct mining zones, each of which may have a different cutoff grade. To determine the optimal cutoff grade in each zone, we develop and employ an enumeration
strategy. The idea of effectively using an enumeration strategy carries over into Chapter 4. For every possible cutoff grade option, we solve the linear programming relaxation of the UG-PSP problem. The linear programming relaxation solutions are sorted based on net present value, and an integer solution to the largest linear programming relaxation is created using a list-ordering heuristic. A mining company used the prescribed cutoff grade and corresponding schedules in their mine plan.

Chapter 4 presents a methodology for determining the appropriate location to transition from open pit to underground mining. The model relies on multiple constraint reformulations to improve the math structure of the problem. For the open pit scheduling portion of the transition model, we reformulate a special knapsack constraint, and inventory balancing constraints to conform to precedence constraint structure within the RCPSP. Ad-hoc branching determines the location at which to transition from open pit to underground mining, and the corresponding schedule.

Our research contributions are: (i) we develop a new methodology to solve the UG-PSP by prohibiting specific pairs of activities from occurring in the same time period; (ii) we show the effectiveness of enumeration strategies when solving problems related to mine scheduling; and (iii) we outline multiple reformulations that allow specific constraints in mine scheduling problems to better conform to RCPSP math structure and to tighten the gap between the linear programming relaxation solution and the optimal integer solution, allowing us to solve large-scale instances in a reasonable amount of time.
In this chapter, we examine how to use integer programming models to solve UG-PSPs which are critical for later our research using real-world data sets. We inform the reader of the properties possessed by UG-PSPs and present two different mathematical formulations that can be used to solve the UG-PSP. Although this chapter is written for the mining engineering community, any scheduling problem that meets our assumptions can be modeled in the two frameworks presented.

2.1 Abstract

We consider an underground production scheduling problem which consists of determining the proper time interval(s) in which to complete each mining activity so as to maximize a mines discounted value, while adhering to precedence, activity durations, and production and processing limits. We present two different integer programming formulations for modeling this optimization problem. Both formulations possess a resource-constrained project scheduling problem structure. The first formulation uses a fine time discretization and is better suited for tactical mine scheduling applications. The second formulation, which uses a coarser time discretization, is better suited for strategic scheduling applications. We illustrate the strengths and weakness of each formulation with an example.

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2.2 Introduction

Project scheduling is an important aspect of underground mine planning that consists of determining the start dates for a given set of activities so as to maximize the value of a project, while adhering to operational and resource-availability constraints. Important activities that require scheduling in underground mine planning include development, drilling, stoping or other ore-extraction techniques, and backfilling. Precedence relationships impose an order in which activities can be carried out. Resource-availability constraints consider production and processing limits determined by capital and equipment availability, among other factors. We define the Underground Mine Project Scheduling Problem, or UG-PSP, as that of scheduling a set of mining activities in such a way as to maximize the net present value of the project, while adhering to precedence and resource-availability constraints. The UG-PSP is a particular case of the Resource-Constrained Project Scheduling Problem (RCPSP). The RCPSP is a class of optimization problems known for their difficulty (Artigues et al., 2013). O’Sullivan et al. (2015) discuss the difficulty of solving underground scheduling problems relative to solving their open-pit counterpart. Trout (1995) first proposed a mixed-integer program to solve a 55-stope UG-PSP over a two-year time horizon using multiple time fidelities. The detailed formulation did not gain widespread adoption due to slow solution times. Others have created case-specific formulations for a variety of underground mines (Carlyle & Eaves, 2001; Epstein et al., 2012; Martinez & Newman, 2011). Newman & Kuchta (2007) provide a model for scheduling the Kiruna mine in which activity duration spans multiple time periods; see also Brickey (2015); O’Sullivan & Newman (2014); Sarin & West-Hansen (2005) for similar models applied to different mines.

2.3 UG-PSP Formulations

We begin by introducing notation for our integer programming (IP) formulations of the UG-PSP, and by noting the assumptions we make in our models. We next present our two, time-indexed formulations, expressed in the “by” form to improve computational tractability
Sets:
\( \mathcal{T} \) uniform time intervals over which scheduling occurs
\( \mathcal{A} \) activities available for scheduling
\( \mathcal{P}_a \) predecessors of activity \( a \in \mathcal{A} \)
\( \mathcal{R} \) scarce resources that are consumed

Parameters:
\( l_{aa} \) number of time intervals that must elapse between the start of activity \( a \in \mathcal{A} \) and the start of its predecessor activity \( \bar{a} \in \mathcal{P}_a \), referred to as lag
\( d_a \) number of time intervals required to complete activity \( a \in \mathcal{A} \) (calculated by rounding up the exact duration to the nearest integer)
\( p_{at} \) objective function value associated with starting activity \( a \in \mathcal{A} \) in time interval \( t \in \mathcal{T} \)
\( q_{ar} \) total quantity of resource type \( r \in \mathcal{R} \) used to complete activity \( a \in \mathcal{A} \)
\( R_{rt} \) total amount of resource type \( r \in \mathcal{R} \) available in time interval \( t \in \mathcal{T} \)

2.3.1 Assumptions

- **A1.** In order to begin an activity \( a \in \mathcal{A} \), it is necessary to have started all activities \( \bar{a} \in \mathcal{P}_a \) at least \( l_{aa} \) time intervals before \( a \). This is not common to all underground production scheduling models. See, for example, O’Sullivan & Newman (2014).

- **A2.** Once an activity is started, it cannot be interrupted.

- **A3.** If the duration \( d_a \) of an activity \( a \in \mathcal{A} \) is greater than one, then the amount of resources consumed per time interval while completing activity \( a \) is equal to \( \frac{q_{ar}}{d_a} \) for all \( r \in \mathcal{R} \).

Note that Assumption A2 can be relaxed for some or all activities in the following way. If preemption is allowed for an activity \( a \in \mathcal{A} \) such that \( d_a > 1 \), then activity \( a \) can be replaced by a set of activities \( a^1, a^2, \ldots, a^{(d_a)} \), each with duration one, such that these smaller activities
correspond to completing portions of the whole. It is necessary to re-define the precedence relationships and relevant parameters accordingly.

2.4 UG-PSP Tactical Formulation

In the UG-PSP tactical formulation, we construct time intervals that are sufficiently short to capture the detail required to accurately model the duration of underground activities. If a mine schedules activities that require a minimum of one day to complete, a daily fidelity model is appropriate.

Variables:

\[ x_{at} \] 1 if activity \( a \) is started by time interval \( t \); 0 otherwise

Objective Function:

\[
\text{UG-PSP tactical max } \sum_{a \in A} \sum_{t \in T} p_{at} (x_{at} - x_{a,t-1}) \tag{2.1}
\]

Constraints:

\[
x_{a,t-1} \leq x_{at} \quad \forall a \in A, t \in T \tag{2.2}
\]
\[
x_{at} \leq x_{\bar{a},t-l_{\bar{a}a}} \quad \forall a \in A, \bar{a} \in P_a, t \in T \tag{2.3}
\]
\[
\sum_{a \in A} \frac{g_{ar}}{d_a} (x_{at} - x_{a,t-d_a}) \leq R_{rt} \quad \forall r \in R, t \in T \tag{2.4}
\]
\[
x_{at} \in \{0, 1\} \quad \forall a \in A, t \in T \tag{2.5}
\]

The objective function, (2.1), cumulates the values associated with starting activities in a specified time interval. This may correspond to discounted metal or discounted cash flow. Constraints (2.2) force a completed activity to remain completed. Constraints (2.3) enforce precedence relationships. Constraints (2.4) impose a bound on the resource consumption in each time interval. Constraints (2.5) restrict all variables to be binary.

The UG-PSP tactical formulation increases in size with the number of time intervals, making instances spanning long horizons using a fine fidelity intractable in practice. In addition, the model assumes detailed knowledge regarding resource and activity attributes.
In practice, when planning activities far in the future, these details are difficult to estimate with precision. For these two reasons, the UG-PSP tactical formulation is very well suited for medium-term scheduling, but another model is required for strategic scheduling.

2.5 UG-PSP Strategic Formulation

In the UG-PSP strategic formulation, we approximate the UG-PSP problem by coarsening the time fidelity; this may be advantageous when a finer time fidelity results in a significant number of time periods and a large number of binary variables. Specifically, we create a set of time intervals by aggregating $\Delta \leq 1$ intervals of $T$. If an activity $a \in A$ exists such that $d_a < \Delta$, the resulting model fails to correctly capture the original precedence relationships.

Consider the following example: Suppose that there are eight development activities which occur along the same heading, each corresponding to advancing 5m in a drift (Figure 2.1). Each of these activities requires one day to complete and is linked with the appropriate precedence. Aggregation into one-week time intervals, i.e., $\Delta = 7$, results in a lag of zero between consecutive development activities, because the aggregated time intervals are long enough for both the predecessor and successor activity to occur in the same aggregated time interval. If the weekly development capacity is 50m, the aggregated model would allow all eight activities to be completed in a week. This is not possible, because completing all eight activities would require eight days. The aggregated model fails to prevent this infeasibility.

![Figure 2.1](image)

Figure 2.1 Development activities are represented by nodes, and the precedence relationships are depicted solid arrows. The selected activity, highlighted in gray, and the successor activity that is 35m away, cannot be completed in the same week, and a delay precedence relationship, shown with dashed arrow, is added.
To address this problem, we add precedence relationships: For every pair of activities (a, ˆ{a}) that cannot be carried out over the course of ∆ consecutive time intervals, we define a precedence relationship such that ˆ{a} ∈ ˆ{P}_a must be completed at least one aggregated time interval in advance of a. We call these delay precedence relationships.

The resulting model formulation follows.

**Sets:**
- ˆ{T} aggregated time intervals which are ∆ time intervals larger than those in T
- ˆ{P}_a delay predecessors of activity a ∈ A

**Parameters:**
- ˆ{l}_{aa} number of aggregated time intervals that must elapse between the start of activity a ∈ A and the start of its predecessor activity ˆ{a} ∈ ˆ{P}_a (If both a and ˆ{a} can be completed in ∆ time intervals, let ˆ{l}_{aa} = 0; else, define ˆ{l}_{aa} = ⌈ ˆ{l}_{aa} / ∆⌉)
- ˆ{d}_a number of aggregated time intervals required to complete activity a ∈ A (If ˆ{d}_a is not an integer multiple of ∆, the value ˆ{d}_a can be obtained by rounding up to the nearest positive integer.)
- ˆ{p}_{af} objective function value associated with starting activity a ∈ A in aggregated time interval ˆ{f} ∈ ˆ{T}
- ˆ{R}_{rt} total amount of resource type r ∈ R available in aggregated time interval ˆ{t} ∈ ˆ{T}

**Variables:**
- ˆ{x}_{af} 1 if activity a is started by time interval ˆ{f}; 0 otherwise

**Objective Function:**
UG-PSP strategic max \( \sum_{a \in A} \sum_{\hat{f} \in \hat{T}} \hat{p}_{af}(\hat{x}_{af} - \hat{x}_{af-1}) \) (2.6)

**Constraints:**
\[ \hat{x}_{af-1} \leq \hat{x}_{af} \quad \forall a \in A, \hat{f} \in \hat{T} \] (2.7)
The objective function, (2.6), cumulates the value associated with starting an activity in a specified time interval. Constraints (2.7) force a completed activity to remain completed. Constraints (2.8) enforce precedence and constraints (2.9) force the pair of activities contained in the delay precedence relationship to occur in different time intervals. Constraints (2.10) bound resource consumption in each time interval. Constraints (2.11) restrict all variables to be binary.

Two important complications arise when using the UG-PSP strategic formulation. The first is that the number of delay precedence relationships grows rapidly as increases. The second is that a feasible solution in this formulation might not necessarily correspond to a solution that is feasible in the UG-PSP tactical formulation. This is the same limitation suffered by integer programming formulations typically used in open pit production scheduling (Johnson, 1968). As such, this formulation is well suited to scheduling large time horizons and making strategic decisions.

2.6 Computational Example

We compare the two formulations by scheduling an artificial open stoping data set.

2.6.1 Data

The single-segment data set is a small section of an underground open stoping mine containing four stopes that can be extracted. In order to extract a stope, the necessary development, stope drilling, and backfilling must have already been completed. Table 2.1 provides an outline of the activities and their attributes that are used, and Figure 2.2 outlines
the precedence structure and activities in the single-segment data set.

Table 2.1 Summary of activity characteristics in the underground mine data set. Resource attributes are given as the total resource consumed. The delay column represents the number of days that must pass after the activity is completed before its successor activity can begin.

<table>
<thead>
<tr>
<th>Activity Type</th>
<th>Quantity</th>
<th>Cost/Profit</th>
<th>Activity Rate</th>
<th>Duration (days)</th>
<th>Delay (days)</th>
<th>Total Resource Consumed by Activity Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal Drift</td>
<td>2</td>
<td>-$10,000/m</td>
<td>5 m/day</td>
<td>10</td>
<td>0</td>
<td>2500</td>
</tr>
<tr>
<td>Cross-Cut</td>
<td>8</td>
<td>$100/t</td>
<td>5 m/day</td>
<td>4</td>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>Stope Drilling</td>
<td>4</td>
<td>-$100/m</td>
<td>120 m/day</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Stope Mucking</td>
<td>4</td>
<td>$250/t</td>
<td>500 t/day</td>
<td>10</td>
<td>1</td>
<td>5000</td>
</tr>
<tr>
<td>Backfilling</td>
<td>4</td>
<td>-$5/t</td>
<td>1000 t/day</td>
<td>4</td>
<td>7</td>
<td>4000</td>
</tr>
</tbody>
</table>

Figure 2.2 Precedence structure for the underground mine (solid arrows). Naming convention: (T-B) indicates whether the activity is located on the top or bottom level of the stope, followed by the activity type, primary resource consumed (), and duration of the activity [].

We also create two larger data sets by copying the single-segment data set. The “triple” and “penta” data sets consist of three and five copies of the single-segment data set, respectively. The triple and penta stopes are differentiated by their stope values: 100%, 80%, and 60%, and 100%, 90%, 80%, 70%, and 60% of the original values, respectively.

The UG-PSP tactical formulation is modeled at daily time fidelity, and the UG-PSP strategic formulation uses a aggregated 14-day time fidelity. The UG-PSP strategic formulation activities are disaggregated, for example, into 10 stoping activities that each require one day, contain 500 ore tons, and are appropriately linked with precedence; delay constraints.
are constructed using the disaggregated activities. Daily production limits are 15 meters of development, 1000 tonnes of total extraction, 1000 tonnes of backfilling, and 240 meters of drilling. The discount rate for UG-PSP strategic formulation is 0.10% for every 14 days, and in the UG-PSP tactical formulation the equivalent daily discount rate of 0.0683% is used, and the objective is to maximize NPV.

### 2.7 Numerical Results

The UG-PSP tactical and UG-PSP strategic formulations for the single-segment, triple, and penta data sets are coded in the algebraic modeling language AMPL (AMPL, 2014) and solved to the default optimality tolerance using CPLEX 12.6.0.1 (IBM CPLEX Optimizer, 2015) on a Dell PowerEdge R410 machine with 16 processors (2.72 GHz each) and 28 GB of RAM. Table 2.2 provides a summary of the different models and the solution times for each formulation and data set.

