LOWER FINITE MODULES OVER COMMUTATIVE RINGS WITH IDENTITY

by

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ABSTRACT

A bounded partially ordered set $(P, 0, 1, \leq)$ is *lower finite* provided P is infinite and for each $x \neq 1$ in P, there are but finitely many elements y in P such that $y < x$. We will call a module M lower finite if the set of proper submodules of M, partially ordered by set-theoretic containment, is lower finite. We will use the (well-studied) class of Jonsson modules (along with other classical results) to classify the lower finite modules over a commutative ring with identity.

CONTENTS

CHAPTER I: INTRODUCTION

An old problem of Irving Kaplansky asked which partially ordered sets can be realized as the posets of prime ideals of a commutative unital ring. We consider a parallel question: given a property $\mathcal{P}$ of posets, for which unitary modules does the poset of submodules have property $\mathcal{P}$? For example, if G is an infinite group which is totally ordered by $\subseteq$, then G is isomorphic to a Prüfer p-group (quasi-cyclic group) $C(p^\infty)$ for some prime p. Many well-known properties of modules are phrased completely in terms of the poset of submodules. To state a few examples, a module M is uniserial if the poset of submodules is totally ordered by $\subseteq$, M is artinian if every nonempty subset of the poset of submodules has a minimal element, and M is noetherian if every nonempty subset of the poset of submodules has a maximal element. Now we shall introduce the property of posets to be studied. Let $\mathbf{P} := (P, 0, 1, \leq)$ be an infinite bounded poset ($0, 1 \in P$ and $0 \leq x \leq 1$ for all $x \in P$). Say $\mathbf{P}$ is lower finite if for all $x \in P$ with $x \neq 1$ there are but finitely many $y \in P$ such that $y \leq x$. We pause to give two simple examples.

(1) Extend the usual order on the natural numbers to include ∞ by defining $n < \infty$ for every natural number n and $\infty \leq \infty$. It is clear that $(\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$ is lower finite.

(2) Let X be an infinite set and $P := \{S \subseteq X | S = X \text{ or } S \text{ is finite}\}$. Since every $S \in P$ with $S \neq X$ has but finitely many subsets, we see that $(P, \varnothing, X, \subseteq)$ is lower finite.

Let R be a commutative, unital ring and M be a unitary left R-module. Note that the poset of R-submodules of M is bounded (by the trivial submodule and M itself). We pose the following question: for which modules is the poset of submodules lower finite? Equivalently, which modules with infinitely many submodules have the property that every proper submodule has but finitely many submodules?

A motivating example is that of $\mathbb{R}^2$ as an $\mathbb{R}$-module. $\mathbb{R}^2$ has infinitely many one-dimensional $\mathbb{R}$-submodules (realized as lines through the origin), but every proper submodule of $\mathbb{R}^2$ (being zero-dimensional or one-dimensional) has finitely many submodules. This makes $\mathbb{R}^2$ lower finite as an $\mathbb{R}$-module.

Using results from G. Oman on the well-studied class of countable Jónsson modules in addition to results from Hirano and Mogami we will classify the infinitely generated (i.e., not finitely generated) lower finite modules. The classification of the finitely generated lower finite modules will follow (primarily) from Nakayama's Lemma and connects nicely to the literature via "Modules with waists" (introduced by Auslander, Green, and Reiten).

Note that the definition of lower finite purposefully excludes modules with finitely many submodules, as these were classified by Akbari, Ghezelahmad, and Yaraneri. The RESULTS section ends with a new proof of their classification of the modules over a commutative ring having but finitely many submodules.

We will close by discussing the "dual notion" of a module being upper finite and other directions for further research.

CHAPTER II: PRELIMINARIES

Note that the results in this chapter are standard, and most of them can be found in Hungerford's Algebra [8].

Rings of Quotients, Localization, and Local Rings.

Definition 1. *A nonempty subset of a ring R is multiplicative provided that $0 \notin S$, $1 \in S$, and $ab \in S$ whenever $a, b \in S$.*

Example 1. *If P is a prime ideal in a commutative ring R, then $R \setminus P := \{r \in R | r \notin P\}$ is a multiplicative set. For if $a, b \in R \setminus P$, then $ab \in R \setminus P$, else primeness of P implies $a \in P$ or $b \in P$. Lastly, since P is a proper ideal, $0 \notin R \setminus P$ and $1 \in R \setminus P$.*

Theorem 1. *Let S be a multiplicative subset of a commutative ring R. The relation*

$$(r, s) \sim (r', s') \iff s_1(rs' - r's) = 0 \text{ for some } s_1 \in S$$

is an equivalence relation.

Proof. $\sim$ is obviously reflexive and symmetric, so we need only show $\sim$ is transitive. Let $r, r', r'' \in R$ and $s, s', s'' \in S$. Suppose $(r, s) \sim (r', s')$ and $(r', s') \sim (r'', s'')$. Then there are $s_1, s_2 \in S$ such that $s_1(rs' - r's) = 0$ and $s_2(r's'' - r''s') = 0$. Multiply both sides of the first equation by s_2s'' to get $s_1rs's_2s'' - s_1r'ss_2s'' = 0$, and multiply both sides of the second equation by $s_1 s$ to get $s_2 r' s'' s_1 s - s_2 r'' s' s_1 s = 0$. Adding these two equations and factoring yields $s_1 s_2 s'(rs'' - r''s) = 0$. Since S is multiplicative, $s_1 s_2 s' \in S$, whence $(r, s) \sim (r'', s'')$. $\square$

Corollary 1. *If R has no nonzero zero divisors, then $(r, s) \sim (r', s') \iff rs' - r's = 0$*

Proof. Follows immediately from the preceding theorem and the definition of nonzero zero divisor. $\square$

Let $\frac{r}{s}$ denote the equivalence class of $(r, s) \in R \times S$ and $S^{-1}R = \left\{ \frac{r}{s} \middle| r \in R,\ s \in S \right\}$, the set of equivalence classes of $R \times S$ modulo $\sim$. Observe that $\frac{tr}{ts} = \frac{r}{s}$ for all $t, s \in S$ and $r \in R$, since $s_1\big((tr)s - r(ts)\big) = s_1(0) = 0$ for all $s_1 \in S$. In particular, we can "reduce" $\frac{r}{s}$ in the same way we reduce elements of $\mathbb{Q}$.

Theorem 2. *Let S be a multiplicative subset of a commutative ring R.*

(i) $S^{-1}R$ is a commutative ring with identity, with addition and multiplication defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \quad \textit{and} \quad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

(ii) If R is a nontrivial ring with no nonzero zero divisors, then $S^{-1}R$ is an integral domain. In particular, if $S = R/\{0\}$, then $S^{-1}R$ is a field.

Proof. (i) We begin by proving that addition and multiplication are well defined. Suppose $\frac{r}{s} = \frac{l}{t}$ and $\frac{r'}{s'} = \frac{l'}{t'}$. So there are $x, y \in S$ such that $x(rt - ls) = 0$ and $y(r't' - l's') = 0$. Since $xy \in S$ and $xy[(rs' + r's)tt' - (lt' + l't)ss'] = s't'[yx(rt -$

$ls)] + st[xy(r't' - l's')] = 0$, we have $\dfrac{rs' + r's}{ss'} = \dfrac{lt' + l't}{tt'}$. This shows addition is well defined.

Observe that $xrt = xls$ and $yr't' = yl's'$. Multiply these equations together and rearrange to obtain $xy(rr'tt' - ll'ss') = 0$. Since $xy \in S$, $\dfrac{rr'}{ss'} = \dfrac{ll'}{tt'}$. This shows multiplication is well defined.

Commutativity of addition and multiplication and associativity of multiplication are easily seen to hold. We will verify associativity of addition and distributivity. Let $r, r', r'' \in R$ and $s, s', s'' \in S$

$\left(\dfrac{r}{s} + \dfrac{r'}{s'}\right) + \dfrac{r''}{s''} = \dfrac{rs' + r's}{ss'} + \dfrac{r''}{s''} = \dfrac{rs's'' + r'ss'' + r''(ss')}{ss's''}$

$= \dfrac{r}{s} + \dfrac{r's'' + r''s'}{s's''} = \dfrac{r}{s} + \left(\dfrac{r'}{s'} + \dfrac{r''}{s''}\right)$. Now for distributivity.

$$\frac{r}{s}\left(\frac{r'}{s'} + \frac{r''}{s''}\right) = \frac{r}{s} \cdot \frac{r's'' + r''s'}{s's''} = \frac{rr's'' + rr''s'}{ss's''} = \frac{rr's''}{ss's''} + \frac{rr''s'}{ss's''} = \frac{r}{s} \cdot \frac{r'}{s'} + \frac{r}{s} \cdot \frac{r''}{s''}$$

Clearly $\dfrac{0}{s} = \dfrac{0}{s'}$ and $\dfrac{s}{s} = \dfrac{s'}{s'}$ for all $s, s' \in S$. By definition of addition and the above observation that $\dfrac{tr}{ts} = \dfrac{r}{s}$, $\dfrac{0}{s}$ is the additive identity of $S^{-1}R$ and $\dfrac{s}{s}$ is the multiplicative identity of $S^{-1}R$. The additive inverse of $\dfrac{r}{s}$ is $\dfrac{-r}{s}$. This completes the proof.

