Thesis

Massive Neutrinos and the See Saw Mechanism

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Abstract

Massive Neutrinos and the See Saw Mechanism

In the current standard model of particle physics, neutrinos are massless and strictly left-chiral. With neutrino oscillations definitively observed, we know experimentally that neutrinos have non-zero mass. The standard model for leptons, including the Higgs mechanism for mass generation will be explored. Extensions to the standard model to give neutrinos mass and the so called see-saw mechanism will then be presented. Finally, one model of the see-saw mechanism purposed by S. F. King will then be compared to recent data from the T2K experiment.

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CHAPTER 1

Introduction

With the recent observation of neutrino flavor oscillation[1], we now know that neutrinos have mass. Furthermore, we know that the masses of the three Standard Model neutrinos are quite small compared to that of the other leptons. However, in the standard model, neutrinos are massless. One attractive Standard Model modification that can give neutrinos non-zero mass, and also predict a rough scale of the observed neutrino masses is the so called see-saw mechanism. Knowledge of the origin of neutrino mass will help complete our understanding of neutrino physics, which as the reader may already know, can help address some very fundamental questions of the universe.

One can argue that ideas within the physics of neutrino oscillations along with some key properties of neutrinos that will be discussed herein, we can gain information as to why we observe a universe made of matter (as opposed to one of anti-matter, or no universe at all). This is a fundamental question indeed.

The main purpose of this paper will be to familiarize the reader with massive neutrinos, specifically how we can get massive neutrinos with slight modifications to the Standard Model. To do this we will explore how particles, specifically leptons, are given mass within the current standard model. We will then look at how, by modifying the standard model, we can give neutrinos mass. We will then explore the so called see-saw mechanism and identify why this proposed theory is favorable to describe the masses of neutrinos given current relevant experimental data.

CHAPTER 2

The Standard Model for Leptons

The Standard Model of Particle Physics is, as the name might suggest, our standard of knowledge on particles and their interactions. It is an extremely successful model that has incited world wide collaborations of physicists to come together to test its predictions.

Within the theory, we find a description of all known fundamental particles (and their corresponding anti-particles) and how they interact through the exchange of force carrying gauge bosons. The portion of the standard model we will be interested in is that of the leptons: the electron, muon, and tau particles along with their corresponding neutrinos. In the accepted standard model, neutrinos are massless, and we only have what are referred to as left-handed neutrinos. Neutrinos are classified by their interactions. We label them by their corresponding charged lepton. We refer to the different pairs of charged leptons and neutrinos as generations of leptons. We only observe left handed neutrinos, classified for example by measuring the spin of an outgoing muon from a pion's decay.

We will later look at some modifications to the Standard Model. However, should we allow the existence of right handed neutrinos, we will still need to address why we do or do not observe interactions like the one mentioned above.

2.1. The Dirac Equation

For our purposes, all of the necessary parts of the standard model for leptons can be formalized using relativistic quantum mechanics for spin $\frac{1}{2}$ particles (as opposed to formal field theory). This theory of relativistic spin $\frac{1}{2}$ particles was proposed by Dirac. He sought a relativistically covariant equivalent of Schrodinger's equation. To be covariant, we make two simple requirements of the equation. First, it must be linear in $\frac{\partial}{\partial t}$, and also in $\vec{\nabla}$. This gives us the general form:

$$H\psi = (\vec{\alpha} \cdot \vec{P} + \beta m)\psi$$

The second requirement is that the solution also satisfy the relativistic energy-momentum relation:

$$H^2\psi = (\vec{P}^2 + m^2)\psi$$

Using only these two equations, it can be shown that[2] the four coefficients, $\alpha_1, \alpha_2, \alpha_3$, and β satisfy the following:

1) All four of the coefficients mutually anti-commute.

2) $\alpha_i^2 = \beta^2 = 1$.

Any choice of coefficients satisfying the above two conditions will be equally valid. It can be shown that the lowest dimensional objects that can possibly satisfy the above are 4×4 matrices. The choice for these matrices is still not uniquely determined. We refer to different choices of the 4×4 matrices as different representations.

An example of a commonly used representation is the so called Weyl representation:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Here, $\vec{\sigma}$ is a vector of the standard 2×2 Pauli spin matrices from non-relativistic quantum mechanics, and I is the 2×2 identity matrix.

The Dirac equation is commonly written using four Dirac matrices, $\gamma^\mu = (\beta, \beta\vec{\alpha})$:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.1.1}$$

The solutions to the above equation, ψ are called spinors, and will take on different forms, depending on choice of representation for the Dirac matrices. As was the case in non-relativistic quantum mechanics, we can obtain a continuity equation, and a conserved quantity:

$$j^\mu = -e\overline{\psi}\gamma^\mu\psi$$

With the $-e$ inserted to associate the conserved current with a charge current density of a particle of charge e.

We will now explore a particular set of solutions to the Dirac equation, working in the Pauli-Dirac representation.

For a free particle, we look for solutions of the form:

$$u(\vec{p})e^{-i\vec{p}\cdot\vec{x}}$$

Upon substitution into the Dirac Equation, we see that there are four linearly independent solutions of this form. Two with positive energy and two with negative energy.

We can express the solution in a compact form:

For the positive energy solutions:

$$u^s = \begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s \end{pmatrix}$$

And for the negative energy solutions:

$$u^{s+2} = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^s \\ \chi^s \end{pmatrix}$$

We interpret the negative energy solutions as anti-particle solutions.

Working in the the Pauli-Dirac representation, we can notice that if we operate on a positive energy state with the operator:

$$C\gamma_0 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We return a negative energy state. The matrix C is called the charge conjugation matrix. We can find a representation of it in any choice of the Dirac matrices. For a spinor ψ, ψ_C will denote $C\gamma_0\psi^*$, the charge conjugate of ψ. The operation $\psi \rightarrow \psi_C$ is called charge conjugation, and can be interpreted as taking a particle to its corresponding anti-particle state.

Another common notation that we will also use in some places is an over-bar:

$$\overline{\psi} \equiv \psi^T C$$

Here, the superscript "T" denotes the transpose of a matrix.

It turns out, there is another operator that commutes with both H and $\vec{P}$, giving us an additional compatible observable to label states with. The operator:

$$\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix}$$

is called the helicity operator, and can be seen explicitly to commute with both H and $\vec{P}$. Helicity can be thought of as the spin angular momentum component along the direction of travel of the free particle. We say that a particle whose spin is parallel to its momentum direction has helicity of $+\frac{1}{2}$ and one whose spin is anti-parallel to it momentum has helicity of $-\frac{1}{2}$.