Table 2.2 Summary of activity characteristics in the underground mine data set. Resource attributes are given as the total resource consumed. The delay column represents the number of days that must pass after the activity is completed before its successor activity can begin.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Number of Variables</th>
<th>Number of Resource Constraints</th>
<th>Number of Precedence and Delay Constraints</th>
<th>Objective Function Value (NPV)</th>
<th>Solution Time (sec)</th>
<th>Time Horizon (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UG-PSP tactical (Single)</td>
<td>2,660</td>
<td>380</td>
<td>3,135</td>
<td>2,901,017</td>
<td>0.08</td>
<td>95</td>
</tr>
<tr>
<td>UG-PSP strategic (Single)</td>
<td>1,099</td>
<td>28</td>
<td>2,359</td>
<td>2,918,144</td>
<td>0.29</td>
<td>98</td>
</tr>
<tr>
<td>UG-PSP tactical (Triple)</td>
<td>10,166</td>
<td>468</td>
<td>11,415</td>
<td>6,196,407</td>
<td>86.83</td>
<td>115</td>
</tr>
<tr>
<td>UG-PSP strategic (Triple)</td>
<td>4,308</td>
<td>32</td>
<td>9,099</td>
<td>6,308,911</td>
<td>7.06</td>
<td>126</td>
</tr>
<tr>
<td>UG-PSP tactical (Penta)</td>
<td>26,740</td>
<td>764</td>
<td>31,515</td>
<td>8,891,858</td>
<td>13,817.74</td>
<td>191</td>
</tr>
<tr>
<td>UG-PSP strategic (Penta)</td>
<td>10,990</td>
<td>56</td>
<td>23,590</td>
<td>9,062,961</td>
<td>99.57</td>
<td>196</td>
</tr>
</tbody>
</table>

The key differences between the UG-PSP tactical and the UG-PSP strategic formulations are the objective function value, solution time, and solution, i.e., schedule. We observe that the NPV is slightly different for a given data set between the two formulations. The UG-PSP strategic always has a higher optimal NPV than that of the UG-PSP tactical, because the UG-PSP strategic formulation is an approximation of UG-PSP. (The discount rate calculations are tied to the time fidelity of the model.) Nonetheless, the NPV difference between the two formulations for the same data set is never greater than 2%, and although the
two formulations produce slightly different NPVs, the overall extraction quantities resulting from the solution of both UG-PSP models are very similar. Figure 2.3 shows the production tonnes associated with the solution for the triple data set using both formulations.

Figure 2.3 Tonnes extracted in every aggregated time period, i.e., 14 days. The production tonnage for the UG-PSP tactical formulation is calculated by summing the production tonnage in each 14-day interval.

The solution time for each formulation also varies drastically; for example, the penta data set using the UG-PSP tactical formulation requires 13,817 seconds, which is much longer than the solution time for the UG-PSP strategic formulation, 99.57 seconds. The UG-PSP tactical formulation does not scale well; solution times for single, triple, and penta data sets are 0.08, 86.8, and 13,817.74 seconds, respectively. Figure 2.4 demonstrates the change in the number of variables and solution time as a function of time horizon length.

On the other hand, the UG-PSP strategic formulation produces solutions with respect to aggregated time periods rather than the more detailed and directly implementable schedules that the UG-PSP tactical formulation produces. The UG-PSP strategic formulation yields a schedule that is feasible relative both to duration and to overall resource consumption, but gives limited information as to how to complete the activities within the time interval. The user must balance solution time with the level of detail required.
Figure 2.4 The number of variables, and solution times for the UG-PSP tactical and UG-PSP.

2.8 Conclusion

We highlight the differences between two formulations applied to the UG-PSP to identify strengths and weakness of each. Although our example is not at the scale of a real-world UG-PSP, Brickey (2015) and King et al. (2016b) solve such problems using formulations similar to the UG-PSP tactical and UG-PSP strategic formulations, respectively, by employing a specialized algorithm (Bienstock & Zuckerberg, 2010) with enhancements (Chicoisne et al., 2012; Muñoz et al., 2016). This highlights that the formulations discussed in this chapter are being applied to problems relevant to the mining industry.

2.9 Chapter Conclusion

This chapter focuses on comparing two different formulations for solving the UG-PSP, and applies them to a synthetic data set. The UG-PSP tactical formulation, or a variant thereof, has been successfully implemented in multiple mining operations. However, our research contribution, the UG-PSP strategic formulation, has not be applied to real-world data sets. The following two chapters focus on extensions and implementation of the UG-PSP strategic formulation.
In this chapter, we transition from a synthetic data set to data set corresponding to a soon-to-be operational underground mine. This chapter presents work that has been used at a major mining company to aid in its decision making. We utilize the UG-PSP strategic formulation in combination with an enumeration strategy to determine which material is to be classified as ore or waste, and, in conjunction with this, a strategic schedule. In addition, we realize that we can employ recently developed algorithms to greatly expedite solutions.

3.1 Abstract

An important decision for any operational mine is the differentiation between ore and waste material, which, in the mining industry, is referred to as the cutoff grade. In underground mining, material classified as waste is left in situ and ore is extracted. Choosing the correct cutoff grade, which, in turn, determines the mine design and value, is a strategic decision. We present a mixed-integer programming optimization framework whose solution determines the cutoff grades in three different zones for a soon-to-be-operational underground mine. Our enumeration strategy, embedded in this optimization framework, provides objective, repeatable solutions, verified by our industry partner, for large-scale problems in less than an hour where manual methods would require six to eight weeks.
3.2 Introduction

Gold has always been a valuable mineral, not only for its use in luxury and commemorative items such as jewelry and Olympic medals, but also as an international monetary standard and as an industrial material. For instance, computers rely on gold’s corrosion resistance and electrical conductivity to function efficiently. In the harsh environment of space, gold is used to efficiently reflect infrared waves that continuously bombard satellites and astronauts. It is estimated that every American born in 2015 will require, on average, 1.59 troy ounces of gold in their lifetime (Mineral Education Coalition, 2015) (Figure 3.1). All of this gold is mined or recycled; not surprisingly, mining produces billions of dollars worth of gold annually.

Figure 3.1 The lifetime total of mineral usage for the average American born in 2015. Each person will require 1.59 troy ounces of gold that was mined at some point in human history. (Mineral Education Coalition, 2015)

Mining is a major part of our global economy, and, in general, is classified as surface or underground mining (Hustrulid, 2001; Hustrulid et al., 2013). Surface mining can be categorized as: strip, open pit, or highwall. Open pit mining, commonly used for gold extraction,
starts from the surface and proceeds downward while maintaining a safe angle of the pit walls, creating a cone shape. When ore is located sufficiently deep below the earth’s surface, underground mining methods such as block caving, room and pillar, stoping, longwall, or drift and fill mining are used, depending on the geometry, size, and host rock characteristics of the ore body. A common underground mining method for gold extracts large rectangular boxes of material, called stopes, from the earth. There exist different variants of stoping, but we narrow our discussion to open stoping, i.e., when a stope is removed, the void is left open, because in-situ rock pillars remain to maintain ground stability.

A major gold producer is planning to open another underground mine in one of two different regions, in which multiple operating mines feed a single processing plant; mining is critical to the local economy. An ore body exists underneath an operating open pit mine with a significant quantity of proven and probable reserves, indicating that ore is present and is economically viable to extract (SME, 2014). The underground mine is to be constructed in this ore body, which has supported open pit mining for over a decade and contains enough gold to justify production for at least a decade and a half more. The construction of an underground mine can be very costly and the decisions regarding the underground mine design and production schedule have large monetary implications.

The cutoff grade of any mine is the degree of mineralization below which material is not processed into a salable good. For instance, a one-troy-ounce-per-tonne cutoff grade indicates that any material containing less than one troy ounce of gold per tonne of material is either not extracted or, if it is extracted, it is sent to a waste dump. In open pit mining, the extraction of material from the surface downward while maintaining safe pit wall angles implies that a vast majority of the material within the ore body is extracted, regardless of the cutoff grade. Therefore, the cutoff grade in an open pit mine primarily determines if material is sent to processing or to a waste dump.

The selectivity of underground mining implies that all material extracted from stopes is at or above a pre-determined cutoff grade; material removed to access the stopes can
be below cutoff grade. The cutoff grade can affect the total extracted tonnes by a factor of two or more. A low cutoff grade yields more ore tonnage, a longer mine life, and more overall metal production, but at the cost of additional development meters, i.e., expensive and time consuming underground construction that provides a route for material haulage to the surface. However, a high cutoff grade implies a shorter mine life that may not justify the large capital cost of starting a mine. Too high a cutoff grade can also result in leaving valuable ore in the ground which cannot be extracted at a later date. An optimal cutoff grade, i.e., the one that maximizes net present value (NPV), must balance the revenue from salable gold, the cost of extraction, and the time value of money.

Underground mines, regardless of the cutoff grade, must establish an extraction schedule for a set of mining activities in such a way as to maximize the NPV of the project while adhering to precedence and resource constraints for a fixed mine design, i.e., for a specific cutoff grade. Precedence constraints define rules regarding the order in which activities can occur. Resource constraints force production capacities to be met.

The purpose of our research is to determine the cutoff grade that maximizes the NPV of the underground mine. Although counterintuitive, we fix all cutoff grades and production capacities a priori, and use an integer programming optimization model to identify the time period in which to complete each stope extraction and development activity so as to maximize the NPV for this restricted case. Constraints include: (i) production capacities for both stope extraction tonnes and development meters are; and (ii) precedence relationships between activities are based on their spatial location, and on activity sequencing rules, e.g., stope extraction cannot occur until the stope has been blasted; and (iii) fixed mining rates that dictate the time required to complete each activity. The cutoff grade can vary depending on the location in the mine; therefore, for each such feasible set of grades, we develop a mine design and construct a schedule to conditionally maximize NPV. The optimization framework identifies an optimum set of feasible grades unconditionally maximizes NPV.
This chapter is organized as follows: The next section presents a Literature Review, and is followed by a discussion on Mine Design and Sequencing. We then provide sections on Solution Strategy, and Computational Results. The last section concludes this study. Appendix A provides information on the integer programming formulation, \((U\!G)\), and details regarding solution times.

### 3.3 Literature Review

Integer programs help to schedule both open pit and underground mines (Johnson, 1968; King et al., 2016b; Newman et al., 2010). Underground mine scheduling has lagged behind its open pit counterpart; most research has occurred within the last two decades (Alford et al., 2007; Martinez & Newman, 2011; Trout, 1995). However, integer programs that are used to schedule underground extraction are making significant impacts on the value of various mining projects (Carlyle & Eaves, 2001; Kuchta et al., 2004; O’Sullivan & Newman, 2014; Topal, 2008). Most research considers the cutoff grade of the underground mine to be fixed a priori.

Certain open pit and underground mine planning problems can be formulated as a resource-constrained production scheduling problem (RCPSP) (King et al., 2016a,b) whose mathematical structure (Artigues et al., 2013) generally leads to long solution times if said structure is not exploited. Bienstock & Zuckerberg (2010) develop a novel algorithm for solving the linear programming (LP) relaxation of RCPSP that greatly expedites solution times. The OMP Solver leverages this algorithm, with improved computational efficiency and methods for creating integer solutions from the linear programming relaxation (Chicoisne et al., 2012; Rivera et al., 2015).

Lane’s (1988) seminal work determines the cutoff grade for an open pit mine by maximizing the NPV using a series of equations. Additional work focuses on determining the open pit cutoff grade through scheduling (Asad & Dimitrakopoulos, 2013; Cullenbine et al., 2011; Osanloo et al., 2008). Hall (2014) provides a methodology for determining the cutoff grade based on evaluating the underground mine’s NPV under a variety of production scenarios.
using rule-based scheduling. Roberts & Bloss (2014) have adapted an open pit optimizer to aide in determining the cutoff grade for an underground mine by differentiating stopes at a finer level of detail and determining whether these “sub-stopes” are to be extracted or left in the earth. Gu et al. (2010) propose a dynamic programming model for selecting the underground cutoff grade.

New software allows mine operators to rapidly change the mine design. Alford & Hall (2009) identify tools to create stope shapes at any cutoff grade, eliminating the tedious task of drawing stopes by hand and allowing the construction of thousands of stopes in a matter of seconds. Therefore, the mining industry is examining more cutoff grades than ever, but is limited in its ability to analyze the NPV of each cutoff grade. We leverage the work of Alford et al. (2007) to create the stopes.

3.4 Mine Design and Sequencing

In the following subsections, we provide background on underground mine design and sequencing for an open stoping mine that utilizes a top-down mining method. We also highlight how changes in the cutoff grade affect each aspect of the mine design, which we now describe.

Figure 3.2 Example of an underground stope grid. Stopes (gray) are shown in their grid locations at various widths. We zoom in on one stope to display the different stope dimensions.
3.4.1 Stope Layout

停止存在于以海平面（图3.2）为基准的z向的定格网格中，定义了一个水平面。一个水平面是停止顶部与上面停止底部相遇的高度。水平间距基于停止的最大高度，且在两个水平面之间的所有停止的高度相等。任何停止的宽度和长度可以在最小值和最大值范围内变化。若停止底部与水平面高度一致，则停止存在。在x方向，网格由宽度由最大停止宽度确定的槽组成，槽按从左到右的顺序编号。若y方向的长度大于特定水平-槽位置的最大停车深度，多个停止可能存在于该位置。随着截断级别增加，大多数停止的体积减少，并在某些情况下停止留在原地。然而，水平高度和最小和最大停止尺寸保持不变，无论截断级别如何。

3.4.2 Mining Zones

地下矿山由许多停止组成，并分为不同的采矿区，其大小和形状由矿体配置决定。更多采矿区允许直接访问更多的矿石，但成本增加。另一方面，安全和地质技术考虑限制停止大型矿体作为一个单区。我们的矿体覆盖2公里的水平距离和750米的垂直距离，太大不能作为单个区开采；而是被分为四个采矿区：上、北、中央（包括中央深部）和南（图3.3）。每个采矿区可以有独特的截断级别，但这些截断级别不改变任何区的边界。

3.4.3 Decline

下降是一个向下倾斜的斜坡，主要通过废石建造，使用橡胶轮设备将矿石运送到地面。一旦斜坡到达地面，它会形成一个循环系统，将矿石运送到地面。一旦斜坡达到地面，它会形成一个循环系统，将矿石运送到地面。
Figure 3.3 Zones in the underground data set. The open pit mine (transparent light blue) is located above most of the underground mine. Each mining zone may have a different cutoff grade. Zone colors are: north (green), south (purple), central (red), central deeps (light blue), upper (orange).

ore body, twin declines, i.e., two corkscrew-shaped declines placed side-by-side that are connected at their closest points, are used for efficiency in the South, Central, and North Zones (Figure 3.4). Precedence dictates that the decline must be completed two levels below a given stope’s level before the stope can be extracted. Each mining zone has its own decline which connects to the main decline that provides access to the surface. The main decline is part of the Upper and Central Zones. A portion of the Central Zone, known as Central Deeps, uses a single decline. Design of the decline remains constant regardless of cutoff grade.

