DISSERTATION

BRIDGELAND STABILITY OF LINE BUNDLES ON SMOOTH PROJECTIVE

SURFACES

Submitted by

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ABSTRACT

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Bridgeland Stability Conditions can be thought of as tools for creating and varying moduli spaces parameterizing objects in the derived category of a variety X. Line bundles on the variety are fundamental objects in its derived category, and we characterize the Bridgeland stability of line bundles on certain surfaces. Evidence is provided for an analogous characterization in the general case. We find stability conditions for $\mathbb{P}^1 \times \mathbb{P}^1$ which can be seen as giving the stability of representations of quivers, and we deduce projective structure on the Bridgeland moduli spaces in this situation. Finally, we prove a number of results on objects and a construction related to the quivers mentioned above.

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This dissertation is typset in LaTeX using a document class designed by Leif Anderson.

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CHAPTER 1

INTRODUCTION

This thesis concerns Bridgeland stability conditions defined on smooth projective surfaces, and more specifically, the Bridgeland stability of line bundles on a given surface. Bridgeland stability conditions (BSCs) are defined on the bounded derived category of a smooth projective variety X, denoted $D^b(X)$. This category contains objects such as line bundles on X, vector bundles on X, and more generally, complexes of vector bundles and their algebraic generalizations called sheaves. A BSC, σ, labels each object as either σ-semistable or σ-unstable, and for a choice of invariants v (e.g. rank) one can consider the algebraic space $\mathcal{M}_\sigma(v)$ parameterizing σ-semistable objects.

BSCs were introduced in [12] and gave a mathematical foundation to Douglas' work on Π-stability of Dirichlet branes in string theory [17]. This physical inception has played a significant role in directing the study of BSCs. For instance, complex varieties called Calabi-Yau 3-folds are of particular interest in string theory, and it is an ongoing effort to construct a (single) BSC on one such space [4, 8, 24, 27]. Furthermore, the set of all BSCs on X carries an action by the autoequivalences of the derived category of X - it is not surprising, then, that BSCs have a meaningful connection to Homological Mirror Symmetry (e.g. [11]).

Moduli spaces parameterizing vector bundles (and coherent sheaves) have classical constructions using, for example, Mumford or Gieseker stability. These stabilities indicate which sheaves to include and exclude in order to form a moduli space with desirable structure. Bridgeland stability has both similarities and meaningful differences from these classical notions of stability.

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Like Mumford or Gieseker stability, one can continuously vary a BSC σ to a new σ'. (The space of all BSCs on X, denoted $\mathrm{Stab}(X)$, is in fact a complex manifold.) However, even in this similarity there is a striking difference - for varieties X of Picard rank 1 (where there is no variation in Mumford or Gieseker stability), one can vary Bridgeland stability non-trivially (see e.g. [2]).

The ability to deform Bridgeland Stability conditions allows objects $E \in D^b(X)$ to change stability, i.e. E may be σ-semistable, but σ'-unstable. For a chosen set of invariants, v, the space $\mathrm{Stab}(X)$ has a wall-and-chamber decomposition, where within a chamber the moduli space $\mathcal{M}_\sigma(v)$ is constant, but the space may change at and across a wall (which are real-codimension 1 subspaces of $\mathrm{Stab}(X)$). As a wall for a chosen set of invariants is crossed, the Bridgeland moduli spaces on either side of the wall typically are birational. This behavior grants a close connection between BSCs and the Minimal Model Program (MMP) and birational geometry.

The connection between Bridgeland stability and the MMP has been completely established for smooth projective surfaces (see [5, 28]) - the sequence of birational transformations connecting a surface X to its minimal model can be understood as a sequence of wall-crossings undergone while following a path in $\mathrm{Stab}(X)$. For smooth projective 3-folds, the situation is more delicate as there are only a select few 3-folds known to support BSCs, i.e. there is not a general construction known to give BSCs on any smooth projective 3-fold. Nevertheless, Toda shows in [29] that the first step of the MMP of X, an extremal contraction, can be realized as a wall-crossing of Bridgeland Moduli spaces, assuming that the construction of [4] yields a BSC on X.

More generally, the connection between BSCs and birational geometry has been given significant attention, e.g. [2, 3, 9, 6, 10, 31, 15]. Speaking broadly, these works look to use the structure inherent in BSCs to study the birational geometry of moduli spaces - often of classical interest - by interpreting them as spaces $\mathcal{M}_\sigma(v)$. The work which primarily inspired this thesis is that of Arcara-Bertram-Coskun-Huizenga [3] which studies the birational geometry of the Hilbert Scheme of points on $\mathbb{P}^2$, denoted $\mathbb{P}^{2[n]}$. By choosing invariants corresponding to ideal sheaves of points on $\mathbb{P}^2$ and selecting a Bridgeland chamber where stability corresponds to Gieseker stability, they interpret $\mathbb{P}^{2[n]}$ as a Bridgeland moduli space. The Bridgeland chambers for the ideal sheaves are studied and a correspondence is found (for "low n") between the Bridgeland chambers and the Mori chambers in the psuedo-effective cone of $\mathbb{P}^{2[n]}$.

1.0.1. STRUCTURE OF SPACES $\mathcal{M}_\sigma(v)$. Unlike the Mumford and Gieseker notions of stability, Bridgeland stability is not a priori connected to a Geometric Invariant Theory (GIT) problem, so very little is known about the structure of the spaces of Bridgeland semistable objects $\mathcal{M}_\sigma(v)$ in general . For example, are these spaces connected? projective? (These are also good questions for the spaces $\mathrm{Stab}(X)$!) However, in [6] Bayer and Marcì associate to a BSC σ a nef divisor on $\mathcal{M}_\sigma(v)$, providing a general approach to understanding the geometry of these spaces. On K3 surfaces, the nef divisors mentioned above are shown to be ample, and it is shown that the spaces $\mathcal{M}_\sigma(v)$ are, in particular, projective.

A different approach is taken in [3]. There, certain BSCs are seen to have a notion of stability which is equivalent to King's notion of stability for representations of a quiver [21]. Geometric Invariant Theory then gives projectivity of the spaces $\mathcal{M}_\sigma(v)$ and projectivity

of the moduli spaces for other BSCs follows after relating them to these "quiver stability conditions."

The quivers involved in the considerations of [3] are associated to certain "exceptional collections" of objects in $D^b(\mathbb{P}^2)$. These exceptional collections exist on other surfaces as well. Del Pezzo surfaces, i.e. $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathrm{Bl}_{p_1,\ldots,p_k}(\mathbb{P}^2)$ for $1 \leq k \leq 8$, are particularly well-suited for this theory (e.g. exceptional collections exist on each), and we have pursued the following conjecture.

Conjecture 1: The program of [3] can be carried out on any Del Pezzo surface S, yielding the projectivity of the spaces $\mathcal{M}_\sigma(v)$.

In Chapter 6, we utilize the results of Theorem 1.2 and carry out the program for $S = \mathbb{P}^1 \times \mathbb{P}^1$. We pause to note that the stability conditions considered here are those constructed on surfaces S in [2], which we denote $\mathrm{Stab}_{div}(S)$.

THEOREM 1.1. *Suitable quiver regions exist in $\mathrm{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$ to conclude that for any invariants v (satisfying the Bogomolov Inequality) and BSC σ, the space $\mathcal{M}_\sigma(v)$ is a projective variety.*

The next step in furthering the program of [3] is to find suitable "quiver regions" in $\mathrm{Stab}_{div}(\mathrm{Bl}_p \mathbb{P}^2)$. As described in Section 1.0.2, line bundles play a key role in describing these quiver regions. Since $\mathrm{Bl}_p \mathbb{P}^2$ has Picard rank 2 and just one irreducible curve of negative self-intersection, the results stated in Section 1.0.2 apply. Thus we understand in what regions line bundles are stable, and can use this information to search for quiver regions.

Preliminary computations show that knowledge of the stability of line bundles will not be enough in this case - to obtain a suitably sized quiver region, the stability of certain torsion sheaves (specifically, line bundles supported on the exceptional curve) will need to be

understood. The author and D. Arcara expect to complete these considerations soon. For higher blow-ups, the Picard rank is > 2 and we do not yet understand the stability of line bundles for these surfaces.

1.0.2. STABILITY OF LINE BUNDLES. Contained in the information of a BSC is a family of subcategories of $D^b(X)$, each of which generate $D^b(X)$ through shifts and extensions. Any one of these subcategories (called *hearts*) can be used to define the BSC, and structural properties of these hearts can be used to deduce structure on the Bridgeland moduli spaces $\mathcal{M}_\sigma(v)$.

For instance, certain stability conditions in $\mathrm{Stab}_{div}(\mathbb{P}^2)$ have finite-length hearts which are equivalent to the representations of a quiver. As described in Section 1.0.1, results on the stability of representations of quivers can then be used to deduce projectivity of the Bridgeland moduli spaces (as is done in [3]).

The generators of these "quiver hearts" are often shifts of line bundles, and understanding the Bridgeland stability of these line bundles is crucial to describing the associated quiver regions. As Pic $\mathbb{P}^2 = \mathbb{Z} \cdot H$ (generated by the class of a line), the stability of line bundles follows relatively quickly from the Bogomolov inequality and Hodge Index Theorem [1, Proposition 3.6].

For surfaces of Picard rank > 1, the algebraic proof of [1] fails. The author and D. Arcara look to settle the following problem:

Problem 1: Characterize the stability of line bundles in $\mathrm{Stab}_{div}(S)$, for S a smooth projective surface.

This problem can be interpreted outside of the motivation given above - namely, as a continuation of the body of work describing a chamber of stability for objects of a certain invariant type (e.g. [13, 7] where chambers corresponding to skyscraper sheaves are described).

Earlier considerations of D. Arcara and A. Bertram suggested that the stability of line bundles is strongly tied to the curves of negative self-intersection (if any) on the surface S. The author's work with D. Arcara has served to explore this connection. Specifically, we look to prove the following conjecture.

Conjecture 2: A line bundle L on S is σ-stable iff it is not destabilized by some $L(-C)$, where C is a curve of negative self-intersection in S.

To study the validity of the conjecture, we adopt the strategy of understanding the walls for destabilizing objects of L. A *wall* for L is a set of stability conditions such that L is semistable on one side of the wall, but unstable on the other. There are certain (half) 3-spaces $\mathcal{S}_{G,H} \subset \mathrm{Stab}_{div}(S)$ in which the walls for L are quadric surfaces, and for "high enough" stability conditions, L is semistable. If C is a curve of negative self-intersection in S then there is always a wall for L corresponding to the destabilizing object $L(-C)$. We must show that these are the highest walls for L.

Maciocia shows [22] that in the 3-spaces $\mathcal{S}_{G,H}$ there are planes in which the walls for L are nested semi-circles. Given a wall for L, this nestedness allows us to apply a result of [3] and in certain cases find a wall higher than our given wall. This higher wall for L will correspond to a destabilizing object of lower rank than the object giving our original wall. With this setup, proof by induction on the rank of (weakly) destabilizing subobjects of L is a strong method, and we employ it regularly.

In what follows, we prove Conjecture 2 in a number of cases, as well as give some partial results.

THEOREM 1.2. *Let S be a smooth projective surface, L be a line bundle on S, and $Stab_{div}(S)$ be the Bridgeland stability conditions described in Section 4.1.2.*

- *If S has no curves of negative self-intersection, then L is σ-stable for all $\sigma \in Stab_{div}(S)$.*

- *If S has Picark rank 2 and one irreducible negative curve C, then L is σ-stable iff it is not destabilized by $L(-C) \hookrightarrow L$.*

In addition to these results, in Chapter 4 we characterize the structure of destabilizing walls and certain invariants of destabilizing objects. In Section 4.4.4 and Chapter 5 we provide evidence supporting Conjecture 2. Each of these results has a dual version which characterizes the stability of the object $L[1]$, where [1] is the "shift-by-1" functor on the derived category.

The author and D. Arcara expect that the partial result of Chapter 5 should be adaptable to give a characterization of the stability of line bundles on any Picard rank 2 surface (the surfaces of note here are those with two irreducible curves of negative self-intersection). For surfaces of Picard rank > 2 the situation is not quite so controlled (e.g. the action of line bundles on $\mathrm{Stab}_{div}(S)$ does not preserve certain 3-spaces of BSCs), but our methods may still be fruitful, as the bounds on actually destabilizing objects [3] and the nestedness of walls in certain slices of the stability manifold [22] still apply.

1.0.3. ORGANIZATION. The organization of this thesis is as follows. Chapter 2 introduces derived categories and related constructions. Chapter 3 defines Bridgeland Stability Conditions and general constructions and results relating to BSCs. Chapter 4 studies the

Bridgeland stability of line bundles on surfaces. Chapter 5 proves a partial result on the stability of line bundles for surfaces with two irreducible curves of negative self-intersection. Chapter 6 identifies quiver regions in $\mathrm{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$, carrying out the program of [3] in this situation. Chapter 7 includes various results on the exceptional collections, quivers, and the "tilting operation" which inform the quiver BSCs used in Chapter 6.

CHAPTER 2

Derived Categories

Derived categories are very large categories (for example, the derived category of an abelian category contains infinitely many copies of the abelian category as subcategories!), and provide a meaningful setting in which to study geometry. For example, they allow one to precisely identify an object with a resolution and hence are the right setting in which to consider derived functors. They also appear in the key statement of homological mirror symmetry, which claims an equivalence between two categories associated to a Calabi-Yau 3-fold and it's mirror pair: $D^b(\operatorname{Coh} S) \cong \operatorname{Fuk}(\hat{X})$. Bridgeland stability conditions give information on the first category and even attach to it a geometric space (the space of stability conditions).

In this chapter, we discuss the appropriate theory leading to derived categories. This builds for us a necessary foundation for working with Bridgeland stability conditions.

2.1. Additive and Abelian Categories

These categories are well-structured and arise in many contexts (e.g. derived categories are additive and Bridgeland stability conditions depend on a choice of abelian subcategory). Note that the defining properties are in fact self-dual.

DEFINITION 2.1. A category $\mathcal{A}$ is called *additive* if the following conditions are satisfied:

(1) $\mathcal{A}$ has a zero object

(2) for any two objects of $\mathcal{A}$, their direct product exists in $\mathcal{A}$

(3) for any $A, B \in \mathcal{A}$, $\operatorname{Hom}_{\mathcal{A}}(A, B)$ is an additive (i.e. abelian) group with an addition that is bilinear with respect to composition

One can show that in an additive category, finite direct products are isomorphic to finite direct sums, so that self-duality does not fail in axiom 2. We will denote the direct product of A and B by $A \oplus B$.

DEFINITION 2.2. An additive category $\mathcal{A}$ is called *abelian* if the following conditions are satisfied:

(1) for any $A \xrightarrow{f} B$, ker f and coker f exist

(2) for any $A \xrightarrow{f} B$, the natural map $(\text{coim } f :=)$ coker $i \to$ ker π $(=: \text{im } f)$ is an isomorphism:

$$
\begin{array}{ccc}
& \text{coker } i \longrightarrow \text{ker } \pi & \\
& \nearrow \qquad \qquad \searrow & \\
A & \xrightarrow{\quad f \quad} & B \\
\nearrow_i & & \searrow^\pi \\
\text{ker } f & & \text{coker } f
\end{array}
$$

EXAMPLE 2.3. Vector bundles over a given topological space X form an additive category, but not an abelian one. Over $\mathbb{P}^1$, this follows from the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}}^1 \to \mathcal{O}_{\mathbb{P}}^1(1) \to \mathcal{O}_x \to 0$ for any $x \in \mathbb{P}^1$. Here $\mathcal{O}_{\mathbb{P}}^1$ and $\mathcal{O}_{\mathbb{P}}^1(1)$ are line bundles, but the skyscraper sheaf $\mathcal{O}_x$ is not.

Note that in the category of abelian groups ($\mathbb{Z}$-**Mod**), axiom 2 of Definition 2.2 asks that the First Isomorphism Theorem hold.