Now, let us consider the operators:

$$\frac{1}{2}(1 \pm \gamma_5) \equiv P_{R(L)}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. It is straight forward to show that $(\gamma_5)^2 = 1$, so that for a spinor ψ:

$$P_R P_L \psi = P_L P_R \psi = 0$$

and we see that these P operators act like a projection operators onto orthogonal directions. The "directions" these operators are projecting onto are so called chiral directions. We say that $P_R\psi = \psi_R$ is the right chiral component of ψ and like wise $P_L\psi = \psi_L$ is the left chiral projection of ψ. For the remainder of this paper, a "right handed field" will refer to the right chiral component of the field, and likewise for left chiral fields. For any spinor, we can write:

$$\psi = \psi_R + \psi_L$$

In the Standard Model, as it stands, we only have left chiral neutrinos. We will see that a simple standard model modification to give neutrinos mass will be to add some number of right chiral neutrinos. An aside that is worth mentioning is as follows.

It can be (fairly easily) shown that, for any field ψ, $(\psi_L^c)_L = 0$. That is: the charge conjugate of a left-chiral field is right chiral. The same is true for the conjugate of a right chiral field.

One can ask the question if there exist purely real solutions to the Dirac equation. Upon inspection of equation (2.1.1), we notice that if there were a choice of purely imaginary Dirac matrices, the solution could automatically be taken to be real. There does exist a choice of purely imaginary Dirac matrices. This was first done by Majorana, and accordingly the choice of the Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}$$

is referred to as the Majorana representation. If we remember the definition of ψ_C, we might anticipate that these real solutions may have the property $\psi_C = \psi$. This intuition is correct. In the Majorana representation, $\psi = \psi^*$ is enough to guaranteethat $\psi_C = C\gamma_0\psi^* = \psi$ (in the Majorana representation). One can show that[2] in any representation, these real solutions can be mapped to solutions with the general property (independent of representation):

$$\psi_C = \psi \tag{2.1.2}$$

Equation (2.1.2) is referred to as the Majorana condition. In any representation, a particle described by a spinor, ψ, satisfying the Majorana condition can be interpreted as a Majorana fermion: a particle that is its own antiparticle. Since the charge conjugation operator changes the sign of the charge of the particle it operates on, only neutral particles have the possibility of being Majorana fermions.

2.2. Lagrangian Formalism

We will see that it is easier to discuss symmetries in the framework of the standard model using a Lagrangian formalism of the dynamics. The reader is likely familiar with discrete Lagrangian systems. Here, we will need a continuous formalization to describe the particle fields of the previous sections. This is a simple generalization found in many graduate level texts on classical mechanics. Instead of a Lagrangian, we use a Lagrangian density, and our discrete set of generalized coordinates becomes a continuous field. If we call that field ϕ, then the familiar Euler-Lagrange equation becomes:

$$\frac{\partial}{\partial x_\mu}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \phi}{\partial x_\mu})} - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \tag{2.2.1}$$

Now, in order to construct a Lagrangian describing a physically realizable system for our purposes, we must make two requirements:

1) The Lagrangian must be Lorentz invariant.

2) The Lagrangian must be quadratic in the fields it describes.

It should be clear why we need the first requirement. If the Lagrangian is Lorentz invariant, any resulting equations of motion will be Lorentz covariant. The second requirement assures that any resulting equations of motion will be linear with plane wave solutions. We impose this condition with a goal in mind: describing free particle fields as was done in the previous sections.

Now, suppose that ψ is a free field as we have dealt with already, and consider the Lagrangian:

$$\mathscr{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} + m)\psi$$

if we substitute this Lagrangian into (2.2.1), it is a straight forward exercise[2] to see that we recover precisely the Dirac equation.

The entirety of the standard model can be represented by a single Lagrangian. We can identify certain terms in a Lagrangian with interactions between different particle fields, and others with so called mass terms. In the interest of brevity, interaction terms will be simply asserted, with a prescription as to how we interpret them. The mass terms will be introduced here, and in the next chapter, we will see how the Higgs mechanism produces them.

An example of an interaction term in a Lagrangian we would be interested in while studying neutrinos is:

$$J_{\mu} \equiv \overline{\nu}\gamma_{\mu}\frac{1}{2}(1-\gamma^{5})e$$

Here, ν and e are single particle spinors corresponding to a neutrino and electron respectively.

We have also the current:

$$J_{\mu}^{\dagger} \equiv \overline{e}\gamma_{\mu}\frac{1}{2}(1-\gamma^{5})\nu$$

The Feynman diagrams of implied interactions corresponding to these currents are given in Figures 2.1 and 2.2.

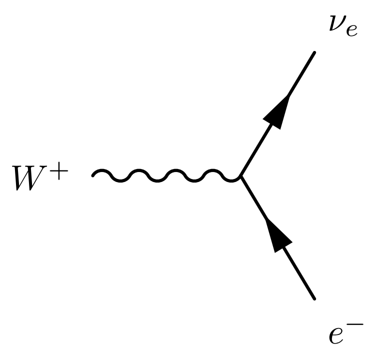

FIGURE 1. Allowed interaction vertex implied by J_μ

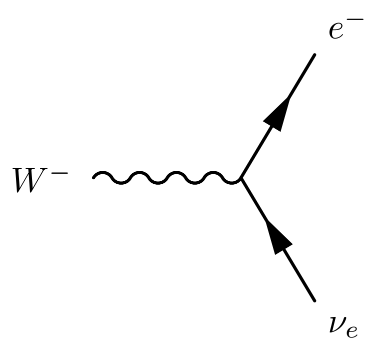

FIGURE 2. Allowed interaction vertex implied by $J_\mu^\dagger$

In general, there are two types of possible mass terms in a given Lagrangian, Dirac and Majorana. We will see that Dirac mass terms arise naturally for particles like the electron using the Higgs Mechanism. They can be classified by the appearance of both the left and right chiral fields in an individual term. For example:

$$\alpha(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)$$

is a Dirac mass term describing the free particle ψ with mass α.

In contrast, Majorana mass terms have only one chiral field:

$$\beta(\overline{\psi_R^c}\psi_R + \overline{\psi_R}\psi_R^c)$$

Here, the superscript "c" indicate charge-conjugation. This notation will be adopted as needed, for example, when we already have a subscript.