(ii) First note that $\dfrac{0}{s} \neq \dfrac{s}{s}$.

Suppose R has no nonzero zero divisors. By Corollary 1, if $\dfrac{r}{s} \cdot \dfrac{r'}{s'} = \dfrac{0}{s}$, then $(rr')s - (ss')0 = rr's = 0$, so $S^{-1}R$ has no nonzero zero divisors. Thus $S^{-1}R$ is an integral domain.

If S is the set of nonzero elements of R, then $\dfrac{s}{r} \in S^{-1}R$ whenever $r \neq 0$. It follows immediately from the definition of multiplication that every nonzero element of $S^{-1}R$ is invertible. Whence $S^{-1}R$ is a field, called the quotient field of R. $\square$

Theorem 3. *Let S be a multiplicative subset of a commutative ring R.*

(i) The map $\varphi_S : R \to S^{-1}R$ given by $r \mapsto \frac{rs}{s}$ (for any $s \in S$) is a well-defined ring homomorphism such that $\varphi_S(s)$ is a unit for each $s \in S$.

(ii) If S has no zero divisors, then φ_S is injective. In particular, an integral domain may be embedded in its quotient field.

Proof. (i) If $, s_0 s, s' \in S$, then $s_0(rs's - rss') = 0$, so $\frac{rs}{s} = \frac{rs'}{s'}$. This shows φ_S is well-defined. Let $x, y \in R$.

$$\varphi_S(x+y) = \frac{(x+y)s}{s} = \frac{xs+ys}{s} = \frac{xs}{s} + \frac{ys}{s} = \varphi_S(x) + \varphi_S(y)$$

$$\varphi_S(xy) = \frac{(xy)ss'}{ss'} = \frac{xs}{s} \cdot \frac{ys'}{s'} = \varphi_S(x)\varphi_S(y)$$

Thus φ_S is a ring homomorphism. If $s \in S$, then $\varphi_S(s) = \frac{ss'}{s'}$ is a unit, since $ss' \in S$.

(ii) Suppose S has no zero divisors. If $\varphi_S(x) = \frac{0}{s}$, then by Corollary 1 and the definition of φ_S we have $(xs')s - (s')0 = 0$. $x = 0$, else $s's \in S$ is a zero divisor. Thus φ_S is injective.

□

For simplicity, we will identify an integral domain D with its embedding $\varphi_S(D)$ in its quotient field. For example, we view the integers $\mathbb{Z}$ as a subring of the rationals $\mathbb{Q}$.

Theorem 4. *Let S be a multiplicative subset of a commutative ring R. If I is an ideal of R, then $S^{-1}I$ is an ideal of $S^{-1}R$.*

Proof. Clearly $S^{-1}I \subseteq S^{-1}R$. Pick $x, x' \in I$ and $s, s' \in S$. $\frac{x}{s} + \frac{x'}{s'} = \frac{xs' + x's}{ss'} \in S^{-1}I$ since I an ideal of R implies $xs' + x's \in I$. Similarly, $r \in R$ implies $rx' \in I$, so $\frac{r}{s} \cdot \frac{x'}{s'} = \frac{rx'}{ss'} \in S^{-1}I$. Thus $S^{-1}I$ is an ideal of $S^{-1}R$. □

Theorem 5. *Let S be a multiplicative subset of a commutative, unital ring R and let I be an ideal of R. $S^{-1}I = S^{-1}R$ if and only if $S \cap I \neq \varnothing$.*

Proof. If $s \in S \cap I$, then $\frac{s}{s} \in S^{-1}I$. Since $\frac{s}{s}$ is the multiplicative identity of $S^{-1}R$, $S^{-1}I = S^{-1}R$. Conversely, if $S^{-1}I = S^{-1}R$, then $\varphi_S(R) = S^{-1}I$. In particular, there are $x \in I$ and $s \in S$ such that $\varphi_S(1_R) = \frac{x}{s}$. But $\varphi_S(1_R) = \frac{1_R s}{s}$, so $s_0(1_R s^2 - xs) = 0$ for some $s_0 \in S$. We rewrite the preceding equation as $s_0 s^2 = s_0 xs$, and note that $s_0 s^2 \in S$ and $s_0 xs \in I$. It follows immediately that $S \cap I \neq \varnothing$. □

Lemma 1. *Let S be a multiplicative subset of a commutative, unital ring R and let I be an ideal of R.*

(i) $I \subseteq \varphi_S^{-1}(S^{-1}I)$.

(ii) If $I = \varphi_S^{-1}(J)$ for some ideal J of $S^{-1}R$, then $S^{-1}I = J$. That is, every ideal of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R.

(iii) If P is a prime ideal of R and $S \cap P \neq \varnothing$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$ and $\varphi_S^{-1}(S^{-1}P) = P$.

Proof. (i) If $x \in I$, then $xs \in I$ for all $s \in S$. Then $\varphi_S(x) = \frac{xs}{s} \in S^{-1}I$, so $x \in \varphi_S^{-1}(S^{-1}I)$. This establishes the desired containment.

(ii) Let $x \in I$ and $s \in S$. Then $xs \in I$ and $\varphi_S(x) = \frac{xs}{s} \in J$. Since R is unital, $\frac{1_R}{s} \cdot \frac{xs}{s} = \frac{x}{s} \in J$. Thus $S^{-1}I \subseteq J$. For the opposite containment, let $\frac{x}{s} \in J$. Then $\varphi_S(x) = \frac{xs}{s} \cdot \frac{s^2}{s} \in J$, which implies $x \in \varphi_S^{-1}(J) = I$. Whence $\frac{x}{s} \in S^{-1}I$.

(iii) Since $S \cap P = \varnothing$, Theorem 5 implies that $S^{-1}P$ is a proper ideal of $S^{-1}R$. Suppose that $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'} \in S^{-1}P$. Then $\frac{rr'}{ss'} = \frac{p}{t}$ for some $p \in P$, $t \in S$. So there exists $s_0 \in S$ such that $s_0 trr' = s_0 pss' \in P$. Since P is prime, $(s_0 t)(rr') \in P$, and $s_0 t \notin P$ (since $S \cap P = \varnothing$), $rr' \in P$. Again, since P is prime, $r \in P$ or $r' \in P$.

Whence $\frac{r}{s} \in S^{-1}P$ or $\frac{r'}{s'} \in S^{-1}P$. Since $S^{-1}R$ inherits commutativity from R, $S^{-1}P$ is prime.

To establish $\varphi_S^{-1}(S^{-1}P) = P$, it suffices to show that $\varphi_S^{-1}(S^{-1}P) \subseteq P$. Let $x \in \varphi_S^{-1}(S^{-1}P)$. Then $\varphi_S(x) = \frac{xs}{s} \in S^{-1}P$, and $\frac{xs}{s} = \frac{p}{t}$ for some $p \in P$ and $t \in S$. So $s_0xst = s_0ps \in P$ for some $s_0 \in S$. Since P is prime and $s_0st \notin P$, $x \in P$. Thus, by (i), $\varphi_S^{-1}(S^{-1}P) = P$.

□

Theorem 6. *Let S be a multiplicative subset of a commutative, unital ring R. There is a one-to-one correspondence between the set of prime ideals of R which are disjoint from S and the prime ideals of $S^{-1}R$, given by $P \mapsto S^{-1}P$.*

Proof. By Lemma 1(iii), the map given by $P \mapsto S^{-1}P$ is injective. It remains to prove that the map is surjective. Let J be a prime ideal of $S^{-1}R$ and $P = \varphi_S^{-1}(J)$. By Lemma 1(ii), $S^{-1}P = J$. We must show that P is prime. Let $xy \in P$, so $\varphi_S(x)\varphi_S(y) = \varphi_S(xy) \in J$. Primeness of J implies that $\varphi_S(x) \in J$ or $\varphi_S(y) \in J$. Thus $x \in P$ or $y \in P$. Since $P \mapsto J$, the map is onto. □

Recall from Example 1 that the complement of a prime ideal P in a commutative ring R is a multiplicative set. The ring of quotients $S^{-1}R$, where $S = R \setminus P$, is called the localization of R at P, and is denoted R_P. Note that by Theorem 6, there is a one-to-one correspondence between the prime ideals of R_P and the prime ideals of R which are contained in P.

Theorem 7. *Let P be a prime ideal in a commutative, unital ring R. The ideal P_P is the unique maximal ideal of R_P.*

Proof. Let M be a maximal ideal of R_P. Since maximal ideals in a commutative, unital ring are prime, M must be prime. Moreover, by Theorem 6, $M = Q_P$ for some prime ideal Q of R with $Q \subseteq P$. But then $Q_P \subseteq P_P$. By Theorem 5, $P_P \neq R_P$, so $Q_P = P_P$, which proves that P_P is the unique maximal ideal of R_P. □

Rings with the property given in Theorem 7 are often used independent of localization.