2.2. TRIANGULATED CATEGORIES

In order to study multiple objects connected by maps as a single object, we work with complexes. Given an abelian category $\mathcal{A}$, the *category of complexes* of $\mathcal{A}$, Kom $\mathcal{A}$ is the

category with

$$\text{Ob Kom } \mathcal{A} = \{\text{cochain complexes } C^\bullet \text{of } \mathcal{A}\}$$

$$\text{Hom}_{\text{Kom } \mathcal{A}}(C^\bullet, D^\bullet) = \{\text{chain maps } f^\bullet : C^\bullet \to D^\bullet\}$$

We will often identify chain maps in the same homotopy class. The category we obtain is the *homotopy category of complexes*, $K(\mathcal{A})$, where

$$\text{Ob } K(\mathcal{A}) = \text{Ob Kom } \mathcal{A}$$

$$\text{Hom}_{K(\mathcal{A})}(C^\bullet, D^\bullet) = \frac{\text{Hom}_{\text{Kom } \mathcal{A}}(C^\bullet, D^\bullet)}{f \equiv g \Leftrightarrow f \sim g}$$

In this transition, we have lost something. Specifically, Kom $\mathcal{A}$ is abelian (in the natural way), but $K(\mathcal{A})$ is not. For example, consider the complexes below:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & = & = & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array}
$$

We obtain chain maps f_0 and f_{id} by choosing $f = 0$ or $f = \text{id}$, respectively. Note that $f_0 \sim f_{\text{id}}$. However, the kernel (in Kom $\mathcal{A}$) of f_0 is $0 \longrightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \longrightarrow 0$, whereas the kernel of f_{id} is $0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$, and these two complexes are not homotopy equivalent since they have different cohomologies. Thus, choosing different representatives of homotopy classes of maps yields different kernels, which implies that $K(\mathcal{A})$ does not have kernels.

However, $K(\mathcal{A})$ is still additive, and we retain a structure similar in some ways to abelian categories by using distinguished triangles. Here are the crucial definitions.

DEFINITION 2.4. Given a chain map $f : X \to Y$, we define the *cone of f*, denoted $C(f)$, to be the complex with $C(f)^i = X^{i+1} \oplus Y^i$ and $d^i_{C(f)}(x^{i+1}, y^i) = (-d^{i+1}_X(x^{i+1}), d^i_Y(y^i) + f(x^{i+1}))$.

Note that in Kom $\mathcal{A}$, the sequence $0 \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{\pi} X[1] \longrightarrow 0$ is short exact for any map $f : X \to Y$.

DEFINITION 2.5. In $K(\mathcal{A})$, a sequence $X \to Y \to Z \to X[1]$ is called a (*distinguished*) *triangle* if it is isomorphic to a sequence of the form $X' \xrightarrow{f} Y' \xrightarrow{i} C(f) \xrightarrow{\pi} X'[1]$, where $[1]$ is the *translation functor*, $C(f)$ is the *cone* of f, and i and π are the natural maps.

Note that in the definition, the objects X, Y, Z and $C(f)$ represent *complexes* of objects of $\mathcal{A}$.

We think of a triangle, $X \to Y \to Z \to X[1]$, often written $X \to Y \to Z \xrightarrow{+1}$ or just $X \to Y \to Z$ as a generalization of a short exact sequence and say that Y is an *extension* of Z by X.

We now extract certain properties of $K(\mathcal{A})$ to obtain the axioms of a triangulated category. These axioms give us a calculus that is much easier to work with than compared to working directly with cochain complexes and chain maps.

DEFINITION 2.6. A *triangulated category* is an additive category $\mathcal{C}$ together with an automorphism $[1] : \mathcal{C} \to \mathcal{C}$ and a family of (distinguished) triangles, such that the following axioms are satisfied:

(TR 0) The set of triangles is closed under isomorphism.

(TR 1) For any $X \in \mathcal{C}$, we have that $X \xrightarrow{id} X \to 0 \to X[1]$ is a triangle

(TR 2) Any $f : X \to Y$ can be embedded in a triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR 3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a triangle.

(TR 4) We can always fill in the following diagram to make all squares commute, given that the rows are triangles and the left square commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle u[1]} \\
X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
$$

(TR 5) (octahedral axiom) Given triangles

$$X \xrightarrow{f} Y \longrightarrow A \longrightarrow X[1],$$

$$Y \xrightarrow{g} Z \longrightarrow C \longrightarrow Y[1],$$

$$X \xrightarrow{g \circ f} Z \longrightarrow B \longrightarrow X[1],$$

then there is a triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

such that the following diagram commutes:

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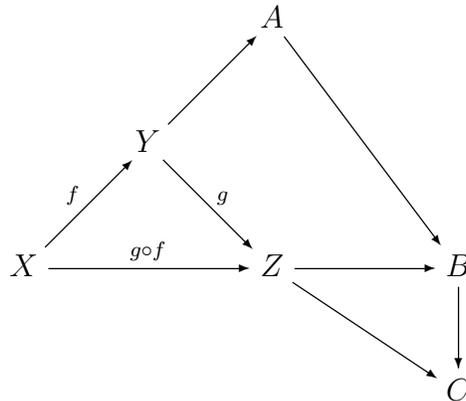
Note that (TR 3) says that triangles can be "rotated" to the left or right. Rotating infinitely, one obtains a "helix" where each set of three consecutive vertices forms a triangle. The octahedral axiom is useful for combining and separating filtrations of objects.

The next propositions show that triangles do retain a number of properties similar to those of short exact sequences.

PROPOSITION 2.7. *The following are true in any triangulated category, $\mathcal{C}$, and for any triangle, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.*

(1) We have $g \circ f = 0$.

(2) ("kernels") If $s : A \to Y$ is such that $gs = 0$, then there exists a (not necessarily unique) map, $\tau : A \to X$ such that $s = f\tau$.

(3) ("cokernels") If $t : Y \to B$ is such that $tf = 0$, then there exists a (not necessarily unique) map, $\sigma : Z \to B$ such that $t = \sigma g$.

(4) If $0 \to A \to B \to 0$ is a triangle, then $A \cong B$.

(5) If $A \xrightarrow{f} B$ is an isomorphism, then $A \xrightarrow{f} B \to 0 \to A[1]$ is a triangle.

(6) If $A \xrightarrow{f} B$ is an isomorphism and $A \xrightarrow{f} B \to C \to A[1]$ is a triangle, then $C \cong 0$.

(7) If $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ is also a triangle, then so is $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} X[1] \oplus X'[1] = (X \oplus X')[1]$.

(8) If $h = 0$, then $Y \cong X \oplus Z$.

PROOF. See [16, pp. 47-68]. □

DEFINITION 2.8. For a functor $F : \mathcal{C} \to \mathcal{D}$, of two additive categories, we say that F is an *additive functor* if, for all $A, B \in \mathcal{C}$ we have that $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ is a group homomorphism.

DEFINITION 2.9. Let $\mathcal{A}$ be an abelian category. An additive functor $F : \mathcal{C} \to \mathcal{A}$ is called a *cohomological functor* if for any triangle, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, the sequence $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ$ is exact in $\mathcal{A}$.

Note that if F is a cohomological functor, then for any triangle, $X \to Y \to Z \to X[1]$, we obtain the long exact sequence:

$$\cdots \to F^{k-1}Z \to F^k X \to F^k Y \to F^k Z \to F^{k+1}X \to \cdots$$

where $F^k = F \circ [1]^k = F \circ [k]$.

PROPOSITION 2.10.

(1) For any $X \in \mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(X, \_)$ and $\mathrm{Hom}_{\mathcal{C}}(\_, X)$ are cohomological.

(2) Let $\mathcal{A}$ be an abelian category. The cohomology functor $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is cohomological.

PROOF. See [20, pp. 39-40]. □

The following is a corollary of 2.10 (1).

COROLLARY 2.11. *Let*

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow u[1] \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
$$

be a morphism of triangles. If u and w are isomorphisms, then so is v.

PROOF. See [18, p. 242]. □

We will see more clearly why 2.10 (1) demonstrates a relationship to short exact sequences when we look at Ext in the derived category.

We now consider an important proposition that will be much used later on. It deals with when a morphism between vertices of triangles can be completed to a morhpism of the two triangles.

PROPOSITION 2.12. *Consider the situation*

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
& & \downarrow{\scriptstyle v} & & & & \\
X' & \longrightarrow & Y' & \xrightarrow{h'} & Z' & \longrightarrow & X'[1]
\end{array}
$$

where the rows are triangles. If $h'vf = 0$, then v can be completed to a morphism of triangles:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
X' & \longrightarrow & Y' & \xrightarrow{h'} & Z' & \longrightarrow & X'[1]
\end{array}
$$

If, moreover, $\mathrm{Hom}_{\mathcal{C}}(X, Z[-1]) = 0$, then the maps u, w in the above diagram are unique.

PROOF. See [18, p. 243]. □

2.3. DERIVED CATEGORIES

One of the main motivations for constructing the derived category is to be able to identify any object of an abelian cateogory $\mathcal{A}$ with a resolution of itself. While the exact structure of a derived category can be difficult to deduce, its construction has a simple presentation. This presentation comes from the construction of a localization of a category. Before we continue, however, a definition.

DEFINITION 2.13. A chain map $f : X \to Y$ is a *quasi-isomorphism* if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism for all n.

One can show that f is a quasi-isomorphism iff $C(f)$, the cone of f, is an exact complex.

We now describe the construction of localization using the specific example of localizing at all quasi-isomorphisms in Kom $\mathcal{A}$. The reader may consult [18, pp. 144-145] for the general construction.

DEFINITION 2.14. The *derived category* $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category where

$$\text{Ob } D(\mathcal{A}) = \text{Ob Kom } \mathcal{A}$$

and

$$\{\text{morphisms of} D(\mathcal{A})\} = \{\text{morphisms of Kom } \mathcal{A}\} \bigcup \{f^{-1} | f : X \to Y \text{ a quasi-isomorphism}\}.$$

This construction should understood as follows: The category Kom$\mathcal{A}$ is a directed graph, where the vertices represent objects. For each morhpism $g : Y \to Z$ in Kom $\mathcal{A}$ there is an arrow pointing from (the vertex) Y to Z in the graph. Now, to obtain $D(\mathcal{A})$, simply add a formal "inverse arrow" $f^{-1} : Y \to X$ for each quasi-isomorphism $f : X \to Y$. The morphisms in $D(\mathcal{A})$ are paths using the original arrows and formal inverse arrows where $f \circ f^{-1} := \text{id}$ and $f \circ f^{-1} := \text{id}$.

The derived category enjoys the following universal property with respect to its natural inclusion functor.

PROPOSITION 2.15. *Let $Q : \text{Kom } \mathcal{A} \to D(\mathcal{A})$ be the functor where $Q(X) = X$ and $Q(f) = f$. Then*

- $Q(f)$ *is an isomorphism for any quasi-isomorphism* f

- *Any functor* $F : \mathrm{Kom}\,\mathcal{A} \to \mathcal{D}$ *sending quasi-isomorphisms to isomorphisms can be uniquely factored through* $D(\mathcal{A})$, *i.e. the following diagram commutes:*

$$
\begin{array}{ccc}
 & D(\mathcal{A}) & \\
 {}^{Q}\nearrow & & \searrow^{!G} \\
 \mathrm{Kom}\,\mathcal{A} & \xrightarrow{\quad F \quad} & D
\end{array}
$$

PROOF. See [18, pp. 144-145]. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The functor G in the above diagram is defined by $G(X) = F(X)$ for all $X \in \mathrm{Ob}\,\mathrm{Kom}\mathcal{A} = \mathrm{Ob}\,D(\mathcal{A})$, $G(f) = F(f)$ for all $f \in \mathrm{Mor}\,\mathrm{Kom}\,\mathcal{A}$, and $G(g^{-1}) = G(g)^{-1}$ for all quasi-isomorphisms $g \in \mathrm{Mor}\,\mathrm{Kom}\,\mathcal{A}$.

Note that for any functor $H : D(\mathcal{A}) \to D$, precomposing with Q gives a functor from $\mathrm{Kom}\,\mathcal{A}$ to D sending quasi-isomorphisms to isomorphisms, so that defining functors as in Proposition 2.15 is "the only way" to define functors from $D(\mathcal{A})$.

As a corollary of Proposition 2.15, we have that the cohomology functors $H^k(\_)$ are well defined on the derived category.

2.3.1. MORPHISMS. Morphisms in the derived category are somewhat mysterious. However, the technique of localizing at a *localizing class of morphisms* allow us to view each morphism as the a double-composition - one map consisting solely of formal inverses, and the other a standard morphism of complexes.

DEFINITION 2.16. A class of morphisms $S \subset \mathrm{Mor}\,\mathcal{C}$ is said to be *localizing* if the following conditions are satisfied:

(1) S is closed under composition: $\mathrm{id}_X \in S$ for any $X \in \mathcal{C}$ and $s \circ t \in S$ for any $s, t \in S$ whenever the composition is defined.

(2) Extension conditions: for any $f \in \mathrm{Mor}\,\mathcal{C}, s \in S$ as in one of the following two diagrams, there exist $g \in \mathrm{Mor}\,\mathcal{C}, t \in S$ such that the corresponding diagram commutes:

$$
\begin{array}{ccc}
W & \xrightarrow{\ g\ } & Z \\
\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
W & \xleftarrow{\ g\ } & Z \\
\ \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle s} \\
X & \xleftarrow{\ f\ } & Y
\end{array}
$$

(3) Let f, g be morphisms from X to Y; the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

Now, in $\mathrm{Kom}\mathcal{A}$, the class is quasi-isomorphisms is *not* localizing; however, in the category $K(\mathcal{A})$, they are. Note that homotopic maps yeild the same map on homology, so that the notion of quasi-isomorphism is well-defined in $K(\mathcal{A})$. It turns out, by localizing $K(\mathcal{A})$ at the class of quasi-isomorphisms, we obtain a category that is, in fact, *isomorphic* to the category $D(\mathcal{A})$ above. We will thus use $D(\mathcal{A})$ for either.

We can now represent morphisms in the derived category as "roofs." Moving all inverses to the right (respectively left) in the strings of morphisms mentioned above, we see that a morphism from X to Y can be represented as

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle s}\swarrow{\scriptstyle\ qis} & & \searrow{\scriptstyle f} \\
X & & Y
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccc}
 & W & \\
{\scriptstyle g}\nearrow & & \nwarrow{\scriptstyle t}{\scriptstyle\ qis} \\
X & & Y
\end{array}
$$

where "qis" denotes a quasi-isomorphism.

Using this formulation and the axioms of a localizing class of morphisms, one can show (by finding a "common denominator") that $D(\mathcal{A})$ is an *additive* category. In fact, $D(\mathcal{A})$ is a *triangulated* category, with distinguished triangles those which are isomorphic to the images of the triangles of $K(\mathcal{A})$ under the functor Q.

Furthermore, we have the following characterization of when a morphism is zero in the derived category.

PROPOSITION 2.17. *In $D(\mathcal{A})$, a morhpism $f = 0 : X \to Y$ iff there exists a quasi-isomorphism $s : Y \to Z$ such that sf is homotopic to zero (iff there exists a quasi-isomorphism $t : W \to X$ such that ft is homotopic to zero)*

PROOF. The proof is relatively straightforward using roofs to represent morphisms. $\square$

The parenthesized iff comes from the axioms of a localizing class of morphisms.

We pause here to give a few helpful implications.

