

# Correspondence

## Estimation and Identification for 2-D Block Kalman Filtering

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**Abstract**—This correspondence is concerned with the development of a recursive identification and estimation procedure for 2-D block Kalman filtering. The recursive identification scheme can be used on-line to update the image model parameters at each iteration based upon the local statistics within a block of the observed noisy image. The covariance matrix of the driving noise can also be estimated at each iteration of this algorithm. A recursive procedure is given for computing the parameters of the higher order models. Simulation results are also provided.

### I. INTRODUCTION

Recursive identification in linear discrete-time dynamic systems has been the topic of a multitude of papers over the past decade [2], [3]. In the image restoration area, recursive identification techniques are used to update the filter parameters based upon the local spatial activities within a processing window. This reduces the smearing effects that are caused otherwise. Kaufman *et al.* [4] proposed an identification and estimation procedure for a nonsymmetric half-plane (NSHP) image model which can be used on-line to evaluate the covariance matrix of the driving noise and also the parameters of the model. This method adjusts the image model parameters for each new pixel of the observed image since it uses a scalar scanning scheme. Keshavan and Srinath [5] developed an adaptive image restoration method for a 1-D interpolative model which is obtained from decorrelating data either row-wise or column-wise. Katayama [6] proposed a method of identifying the parameters of a 2-D autoregressive moving average (ARMA) image model and the filter gain simultaneously. The method proposed by Azimi-Sadjadi [7] uses the stochastic Newton approach for updating the parameters of a 2-D block recursive model in the block Kalman filter structure.

In this correspondence, a recursive identification process for 2-D adaptive block Kalman filtering is proposed which can be used on-line to estimate the image model parameters based upon each new block of the noisy image. Unlike the method in [7], the identification scheme in this correspondence is derived using a nongradient approach [8], [9] and by taking into account the particular structure of the dynamic model in the block Kalman filter [1]. A recursive scheme for computing the parameters of higher order models is introduced which can generate a 2-D vector autoregressive (AR) model with minimized correlation mismatch. Finally, simulation results are presented.

### II. IMAGE MODEL PARAMETER IDENTIFICATION

Consider an image of size  $N \times N$  which is scanned vectorially from left to right and top to bottom. The image is assumed to be

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represented by a zero mean vector Markov field and modeled, within each two adjacent strips of size  $M \times N$ , by an  $M$ th order (along the horizontal direction) 2-D vector AR model with causal quarter-plane region of support (ROS). Let us define a vector of pixels in the  $i$ th strip of the original image by

$$\mathbf{z}(i, k) = [z(iM, k)z(iM + 1, k) \cdots z(iM + M - 1, k)]', \quad (i, k) \in \mathbf{R} \quad (1)$$

where  $\mathbf{R} = \{(i, k): 0 \leq i \leq P - 1, 0 \leq k \leq N - 1\}$ ,  $P := N/M$ , and  $z(m, n)$  represents the intensity of the pixel at  $(m, n)$ th location. The 2-D vector AR model with a ROS shown in Fig. 1 can then be written as

$$\begin{aligned} \mathbf{z}(i, k) = & \Phi_1 \mathbf{z}(i, k - 1) + \Phi_2 \mathbf{z}(i, k - 2) + \cdots \\ & + \Phi_M \mathbf{z}(i, k - M) + \theta_0 \mathbf{z}(i - 1, k) \\ & + \theta_1 \mathbf{z}(i - 1, k - 1) + \cdots \\ & + \theta_M \mathbf{z}(i - 1, k - M) + \mathbf{u}(i, k) \end{aligned} \quad (2)$$

where  $\Phi_m$ 's and  $\theta_m$ 's are the coefficient matrices of the model that have to be identified and the driving noise sequence  $\{\mathbf{u}(i, k)\}$  is a zero mean white Gaussian vector process with covariance matrix  $Q_u$ . Note that although  $\{\mathbf{u}(i, k)\}$  is a white vector process, the elements within each vector can be mutually correlated. The orthogonality property of the estimator gives [1]

$$\begin{aligned} E[\mathbf{u}(i, k) \mathbf{z}'(i - p, k - q)] \\ = Q_u \delta(p, q) \quad (i, k) \in \mathbf{R}, \quad (p, q) \in \mathbf{S} \end{aligned} \quad (3)$$

where  $\delta(p, q)$  represents the Kronecker delta function;  $E$  is the expectation operator, and  $\mathbf{S} = \{(p, q): p = 0, 1, q \in [0, M]\}$  is the ROS.