We notice that if we introduce:

$$\phi = \psi_R + \psi_R^c$$

we get:

$$\beta(\overline{\psi_R^c}\psi_R + \overline{\psi_R}\psi_R^c) = \beta\overline{\phi}\phi$$

and we conclude that $\beta(\overline{\psi_R^c}\psi_R + \overline{\psi_R}\psi_R^c)$ is a Majorana mass term describing free particle ϕ with mass β.

We can see why this is called a Majorana mass term as we clearly have:

$$\phi = \phi^c$$

The particle described by ϕ is a Majorana fermion. There is something else worth noting here. For and arbitrary field, ψ, we were able to construct a Majorana field by taking a sum of the original field and its charge conjugate. If we were to try this out on the left chiral neutrino field that appeared in the interaction currents discussed above, we would see that the interaction currents are completely unchanged by the substitution: $\nu \to \nu_L + \nu_L^c$. This comes from the fact that ν_L^c is a right chiral field and we see a P_L operator in the currents.

So, only the ν_L field takes place in the interactions, and we conclude that for a Majorana neutinrio described by $\nu_L + \nu_L^c$, the physics of its interactions are unchanged from that of the ordinary Dirac neutrino.

2.3. Symmetry Groups

The standard model can be discussed in the context of symmetry groups. Such a formalism gives natural rise to conserved quantities and allows a transparent introduction of the Higgs mechanism to generate mass.

If we recall the two interaction currents presented in section 2.2, and introduce a doublet of left chiral fields:

$$\chi_L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}$$

along with two "step" operators:

$$\tau_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$$

we can express the interaction terms as:

$$J_\mu = \overline{\chi_L}\gamma_\mu \tau_+ \chi_L$$

and likewise:

$$J_\mu^\dagger = \overline{\chi_L}\gamma_\mu \tau_- \chi_L$$

If we had a current of the form:

$$J_\mu^3 = \overline{\chi_L}\gamma_\mu \sigma_3 \chi_L$$

we would notice that for the currents above, the corresponding "charges", $T^i = \int J_0^i d^3x$ form an SU(2) algebra:

$$[T^i, T^j] = \epsilon_{ijk} T^k$$

As it turns out, the introduced current, J_μ^3 is not exactly what we have, but is pretty close. In order to formalize the interactions of the leptons, we must include the electromagnetic current as well[2], and by enlarging the symmetry group to SU(2) $\times$ U(1), we get a valid third current to produce the necessary algebra.

Casting the theory in the context of symmetry groups allows us to immediately ask if some of our physical quantities are invariant under the natural symmetry present in the group. For example, it can be seen that if we transform the lepton doublets:

$$\chi_L \rightarrow e^{\frac{i}{2}\sigma_j \Lambda_j} \chi_L$$

we find that all of the currents we had before are unchanged. We call this invariance of the physics under the above transformation global SU(2) symmetry. It can be shown that the presence of this symmetry gives direct rise to a conserved quantity, Lepton number.

For our purposes here, we really only need to note a few main points. First, we represent left handed lepton fields as SU(2) doublets like χ_L. Right handed fermion fields are represented by singlets. Second, the presence of the overlying global symmetry gives rise to a conserved quantity.

As it turns out, in order to construct the theory using the symmetry group SU(2) $\times$ U(1), we cannot have right handed neutrino singlets (if we wish to preserve the global SU(2) symmetry).

2.4. Summary

In summary, the standard model for the interaction of leptons starts with solutions to the Dirac equation. We can then generalize the theory to that of Lagrangians where we have interaction terms to describe all the allowed interactions of the particles, and mass terms to identify the masses of the particles.

We can cast the theory in a convenient $SU(2) \times U(1)$ framework with the following important properties:

1) The left handed fermion fields are represented as doublets.
2) The right handed fermion fields are represented as singlets.
3) There are no right handed neutrino fields.

It is worth while mentioning here a possible source of confusion when reading other sources relevant to neutrino mass: the three different "Majorana's". We have seen the term describe a representation of the Dirac matrices, a kind of mass term in Lagrangians, and a type of fermion that is its own anti-particle.

We saw that the Majorana representation is a particular choice of purely imaginary Dirac matrices. We are always free to work in whatever representation we like, and working in the Majorana representation is in no way related to the idea of Majorana fermions, or mass terms.

Majorana mass terms were ones that coupled only one chiral fermion field to itself, as opposed to a Dirac mass terms that coupled a left chiral fermion field to a right one. We can have both Majorana and Dirac mass terms in any representation of the Dirac matrices, and independent of whether the massive particle is a Dirac or a Majorana fermion.

Majorana particles are particles that are their own anti-particles. They are classified mathematically by equation (2.1.2). We saw that we can have the Majorana condition in any representation of the Dirac matrices we like, and do not need to reference mass terms.

These three uses of “Majorana” are a common source of confusion while reading into the subject of massive neutrinos. The three uses are very much so de-coupled, and it is very important to note that the presence of a Majorana mass term does not imply, a priori, that the massive particle is a Majorana fermion.

In the next chapter, we will work in the $SU(2)\times U(1)$ formalism, and consider Lagrangians satisfying the two requirements presented at the beginning of of section 2.2, but also insist that the Lagrangians be invariant under so called local gauge transformations of the $SU(2)\times U(1)$ group. We will see that by insisting on this symmetry initially, but then ”breaking” it, we can generate the masses of the particles described by a particular Lagrangian.

CHAPTER 3

Higgs Mechanism

In this chapter, the Higgs mechanism within the standard model is presented[3]. While we will see that we cannot use the Higgs mechanism within the current standard model to generate neutrino masses, understanding how the usual Higgs mechanism works will allow us to more clearly explore how generalizations out side the standard model can give us massive neutrinos.

To understand how the Higgs mechanism generates mass, we will need two main ideas. First, we will explore SU(2) local gauge invariance. We will then explore the spontaneous "hiding," also referred to as "breaking" of the symmetry induced by the requirement of local gauge invariance. The second point is what is commonly called the Higgs mechanism.

3.1. SU(2) Local Gauge Invariance

We will start with a two-component complex field , transforming as a doublet of the SU(2) group.

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$$

where ψ_a and ψ_b are complex fields.