PROPOSITION 2.18. *The implications labeled $\overset{s}{\Longrightarrow}$ are strict:*

(1) $f = 0$ in $Kom\,\mathcal{A} \overset{s}{\Longrightarrow} f = 0$ in $K(\mathcal{A}) \overset{s}{\Longrightarrow} f = 0$ in $D(\mathcal{A}) \overset{s}{\Longrightarrow} H^n(f) = 0$ for all n

(2) A and B are homotopy equivalent $\overset{s}{\Longrightarrow} A$ and B are quasi-isomorphic $\Longrightarrow A \cong B$ in

    $D(\mathcal{A}) \Longrightarrow H^n(A) \cong H^n(B)$ in $\mathcal{A}$ for all n

(3) f and g are homotopic $\overset{s}{\Longrightarrow} H^n(f) = H^n(g)$ for all n

(4) A is exact (i.e. $H^n(A) = 0$ for all n) $\Longleftrightarrow H^n(id_A) = H^n(0)$ for all n. In particular,

    the following implication is stict: id_A is homotopic to $0 \Longrightarrow A$ is exact.

The author conjectures that the last two implications in (2) are strict.

PROOF. Here we give examples showing the implications are strict:

(4) For $A = 0 \longrightarrow \mathbb{Z} \overset{2}{\longrightarrow} \mathbb{Z} \overset{\pi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$, we have A is exact, but $id_A \not\simeq 0$.

(1) For the first implication, consider the example of Section 2.2 showing that $K(\mathcal{A})$ is

    not abelian. For the second, use A from above and note that A exact implies $A = 0$ in

$D(\mathcal{A})$ implies $id_A = 0$ in $D(\mathcal{A})$. For the third, consider this example from [18, p. 163]:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\ 2\ } & \mathbb{Z} \\
\Big\| & & \Big\downarrow{\scriptstyle \pi \circ 2} \\
\mathbb{Z} & \xrightarrow{\ \pi\ } & \mathbb{Z}/3\mathbb{Z}
\end{array}
$$

(2) Consider the example:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\ 2\ } & \mathbb{Z} \\
\Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \pi} \\
0 & \longrightarrow & \mathbb{Z}
\end{array}
$$

(3) Follows from (1).

$\square$

2.3.2. EXTENSIONS. The Ext groups have a strong connection to the derived category. To see the nature of this connection, we begin with a lemma.

LEMMA 2.19. *Let I be a bounded below complex of injectives, i.e. $I^j = 0$ for all $j \leq N$. Then every quasi-isomorphism $t : I \to Z$ is a split injection in $K(\mathcal{A})$, i.e. there exists an $s : Z \to I$ with ts homotopic to id_I.*

PROOF. Here we use from [30, p. 18] a slightly modified definition of the cone of a morphism with differential $d^i_{C(f)}(x^{i+1}, z^i) = (-d^{i+1}_I(x^{i+1}), d^i_Z(z^i) - f(x^{i+1}))$. This retains the fact that t is a quasi-isomorphism iff $C(t)$ is an exact complex. Now, there is a natural map $\pi : C(t) \to I[1]$, and using a result from homological algebra, we have that $\pi \sim 0$. The second coordinate of these homotopy maps gives maps $s_i : Z_i \to I_i$. Now, writing out explicitly the equation that $\pi \sim 0$ gives and then resticting π to each coordinate, we see that

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the maps s_i form a chain map s and that $st \sim \mathrm{id}_I$, i.e. $st = \mathrm{id}_I$ in $K(\mathcal{A})$. For the details of the proof, see [30, p. 387]. $\qquad \square$

The next result shows two situations where maps in the derived category are the same as those in the homotopy category.

PROPOSITION 2.20. *Let I be a bounded below complex of injectives, then $\mathrm{Hom}_{D(\mathcal{A})}(X, I) \cong$ $\mathrm{Hom}_{K(\mathcal{A})}(X, I)$ for every X. Dually, if P is a bounded above complex of projectives, then $\mathrm{Hom}_{D(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{K(\mathcal{A})}(P, X)$. For the details of the proof, see [30, p. 388].*

PROOF. We have the map $Q : \mathrm{Hom}_{K(\mathcal{A})}(X, I) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(X, I)$. To show this map is surjective, represent morphisms in $D(\mathcal{A})$ as right fractions and use Proposition 2.20. To show injectivity, use Propositions 2.17 and 2.20. $\qquad \square$

We now can now prove the relationship between Ext groups and certain Hom groups in the derived category. We do this for $\mathcal{A} = R\text{-}\mathbf{Mod}$ but an analogous result holds for $\mathcal{A} = \mathrm{Coh}\, X$.

PROPOSITION 2.21. *Let X, Y be objects in the category $R\text{-}\mathbf{Mod}$ and denote also by X and Y the complexes with X and Y in position 0 and the zero object elsewhere. We have $\mathrm{Ext}_R^n(X, Y) \cong \mathrm{Hom}_{D(R\text{-}\mathbf{Mod})}(X, Y[n])$.*

PROOF. We let $\mathcal{A} = R\text{-}\mathbf{Mod}$. Let $\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A$ be a projective resolution of X. Then, by definition,

$$\mathrm{Ext}_R^i(X, Y) = \frac{\ker d_{i+1}^\star}{\mathrm{im}\, d_i^\star} = \frac{\{f : P_i \to Y \mid f \circ d_{i+1} = 0\}}{f \equiv g \iff f - g = h \circ d_i \text{ for some } h : P_{i-1} \to Y}$$

Now, since $X \cong P_\bullet$ in $D(\mathcal{A})$, we have $\operatorname{Hom}_{D(\mathcal{A})}(X, Y[i]) \cong \operatorname{Hom}_{D(\mathcal{A})}(P_\bullet, Y[i])$, and then by Proposition 2.20 we have $\operatorname{Hom}_{D(\mathcal{A})}(P_\bullet, Y[i]) = \operatorname{Hom}_{K(\mathcal{A})}(P_\bullet, Y[i])$. Considering the definition of morphisms in $K(\mathcal{A})$ and the diagram below

$$
\begin{array}{ccccccccccc}
\cdots \longrightarrow & P_{i+1} & \xrightarrow{d_{i+1}} & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \quad : P_\bullet \\
& & & \Big\downarrow{\scriptstyle f} & {\scriptstyle h} & & & & & & \\
\cdots \longrightarrow & 0 & \longrightarrow & Y & & & & & & \quad : Y[i]
\end{array}
$$

we see that

$$
\operatorname{Hom}_{K(\mathcal{A})}(P_\bullet, Y[i]) = \frac{\{f : P_i \to Y \mid f \circ d_{i+1} = 0\}}{f \equiv g \iff f - g = h \circ d_i \text{ for some } h : P_{i-1} \to Y}(= \operatorname{Ext}_R^i(X, Y)).
$$

$\square$

Because of this result, we give the general notation $\operatorname{Ext}^n(X, Y) := \operatorname{Hom}_{D(\mathcal{A})}(X, Y[n])$ for complexes X and Y.

CHAPTER 3

Introduction to Stability Conditions

If we restrict to bounded objects in the derived category, then for every object we can obtain a unique filtration by breaking off its homology, one position at a time. The way we will do this is by truncation functors, and generalizing the crucial properties of these functors will yield the axioms of a t-structure.

In this sense, objects in the bounded derived category are built from of objects of $\mathcal{A}$. We will see that $\mathcal{A}$ is *the heart* of the bounded derived category. However, some crucial information is lost in the deconstruction of a complex to its homology objects. For example, there are other hearts $\mathcal{B}$ that one can decompose complexes into, whose associated derived categories are not equivalent to the original derived category.

One function of Bridgeland stability conditions will be to interpolate between these hearts, giving a finer collection of hearts of the derived category at hand.

3.1. t-structures

First, we motivate further the idea of an object in the derived category being built from objects in the abelian category.

DEFINITION 3.1. A *0-complex* of Kom $\mathcal{A}$ is one with the zero object in all nonzero positions.

Note that we have a category of 0-complexes in Kom $\mathcal{A}$ and that, using the natural inclusion functor, the category of 0-complexes in Kom $\mathcal{A}$ is isomorphic to the category of 0-complexes in $K(\mathcal{A})$ (since the only homotopy of 0-complex maps is the zero homotopy). We also have the following result.

PROPOSITION 3.2. *The inclusion functor Q gives an equivalence between the category of 0-complexes in $K(\mathcal{A})$ and the H^0-complexes in $D(\mathcal{A})$, where an H^0-complex is one with zero homology in all nonzero positions.*

PROOF. See [18, p. 164]. □

We now define the functors that we will later use to filter the objects of the bounded derived category.

DEFINITION 3.3. We have the following functors, called *truncation functors*, on the category of complexes. If

$$X \quad = \quad \cdots \longrightarrow X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \longrightarrow \cdots$$

then we have

$$\tau_{\leq n} X \quad = \quad \cdots \longrightarrow X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} \ker d^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\tau_{\geq n} X \quad = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Coker}\, d^{n-1} \xrightarrow{\overline{d^n}} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \longrightarrow \cdots$$

Note that there are natural maps $\tau_{\leq n} X \to X$ and $X \to \tau_{\geq n} X$, and that since keeping the kernel (respectively Cokernel) in position n keeps the necessary homology information there, we have that the map $\tau_{\leq n} X \to X$ is a quasi-isomorphism if $H^i(X) = 0$ for $i \geq n+1$ and similarly the map $\tau_{\geq n} X \to X$ is a quasi-isomorphism if $H^i(X) = 0$ for $i \leq n-1$. Finally, we have $\tau_{\geq n} \tau_{\leq n} = \tau_{\leq n} \tau_{\geq n} = H^n(\_)$ as functors.

The following properties of the derived category are strongly tied to the truncation functors.

PROPOSITION 3.4. *Let $\mathcal{D} = D(\mathcal{A})$ and set $\mathcal{D}^{\leq 0} = \{X \in D(\mathcal{A})|H^i(X) = 0, \text{for all } i > 0\}$ and $\mathcal{D}^{\geq 0} = \{X \in D(\mathcal{A})|H^i(X) = 0, \text{for all } i < 0\}$. Denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$. Note that, for example, $\mathcal{D}^{\leq n} = \{X \in D(\mathcal{A})|H^i(X) = 0, \text{for all } i > n\}$. We have the following properties:*

(1) $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ are both strictly full subcategories (i.e. isomorphism-closed full subcategories) of $\mathcal{D}$

(2) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$

(3) $\text{Hom}(X, Y) = 0$ for any $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$

(4) For any $X \in \mathcal{D}$ there exists a triangle $\tau_{\leq 0} X \to X \to \tau_{\geq 1} X \to (\tau_{\leq 0} X)[1]$

- *Note: $\tau_{\leq 0} X$ is in $\mathcal{D}^{\leq 0}$ and $\tau_{\geq 1} X$ is in $\mathcal{D}^{\geq 1}$.*

(5) The category $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is equivalent to $\mathcal{A}$

PROOF. Here, (1) follows from Proposition 2.18, (2) is straightforward, (4) requires a straightforward calculation and (5) follows from Proposition 3.2. We prove (3):

Let $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$ and let a morphism $X \to Y$ be represented by the roof $X \xleftarrow{s} Z \xrightarrow{f} Y$. Since s is a quasi-isomorphism and $X \in \mathcal{D}^{\leq 0}$, we have $Z \in \mathcal{D}^{\leq 0}$ and thus the natural map $r : \tau_{\leq 0} Z \to Z$ is a quasi-isomorphorphism. Hence $\tau_{\leq 0} Z \cong Z$ in $D(\mathcal{A})$ and $X \xleftarrow{sr} \tau_{\leq 0} Z \xrightarrow{fr} Y$ also represents our morphism. Now, since $Y \in \mathcal{D}^{\geq 1}$, we have that $k : Y \to \tau_{\geq 1} Y$ is a quasi-isomoriphism and thus $Y \cong \tau_{\geq 1} Y$ in $D(\mathcal{A})$. Finally, we have $(\tau_{\leq 0} Z)_i = 0$ for $i \geq 1$ and $(\tau_{\geq 1} Y)_i = 0$ for $i \leq 0$. Thus, $rfk = 0 : \tau_{\leq 0} Z \longrightarrow \tau_{\geq 1} Y$ in $\text{Kom } \mathcal{A}$ and hence in $D(\mathcal{A})$. But k an isomorphism in $D(\mathcal{A})$ implies that $rf = 0$ in $D(\mathcal{A})$, and so our original morphism is zero. $\square$

In fact, one can show that for any $X \in D(\mathcal{A})$ and any n, we have the triangle $\tau_{\leq n} X \to X \to \tau_{>n} X$.

We can now obtain the aforementioned filtration of an object of the derived category by slicing off one homology at a time, starting from the right. In order to obtain a finite filtration, we need to restrict our attention.

DEFINITION 3.5. The *bounded derived category* of $\mathcal{A}$, denoted $D^b(\mathcal{A})$, is the full subcategory of $D(\mathcal{A})$ consisting of the objects X for which $H^i(X) = 0$ for all $|i| >> 0$.

PROPOSITION 3.6. *Let $X \in D^b(\mathcal{A})$. If $H^i(X) = 0$ for $|i| > N$, we have the filtration*

$$0 = E_{-N-1} \longrightarrow E_{-N} \longrightarrow E_{-N+1} \longrightarrow \cdots \longrightarrow E_{N-1} \longrightarrow E_N = X$$

$$A_{-N} \qquad A_{-N+1} \qquad\qquad A_N$$

where $E_i = \tau_{\leq i} X$ and $A_i = \tau_{\geq i} E_i = H^i(X)[-i]$.

PROOF. This is straightforward, using Definition 3.3 and the note following 3.4. $\square$

Note that in the filtration above that each $E_{i-1} \to E_i \to A_i$ is a traingle and each A_i in is in $\mathcal{A}[-i]$, where $\mathcal{A}$ is identified with the H^0-complexes as in 3.2. Also, we cut out any triangles with $A_i = 0$, since then $E_i = E_{i-1}$.

We now abstract these properties so that we can apply them in other situations.

DEFINITION 3.7. A *t-structure* on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories (i.e. isomorphism-closed full subcategories) $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the conditions below. Denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

(1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$

(2) $\mathrm{Hom}(X, Y) = 0$ for any $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$

(3) For any $X \in \mathcal{D}$ there exists a triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{D}^{\leq 0}, B \in \mathcal{D}^{\geq 1}$.

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The *heart* of the t-structure is the full subcategory $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Note that $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ implies that $\mathcal{D}^{\leq N} \subset \mathcal{D}^{\leq N+1}$ for all N and similarly $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$ implies that $\mathcal{D}^{\geq N} \supset \mathcal{D}^{\geq N+1}$ for all N

EXAMPLE 3.8. Proposition 3.4 above shows that if $\mathcal{D} = D(\mathcal{A})$, then setting $\mathcal{D}^{\leq 0} = \{X \in D(\mathcal{A}) | H^i(X) = 0, \text{for all } i > 0\}$ and $\mathcal{D}^{\geq 0} = \{X \in D(\mathcal{A}) | H^i(X) = 0, \text{for all } i < 0\}$ gives a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with heart $\mathcal{A}$.

The following proposition shows that general t-structures behave similarly to the natural t-structure of Example 3.8: they admit truncations of complexes and even a notion of cohomology.

PROPOSITION 3.9.

(1) *For a given X, any two triangles as in Definition 3.7 (3) are canonically isomorphic.*

(2) *The triangles in Definition 3.7 (3) give functors, $\tau_{\leq 0}$ and $\tau_{\geq 1}$ where $\tau_{\leq 0} X = A$ and $\tau_{\geq 1} X = B$. We obtain $\tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$ and $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$ by setting $\tau_{\leq n} = [-n]\tau_{\leq 0}[n]$ and $\tau_{\geq n} = [-n]\tau_{\geq 0}[n]$.*

(3) *The functors in (2) are left (resp. right) adjoint to the corresponding embedding functors.*

(4) *For all n, $\tau_{\leq n}\tau_{\geq n} \simeq \tau_{\geq n}\tau_{\leq n} =: \tau_{[n,n]}$*

(5) *Let $H^0 := \tau_{[0,0]} : \mathcal{D} \to \mathcal{A}$ and $H^i(X) = H^0(X[i])$. Then H^0 is a cohomological functor.*

PROOF. See [18, pp. 279-280,283]. $\square$

As in the case of the derived category, we need a definition to obtain finite filtrations.