A vector of the observed noisy image is given by

$$\mathbf{y}(i, k) = \mathbf{z}(i, k) + \mathbf{v}(i, k), \quad (i, k) \in \mathbf{R} \quad (4)$$

where  $\mathbf{v}(i, k)$  represents a vector of the scalar additive noise sequence  $\{v(m, n)\}$  which is a white Gaussian process with zero mean and variance  $\sigma_v^2$ . Thus, the vector  $\mathbf{v}(i, k)$  which is defined similar to  $\mathbf{z}(i, k)$  in (1) forms a vector process with zero mean and covariance matrix  $Q_v = \sigma_v^2 I_M$  where  $I_M$  is an identity matrix of size  $M \times M$ . It is assumed that  $\{\mathbf{v}(i, k)\}$  is independent of  $\{\mathbf{u}(i, k)\}$  and  $\{\mathbf{z}(i, k)\}$ .

The 2-D vector AR model in (2) and the observation model in (4) can be arranged in a block state-space form and the corresponding Kalman filter equations can then be derived [1] to generate the block filtered estimates at each iteration. However, this requires identification and estimation of the model parameters and the covariance matrix of the driving noise sequence from the observed image.

The parameters of the 2-D vector AR model in (2) can be obtained by solving the corresponding set of normal equations. The normal equation is obtained by postmultiplying  $\mathbf{z}(i, k)$  by  $\mathbf{z}'(i - p,$

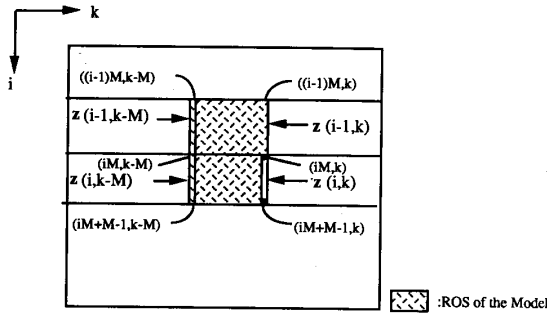


Fig. 1. Region of support for the vector AR model.

$k - q$ ) and then taking the expectation of the result, i.e.,

$$\begin{aligned} \rho_{p,q} &:= E[z(i, k)z'(i - p, k - q)] \\ &= \rho_{p,q-1}\Phi_1^t + \rho_{p,q-2}\Phi_2^t + \cdots + \rho_{p,q-M}\Phi_M^t \\ &\quad + \rho_{p-1,q}\theta_0^t + \rho_{p-1,q-1}\theta_1^t + \cdots \\ &\quad + \rho_{p-1,q-M}\theta_M^t + Q_u\delta(p, q). \end{aligned} \quad (5)$$

However, since the original image is not generally given  $\rho_{p,q}$  should be estimated from the observed noisy image. Taking into account the statistical properties of the sequence  $\{v(i, k)\}$  and using (4) we can write

$$\begin{aligned} r_{p,q} &:= E[y(i, k)y'(i - p, k - q)] \\ &= \rho_{p,q} + Q_v\delta(p, q), \quad (p, q) \in S. \end{aligned} \quad (6)$$

Having known  $Q_v$ , the matrices  $\rho_{p,q}$ 's can readily be estimated from  $r_{p,q}$ 's. Now, writing (5) for all the values of  $(p, q)$  in  $S$  gives the following vector Yule-Walker equation which should be solved for  $\Phi_m$ 's and  $\theta_m$ 's:

$$\begin{bmatrix} R_0 & R_1^t & \cdots & R_M^t \\ R_1 & R_0 & \cdots & R_{M-1}^t \\ \cdots & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ R_M & R_{M-1} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_M \end{bmatrix} = \begin{bmatrix} \Omega \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad (7)$$

where

$$R_k = \begin{bmatrix} \rho_{0,k} & \rho_{1,-k}^t \\ \rho_{1,k} & \rho_{0,k} \end{bmatrix}, \quad S_l = [-\Phi_l \quad -\theta_l^t], \quad (8a)$$

$$S_0 = [I_M \quad -\theta_0^t]^t, \quad \text{and} \quad \Omega = [Q_u^t \quad \mathbf{0}]^t. \quad (8b)$$

