THESIS

BIFURCATION OF SEMIALGEBRAIC MAPS

Submitted by

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ABSTRACT

BIFURCATION OF SEMIALGEBRAIC MAPS

A semi-algebraic map is a function from a space to itself whose domain and graph are unions of solutions to systems of polynomial equations and inequalities. Thus it is a very general object with many applications, some from population genetics. The isoclines of such a map are semi-algebraic sets, which enjoy many striking properties, the most consequential of which here is that there is an algorithm to compute a "cylindrical decomposition" adapted to any finite family of semi-algebraic sets. The main subject of this paper is that a cylindrical decomposition adapted to the isoclines of a semi-algebraic map partitions parameter space into a tree which isolates bifurcations.

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INTRODUCTION

Bifurcation theory is a subject of dynamics and a dynamic subject. Steven Strogatz said in 1994 that “Bifurcation theory is rife with conflicting terminology. The subject really hasn't settled down yet, and people use different words for the same thing.” [1] This paper will have yet more new terminology. In doing so, it is really just going with the flow of bifurcation theory and is not a dramatic, fundamental change in how the theory has been developing. Speaking of dramatic, fundamental change, that's what a bifurcation is! It is natural to wonder though what a dramatic, fundamental change is. A usual definition of bifurcation is the change, as a parameter is varied, in equivalence class of a dynamical system--like topological conjugacy class, topological equivalence class, smooth equivalence class, or some other notion of equivalence class. This paper takes up two conflicts with these notions of equivalence:

1. Usually the notion of equivalence involves homeomorphisms or diffeomorphisms between orbits. All manner of smooth changes of coordinates can be freely performed by these morphisms; however, in many applications the dynamic variables provided by a problem play a special role in the model and should not be freely changed!
2. The main subject of this paper is to partition parameter space into subsets where a dynamical system is of constant equivalence class. Such “bifurcation diagrams” adapted to usual notions of equivalence do not have algorithms to compute them.

A new notion of equivalence “isoclinic cylindrical equivalence”, which respects coordinate directions and has a computable bifurcation diagram adapted to it, will be defined in Chapter 3.

To put this into practice, the universe of dynamical systems to be studied will have to be reduced to the set of “semi-algebraic maps”. However, semi-algebraic maps are very general objects with many applications, and the study of many ordinary differential equations can be reduced to the study of semi-algebraic maps [2].

CYLINDRICAL DECOMPOSITION

[3] defines the semi-algebraic subsets of $\mathbb{R}^k$ as the smallest family of sets that

- contains sets of the form $\{x \in \mathbb{R}^k | P(x) = 0\}$ for some $P \in \mathbb{R}[X_1, \dots, X_k]$,
- contains sets of the form $\{x \in \mathbb{R}^k | P(x) > 0\}$ for some $P \in \mathbb{R}[X_1, \dots, X_k]$,
- and is closed under boolean operations (complementation, finite unions, and finite intersections).

A function $f: S \to T$ is a "semi-algebraic function" if and only if

- S is semi-algebraic,
- T is semi-algebraic,
- and the graph of f is semi-algebraic.

A cylindrical decomposition $\mathcal{S}$ is a tree, a set of "nodes", one of which is the "root". The root is $\{(\,), \mathbb{R}^0\}$. Each node contains a sequence of positive integers and a semi-algebraic set. If a node $N = \{(p_1, \dots, p_l), S\} \in \mathcal{S}$ is not a "leaf", there are semi-algebraic continuous functions

$$\xi_1 < \cdots < \xi_l : S \to \mathbb{R}$$

such that

- $\{(p_1, \dots, p_l), \{(x, y) \in \mathcal{S} \times \mathbb{R} | y < \xi_1(x)\}\}$,
- $\{(p_1, \dots, p_l), \{(x, y) \in \mathcal{S} \times \mathbb{R} | y = \xi_1(x)\}\}$,
- $\{(p_1, \dots, p_l), \{(x, y) \in \mathcal{S} \times \mathbb{R} | \xi_1(x) < y < \xi_2(x)\}\}$,

$$\vdots$$

- $\{(p_1, \dots, p_l), \{(x,y) \in \mathcal{S} \times \mathbb{R} | \xi_{l-1}(x) < y < \xi_l(x)\}\}$,
- $\{(p_1, \dots, p_l), \{(x,y) \in \mathcal{S} \times \mathbb{R} | y = \xi_l(x)\}\}$,
- $\{(p_1, \dots, p_l), \{(x,y) \in \mathcal{S} \times \mathbb{R} | \xi_l(x) < y\}\}$

are nodes of $\mathcal{S}$.

A cylindrical decomposition is adapted to semi-algebraic sets $T_1, \dots, T_l$ if and only if every T_i is a union of semi-algebraic sets contained in its nodes. Corollary 5.7 of [3] implies that every finite family of semi-algebraic sets has a cylindrical decomposition adapted to them.