Suppose that the masses of each component field is m. Then, the free Lagrangian field density will be given by:

$$\mathscr{L}_{free} = \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi \tag{3.1.1}$$

Equation (3.1.1) can be seen to be invariant under the following transformations:

$$\psi \to U\psi = \psi' \tag{3.1.2}$$

and like wise for $\bar{\psi}$:

$$\bar{\psi} \to \bar{\psi}U^\dagger = \bar{\psi}'$$

Here, U is an arbitrary, constant unitary matrix, which we can parametrize by $U = e^{\frac{i}{2}\sigma_j \Lambda_j}$, with σ_j the standard Pauli Matrices, Λ_j arbitrary complex numbers, and summation implied over j.

We call the invariance of equation (3.1.1) under the above transformations Global SU(2) Gauge Invariance.

We now wish to make equation (3.1.1) invariant under a more general set of transformations. Namely, we will let the coefficients Λ_j vary with coordinate. That is: $\Lambda_j = \Lambda_j(x)$.

We immediately see a problem, as the free field Lagrangian has derivatives ∂_μ in it. If we simply take a derivative of the transformation (3.1.2), we get:

$$\partial_\mu \psi = U^\dagger(x)U(x)\partial_\mu U^\dagger(x)\psi' = U^\dagger(x)(\partial_\mu + U(x)\partial_\mu U^\dagger(x))\psi'$$

If we consider and infinitesimal transformation $U \to 1 + \frac{i}{2}\sigma_j\Lambda_j$, the above becomes:

$$\partial_\mu \psi' = (1 + \frac{i}{2}\sigma_j\Lambda_j)\partial_\mu\psi + \frac{i}{2}\sigma_j(\partial_\mu\Lambda_j)\psi$$

We see that ψ and $\partial_\mu\psi$ do not transform in such a way to make (3.1.1) invariant under the more general transformations. There is a way to develop a theory that is invariant under the local transformations. We can get a hint at what modifications might be necessary by looking at the operator:

$$D_\mu = \partial_\mu + g\frac{i}{2}\sigma_j A^j_\mu(x) \tag{3.1.3}$$

Here, g is some number, and the $A^\mu_j(x)$ are three vector fields. If we look at how $D_\mu\psi$ transforms:

$$D_\mu\psi = U^\dagger(x)U(x)(\partial_\mu + g\frac{i}{2}\sigma_j A^j_\mu(x))U^\dagger(x)\psi'$$

considering again infinitesimal transformations:

$$U(x)(\partial_\mu + g\frac{i}{2}\sigma_j A^j_\mu(x))U^\dagger(x) = \partial_\mu - \frac{i}{2}\sigma_j\partial_\mu\Lambda_j + g\frac{i}{2}U(x)\sigma_j A^j_\mu U^\dagger(x)$$

exploiting the fact that for the Pauli matrices, $[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j] = i\epsilon_{ijk}\sigma_k$, we get:

$$\frac{1}{2}U(x)\sigma_j A^j_\mu U^\dagger(x) = \frac{1}{2}\sigma_j A^j_\mu + i[\frac{1}{2}\sigma_j\Lambda_j, \frac{1}{2}\sigma_j A^j_\mu]$$

and

$$i[\frac{1}{2}\sigma_j\Lambda_j, \frac{1}{2}\sigma_j A^j_\mu] = -\frac{1}{2}\sigma_j(\vec{\Lambda} \times \vec{A_\mu})_j$$

This gives us:

$$U(x)(\partial_\mu + g\frac{i}{2}\sigma_j A^j_\mu(x))U^\dagger(x) = \partial_\mu + \frac{ig}{2}\sigma_j(A^j_\mu - \frac{1}{g}\partial_\mu\Lambda_j - (\vec{\Lambda} \times \vec{A_\mu})_j)$$

so if the vector field $\vec{A^\mu}$ transforms as:

$$\vec{A_\mu} \to \vec{A_\mu} - \frac{1}{g}\partial_\mu\vec{\Lambda} - \vec{\Lambda} \times \vec{A_\mu} = \vec{A}'$$

then we see that:

$$UD_\mu U^\dagger = D'_\mu$$

and

$$D_\mu\psi = U^\dagger D'_\mu\psi'$$

and we conclude that $D_\mu\psi$ and ψ transform in a way such that the Lagrangian:

$$\mathscr{L}_{cov} = \bar{\psi}(i\gamma^{\mu}D_{\mu} + m)\psi \tag{3.1.4}$$

will be invariant under the local gauge transformations. We can recognize (3.1.4) as the sum of the original free Lagrangian density, (3.1.1) and a so called interaction term:

$$\mathscr{L}_{int} = \frac{g}{2}\bar{\psi}\gamma_{\mu}\sigma_{j}\psi A^{\mu}_{j} \tag{3.1.5}$$

with a sum on j and μ implied. This interaction term physically describes the interaction of the original field, ψ with the $\vec{A_{\mu}}$ fields we introduced for local SU(2) gauge invariance. When we use the standard SU(2) theory for weak interactions, the particles corresponding to the $\vec{A_{\mu}}$ fields are the familiar gauge bosons, and the ψ field corresponds to a lepton SU(2) doublet.

In summary, we have developed a locally gauge invariant SU(2) theory that can describe weak interaction. In the next sections, we will see that upon consideration of a complex scalar field (the Higgs field), and a "breaking" of the symmetry developed, we can generate masses for the gauge bosons. It is a straight forward generalization to generate the masses of the leptons.

3.2. Spontaneous Symmetry Breaking

In this section, we will see in a simple example how we generate masses by breaking a symmetry. We will consider a simple, single component scalar field, ϕ that obeys the Lagrangian density:

$$\mathscr{L} = \frac{1}{2}(\partial_\alpha \phi)^2 - (\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4)$$

we see that the Lagrangian is invariant under the transformation $\phi \to -\phi$. We will take $\lambda > 0$. Now, if $\mu^2 > 0$ this is simply a Lagrangian density for a scalar field with mass μ. Let us examine the case when $\mu^2 < 0$. The potential now does not have a minimum at $\phi = 0$, but rather two minima at $\pm\sqrt{-\frac{\mu^2}{\lambda}}$. So, should we wish to proceed with perturbative calculations, we cannot use our original field, ϕ, but instead should consider fluctuations about the minima: $\phi(x) = \sqrt{-\frac{\mu^2}{\lambda}} + \eta(x)$. Here it is important to notice that we had a choice in where to preform our perturbative calculation. We will see that this choice breaks the reflection symmetry in our scalar field that we started with. Substituting into the Lagrangian above gives:

$$\mathscr{L} = \frac{1}{2}(\partial_\alpha \eta)^2 + \mu^2\eta^2 - \lambda\sqrt{-\frac{\mu^2}{\lambda}}\eta^3 - \frac{1}{4}\lambda\eta^4$$

As mentioned, the symmetry is no longer present (directly). This is clearly because we essentially translated our field, "breaking" the previously had symmetry. In reality, we just "hid" the symmetry from our selves, as the two Lagrangians considered above need to be physically equivalent. So, what is going on here? One interpretation is that in practice, we do not solve such problems exactly, but rather by perturbative methods so that any calculations done about the original $\phi = 0$ would not have converged. So, if we want to do real calculations that will produce physically realizable results, we must choose the second Lagrangian. If we could have solved both Lagrangians exactly, we would have needed to obtain the same physical results.