Definition 2. *Let R be a commutative, unital ring. R is local if and only if R has a unique maximal ideal.*

Definition 3. *Let R and S be rings, and suppose that M is simultaneously an R-module and an S-module. We say the R-module structure of M and the S-module structure of M are essentially the same provided $Rm = Sm$ for every $m \in M$.*

Example 2. *Let R be a ring, M an R-module, and $Ann_R(M) = \{r \in R | rM = \{0\}\}$ (the annihilator of M in R). Recall that $Ann_R(M)$ is an ideal of R. M is naturally an $R/Ann_R(M)$-module via the scalar multiplication $\overline{r} \cdot m := rm$. It is immediate that the structures of ${}_{R/Ann_R(M)}M$ and ${}_RM$ are essentially the same.*

Definition 4. *Let R be a ring, J a maximal ideal, and M an R-module. M is said to be J-primary provided that for every $m \in M$, there is a positive integer n such that $J^n m = \{0\}$. In this case, we write $M = M[J]$.*

Lemma 2. *Let R be a ring, J a maximal ideal of R, and suppose that M is a J-primary R-module. Then M has a natural module structure over the local ring R_J. Moreover, the structure of M as an R-module is essentially the same as the structure of M as an R_J-module.*

Proof. Let $s \in R - J$ and $m \in M$. Since M is J-primary, there is a positive integer k for which $J^k m = \{0\}$. If (J^k, s) is proper, then it is contained in some maximal ideal M. But then $J^k \subseteq M$ and maximal ideals are prime, so $J = M$. This contradicts $s \notin J$. Thus $(J^k, s) = R$. We have $y + sx = 1$, for some $y \in J^k$ and $x \in R$. Multiplying through by m yields $s(xm) = m$. For uniqueness, suppose $m = sm' = sm''$. Then $s(m' - m'') = 0$. As above, there is a positive integer k such that $J^k(m' - m'') = \{0\}$. But then $R = (J^k, s) \subseteq Ann(m' - m'')$, which implies $1(m' - m'') = 0$. So $m' = m''$. We note that the unique m' is always a multiple of m. Thus M is naturally an R_J-module via $\frac{r}{s} \cdot m = \frac{rm}{s} = m'$, the unique element for which $rm = sm'$.

Now we show that ${}_R M$ and ${}_{R_J} M$ have essentially the same structure. Let $m \in M$. If $x \in R_J m$, then x is the unique element such that $rm = sx$, where $r \in R$ and $s \in R - J$. By a preceding proof, $x = ym$ for some $y \in R$. Whence $x \in Rm$. Conversely, if $x \in Rm$, then $x = rm = \frac{rm}{1} \in R_J m$. Thus $Rm = R_J m$. □

The Quasi-cyclic Groups and Discrete Valuation Rings.

Definition 5. *Let p be a prime integer. The quasicyclic p-group (Prüfer p-group) $C(p^\infty)$ is the additive subgroup of $\mathbb{Q}/\mathbb{Z}$ whose elements have additive order a power p. In particular, $C(p^\infty) = \left\{ \frac{a}{p^n} \middle| a, n \in \mathbb{Z},\, n \geq 0 \right\}$.*

Lemma 3. *(i) $C(p^\infty)$ contains an isomorphic copy of $\mathbb{Z}/(p^n)$ for each n, and (ii) every proper subgroup of $C(p^\infty)$ is isomorphic to some $\mathbb{Z}/(p^n)$.*

Proof. (i) Clearly the subgroup of $C(p^\infty)$ generated by $\frac{1}{p^n} + \mathbb{Z}$ has order p^n, and is therefore isomorphic to $\mathbb{Z}/(p^n)$.

(ii) Let G be a proper subgroup of $C(p^\infty)$. If G is trivial, then $G \cong \mathbb{Z}/(p^0)$. If G is nontrivial, then we can pick a nonzero $\frac{a}{p^n} + \mathbb{Z} \in G$ with $\gcd(a, p^n) = 1$. By Bézout's identity, there are integers r, s such that $ra + sp^n = 1$. Thus $\frac{ra}{p^n} + \mathbb{Z} = \frac{1 - sp^n}{p^n} + \mathbb{Z} = \frac{1}{p^n} - s + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z} \in G$. Let N be the largest natural number for which $\frac{1}{p^N} + \mathbb{Z} \in G$ (N exists, else G is not proper). We claim that $\left(\frac{1}{p^N} + \mathbb{Z}\right) = G$.

If $\frac{a}{p^n} + \mathbb{Z} \in G$ with $\frac{a}{p^n}$ in lowest terms, then maximality of N implies $n \leq N$. So $\frac{a}{p^n} + \mathbb{Z} = \frac{ap^{N-n}}{p^N} + \mathbb{Z} \in \left(\frac{1}{p^N} + \mathbb{Z}\right)$. This establishes $G \subseteq \left(\frac{1}{p^N} + \mathbb{Z}\right)$. The opposite containment is clear. □

Proposition 1. *If p is a prime integer, then $C(p^\infty) \cong \mathbb{Q}/\mathbb{Z}_{(p)}$.*

Proof. Recall that $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p). Define $\varphi : C(p^\infty) \to \mathbb{Q}/\mathbb{Z}_{(p)}$ by $\frac{a}{p^n} + \mathbb{Z} \to \frac{a}{p^n} + \mathbb{Z}_{(p)}$. φ is well-defined, since $\mathbb{Z} \subseteq \mathbb{Z}_{(p)}$. It is clear that φ is a homomorphism. Suppose that $\frac{a}{p^n} \in \mathbb{Z}_{(p)}$ is in lowest terms. Then $\frac{a}{p^n} = \frac{a'}{b'}$ where $b' \notin (p)$, so $ab' = a'p^n$. If $n > 0$, then $p|ab'$ and primeness of p implies that $p|a$ or $p|b'$, both of which are contradictions. Thus $n = 0$, and $\frac{a}{p^n} + \mathbb{Z} = \mathbb{Z}$, which shows φ is injective.

Let $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$. Factor b as $b'p^m$, where $m \geq 0$ and $\gcd(b', p) = 1$. By Bézout's identity, there are integers r and s such that $rp^m + sb' = 1$. It follows that $\frac{a}{b} - \frac{as}{p^m} = \frac{a(1 - sb')}{b'p^m} = \frac{arp^m}{b'p^m} = \frac{ar}{b'}$, which is an element of $\mathbb{Z}_{(p)}$ since b' and p are relatively prime. Thus $\varphi\left(\frac{as}{p^m} + \mathbb{Z}\right) = \frac{a}{b} + \mathbb{Z}_{(p)}$, which shows φ is surjective.

□

It turns out that the localization of $\mathbb{Z}$ at (p) is an example of a type of ring that fundamental in the results chapter.

Definition 6. *A discrete valuation ring (DVR) is a principal ideal domain with a unique nonzero prime ideal* $\mathfrak{m} := (m)$*. Note that* $\mathfrak{m}$ *must be maximal.*

Proposition 2. $\mathbb{Z}_{(p)}$ *is a DVR and* $\mathbb{Q}$ *is its quotient field (up to isomorphism).*

Proof. By Theorem 7, $P = (p)_{(p)} = \mathbb{Z}_{(p)}p$ is the unique maximal ideal of $\mathbb{Z}_{(p)}$. In fact, by Theorem 7, P is prime. If Q is a nonzero prime ideal of $\mathbb{Z}_{(p)}$, then $Q \subseteq P$. But then Q corresponds to some nonzero prime ideal of $\mathbb{Z}$ which is contained in (p). Since the only nonzero prime ideal contained in (p) is (p) itself, we must have $Q = P$. Lastly, let I be a proper, nonzero ideal of $\mathbb{Z}_{(p)}$. Then $I \subseteq P$. Let p^k be the smallest power of p that is contained in I. If $k = 1$, then $I = P$ and so I is principal. If $k > 1$, then consider the ideal $\mathbb{Z}_{(p)}p^k \subseteq I$. Let $\frac{a}{b} \in I$ be in lowest terms. Then $\frac{a}{b} = \frac{a'p^n}{b}$ with $\gcd(a', p) = 1$. By minimality of k, $k \leq m$. So $\frac{a}{b} = \frac{(a'p^{n-k})p^k}{b} \in \mathbb{Z}_{(p)}p^k$. Thus $I = \mathbb{Z}_{(p)}p^k$, which proves that $\mathbb{Z}_{(p)}$ is a principal ideal domain. In fact we have shown that $\mathbb{Z}_{(p)}$ is a DVR.
Let K be the quotient field of $\mathbb{Z}_{(p)}$. If $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$, then $\frac{a}{b} = \frac{a/1}{b/1} \in K$, since $\frac{a}{1}, \frac{b}{1} \in \mathbb{Z}_{(p)}$. Conversely, if $\frac{a/b}{c/d} \in K$, then $c \neq 0$ and $\frac{a/b}{c/d} = \frac{ad}{bc} \in \mathbb{Q}$. So $K = \mathbb{Q}$. □

Theorem 8. *Every principal ideal domain is a unique factorization domain. That is, every nonzero nonunit in a principal ideal domain can be factored as a product of prime elements.*