3.4.4 Horizontal Development

Once the decline reaches the ore body, horizontal development is constructed on each underground level, designed to pass through each stope along its width (Figure 3.4). This development begins at the decline and proceeds towards the edge of the mining zone. The cutoff grade significantly influences the amount of horizontal development that is required
on each level, because it is correlated with the size and number of stopes on the level. The total amount of horizontal development required differs by multiple kilometers depending on the cutoff grade. Therefore, an estimate of the horizontal development distance for each cutoff grade is required.

3.4.5 Extraction Sequencing

Stopes on each level are separated into a left and a right mining corridor, typically divided by a decline. Before a stope is extracted, drilling and blasting must have occurred. Drilling is done from the bottom of the stope upward, creating large columnar holes, which are then filled with explosives and blasted. This fragments the rock so that it can be extracted with equipment from the bottom of the stope. The mining sequence forces all horizontal development to be completed on the levels above and below the stopes in a corridor before any extraction in the corridor can occur.
Stopes within a mining zone have three sequencing rules (Figure 3.5), invariant of cutoff grade: (i) stopes in the left corridor are extracted increasing by slot number and stopes in the right corridor are extracted decreasing by slot number; (ii) if multiple stopes exist at the same level-slot location, they are extracted in decreasing order of economic value; (iii) a stope may not be extracted unless the stope(s) directly above have been completely extracted. If no stope(s) exist(s) in the same slot location on the level above, the stope below can be extracted after (i) and (ii) are satisfied. A rib pillar exists between each pair of stopes to ensure stability. A fixed mining rate determines the time required to extract each stope.

Figure 3.5 The extraction sequencing for an open stoping underground mine. First, development must be constructed on the levels above and below the stope. Then, drilling and blasting fragments the rock. Finally, stopes are extracted, leaving open voids in the ground. Rib pillars maintain stability of the host rock. All stopes exist in the same corridor and all stope extraction proceeds from left to right. Stopes on the level above must always be extracted farther to the right.
3.5 Mine Scheduling

In our framework, mine scheduling is categorized as strategic or tactical. A strategic schedule provides guidance as to which areas of the mine should be extracted in a given year, when access to a new mining zone should be constructed, and how long the remaining life expectancy of the mine might be. Tactical schedules provide specified starting dates for mining activities, e.g., decline construction, horizontal development, and stoping, and this schedule is updated frequently based on activity completion. The purpose of our work is to make strategic decisions that can then be implemented in the tactical schedule.

Determining the cutoff grade for a mine is typically done before a strategic schedule is created, which may result in a suboptimal NPV for the mine. Therefore, combining cutoff grade selection into the strategic mine schedule, while difficult, is valuable to mining companies. Our framework determines a strategic production schedule for the underground mine described in the Mine Design and Sequencing Section while optimizing cutoff grade. To expedite solutions, we leverage new algorithms mentioned in the Literature Review that exploit the mathematical structure of the formulation. In the Solution Strategy Section, we outline an optimization-based enumeration strategy to simultaneously determine an optimal production schedule and cutoff grade.

3.5.1 Current Practice at the Mining Company

The mining company uses the following analysis to determine the cutoff grade for each zone: (i) for a single zone, e.g., the Central Zone, they begin by constructing a full mine design in computer-aided design software at a single cutoff grade; (ii) for each relevant cutoff grade, stopes shapes are altered in the mine design and the horizontal development is adjusted to match; (iii) a genetic algorithm is used to create a schedule for each cutoff grade option and stope extraction capacity, i.e., production capacity; (iv) the schedule is post-processed via spreadsheet analysis to obtain a more accurate NPV based on specific costs not considered in the scheduling model; (v) the procedure iterates for each operationally feasible cutoff grade.
option and stope extraction rate. After steps (i)-(v) are completed, the result is tailored to the other zones.

The mine planner requires approximately one day to complete a schedule for each cutoff grade option and stope extraction capacity, resulting in a six- to eight-week exercise associated with evaluating a single zone’s NPV with respect to all cutoff grade options and production levels. This excessive planning time results in only approximate solutions for the other zones based on simplified analysis. This paradigm precludes the evaluation of alternate scenarios due to time limitations.

3.6 Data and Model Description

For our integer programming model, \( (UG) \), we use the same computer-generated stope shapes for each cutoff grade as are used in the manual method. The tonnage, average grade, and location of the stopes populate the model parameters. The quantity of extracted tonnes is given by the stope shape, and the development meters are estimated based on the number and size of the stopes in each zone. The cutoff grade drastically changes many attributes of the data, e.g., at 7-units-per-tonne cutoff grade, the total ounces are 83% less than at the 1-unit-per-tonne cutoff grade (Figure 3.6). The total number of stopes also decreases by 63% between the lowest and highest cutoff grade. Each scheduled activity is associated with cost or profit, quantity of extraction tonnes, and development meters. We schedule at a yearly fidelity and 9% discount rate. Variable costs consist of extraction, development, processing, and haulage costs on a per-tonne or per-meter basis. There exists an initial capital and annual fixed cost for each zone.

The integer programming (IP) model determines which underground mining activities to complete in each time period. The model maximizes the present value of a mine instance subject to the resource constraints, precedence relationships, and fixed mining rates. The NPV is then calculated by post-processing the fixed and capital costs. For an activity to be scheduled in a given time period, all of its predecessors must be scheduled in the same or in a previous time period. Delay constraints require an activity to be completed at least
Figure 3.6 Number of stopes per grid location; lighter areas represent fewer stopes. The top image shows the quantity of stopes available for extraction at a 1 unit-per-tonne cutoff grade, and the bottom image shows the quantity of stopes available for extraction at a 7 unit-per-tonne cutoff grade. An underground mine design changes significantly based on the number of stopes that are to be extracted.

one time period in advance of the given activity, and only allow activities to be scheduled if there remains enough time in a period to complete the activity. Stope extraction tonnage and development meters are limited by the resource capacity constraints. See Appendix A for the mathematical model (UG).

3.6.1 Solution Strategy

The integer programming model, (UG), only determines the optimal schedule for a single cutoff grade in each zone and for a fixed set of production capacities; we refer to each such combination of cutoff grades as a cutoff-grade triple. Because the zones compete for overarching production capacity, we must consider all of them within a single model. To obtain a globally optimal solution across all zones considering all cutoff-grade triples, we exploit an enumeration strategy whose effectiveness relies on two key features: (i) fast solutions times for each cutoff-grade triple, and (ii) the ability to bound the objective function value for
To address the first key feature, we employ the OMP Solver (Rivera et al., 2015) to quickly solve the linear programming relaxation of each cutoff-grade triple and to create an integer feasible solution from the corresponding linear programming relaxation. The OMP Solver is academic and tailored to solve RCPSP problems for which a vast majority of the constraints are precedence relationships, which is the case for our formulation. The algorithms used by this solver have been shown empirically to be two to three orders of magnitude faster than simplex-based methods for solving the linear programming relaxation of mine scheduling problems. A heuristic used to create near-optimal integer solutions to mine scheduling problems with our mathematical structure solves in seconds (Chicoisne et al., 2012; King et al., 2016b).

To address the second key feature, for each cutoff-grade triple, we solve the linear programming relaxation of $(UG)$. The resulting objective function value provides an upper bound for the net present value gained from the schedule across all zones for a specific cutoff-grade triple. If the heuristic yields an integer solution with an objective function value close to that of the linear programming relaxation, that integer solution is (near-)optimal. If the integer solution for a given cutoff-grade triple has an objective function value (i.e., a lower bound) that is greater than the linear programming relaxation’s objective function value (i.e., the upper bound) of another cutoff-grade triple, we can guarantee that cutoff-grade triple cannot results in the optimal schedule by dominance.

With a reasonable number of cutoff-grade triples, fast solution times, and strong bounds, enumeration is a viable strategy for determining the cutoff grade for an underground mine.

### 3.7 Computational Results

In this section, we provide computational results associated with determining the optimal cutoff grade, first for the Central Zone and then for the entire mine with the exception of the Upper Zone because its cutoff grade is already fixed by the mining company. The Central Zone-only schedule (i) calibrates our model parameters, (ii) shows the advantages of our
optimization framework, and (iii) provides a solution for the most profitable part of the orebody a priori. We use a deterministic approach because the mining company bases its decisions on a standardized geological model of the deposit, which is updated frequently as new information becomes available. The speed with which we are able to solve our instances makes these resolves now possible. All of the computational results are completed using a Dell PowerEdge R410 with 16 processors (2.72 GHz each) and 28 GB of RAM with OMP Solver version 1854.

3.7.1 Single-Zone Scheduling

We create a schedule for a set of cutoff grades for a series of production capacities, each instance of which is referred to as a cutoff grade-production capacity option, in the Central Zone to determine that above which the NPV does not significantly increase, i.e., the mine becomes limited by precedence and production rates and not by production capacities. Various cutoff grades must be explored because although the lowest one increases the amount of metal extracted, the marginal cost of said extraction eventually outpaces the revenue from the additional metal ounces. Additionally, this potential increase in ounces is spread over multiple years due to production constraints, and may not yield the highest discounted profit. Specifically, we vary the cutoff grade used to create the stope shapes from 1 unit per tonne to 7 units per tonne, inclusive, by 0.4 units-per-tonne increments. From a practical standpoint, 0.4 units per tonne is very detailed for a strategic mine schedule. Any finer fidelity would result in unnecessary computation, and add insignificant value to the mine plan. The detail in our cutoff grade increments allows for the construction of an “NPV curve” to identify favorable cutoff grades. Since these curves have been observed to be unimodular in practice, finer refinement need only occur near the peak of the curve.

The cutoff grade and NPV change significantly when we alter the production rate until we reach 100% of annual stope extraction. The optimal cutoff grade ranges from 3.0 to 4.2 units/tonne depending on the annual stope extraction. As the cutoff grade increases, the NPV curves for different production capacities become virtually identical, providing an
indication of the maximum stope extraction capacity for each cutoff grade. For example, if the mine operates with a 6.2 unit-per-tonne cutoff grade, an annual stope extraction capacity greater than 50% of full capacity does not add value (Figure 3.7). There is no significant increase in NPV at any cutoff grade beyond 100% of production capacity. These solutions provide a good bound on minimum and maximum cutoff grades which are likely to be optimal, and, in practice, if computational time is major concern when scheduling the entire mine, cutoff grades below 3 units per tonne or higher than 4.2 units per tonne may be omitted when scheduling the entire mine.

The solution times for each cutoff grade-production capacity option are fewer than 10 seconds, and the time to enumerate all of the Central Zone linear programming relaxations and create an integer solution is under 25 seconds for an annual production capacity of 100%. For a given production capacity curve in Figure 3.7, the heuristic is able to produce an integer solution that dominates all others, i.e., its NPV is greater than the NPV associated with any linear programming relaxation of a different cutoff grade; this allows us to mathematically guarantee that there exits a single optimal cutoff grade for a given production capacity for these numerical experiments. The heuristic finds solutions within 1% of optimality for our Central-Zone-only instances, demonstrating empirically that the bounds provided by the linear programming relaxation are tight. Detailed computational results are provided in the Appendix A.

3.7.2 Entire Mine Scheduling

Mine planners provide an overarching production capacity for all zones throughout the expected 20-year mine life. Within a zone, we establish an optimal, maximum capacity based on the same type of analysis as was conducted in the Single-Zone Scheduling section. The Central Zone may begin production in the first year, but the North and South Zones cannot begin until the fifth year (where development to begin said extraction may start as soon as necessary). With the production capacities fixed for the entire mine and each zone, we identify the highest-value NPV from 16 cutoff-grade options, any of which might occur
in the three mining zones: Central, South and North. The enumeration of this complete set results in \(16^3\) (4,096) cutoff grade triples for the mine in its entirety.

We use the heuristic to create an integer-feasible schedule associated with the linear programming solution corresponding to the highest NPV among all cutoff grade triples, and refer to this as the Original Schedule with a corresponding scaled NPV of 85.53. (We use the same scale as the Central Zone-only schedule.) The resulting relative difference between the linear programming bound and the integer-solution NPV is 0.11%. The cutoff grades corresponding to the highest NPV schedule are 3.8, 3.4, and 3.0 units per tonne for the South, Central, and North Zones, respectively. Unfortunately, our initial solution fails to consider some operational details. Specifically, the production profiles from the individual zones fluctuate unacceptably (Figure 3.8).

Specifically, the South Zone production peaks and then trends downwards, while the North and Central Zone production appears either in the shape of a single- or triple-hump,
Figure 3.8 Production rates for the entire mine and for each of the three zones as a fraction of the total maximum production level, 100, for each year in the planning horizon. The production rates in every zone vary too greatly to be operationally feasible.

respectively; in no case does the production level off at some consistent value. These features preclude the mine planner’s ability to coordinate production because, under this schedule, employee and production equipment would be oscillating between zones. We must therefore post-process our integer-programming generated schedules while sacrificing as little NPV as possible.

The mining company desires that only two of the three zones operate at one time, unless one zone is ending and another zone is beginning. Both the North and the South Zones start stope extraction as soon as possible in the Original Schedule due to the fact that stope value decreases with depth, and mining at the top of both the North and South Zones provides the most value. Mining in the South Zone first is preferred because of ventilation considerations; therefore, we adjust the production constraints for the South Zone to start
in the fifth year and delay extraction in the North Zone until the South Zone extraction is nearly complete. In this way, we alter the extraction and development capacities for the North Zone by preventing any stope extraction until year 9.

We re-run the entire enumeration procedure with this restriction, which results in an integer solution constructed from the linear programming relaxation with the highest objective function value of 83.9, only 1.8% lower than that from the Original Schedule. Cutoff grades of 3.8, 3.4, and 3.0 units per tonne for the South, Central, and North zones, respectively, remain the same. The South Zone has its development constructed in time for full stope extraction to occur in years 5 through 8 and end by year 11. The North Zone begins stope extraction in year 9 and operates at a constant production rate in years 10 through 13 before decreasing for the remaining zone life. The Central Zone slowly increases stope extraction in years 2 through 4 and levels off in years 5 through 9. Once the North Zone reaches full production, the Central Zone has a lower, but consistent, production rate in years 10 through 14, before quickly dropping off. (During the Central Zone’s ramp-up phase, the Upper Zone augments total ore tonnage.) These profiles yield a desirable total production for the mine. The adjusted production schedule is shown in Figure 3.9.

Not only are the production schedules desirable, but so is their solution time. On average, the cut-off grade triple instances contain 31,575 variables and 230,117 constraints; solution times for the linear programming relaxation of each cutoff-grade triple average 8.10 seconds. Similar to simplex-based methods, the OMP Solver only utilizes one core when solving the linear programming. Therefore, we expedite solutions by enumerating all 256 of the Central and North Zone cutoff-grade doubles for a given cutoff grade in the South Zone, and running the 16 different South Zone cutoff-grade options for each of the 256 doubles on 16 different cores. With this type of parallel computation, we are able to solve all of the linear programming relaxations in 2,510 seconds (see Appendix A). The minimum and maximum linear programming relaxation solution times across all 16 cores are 1.59 and 19.02 seconds, respectively. We obtain an integer solution for the top ten highest cutoff grade triples (with
Figure 3.9 Production schedule for the entire mine and for each of the three zones as a percentage of the total maximum production capacity. Although, the Central Zone’s stope extraction fluctuates slightly, the North and South Zone are level, and this is desirable. The overall capacity is reasonably smooth.