Now, looking at our new Lagrangian, we can recognize how the first two terms relate to the original Lagrangian and recognize the η field now has a mass of $m = \sqrt{-2\mu^2}$.

So, we say that by "breaking" the symmetry of the original Lagrangian, we have "generated" mass for the η field.

3.3. Breaking the Local SU(2) Symmetry

We will now take the idea developed in the previous section and break the symmetry of a SU(2) local gauge invariant Lagrangian.

Let us now consider an SU(2) doublet of complex scalar fields:

$$\phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}$$

with

$$\phi_a = \phi_1 + i\phi_2$$

,

$$\phi_b = \phi_3 + i\phi_4$$

for four real scalar fields, $\phi_1, \phi_2, \phi_3, \phi_4$.

Consider the Lagrangian density:

$$\mathscr{L} = \partial_\alpha \phi^\dagger \partial^\alpha \phi - V(\phi)$$

with

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$$

for $\mu^2 < 0$ and $\lambda > 0$.

It can be seen that this Lagrangian density is invariant under global gauge transformations. If we repeat the steps in the previous section as to construct a Lagrangian that is invariant under local gauge transformations, we get that the invariant Lagrangian is given by:

$$\mathscr{L} = (\partial_\alpha \phi + g\frac{i}{2}\vec{\sigma} \cdot \vec{A_\alpha})^\dagger (\partial^\alpha \phi + g\frac{i}{2}\vec{\sigma} \cdot \vec{A^\alpha}) - V(\phi) + K(\vec{A})$$

with the A_j fields the gauge fields as before, and $K(\vec{A})$ a collection of terms corresponding to the kinetic energy of the gauge fields, which we will not be interested in here. It turns out these terms will give us the generated masses of the gauge bosons.

Now that we have developed a symmetric Lagrangian, let us generalize the idea presented in the previous section to break the symmetry and generate the masses of the gauge bosons.

The potential $V(\phi)$ is minimized when:

$$\frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{\lambda}$$

As was the case in the previous section, we now have a choice for where to preform our perturbative calculations. Let us take:

$$\phi_1 = \phi_2 = \phi_4 = 0$$

and

$$\phi_3^2 = -\frac{\mu^2}{\lambda} = \gamma^2$$

So, the minimum ϕ is given by:

$$\phi = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

It can be seen that[2] a perfectly good choice of ϕ to put back into our symmetric Lagrangian is:

$$\phi = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ \gamma + h(x) \end{pmatrix}$$

As was mentioned before, the term that will give us the mass terms of the Lagrangian corresponding to the gauge bosons is $g\frac{i}{2}\vec{\sigma} \cdot \vec{A_\alpha}$.

Upon evaluation of these terms with our perturbative ϕ gives mass terms:

$$\frac{g^2\gamma^2}{8}((A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^3)^2)$$

corresponding to the three gauge bosons of the standard weak theory.

3.4. Masses of the Fermions

To generate the masses of the Fermions, we can add a term into the overall Lagrangian[2]:

$$-\begin{pmatrix} \overline{\nu_e} & \overline{e} \end{pmatrix}_L \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} e_R \;-\; \overline{e_R} \begin{pmatrix} \phi_a & \phi_b \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{3.4.1}$$

This term can be seen to satisfy the conditions of section 2.2, and it can also be seen to be $SU(2) \times U(1)$ gauge invariant.

Upon breaking the gauge symmetry, and using the same notation as before, this term becomes:

$$-\frac{\gamma}{\sqrt{2}}(\overline{e_L}e_R + \overline{e_R}e_L)$$

The mass term that appears in the above is a Dirac mass terms mentioned in the previous chapter. It is so named because massive Dirac particles naturally get mass terms of this form from the standard model Higgs. We should also note that the Lagrangian term above is globally gauge invariant, so these terms will conserve Lepton number.

3.5. Summary

While the actual working of the Higgs mechanism for leptons is pretty intricate mathematically, we can summarize what we need to proceed as follows:

To generate the mass of leptons, we add a term to the total Lagrangian similar to (3.4.1) describing the interaction of the lepton with the Higgs field. We then make that term locally gauge invariant under the relevant gauge group, and break the symmetry. The resulting terms in the Lagrangian are then the mass terms we will be interested in. Which terms we write down initially to generate the mass are constrained by our assumptions of which particles exist, and how they interact. In the next chapter, we will look at how these different assumptions give different mass terms for neutrinos after breaking the symmetry.

CHAPTER 4

Neutrino Mass

4.1. Neutrino Mass Terms

In the Standard Model, massive spin $\frac{1}{2}$ particles all have corresponding Dirac mass terms in the total Lagrangian. These terms always couple the left handed and right handed fields of the massive particle. It is assumed in the Standard Model that only left handed neutrinos exist, and therefore there can be no Dirac mass terms for neutrinos. If we were, however to introduce three right handed, singlet fields similar to those of the charged leptons, we can still generate mass.

Following the same prescription as was presented for the charged leptons in Section 3.4, we introduce a term into to total Lagrangian[4]:

$$-\begin{pmatrix} \overline{\nu_e} & \overline{e} \end{pmatrix}_L \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} \nu_R \; - \; \overline{\nu_R} \begin{pmatrix} \phi_a & \phi_b \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$$

and upon breaking the symmetry we get:

$$-\frac{\gamma}{\sqrt{2}}(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L)$$

We can see the appearance of a Dirac Mass term just as we saw for the charged leptons. However, upon inserting the right handed singlets, we end up unintentionally breaking the global SU(2) symmetry of select other terms in the total Lagrangian. We therefore lose conservation of Lepton number. We instead get a more relaxed condition that total lepton number is conserved (sum over flavors).