Proof. Let R be a PID and $S \subsetneq R$ be the set of all nonzero nonunits that cannot be expressed as a finite product of prime elements. Suppose $a \in S$. Since a is a nonunit, $(a) \neq R$. Because PIDs always have an identity, there is a maximal ideal (c) that contains (a). Maximal ideals in a commutative, unital ring are prime, so c must be prime. Since $(a) \subseteq (c)$, $a = ca_1$ for some $a_1 \in R$. Note that $a_1 \in S$, else a is a finite product of primes. Proceeding as above, we find a prime $c_1 \in R$ such that $a_1 = c_1 a_2$, where $a_2 \in S$. Thus we form a chain $(a) \subsetneq (a_1) \subsetneq (a_2) \subseteq \cdots$ of ideals of R. But this contradicts the fact that every principal ideal domain is noetherian, so S is empty. Finally, we prove that any product of primess that equals a is unique up to order and associates. Suppose $c_1^{k_1} c_2^{k_2} \cdots c_n^{k_n} u = a = d_1^{\ell_1} d_2^{\ell_2} \cdots d_m^{\ell_m} v$, where u, v are units and the c_i and d_j are prime and pairwise nonassociates. By definition of prime, $c_1 | d_i$, for some i, so $(d_i) \subseteq (c_1)$. There is a maximal ideal (m) which contains (d_i). Whence $d_i = mx$. Primeness of d_i implies $d_i | m$ or $d_i | x$. The latter cannot happen, else $d_i = 0$ or m is a unit. Thus (d_i) is maximal and $(d_i) = (c_1)$. It follows immediately that c_1 and d_i are associates. Suppose $c_j | d_{i_j}$ for each $j = 1, \ldots, k$ with $k < n$. Note that $l \neq j \implies d_{i_l} \neq d_{i_j}$, else the fact that the c_j are pairwise nonassociates is contradicted. We claim that $c_{k+1} | d_{i_{k+1}}$ with $d_{i_{k+1}} \neq d_{i_j}$ for $j \neq k+1$. If not, then the pairwise nonassociate condition is again contradicted. Then by induction, we have that $c_j | d_{i_j}$ for $j = 1, \ldots, n$, so each pair c_j, d_{i_j} are associates. Furthermore, $n \leq m$ and $k_i \leq \ell_i$. A symmetric argument with the d_i shows that $m \leq n$ and $\ell_i \leq k_i$. Whence, $m = n$, $\ell_i = k_i$, and each c_j has some d_j as an associate (WLOG suppose $i = j$). Now, let $c = c_1^{k_1} c_2^{k_2} \cdots c_n^{k_n}$ and $d = d_1^{\ell_1} d_2^{\ell_2} \cdots d_m^{\ell_m}$. For each j, there exist $x_j, y_j \in R$ with $c_j x_j = d_j$ and $d_j y_j = c_j$. It follows that $d = c(x_1 x_2 \cdots x_n)$. Moreover, $d_j(y_j x_j) = d_j$ and the fact that R is a domain implies $y_j x_j = 1_R$. Whence $x_1 x_2 \cdots x_n$ is a unit. This proves that each nonzero nonunit of R can be expressed as a product of primes and units. □

The next lemma collects some useful results on DVRs and their quotient fields.

Lemma 4. *Let V be a DVR with maximal ideal $\mathfrak{m} := (m)$ and quotient field K.*

(i) Every nonzero element of V is of the form um^k, where $u \in V$ is a unit and k is a nonnegative integer.

(ii) K/V is a unitary, faithful V-module.

(iii) The proper, nontrivial V-submodules of K/V are $M_k := \left(\frac{1}{m^k} + V\right)$

(iv) For each nonnegative integer k, $M_k \cong_V V/\mathfrak{m}^k$.

(v) The M_k are distinct, and linearly ordered by set-theoretic containment.

(vi) K/V is infinitely generated over V.

Proof. (i) Let $a, b \in V$ and suppose $m|ab$. Then $ab \in (m)$ and primeness of (m) implies $a \in (m)$ or $b \in (m)$. Thus $m|a$ or $m|b$ which proves m is prime. If $x \in V\backslash(m)$, then $(x) = V$ and so x is a unit, else maximality of (m) is contradicted. So if $q \in V$ is prime, then q is not a unit, so $m|q$. Since the only divisors of a prime are units and its associates, m and q are associates. Thus m is the unique prime (irreducible) element of V up to multiplication by a unit. By Theorem 8, every nonzero element of V factors into um^k, where u is a unit and k is a nonnegative integer.

(ii) Define the module multiplication on K/V by $v \cdot \left(\frac{a}{b} + V\right) = \frac{va}{b} + V$. This multiplication is obviously well defined, and $1_V \cdot \left(\frac{a}{b} + V\right) = \frac{a}{b} + V$. Lastly, suppose that $v(K/V) = \{0\}$ for some nonzero $v \in V$. v is not a unit, else $\frac{v}{m} + V \in K/V$ is nonzero. So $v = um^k$, where u is a unit and k is a positive integer. By hypothesis, $vK \subseteq V$. But then $\frac{v}{m^k + 1} = \frac{um^k}{m^{k+1}} = \frac{u}{m} \in V$, which implies $m|u$. This is a contradiction, since any divisor of a unit is also a unit. Hence, $v = 0$, which proves K/V is faithful over V.

(iii) Observe that, by (i), every nonzero element of K/V is of the form $\frac{u}{m^k}$, where u is a unit and k is a positive integer. Let M be a proper, nontrivial submodule of

K/V. If $\dfrac{u}{m^n} + V \in M$ is nonzero with u a unit, then by Bézout's identity, there are $v_1, v_2 \in V$ such that $uv_1 + m^n v_1 = 1$. So $\dfrac{uv_1}{m^n} = \dfrac{1 - m^n v_2}{m^n} = \dfrac{1}{m^n} - v_2$. Since M is a V-module, $\dfrac{1}{m^n} + V \in M$. Let N be the largest integer for which $\dfrac{1}{m^N} + V \in M$ (N exists since $M \neq K/V$). We claim that M is generated by $\dfrac{1}{m^N} + V$. If $\dfrac{v}{m^k} + V \in M$ with v a unit, then $\dfrac{1}{m^k} + V \in M$, so $k \leq N$. Thus $\dfrac{v}{m^k} = \dfrac{vm^{N-k}}{m^N} \in \left(\dfrac{1}{m^N} + V\right)$. The opposite containment is clear, whence $M = \left(\dfrac{1}{m^N} + V\right)$.

(iv) First, note that $\mathfrak{m}^k = (m^k)$. Consider the obvious V-epimorphism $f : V \twoheadrightarrow M_k$ defined by $f(v) = \dfrac{v}{m^k} + V$. The kernel of f is $\left\{vm^k | v \in V\right\} = (m^k)$, whence $M_k \cong_V V/\mathfrak{m}^k$.

(v) Let k, t be positive integers with $k > t$. $\dfrac{v}{m^t} = \dfrac{vm^{k-t}}{m^k}$, so $x \in M_t \implies x \in M_k$. Thus the M_k are linearly ordered by $\subseteq$. Now, if $\dfrac{1}{m^k} + V \in M_t = \left(\dfrac{1}{m^t} + V\right)$, then $\dfrac{1}{m^k} - \dfrac{v}{m^t} = \dfrac{1 - vm^{k-t}}{m^k} \in V$ for some $v \in V$. So $1 - vm^{k-t} = um^\ell$, where u is a unit or $u = 0$, and ℓ is a nonnegative integer. In all cases, it follows that m is a unit, which contradicts primeness of m. Thus $M_t \neq M_k$, and so the proper submodules of ${}_V K/V$ are $\{0\} \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$.

(vi) Clearly $\displaystyle\bigcup_{k \geq 1} M_k = K/V$. Any finite collection of elements of K/V is contained in some M_k. Thus, by (v), K/V is not finitely generated. □

Chain Conditions on Modules.

Artinian Modules.

Definition 7. *Let R be a ring and M an R-module. M is artinian provided every infinite descending chain of R-submodules stabilizes. We say that a ring R is artinian provided R is artinian as a module over itself.*

Lemma 5. *Let R be a ring and consider R as a module over itself, so the R-submodules of R are the ideals of R. If R is artinian, then R has finitely many maximal ideals.*

Proof. Suppose by way of contradiction R has infinitely many (distinct) maximal ideals $M_1, M_2, \ldots$. Form the descending chain $(*)$ $M_1 \supseteq M_1 \cap M_2 \supseteq M_1 \cap M_2 \cap M_3 \supseteq \cdots$. Since R is artinian, there is a positive integer k such that $M_1 \cap \cdots \cap M_{k+1} = M_1 \cap \cdots \cap M_k$, which implies $M_1 \cap \cdots \cap M_k \subseteq M_{k+1}$. But maximal ideals in a commutative, unital ring are prime, so $M_j = M_{k+1}$ for some $1 \leq j \leq k$. This contradicts our claim that the M_i are distinct. Thus R has but finitely many maximal ideals. □

Noetherian Modules.