We examine the effect of fixing the cutoff grade in each zone on the overall NPV of the mine by fixing the cutoff grade for a selected zone and allowing that in the other two zones to vary (Figure 3.10). The Central Zone has the largest effect on NPV; setting the cutoff grade in this zone to 7.0 units per tonne reduces the maximum attainable NPV by 33%. An NPV within 0.48% of optimality is attainable if the Central Zone’s cutoff grade is between 3.0 and 3.8 units per tonne. Although the South Zone begins stope extraction earlier than the North Zone, the choice of a suboptimal cutoff grade is buffered by the choice of the correct cutoff grade in the other zones. The North zone has a small peak in NPV at 3.0 units-per-tonne cutoff grade, which corresponds to a solution in almost all of the 20 highest-NPV generating linear programming relaxations.
3.8 Conclusions

Mining companies must make a variety of decisions when starting an underground mine, and determining the cutoff grade has a significant impact on the NPV and mine design. After fixing a reasonable production capacity for each zone, we outline an optimization framework for producing the optimal cutoff grade for multiple mining zones by using an integer program. By examining linear programming relaxation bounds, it is possible to eliminate many cutoff-grade options from being optimal. Although it is possible to model the cutoff-grade decision implicitly, our enumeration strategy exploits the mathematical structure of the problem to solve the linear programming relaxation efficiently using the OMP Solver. An associated integer solution allows us to create value curves indicating the maximum NPV for every cutoff grade. These curves can be used to identify the monetary value associated with choosing a cutoff grade for a fixed production capacity in any mining zone.

Although our optimization framework follows the same structure as that of the mining company, we provide three distinct improvements: (i) We solve a different linear program-
ming relaxation in seconds for each instance to determine an upper bound on NPV as opposed to using a manual technique that requires a significant amount of time to create a solution; (ii) our optimization framework is able to be updated and rerun in a few days if any aspect of the mine changes, contrary to the six to eight weeks it takes to rerun the mining company’s model; (iii) it is possible to schedule the entire mine and select the optimal cutoff grade for multiple zones rather than examining a single zone.

Future work might include incorporating fixed and capital cost within the integer programming model to eliminate the post-processing step and to more accurately reflect a mine’s cost structure. Although we examine an open stoping operation, this optimization framework can be used in most stoping, drift and fill, and room and pillar mines or for a combination of mining methods, which may provide value to a mine operator, as the cutoff grade for each mining method is likely to be different.

3.9 Chapter Conclusion

This chapter expands the usage of the UG-PSP strategic formulation, considering only underground mining methods. In the next chapter, we examine the impacts of scheduling an open pit mine in concert with an underground mine. We borrow the UG-PSP strategic formulation and the idea of enumeration.
CHAPTER 4

OPTIMIZING THE OPEN PIT-TO-UNDERGROUND MINING TRANSITION

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We expand on the ideas presented in Chapters 2 and 3 to include open pit mine scheduling. A number of the constraints in the open pit scheduling model do not fit into a mathematical structure which we desire, but we show that this can be corrected with careful reformulation, which allows us to convert a “warehouse”-style inventory constraint into a precedence constraint and to identify a subset of variables to enumerate, i.e., branch upon. This chapter is also constructed for a mathematical audience and provides a more in-depth discussion of the mathematical properties of our formulations.

4.1 Abstract

A large number of metal deposits are initially extracted via surface methods, but then transition underground without necessarily ceasing to operate above ground. Currently, most mine operators schedule the open pit and underground operations independently and then merge the two, creating a myopic solution. We present a methodology to maximize the NPV for an entire metal deposit by determining the spatial expanse and production quantities of both the open pit and underground mines while adhering to operational production and processing constraints. By taking advantage of a new linear programming solution algorithm and using an ad-hoc branch-and-bound scheme, we solve real-world scenarios of our transition model to near optimality in a few hours, where such scenarios were otherwise completely unfeasible.

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intractable. The decision of where and when to transition changes the net present value of the mine by hundreds of millions of dollars.

4.2 Introduction and Literature Review

The mining industry contributes trillions of dollars annually to the global economy by providing minerals, metals, and aggregates. This, and volatile metal prices, make it critical that mines possess an efficient production schedule, which can be categorized as: (i) short-term (days to months), (ii) long-term (years), and (iii) strategic (life-of-mine) (Gershon, 1983). A short-term schedule might determine what material to process on a given day; a long-term schedule may examine production rate changes (Alonso-Ayuso et al., 2014; Epstein et al., 2012). Finally, a strategic schedule is used to evaluate large capital investments, and other decisions that have long-ranging impacts. Because the transition from open pit to underground extraction affects a mine for the remainder of its operational life, it falls into the category of strategic scheduling.

At the time of this writing, a large number of metal deposits are being extracted via surface methods, but plan to transition to concurrently or exclusively extracting ore via underground mining methods. For safety reasons, the underground mine must be sufficiently geographically separated, with horizontally positioned in situ rock, from the open pit mine via what is typically referred to as a crown pillar. Current industry practice places the crown pillar based on: (i) largest economically viable open pit mine, or (ii) the extraction method that results in the largest undiscounted profit for each three-dimensional discretization of the ore body and surrounding rock. Mine operators tend to delay the transition, leading to NPV losses of up to hundreds of millions of dollars. We provide a systematic means by which a mine operator can determine the highest value of a combined open pit and underground design.

The most common method used to extract material is open pit, or surface, mining. Open pit mines vary in both shape and size, and their design is based on the deposit’s block model, a model which discretizes the orebody and surrounding rock, and assigns a series of
attributes, including mining cost, degree of mineralization (referred to as grade), location, and the cost or profit associated with processing the specific block. Blocks can be categorized using a minimum cutoff grade; blocks at or above the cutoff grade are sent to the processing plant, referred to as a mill, while those below the cutoff grade are sent to a waste dump. The slope angle for the open pit mine, resulting from geotechnical constraints of the host rock, ensures the stability of the pit’s walls (Hustrulid et al., 2013).

Given the block attributes and slope angle, mine planners determine the largest economically viable pit for a given deposit, i.e., the ultimate pit limit (Lerchs & Grossman, 1965; Underwood & Tolwinski, 1998). However, while the solution to the ultimate pit limit problem yields the size of the open pit mine, it provides no indication of the extraction sequence required to maximize its discounted value. Johnson (1968) originally formulated the open pit block sequencing problem as an integer program that schedules the extraction of blocks such that the open pit’s value is maximized subject to resource and precedence constraints.

Solution techniques for open pit block sequencing problems are still widely studied (Chicoisne et al., 2012; Osanloo et al., 2008; Ramazan, 2007; Shishvan & Sattarvand, 2015; Souza et al., 2010; Topal & Ramazan, 2010). One such recent significant advance for the linear programming relaxation of a general version of the so-called precedence constrained production scheduling problem (PCPSP), i.e., the open pit block sequencing problem, is with the use of an algorithm outlined in Bienstock & Zuckerberg (2010), which exploits the problem structure (Muñoz et al., 2016). Lambert et al. (2014) present a guide to formulating and efficiently solving monolithic instances of the open pit block sequencing problem, i.e, without decomposition.

Underground mining is used when an economically viable deposit is situated sufficiently deep such that open pit mining is cost prohibitive. There exist many underground mining techniques: (i) open stoping (Figure 4.1), (ii) room-and-pillar, (iii) sublevel caving, (iv) drift-and-fill, (v) longwall, and (vi) block caving. Determining which method(s) to use is typically based on geotechnical constraints, size, and shape of the deposit (Qinglin et al.,
1996). For the purpose of this chapter, we confine our discussion to open stoping mining and its associated sequencing options.

A stope is a large, three-dimensional, mineable volume whose maximum size is correlated with the geotechnical properties of the host rock, and is the basic unit for stoping methods. The void left by an extracted stope is sometimes filled with an aggregate to provide structural stability, a process referred to as backfilling. Most underground stoping mines are separated into vertically spaced levels based on the maximum stope height, creating a near-regular grid of possible stope positions (Alford, 2006).

After determining possible locations from which the ore can economically be extracted, i.e., possible stope locations, mine planners design the development (Alford et al., 2007; Brazil, 2007), which is required to gain access to the ore, provide haulage routes, and maintain proper ventilation within the underground mine. All stoping activities require the completion of a specific set of development activities before that stope’s extraction can commence. Underground sequencing constraints are created after the design, and provide rules for the order in which the development and stopes are extracted. Given a fixed design and sequencing method, we can schedule the underground mining activities to, e.g., maximize NPV, or

Figure 4.1 (Open Stoping) In this mining method, rib pillars provide stability, as does the backfilling of open voids left by extracted stopes. Stope advance shows the direction in which mining proceeds.
minimize deviation from production targets (Brickey, 2015; Carlyle & Eaves, 2001; Martinez & Newman, 2011; Newman & Kuchta, 2007; O’Sullivan, 2013). Trout (1995) provides one of the first generalized formulations for underground stope scheduling; our formulation is a bit more streamlined than his in that we do not differentiate between scheduled and actual decisions, and because we assume that once an activity commences, it must continue at a prescribed rate until finished. The latter characteristic implies that our model contains no continuous variables. On the other hand, we determine sill pillar placement, i.e., locations in which material is left in situ to allow for a change in mining direction, which adds a layer of complexity.

An early transition model assigns large aggregated blocks to be extracted via open pit or underground mining methods in order to maximize value of the deposit (Bakhtavar et al., 2008). This idea was later improved to include the time element and to capture underground capital costs (Newman et al., 2013). In both previous transition models, there is little differentiation between the mining units used above and below ground. The mining industry comments on the difficulty of modeling the transition correctly (Finch, 2012); however, decisions regarding the transition are becoming increasingly relevant (Araneda, 2015). Figure 4.2 shows an open pit atop an underground mine. The transition zone is depicted as the material that would be extracted were it done via underground methods; the corresponding amount of material would greater were open pit methods used in the transition zone.

We present a new model and corresponding solution techniques to determine the timing of a transition from open pit to underground mining in both a spatial and a temporal sense. This transition incorporates a crown pillar placement that separates the open pit from the underground mine, and of the sill pillars, i.e., levels left in situ that can grant earlier access to stopes by creating a false bottom. Our methodology is based on an ad-hoc branch-and-bound approach that incorporates decomposition methods for solving PCPSP linear programming relaxations, and that includes rounding heuristics. We outline underlying models for the transition in Section 4.3. Mathematical reformulations to enhance tractability are presented.
Figure 4.2 (Transition Zone) The transition zone is an area where it is economically viable to extract material via open pit or underground mining methods. We see the open pit, black, encroaching on the underground mine, gray, in the transition zone.

in Section 4.4, and the solution strategy in Section 4.5. Sections 4.6 and 4.7 provide the numerical results and conclusions, respectively.

4.3 Underlying Models

In this section, we introduce three models that underlie our computationally tractable transition model. We first present a surface extraction formulation, followed by an underground formulation, and conclude with a preliminary transition formulation which is essentially a combination of the two.

4.3.1 Surface Model

We consider a surface model based on open pit mining with a multi-phase pit design (Figure 4.3), in which a phase corresponds to a sub-region of the pit. A block within a phase consists of all of the material in the phase that resides within a predefined vertical distance. (Note that some mine operators refer to our blocks as benches.) Inside each block, there exists a series of bins that are differentiated by grade, categorized as waste, low, medium, or high, and geological properties. This type phase-block-bin scheduling is common in the mining industry and is the basis for the Minemax (2013) software package. Whittle
Consulting (2013) have developed multiple products for this type of scheduling.

Figure 4.3 (Phase-block-bin Data Aggregation) The phases are shown as sub-pits with a block occupying a small vertical space within the phase. Each block is separated into bins based on grade. Material in the low- (stripes), medium- (checkered), and high- (waves) grade bins may go directly to the mill (dashed arrows), or to an individual stockpile (solid arrows). Waste is sent to the dump. Although naturally occurring material of different grades is scattered within the block, for stylistic purposes, we group material of each grade.

The objective of the surface model, \((S)\), is to schedule the extraction, stockpiling, and mill feed in such a way that the NPV is maximized, while adhering to annual extraction and milling capacity constraints. In addition, the desired shape of the open pit is maintained by precedence constraints, which can be categorized into two types: (i) intra-phase precedence expresses that the blocks inside the phase be extracted from the surface down and (ii) inter-phase precedence expresses that blocks inside a phase be extracted only after a specific block in the predecessor phase has been fully extracted. A maximum sinking rate restricts the number of blocks in each phase one can mine in a given time period based on operating constraints. We also require that the material contained in each block and bin be extracted in equal proportion to prevent selective extraction within the block. Once extracted, individual bins can be directed to one of three destinations: waste dump, stockpile, or mill. All material that is below cutoff grade is sent to the waste dump.
Stockpiling is important in our application, because processing stockpiled material augments the underground production to ensure that the mill remains at maximum capacity after extraction ceases in the open pit mine. Bley et al. (2012) outline the formulation from which we construct our “warehouse-style” stockpiling strategy, i.e., each stockpile contains only one block-bin combination; the objective function value corresponding to an optimal solution subscribing to this strategy provides an upper bound on the NPV that can be obtained. Material retrieved from the stockpile is identical to material placed in the stockpile. Some authors have attempted to use “mixing constraints” to more accurately model the characteristics of material retrieved from a stockpile (Bley et al., 2012), but Moreno et al. (2016) show that there are both more accurate and more tractable methods for modeling inventory; they also show that the warehouse model indeed provides a reasonable approximation of reality, within a few percentage points of the “real” net present value for representative data sets (not unlike our own). For ease of exposition and to enable us to use a special solution strategy, we omit rehandling costs from our formulations. For our transition model, these costs prove to be insignificant; post-processing them into the objective function value results in hundredths of a percentage change, an amount that is not likely to significantly increase were we to impose the rehandling costs a priori and certainly not sufficiently substantial to consider as part of strategic planning costs.

We define the notation below. In general, use of lower case letters is reserved for indices and parameters. Upper case letters in Roman font represent variables, and sets are given in calligraphic font. An $S$ superscript on a parameter or variable denotes notation specific to the surface model; we use hats to differentiate parameters and variables that represent similar entities.