DEFINITION 3.10. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{D}$ is called *bounded* if

$$\mathcal{D} = \bigcup_{n,m \in \mathbb{Z}} \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$$

or equivalently, if $\cap_n \mathrm{Ob}\, \mathcal{D}^{\geq n} = \{0\}$ and for any $X \in \mathcal{D}$, only a finite number of objects $H^i(X) \in \mathcal{A}$ is nonzero.

The following proposition gives a characterization of hearts of bounded t-structures.

PROPOSITION 3.11. *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ if and only if the following two conditions hold:*

(1) if $k_1 > k_2$ are integers and $A, B \in \mathcal{A}$ then $Hom_{\mathcal{D}}(A[k_1], B[k_2]) = 0$

(2) for every nonzero object $E \in \mathcal{D}$ there is a finite sequence of integers

$$k_1 > k_2 > \cdots > k_n$$

and a filtration through triangles

with $0 \neq A_i \in \mathcal{A}[k_i]$ for all i.

PROOF. The statement is found here: [12, p. 326], but no proof is given. We give some of the main points below. □

In the filtration above, we may assume that $E_i \neq 0$ for all $i > 0$. Also, one can show by induction that the horizontal map $E_1 \to E$ is nonzero, and thus all the horizontal maps

are nonzero. Finally, all vertical maps are nonzero as well. Note that since $E_0 = 0$, we have $E_1 = A_1$.

An important fact about these filtrations (whose proof uses the following lemma) is that they are unique up to canonical isomorphism. In other words, we have the following strict implication: if F and F' are two filtrations of E as in Proposition 3.11, then $F \cong F'$, where the isomorphism of filtrations means an isomorphism at each level such that all squares commute.

LEMMA 3.12. *If $X \to E \to Y$ is a triangle and $E \xrightarrow{\neq 0} B$ then there exists either $X \xrightarrow{\neq 0} B$ or $Y \xrightarrow{\neq 0} B$.*

PROOF. The proof is straightforward using Proposition 2.7 (2) and (3). $\square$

Similarly, If $S \to B \to T$ is a triangle and $E \xrightarrow{\neq 0} B$ then there exists either $E \xrightarrow{\neq 0} S$ or $E \xrightarrow{\neq 0} T$.

It follows by induction on the lemma, that if E is "built up" from extensions of $A_1, \ldots, A_n$ and B is "built up" from extensions of $F_1, \ldots, F_m$, then any nonzero map $E \xrightarrow{\neq 0} B$ gives a nonzero map $A_i \xrightarrow{\neq 0} F_j$ for some i, j. By "built up," we mean that E is in $\langle A_1, \ldots, A_n \rangle$ as defined below.

DEFINITION 3.13. Let $\mathcal{D}$ be a triangulated category and S be a set of objects of $\mathcal{D}$. The *extension closed subcategory of $\mathcal{D}$ generated by S*, denoted $\langle S \rangle$, is the full subcategory of $\mathcal{D}$ defined as follows:

Let $S_0 = S$. For all $i \geq 1$, set

$$S_i = \{X \in \mathcal{D} \mid A_{i-1} \to X \to B_{i-1} \text{ is a triangle for some } A_{i-1}, B_{i-1} \in S_{i-1}\}.$$

Then $\langle S \rangle := \bigcup_{i \geq 0} S_i$.

In proving Proposition 3.11, one uses the bounded derived category as an example and guide for both directions. Abstracting the proof from this category yields the proof in general. For instance, the property that a complex X has $H^i(X) = 0$ for $i > N$ becomes the property $X[N] \in \mathcal{D}^{\leq 0}$ but $X[N-1] \notin \mathcal{D}^{\leq 0}$. Similarly, X has $H^i(X) = 0$ for $i < M$ becomes the property $X[M+1] \in \mathcal{D}^{\geq 1}$ but $X[M] \notin \mathcal{D}^{\geq 1}$. To obtain the filitration in the proposition, one uses this analogy and Proposition3.14 (3) below to pull off each homology object (complex) in turn. Going backwards, one shows that the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, is given by $\mathcal{D}^{\leq 0} = \langle \mathcal{A}[i] \mid i \leq 0 \rangle$, $\mathcal{D}^{\geq 0} = \langle \mathcal{A}[i] \mid i \geq 0 \rangle$ and also that $\mathcal{D}^{m \leq n} := \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m} = \langle \mathcal{A}[i] \mid m \leq i \leq n \rangle$.

The octahedral axiom, Definition 2.6 (TR 5), becomes quite useful in the backwards direction. It was mentioned in the previous chapter that the octahedral axiom can be used for gluing together or breaking apart filtrations. Here, given a filtration of X, one uses the octahedral axiom to obtain the extension of Definition3.7 (3) by setting $A = E_i$ where $i = \max\{i \mid k_i \geq 0\}$. Furthermore, the octahedral axiom shows that we have triangles $A_{i-1} \to Z \to A_i$ for all i. This follows from the diagram below (whose form mimics that of the diagram in Definition 2.6 (TR 5)):
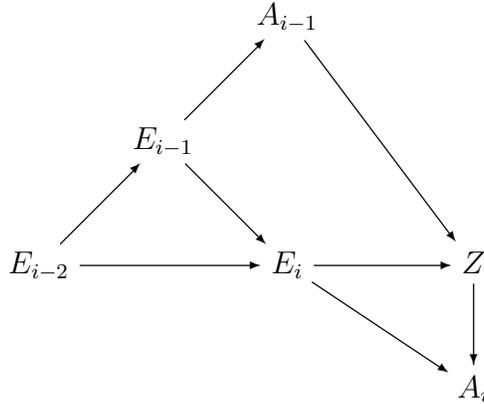
We summarize here a few important facts concerning truncation functors and their associated subcategories.

PROPOSITION 3.14.

(1) If $X \in \mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$), then the morphism $\tau_{\leq n}X \to X$ (resp. $X \to \tau_{\geq n}X$) is an isomorphism.

(2) Let $X \in \mathcal{D}$. Then $X \in \mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$) if and only if $\tau_{\geq n+1}X = 0$ (resp. $\tau_{\leq n-1}X = 0$).

(3) If $A \to X \to B$ is a triangle and A and B belong to $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$), then so does X.

(4) If $A \to X \to B$ is a triangle and A and B belong to $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, then so does X, i.e. the heart of a t-structure is closed under extensions.

(5) The heart $\mathcal{A}$ of a t-structure is an abelian category, with short exact sequences given by triangles in $\mathcal{D}$ with all vertices lying in $\mathcal{A}$.

(6) In fact, if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$, then there exists a unique $h : Z \to X[1]$ such that $X \to Y \to Z \xrightarrow{h} X[1]$ is a triangle in $\mathcal{D}$.

PROOF. See [20, pp. 413-415]. □

3.2. STABILITY CONDITIONS

Bridgeland stability conditions were introduced in [12] and set in a precise mathematical framework many concepts considered in Douglas' work on the Π-stability of D-branes [17]. Bridgeland stability conditions yield a geometric object (the space of stability conditions) associated to a triangulated category , and provide a finer collection of hearts indexed not just by the integers (e.g. as in Proposition 3.11), but by the reals.

Here we introduce Bridgeland stability conditions in general, and discuss deforming stability conditions. In the next chapter we consider the Bridgeland stability of line bundles for a certain class of stability conditions defined for surfaces.

DEFINITION 3.15. A *slicing* $\mathcal{P}$ of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying the following axioms:

(1) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(2) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $\operatorname{Hom}_\mathcal{D}(A_1, A_2) = 0$,

(3) for each nonzero object $E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a filtration through triangles

with $0 \neq A_i \in \mathcal{P}(\phi_i)$ for all i.

As in Proposition 3.11, these filtrations are uniquely defined up to isomorphism and all maps and objects are nonzero. For any $0 \neq E \in \mathcal{D}$ we may thus define $\phi_\mathcal{P}^+(E) = \phi_1$ and $\phi_\mathcal{P}^-(E) = \phi_n$. We then have $\phi_\mathcal{P}^-(E) \leq \phi_\mathcal{P}^+(E)$ with equality if and only if $E \in \mathcal{P}$ for some $\phi \in \mathbb{R}$. Note that a slicing with information concentrated only on the integers is the same information as a heart.

For any interval $I \subset \mathbb{R}$, we define $\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle$. A useful fact is that

$\mathcal{P}([a, b)) =: \mathcal{P}[a, b) = \{0 \neq E \in \mathcal{D} \mid a \leq \phi^-(E) \leq \phi^+(E) < b\}$. To show this, one uses the

filtrations above, as well as Lemma 3.12 and the definition of an extension closed subcategory.

Now, for any $\phi \in \mathbb{R}$, the pair $(\mathcal{P}(> \phi), \mathcal{P}(\leq \phi + 1))$ is a t-structure of $\mathcal{D}$ by the axioms

in Definition 3.15 (note that if $\mathcal{D}^{\geq 0} = \mathcal{P}(\leq \phi + 1)$, then $\mathcal{D}^{\geq 1} = \mathcal{P}(\leq \phi)$). Also, the pair

$(\mathcal{P}(\geq \phi), \mathcal{P}(< \phi + 1))$ is a t-structure. These t-structes give the hearts $\mathcal{P}(\phi, \phi + 1]$ (resp.

$\mathcal{P}[\phi, \phi + 1))$, where ϕ is any real number. For convention's sake, we define *the heart of the*

slicing $\mathcal{P}$ as $\mathcal{P}(0, 1]$.

The following proposition from [23, p. 658] shows that the categories $\mathcal{P}[\phi, \phi + 1)$ are

minimal in some sense.

PROPOSITION 3.16. *Let $\mathcal{P}$ be a slicing of the triangulated category $\mathcal{D}$. Assume that $\mathcal{A}$ is*

a full abelian subcategory of $\mathcal{P}[\phi, \phi + 1)$ and the heart of a bounded t-structure on $\mathcal{D}$. Then

$\mathcal{A} = \mathcal{P}[\phi, \phi + 1)$.

In Definition 3.18, we define a stability condition as a slicing together with the information

of a related additive map. We will see in Proposition 3.20 that the additive map will allow

us to create an appropriate slicing given just a heart of the triangulated category. We first

give a preliminary definition.

DEFINITION 3.17. The *Grothendieck group* $K(\mathcal{D})$ of a triangulated category $\mathcal{D}$, is the

free abelian group generated by the objects of $\mathcal{D}$ with the relations $B = A + C$ whenever

$A \to B \to C$ is a triangle in $\mathcal{D}$.

DEFINITION 3.18. A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ and a slicing $\mathcal{P}$ of $\mathcal{D}$ such that if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.

The map Z is called the *central charge* of the stability condition. The nonzero objects of $\mathcal{P}(\phi)$ are said to be *semistable* in σ of *phase* ϕ, and are called *stable* if they are also simple (i.e. no subobjects). One can show that each $\mathcal{P}(\phi)$ is an abelian category (see [12, p. 331]).

The definition of a stability condition above seems undesirable because it appears that we would have to hand pick the semistable objects in order to create one. However, the next result shows that there is a natural process that yields a stability condition (and is in fact equivalent to the above definition). Before we give it, however, we must give a few definitions.

DEFINITION 3.19.   (1) A *slope function* Z on a heart $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for $0 \neq E \in \mathcal{A}$, $Z(E)$ lies in $H := \{r\exp(i\pi\phi) \mid r > 0, \text{ and } 0 < \phi \leq 1\}$.

(2) We define the *slope* of $E \neq 0$ to be $\phi(E) = \frac{1}{\pi}arg\, Z(E) \in (0, 1]$.

(3) We say that $0 \neq A \in \mathcal{A}$ is $Z-semistable$ if for all subobjects $0 \neq C \subset A$ we have

$\phi(A) \geq \phi(C)$.

(4) Finally, we say that Z has the *Harder-Narasimhan (HN-) property* if for all $E \neq 0$ in $\mathcal{A}$ we have a finite filtration of short exact sequences in $\mathcal{A}$ (which we still draw as triangles)

such that $0 \neq A_i$ is $Z-$semistable for all i and $\phi(A_1) > \cdots > \phi(A_n)$.

PROPOSITION 3.20. *To give a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a centered slope-function on its heart with the Harder-Narasimhan property.*

Bridgeland gives a proof in [12] - we make a few notes here: Going left to right in the above proof, we use the heart $\mathcal{P}(0, 1]$ and the induced function Z gives the the slope fuction with HN-property. It turns out that for all $\phi \in (0, 1]$, we have $\mathcal{P}(\phi) = \{Z-$semistable objects of slope $\phi\}$. Going right to left (the "more useful" direction), we *define* for all $\phi \in (0, 1]$ that $\mathcal{P}(\phi) = \{Z$-semistable objects of slope $\phi\}$. Shifting these $\mathcal{P}(\phi)$ around with the shift functor gives the slicing. In order to obtain the filtrations needed for a stability condition, we first use the heart to obtain a filtration, then we use the octahedral axiom to combine this filtration with the filtrations in $\mathcal{A}$ of each of the $A_i \in \mathcal{A}[k_i]$. This proof is another that would be very useful to work through. Note that, for a heart $\mathcal{A}$, using the filtrations with the heart, we see that $K(\mathcal{A}) = K(\mathcal{D})$.

A useful property that slope functions have is what is called the *see-saw property*. What this says is that, if $A \to E \to B$ is a short exact sequence in $\mathcal{A}$, then $\phi(A) < \phi(E) \Leftrightarrow \phi(E) < \phi(B)$ and $\phi(A) > \phi(E) \Leftrightarrow \phi(E) > \phi(B)$. The fact that centered slope-functions have this can be seen easily by how complex numbers are added. Also, we can replace the the inequalities above with equalities. Using the see-saw property, we see that we may equivalently define an object $0 \neq E \in \mathcal{A}$ to be $Z-$semistable if, for every quotient $E \twoheadrightarrow B$ we have $\phi(B) \geq \phi(E)$.

Showing that a centered slope-function has HN-propterty can be challenging, but can also come for free, for instance, if $\mathcal{A}$ has finite length, as the next proposition shows. It's

proof uses maximally destabilizing quotients (mdqs) to iteratively construct the HN-filtration (find the mdq of E, take the kernel of the map, find the mdq of the kernel, etc.). The reader should consult [12, p. 324] for the precise definition of an mdq.

PROPOSITION 3.21. *Suppose $\mathcal{A}$ is an abelian category with a centered slope-function $Z :$ $K(\mathcal{A}) \to \mathbb{C}$ satisfying the chain conditions*

(1) there are no infinite sequences of subobjects in $\mathcal{A}$

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\phi(E_{j+1}) > \phi(E_j)$ for all j,

(2) there are no infinite sequences of quotients in $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

with $\phi(E_{j+1}) < \phi(E_j)$ for all j.

Then $\mathcal{A}$ has the HN-property.

PROOF. [12, p. 323] □

In order to throw out some undesirable stability conditions, we make the following definition.

DEFINITION 3.22. A stability condition $\sigma = (\mathcal{P}, Z)$ is called locally finite if there is an $\epsilon > 0$ such that $\mathcal{P}(\phi - \epsilon, \phi + \epsilon)$ is a category of finite length for all ϕ in $\mathbb{R}$.

DEFINITION 3.23. For a triangulated category $\mathcal{D}$, $\mathrm{Stab}(\mathcal{D})$ is the set of all locally-finite stability conditions on $\mathcal{D}$.

In order to obtain a well-behaved wall and chamber structure for chosen invariants in the space of stability conditions, in practice one often imposes the extra condition of a stability condition being "full" or satisfying the "support property." See Remark 4.3 for details.