Definition 8. *Let R be a ring and M an R-module. M is noetherian provided every infinite ascending chain of R-submodules stabilizes. We say that a ring R is noetherian provided R is noetherian as a module over itself.*

Proposition 3. *Let M be an R-module. M is noetherian if and only if every submodule of M is finitely generated.*

Proof. Suppose that M is noetherian and N is a submodule of M generated by $n_1, n_2, n_3, \ldots$. Then form the ascending chain $Rn_1 \subseteq Rn_1 + Rn_2 \subseteq Rn_1 + Rn_2 + Rn_3 \subseteq \cdots$. Since M is noetherian, this chain must stabilize. Whence N is finitely generated. Conversely, suppose that every submodule of M is finitely generated. Consider the chain of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$. Clearly $N = \bigcup_{i \geq 1} N_i$ is a submodule of M. By hypothesis, N is finitely generated. But then all of the generators of

N are contained in some N_k, so $N = N_k = N_{k+1} = \cdots$. This shows that M is noetherian. $\square$

Lemma 6. *Suppose that M is a faithful torsion module over a domain R. If N is a finitely generated submodule of M, then M/N is also faithful over R.*

Proof. Suppose N is finitely generated by $n_1, \ldots, n_k$. Since M is torsion, there exist nonzero $r_1, \ldots, r_k \in R$ such that $r_i n_i = 0$. If $r(M/N) = N$, then $rm \in N$ for all $m \in M$. But then $(r_1 \cdots r_k r)m = 0$ for all $m \in M$. Since M is faithful, $r_1 \cdots r_k r = 0$. But R is a domain and each $r_i \neq 0$, so $r = 0$. Whence M/N is a faithful R-module. $\square$

Lemma 7. *Let M be a finitely generated module over a ring R. M is noetherian (artinian) if and only if $R/Ann_R(M)$ is noetherian (artinian).*

Proof. Suppose M is generated by $m_1, \ldots, m_n$. Then observe that the annihilator of M, $A = Ann_R(M)$, is the intersection of the annihilators $A_i = Ann_R(m_i)$. Define $\theta : R/A \to R/A_1 \oplus \cdots \oplus R/A_n$ by $r + A \mapsto (r + A_1, \ldots, r + A_n)$. Clearly θ is a R-monomorphism. Moreover, the map given by $R/A_i \to Rm_i$ is an R-isomorphism. Since each $Rm_i \subseteq M$ is necessarily noetherian (artinian), so is each R/A_i. Since finite direct sums preserve chain conditions, $R/A_1 \oplus \cdots \oplus R/A_n$ is noetherian (artinian). Consequently, $Im(\theta) \cong R/A$ is noetherian (artinian) as an R-module. Now, observing that the R submodules of R/A coincide with the ideals of R/A concludes this part of the proof.

Conversely, suppose R/A is noetherian (artinian). By Example 2, M is naturally an R/A-module, and the structure of M as an R/A module is essentially the same as the structure of M as an R-module. Since M is finitely generated, there is an epimorphism from a finite direct sum of copies of R/A onto M. We deduce that M is a noetherian (artinian) R-module. $\square$

Note that the following theorem is the commutative case of a well-known corollary of the Hopkins-Levitzki Theorem which states that any left artinian ring is also left noetherian. The following proof is due to Karamzadeh (1994).

Theorem 9. *Let R be a commutative ring with identity. If R is artinian, then R is noetherian.*

Proof. Suppose by way of contradiction that R is artinian but not noetherian. Then R has ideals which are infinitely generated (not finitely generated). Since R is artinian, there is a minimal infinitely generated ideal I. We claim that for all $x \in R$ either $xI = I$ or $xI = \{0\}$. Suppose $xI \neq I$. By minimality of I, xI is finitely generated. Considering the obvious surjection $I \to xI$, we have $I/K \cong xI$ where K is the kernel of the surjection. Observing that I is infinitely generated but I/K is finitely generated, it must be that K is infinitely generated. But then $K = I$, and so $xI = \{0\}$. Now, suppose $ab \in Ann_R(I)$ and $b \notin Ann_R(I)$. By the above claim, $\{0\} = (ab)I = a(bI) = aI$, so $a \in Ann_R(I)$. This shows that $P := Ann_R(I)$ is a prime ideal of R, and so R/P is an artinian domain. If $x \in R/P$ is a nonzero nonunit, then $x^m \neq x^n$ whenever $m \neq n$. But then the chain $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$ does not stabilize, a contradiction. So every nonzero element of R/P is a unit, so $F := R/P$ is a field. With the natural scalar multiplication, I becomes an infinite dimensional F-vector space whose F-subspaces coincide with the ideals of R that are contained in I. Then minimality of I implies that every F-subspace of I is finitely generated. This cannot happen since every infinite dimensional vector space contains proper infinite dimensional subspaces. ☐

Theorem 10. *Let R be a ring and M be an artinian R-module. Then there is a finite, nonempty collection $\{J_1, \ldots, J_n\}$ of n (distinct) maximal ideals of R such that $M = M[J_1] \oplus \cdots \oplus M[J_n]$, where the $M[J_i]$ are defined as in Definition 4.*

Proof. First we will show that every element of M is annihilated by a finite product of maximal ideals. We claim $(*)$ that if $\{0\} = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_K$ is a sequence of R-submodules of M with simple factors, then there are maximal ideals $J_1, J_2, \ldots, J_k$ such that $(J_1 J_2 \cdots J_k)N_k = \{0\}$. We proceed by induction. If $k = 1$, then N_1 is simple (cyclic) and so $R/Ann_R(N_1) \cong_R N_1$, whence $J = Ann_R(N_1)$ is a maximal ideal. Now suppose $(*)$ holds for some $k \geq 1$. If $\{0\} = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_K \subsetneq N_{k+1}$ is a sequence of R-submodules of M with simple factors, then (reusing the above argument) $J_{k+1} = Ann_R(N_{k+1}/N_k)$ is maximal. So $J_{k+1}N_{k+1} \subseteq N_k$. But then by inductive hypothesis, there are maximal ideals $J_1, \ldots, J_k$ such that $\{0\} = (J_1 \cdots J_k)N_k \supseteq (J_1 \cdots J_k J_{k+1})N_{k+1}$. This establishes $(*)$.
Let $m \in M$. From the obvious surjection $R \to Rm$, we see that $R/Ann_R(m) \cong Rm$ as R-modules, where the scalar multiplication on $R/Ann_R(m)$ is given by $r \cdot \overline{x} = rx$. It follows easily that $R/Ann_R(m)$ is an artinian ring. By Theorem 9, $R/Ann_R(m)$ is a noetherian ring, whence Rm is a noetherian R module. So R has a composition series. By $(*)$, Rm is annihilated by a finite product of maximal ideals of R. In particular (since R is unital), m is annihilated by this finite product of maximal ideals. We claim that $J_1^{k_1} + \cdots + J_n^{k_n} = R$, each J_i a maximal ideal. If not, then $J_1^{k_1} + \cdots + J_n^{k_n}$ is contained in some maximal ideal I of R. But then $J_1^{k_1}$ and $J_2^{k_2}$ are

contained in I, and (since maximal ideals in a commutative unital ring are prime) $J_1 = I = J_2$, which contradicts uniqueness of the J_i.

Now, in light of the preceding claim, $1 = e_1 + \cdots + e_n$, where $e_i \in J_i^{k_i}$. Lastly, we claim (**) that if $(J_1^{k_1} \cdots J_n^{k_n})m = \{0\}$ and $m = e_1m + \cdots + e_nm$, then each e_im is annihilated by some $J_\ell^{k_\ell}$. Again by induction, if $n = 1$, then we are done. If $n = 2$, then $m = e_1m + e_2m$ and $J_2^{k_2}(e_1m) \subseteq (J_1^{k_1}J_2^{k_2})m = \{0\}$ (similarly, $J_1^{k_1}(e_2m) = \{0\}$). Suppose that (**) holds for some $n \geq 1$. So $(J_1^{k_1} \cdots J_n^{k_n}J_{n+1}^{k_{n+1}})m = \{0\}$ and $m = e_1m + \cdots + e_nm + e_{n+1}m$. But by our inductive hypothesis $(J_1^{k_1} \cdots J_n^{k_n})m = (J_1^{k_1} \cdots J_n^{k_n})e_{n+1}m = \{0\}$, and so $e_{n+1}m$ is annihilated by some $J_\ell^{k_\ell}$, $1 \leq \ell \leq n$. This proves (**).