This is the simplest standard model extension to generate neutrino mass. Lepton number conservation is violated, but most theories beyond the standard model agree that Lepton number is violated at energy scales higher than where the current standard model has been tested.

We now see how a Dirac mass term can arise for neutrinos, but the reader may be wondering why we couldn't have simply had a Majorana mass term for the left-chiral neutrino fields. After all, the Majorana mass terms need only one chiral field, so we may have been able to generate neutrino mass without adding these right handed singlets. This, unfortunately cannot be done. The reason for this is that the current Higgs doublet is incapable of coupling the ν_L field to the $\overline{\nu_L}$ field. So, we see that the addition of these right handed singlet fields really is the easiest modification we could hope to make.

Furthermore, if we insist that these right handed singlet fields do not take place in the weak interaction (we are actually forced to assume this unless we wish to completely re-formalize how the weak interaction works), there is nothing preventing the right handed singlets from having a Majorana mass term. So, with only the addition of three right handed neutrino singlets we are able to generate both a standard Dirac mass term and a right chiral Majorana mass term for neutrinos. This procedure is the most widely accepted route to a see-saw mechanism[4].

It turns out, we can get around the problem of the current Higgs doublet forbidding a left handed Majorana mass term by modifying the Higgs sector of the standard model. If instead of a doublet of scalar fields like we say in the previous chapter, we had introduced a triplet of scalar fields, such left handed Majorana terms for neutrinos would be allowed. This is not generally considered an attractive modification for the see-saw mechanism as it requires a re-formulation of the standard Higgs that works so well for all the gauge bosons and other leptons. We can summarize how we can generate neutrino mass as follows:

We must modify the current standard model. We can either modify the lepton part of the standard model, the Higgs part, or both. A very general treatment of possible modifications and how they can lead to a see-saw mechanism can be found in[5]. The easiest modification is the addition of three right handed neutrinos that do not take part in the weak interaction. Using this modification, we can generate Dirac mass terms for neutrinos and a right chiral Majorana mass term. In the next section, we will see how we can use the mass terms obtained from the standard model modifications to get light left handed neutrinos (the trade off for which is heavy right handed neutrinos). This is the see-saw mechanism.

We saw in the last section the different possible mass terms we can generate for neutrinos by altering the assumptions of the standard model. Let us now consider one generation of neutrino with all of the previously discussed mass terms present. The mass term in the overall Lagrangian will then be:

$$\mathscr{L}_{mass} = -\frac{1}{2}m_L\overline{\nu_L}\nu_L^c - m_D\overline{\nu_L}\nu_R - \frac{1}{2}m_R\overline{\nu_R^c}\nu_R + h.c.$$

The "h.c." stands for the hermitian conjugate of all the terms before it. We can express this in terms of a mass matrix by introducing the notation:

$$\nu = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}$$

and the mass terms become:

$$\mathscr{L}_{mass} = -\frac{1}{2}\overline{\nu}M\nu + h.c.$$

with:

$$M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

We can diagonalize this matrix with an orthogonal transformation:

$$M = O\tilde{m}O^T$$

with the diagonal elements of $\tilde{m}$ given by:

$$\tilde{m}_{22(11)} = \frac{1}{2}(m_R + m_L) \pm \sqrt{(m_R - m_L)^2 + 4m_D^2} \tag{4.2.1}$$

these have the possibility of being negative, if we introduce

$$m'_{ij}\eta_i = m_{ij}$$

with $\eta_i = \pm 1$ so that the elements of m are positive. We can express the mass terms as:

$$\mathscr{L}_{mass} = -\frac{1}{2}\overline{\nu_M} m \nu_M + h.c.$$

with:

$$\nu_M = (O\sqrt{\eta})^\dagger \nu + ((O\sqrt{\eta})^\dagger \nu)^c$$

and we immediately see that:

$$\nu_M = \nu_M^c$$

and conclude that we have two Majorana neutrino fields with masses m_{11} and m_{22}.

Perhaps the most widely accepted version of the see-saw mechanism[4], usually referred to as a Type I see-saw mechanism, works under the following assumptions:

1) There is no left chiral Majorana mass term.

2) We generate the Dirac mass terms through the standard Higgs mechanism.

3) $m_R \gg m_D$

These assumptions are well justified. We make the first so that the standard weak interaction theory remains unchanged. The second ensures us that the Dirac neutrino mass terms are on the order of that of the charged leptons, and that we do not alter the Higgs sector if the standard model. The third guarantees us that although the right chiral Majorana mass term was seen previously to violate lepton number, this violation will occur at very high energy scales, so that lepton number is still effectively conserved at energies where the standard model has been seen to be valid.

With these assumptions, and equation (4.2.1) we see that the masses of the two Majorana neutrinos are given by:

$$m_{11} \cong \frac{m_D^2}{m_R}$$

$$m_{22} \cong m_R$$

By following the procedure above, we end up with a heavy neutrino (on the order of the GUT scale) and a light neutrino. We can see from the above equations that by "tuning" m_R, we can make one neutrino lighter and the other heavier. This is where we get the name: see-saw mechanism.

We can summarize what we have seen here. If we adopt the above see-saw mechanism in our description of neutrinos, we see that:

1) Neutrinos are Majorana particles

2) We can get a neutrino mass that is much smaller than that of the charged leptons

3) For each light neutrino, we have a heavy, see-saw counterpart neutrino.

4.3. Three Generation See-Saw

In this section, we will look at a particular three neutrino see-saw mechanism outlined in Neutrino Mass by S. F. King [6]. It is a straight forward, however tedious extension of the previous to consider three generations of neutrinos. The mass terms m_L, m_R, and m_D become 3×3 matrices, and then the process proceeds in exactly the same fashion. Once three generations are included, it can be shown that[6] the orthogonal transformation, O from before can be directly related via a unitary transformation to the PMNS matrix of neutrino oscillation physics.