3.2.1. STAB(D) AS A TOPOLOGICAL SPACE. Bridgeland puts a topology on Stab($\mathcal{D}$) using the following generalized metric. Here, for $E \neq 0$, its *mass*, $m(E)$, is defined to be $\sum_i |Z(A_i)|$, where the A_i are the semistable quotients in the HN-filtration of E. The metric is

$$d(\sigma, \gamma) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_\sigma^+(E) - \phi_\gamma^+(E)|, |\phi_\sigma^-(E) - \phi_\gamma^-(E)|, |\log \frac{m_\sigma(E)}{m_\gamma(E)}| \right\}$$

Bridgeland then shows that Stab($\mathcal{D}$) is in fact a manifold. To obtain finite dimensionality, we now assume that for $\mathcal{D}$, either $K(\mathcal{D})$ has finite rank, or that the numerical Grothendieck group $N(\mathcal{D})$ has finite rank and $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through $N(\mathcal{D})$. See [12, p. 319] for the definition of $N(\mathcal{D})$. Set $K = K(\mathcal{D})$ or $N(\mathcal{D})$ appropriately, and note that $\text{Hom}_\mathbb{Z}(K, \mathbb{C})$ is thus a finite dimensional vector space. We then have the following.

THEOREM 3.24. *For each connected component* $\Sigma \subset Stab(\mathcal{D})$ *there is a subspace* $V(E) \subset Hom_\mathbb{Z}(K, \mathbb{C})$ *and a local homeomorphism* $\mathfrak{Z} : \Sigma \to V(E)$ *that sends a stability condition to its central charge* Z. *In particular,* Σ *is a finite-dimensional complex manifold.*

The point is that Stab($\mathcal{D}$) is a manifold that one that one can move around on by deforming the central charge. The following discussion shows in more detail how this works, i.e. how to find a matching slicing after slightly deforming the central charge.

The idea is to deform your central charge slightly, then look at sectors of the plane and see which objects now are or are not semistable, using the objects that were semistable before the deformation as a starting place.

More specifically, let $\sigma = (\mathcal{P}, Z)$ be a stability condition and let Z' be the central charge we get by deforming Z slightly. We wish to find the corresponding slicing, $\mathcal{P}'$. Let $\epsilon > 0$ be "small" and $\phi \in \mathbb{R}$. We define $\mathcal{A}_\epsilon^\phi := \mathcal{P}(\phi - \epsilon, \phi + \epsilon)$. Let us assume that Z' sends $\mathcal{A}_\epsilon^\phi$ to some slightly bigger sector (i.e. that our deformation only moves the sector $\mathcal{A}_\epsilon^\phi$ slightly). We can now update our slicing near the phase ϕ to account for our new central charge.

To do this, for ϕ' near ϕ we let $\mathcal{P}'(\phi')$ be the objects of $\mathcal{A}_\epsilon^\phi$ that are $Z'-$semistable and now have slope ϕ' under Z'. (Recall the definition of $Z'-$semistability from Definition 3.19 (4)). More precisely, we consider $\mathcal{A}_\epsilon^\phi$, map it under Z', and consider Z' as a slope function on $\mathcal{A}_\epsilon^\phi$. That is, for any object $0 \neq E \in \mathcal{A}_\epsilon^\phi$ define its slope under Z', $\phi'(E)$ to be the appropriate slope near ϕ. Then, for ϕ' near ϕ, we let $\mathcal{P}'(\phi')$ be the subcategory of objects of $\mathcal{A}_\epsilon^\phi$ that have slope ϕ' under Z' and are $Z'-$semistable.

We must be careful about what we mean by "subobject" in the definition of $Z'-$semistable, as $\mathcal{A}_\epsilon^\phi$ is *not* in general an abelian category. It is, however, a *quasi-abelian category* (see [12, p. 328-331]). In these categories, we have a replacement for subobject and quotient: we say that, for $0 \neq E \in \mathcal{A}_\epsilon^\phi$ that A is a (strict) subobject of E (and that B is a (strict) quotient of E) if $A \to E \to B$ is a triangle in $\mathcal{D}$ and $A, B \in \mathcal{A}_\epsilon^\phi$.

3.3. Actions on Stab($\mathcal{D}$)

We now consider two actions on the stability manifold from [12, pp. 342-343]. The first amounts to combinations of rotation, stretching and shearing *without* changing orientation. The second uses the symmetry of the triangulated category $\mathcal{D}$ to exchange elements in a given stability condition. Because these actions do not yield substantially different stability conditions, it is reasonable to mod out by them and this can simplify the space of stability conditions.

PROPOSITION 3.25. *The generalized metric space* $Stab(\mathcal{D})$ *carries the following two actions. These actions commute.*

(1) A right action of the group $\widetilde{GL}^+(2,\mathbb{R})$, *the universal covering space of* $GL^+(2,\mathbb{R})$, *that is* $\{(T,f) \mid f : \mathbb{R} \to \mathbb{R}$ *an increasing map with* $f(\phi+1) = f(\phi)+1$, *and* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is an orientation-preserving linear isomorphism, such that their induced maps on* $S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2/\mathbb{R}_{>0}$ *are the same.} Let* $(Z,\mathcal{P}) \in Stab(\mathcal{D})$ *and* $(T,f) \in \widetilde{GL}^+(2,\mathbb{R})$. *Then* $(T,f) \bullet (Z,\mathcal{P}) := (T^{-1} \circ Z, \mathcal{P}')$, *where* $\mathcal{P}'(t) = \mathcal{P}(f(t))$.

(2) A left action by $Aut(\mathcal{D})$ *of exact autoequivalences (i.e. triangles are sent to triangles) of* $\mathcal{D}$. *Note that* $\Psi \in Aut(\mathcal{D})$ *induces an automorphism* ψ *of* $K(\mathcal{D})$. *Let* $\Psi \in Aut(\mathcal{D})$. *Then* $\Psi \bullet (Z,\mathcal{P}) := (Z \circ \psi^{-1}, \widetilde{\mathcal{P}})$, *where* $\widetilde{\mathcal{P}}(t) = \Psi(\mathcal{P}(t))$.

Note that the action of $\widetilde{GL}^+(2,\mathbb{R})$ only relabels the phases of semistable objects, it does not change which objects are semistable. In our study of the Bridgeland stability of line bundles on surfaces, we use the action of autoequivalences given by tensoring by line bundles (Lemma 4.7), and use rotation to identify certain stability conditions with descriptions in terms of quivers (Section 6.2).

CHAPTER 4

BRIDGELAND STABILITY OF LINE BUNDLES ON SURFACES

Let S be a smooth projective surface. In this paper, we study Bridgeland stability for line bundles on S using the geometric Bridgeland stability conditions introduced in [2] (see Section 4.1.2 for a precise definition). Bridgeland stability conditions can be seen as an extention of Mumford μ-stability for sheaves to complexes of sheaves in the derived category, $D^b(\operatorname{Coh} S)$. Line bundles are always Mumford slope-stable, as their only subobjects are ideal sheaves, but the situation is less constrained in the derived setting. For example, in the abelian subcategories we consider, a subobject of a line bundle is a sheaf, but may a priori have arbitrarily high rank. The quotient is a possibly two-term complex.

One might still expect line bundles to always be Bridgeland stable, and this is correct if S has no curves C of negative self-intersection (see the first part of Theorem 4.1 below). However, if there exists a curve C on S of negative self-intersection, then $L(-C)$ destabilizes L for some Bridgeland stability conditions, and $L(C)|_C$ destabilizes $L[1]$. We make the following conjecture.

CONJECTURE. *Given a surface S and a stability condition $\sigma_{H,D}$ as in [2],*

- *the only objects that could destabilize a line bundle L are line bundles of the form $L(-C)$ for a curve C of negative self-intersection, and*

- *the only objects that could destabilize $L[1]$ are torsion sheaves of the form $L(C)|_C$ for a curve C of negative self-intersection.*

The goal of this paper is to prove the conjecture in several cases, and provide evidence for the conjecture for others. Specifically, we prove the following.

THEOREM 4.1. *The conjecture is true in the following cases:*

- *If S does not have any curves of negative self-intersection.*

- *If the Picard rank of S is 2, and there exists only one irreducible curve of negative self-intersection.*

In particular, the conjecture is true for Hirzebruch surfaces. We cite Propositions 4.21 and 4.26 as further evidence for our conjecture in general. Proposition 4.21 estabilishes some structure of actually destabilizing subobjects for line bundles and their walls for surfaces of any Picard rank, while Proposition 4.26 proves a stronger version of the conjecture for a subset of stability conditions when S has Picard rank 2 and two irreducible curves of negative self-intersection.

In [2] the stability of line bundles is proven for stability conditions $\sigma_{D,H}$ with $D = sH$, and is utilized in [3] and [9] while classifying destabilizing walls for ideal sheaves of points on surfaces. When $D \neq sH$ the more algebraic proof of [2] using the Bogomolov inequality and Hodge Index Theorem fails and new techniques are required. We use Theorem 3.1 (Bertram's Nested Wall Theorem) from [22] and Lemma 6.3 (which we refer to as Bertram's Lemma) from [3] along with an analysis of the relative geometry of relevant walls in certain three-dimensional slices of the space of stability conditions. This technique lends itself well to induction, which is the primary method of proof used here.

We begin in Section 4.1 by introducing Bridgeland stability conditions, the stability conditions $\sigma_{D,H}$ of interest, as well as important slices of the space of stability conditions. In Section 4.2 we present an action by line bundles which allows us to consider only $\mathcal{O}_S$ in our questions of the stability of line bundles. In Section 4.3 we consider the basic structure of subobjects of $\mathcal{O}_S$ as well as present two already known results which will serve as important

tools in the remainder. In Section 4.4 we prove our main results, but first consider the rank 1 subobjects of $\mathcal{O}_S$. The rank 1 subobjects form the base case for our main results, all of which use induction. The case of $\mathcal{O}_S[1]$ is then completed primarily using duality, which allows us to use our results for $\mathcal{O}_S$ except when $D.H = 0$.

4.1. Bridgeland Stability Conditions

Bridgeland stability conditions (introduced in [12]) give a notion of stability on the derived category of a variety. They generalize other classical notions of stability conditions, e.g., Mumford-slope stability. As with slope stability, we may deform our stability conditions (Bridgeland showed that the space of all stability conditions is a complex manifold) and the stability of objects can change. We first introduce these stability conditions in general, and then restrict our attention to surfaces in the next section. Our goal is to study Bridgeland stability of line bundles.

4.1.1. General Definition of Bridgeland Stability Conditions. Let X be a smooth projective variety, $D(X) = D^b(\mathrm{Coh}\,S)$ the bounded derived category of coherent sheaves on X, $K(X)$ its Grothendieck group, and $K_{num}(X)$ its quotient by the subgroup of classes F such that $\chi(E, F) = 0$ for all $E \in D(X)$.

Definition 4.2. A *full numerical stability condition* on X is a pair $\sigma = (Z, \mathcal{A})$ where

- $\mathcal{A}$ is a *heart* of $D(X)$

- $Z : K_{num}(X) \to \mathbb{C}$ a group homomorphism called the *central charge*

satisfying properties 1,2 and 3 below.

1 (Positivity): For all $0 \neq E \in \mathcal{A}$, $Z(E) \in \{re^{i\pi\varphi} \mid r > 0, 0 < \varphi \leq 1\}$.

To discuss stability for a given stability condition, we define for each $E \in D(X)$

$$\beta(E) = -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)} \in (-\infty, \infty]$$

For example, if $Z(E) = -1$ then $\beta(E) = \infty$, and if $Z(E) = \sqrt{-1}$ then $\beta(E) = 0$.

We say that $E \in \mathcal{A}$ is σ-stable (resp. σ-semistable) if for all nontrivial $F \hookrightarrow E$ in $\mathcal{A}$ we have $\beta(E) > \beta(F)$ (resp. $\beta(E) \geq \beta(F)$).

2 (Harder-Narasimhan Filtrations): For all $E \in \mathcal{A}$ there exist objects $E_1, \ldots, E_{n-1} \in \mathcal{A}$ such that

- $0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = E$ in $\mathcal{A}$

- E_{i+1}/E_i is σ-semistable for each i

- $\beta(E_1/E_0) > \beta(E_2/E_1) > \cdots > \beta(E_n/E_{n-1})$

3 (Support Property): Choose a norm $\|.\|$ on $K_{num}(X) \otimes \mathbb{R}$. There exist a $C > 0$ such that for all σ-semistable $E \in D(X)$ we have $C\|E\| \leq |Z(E)|$.

REMARK 4.3. The support property guarantees us a nicely behaved wall and chamber structure for classes of objects - namely, the walls are locally finite, real codimension 1 submanifolds of the stability manifold and deleting the walls gives chambers where Bridgeland stability is constant (see [7, Proposition 3.3]). The support property is equivalent to Bridgeland's notion of *full*, see [7, Proposition B.4].

We will say **stability condition** to mean full numerical stability condition.

4.1.2. BRIDGELAND STABILITY CONDITIONS ON A SURFACE. Let S be a smooth projective surface. The stability conditions that we are going to consider were defined in [2]. They form a subset $\operatorname{Stab}_{div}(S)$ of stability conditions that depend on a choice of ample and

general divisor (the "div" stands for "divisor"), and are well suited to computations. Let us recall their definition.

Let S be a smooth projective surface. Given two $\mathbb{R}$-divisors D, H with H ample, we define a stability condition $\sigma_{D,H} = (Z_{D,H}, \mathcal{A}_{D,H})$ on S as follows:

Consider the H-Mumford slope

$$\mu_H(E) = \frac{c_1(E).H}{\mathrm{rk}(E)H^2}.$$

Let $\mathcal{A}_{D,H}$ be the tilt of the standard t-structure on $D(S)$ at $\mu_H(D) = \dfrac{D.H}{H^2}$ defined by $\mathcal{A}_{D,H} = \{E \in D(S) \mid H^i(E) = 0 \text{ for } i \neq -1, 0, \ H^{-1}(E) \in \mathcal{F}_{D,H}, \ H^0(E) \in \mathcal{T}_{D,H}\}$ where

- $\mathcal{T}_{D,H} \subset \mathrm{Coh}(S)$ is generated by torsion sheaves and μ_H-stable sheaves E with $\mu_H(E) > \dfrac{D.H}{H^2}$.
- $\mathcal{F}_{D,H} \subset \mathrm{Coh}(S)$ is generated by μ_H-stable sheaves F with $\mu_H(F) \leq \dfrac{D.H}{H^2}$.

Now define $Z_{D,H}$ by $Z_{D,H}(E) = -\int e^{-(D+iH)}\mathrm{ch}(E)$. It is equal to

$$Z_{D,H}(E) = \left(-\mathrm{ch}_2(E) + c_1(E).D - \frac{\mathrm{rk}(E)}{2}(D^2 - H^2)\right) + i\left(c_1(E).H - \mathrm{rk}(E)D.H\right)$$

By [2, Corollary 2.1] and [29, Sections 3.6 & 3.7], $\sigma_{D,H}$ is a stability condition on S. Let $\mathrm{Stab}_{div}(S)$ be the set of all such stability conditions. By the support property, $\mathrm{Stab}_{div}(S) \cong (\mathrm{Amp}(S) \oplus \mathrm{Pic}(S))_{\mathbb{R}}$ is a submanifold of the space of all stability conditions.

REMARK 4.4. These are *geometric* stability conditions since for all $p \in S$, the skyscraper sheaf $\mathbb{C}_p \in \mathcal{A}_{D,H}$ is $\sigma_{D,H}$-stable with $Z_{D,H}(\mathbb{C}_p) = -1$ (the proof is the same as [3, Proposition 6.2.a]).

Note. When the D and H divisors have been fixed, we will often drop the D, H subscript from σ, Z, $\mathcal{A}$, $\mathcal{T}$, and $\mathcal{F}$.

4.1.3. SLICES OF $\mathrm{STAB}_{div}(S)$. One of the features that makes $\mathrm{Stab}_{div}(S)$ well suited to computations is its decomposition into well-behaved 3-spaces, each given by a choice of ample divisor and another divisor orthogonal to it. In these 3-spaces, walls of interest will be quadric surfaces and most of our work will begin by first choosing a particular 3-space to live in. Most of these concepts where introduced in [22].