We have shown that $M = \sum_{J \text{ maximal}} M[J]$. If I and J are distinct maximal ideals and $x \in M[J] \cap M[I]$, then there are positive integers n,m such that J^n and I^m are contained in $Ann_R(x)$. If $Ann_R(x) \neq R$, then $Ann_R(x)$ is contained in some maximal ideal of R. But then primeness of I and J implies $I = J$, a contradiction. So $Ann_R(x) = R$, which implies $x = 0$. This shows that $M = \bigoplus_{J \text{ maximal}} M[J]$. Finally, since M is artinian, the preceding direct sum must be finite. This completes the proof. $\square$

Lemma 8. *Let R be a ring and M an artinian R-module. Further, suppose $M = \oplus_{i=1}^n M[J_i]$ (as in Theorem 10). If each $M[J_i]$ has but finitely many R-submodules, then M has but finitely R-submodules.*

Proof. Let N be an R-submodule of M. Since M is artinian, so is N. Thus, by Theorem 10, there are distinct maximal ideals $Q_1, \ldots, Q_m$ such that each $N[Q_i]$ is nontrivial and $N = \oplus_{i=1}^m N[Q_i]$. We claim that each Q_i is equal to some J_k. Pick a nonzero $x \in N[Q_i]$. $Ann_R(x)$ is a proper ideal of R, else $x = 1_R x = 0$. Thus there is a maximal ideal I which contains $Ann_R(x)$. By definition of $N[Q_i]$, there is a positive integer ℓ such that $Q_i^\ell \subseteq Ann_R(x) \subseteq I$. Since maximal ideals in a commutative, unital ring are prime, $Q_i \subseteq I$. Then maximality implies $Q_i = I$. Now, since $N[Q_i]$ is a submodule of M, $x \in M$ and so there is a positive integer power of each J_k which annihilates n. That is, $(J_1^{k_1} \cap \cdots \cap J_n^{k_n})n = \{0\}$. Reusing the argument above and observing that the product of two ideals is contained in their intersection, we see that $J_1^{k_1} \cdots J_n^{k_n} \subseteq J_1^{k_1} \cap \cdots \cap J_n^{k_n} \subseteq I$. By primeness and maximality, $J_k = I$ for some k. So $Q_i = J_k$. Without loss of generality, we let $Q_i = J_i$ for $1 \leq i \leq m$. Thus $N = \oplus_{i=1}^m N[J_i]$. In general, every submodule of M is $M_1 \oplus \cdots \oplus M_n$, where each M_i is a submodule of $M[J_i]$. Since each $M[J_i]$ has finitely many submodules, we deduce that M itself has but finitely many submodules. □

Lemma 9 (Nakayama). *If J is an ideal in a commutative, unital ring R, then the following are equivalent:*

(i) J is contained in every maximal ideal of R;

(ii) $1_R - j$ is a unit for every $j \in J$;

(iii) If N is a finitely generated R-module such that $JN = N$, then $N = \{0\}$;

(iv) If N is a submodule of a finitely generated R-module M such that $M = JM + N$, then $M = N$.

Proof. (i) $\Longrightarrow$ (ii) If $j \in J$ and $1_R - j$ is not a unit, then $(1_R - j)$ is a proper ideal of R. But then there is a maximal ideal M which contains $(1_R - j)$. Since J is contained in every maximal ideal, $j \in M$ and $(1_R - j) + j = 1_R \in M$. This contradicts $M \neq R$, so $1_R - j$ is a unit.

(ii) $\implies$ (iii) Suppose N is minimally generated by $X = \{n_1, \ldots, n_k\}$. If $N \neq \{0\}$, then one of its generators is nonzero, say $n_1 \neq 0$. Since $JN = N$, there are $j_1, \ldots, j_k \in J$ such that $n_1 = j_1 n_1 + \cdots + j_k n_k$. $k \neq 1$, else invertibility of $1_R - j_1$ implies $n_1 = 0$. If $k > 1$, then $n_1 = (1_R - j_1)^{-1} j_2 n_2 + \cdots + (1_R - j_1)^{-1} j_k n_k$, which contradicts our claim that N is minimally generated by X. Thus $N = \{0\}$.

(iii) $\implies$ (iv) If $x \in (JM + N)/N$, then $x = (\sum j_i m_i) + n + N = (\sum j_i m_i) + N \in (JM)/N$. The opposite containment is obvious, so $(JM + N)/N = (JM)/N$. Now observe that $(JM)/N = J(M/N)$ (using the obvious scalar multiplication), and so $J(M/N) = M/N$. Since M is finitely generated, so is M/N. Thus $M/N = \{0\}$, which means $M = N$.

(iv) $\implies$ (i) If I is a maximal ideal of R, then the ideal $I \subseteq JR + I$. If $JR + I = R$, then $I = R$, a contradiction. Thus $JR + I = I$. Since R is unital, $J \subseteq JR + I = I$. $\square$

CHAPTER III: RESULTS

Definition 9. *Let R be a ring and M an R-module. Say M is lower finite provided M has infinitely many submodules, but every proper submodule of M has but finitely many submodules.*

It is clear that every lower finite module M is artinian. Whence, by Theorem 9, $M = \oplus_{i=1}^{n} M[J_i]$ for some $n \geq 1$. Suppose by way of contradiction that $n \neq 1$, then Lemma 6 implies that M has finitely many submodules. This contradicts Definition 9. Thus $M = M[J]$ for some maximal ideal J of R. Without loss of generality, Lemma 2 (${}_{R}M$ has essentially the same structure as ${}_{R_J}M$) allows us to assume that R is local. Moreover, since the quotient ring $R/Ann_R(M)$ remains local, we need only consider lower finite modules which are faithful over local rings.

Our first result shows that lower finiteness is a strictly module-theoretic property.

Corollary 2. *No ring is lower finite as a module over itself.*

Proof. Let R be a ring. Suppose by way of contradiction that R is a lower-finite R-module. Since R is artinian, Lemma 5 implies that R has finitely many maximal ideals. But then lower finitness of R implies that R has finitely many ideals. This contradicts the definition of lower finite. □

The following lemma extends the example given in the introduction that $\mathbb{R} \oplus \mathbb{R}$ is a lower finite $\mathbb{R}$-module.

Lemma 10. *Let F be an infinite field. Then the F-vector space $F \oplus F$ is lower finite.*

Proof. For each $a \in F$, define $S_a := \{t(1,a)|t \in F\}$, the subspace generated by $(1,a)$. If $a \neq b$, then $(1,a) \in S_a \setminus S_b$, so $S_a \neq S_b$. Since F is infinite, $F \oplus F$ has infinitely many subspaces. Proper subspaces of $F \oplus F$ are either one-dimensional or trivial, and a one-dimensional vector space only has two subspaces. Hence ${}_F F \oplus F$ is lower finite. □

Proposition 4. *Let R be a local ring with unique maximal ideal J, and suppose that M is a faithful lower finite R-module. Suppose further that the residue field R/J is infinite. Then either $M \cong_R R \oplus R$ and R is a field or there is a simple essential R-submodule N of M.*

Proof. We distinguish two cases.

Case 1. M has two R-submodules with trivial intersection. Observe that since M is artinian, every nonzero R-submodule contains a simple (cyclic) submodule. So without loss of generality there exist simple R-submodules Rm and Rn such that $Rn \cap Rm = \{0\}$. Since R is local, $Ann_R(n) \subseteq J$. If $Ann_R(n) \neq J$, then there exists $j \in J$ such that $jn \neq 0$. But then $R(jn) = Rn$, so $(rj)n = n$ for some $r \in R$. This implies that $1 - rj \in J$, a contradiction. Thus $Ann_R(n) = J$. Similarly, $Ann_R(m) = J$. Note that Rn and Rm are now naturally one-dimensional vector spaces over R/J. Whence $Rn \cong Rm \cong R/J$, and M contains an isomorphic copy of $R/J \oplus R/J$. Moreover, since $Ann_R(R/J \oplus R/J) = J$, $R/J \oplus R/J$ as an R-module has essentially the same structure as $R/J \oplus R/J$ as an R/J-vector space. By Lemma 10, $R/J \oplus R/J$ has infinitely many R/J-subspaces, so lower finiteness of M implies

$M \cong_R R/J \oplus R/J$. Lastly, since M is faithful over R, $AnnR(M) = J = \{0\}$. This proves that $M \cong_R R \oplus R$ and R is a field.

Case 2. Any two nonzero submodules of M have nontrivial intersection. By the observation in Case 1, M contains a simple R-submodule N. N is immediately essential. □

We will classify the lower finite R-modules by dividing them into three classes and treating the classes separately.

Infinitely generated lower finite J-primary R-modules.

***Infinite residue field* R/J.** We rely heavily on the following nontrivial* result from Hirano and Mogami in [7]:

Lemma 11. *Let R be a ring and M an R-module. Suppose further that the poset of R-submodules of M is isomorphic to $\omega + 1$. Let $S := End_R(M)$ be the endomorphism ring of M over R. Then the structure of M as an R-module is essentially the same as the structure of M as an S-module. Moreover, S is a DVR, and if K is the quotient field of S, then $M \cong_S K/S$.*

*Note that Lemma 11 is the primary result of the paper *Modules whose proper submodules are non-Hopf kernels*.