In a simpler notation (7) can be rewritten as

$$\Gamma_M \Psi_M = \Delta \quad (9)$$

where matrix  $\Gamma_M$  is a Hermitian doubly Toeplitz, though its individual subelements  $R_k$  are not in general Hermitian, i.e.,  $R_k^t \neq R_k$ ,  $k \neq 0$ . From (7) or (9) we obtain

$$Q_u = (\tilde{I}_1^t \Gamma_M^{-1} \tilde{I}_1)^{-1} \quad (10a)$$

$$\Phi_k = (\tilde{I}_{2k+1}^t \Gamma_M^{-1} \tilde{I}_1) Q_u \quad (10b)$$

$$\theta_k = (\tilde{I}_{2k+2}^t \Gamma_M^{-1} \tilde{I}_1) Q_u \quad (10c)$$

where

$$\tilde{I}_i = [\mathbf{0} \cdots \mathbf{0} \quad I_M \quad \mathbf{0} \cdots \mathbf{0}].$$

$i$ th entry

Note that premultiplying matrix  $\Gamma_M^{-1}$  by operator  $\tilde{I}_i^t$  and postmultiplying it by  $\tilde{I}_j$  extracts the  $(i, j)$ th block of this matrix. Although fast algorithms such as Levinson method can be used to solve this system of equations efficiently, the procedure becomes computationally very laborious when the parameters of the model are to be estimated for each new block of the noisy image. Thus, it would be very useful to develop a recursive identification process which requires neither matrix inversion operation nor solving the system of (7) at every iteration. Now, in order to develop such a procedure, let us assume that the image field is column-wise sense stationary. Using ergodic property of the image field, a reasonable estimate for  $r_{p,q}$  can be obtained from

$$r_{p,q} = \frac{1}{\|\mathbf{R}\|} \sum_{(l,k) \in \mathbf{R}} y(l, k)y'(l - p, k - q), \quad (p, q) \in T \quad (11)$$

where  $T = \{(p, q): p = 0, 1, q \in [-M, M]\}$ , and  $\|\mathbf{R}\|$  is the size of  $\mathbf{R}$ . Using a similar approach as suggested in [8]-[11]  $r_{p,q}$  can recursively be updated at each iteration based upon the information in the current block of the image. Let us denote the estimate of  $r_{p,q}$  based upon all the blocks up to and including block  $n$  by  $r_{p,q}(n)$ , where the block index  $n$  is obtained by mapping the 2-D array  $(i, j)$  to a 1-D array  $n$  with  $n = iP + j + 1$ . And block  $(i, j)$  or  $n$  of the observed image is defined as

$$Y_{i,j} = [y'(i, jM)y'(i, jM + 1) \cdots y'(i, jM + M - 1)]^t. \quad (12)$$

Then the estimate of  $r_{p,q}$  at the  $n$ th iteration is

$$r_{p,q}(n) = \frac{1}{nM} \sum_{(l,k) \in \mathbf{W}_n} y(l, k)y'(l - p, k - q), \quad (p, q) \in T \quad (13a)$$

where

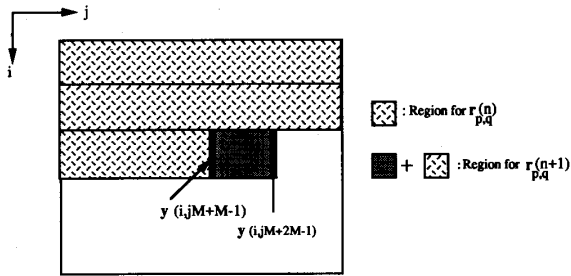
$$\begin{aligned} \mathbf{W}_n &= \{(l, k): \{l = 0, 1, \cdots, i - 1, k \in [0, N - 1]\} \\ &\quad \cup \{l = i, k = 0, 1, \cdots, jM - 1\}\}, \quad i, j \in [0, P - 1]. \end{aligned} \quad (13b)$$

The region in which  $r_{p,q}(n)$  is evaluated is shown in Fig. 2. The estimate based upon  $(n + 1)$  blocks can be obtained using

$$r_{p,q}(n + 1) = \frac{1}{(n + 1)M} \sum_{(l,k) \in \mathbf{W}_{n+1}} y(l, k)y'(l - p, k - q), \quad (p, q) \in T \quad (14a)$$

or

$$\begin{aligned} r_{p,q}(n + 1) &= \frac{1}{(n + 1)M} \left[ \sum_{l=0}^{i-1} \sum_{k=0}^{N-1} y(l, k)y'(l - p, k - q) \right. \\ &\quad \left. + \sum_{k=0}^{jM-1} y(i, k)y'(i - p, k - q) \right. \\ &\quad \left. + \sum_{k=jM}^{jM+M-1} y(i, k)y'(i - p, k - q) \right], \quad (p, q) \in T \\ &= r_{p,q}(n) + \frac{1}{(n + 1)} \\ &\quad \cdot \left[ \frac{1}{M} \sum_{k=jM}^{jM+M-1} y(i, k)y'(i - p, k - q) - r_{p,q}(n) \right], \quad (p, q) \in T. \end{aligned} \quad (14b)$$