Let H be an ample divisor such that $H^2 = 1$. If S has Picard rank 1, then the stability conditions in $\mathrm{Stab}_{div}(S)$ are all of the form $\sigma_{sH,tH}$. It was already proved in [2] that line bundles are always Bridgeland stable for these stability conditions.

Assume from now on that S has Picard rank greater than 1.

DEFINITION 4.5. Choose a divisor G with $G.H = 0$ and $G^2 = -1$ (note that $G^2 \leq 0$ by the Hodge Index Theorem with $G^2 = 0$ iff $G = 0$). Then, define $\mathcal{S}_{H,G} := \{\sigma_{sH+uG,tH} \mid s, u, t \in \mathbb{R}, t > 0\} \subset \mathrm{Stab}_{div}(S)$.

From now on we assume that any divisors H, G are as above. We identify $\mathcal{S}_{H,G}$ with $\{(s, u, t) \mid t > 0\}$ by $(s, u, t) \leftrightarrow \sigma_{sH+uG,tH}$.

Each of the stability conditions $\sigma_{D,H}$ defined in 4.1.2 can be seen as an element of a particular 3-dimensional slice. Indeed, we can just scale the H to ensure that $H^2 = 1$, and then choose G such that $D = sH + uG$ so that $\sigma_{D,H} \in \mathcal{S}_{H,G}$. Thus these slices cover all of $\mathrm{Stab}_{div}(S)$.

Note. Though these spaces do not contain the plane $t = 0$, for convenience we will treat them as if they do. Also, we write (s, u) to mean $(s, u, 0)$ and identify $\sigma = \sigma_{sH+uG,tH} = (s, u, t)$.

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Let $E \in D(S)$ and set $\mathrm{ch}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)) = (r, c_1(E), c)$. We may write $c_1(E) = d_h H + d_g G + \alpha$ where $\alpha.H = \alpha.G = 0$ and $d_h, d_g \in \mathbb{R}$. Specifically, we have $d_h = c_1(E).H$ and $d_g = -c_1(E).G$.

REMARK 4.6. Given $\mathcal{S}_{H,G}$, the equality $\mu_H(sH + uG) = \mu_H(E)$ is equivalent to $s = \dfrac{\mathrm{ch}_1(E).H}{rH^2} = \mu_H(E)$, and $\mu_H(sH + uG) < \mu_H(E)$ iff $s < \mu_H(E)$.

The vertical plane $s = \mu_H(E)$ is very important, because, if E is a μ_H-semistable sheaf, then $s < \mu_H(E)$ iff $E \in \mathcal{A}_{sH+uG,H}$, and $s \geq \mu_H(E)$ iff $E[1] \in \mathcal{A}_{sH+uG,H}$.

For each fixed value of u, we denote by Π_u the vertical plane of stability conditions (s, u, t) of fixed u-value. It is parametrized by (s, t), and it will play a special role in our work (see Section 4.3.1 below for more details).

The central charge of a stability condition $\sigma = \sigma_{sH+uG,tH}$ for an object E with $\mathrm{ch}(E) = (r, d_h H + d_g G + \alpha, c)$ is equal to

$$Z(E) = \left(-c + s d_h - u d_g - \frac{r}{2}(s^2 - u^2 - t^2) \right) + i\,(t d_h - rst)\,.$$

4.2. REDUCTION TO THE CASE OF $\mathcal{O}_S$

The action of tensoring stability conditions by line bundles will allow us to restrict our attention from the stability of all line bundles to that of $\mathcal{O}_S, \mathcal{O}_S[1]$. Proving the action on $\mathrm{Stab}(S)$ descends to $\mathrm{Stab}_{div}(S)$ is straightforward but we provide it here for completeness.

LEMMA 4.7. *Let S be a smooth projective surface with line bundle $\mathcal{O}_S(D')$. For any $\sigma_{D,H} \in \mathrm{Stab}_{div}(S)$, we have $\mathcal{O}_S(D') \otimes \sigma = \sigma_{D+D',H}$.*

PROOF. For an autoequivalence Φ and stability condition $\sigma = (Z, \mathcal{A})$, let $\Phi\sigma = (\Phi Z, \Phi \mathcal{A})$.

We first consider the central charge: We have $(\Phi Z)(E) = Z(\Phi^{-1}E)$. Thus, for $\Phi = \mathcal{O}_S(D') \otimes -$, we have

$$(\Phi Z)(E) = Z_{D,H}(\Phi^{-1}E)$$

$$= Z_{D,H}(\mathcal{O}_S(-D') \otimes E)$$

$$= -\int e^{-(D+iH)}\mathrm{ch}(\mathcal{O}_S(-D') \otimes E)$$

$$= -\int e^{-(D+iH)}\mathrm{ch}(\mathcal{O}_S(-D')).\mathrm{ch}(E)$$

$$= -\int e^{-(D+iH)}\left(1[S] - D' + \frac{(-D')^2}{2}[pt]\right).\mathrm{ch}(E)$$

$$= -\int e^{-(D+iH)}e^{-D'}.\mathrm{ch}(E)$$

$$= -\int e^{-(D+D'+iH)}\mathrm{ch}(E)$$

$$= Z_{D+D',H}(E)$$

Next, the heart: we have $\Phi\mathcal{A} = \Phi(\mathcal{A})$, i.e. the image of $\mathcal{A}$ under the autoequivalence Φ. Since

$$E \in \mathrm{Coh}(S) \text{ is } \mu_H - \text{stable with } \mu_H(E) > (\text{resp. } \geq) \, \mu_H(D) \text{ iff}$$

$$F := \mathcal{O}(D') \otimes E \text{ is } \mu_H - \text{stable with } \mu_H(F) > (\text{resp. } \geq) \, \mu_H(D + D')$$

we have that

$$\Phi\mathcal{A} = \Phi(\mathcal{A}_{D,H})$$

$$= \langle E \in \mathrm{Coh}(S), \mu_H\text{-stable with } \mu_H(E) > \mu_H(D + D');$$

$$F[1], \text{ where } F \in \mathrm{Coh}(S) \text{ is } \mu_H\text{-stable with } \mu_H(E) \leq \mu_H(D + D'))\rangle$$

$$= \mathcal{A}_{D+D',H}$$

We have shown $\mathcal{O}_S(D') \otimes \sigma_{D,H} = \sigma_{D+D',H}$. $\qquad\qquad\square$

As an immediate corollary we have that $\mathcal{O}_S(D')$ is (semi)stable at $\sigma_{D,H}$ iff $\mathcal{O}_S$ is (semi)stable at $\sigma_{D-D',H}$ (and similarly for $\mathcal{O}_S[1]$).

4.3. Preliminaries on the Stability of $\mathcal{O}_S$

Walls are subsets of the stability manifold where the stability of objects can change. Our main interest lies in in describing the chambers of stability for $\mathcal{O}_S$, which are bounded by walls corresponding to certain destabilizing objects. Let us start with a few definitions, and a description of the possible walls.

4.3.1. Subobjects of $\mathcal{O}_S$ and their walls. First of all, here is our generic definition of a wall.

Definition 4.8. Given two objects $E, B \in D(S)$, with B Bridgeland-stable for at least one stability condition, we define the wall $\mathcal{W}(E, B)$ as $\{\sigma \in \mathrm{Stab}_{div}(S) \mid (\mathrm{Re}\, Z(E))(\mathrm{Im}\, Z(B)) - (\mathrm{Re}\, Z(B))(\mathrm{Im}\, Z(E)) = 0\}$. If at some $\sigma \in \mathcal{W}(E, B)$ we have $E \subset B$ in $\mathcal{A}$, we say that $\mathcal{W}(E, B)$ is a **weakly destabilizing wall** for B. If at some $\sigma \in \mathcal{W}(E, B)$ we have $E \subset B$

in $\mathcal{A}$, and B is Bridgeland σ-semistable, we say that $\mathcal{W}(E, B)$ is an **actually destabilizing wall** for B.

Note that if $\operatorname{Im} Z(E) \neq 0 \neq \operatorname{Im} Z(B)$ then the defining condition is just $\beta(E) = \beta(B)$.

We are interested in the walls for $\mathcal{O}_S$ and $\mathcal{O}_S[1]$, and we start by studying the walls for $\mathcal{O}_S$. Note that, given a Bridgeland stability condition σ, we have $\mathcal{O}_S \in \mathcal{A}$ iff $s < 0$, and $\mathcal{O}_S[1] \in \mathcal{A}$ iff $s \geq 0$.

At each fixed value of u, Maciocia showed in [22, Section 2] that all walls for $\mathcal{O}_S$ in Π_u are nested semicircles centered on the s-axis. Therefore, given two objects E_1 and E_2, and a fixed value of u, we have that $\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u$ and $\mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u$ are both semicircles, with one of them inside the other, unless they are equal.

DEFINITION 4.9. We say that the wall $\mathcal{W}(E_1, \mathcal{O}_S)$ is **inside** the wall $\mathcal{W}(E_2, \mathcal{O}_S)$ at u if the semicircle $\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u$ is inside the semicircle $\mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u$ or equal to it. We will use the notation

$$\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u.$$

LEMMA 4.10. *Let $\sigma \in Stab_{div}(S)$, and let $0 \to E \to \mathcal{O}_S \to Q \to 0$ be a short exact sequence in $\mathcal{A}$. Then E is a torsion-free sheaf, $H^0(Q)$ is a quotient of $\mathcal{O}_S$ of rank 0, and the kernel of the map $\mathcal{O}_S \to H^0(Q)$ is an ideal sheaf $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional scheme Z.*

PROOF. The long exact sequence in cohomology associated to the short exact sequence shows that E must be a sheaf, while Q may have cohomologies in degrees -1 and 0:

$$0 \longrightarrow H^{-1}(Q) \longrightarrow E \longrightarrow \mathcal{O}_S \longrightarrow H^0(Q) \longrightarrow 0.$$

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If $H^0(Q)$ had rank 1, then it would have to be equal to $\mathcal{O}_S$, and we would have that $H^{-1}(Q) = E = 0$, which is impossible. Therefore, $H^0(Q)$ is a quotient of $\mathcal{O}_S$ of rank 0. Since $\mathcal{I}_Z(-C)$ and $H^{-1}(Q)$ are both torsion-free sheaves, E is also a torsion-free sheaf. $\square$

Here we study which forms the walls $\mathcal{W}(E, \mathcal{O}_S)$ can take. The intersection of a wall with the $t = 0$-plane is a conic through the origin, and we classify the wall based on invariants associated to E.

If $\mathrm{ch}(E) = (r, d_h H + d_g G + \alpha_E, c)$, then we saw above that

$$Z(E) = \left(-c + sd_h - ud_g - \frac{r}{2}(s^2 - u^2 - t^2)\right) + i\,(td_h - rst),$$

and the equation of the wall $\mathcal{W}(E, \mathcal{O}_S)$ is

$$\frac{t}{2}(-d_h(s^2 + t^2 + u^2) + 2d_g su + 2cs) = 0.$$

Since $t \neq 0$, this is equivalent to

$$-d_h(s^2 + t^2 + u^2) + 2d_g su + 2cs = 0.$$

It is a quadric, and we will start by studying its intersection with the $t = 0$ plane:

$$-d_h(s^2 + u^2) + 2d_g su + 2cs = 0.$$

We will abuse notation, and still refer to this equation as the wall $\mathcal{W}(E, \mathcal{O}_S)$. The determinant is equal to

$$\Delta = 4(d_g^2 - d_h^2).$$

If $\Delta = 0$, then the wall is a parabola. Since $s < 0$, the wall can only be a weakly destabilizing wall if $c > 0$, in which case the equation of the parabola is $-d_h(s \pm u)^2 + 2cs = 0$.

If $\Delta \neq 0$ and $c \neq 0$, straightforward calculations show the following:

- $0P$ and $2P$ are on the wall, where

$$P = -\frac{c}{d_g^2 - d_h^2}(d_h, d_g).$$

- The tangent line to the wall at $0P$ and $2P$ is vertical, i.e. is $s = \text{constant}$.

- The tangent line to the wall is horizontal (i.e. is $u = \text{constant}$) at the points where the conic intersects $u = s$ and $u = -s$. These are

$$\left(\frac{c}{d_h - d_g}, \frac{c}{d_h - d_g}\right) \text{ and } \left(\frac{c}{d_h + d_g}, -\frac{c}{d_h + d_g}\right).$$

If $\Delta < 0$, then the wall is a weakly destabilizing wall only if $c > 0$, and it is an ellipse.

If $\Delta > 0$, then there are three possibilities:

- If $c = 0$, then the wall is a cone centered at $(0, 0)$.

- If $c > 0$, then the wall is a hyperbola with center P to the right of $s = 0$.

- If $c < 0$, then the wall is a hyperbola with center P to the left of $s = 0$. In this case, the asymptotes have slope

$$\frac{d_g \pm \sqrt{d_g^2 - d_h^2}}{d_h}.$$

Here is a summary of all possible weakly destabilizing walls (pictures drawn for $d_h < 0$ and $d_g > 0$):

Parabola.: When $d_g^2 - d_h^2 = 0$ and $c > 0$.

Ellipse.: When $d_g^2 - d_h^2 < 0$ and $c > 0$.

Cone.: When $d_g^2 - d_h^2 > 0$ and $c = 0$.

Right Hyperbola.: When $d_g^2 - d_h^2 > 0$ and $c > 0$.

Left Hyperbola.: When $d_g^2 - d_h^2 > 0$ and $c < 0$.
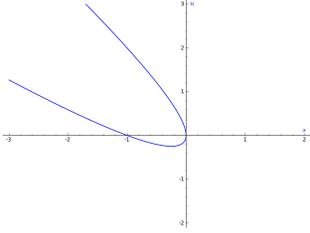
FIGURE 4.1. Parabola
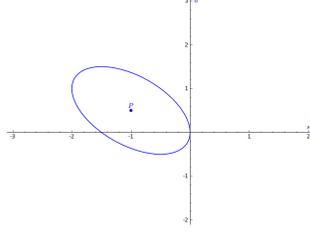
FIGURE 4.2. Ellipse
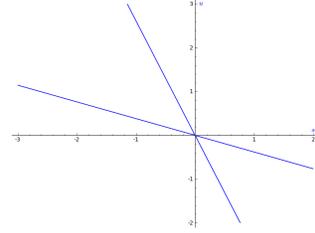
FIGURE 4.3. Cone
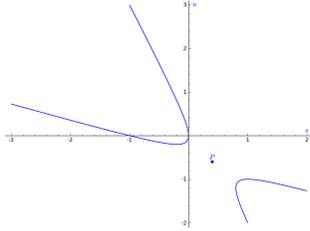
FIGURE 4.4. Right hyperbola
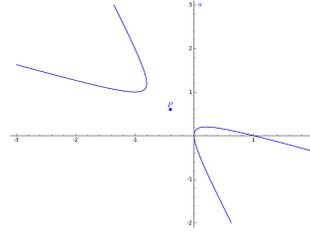
FIGURE 4.5. Left hyperbola

We end this section by pointing out an important geometric property of these walls that will be useful in various proofs later in the paper.

LEMMA 4.11. *Given two subobjects E_1 and E_2 of $\mathcal{O}_S$ in $\mathcal{A}$, there exists at most one value of $u \neq 0$ such that*

$$\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u,$$

unless the two walls coincide everywhere.