Theorem 11. *Let R be a local ring with maximal ideal J and infinite residue field R/J. Suppose that M is an R-module. M is an infinitely generated faithful lower finite R-module if and only if $S := End_R(M)$ is a DVR, the structure of M as an R-module is essentially the same as the structure of M as an S-module, and $M \cong_S K/S$, where K is the quotient field of S.*

Proof. ($\Rightarrow$) By Lemma 11, it suffices to prove that the poset of R-submodules of M is (order) isomorphic to $\omega + 1$. Invoking Proposition 4, either $M \cong_R R \oplus R$ and R is a field or M contains a simple essentially R-submodule. The former implies that M is two-generated, a contradicition. So M has a simple essential R-submodule N_1. In particular, N_1 is contained in every proper nonzero R-submodule of M. Consequently, M/N_1 remains lower finite. Because M is infinitely generated and N_1 is finitely generated (cyclic), M/N_1 is also infinitely generated. Reusing the argument above, we see that M/N_1 has a simple essential submodule N_2/N_1. Moreover, simplicity of N_2/N_1 implies that every R-submodule of M which properly contains N_1 also contains N_2. Observing that $(M/N_1)/(N_2/N_1) \cong_R M/N_2$, we proceed recursively to construct a strictly ascending chain $N_0 := \{0\} \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots$ of R-submodules of M with the property that N_{i+1}/N_i is simple for all $i \geq 0$. Now, set $N := \cup_{i \geq 0} N_i$. Since the N_i form a chain, N is an R-submodule of M. Clearly N has infinitely many submodules (the N_i), whence lower finiteness of M implies $N = M$. We conclude the proof by showing that the N_i are precisely the proper R-submodules of M. For each i, $N_i \neq N_{i+1}$, so N_i is proper. Conversely, if K is a proper R-submodule of M, then there is a largest i for which $N_i \subseteq K$. Else $M = \cup_{i \geq 1} N_i \subseteq K$, which contradicts K being proper. We claim that $N_i = K$. Otherwise, $N_i \subsetneq K$ and, since N_{i+1}/N_i is simple, we have $N_{i+1} \subseteq K$. This contradicts maximality of i, so $K = N_i$. Thus the R-submodules of M can be arranged as $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots M$.

($\Leftarrow$) $M \cong_S K/S$ is an infinitely generated lower finite faithful S-module by (iii), (v), and (vi) of Lemma 4. Then since the structure of M as an R-module is essentially the same as the structure of M as an S-module, M is infinitely generated and lower finite. $\square$

***Finite residue field* R/J**. For this class, we will invoke the theory of Jónsson modules.

Definition 10. *A module M over a ring R is a Jónsson module provided M is infinite and $|N| \neq |M|$ for every proper submodule N of M.*

Since any proper submodule of a countable Jónsson module is finite, lower finite modules extend the notion of countable Jónsson modules.

We pause to collect the results (Oman, [12]) which will be used in our classification.

Lemma 12. *Let R be a ring and M an R-module.*

(i) If M is a Jónsson R-module, then $Ann_R(M)$ is a prime ideal of R.

(ii) If M is a faithful, infinitely generated artinian Jónsson module, then M is countable.

(iii) Let M be infinitely generated, countable, and faithful over a ring R. Then M is a Jónsson R-module if and only if R is a domain (say with quotient field K) and there exist both a DVR overring $(V, \mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$.

Lemma 13. *Let R be a ring and suppose that I is a finitely generated ideal of R. If R/I is finite, then R/I^n is finite for every positive integer n.*

Proof. First observe that since $I^n \subseteq I$, $R/I \cong (R/I^n)/(I/I^n)$. Thus $|R/I^n| = |R/I| \cdot |I/I^n|$. Since R/I is assumed to be finite, it suffices to prove that I/I^n is finite for each n. We proceed by induction. The base case ($n = 1$) is obvious. Now suppose that R/I^{k-1} is finite for some $k \geq 1$. I/I^k is an R/I^{k-1}-module with scalar multiplication given by $(r + I^{k-1}) \cdot (i + I^k) = ri + I^k$. To see that this multiplication is well defined, suppose $r+I^{k-1}=s+I^{k-1}$ and $i+I^k=j+I^k$. Then $ri-sj=ri-si+si-sj=(r-s)i+s(i-j)\in I^k$ since $r-s\in I^{k-1}$, $i\in I$, and $i-j\in I^k$. Since I is finitely generated, so is I/I^k. Finiteness of R/I^{k-1} then implies that I/I^k is finite. Consequently, R/I^k is finite. □

Theorem 12. *Suppose that R is a local ring with maximal ideal J, M is an R-module, and the residue field R/J is finite. Then M is an infinitely generated faithful lower finite R-module if and only if R is a domain (say with quotient field K) and there is a DVR overring $(V,\mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V\subseteq N\subsetneq K$ and $M\cong_R K/N$.*

Proof. ($\Rightarrow$) We claim that M is a Jónsson R-module. To this end, note that lower-finiteness of M implies that M has infinitely many R-submodules and, therefore, is infinite. Since M is artinian, there is an R-submodule N of M which is minimal with respect to being infinite. N is clearly Jónsson. Suppose by way of contradiction that $N\neq M$. Then N must be finitely generated (say by $n_1,\dots,n_k$), lest M contain a proper submodule with infinitely many submodules. There are two cases.

If $k>1$, then minimality of N implies that each Rn_i is finite, and the infinitude of N is contradicted. If $k=1$, then N is cyclic. Recalling that M is J-primary, we note that $J^\ell N=\{0\}$ for some positive integer ℓ. By Lemma 12(i), $Ann_R(N)$ is a prime ideal, so the preceding note along with the maximality of J imply that $J=Ann_R(N)$. But then $N\cong_R R/J$, which is finite. This contradicts the fact that N is Jónsson (infinite, in particular).

Thus $M=N$, which establishes that M is Jónsson. Now, since M is faithful over R, Lemma 12 (i) implies $Ann_R(M)=\{0\}$ is a prime ideal, whence R is a domain. By

Lemma 12 (ii), M is countable. Finally, an application of (iii) of Lemma 12 yields the result.

($\Leftarrow$) Suppose R is a domain with quotient field K, $(V, \mathfrak{m})$ is a DVR overring (of R) with $V/\mathfrak{m}$ finite, and N is an R module such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$. Recall from Lemma 5 that each proper, nontrivial V-submodule of K/V is cyclic and isomorphic to $V/\mathfrak{m}^k$ for some positive integer k. By Lemma 13, every proper V-submodule of K/V is finite. Observing that K/V is equal to the strictly increasing union of its proper submodules, we see that K/V is countable. Whence K/V is a countable Jónsson V-module. Invoking Lemma 5 again, we see that K/V is infinitely generated and faithful over V. Since $V \supseteq R$, K/V as an R-module inherits the preceding properties. Now, $M \cong_R K/N \cong_R (K/V)/(N/V)$. Since K/V is countable and Jónsson, and N/V is an R-submodule of K/V, N/V is finite. Moreover, since K/V is infinitely generated, $M \cong_R (K/V)/(N/V)$ remains infinitely generated. Observe that M is an R-homomorphic image of the countable R-module K/V, and so M is countable. Finally, (recalling the ${}_RK/V$ is faithful) K/V is a torsion V-module and N/V is finitely generated (because it is finite), so Lemma 6 implies that $M \cong_R K/N \cong_R (K/V)/(N/V)$ is faithful over R. By Lemma 12(iii), M is a Jónsson R-module. In conclusion, M has infinitely many submodules because M is infinitely generated, and every proper submodule has finitely many submodules (is finite, in particular) because M is countable and Jónsson. Thus M is a lower finite R module. □

Finitely generated lower finite J-primary R-modules. For the remaining class, we introduce the notion of a module with a waist. Let M be an R-module and K

be an R-submodule of M. We call K a *waist* of M provided that for every R-submodule N of M, either $K \subseteq N$ or $N \subseteq K$. Note that if M is uniserial, then every R-submodule of M is a waist. Moreover, any simple essential R-submodule of M is also a waist.

Theorem 13. *Let R be a local ring with maximal ideal J and M a faithful R-module. Then M is a finitely generated lower finite R-module if and only if R/J is infinite and there exists a finitely generated uniserial artinian waist N of M such that $M/N \cong_R R/J \oplus R/J$.*

Proof. ($\Rightarrow$) Seeking a contradiction, suppose that R/J is finite. Now, since every proper submodule of M has finitely many submodules, every proper submodule of M must be finitely generated. Further, since M is finitely generated, M is noetherian by Proposition 3. We claim that every maximal R-submodule of M contains the R-submodule JM. If not, then there is an R-submodule K of M such that $JM \not\subseteq K$. But then maximality of K implies that $JM + K = M$. Whence Nakayama's Lemma implies that $K = M$, a contradiction. It is easy to verify that M/JM is a finite dimensional R/J-vector space with scalar multiplication $(r + J)(m + JM) = rm + JM$. Since R/J was assumed finite, we see that M/JM is finite (as it is a finite sum of copies of a finite field). Consequently, there are but finitely many R-submodules of M which contain JM. We deduce that M has finitely many maximal R-submodules. Because M is noetherian, every proper R-submodule is contained in a a maximal submodule. Then lower finiteness of M implies that M has finitely many submodules, a contradiction. This establishes that the residue field R/J is infinite.