King explores a see-saw mechanism where one of the right chiral neutrinos is sufficiently light so that it dominates in the see-saw mechanism. He assumes that the m_R is diagonal, and dominated by a single term we will call Y. He leaves the Dirac mass matrix general. This seems good, as a single neutrino dominated see-saw mechanism will only minimally modify the current standard model, and the general Dirac mass matrix leaves the theory somewhat general. The mass matrices are:

$$m_R = \begin{pmatrix} Y & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Z \end{pmatrix}$$

$$m_D = \begin{pmatrix} d & a & a' \\ e & b & b' \\ f & c & c' \end{pmatrix}$$

In King's paper, he then imposes what he refers to as sequential dominance[6]. King claims that sequential dominance here gives only the possibility of a normal mass hierarchy,

$m_3 > m_2 > m_1$. Combining this assumption along with some algebra, King claims to achieve the following[6]:

$$m_3 \cong \frac{e^2 + f^2}{Y}, \quad m_2 \cong \frac{a^2}{X \sin^2(\theta_{12})}$$

$$\tan(\theta_{23}) \cong \frac{e}{f}, \quad \tan(\theta_{12}) \cong \frac{a}{b\cos(\theta_{23}) - c\sin(\theta_{23})}, \quad \tan(\theta_{13}) \cong \frac{a(b\sin(\theta_{23}) + c\cos(\theta_{23}))}{m_3 X}$$

We can use:

$$m_2 \cong \frac{a^2}{X \sin^2(\theta_{12})}, \quad \tan(\theta_{12}) \cong \frac{a}{b\cos(\theta_{23}) - c\sin(\theta_{23})}, \quad \tan(\theta_{13}) \cong \frac{a(b\sin(\theta_{23}) + c\cos(\theta_{23}))}{m_3 X}$$

to eliminate the parameters a and b and obtain:

$$m_3 \cong \sqrt{\frac{m_2}{X}} \frac{\sin\theta_{12}}{\tan\theta_{13}} \left(\tan\theta_{23} \left(\sqrt{m_2 X}\cos\theta_{12} + c\sin\theta 23\right) + c\cos\theta_{23}\right)$$

neglecting the term proportional to $\frac{c}{\sqrt{X}}$ and combining with $m_2 = \sqrt{m_3^2 - \Delta m_{32}^2}$ (both valid if King is correct in assumption of sequential dominance and its implication of normal hierarchy) gives:

$$m_3 \cong \sqrt{m_{23}^2}((\frac{\cos(\theta_{13})}{\sin(\theta_{12})\tan(\theta_{23})\cos(\theta_{12})})^2 - 1)^{-\frac{1}{2}}$$

allowing us to get an explicit prediction of m_3 in terms of the experimentally measured mixing angles of neutrino oscillation physics. Plugging in the best known values of these parameters as measured by the T2K experiment gives:

$$m_3 \cong 0.026\ eV$$

But, this is not so good. While working in this model, King claims that only a normal hierarchy can exist, which we have also assumed here. If we set $m_1 = 0$, we can use the measured mass-squared splittings to get a minimum allowed value of m_3:

$$m_3 > 0.05\ eV$$

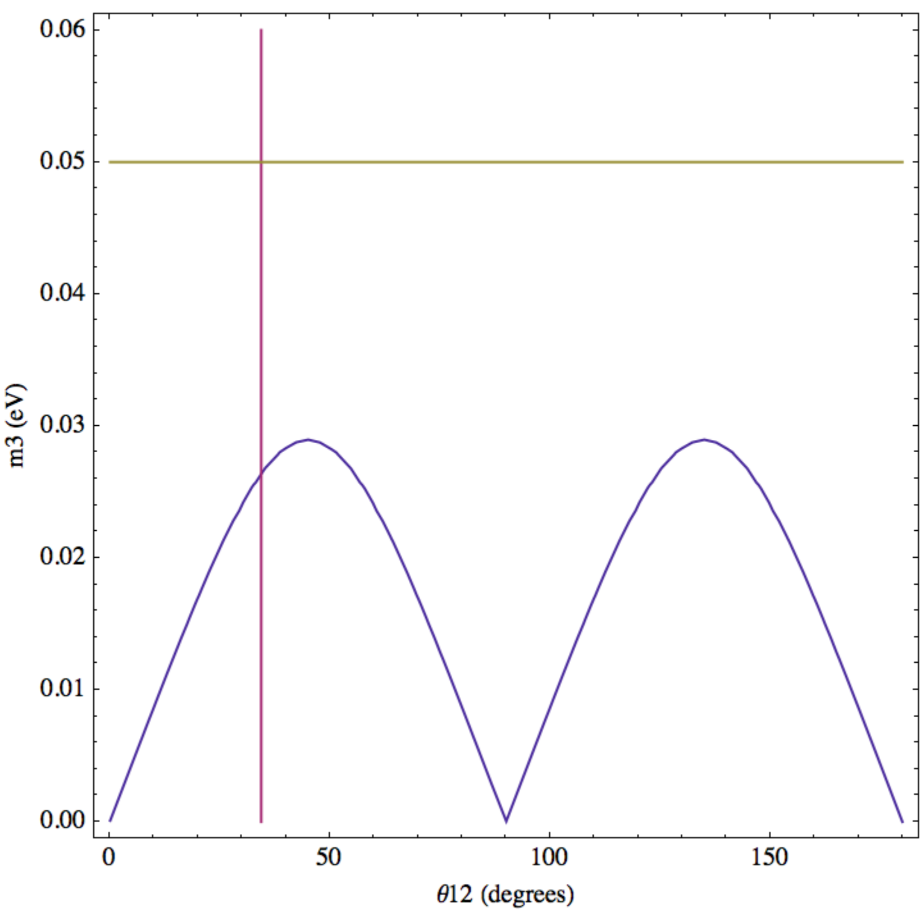

FIGURE 1. $\theta_{12} - m_3$ contour. The horizontal line corresponds to the minimum m_3 value. The vertical line is the best fit parameter for θ_{12} as measured by the T2K experiment

contradicting some assumption made in the above.