PROOF. Suppose that $\mathcal{W}(E_1, \mathcal{O}_S)$ and $\mathcal{W}(E_2, \mathcal{O}_S)$ do not coincide everywhere. Looking at the intersection of the walls with the $t = 0$ plane, we see that they are two conics that intersect at $(0, 0)$ with multiplicity two. Therefore, they can only intersect at at most two

other points there. Since the walls $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ are nested semicircles centered on the s-axis, they intersect the $t = 0$ plane. Therefore, if $\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u$, the walls coincide at the two points where the semicircle intersects the $t = 0$ plane, and cannot intersect anywhere else there except at $(0, 0)$. Therefore, the walls cannot coincide at any other value of $u \neq 0$. $\qquad\square$

4.3.2. BRIDGELAND STABILITY OF $\mathcal{O}_S$ FOR $t \gg 0$. In this section, we prove that $\mathcal{O}_S$ is Bridgeland stable in $\mathcal{S}_{H,G}$ for $t \gg 0$. This fact is already known. For example, it follows from the result in [22] that walls in each plane of the form $u = $ constant are disjoint circles that are bounded above. We give however a new proof that would easily generalize to the case of an object of the form E or $E[1]$ with E a μ-stable sheaf.

PROPOSITION 4.12. *Let H and G be as above, and fix a divisor $sH + uG$ with $s < 0$. Then $\mathcal{O}_S$ is Bridgeland stable for the stability condition $\sigma_{sH+uG,tH}$ for $t \gg 0$.*

PROOF. Let $\mathcal{A} = \mathcal{A}_{sH+uG,tH}$ (note that $\mathcal{A}_{sH+uG,tH}$ is independent of t), and let E be subobject of $\mathcal{O}_S$ in $\mathcal{A}$. As we saw above, E is torsion-free. Moreover, since $0 < \operatorname{Im} Z(E) < \operatorname{Im} Z(\mathcal{O}_S)$ for all t, we have that $\mu_H(E) < \mu_H(\mathcal{O}_S) = 0$. This implies that $\beta(E) < \beta(\mathcal{O}_S)$ for $t \gg 0$. From our analysis of the possible walls above, we know that, for a fixed (s, u), there is at most one value of t such that $\beta(E) = \beta(\mathcal{O}_S)$. Moreover, if $\beta(E) < \beta(\mathcal{O}_S)$ at a given σ_{s,u,t_0} then the same inequality holds for all $t > t_0$.

Hence the only way $\mathcal{O}_S$ could not be Bridgeland stable for large t is if there are infinitely many chern characters corresponding to subobjects E of $\mathcal{O}_S$ with $\beta(E) \geq \beta(\mathcal{O}_S)$.

Suppose this is so and fix a value of t (in particular, we can now assume that the subobjects in question are Bridgeland semistable). Now, for the fixed stability condition $\sigma_{sH+uG,tH}$,

$\mathcal{O}_S$ has a HN-filtration of Bridgeland semistable objects $\mathcal{O}_S = \langle L_i \rangle_1^l$ with L_1 torsion-free and thus $\operatorname{Im} Z(L_1) > 0$ and $\beta(L_1) < \infty$. We have that $E \in \mathcal{A}$ and $E \subset \mathcal{O}_S$ imply $\beta(L_1) \geq \beta(E)$ and since we also have $\operatorname{Im} Z(E) < \operatorname{Im} Z(\mathcal{O}_S)$ the images of these semistable subobjects are forced into a finite triangular region of the upper half-plane, specifically the region bounded by the ray through $Z(\mathcal{O}_S)$, the ray through $Z(L_1)$, and the horizontal line $\operatorname{Im} z = \operatorname{Im} Z(\mathcal{O}_S)$. But this is impossible by the support property. Thus $\mathcal{O}_S$ is Bridgeland stable for large t. $\quad\square$

REMARK 4.13. Even when fixing H and G, there is not necessarily a t_0 such that $\mathcal{O}_S$ is Bridgeland stable for all (s, u, t) with $t > t_0$. As a matter of fact, due to the nature of the rank 1 walls, which are hyperboloids, whenever $\mathcal{O}_S$ is not Bridgeland stable somewhere in a space $\mathcal{S}_{H,G}$, there will be stability conditions for any t for which $\mathcal{O}_S$ is not Bridgeland stable. However, for every fixed value of u, Maciocia proved in [22] that there exists a value $t_0(u)$ such that $\mathcal{O}_S$ is Bridgeland stable for $\sigma_{sH+uG,tH}$ for all $s < 0$ and $t > t_0(u)$. What happens is that the $t_0(u)$, if it is not equal to 0, goes to ∞ as u goes to $\pm\infty$.

4.3.3. BERTRAM'S LEMMA. The following lemma is a key tool in the proof of the main theorem. The lemma is essentially Lemma 6.3 in [3], adapted to our situation. It allows us to, in some situations, find walls higher than a given wall for $\mathcal{O}_S$ by removing a Mumford semistable factor from the subobject or quotient. In either case, rank strictly drops and this sets the stage for our induction proof characterizing the stability of $\mathcal{O}_S$.

Let E be a subobject of $\mathcal{O}_S$ of rank ≥ 2 in $\mathcal{A}$, and let Q be the quotient.

Let $0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$ be the Harder-Narasihman filtration of E, and let $K_i = E_i/E_{i-1}$ (so that $K_1 = E_1$ and $K_n = E/E_{n-1}$). We have that $\mu_H(K_1) > \mu_H(K_2) > \cdots > \mu_H(K_n)$. Also, let $\underline{K} = K_n$.

Similarly, let $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{m-1} \subseteq F_m = H^{-1}(Q)$ be the Harder-Narasihman filtration of $H^{-1}(Q)$, and let $J_i = F_i/F_{i-1}$ with $\overline{J} = J_1$.

We have that $E \subseteq \mathcal{O}_S \in \mathcal{A}$ iff $\mu_H(\overline{J}) \leq s < \mu_H(\underline{K})$.

At $s = \mu_H(\underline{K})$, we will consider the natural subsheaf $E_{n-1} \subseteq E$. At $s = \mu_H(\overline{J})$, we will consider the natural quotient sheaf $E \twoheadrightarrow E/\overline{J}$ (note that, as sheaves, $\overline{J} = J_1 = F_1 \subseteq H^{-1}(Q) \subseteq E$).

LEMMA 4.14 (Bertram's Lemma). *Fix H and G as above, and let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ for some $\sigma = \sigma_{sH+uG,tH} = (Z, \mathcal{A})$ such that $\sigma \in \mathcal{W}(E, \mathcal{O}_S)$.*

(1) If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects the line $s = \mu_H(\underline{K})$ for $t > 0$, then $\beta(E_{n-1}) > \beta(E)$ at σ, with $E_{n-1} \subseteq \mathcal{O}_S$ in $\mathcal{A}$ (in particular, E is not μ_H-semistable).

(2) If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects the line $s = \mu_H(\overline{J})$ for $t > 0$, then $\beta(E/\overline{J}) > \beta(E)$ at σ, with $E/\overline{J} \subseteq \mathcal{O}_S$ in $\mathcal{A}$.

PROOF. (1) Maciocia proved in [22] that, if E is μ_H-semistable, then $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ does not intersect the line $s = \mu_H(E)$. Therfore, if $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects $s = \mu_H(\underline{K})$, E cannot be μ_H-semistable. Since $E \in \mathcal{A}_\sigma$, we have that $s(\sigma) < \mu_H(\underline{K})$. Because $\underline{K} \in \mathcal{A}$ iff $s < \mu_H(\underline{K})$, it follows that for all values of s between $s(\sigma) \leq s < \mu_H(\underline{K})$ we have that $0 \to E_{n-1} \to E \to \underline{K} \to 0$ in $\mathcal{A}$. At $s = \mu_H(\underline{K})$, we have that $\mathrm{Im}\, Z(\underline{K}) = 0$, and $\beta(\underline{K}) = -\infty$. Therefore, approaching $s = \mu_H(\underline{K})$ from σ along $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$, we have that $\beta(\underline{K}) \to -\infty$, and $\beta(E_{n-1}) > \beta(E) > \beta(\underline{K})$. Since the walls are nested in Π_u, we must have $\beta(E_{n-1}) > \beta(E) = \beta(\mathcal{O}_S)$ at σ as well.

(2) Since $E \subseteq \mathcal{O}_S \in \mathcal{A}_\sigma$, we have that $\mu_H(\overline{J}) \leq s(\sigma) < \mu_H(\underline{K})$. Therefore, for all values of s between $\mu_H(\overline{J}) \leq s \leq s(\sigma)$, we have that $E, \overline{J}[1] \in \mathcal{A}$, and there exists a short exact sequence $0 \to E \to E/\overline{J} \to \overline{J}[1] \to 0$ in $\mathcal{A}$.
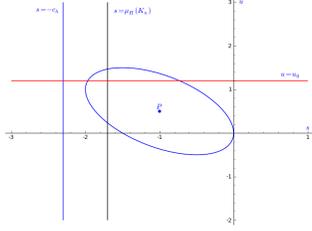
FIGURE 4.6. $t = 0$ plane
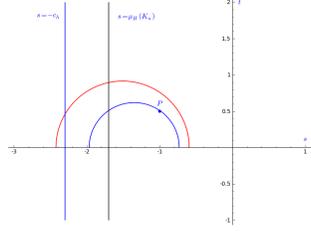
FIGURE 4.7. Π_u plane at $u = u_0$

At $s = \mu_H(\overline{J})$, we have that $\operatorname{Im} Z(\overline{J}[1]) = 0$, and $\beta(\overline{J}[1]) = \infty$. Therefore, approaching $s = \mu_H(\overline{J})$ from σ along $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$, we have that $\beta(\overline{J}[1]) \to \infty$, and $\beta(E) < \beta(E/\overline{J}) < \beta(\overline{J}[1])$. Since the walls are nested in Π_u, we must have $\beta(E/\overline{J}) > \beta(E) = \beta(\mathcal{O}_S)$ at σ as well. $\qquad\square$

REMARK 4.15. Since the surface S may have Picard rank larger than 1, we are forced to strengthen the hypotheses from those of [3], but *the proof is exactly the same.* We have restricted our attention to $\mathcal{O}_S$, but the proof holds for any sheaf satisfying the Bogomolov inequality (even a torsion sheaf).

REMARK 4.16. Let us point out an important fact that will be needed later in the paper: When looking at $E/\overline{J} \subseteq \mathcal{O}_S$ in $\mathcal{A}$, if we call Q' the quotient of $\mathcal{O}_S$ by $E/\overline{J}$ in $\mathcal{A}$, we have that $H^{-1}(Q') = H^{-1}(Q)/\overline{J}$ and $H^0(Q') = H^0(Q)$.

4.4. BRIDGELAND STABILITY OF $\mathcal{O}_S$

We prove Conjecture 4 for surfaces with (any Picard rank and) no curves of negative self-intersection as well as surfaces with Picard rank 2 and one irreducible curve of negative self-intersection. Proposition 4.26 serves as evidence for the conjecture on surfaces with Picard rank 2 and two irreducible curves of negative self-intersection.

4.4.1. SUBOBJECTS OF $\mathcal{O}_S$ OF RANK 1. Understanding the rank 1 weakly destabilizing subobjects of $\mathcal{O}_S$ is crucial to our main results, all of which use induction. If a subobject $E \subset \mathcal{O}_S$ has rank 1, then Lemma 4.10 shows that E must be equal to $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional scheme Z. We show here that for $\mathcal{I}_Z(-C)$ to weakly destabilize $\mathcal{O}_S$, we must have $C^2 < 0$.

PROPOSITION 4.17. *If* $C^2 \geq 0$, *then* $\mathcal{I}_Z(-C)$ *does not weakly destabilize* $\mathcal{O}_S$ *for any* σ, *i.e., there does not exist any* σ *such that* $\mathcal{I}_Z(-C) \subseteq \mathcal{O}_S$ *in* $\mathcal{A}$ *and* $\beta(\mathcal{I}_Z(-C)) = \beta(\mathcal{O}_S)$.

PROOF. We have that $\mathrm{ch}(\mathcal{I}_Z(-C)) = (1, -C, C^2/2 - l(Z))$, where $l(Z)$ is the length of Z. If $C = c_h H + c_g G + \alpha_C$, with $\alpha_C.H = \alpha_C.G = 0$, then $C^2 = c_h^2 - c_g^2 + \alpha_C^2$, and $\alpha_C^2 \leq 0$ by the Hodge Index Theorem. Therefore, if $C^2 \geq 0$, then $c_h^2 - c_g^2 \geq 0$. The equation for the wall $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ simplifies to

$$c_h(s^2 + t^2 + u^2) - 2c_g su + (c_h^2 - c_g^2)s + 2\alpha_C^2 s - 2l(Z)s = 0.$$

If $c_h^2 - c_g^2 > 0$, then the wall is an ellipse going through $0P$ and $2P$ with

$$P = \frac{C^2/2 - l(Z)}{c_g^2 - c_h^2}(c_h, c_g).$$

The s-coordinate of $2P$ is

$$\frac{c_h^2 - c_g^2 + 2\alpha_C^2 - 2l(Z)}{c_g^2 - c_h^2}c_h \geq -c_h + \frac{2\alpha_C^2 - 2l(Z)}{c_g^2 - c_h^2}c_h \geq -c_h(<0).$$

Therefore, the ellipse is contained in the region $s \geq -c_h$.

Since $\mathcal{I}_Z(-C) \in \mathcal{A}$, we have that $s < -c_h$, and therefore $\mathcal{I}_Z(-C)$ cannot weakly destabilize $\mathcal{O}_S$.
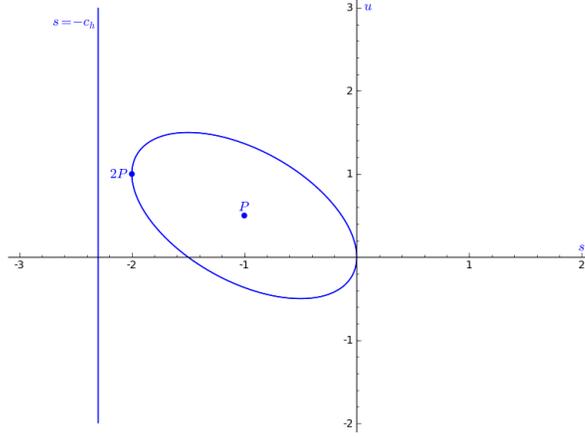
FIGURE 4.8. $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ when $c_h^2 - c_g^2 > 0$

If $c_h^2 - c_g^2 = 0$, then $C^2 \geq 0$ implies that $C^2 = 0$. Therefore, $\mathrm{ch}_2(\mathcal{I}_Z(-C)) = -l(Z) \leq 0$, and, as we saw in Section 4.3.1 where we listed the possible weakly destabilizing walls for $\mathcal{O}_S$, this wall cannot be a weakly destabilizing wall. $\qquad \square$

REMARK 4.18. Since $\mathcal{O}_S$ is Bridgeland stable for $t >> 0$ (by Proposition 4.12), we have from Proposition 4.17 that if $C^2 \geq 0$, then $\beta(\mathcal{I}_Z(-C)) < \beta(\mathcal{O}_S)$ whenever $\mathcal{I}_Z(-C) \subseteq \mathcal{O}_S$ in $\mathcal{A}$.

REMARK 4.19. For C a curve of negative self-intersection, we a have two possibilities.

(1) If $C^2 < 0$ and $c_g^2 - c_h^2 \leq 0$ then $\mathcal{I}_Z(-C)$ does not weakly destabilize $\mathcal{O}_S$ in $\mathcal{S}_{H,G}$ (note that this is only possible for S with Picard rank ≥ 3).

(2) If $C^2 < 0$ and $c_g^2 - c_h^2 > 0$ then $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ is a Left Hyperbola and is a weakly destabilizing wall for $\mathcal{O}_S$.

4.4.2. SURFACES WITH NO CURVES OF NEGATIVE SELF-INTERSECTION. We characterize the stability of $\mathcal{O}_S$ when S has no curves of negative self-intersection. This implies the first part of Theorem 4.1.