As in the proof of Theorem 11, we construct a chain $N_0 = \{0\} \subseteq N_1 \subseteq N_2 \subseteq \cdots$ of R-submodules of M such that N_{i+1}/N_i is simple for each $i \geq 0$. Observe that M/N_i remains lower finite, since the R-submodules of N_i are precisely the N_j where $j \leq i$. Since M is noetherian, the chain must stabilize; say $\cup_{i\geq 0} N_i = N_k$. Now, by (the proof of) Proposition 4, $M/N_k \cong_R R/J \oplus R/J$. It is clear that N_k is a finitely generated uniserial artinian waist of M.

($\Leftarrow$) Suppose that N is a finitely generated uniserial artinian waist of M such that $M/N \cong_R R/J \oplus R/J$ and R/J is infinite. First note that infinitude of R/J implies that M/N (and therefore M) has infinitely many R-submodules. By Lemma 7, $R/Ann_R(N)$ is an artinian ring. Then Theorem 9 implies that $R/Ann_R(N)$ is noetherian, and then another application of Lemma 7 shows that N is noetherian. Thus N has a (finite) composition series. In fact, since N is uniserial, the modules in this composition series are precisely the R-submodules of N. Thus, N has finitely many submodules. It remains to prove that any proper R-submodule K of M has but finitely many submodules. Since N is a waist, $K \subseteq N$ or $N \subseteq K$. If the former holds, then K has finitely many R-submodules since N has finitely many. In the latter case, K/N is a proper, nontrivial R-submodule of $M/N \cong R/J \oplus R/J$. Since $Ann_R(M/N) = Ann_R(R/J \oplus R/J) = J$, M/N is an R/J-vector space with essentially the same structure as M/N as an R-module. We deduce that K/N is a one-dimensional R/J-subspace of M/N. In particular, there are no R-submodules of M that are properly contained in K and properly contain N. Thus if L is an R-submodule of K, then either $L = K$ or $L \subseteq N$. Now, since N has finitely many submodules, so does K. This proves that M is a lower-finite R-module. Lastly, we show that M is finitely generated. Observe that $M/N \cong R/J \oplus R/J$ is two-generated, say by $m_1 + N$ and $m_2 + N$. Since N is a waist and $m_1, m_2 \notin N$, $N \subseteq Rm_1$ and $N \subseteq Rm_2$. Thus $M = Rm_1 + Rm_2$. □

Note that we cannot dispense with the requirement that the finitely generated artinian uniserial submodule N is a waist for M. For $M = N \oplus R/J \oplus R/J$ has the property that $M/N \cong R/J \oplus R/J$, but (assuming R/J infinite) M is not lower finite.

Classification of the lower finite modules. We now collect the preceding results into a single theorem, the main result of this dissertation.

Theorem 14. *Let R be a local ring with maximal ideal J and let M be a faithful R-module. Then M is lower finite if and only if one of the following holds:*

(i) *$S := \mathrm{End}_R(M)$ is a discrete valuation ring, the structure of M as an S-module is essentially the same as the structure of M as an R-module, and $M \cong_S K/S$, where K is the quotient field of S,*

(ii) *R is a domain; let K be the quotient field of D. There is a DVR overring $(V, \mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$, or*

(iii) *R/J is infinite and there exists a finitely generated, artinian waist N of M such that $M/N \cong_R R/J \oplus R/J$. (in this case, M is finitely generated).*

Modules with finitely many submodules. We close this section by presenting a new proof of the classification ([17], [18]) of the unitary modules over a commutative ring which have but finitely many submodules. Modules of finite cardinality are trivially in this category, so we consider only the infinite modules with finitely many submodules.

Theorem 15. *Let R be a commutative, unital ring and M an R-module. M is infinite and has finitely many submodules if and only if M is a finite direct sum a finite module and infinite modules which are artinian, noetherian, and uniserial.*

Proof. Suppose that M is an R-module with finitely many submodules, and let J be a maximal ideal of R. By Lemma 2, $M[J]$ as an R-module has essentially the same structure as $M[J]$ as an R_J-module. Further, by Example 2, the structure of $M[J]$ as an R_J-module is essentially the same as the structure of $M[J]$ as an $R_J/Ann_R(M[J])$-module, and $R_J/Ann_R(M[J])$ remains local. Thus, without loss of generality, we may assume that R is local and $M[J]$ is faithful over R. Lemma 8 implies $R/Ann_R(M[J]) \cong R$ is artinian, and then Theorem 9 implies that R is noetherian. By Proposition 3, the maximal ideal J is finitely generated. We claim that the residue field R/J is infinite. If not, then it follows from Lemma 13 that R/J^n is finite for each $n \geq 1$. Observe that for any $m \in M[J]$ there is a positive integer n such that $R/J^n \cong_R Rm$. Now, since $M[J]$ is the sum of its cyclic submodules (of which there are finitely many) and each cyclic submodule is finite, $M[J]$ must be finite. This contradicts the fact that all finite summands of $M[J_1] \oplus \cdots \oplus M[J_n]$ were removed. Thus the residue field R/J is infinite. Invoking Proposition 4, we see that $M[J]$ has a simple essential submodule M_1 ($M[J]$ cannot contain an isomorphic copy of $R/J \oplus R/J$, lest it have infinitely many submodules). Proceeding as in the proof of Theorem 11 and noting that $M[J]$ is noetherian, we construct a composition series $\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M[J]$ which contains all of the submodules of

$M[J]$. Now, since M is artinian, Theorem 10 implies that there are distinct maximal ideals $J_1, \ldots, J_n$ of R such that $M = M[J_1] \oplus \cdots \oplus M[J_n]$. We complete the proof by noting that the infinitude of M implies that some $M[J_i]$ is infinite, and any infinite $M[J_i]$ is artinian, noetherian, and uniserial (has a unique composition series).

Conversely, $M \cong F \oplus N$ where F is a finite module and N is a finite direct sum of infinite modules which are artinian, noetherian, and uniserial. The artinian condition passes to finite direct sums, so N is artinian. Whence, by Theorem 10, there are maximal ideals $J_1, \ldots, J_k$ such that $N = N[J_1] \oplus \cdots N[J_k]$. Without loss of generality, we may suppose each $N[J_i]$ is infinite. From the proof of Lemma 8, we see that the R-submodules of N are precisely direct sums of R-submodules of the $N[J_i]$. Since each $N[J_i]$ has but finitely many R-submodules (proved above), N has finitely many R-submodules. Thus M is infinite and has finitely many submodules, as desired. $\square$

CHAPTER IV: DIRECTIONS FOR FURTHER RESEARCH

We close with three natural lines of investigation for further research. The first two questions we leave completely open, though we have some comments on the last.

Open Problem 1. *Investigate lower finite modules over noncommutative rings.*

The following question can be viewed as a generalization of lower finite modules.

Open Problem 2. *Let κ and μ be cardinal numbers with $\kappa \leq \mu$. Investigate the R-modules M with μ-many submodules such that each proper R-submodule of M has κ-many submodules.*

A third question is the following "dual" of the question investigated in this article. We are agnostic on whether to assume commutativity of the operator ring, so as to state in more generality.

Open Problem 3. *Let R be a ring and let M be a left R-module. Say that M is* upper finite *provided M has infinitely many left R-submodules, but for every nonzero left R-submodule N of M, there are but finitely many left R-submodules of M which contain N. Study the upper finite left R-modules.*

We have a bit to say about this problem. First, it is not hard to see that the ring $\mathbb{Z}$ of integers is upper finite as a module over itself; this follows more or less immediately from the fact that every nonzero integer has but finitely many divisors.

In fact, more generally, if D is any Dedekind domain (a domain for which every proper, nonzero ideal factors as a product of prime ideals), then D is upper finite. A curious fact is that for "sufficiently large" commutative domains D, D is Dedekind if and only if D is upper finite. This completely characterizes the Dedekind domains of size strictly greater than $2^{\aleph_0}$. On the other hand, if κ is a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, then it can be shown (in ZFC) that there is an upper finite domain D of cardinality κ which is not Dedekind, and so the strict lower bound of $2^{\aleph_0}$ is sharp. These results can be found in [11]. These results show that unlike the case for lower finite modules, being upper finite is not strictly a proper module-theoretic property in the sense that there are rings which are upper finite as modules over themselves. We close with an upper-finite characterization of the abelian group $(\mathbb{Z}, +)$.

Proposition 5. *Let G be an abelian group. Then G is infinite cyclic if and only if G is upper finite as a $\mathbb{Z}$-module.*

Proof. We have already explained that $\mathbb{Z}$ is upper finite. Conversely, suppose that G is an upper finite abelian group. Upper finiteness clearly implies that G is a Noetherian $\mathbb{Z}$-module, and so G is finitely generated. The fact that G is infinite along with the Fundamental Theorem of Finitely Generated Abelian Groups implies that $G \cong H \oplus F$, where H is a finite abelian group and F is a nontrivial free abelian group. Clearly F has rank one, lest a summand be contained in infinitely many subgroups of G. Hence $G \cong H \oplus \mathbb{Z}$. Now, H is a subgroup of $H \oplus K$ for every subgroup K of $\mathbb{Z}$. It follows from upper finiteness that H must be trivial, whence $G \cong \mathbb{Z}$, as claimed. □

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