**Indices and sets:**

$b \in B$ blocks $b$

$n \in N_b$ bins in block $b$

$\hat{b} \in \hat{B}_b$ blocks that must be mined directly before block $b$
$b \in B_p$ blocks in phase $p$

$d \in D$ bin destination (1 = mill, 2 = stockpile, 3 = waste)

$p \in P$ phases $p$

$r \in R$ resources (1 = mine, 2 = mill, 3 = sinking rate)

$t \in T$ time periods

Data:

$\ell_{nb}^-$ mining cost for bin $n$ in block $b$ [$\$]

$\ell_{nb}^+$ revenue generated after having milled bin $n$ of block $b$ [$\$]

$q_{rbn}$ quantity of resource $r$ consumed by bin $n$ of block $b$ [1 & 2 = tonnes]

$q_{r}, \bar{q}_{r}$ minimum, maximum amount of resource $r$ available in time $t$ [1 & 2 = tonnes, 3 = blocks]

$\delta_t$ discount factor for time period $t$ (fraction)

Decision variables:

$X_{bt}^S$ 1 if block $b$ has finished being extracted by the end of time $t$; 0 otherwise

$Y_{nbdt}^S$ fraction of bin $n$ in block $b$ extracted by the end of time $t$ and sent to destination $d$

$I_{nbt}^S$ fraction of bin $n$ from block $b$ in the stockpile at the end of time $t$

$I_{nbt}^-$ fraction of bin $n$ from block $b$ sent to the mill from stockpile at the beginning of time $t$

\[
\max \sum_{b \in B} \sum_{n \in N_b} \sum_{t \in T} \delta_t \ell_{nb}^+ (Y_{nb1}^S - Y_{n, b, t-1}^S) + I_{nbt}^- - \sum_{b \in B} \sum_{n \in N_b} \sum_{d \in D} \sum_{t \in T} \ell_{nb}^- (Y_{nbdt}^S - Y_{n, b, t-1}^S) \quad (4.1a)
\]

s.t.

$\sum_{d \in D} \sum_{n \in N_b} \sum_{t \in T} Y_{n, b, t-1}^S \leq \sum_{d \in D} \sum_{n \in N_b} Y_{nbdt}^S \forall b \in B, n \in N_b, t \in T \quad (4.1b)$

$X_{b, t-1}^S \leq X_{bt}^S \forall b \in B, t \in T \quad (4.1c)$

$\sum_{d \in D} Y_{n-1, b, d}^S = \sum_{d \in D} Y_{nbdt}^S \forall b \in B, n \in N_b, t \in T \quad (4.1d)$
The objective (4.1a) maximizes discounted revenue associated with mill profits, and mining costs. Constraints (4.1b) and (4.1c) ensure that once a bin-block combination is completed, it remains completed. Constraints (4.1d) preclude selective mining of any bin in a block, i.e., the constraint forces all bins to be mined in equal proportion. Constraints (4.1e) relate the fractional and binary extraction variables. Constraints (4.1f) enforce precedence by preventing the extraction of a block until its predecessors’ blocks have been fully extracted. Constraints (4.1g) balance the inventory in the stockpile at the end of every time period. Constraints (4.1h) limit the capacity for extraction tonnes in each time period. Constraints (4.1i) bound processing at the mill in each time period. Constraints (4.1j) prevent mining too rapidly in one phase. Constraints (4.1k) enforce nonnegativity and integrality of the decision variables, as appropriate.

### 4.3.2 Underground Model

Our underground formulation incorporates a mine design based on pre-constructed stope shapes, organized into vertical levels. Drifts, i.e., tunnels that are only open at one end, are used to access the mine. A vertical decline is a drift that descends from the surface to the lowest underground level. On each level, horizontal drifts are constructed from the decline to
the stope locations. Our method sequences stopes from the bottom up such that extraction and backfilling on the level underneath the given level must be completed before extraction on the given level can begin. The method advances such that mining proceeds away from an initial stope determined a priori. The ore contained within a sill pillar (Figure 4.4) is partially sterilized and can only be recovered, with significant dilution, at the end of the mine life. Sill pillar placement must balance the sterilization of ore with the increase in net present value gained by earlier access to stopes.

![Figure 4.4 (Regular Grid of Stopes)](image)

Figure 4.4 (Regular Grid of Stopes) Sill pillar levels are black, and each block constitutes a designed stope. Any horizontal level of stopes shown in the figure could act as a sill pillar.

The precedence relationships for an underground mine that uses this sequencing method can be categorized as follows: (i) Fixed predecessors include the development required to access the stope, and stopes on the same level. These predecessors ensure that with respect to the mining direction, the adjacent stope on the same level be fully extracted and backfilled before extraction of the next stope can commence. If no adjacent stope on the level exists, then the stope has only development activities as predecessors. (ii) Conditional stope predecessors require that the stope directly below and the stopes on either side of the given stope on the level below must be fully extracted and backfilled before the given stope can be extracted. If the stopes on the level below act as a sill pillar, then the conditional predecessors are omitted (Figure 4.5). In addition, every underground activity has a set of predecessor activities that are dictated by the mine design.
Figure 4.5 (Sequencing Method) The initial stope location is labeled, with a black arrow, and Stope X’s predecessors are denoted by gray arcs. On the left is an example of standard precedence, i.e., both fixed and conditional predecessors. On the right is a stope that has only fixed predecessors, because the sill pillar eliminated all of the conditional predecessors.

The underground mine scheduling model, \((\mathbb{U})\), determines sill pillar placement and a life-of-mine schedule consisting of development, stoping, and backfilling activities to maximize the underground mine’s NPV. This model precludes specific pairs of activities from being completed in the same time period, and resource constraints limit development, extraction, and backfilling. We assume fixed activity rates.

We maintain the same notation style as in the surface model, but use the superscript \(\mathbb{U}\) to represent underground-specific parameters and variables; we use checks and bars as accents.

Indices and sets:

\(a \in \mathcal{A}\) set of all activities
\(s \in \mathcal{S} \subset \mathcal{A}\) set of stoping activities
\(\bar{a} \in \mathcal{A}_a\) set of fixed predecessors for activity \(a\)
\(\bar{\bar{a}} \in \mathcal{A}_a\) set of fixed predecessor activities \(\bar{a}\) that must be completed one time period in advance of activity \(a\)
\( \tilde{s} \in \tilde{S} \subset \tilde{A} \) \( s \) \( \tilde{s} \in S \subset A \) set of conditional predecessors \( \tilde{s} \) below stope \( s \) set of conditional predecessor stopes \( \tilde{s} \) that must be completed one or more time periods in advance of stope \( s \)

\( l \in \mathcal{L} \) levels in the mine

\( l \in \tilde{\mathcal{L}}_s \) level on which stope \( s \) exists (set has cardinality of one)

\( r \in \mathcal{R} \) resources (4 = mine/mill capacity, 5 = backfill capacity, 6 = development capacity)

**Data:**

- \( c^U_a \) monetary value associated with completing activity \( a \) [\$]
- \( q^U_{ra} \) quantity of resource \( r \) associated with activity \( a \) [2, 4 & 5 = tonnes, 6 = meters]
- \( \underline{L}_{rt}, \bar{r}_{rt} \) minimum, maximum level of resource \( r \) in time \( t \) [4–5 = Mtonnes/yr, 6 = m/yr]
- \( \delta_t \) discount factor for time period \( t \) (fraction)

**Decision variables:**

- \( X^U_{at} \) 1 if activity \( a \) is finished by the end of time \( t \); 0 otherwise
- \( W^U_l \) 1 if level \( l \) serves as a sill pillar; 0 otherwise

\[
\begin{align*}
\text{(U)} & \quad \max \sum_{a \in A} \sum_{t \in T} \delta_t c^U_a (X^U_{at} - X^U_{a,t-1}) \quad (4.2a) \\
\text{s.t.} & \quad X^U_{a,t-1} \leq X^U_{at} \quad \forall a \in \mathcal{A}, t \in T \\
& \quad X^U_{at} \leq X^U_{\tilde{a},t} \quad \forall a \in \mathcal{A}, \tilde{a} \in \tilde{\mathcal{A}}, t \in T \\
& \quad X^U_{at} \leq X^U_{\tilde{s},t} + W^U_l \quad \forall s \in S, \tilde{s} \in \tilde{S}_s, l \in \tilde{\mathcal{L}}_s, t \in T \\
& \quad X^U_{at} \leq X^U_{\tilde{a},t-1} \quad \forall a \in \mathcal{A}, \tilde{a} \in \tilde{\mathcal{A}}, t \in T \\
& \quad X^U_{at} \leq X^U_{\tilde{s},t-1} + \sum_{i \leq j \leq k} W^U_j \quad \forall s \in S, \tilde{s} \in \tilde{S}_s, i \in \tilde{\mathcal{L}}_s, j \in \tilde{\mathcal{L}}_s, k \in \tilde{\mathcal{L}}_s, t \in T \\
& \quad X^U_{at} + W^U_l \leq 1 \quad \forall s \in S, l \in \tilde{\mathcal{L}}_s, t \in T \\
& \quad L_{rt} \leq \sum_{a \in \mathcal{A}} q^U_{ra} (X^U_{at} - X^U_{a,t-1}) \leq \bar{r}_{rt} \quad \forall r \in \mathcal{R} \exists r \geq 4, t \in T \\
& \quad X^U_{at}, W^U_l \text{ binary} \quad \forall a \in \mathcal{A}, l \in \mathcal{L}, t \in T \\
\end{align*}
\]
The objective function (4.2a) maximizes net present value. Constraints (4.2b) ensure that once an activity is completed, it remains completed. Constraints (4.2c) enforce fixed precedence, and constraints (4.2d) enforce conditional precedence based on sill pillar placement. Constraints (4.2e) ensure that at least one time period elapses between the completion of the specified pair of activities. Constraints (4.2f) force at least one time period to elapse between the completion of two given stoping activities unless a sill pillar exists on a level inbetween them; these stoping activities need not be on consecutive levels because the precedence might actually allow stopes on consecutive levels to be mined in the same time period. Constraints (4.2g) prevent mining the stopes on level \(l\), if level \(l\) acts as a sill pillar. Constraints (4.2h) bound extraction, backfill, and development resource use. (Mill processing capacity is essentially unconstrained underground because of the low production rate.) All variables are required to be binary by (4.2i).

Delay constraints, (4.2e) and (4.2f), capture sub-annual detail in our model with annual time fidelity, and exclude two specified activities \((a', a)\) from being completed in the same time period, where \(a'\) is a predecessor of \(a\), and the minimum time required to elapse between the start of \(a'\) and the completion of \(a\) is greater than the time fidelity of the model. It is important to construct a minimal number of delay constraints so as not to unnecessarily increase the number of precedence constraints.

### 4.3.3 Basic Transition Model

The basic transition model, \((T^b)\), may be formulated by combining of the surface, \((S)\), and underground, \((U)\), models. The objective function maximizes the NPV of the combined open pit and underground operations, i.e., the sum of (4.1a) and (4.2a). A vast majority of the constraints, (4.1b)-(4.1h), (4.1j), (4.1k), and (4.2b)-(4.2i), remain the same as in their respective models. The precedence constraints in both the open pit and underground mine do not need to be changed based on the crown pillar location, because of the direction of extraction for each method. Any non-zero lower bounds on the underground mine’s resource constraints are removed to allow for a delayed start of the underground mine.
The resource constraints must be altered to accurately reflect that the open pit, stockpile, and/or underground mine may be sending material to the mill in the same time period. Constraints associated with the additional variables, i.e., that represent the crown pillar location, preclude any open pit or underground extraction of material located in the crown pillar. Additional notation is shown below with the superscript $T^b$ representing transition model-specific variables.

Indices and sets:

$v \in \mathcal{V}$ set of crown pillar elevations

$\tilde{b} \in \tilde{\mathcal{B}}_v$ set of blocks that exist below the crown pillar if the crown pillar is located at elevation $v$

$\tilde{a} \in \tilde{\mathcal{A}}_v$ set of activities that exist above the crown pillar if the crown pillar is located at elevation $v$

Decision variables:

$W_{T^b}^{T^b}$ 1 if the crown pillar is located a elevation $v$; 0 otherwise

\begin{equation}
\begin{aligned}
\max \left\{ \sum_{b \in B} \sum_{n \in N_b} \sum_{t \in T} \delta_t c_n^{S-} \left( (Y_{nb1,t}^S - Y_{nb1,t-1}^S) + I_{nb1,t}^S \right) - \right. \\
\sum_{b \in B} \sum_{n \in N_b} \sum_{d \in D} \sum_{t \in T} \delta_t c_n^{S-} \left( Y_{nbdt}^S - Y_{nbdt-1}^S \right) \\
+ \sum_{a \in A} \sum_{t \in T} \delta_t c_a^U \left( X_{at}^U - X_{a,t-1}^U \right) \\
\left. \right. \\
\text{s.t.}
X_{bt}^S \leq 1 - W_{T^b}^{T^b} \quad \forall \tilde{b} \in \tilde{\mathcal{B}}_v, v \in \mathcal{V}, t \in \mathcal{T} \\
X_{at}^U \leq 1 - W_{T^b}^{T^b} \quad \forall \tilde{a} \in \tilde{\mathcal{A}}_v, v \in \mathcal{V}, t \in \mathcal{T} \\
\sum_{v \in \mathcal{V}} W_{T^b}^{T^b} = 1 \\
\sum_{b \in B} \sum_{n \in N_b} q_{rb}(Y_{nb1,t}^S - Y_{nb1,t-1}^S) + q_{nb1,t}^S + \\
\sum_{a \in A} q_a^U (X_{at}^U - X_{a,t-1}^U) \leq r_{rt} \quad \forall r \in \mathcal{R} \ni r = 2, t \in \mathcal{T} \\
W_{T^b}^{T^b} \text{ binary} \quad \forall v \in \mathcal{V}
\end{aligned}
\end{equation}

Retained constraints from ($S$): (4.1b), (4.1c), (4.1d), (4.1e), (4.1f),
The objective function (4.3a) maximizes net present value of the entire deposit and replaces (4.1a) and (4.2a). Constraints (4.3b) allow for open pit mining to only occur above the crown pillar. Constraints (4.3c) restrict underground mining to only occur below the crown pillar. Constraint (4.3d) forces the placement of a crown pillar. Constraints (4.3f) replace constraints (4.1i) with respect to mill capacity.

4.4 Reformulations

The basic transition model, \((T^b)\), is theoretically NP-hard, and, in practice, real-world size problems are intractable with current computer hardware and software. Our scenarios contain nearly 50,000 variables and more than 1.5 million constraints, even after efficient variable elimination techniques are used (Lambert et al., 2014; O’Sullivan, 2013).

Bienstock & Zuckerberg (2010) provide an algorithm, the “BZ algorithm,” for efficiently solving the LP relaxation of problems with the math structure seen in PSPCP, i.e., a model in which a majority of the constraints are precedence, rather than “side,” e.g., knapsack, constraints. In practice, the BZ algorithm’s solution time is more sensitive to the latter type of constraints than to the number of precedence constraints. Muñoz et al. (2016) provide an implementation framework for solving the LP relaxation of open pit mining problems using the BZ algorithm, and show that it is possible to obtain LP relaxation solutions to PCPSPs with millions of variables and precedence constraints, but fewer than 200 “side” constraints, orders of magnitude faster than simplex-based methods. With reformulation and an ad-hoc branch-and-bound strategy, we are able to identify open pit-to-underground transition options with near-optimal NPVs.

We reformulate the basic transition model \((T^b)\) by transforming some of the side constraints into precedence constraints, specifically, a special knapsack, (4.1j), and the “warehouse-style” inventory, (4.1g), constraints in the surface model, \((S)\). Mathematical proofs showing
that these reformulations are no weaker than the original formulations can be found in Appendix B.