NONPARAMETRIC SEMI-ALGEBRAIC MAPS

A semi-algebraic function $f: S \to T$ is a "semi-algebraic map" if and only if $S = T$. The semi-algebraic set $S = T$ is the "phase space".

[3] defines the sign of a real number a as

$$\begin{array}{ll} sign(a) = 0 & if\ a = 0, \\ sign(a) = 1 & if\ a < 0, \\ sign(a) = -1 & if\ a > 0. \end{array}$$

To gain information about a map $: M \to M \subset \mathbb{R}^n$, it is useful to consider the isoclines: subsets of M on which the sign of the change in a single component is constant. For each $\alpha \in \{-1,0,1\}$ and each $i \in \{1, \dots, n\}$,

$$I_i^\alpha(f) = \{(x_1, \dots, x_n) \in M | (\exists y_1) \cdots (\exists y_n) f(x_1, \dots, x_n) = (y_1, \dots, y_n) \wedge sign(y_i - x_i) = \alpha\}$$

is an isocline.

A cylindrical decomposition is an "isoclinic cylindrical decomposition" of a map f if and only if it is adapted to the isoclines of f.

THE STRUCTURE OF DECOMPOSITIONS

The “structure” of a cylindrical decomposition $\mathcal{S}$ is

$$\{(p_1, \ldots, p_l), |(\exists S)\{(p_1, \ldots, p_l), S\} \in \mathcal{S}\}.$$

Let $\{(p_1, \ldots, p_l), S\}$ be a node of a cylindrical decomposition $\mathcal{S}$. Let $x = (x_1, \ldots, x_l) \in S$. The “cylindrical decomposition induced by $\mathcal{S}$ above x” [3], called $\mathcal{S}_x$, is

$$\{(p_{l+1}, \ldots, p_n), \{(x_{l+1}, \ldots, x_n) | (x_1, \ldots, x_n) \in T\} | \{(p_1, \ldots, p_n), T\} \in \mathcal{S}\}.$$

For each $i \in \{1, \ldots, n\}$ and each $S \subseteq M$ contained in $\mathcal{I}$, define $sign_i(S)$ as

$$\begin{aligned} sign_i(S) &= 0 \quad && if\ S \subseteq I_i^0(f), \\ sign_i(S) &= 1 \quad && if\ S \subseteq I_i^1(f), \\ sign_i(S) &= -1 \quad && if\ S \subseteq I_i^{-1}(f). \end{aligned}$$

The “sign structure” of $\mathcal{I}$ is

$$\Big\{\{(p_1, \ldots, p_n), dim(S), \big(sign_1(S), \ldots, sign_n(S)\big)\} | \{(p_1, \ldots, p_n), S\} \in \mathcal{I} \wedge\ S \subseteq M\Big\}.$$

Two algebraic maps are of the same isoclinic cylindrical equivalence class if there are isoclinic cylindrical decompositions of each with equal sign structure.

PARAMETRIC SEMI-ALGEBRAIC MAPS

In many maps $: M \to M \subset \mathbb{R}^n$, it is natural that the first d coordinates be parameters, representing that $I_1^0(f) = \cdots = I_d^0(f) = M$. Such a map is called a parametric map with d parameters. The projection

$$\{x_1, \dots, x_d \in \mathbb{R}^d | (\exists x_{d+1}) \cdots (\exists x_n)(x_1, \dots, x_n) \in M\}$$

is the parameter space of f.

Let $u = (u_1, \dots, u_d)$ be a value of parameters of a map $f : M \to M \subset \mathbb{R}^n$ with d parameters. Consider the semi-algebraic set obtained by intersecting M with $\{u\} \times \mathbb{R}^{n-d}$. This subset of M is important dynamically because it is an invariant subspace [4]—for every $v \in M \cap (\{u\} \times \mathbb{R}^{n-d})$, it follows that $f(v) \in M \cap (\{u\} \times \mathbb{R}^{n-d})$. Denote its projection forgetting the first d coordinates

$$M_u = \{(x_{d+1}, \dots, x_n) \in \mathbb{R}^{n-d} | (u_1, \dots, u_d, x_{d+1}, \dots, x_n) \in M\}.$$

This could be called the "phase space induced by f above u".

Denote by π the projection forgetting the first d coordinates. Denote f_u as the map from M_u to M_u:

$$f_u(x_{d+1}, \dots, x_n) = \pi \circ f(u_1, \dots, u_d, x_{d+1}, \dots, x_n).$$

This map could be called the "map induced by f above u".

DISCUSSION AND CONCLUSION

The result that every finite set of polynomials and every finite family of semi-algebraic sets have cylindrical decompositions adapted to them is quite striking and has many consequences. The proof is complicated and does not essentially imply that there is an algorithm for computing these decompositions, but in fact there is such an algorithm, and it has been implemented many times [6,7,8]. The computer algebra system *Mathematica* has implementations of algebraic functions, algebraic numbers, and cylindrical decomposition built into its C Kernel [9,10].

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