 Fig. 2. Regions for computing  $r_{p,q}(n)$  and  $r_{p,q}(n+1)$ .

Let us denote the term in the bracket in (14b) by  $\delta r_{p,q}(n)$ . Note that we can assume that  $\delta r_{p,q}(n) = \delta \rho_{p,q}(n)$ . Then applying this to all the elemental blocks of  $\Gamma_M(n+1)$  gives

$$\Gamma_M(n+1) = \Gamma_M(n) + \delta \Gamma_M(n) \quad (15a)$$

where

$$\delta \Gamma_M(n) = \begin{bmatrix} \delta R_0 & \delta R_1' & \cdots & \delta R_M' \\ \delta R_1 & \delta R_0 & \cdots & \delta R_{M-1}' \\ \cdots & \cdots & \cdots & \cdots \\ \delta R_M & \delta R_{M-1} & \cdots & \delta R_0 \end{bmatrix} \quad (15b)$$

and

$$\delta R_k = 1/(n+1) \begin{bmatrix} \delta \rho_{0,k} & \delta \rho_{1,-k}' \\ \delta \rho_{1,k} & \delta \rho_{0,k} \end{bmatrix} \quad k \in [0, M]. \quad (15c)$$

Similarly,  $\Delta(n+1) = \Delta(n) + \delta \Delta(n)$ . Now, using a similar method developed in [8]–[11] the first-order approximation  $\Gamma_M^{-1}(n+1)$  can be given by

$$\Gamma_M^{-1}(n+1) \approx \Gamma_M^{-1}(n)[I_{2M(M+1)} - \delta \Gamma_M(n)\Gamma_M^{-1}(n)]. \quad (16)$$

However, the first-order approximation may result in a poor estimation. An iterative procedure is developed in [12] which allows generation of higher order approximation of this matrix. Using this scheme the  $k$ th order approximation of  $\Gamma_M^{-1}(n+1)$  is given by

$$\Gamma_M^{(k)-1}(n+1) = [I_{2M(M+1)} - \Gamma_M^{(k-1)-1}(n+1)\delta \Gamma_M(n)]\Gamma_M^{-1}(n), \quad \forall k \geq 1 \quad (17a)$$

with

$$\Gamma_M^{(0)-1}(n+1) = \Gamma_M^{-1}(n). \quad (17b)$$

Having computed  $\Gamma_M^{-1}(n+1)$  recursively, the solution of the vector Yule-Walker equation in (9) for the  $(n+1)$ th block can then be written as

$$\begin{aligned} \Psi_M(n+1) &= \Gamma_M^{-1}(n+1)\Delta(n+1) \\ &= \Psi_M(n) - \Gamma_M^{(k^*-1)-1}(n+1)\delta \Gamma_M(n)\Delta(n) \\ &\quad + \Gamma_M^{-1}(n+1)\delta \Delta(n) \end{aligned} \quad (18a)$$

where  $k^*$  is a value of  $k$  at which an acceptable solution for  $\Gamma_M^{-1}(n+1)$  is achieved. That is

$$\Gamma_M^{(k^*)-1}(n+1) = \Gamma_M^{-1}(n+1). \quad (18b)$$

Note that  $\delta \Delta(n)$  is computed using (10a) and (16).

The recursive equation (18) provides a method for estimating the 2-D vector AR model parameters from the noisy observations. The advantage of using this nongradient approach over that in [7] is that the proof of convergence is easily derived [12]. An iterative procedure for computing the Kalman gain matrix and the error covariance matrices can also be derived which does not require any matrix inversion operation. The interested reader is referred to [12].