### 4.4.1 Special Knapsack Reformulations

We show how to transform sinking rate constraints, (4.1j), into precedence constraints by exploiting the facts that: (i) the blocks within a phase are required to be completed in a fixed order, i.e., blocks must be extracted in sequential order from the surface downwards, and (ii) the left-hand side is 0 for all time periods in all scenarios. Constraints (4.1j) from the initial surface model ($S$) appear as follows:

$$
\sum_{b \in B_p} (X^S_{bt} - X^S_{b,t-1}) \leq \bar{r}_rt \quad \forall p \in P, r \in R \ni r = 3, t \in T
$$

(4.4)

The reformulation of constraint (4.4) prevents a block $b$ that is $\bar{r}_3t$ successor blocks away from the selected block $\tilde{p}$ in the phase from being completed in the same time period as block $b$. This requires the following set definition:

$\tilde{p} \in \tilde{P}_b$ predecessors for block $b$ that must be completed at least one time period prior to block $b$

The reformulation is shown in (4.5):

$$
X^S_{bt} \leq X^S_{\tilde{p},t-1} \quad \forall b \in B, \tilde{p} \in \tilde{P}_b, t \in T
$$

(4.5)

This constraint set, (4.5), has far greater cardinality than (4.4), but possesses precedence structure. Figure 4.6 shows an example of the constraint construction.

### 4.4.2 Inventory Balance Reformulations

In this section, we present a reformulation of the inventory balance constraints (4.1g):

$$
I^S_{nb,t+1} = I^S_{nbt} - I^S_{nbt} + (Y^S_{nb2t} - Y^S_{nb2,t-1}) \quad \forall b \in B, n \in N_b, t \in T
$$

(4.6)
Figure 4.6 (Special Knapsack Reformulation) An additional precedence arc, dashed, is added to prevent block $\tilde{p}$ and the successor block $b$ from being completed in the same time period, because their precedence separates them by more blocks than can be completed in a time period. Immediate precedence is shown with solid arcs.

Our reformulation implies that material must be placed in inventory before it is processed in the same or in a later time period, and is mathematically equivalent to the original under the assumption that there is no value lost for placing material in inventory, i.e., there is no mixing, degradation, or rehandling cost associated with placing or retrieving material. We require the following variable definitions:

\[ \tilde{Y}_{Sbt} \] fraction of block $b$ extracted and able to be processed by the end of time $t$

\[ Z_{Snb} \] fraction of bin $n$ in block $b$ sent to the mill by the end of time $t$

We replace all instances of variables $Y_{nbdt}$ with the appropriate $\tilde{Y}_{Sbt}$ or $Z_{Snb}$ variables, where the former newly introduced variable represents the fraction of a block extracted in time $t$, without recognizing the destination. The latter newly introduced variable $Z_{Snb}$ tracks both the processing time period and destination of each bin-block combination. If both variables for a given bin-block combination assume a value of 1 in the same time period, that bin-block combination is immediately sent to the mill for processing after extraction. For all periods in which a specific bin-block combination is in the stockpile, the variable representing that block’s extraction, $\tilde{Y}_{Sbt}$, assumes a value of 1 and the corresponding variable representing processing, $Z_{Snb}$, assumes a value of 0. Any bin-block combination that is extracted and not processed is sent to the waste dump, resulting in all corresponding $Z_{Snb}$ variables possessing a value of 0. The reformulation of (4.6) is shown in (4.7):
\[ Z_{nb,t}^S \leq \hat{Y}_{bt}^S \quad \forall b \in B, n \in N_b, t \in T \] (4.7)

Constraints (4.7) allow only material that has been extracted to be sent to the mill. This reformulation also requires substituting the variables \( Y_{nb,t}^S \) and \( I_{nb,t}^S \) in the objective function and mill capacity constraints with \( \hat{Y}_{bt}^S \) and \( Z_{nb,t}^S \):

\[
\text{max} \sum_{b \in B} \sum_{n \in N_b} \sum_{t \in T} \delta_t c_{nb}^S (Z_{nb,t}^S - Z_{nb,t-1}^S) - \sum_{b \in B} \sum_{n \in N_b} \sum_{t \in T} \delta_t c_{nb}^S (\hat{Y}_{bt}^S - \hat{Y}_{b,t-1}^S) \quad (4.8)
\]

and

\[
L_{rt} \leq \sum_{b \in B} \sum_{n \in N_b} q_{nb}(Z_{nb,t}^S - Z_{nb,t-1}^S) \leq \bar{r}_{rt} \quad \forall r \in R \ni r = 2, t \in T \quad (4.9)
\]

For all other constraints, the variable \( \hat{Y}_{bt}^S \) replaces \( Y_{nb,t}^S \) and, the variables \( I_{nb,t}^S \) and \( I_{nb,t}^S \) are eliminated.

### 4.4.3 Enhanced Transition Model

The enhanced transition model is the combination of the reformulated surface (\( \mathcal{S} \)) and underground (\( \mathcal{U} \)) mine scheduling models, in which the crown and sill pillar placements are fixed a priori. All other constraints are similar to those in the basic transition model, (\( \mathcal{T}^b \)). The hat and tilde accents are reserved for open pit sets, and bar accents for underground sets. Additional notation and the enhanced transition model (\( \mathcal{T}^e \)) follow:

**Indices and sets:**

- \( \bar{P} \): predecessors for activity \( a \) that must be completed at least one time period in advance of activity \( a \)

\[
(\mathcal{T}^e) \quad \text{max} \sum_{b \in B} \sum_{n \in N_b} \sum_{t \in T} \delta_t c_{nb}^S (Z_{nb,t}^S - Z_{nb,t-1}^S) - \sum_{b \in B} \sum_{t \in T} \delta_t c_{b,t}^S (\hat{Y}_{bt}^S - \hat{Y}_{b,t-1}^S) \quad (4.10a)
\]

\[
+ \sum_{a \in A} \sum_{t \in T} \delta_t c_{a^U} (X_{a,t}^U - X_{a,t-1}^U)
\]

55
The objective function (4.10a) maximizes net present value, and replaces the objective function (4.3a). Constraints (4.10b), (4.10c), (4.10d), and (4.10e) ensure that each completed activity or block remains completed, and are a substitute for (4.1b), (4.1c), and (4.2b). Constraints (4.10f) and (4.10g) enforce the precedence structure for the open pit mine, and replace (4.1e) and (4.1f). Note that the replacement constraints do not sum on the destination index because all material is sent to a stockpile, even if just instantaneously. Constraints (4.10h) ensure that the fraction of a bin that is sent to the mill is no greater than the fraction extracted from the corresponding block, and is a reformulation of (4.1g). Constraint (4.10i) enforces underground mine precedence, and is used instead of constraints (4.2c), (4.2d), and (4.2g). Constraints (4.10j) and (4.10k) ensure that one time period elapses between the completion of two specific activities or blocks, and are a replacement for
constraints (4.1j), (4.2e), and (4.2f). Constraint (4.10l) bounds open pit-specific resource use, and is a substitute for constraints (4.1h). Constraint (4.10m) bounds the mill capacity, and is a substitute for constraints (4.3f). Constraint (4.10n) bounds underground-specific resource consumption, and is equivalent to constraints (4.2h). Constraints (4.10o) and (4.10p) enforce binary and variable bounds, where appropriate.

4.5 Solution Strategy

We obtain near-optimal solutions for the enhanced transition model, \( T^e \), presented in §4.4.3, by: (i) exhaustively searching possible crown and sill pillar placement options using an ad-hoc branch-and-bound strategy and solving the resulting LP relaxations, (ii) using a rounding heuristic to convert the LP relaxation solutions with favorable objective function values into integer solutions, and (iii) using integer solutions to eliminate a number of possible crown and sill pillar placement options to reduce the amount of computation required in (ii).

The reformulation in Subsections 4.4.1 and 4.4.2 reduces the number of side constraints in the basic transition model, \( T^b \), but the model is still not in the desired form for obtaining an efficient LP relaxation solution using the BZ algorithm. By fixing, i.e., branching on, all of the variables associated with the placement of the crown and sill pillars, \( W_i^U \) and \( W_v^T \), respectively, we convert all of the conditional precedence constraints, (4.2d) and (4.2f), in the underground model, \( U \), to standard precedence constraints, and the basic transition model at each node to a model with a PCPSP mathematical structure. We branch as follows: For each possible crown pillar placement (which, for our data set, is twelve), we consider all sill pillar placements consisting of between zero and three such pillars, where three would be a maximum operationally feasible number. The total number of viable crown and sill pillar placement options numbers in the thousands.

We first solve the LP relaxation of the transition model for each set of reasonable crown and sill pillar placements, i.e., for each branch, using the BZ algorithm. We then sort these LP relaxation solutions, decreasing by objective function value. Solutions with the best LP relaxation objective function values are transformed into IP solutions using TopoSort
(Chicoisne et al., 2012), which has been shown to provide near-optimal solutions quickly for open pit mine scheduling problems that only have non-zero upper bounds on resource constraints, and which is based on the premise that the earlier the expected completion time of a block or activity in the LP solution, the earlier the block or activity is scheduled in the IP solution. The algorithm maintains precedence constraints by the fact that the expected completion time of a block or activity in the LP relaxation is always greater than or equal to that of its predecessors. Also employed in our variant of TopoSort is an “alpha points” procedure in which activities are ordered not by their expected completion time, but by the time period in which a specified fraction, i.e., alpha point, of the activity has been completed. Therefore, an alpha point of 0.7 would set the order based on the first time period in which the “by” variable obtains a value larger than 0.7. The TopoSort heuristic allows for us to match the \((\mathbb{T}^e)\) formulation exactly, i.e., create a mixed integer solution to the open pit portion, and a fully integral solution to the underground portion. Once an IP solution is obtained from the LP relaxations with the largest objective function values, we use bound dominance to eliminate a significant number of the crown and sill pillar placement options. Specifically, every LP relaxation whose objective function value is less than that of an existing feasible IP solution’s cannot correspond to an optimal integer placement of the crown and sill pillar.

4.6 Data and Numerical Results

We introduce the data required for the enhanced transition model. Computational results highlight the speed, effectiveness, and robustness of the methodology which yields consistent near-optimal solutions to our multiple scenarios of the transition model.

4.6.1 Data

Our industry partner provided all of the data required for the transition model from an active mine; grade and cost data are confidential. The deposit is known to extend over a large vertical expanse, and the overlap between the upper-most designed stope and lowest-planned
open pit extraction elevation is over 400 meters; nearly 80% of the remaining recoverable material is located in this overlap, or transition zone.

The open pit dataset consists of a four-phase design for a partially extracted open pit mine with a total of 336 blocks ranging in weight from 20,000 to 5,500,000 tonnes. Blocks may contain a high-, medium-, and low-grade bin for two material types based on processing properties, and a waste bin. This results in a total of 1,312 bin-block combinations ranging from 250 to 1,100,000 tonnes. Extraction activities may possess as many as three immediate predecessor activities. The cost of extraction increases as the depth of the open pit increases. Subsection 4.3.1 provides a detailed description of the open pit precedence and physical representation of the data.

Our basic underground model dataset consists of 1,123 development activities, 351 stopping activities and an equal number of backfilling activities. Stopes range from approximately 5,000 to 40,000 tonnes, resulting in 17 levels in the underground mine that are a maximum of 40 meters in height. The required development and backfilling is estimated based on the stope properties. Each activity has up to 12 immediate predecessors and up to 100 delay constraints. Subsection 4.3.2 provides a description of the underground mine’s precedence structure. For our analysis, we construct ten distinct scenarios, each for a 24-year time horizon with decisions made at yearly fidelity, and each defined as a set of upper bounds on the resource constraints and a given discount rate (Table 4.1). We set the underground backfilling capacity equal to the underground extraction capacity.

### 4.6.2 Numerical Results

We compare the performance of the OMP Solver (Muñoz et al., 2016) to that of AMPL/CPLEX, (AMPL, 2014; IBM CPLEX Optimizer, 2015), using a Dell PowerEdge R410 with 16 processors (2.72 GHz each) and 28 GB of RAM. OMP, Version 1509 is an academic, customized solver that uses standard preprocessing and exploits the mathematical structure of PCPSP to solve the LP relaxation quickly using the BZ algorithm; then, we execute the TopoSort heuristic eleven times, each with a different alpha point value between
Table 4.1 (Scenario Summary) Capacities and discount rates used in each scenario, with constraint numbers from the enhanced transition model also given in the column headers.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Annual Capacities</th>
<th>Annual Discount Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Open Pit (t) (4.10)</td>
<td>Underground (t) (4.10)</td>
</tr>
<tr>
<td>1</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>2</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>3</td>
<td>50,000,000</td>
<td>1,500,000</td>
</tr>
<tr>
<td>4</td>
<td>40,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>5</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>6</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>7</td>
<td>50,000,000</td>
<td>1,500,000</td>
</tr>
<tr>
<td>8</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
<tr>
<td>9</td>
<td>50,000,000</td>
<td>1,500,000</td>
</tr>
<tr>
<td>10</td>
<td>50,000,000</td>
<td>2,000,000</td>
</tr>
</tbody>
</table>

0 and 1, inclusive, incremented by 0.1; this procedure transforms the LP relaxation to an integer-feasible solution, of which we choose the best one. All other parameter settings are default. CPLEX 12.6.0.0 uses default parameter settings other than memory emphasis, and a 40,000-second time limit. Variable elimination techniques are employed before passing the model to CPLEX (Lambert et al., 2014; O’Sullivan, 2013). Both solvers provide solutions to the enhanced transition model, \( T_e \), with the same crown and sill pillar placement options available in each scenario. Depending on the crown and sill pillar placement, for our dataset, the enhanced transition model, \( T_e \), averages 50,000 variables and 1.5 million constraints. (The numerical performance of \( T_b \) is dominated by that of \( T_e \) using our methodology; see Appendix B.)

We first compare the performance using CPLEX to solve the enhanced transition model \( T_e \) for a fixed crown and sill pillar location (giving CPLEX the benefit of the faster LP solver) against that of the OMP Solver. For a representative crown and sill pillar placement given as the ordered pair \([(820), (460)]\), where these elevations are relative to sea level, the enhanced transition model, \( T_e \), contains approximately 60,000 variables and 1,200,000 constraints, of which 120 are “side” constraints. CPLEX averages 163.77 seconds with the faster LP solver for each LP relaxation over the ten scenarios, and produces slightly
better integer solutions in only two of the ten scenarios (while CPLEX is unable to find an
integer-feasible solution in the other scenarios due either to memory or time limitations).
By contrast, OMP is able to solve the LP relaxations in fewer than ten seconds, regardless
of the scenario, and produces an integer solution within 6% of optimality or better in just a
few additional seconds (Table 4.2).