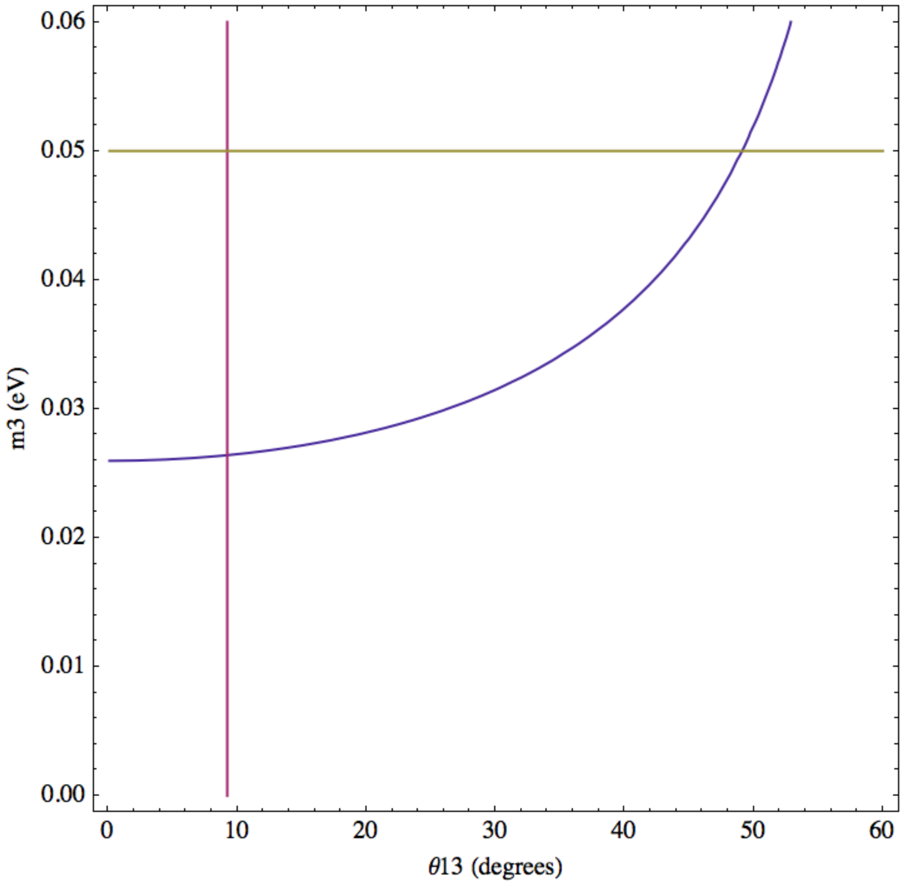

FIGURE 2. $\theta_{13} - m_3$ contour. The horizontal line corresponds to the minimum m_3 value. The vertical line is the best fit parameter for θ_{13} as measured by the T2K experiment

We can ask if varying the measured parameters around could possibly fix this. To address this question, we can look at a couple of contour plots of m_3 vs. θ_{ij} to see if we can get an allowed solution for m_3 with different values for the mixing angles. These contour plots are displayed in figures 4.1 through 4.3. For each of the plots, two of the mixing angles are fixed at the central, measured value, and the last is allowed to vary. We can see that varying the mixing angles from where we think they are does not look like a viable solution to this inconsistency. We can conclude that given the recent measurements for the neutrino oscillation parameters, this particular model for the see-saw mechanism does not look so attractive.

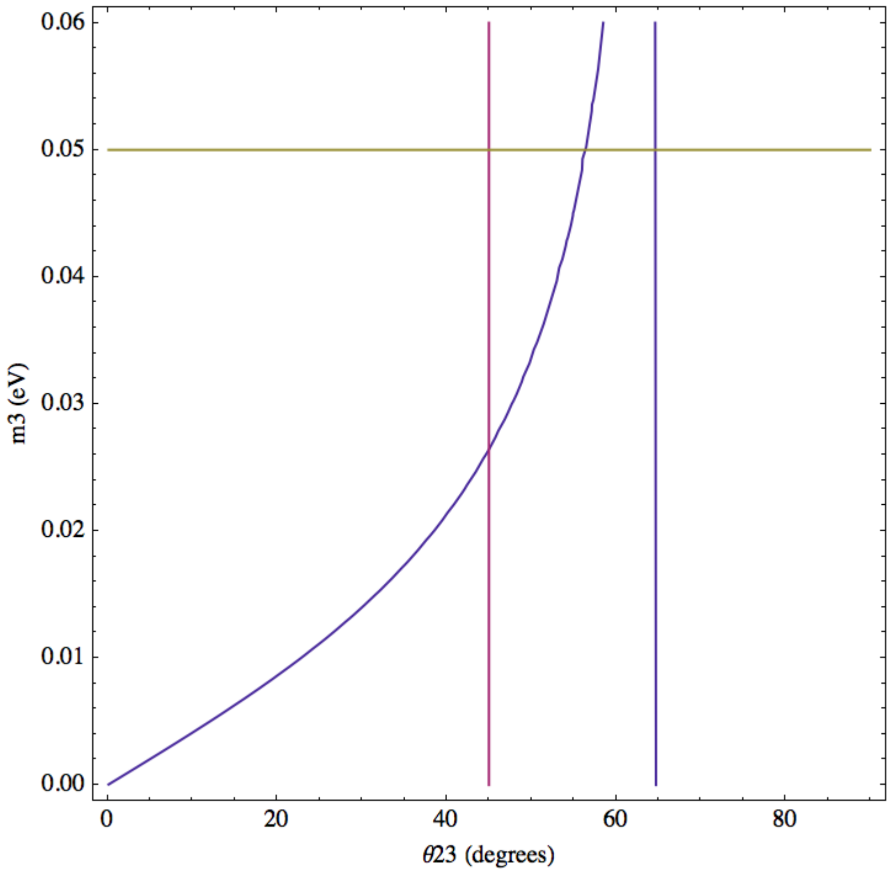

FIGURE 3. $\theta_{23} - m_3$ contour. The horizontal line corresponds to the minimum m_3 value. The vertical line is the best fit parameter for θ_{23} as measured by the T2K experiment

4.4. Summary

We have now seen that, by making some very simple modifications to the current standard model of particle physics, we can generate neutrino mass. Furthermore, via the see-saw mechanism, we can predict the smallness of the observed left chiral neutrino masses without disturbing the current understanding of the weak interaction at low energy scales. We have seen that in order for all of this to work, neutrinos are Majorana fermions. They are their own anti-particles. The right handed, see-saw counter part neutrinos are very heavy, explaining why we have not observed them at the energy scales probed by current experiments. In the near future, one big step in deciding experimentally whether or not the see-saw mechanism described here is viable, will be determining if neutrinos are in fact Majorana particles. There are several current efforts to measure what is known as neutrino-less double beta decay. Should this process be observed, we will have observed that neutrinos are in fact their own anti-particles, and the see-saw mechanism may seem even more attractive.

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