### III. ORDER DETERMINATION OF THE IMAGE MODEL

In Section II, the order of the 2-D vector AR model along the horizontal direction, i.e.,  $M$  was assumed to be known. However, if  $M$  is unknown we can fit models of increasing orders and test their goodness of fit by measuring the correlation mismatch between the actual correlation values and those generated by the model. The appropriate order is determined when the correlation mismatch is minimized. An alternative criterion will be to check the optimality of the Kalman filter and test for the whiteness of the innovation sequence [8], [9]. For both criteria the approach would be similar and is given below. Let us denote the parameters of the  $(M+1)$ th order system by  $\Phi_k$ 's and  $\theta_l$ 's,  $k \in [1, M+1]$ ,  $l \in [0, M+1]$ . Then writing (7) for this system gives

$$\begin{bmatrix} R_0 & R_1' & \cdots & R_M' & R_{M+1}' \\ R_1 & R_0 & \cdots & R_{M-1}' & R_M' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{M+1} & R_M & \cdots & R_1 & R_0 \end{bmatrix} \begin{bmatrix} S_0' \\ S_1' \\ \vdots \\ S_{M+1}' \end{bmatrix} = \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (19a)$$

or

$$\begin{bmatrix} \Gamma_M & \Pi' \\ \Pi & R_0 \end{bmatrix} \begin{bmatrix} \Psi_M' \\ S_{M+1}' \end{bmatrix} = \begin{bmatrix} \Delta \\ 0 \end{bmatrix} \quad (19b)$$

where

$$\Pi = [R_{M+1} R_M \cdots R_1]. \quad (19c)$$

$S_k$ 's are defined in a manner similar to (8a) with submatrices  $\Phi_k$ 's and  $\theta_l$ 's. Using (19b) we obtain

$$\Gamma_M \Psi_M' + \Pi' S_{M+1}' = \Delta \quad (20a)$$

$$\Pi \Psi_M' + R_0 S_{M+1}' = 0 \quad (20b)$$

which give

$$\Psi_M' = \Psi_M - \Lambda_M S_{M+1}' \quad (21a)$$

where

$$\Lambda_M = [\lambda_0 \lambda_1 \cdots \lambda_M]' := \Gamma_M^{-1} \Pi'$$

and

$$\begin{aligned} S_{M+1}' &= [-R_0 + R_1 \lambda_M + \cdots + R_{M+1} \lambda_0]^{-1} \\ &\quad \cdot [R_1 S_M + \cdots + R_{M+1} S_0]. \end{aligned} \quad (21b)$$

Equations (21a) and (21b) provide a tool for computing the new  $(M+1)$ th order model parameter matrix  $\Psi_{M+1} = [\Psi_M' S_{M+1}']'$  from the old  $M$ th order parameter matrix  $\Psi_M$ .

### IV. IMPLEMENTATION

The recursive parameter identification method proposed in this paper is implemented on the girl (Lena) image shown in Fig. 3. This image has a resolution of  $512 \times 512$  and a number of gray



Fig. 3. Original girl (Lena) image.



Fig. 5. Restored girl image (SNR = 8.5 dB).



Fig. 4. Degraded girl image (SNR = 2 dB).

levels that is 256. The image is corrupted by adding a white Gaussian noise with zero mean and variance  $\sigma_v^2 = 1444$  to produce a SNR = 2 dB. The resultant degraded image is shown in Fig. 4. The image process is modeled by a 2-D vector AR model of order 4 ( $M = 4$ ) with a ROS extended to two adjacent strips, as shown in Fig. 1. The size of each vector or the width of each strip is also 4. Thus, the processing window consists of 4 blocks of image each of size  $4 \times 4$ . The AR model and the observation equation are then arranged into a block state-space form [1]. The block Kalman filtering equations in [1] together with recursive equations (14)–(18) are used to generate the block filtered estimates of the image at

each iteration. The filtered image is shown in Fig. 5. The SNR for this image is measured to be 8.5 dB which indicates considerable improvement in the quality of the processed image. The result of applying the direct block Kalman filtering [1] to this severely degraded image exhibits blocking effects which can be reduced by updating the parameters of the 2-D vector AR model using the recursive identification scheme introduced in this paper.

#### V. CONCLUSION

Two-dimensional adaptive block Kalman filtering for restoration of noisy images is considered. A recursive parameter identification scheme is developed for this filtering operation. Using this method the parameters of a 2-D vector AR model can be updated on-line based upon the spatial activities within each new block of the observed image. This scheme is derived using a nongradient approach. A recursive algorithm for finding the parameters of higher order AR models is also proposed.