Table 4.2 (OMP and CPLEX comparison) Comparison of solution times and optimality gaps
between the OMP Solver and CPLEX for (Tc). All scenarios are run with a crown pillar
located at elevation 820 and a sill pillar located at level 460.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>CPLEX</th>
<th>OMP Solver</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LP Solution Time (sec)</td>
<td>IP Solution Time (sec)</td>
</tr>
<tr>
<td>1</td>
<td>Barrier</td>
<td>Simplex</td>
</tr>
<tr>
<td>2</td>
<td>163.75</td>
<td>488.41</td>
</tr>
<tr>
<td>3</td>
<td>177.85</td>
<td>1631.15</td>
</tr>
<tr>
<td>4</td>
<td>183.08</td>
<td>577.07</td>
</tr>
<tr>
<td>5</td>
<td>146.24</td>
<td>613.93</td>
</tr>
<tr>
<td>6</td>
<td>172.88</td>
<td>783.70</td>
</tr>
<tr>
<td>7</td>
<td>152.41</td>
<td>416.32</td>
</tr>
<tr>
<td>8</td>
<td>152.00</td>
<td>1300.20</td>
</tr>
<tr>
<td>9</td>
<td>163.26</td>
<td>707.38</td>
</tr>
<tr>
<td>10</td>
<td>162.59</td>
<td>653.63</td>
</tr>
</tbody>
</table>

*CPLEX is allowed to choose the variant of simplex to use, which results in employing dual
simplex on the dual problem
† CPLEX is unable to produce an integer solution within a 5% gap before running out of memory
‡ CPLEX is unable to produce an integer solution within a 5% gap before the 40,000-second limit
Note: Optimality gaps are calculated as 100% · \left(1 - \frac{\text{IP Obj. Func. Value}}{\text{LP Relaxation Value}}\right)

Figure 4.7 depicts the LP relaxation objective function value and the best known IP
objective function value for each reasonable set of crown and sill pillar placements for Scenario
1 using the OMP Solver. Both the LP relaxation objective function value and the best-known
IP objective function value follow the same trend as we exhaustively enumerate all of the
3,500 crown and sill pillar placement options. The average gap between the LP relaxation
objective function value and the best-known IP objective function value is, on average, 3.91%,
and the time to obtain the integer solution is, on average, 9.78 seconds. This gap also appears
to be relatively consistent across all of the crown and sill pillar placement options for this
scenario. The LP relaxation with the largest objective function value produces the largest IP objective function value, suggesting empirically that our solution methodology provides consistently high-quality IP solutions relative to the LP solutions for the enhanced transition model, \( T^e \).

Our ad-hoc branch-and-bound strategy supplies a wealth of information for the mine operator: Crown pillar placement affects the NPV significantly more than sill pillar placement. Additional insights might involve geology: if it is undesirable to have a crown pillar located at elevation 820, moving the crown pillar to elevation 780 would have the least impact on the mine’s NPV (Figure 4.7).

![LP Relaxation Objective Function Values](image1)

![Best Known IP Objective Function Values](image2)

Figure 4.7 (LP and IP Comparison) Left: LP relaxation values for all feasible crown and sill pillar placement options. Right: Best-known IP objective function value for the corresponding crown and sill pillar placement options. The vertical band of crosses at each crown pillar elevation is associated with the scaled NPV corresponding to all viable sill pillar location combinations. A horizontal line indicates overall best-known IP objective function value for Scenario 1.

It is possible to heavily prune our ad-hoc branch-and-bound tree using *bound dominance*. For example, for Scenario 1, after we obtain an IP objective function value associated with the crown and sill pillar placement option that has the highest LP relaxation objective
function value, we can eliminate solving the integer program corresponding to all crown and sill pillar placements whose LP relaxation objective function value is lower. Only 40 of the over 3,500 crown and sill pillar placement options have an LP relaxation objective function value greater than the best known IP objective function value (Figure 4.8). The mine planner interested in robust solutions might note that of those options, only one of them is not associated with a crown pillar located a elevation 820, and the corresponding LP relaxation’s objective function value is only 0.13% greater than the best-known IP objective function value.

Figure 4.8 (Zoom Comparison) Left: LP relaxation values associated with crown and sill pillar placements that produce a high objective function relaxation value, and a horizontal line representing the best-known IP objective function value. Right: Corresponding IP objective function values for models whose LP relaxation objective function value is greater than the best known IP objective function value. Note: Circled is the best LP relaxation objective function value and its corresponding IP objective function value.

Table 4.3 summarizes the near optimal crown and sill pillar placement options associated with each scenario. The average gap between the LP and IP objective function values is 5.55%. For any scenario with a discount rate of 9%, the crown pillar associated with the highest IP objective function value is located at the same elevation, 820. (Changing the
discount rate affects the best-known crown and sill pillar locations.) Techniques may be employed to reduce the gaps, but, for the scenarios we tested, it is unlikely that such refinements would lead to solutions with a change to the crown pillar placement, because of the 40 solutions not eliminated by bound dominance, only one had a placement at a level other than 820 (Figure 4.8). We report solution time as the CPU time required to solve the LP relaxations associated with all reasonable crown and sill pillar placement options, plus that required to solve the necessary integer programs, i.e., those not excluded by bound dominance. However, our procedure is massively parallelizable in that all LPs can be solved simultaneously, as can all relevant IPs. Hence, on average, even the longest-running scenarios would require fewer than ten seconds to solve with the appropriate hardware; our methodology efficiently provides a way to identify near-optimal crown and sill pillar placements where no such methodology had existed.

Table 4.3 (Scenario Summary) Optimal solution, integrality gap, and total solution time for each scenario if enumerated crown and sill pillar placement solves are performed in serial.

<table>
<thead>
<tr>
<th>Scenario Number</th>
<th>Optimal Crown and Sill Pillar Placement</th>
<th>Integrality Gap</th>
<th>Total Solution Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[(820), (500)]</td>
<td>2.51%</td>
<td>29,652</td>
</tr>
<tr>
<td>2</td>
<td>[(820), (420)]</td>
<td>3.28%</td>
<td>28,220</td>
</tr>
<tr>
<td>3</td>
<td>[(820), (420)]</td>
<td>3.25%</td>
<td>30,642</td>
</tr>
<tr>
<td>4</td>
<td>[(820), (660)]</td>
<td>3.42%</td>
<td>31,426</td>
</tr>
<tr>
<td>5</td>
<td>[(820), (500)]</td>
<td>4.40%</td>
<td>25,993</td>
</tr>
<tr>
<td>6</td>
<td>[(820), (420)]</td>
<td>4.97%</td>
<td>27,404</td>
</tr>
<tr>
<td>7</td>
<td>[(820), (460)]</td>
<td>4.76%</td>
<td>36,312</td>
</tr>
<tr>
<td>8</td>
<td>[(820), (500)]</td>
<td>3.45%</td>
<td>41,652</td>
</tr>
<tr>
<td>9</td>
<td>[(700), (420)]</td>
<td>0.70%</td>
<td>26,016</td>
</tr>
<tr>
<td>10</td>
<td>[(820), (500), (660)]</td>
<td>3.65%</td>
<td>23,825</td>
</tr>
</tbody>
</table>

Notes: Crown and sill pillar placement option format is [(Crown Pillar Elevation), (Sill Pillar Elevation(s))]

Total solution time is the time required for the LPs associated with all possible crown and sill pillar locations, and the additional time to obtain an IP solution for the non-dominated LP relaxations.

Qualitatively, we can establish some generalities about the schedules for each scenario. The crown pillar location that is chosen in a majority of the scenarios, 820, contains the
fifth-highest amount of metal, and, as such, does not correspond to an intuitive solution of minimizing lost metal. Additionally, for the best schedule we report for each scenario, underground construction and production begins as soon as possible owing to that fact that all underground mine production is sufficiently high grade that it displaces material from the open pit at the mill. We do observe some fluctuations in both the open pit and underground production, which is undesirable from an operational standpoint, and would require smoothing to create an operationally feasible schedule. However, these fluctuations are not uncommon in a strategic plan.

4.7 Conclusions and Future Work

The methodology developed in this chapter provides a robust framework for solving a linear-integer program representing an open-pit-to-underground transition model involving scenarios that contain 50,000 variables and over 1.5 million constraints. An ad-hoc branch-and-bound scheme fixes the variables that destroy the PCPSP structure without compromising optimality. This methodology permits us to test a wide variety of scenarios quickly and provides a better understanding of how crown and sill pillar placement affects NPV.

With our specialized technique, we are able to solve a relevant and economically significant problem for the mining industry. As current open pit mines are required to extract an increasing number of tons of waste material for every ton of ore, it becomes crucial to identify the proper transition location. Although, for confidentiality reasons, the exact NPVs are not given, our results show that the NPV can change by hundreds of millions of dollars depending on the crown pillar placement, and by tens of millions based on the sill pillar placement. Our model provides not only a near-optimal solution, but identifies the economic outcome of all possible crown and sill pillar placements.

Many mine operators defer underground mining until the open pit has finished production, resulting in insufficient cash flow to justify an underground mine and an unmined portion of the deposit that could have been extracted economically. By developing an efficient and tractable solution methodology, we can provide mine operators with a tool to
better understand the benefits of each transition elevation, and the ability to confidently make a timely decision.

Additional work could address: (i) accuracy, (ii) applicability, and (iii) optimality gap. The accuracy of the model would be improved with a better representation of the stockpiles. Since stockpiles contribute significantly to the NPV, it would be beneficial to include mixing of ore in the stockpile and the degradation of ore grade over time. The applicability of the model could be improved by adding blending requirements at the mill and non-zero lower bounds on the knapsack constraints, which can be vital to maintain proper mill feed, but that would destroy the mathematical structure that the TopoSort heuristic relies on. Finally, we wish to incorporate a branch-and-bound algorithm within the OMP Solver to reduce the optimality gap for a fixed crown and sill pillar placement option.

4.8 Chapter Conclusion

This chapter concludes the advancements made is this thesis. We create a new formulation for strategic underground mine scheduling and, by pairing that formulation with a new solver, enumeration strategies, and constraint reformulations, we have made a significant contribution to the application of integer programming in underground mine scheduling.
CHAPTER 5
GENERAL CONCLUSIONS

At the time of this writing, the UG-PSP has lacked a generalized integer programming formulation for strategic scheduling. We present a formulation for solving a close approximation to the UG-PSP that has fast solution times when using recently developed algorithms. This formulation is able to aid with both scheduling and strategic mining decisions, such as cutoff grade, and crown and sill pillar placement. We show the flexibility of the formulation and its value to mining companies.

5.1 Research Contributions

We outline our three major research contributions; the following paragraphs provide a detailed explanation.

- We develop a new methodology to solve the UG-PSP by prohibiting specific pairs of activities from occurring in the same time period.
- We show the effectiveness of enumeration strategies when solving problems related to mine scheduling.
- We outline multiple reformulations that allow specific constraints in mine scheduling problems to better conform to RCPSP math structure and to tighten the gap between the linear programming relaxation solution and the optimal integer solution, enabling us to solve large-scale instances in a reasonable amount of time.

Chapter 2 compares two different formulations to solve the UG-PSP using integer programming. The UG-PSP tactical formulation, i.e., a formulation with time fidelity sufficiently small to capture activity durations, exists in previous literature, but quickly becomes intractable as the time horizon increases. Therefore, we propose a strategic model, which
is a slight approximation of the UG-PSP, to solve models with coarse fidelity time periods over a long time horizon. This formulation hinges on delay constraints that restrict specific pairs of activities from occurring in the same time period. We construct these constraints by determining whether it is feasible to complete a predecessor and successor activity in the same time period, based on the time required relative to the model fidelity. Infeasible activity pairs may be separated by an arbitrary number of intermediate activities. Although delay constraints are typically numerous, their math structure conforms to that of a precedence relationship, which is desirable in state-of-the-art linear programming algorithms. By comparing and contrasting the UG-PSP tactical and strategic formulations, we show the benefits and weakness of each.

Chapter 3 utilizes the UG-PSP strategic formulations to aid in the determination of the cutoff grade for an underground mine. This chapter highlights how employing new linear programming algorithms, in combination with our formulations, can result in rapid solution times. We enumerate over 4,000 different linear programming relaxations and solve the integer program corresponding to the linear program with the largest objective function value, which results in a near-optimal solution in hours. This shows that optimization-based enumeration is viable for obtaining strategic decisions related to underground mining using the UG-PSP strategic formulation.

Chapter 4 expands the ideas presented in Chapters 2 and 3 by including an open pit mine scheduling formulation with the UG-PSP strategic formulation and solving the resulting model using ad-hoc branching within a specialized optimization framework; by reformulating two “side” constraints as precedences, we are able to obtain a math structure that significantly reduces the number of non-precedence constraints. This allows us to lower the gap between the linear programming relaxation objective function value and that of the optimal integer solution. The result enables us to determine the location at which to transition from open pit to underground mining, and how to design the underground mine to achieve a maximal net present value.
5.2 Suggested Further Research

A significant improvement to our solutions could include explicitly capturing fixed costs within the model. Our methodology only applies a variable cost to mining activities, which may result in solutions that are not optimal when considering all financial aspects. Models that include fixed costs within a portion of the variable costs provide solutions that are inferior to those in which fixed costs are considered explicitly in the model. To add fixed costs into the optimization framework, we suggest modeling them in conjunction with a binary variable, which must assume a value of one if any activities are completed in that time period. It may also be computationally advantageous to initially branch on the variables that represent fixed cost, because once those variables are fixed, the model reverts back to the RCPSP math structure.

Creating an accurate stochastic model for the UG-PSP would represent a significant advance in risk mitigation. However, because every underground mine design differs depending on the grade distribution model, the resulting two-stage stochastic model would be difficult to construct. Open pit stages are: (i) what material to extract, and (ii) to what destination the material is sent based on the realization of the grade. In underground mining, only a single decision exists, because material contained within a mining unit is designated to be sent to a single destination. Ideally, one would develop a way in which to introduce a second-stage decision, but, at the time of this writing, it is unclear what that stage would be. Designing a stochastic model for underground mine scheduling may begin with a methodology that schedules each different geological realization and searches for commonalities between the schedules.

Another interesting aspect of underground mine scheduling is that stockpiling is not commonly done in practice. There exists no academic literature evaluating the change in net present value when stockpiling underground material is allowed. If the mill is sufficiently small and there is large quantity of material, stockpiling may increase NPV.
This work lends itself well to being implemented at many mining operations. The UG-PSP strategic formulation is flexible and, as shown in this thesis, with the addition of enumeration and reformulation extensions, we are able to quickly make more accurate strategic mine planning decisions. Mining companies and consultants will be able to identify additional uses for our research.
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We present the model formulation and detailed computational results for Chapter 3. The model is presented to match the UG-PSP strategic formulation presented in Chapter 2. In addition, we provide detailed computational numbers for both the Central Zone and the entire mine.

A.1 Underground Mine Scheduling Formulation

Our underground mine scheduling problem is formulated to possess a resource constrained project scheduling problem mathematical structure, with a majority of the constraints representing precedence relationships. This structure is well suited for the OMP solver and the TopoSort heuristic (Rivera et al., 2015). TopoSort (Chicoisne et al., 2012) provides near-optimal solutions quickly for resource constrained project scheduling problems, but is only guaranteed to provide a feasible solution if all of the coefficients in the resource constraints are non-negative and the lower bounds are zero. It is a list ordering heuristic based on the premise that the earlier the expected completion time of a block or activity in the linear programming solution, the earlier the block or activity is scheduled in the integer programming solution. King et al. (2016b) show the effectiveness of TopoSort for a model containing an underground mine scheduling formulation.

Indices and sets:

\[ a \in A \] set of all activities
\[ \tilde{a} \in \tilde{A}_a \] set predecessors for activity \( a \)
\[ \bar{a} \in \bar{A}_a \] set of predecessor activities \( \bar{a} \) that must be completed one time period in advance of activity \( a \)
\[ r \in R \] set of resources, such as stope extraction and development capacity
\[ t \in T \] time periods
Data:
\[c_a\] monetary value associated with completing activity \(a\) [$]
\[q_{ra}\] quantity resource \(r\) consumed when completing activity \(a\) [tonnes, meters]
\[\bar{r}_{rt}\] maximum amount of resource \(r\) available in time \(t\) [tonnes, meters]
\[\delta_t\] discount factor for time period \(t\) [fraction]

Decision variables:
\[X_{at}\] 1 if activity \(a\) is completed by the end of time \(t\); 0 otherwise

\begin{align*}
\text{(UG)} & \quad \max \sum_{a \in A} \sum_{t \in T} \delta_t c_a (X_{at} - X_{a,t-1}) \quad (A.1a) \\
\text{s.t.} & \quad X_{a,t-1} \leq X_{at} \quad \forall a \in A, t \in T \quad (A.1b) \\
& \quad X_{at} \leq X_{\bar{a}t} \quad \forall a \in A, \bar{a} \in \bar{A}_a, t \in T \quad (A.1c) \\
& \quad X_{at} \leq X_{\bar{a},t-1} \quad \forall a \in A, \bar{a} \in \bar{A}_a, t \in T \quad (A.1d) \\
& \quad \sum_{a \in A} q_{ra} (X_{at} - X_{a,t-1}) \leq \bar{r}_{rt} \quad \forall r \in R, t \in T \quad (A.1e) \\
& \quad X_{at} \text{ binary} \quad \forall a \in A, t \in T \quad (A.1f)
\end{align*}

The objective function (A.1a) maximizes net present value. Constraints (A.1b) ensure that once an activity is completed, it remains completed. Constraints (A.1c) enforce precedence. Constraints (A.1d) ensure that at least one time period elapses between the completion of the specified pair of activities. Constraints (A.1e) bound stope extraction rate and development construction. All variables are required to be binary by (A.1f).

A.2 Central Zone Computational Results

Table A.1 provides a summary of the Central Zone solutions at different stope extraction capacities. The first column indicates the production capacity and the second contains the cutoff grade that produces the highest net present value. The third column represents the total time required to solve of the linear programming relaxations associated with all
economically feasible cutoff grades for the production capacity specified in the first column, and to obtain an integer solution. The final column represents the gap between the linear programming objective function value and the objective function value corresponding to the integer solution. For the Central Zone, all of the gaps are sufficiently small such through dominance that we can show that there exists one optimal cutoff grade through dominance. A similar procedure is completed for the South and North Zone.

Table A.1 Central Zone scheduling computational summary.

<table>
<thead>
<tr>
<th>Production Capacity</th>
<th>Highest NPV Cutoff</th>
<th>Total Solution Time (sec)</th>
<th>Optimality Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>112.50%</td>
<td>3.4</td>
<td>20.42</td>
<td>0.000%</td>
</tr>
<tr>
<td>100.00%</td>
<td>3.4</td>
<td>24.11</td>
<td>0.000%</td>
</tr>
<tr>
<td>87.50%</td>
<td>3.8</td>
<td>32.2</td>
<td>0.000%</td>
</tr>
<tr>
<td>75.00%</td>
<td>4.2</td>
<td>43.89</td>
<td>0.001%</td>
</tr>
<tr>
<td>62.50%</td>
<td>4.2</td>
<td>50.99</td>
<td>0.684%</td>
</tr>
<tr>
<td>50.00%</td>
<td>4.2</td>
<td>67.38</td>
<td>0.348%</td>
</tr>
</tbody>
</table>

Notes: Optimality gaps are calculated using $100\% \cdot \left(1 - \frac{\text{IP Obj. Func. Value}}{\text{LP Relaxation Value}}\right)$

A.3 Parallel Computing Results

We highlight the effectiveness of parallelization by separating the enumerations across multiple cores (Table A.2). By dividing the work across 16 cores based on the South Zone’s cutoff grade, we are able to solve all of the linear programming relaxations in 2,510.23 seconds, the maximum value in column two of Table A.2, as opposed to 33,186.30 seconds for a serial execution. A significant correlation between cutoff grade and solution times exist, owing to the fact that lower cutoff grades contain more stopes and horizontal development, thus generally increasing the number of decisions that need to be made.
Table A.2 Entire mine scheduling parallel computational times.

<table>
<thead>
<tr>
<th>South Zone Cutoff Grade</th>
<th>LP Solve Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2489.16</td>
</tr>
<tr>
<td>1.4</td>
<td>2510.23</td>
</tr>
<tr>
<td>1.8</td>
<td>2447.53</td>
</tr>
<tr>
<td>2.2</td>
<td>2386.63</td>
</tr>
<tr>
<td>2.6</td>
<td>2347.86</td>
</tr>
<tr>
<td>3</td>
<td>2174.90</td>
</tr>
<tr>
<td>3.4</td>
<td>2068.31</td>
</tr>
<tr>
<td>3.8</td>
<td>1973.00</td>
</tr>
<tr>
<td>4.2</td>
<td>2165.04</td>
</tr>
<tr>
<td>4.6</td>
<td>1852.51</td>
</tr>
<tr>
<td>5</td>
<td>1869.71</td>
</tr>
<tr>
<td>5.4</td>
<td>1829.00</td>
</tr>
<tr>
<td>5.8</td>
<td>1835.19</td>
</tr>
<tr>
<td>6.2</td>
<td>1777.48</td>
</tr>
<tr>
<td>6.6</td>
<td>1707.09</td>
</tr>
<tr>
<td>7</td>
<td>1752.66</td>
</tr>
</tbody>
</table>
APPENDIX B - CHAPTER 4

We demonstrate here computationally that, for the scenarios we examine, the LP relaxation of the basic transition model ($T^b$) is weak relative to that of the enhanced transition model ($T^e$), and we compare LP solution times across standard algorithms. We also show that CPLEX is unable to solve the basic transition model ($T^b$) for any scenario, i.e., that the basic transition model is intractable when solved with CPLEX. Furthermore, two proofs – one each for the special knapsack and inventory balance constraints (see §4.4) – show that our reformulations are no weaker than the original ones; moreover, computational results show that the reformulations are strictly stronger.

B.1 Solutions to the Basic Transition Model ($T^b$)

Table B.1 provides specific information regarding algorithmic performance and solution quality for our ten test scenarios. We first compare solution times from standard LP algorithms when used to solve ($T^b$). All computations are run on the same server as the enhanced transition model ($T^e$). The barrier outperforms the best version of simplex in all cases; the academic nature of the OMP Solver precludes it from handling the complexity of the basic transition model, ($T^b$), in particular, the decisions regarding crown and sill pillar placement (hence, our ad-hoc branch-and-bound strategy). Despite the ability of CPLEX (Version 12.6.0.0 using default parameter settings other than turning on memory emphasis) to solve the LP relaxation of ($T^b$), the problem proves intractable when seeking a good, integer-feasible solution. For all our scenarios, CPLEX exhausts a 40,000-second time limit or runs out of memory before a solution within 5% of optimality is found. We attribute this poor performance, in part, to the weak LP bound (as seen with a comparison of the scaled net present values in the penultimate and last columns of Table B.1). By contrast, the results we obtain from using our reformulations and solution procedure on ($T^e$) result not only in tighter bounds, but also in near-optimal integer solutions, the latter stemming
in large part from the mathematical structure of \((T^e)\) our TopoSort heuristic is able to exploit. The values for the tight LP relaxation solutions we obtain from \((T^e)\) in Table 4.2 result from the fixed crown and sill pillar combination that gives the best objective function value for that scenario.

Table B.1 (Basic and Enhanced Comparison) Comparison of solution times and gaps between the basic transition model \((T^b)\) and the enhanced transition model \((T^e)\) using CPLEX.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>LP Solution Time for ((T^b)) (sec)</th>
<th>IP Solution Time (sec)</th>
<th>((T^b)) LP Value</th>
<th>Largest ((T^e)) LP Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>373.98</td>
<td>10343.09</td>
<td>3.20</td>
<td>2.72</td>
</tr>
<tr>
<td>2</td>
<td>359.53</td>
<td>8222.94</td>
<td>3.11</td>
<td>2.66</td>
</tr>
<tr>
<td>3</td>
<td>992.07</td>
<td>13821.83</td>
<td>2.71</td>
<td>2.55</td>
</tr>
<tr>
<td>4</td>
<td>333.09</td>
<td>13163.29</td>
<td>3.11</td>
<td>2.70</td>
</tr>
<tr>
<td>5</td>
<td>300.37</td>
<td>7451.82</td>
<td>3.20</td>
<td>2.70</td>
</tr>
<tr>
<td>6</td>
<td>338.96</td>
<td>8018.63</td>
<td>3.09</td>
<td>2.64</td>
</tr>
<tr>
<td>7</td>
<td>407.66</td>
<td>7792.96</td>
<td>3.01</td>
<td>2.58</td>
</tr>
<tr>
<td>8</td>
<td>756.81</td>
<td>25526.81</td>
<td>2.71</td>
<td>2.55</td>
</tr>
<tr>
<td>9</td>
<td>383.78</td>
<td>8240.55</td>
<td>6.24</td>
<td>4.86</td>
</tr>
<tr>
<td>10</td>
<td>369.52</td>
<td>7738.03</td>
<td>2.19</td>
<td>2.06</td>
</tr>
</tbody>
</table>

*CPLEX is allowed to choose the variant of simplex to use, which results in employing dual simplex on the dual problem
† CPLEX is unable to produce an integer solution within a 5% gap before running out of memory
‡ CPLEX is unable to produce an integer solution within a 5% gap before the 40,000-second limit

B.2 On the strength of the Special Knapsack reformulations

In this section, we prove that the reformulation of the special knapsack constraints \((4.1j)\), presented in Section 4.4.1, is not weaker than the original formulation.

For simplicity, but without loss of generality, we consider the case of a single phase. For notation purposes, assume that the blocks in the phase are numbered 1, \ldots, \(|\mathcal{B}|\) and that there is a limit of \(k\) blocks that can be extracted in a given time period. (In Section 4.4.1, this term is represented as \(\bar{r}\).) Variables \(x_{bt}\) are defined as before:

\[
x_{bt} = \begin{cases} 
1 & \text{if block } b \text{ is extracted by time } t \\
0 & \text{otherwise.}
\end{cases}
\]
We can model the bound of $k$ extractable blocks in a time period either by adding the knapsack constraint:

$$\sum_{b=1}^{\vert B \vert}(x_{bt} - x_{b,t-1}) \leq k \quad \forall t \in T \ni t > 1 \quad \text{(B.1)}$$

or by adding precedence constraints:

$$x_{bt} \leq x_{b-k,t-1} \quad \forall b \in B \ni b \geq k + 1; t \in T \ni t > 1. \quad \text{(B.2)}$$

We now show that (B.2) is at least as strong as (B.1).

**Lemma** Let $\mathcal{X}$ be the set of $x_{bt}$ variables such that:

$$x_{bt} \leq x_{b-1,t} \quad \forall b \in B \ni b \geq 1; t \in T \quad \text{(B.3)}$$
$$x_{bt} \leq x_{b,t+1} \quad \forall b \in B, t \leq \vert T \vert - 1 \quad \text{(B.4)}$$
$$0 \leq x_{bt} \leq 1 \quad \forall b \in B, t \in T. \quad \text{(B.5)}$$

For sets:

$$\mathcal{P}_1 = \{ x \in \mathcal{X} : \sum_{b=1}^{\vert B \vert}(x_{bt} - x_{b,t-1}) \leq k \quad \forall t \in T \}$$

$$\mathcal{P}_2 = \{ x \in \mathcal{X} : x_{bt} \leq x_{b-k,t-1} \quad \forall b \in B \ni b \geq k + 1; t \in T \}$$

we wish to show that $\mathcal{P}_2 \subseteq \mathcal{P}_1$.

**Proof.**

We show that if $x_{bt}$ satisfies (B.2), then $x_{bt}$ satisfies (B.1). For each $t \in T$:

$$\sum_{b=1}^{\vert B \vert}(x_{bt} - x_{b,t-1}) = \sum_{b=1}^{k}x_{bt} - \sum_{b=1}^{\vert B \vert-k}x_{b,t-1} + \sum_{b=k+1}^{\vert B \vert-k+1}x_{b,t-1} - \sum_{b=\vert B \vert-k+1}^{\vert B \vert}x_{b,t-1}$$

$$\leq \sum_{b=1}^{k}x_{bt} - \sum_{b=1}^{\vert B \vert-k}x_{b,t-1} + \sum_{b=k+1}^{\vert B \vert-k}x_{b-k,t-1} - \sum_{b=\vert B \vert-k+1}^{\vert B \vert}x_{b,t-1}$$
Combining the first and last expressions implies that \( \sum_{b=1}^{k} (x_{bt} - x_{b,t-1}) \leq k \). The reformulation of the special knapsack constraints is done not only to enable the model to be more easily solved within a framework the OMP Solver can handle, but also to improve the upper bound. For (S), the reformulation does not improve the LP solution time over that obtained with the original model when using the OMP Solver for both formulations; however, for the scenarios we test, the LP bound for (S) improves with the reformulation by approximately 10%.

### B.3 Inventory Balance Reformulation as Variable Substitution

In this subsection, we show that the reformulation of the inventory balance constraints (4.6) and the corresponding variable substitutions into expressions (4.7)-(4.9) presented in Section 4.4.2, is not weaker than the original formulation. To this end, it suffices to show that for every integer-feasible solution of the reformulation, there exists a corresponding feasible solution to the original formulation having the same objective function value. Given a solution \( \hat{Y}_{bt}^S, Z_{nbt}^S \) of the reformulation, we construct a feasible solution in the original space via the following linear mapping:

\[
I_{nbt}^S = \hat{Y}_{b,t-1}^S - Z_{nb,t-1}^S \quad \forall b \in B, n \in N_b, t \in T
\]

\[
I_{nbt}^{S-} = Z_{nbt}^S - Z_{nb,t-1}^S \quad \forall b \in B, n \in N_b, t \in T
\]

\[
Y_{nb2t}^S = \hat{Y}_{bt}^S \quad \forall b \in B, n \in N_b, t \in T
\]
$Y_{nblt}^S = 0 \quad \forall b \in B, n \in \mathcal{N}_b, t \in \mathcal{T}$

$Y_{nblt}^S = 0 \quad \forall b \in B, n \in \mathcal{N}_b, t \in \mathcal{T}$

That is, substituting into (S) the expressions on the right-hand-side of the mapping for the variables listed on the left-hand side results in true statements for each relevant set of constraints. (Constraints containing only variables not involved in the mapping remain unchanged.) This implies that the solution involving $\hat{Y}_{bt}^S$ and $Z_{nbt}^S$ is feasible, and therefore valid, for (S). The same type of substitution, and the correct interpretation of the new variables $\hat{Y}_{bt}^S$ and $Z_{nbt}^S$, yields the same objective function value as in the original formulation.