The results developed in this paper are only concerned with the case of identification from a noisy image. The degradation process can, in general, include additive noise and blur. To develop an identification method for this case, the available 1-D methods for multivariable stochastic systems should be extended to the 2-D case. The problem in the 1-D case which is studied in a number of papers such as [13], [14], is a complicated one due to lack of a unique canonical realization for these systems. The problem in the 2-D case is even more difficult since there is no relation between the local controllability and observability and minimal realization for a given 2-D transfer function. This problem should further be investigated.

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### Design of the Unimodular Shaping Filter

Vijay P. Dimri

**Abstract**—A unimodular shaping filter for noisy input is designed. It is stable and also shapes the wavelet into a) the desired output with least squares error and b) the actual output with a unit area. Stability was achieved by using the prewhitening parameter. A numerical example shows that the prewhitening parameter reduces the error energy in the unimodular constrained shaping filter as compared to the unconstrained shaping filter.

#### I. INTRODUCTION

There are many techniques in geophysical literature for inverting the observed data when the source function is known [2], [7]. The principle here is to design the inverse filter which, when convolved (in time domain) with the observed data, produces the source function close to the spike or other close narrow function with different criteria of closeness, e.g., the least squares criterion results in the design of the optimum shaping filter. It produces a minimum error between the desired and the actual output in the least squares sense. To modify it further, one can impose constraints on the design prin-

ciple for better performance of the shaping filter, e.g., the unimodular constraint forces the area of the actual output to be unity [8]. Such a filter is required if one wants to compare the area of the desired output [6], which has been designed in [8]. The other area of application of the unimodular constraint is in the spectroscopic studies [6] and in the optimization problems. In this correspondence, we design a unimodular constrained filter which is stable. The stability has been achieved by adding the prewhitening parameter to the principal diagonal of the matrix to be inverted in the solution of the unimodular constrained shaping filter. A numerical example is presented to compare the performance of the unimodular constrained filter with the unconstrained shaping filter.

#### II. STABILITY OF THE UNIMODULAR FILTER

We intend to design a filter which minimizes the error energy between the actual output and the desired output in the least squares sense, shapes the actual output with unit area, and is stable for noisy input.

The input–output equation for such a system, in the vector notation, can be written

$$Y = (K + N)f \tag{1}$$

where

- $Y$  is col  $(Y_0, Y_1, \dots, Y_{m+n})$ : the actual output,
- $f$  is col  $(f_0, f_1, \dots, f_m)$ : the shaping filter, to be determined,
- $K$  is  $(m + n + 1) \times (m + n + 1)$  matrix obtainable from  $k$  wherein
- $k$  is col  $(k_0, k_1, \dots, k_n)$ : source function,
- $N$  is  $(m + n + 1) \times (m + n + 1)$  matrix obtainable from  $n$  wherein
- $n$  is col  $(n_0, n_1, \dots, n_n)$ : noise with zero mean and variance defined as

$$E\{n_i\} = 0, \quad \text{and } E\{n_i^2\} = \sigma_0^2$$

$E$  and  $\sigma_0^2$  are the expectation and noise power, respectively.

The error energy which is the difference between the actual output and the desired output in the least squares sense can be written

$$\|e\|^2 = \|d - K\hat{f} - N\hat{f}\|^2 \tag{2}$$

where  $d$  is col  $(d_0, d_1, \dots, d_{m+n})$ : the desired output.

The unimodular constraint for a noise free case is given in [8] as

$$U^T K f = 1 \tag{3}$$

where  $U$  is col  $(1, 1, \dots, 1)$  of length  $(m + n + 1)$ : unit column vector, and  $T$  is transpose.

The constraint for (1) becomes

$$U^T (K + N) \hat{f} = 1. \tag{4}$$

Equation (2) is minimized subject to (4) using the well-known method of the Lagrange multiplier

$$\begin{aligned} \|e\|^2 &= \|d - K\hat{f} - N\hat{f}\|^2 + \lambda (U^T (K\hat{f} + N\hat{f}) - 1) \\ &= \|d - K\hat{f}\|^2 + \|N\hat{f}\|^2 - 2(N\hat{f})^T (d - K\hat{f}) \\ &\quad + \lambda U^T K\hat{f} + \lambda U^T N\hat{f} - \lambda \end{aligned} \tag{5}$$

where  $\lambda$  is the Lagrange multiplier.

Under the assumption of random noise, the cross-correlation term with noise in (5) tends to zero. The autocorrelation and cross-cor-

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