THEOREM 4.20. *If S does not contain any curves of negative self-intersection, then $\mathcal{O}_S$ is always Bridgeland stable (whenever in $\mathcal{A}$).*

PROOF. Let $E \subseteq \mathcal{O}_S$ be a proper subobject in $\mathcal{A}$ for some σ. We prove that $\beta(E) < \beta(\mathcal{O}_S)$ for all σ by induction on the rank of E.

If E has rank 1, then the result follows from Proposition 4.17 and 4.12. Assume that E has rank r, and that the result holds for any proper subobject of rank less than r. Choose a pair H, G as above so that $\sigma \in \mathcal{S}_{H,G}$. From our study of the walls above, we know that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is an Ellipse, a Cone, a Left or Right Hyperbola, or a Parabola. Using the same notation as in Section 4.3.3, we know that $E \subseteq \mathcal{O}_S \in \mathcal{A}$ iff $\mu_H(\overline{J}) \leq s < \mu_H(\underline{K})$. If we had that $\beta(E) \geq \beta(\mathcal{O}_S)$ for some σ in that range, then E would weakly destabilize $\mathcal{O}_S$. If the wall were an Ellipse, it would have to intersect $s = \mu_H(\underline{K})$, because these walls are connected and pass through $(0, 0)$. If it were a Left Hyperbola, it would have to intersect $s = \mu_H(\overline{J})$. If it were a Right Hyperbola, a Parabola, or a Cone it would have to intersect $s = \mu_H(\underline{K})$ and/or $s = \mu_H(\overline{J})$. Regardless of the type of wall, we would have that, by Lemma 4.14, there would exist a proper subobject of $\mathcal{O}_S$ of higher β and lower rank, contradicting our induction hypothesis. $\qquad\square$

4.4.3. ACTUAL WALLS ARE LEFT HYPERBOLAS. There are characteristics of actually destabilizing walls and subobbjects which persist regardless of the Picard rank of S or the composition of curves of negative self-intersection within S. We prove the following, and apply it in our study of surfaces with Picard rank 2 (Section 4.4.4).

PROPOSITION 4.21. *Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ with quotient Q. If the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a destabilizing wall, then it is a Left Hyperbola. Moreover, $C = c_1(H^0(Q))$ is a curve of*

negative self-intersection such that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also a Left Hyperbola, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ for all $|u| >> 0$.

The proof follows mostly from two basic lemmas in which we prove the following two basic results:

- Every weakly destabilizing wall that is not a Left Hyperbola is inside a higher wall which is a Left Hyperbola.

- If $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ with quotient Q has a weakly destabilizing wall that is a Left Hypberbola, then either $C = c_1(H^0(Q))$ is a curve of negative self-intersection or the wall is inside a higher wall that is not a Left Hyperbola.

Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$, and let Q be the quotient. We use the same notation as in Section 4.3.3 for the Harder-Narasihman filtrations of E and $H^{-1}(Q)$ with respect to the Mumford slope μ_H.

LEMMA 4.22. *Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a weakly destabilizing wall that is not a Left Hyperbola. Then, for some E_i in the Harder-Narasihman filtration of E, the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola such that the following is true: If there exists a stability condition σ such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E_i \subset \mathcal{O}_S$ in $\mathcal{A}_\sigma$, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ at $u(\sigma)$. In particular, E cannot actually destabilize $\mathcal{O}_S$.*

PROOF. We prove this by induction on the number of terms n in the Harder-Narasihman filtration of E.

If $n = 1$ (i.e. E is μ_H-semistable), then the wall $\mathcal{W}(E, \mathcal{O}_S)$ cannot be a weakly destabilizing wall without being a Left Hyperbola, because all other type of walls would intersect the line $s = \mu_H(E)$, contradicting Lemma 4.14.

Let now $n > 1$, and assume that the statement is true for all subobjects of $\mathcal{O}_S$ which have a weakly destabilizing wall that is not a Left Hyperbola, and have a Harder-Narasihman filtration of length $< n$.

Since the wall $\mathcal{W}(E, \mathcal{O}_S)$ is not a Left Hyperbola, it will intersect the line $s = \mu_H(\underline{K})$. Consider a value of u such that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects $s = \mu_H(\underline{K})$ at $t > 0$. By Lemma 4.14, we have that $\beta(E_{n-1}) > \beta(E)$ at the stability conditions $\sigma_{s,u}$ on $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ such that $s < \mu_H(\underline{K})$. Therefore, the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is also a weakly destabilizing wall.

If the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is not a Left Hyperbola, we can conclude by induction that there exists an E_i such that the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola and the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ for all $u(\sigma)$ for which $E_{n-1} \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$. If the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is a Left Hyperbola, let $i = n - 1$.

Let σ be a stability condition such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$. We need to prove that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ at $u(\sigma)$. First of all, notice that, if $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E_{n-1} \subseteq E \subseteq \mathcal{O}_S$ is $\mathcal{A}_\sigma$, because if $E \in \mathcal{A}_\sigma$, then $E_{n-1} \subseteq E \in \mathcal{A}_\sigma$.

Suppose now that E weakly destabilizes $\mathcal{O}_S$ at σ (i.e. $\beta_\sigma(E) \geq \beta_\sigma(\mathcal{O}_S)$), and suppose moreover that $u(\sigma) > 0$ (the proof for $u(\sigma) < 0$ is similar). Since the wall $\mathcal{W}(E, \mathcal{O}_S)$ is not a Left Hyperbola, its intersection with the planes Π_u is non-empty for all $0 \leq u \leq u(\sigma)$. Since the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola, on the other hand, its region in the $s < 0$ half-plane does not reach the s-axis. Therefore, for small positive values of u, $\mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_u$ is empty in the $s < 0$ region, and $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ is non-empty.

For every value of u such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(\underline{K})$ at $t > 0$, we know that

$$\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_u$$

by Lemma 4.14 and the induction hypothesis, respectively. The statement is still true at u_0 by continuity, where we denote by u_0 be the smallest value of u such that $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(\underline{K})$ (the intersection will be at $t = 0$).

Since the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ for small positive values of u, while the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ at u_0, there must exist a value u_1 with $0 < u_1 < u_0$ such that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_1} = \mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_{u_1}$ (recall that, for each u, the intersections of the walls with the plane Π_u are nested semicircles). Moreover,

$$\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_u$$

for all $u \geq u_1$ by Lemma 4.11. Since $E \in \mathcal{A}_\sigma$ only if $s < \mu_H(\underline{K})$, we have that the part of the wall $\mathcal{W}(E, \mathcal{O}_S)$ where $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$ is contained in the region $u \geq u_0$, which is contained in $u \geq u_1$ where the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$. $\qquad\square$

LEMMA 4.23. *Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ with quotient Q such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a weakly destabilizing Left Hyperbola. Then:*

- *Either $C = c_1(H^0(Q))$ is a curve of negative self-intersection, and the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is a Left Hyperbola, or*

- *For some F_j in the Harder-Narasihman filtration of $H^{-1}(Q)$, the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ is not a Left Hyperbola, and the following is true: If there exists a stability condition σ such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E/F_j \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ at $u(\sigma)$. In particular, E cannot actually destabilize $\mathcal{O}_S$.*

PROOF. We prove this by induction on the number of terms m in the Harder-Narasihman filtration of $H^{-1}(Q)$ (including the case $m = 0$ corresponding to $H^{-1}(Q) = 0$).

If $m = 0$ and $H^{-1}(Q) = 0$, then $E = \mathcal{I}_Z(-C)$, and by Proposition 4.17, $C = c_1(H^0(Q))$ must be a curve of negative self-intersection for the wall $\mathcal{W}(E, \mathcal{O}_S)$ to be a weakly destabilizing wall.

Let now $m > 0$, and assume that the statement is true for all subobjects of $\mathcal{O}_S$ which have a weakly destabilizing wall that is a Left Hyperbola, and a Harder-Narasihman filtration for H^{-1}(quotient) of length $< m$.

Since the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola, it will intersect the line $s = \mu_H(\overline{J}) = \mu_H(F_1)$. Consider a value of u such that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects $s = \mu_H(F_1)$ at $t > 0$. By Lemma 4.14, we have that $\beta(E/F_1) > \beta(E)$ at those stability conditions. Therefore, the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ is also a weakly destabilizing wall.

Let Q_1 be the quotient for $E/F_1 \subseteq \mathcal{O}_S$. By Remark 4.16, we have that $H^0(Q_1) = H^0(Q)$ and $H^{-1}(Q_1) = H^{-1}(Q)/F_1$. In particular, the Harder-Narasihman filtration of $H^{-1}(Q_1)$ is simply F_j/F_1 ($1 \leq j \leq m$), which has length $m-1$. Moreover, the quotients $(E/F_1)/(F_j/F_1)$ are the quotients E/F_j.

If the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ is a Left Hyperbola, we have by induction that at least one of the following two options is true:

(1) $C = c_1(H^0(Q_1))$ is a curve of negative self-intersection, and the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is a Left Hyperbola. Since $H^0(Q_1) = H^0(Q)$, we are done.

(2) There exists an F_j/F_1 in the Harder-Narasihman filtration of $H^{-1}(Q_1)$ such that the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ is not a Left Hyperbola, and the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ for all $u(\sigma)$ for which $E/F_1 \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, with $E/F_j \subseteq \mathcal{O}_S$ there.

If the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ is not a Left Hyperbola, let $j = 1$.

Let σ be a stability condition such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$. We now need to prove that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ at $u(\sigma)$.

First of all, notice that, if $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E/F_1 \subseteq \mathcal{O}_S \in \mathcal{A}_\sigma$.

As in the previous proof, assume that E weakly destabilizes $\mathcal{O}_S$ at σ, and that $u(\sigma) > 0$. We know that $\mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u$ is not empty for small positive values of u, while $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ is empty in the $s < 0$ region for u positive and sufficiently small.

Let u_0 be the largest value of u such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(\overline{J})$ (the intersection will be at $t = 0$). As in the previous proof we can use Lemma 4.14, the induction hypothesis, and continuity to show that

$$\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_0} \preceq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_0} \preceq \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_{u_0}.$$

Since $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ is inside than $\mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u$ for small positive values of u and at u_0, it must be inside of it for all $0 < u \leq u_0$. Otherwise, there would have to exist two values of u between 0 and u_0 where $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u$ which is not possible by Lemma 4.11.

Since $E \subseteq \mathcal{O}_S \in \mathcal{A}_\sigma$ only if $s \geq \mu_H(\overline{J})$, we have that the part of the wall $\mathcal{W}(E, \mathcal{O}_S)$ where $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$ is contained in the region $u \leq u_0$, where the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$. □

We can now prove Proposition 4.21. The only part of Proposition 4.21 that does not follow directly from the two lemmas is the part where we claim that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ for all $|u| >> 0$.

We will prove this by actually proving the following statement:

LEMMA 4.24. *Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ be an actually destabilizing object with quotient Q. For all F_j in the Harder-Narasihman filtration of $H^{-1}(Q)$, the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ is a Left Hyperbola, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ for all $|u| \gg 0$.*

The statement of the proposition follows from this lemma because, since $F_m = H^{-1}(Q)$, $E/F_m = \mathcal{I}_Z(-C)$ for some zero-dimensional scheme Z. Then, for $|u| \gg 0$, the wall $\mathcal{W}(E, \mathcal{O}_S)$ must be inside the wall $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$, which is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$.

PROOF OF LEMMA 4.24. Assume that E actually destabilizes $\mathcal{O}_S$ at a stability condition σ, and that $u(\sigma) > 0$. We know that all of the walls $\mathcal{W}(E/F_j, \mathcal{O}_S)$ must be Left Hyperbolas by the proof of the previous lemma. Indeed, if they were not Left Hyperbolas, then there would exist a j such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ would be inside the wall $\mathcal{W}(E/F_j, \mathcal{O}_S)$ at $u(\sigma)$ whenever $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, making it impossible for the wall $\mathcal{W}(E, \mathcal{O}_S)$ to be an actually destabilizing wall.

Let u_0 be the largest value of u such that $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(\overline{J})$ (the intersection will be at $t = 0$). Then we know that the region where $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ and $\beta(E) > \beta(\mathcal{O}_S)$ is contained within $s \geq \mu_H(F_1)$ and $0 < u \leq u_0$. Moreover, as above, we have that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_0} \preceq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_0}$.

For E to actually destabilize $\mathcal{O}_S$ at σ, we must have that $0 < u(\sigma) < u_0$, and $\mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u(\sigma)} \preceq \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$. Using Lemma 4.11 as above, we see that since the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ at $u(\sigma)$, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ at $u_0 > u(\sigma)$, we know that the wall $\mathcal{W}(E, \mathcal{O}_S)$ must stay inside for all $u \geq u_0$. Thus, if $m = 1$ we are done.

If $m > 1$, let $u_1 > u_0$ be the largest value of u such that $\mathcal{W}(E/F_1, \mathcal{O}_S)$ intersects $s = \mu_H(F_2/F_1)$. Then, $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_1} \preceq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_1} \preceq \mathcal{W}(E/F_2, \mathcal{O}_S) \cap \Pi_{u_1}$ as above.

Since the wall $\mathcal{W}(E/F_2, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ at $u(\sigma)$, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E/F_1, \mathcal{O}_S)$ at $u_1 > u_0 > u(\sigma)$, we know that the wall $\mathcal{W}(E, \mathcal{O}_S)$ must stay inside for all $u \geq u_1$.

Continuing in a similar manner proves the statement for all $1 \leq j \leq m$. $\qquad\square$

4.4.4. SURFACES OF PICARD RANK 2. We now restrict our attention to surfaces of Picard Rank 2, where we can describe the actually destabilizing walls for $\mathcal{O}_S$ more precisely. We pause to illuminate a fact which is helpful in this situation.

REMARK 4.25. Let S have Picard rank 2. Then for any line bundle $\mathcal{O}_S(D')$ and $\sigma_{D,H} \in \mathcal{S}_{H,G}$ we have $\mathcal{O}_S(D') \otimes \sigma_{D,H} = \sigma_{D+D',H} \in \mathcal{S}_{H,G}$.

Let S be a surface of Picard Rank 2, and let H and G be as above. Moreover, let C_1 and C_2 be the generators of the cone of effective curves on S. Since H is ample, we must have that $H = eC_1 + fC_2$ for some $e, f > 0$. Then, $H.G = 0$ implies that $fC_2.G = -eC_1.G$. Therefore, $(C_1.G) \cdot (C_2.G) < 0$. Assume that $C_1.G > 0$ and $C_2.G < 0$.

We saw in Proposition 4.21 that every destabilizing wall is a Left Hyperbola which has to be inside a rank 1 weakly destabilizing wall for $|u| >> 0$. In the case of Picard rank 2, we can make $|u| >> 0$ more precise as follows:

PROPOSITION 4.26. If $u \geq C_1.G$, then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$, and if $u \leq C_2.G$, then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_2)$.

PROOF. We prove the following statement by induction on the rank of E: If $E \subseteq \mathcal{O}_S$ for some stability condition σ with $u(\sigma) \geq C_1.G$, and $\beta(E) \geq \beta(\mathcal{O}_S)$, then the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$. (A similar proof would work with C_2 in place of C_1 if $u(\sigma) \leq C_2.G$.)

If the rank of E is 1, and $\beta(E) \geq \beta(\mathcal{O}_S)$ at σ, then $E = \mathcal{I}_Z(-C)$ for some curve C of negative self-intersection and some zero-dimensional scheme Z. Since $\beta(\mathcal{O}_S(-C)) \geq \beta(\mathcal{I}_Z(-C)) \geq \beta(\mathcal{O}_S)$ at σ with $u(\sigma) \geq C_1.G > 0$, we must have that $C = aC_1 + bC_2$ with $a > 0$. Therefore, $\mathcal{O}_S(-C) \subseteq \mathcal{O}_S(-C_1)$.

Since the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ for $u >> 0$ because of the slope of their asymptotes (see Section 4.3.1), and it is inside of it at $u = C_1.G$ because $\mathcal{O}_S(-C_1)$ is always Bridgeland-stable there, it must be inside of it for all $u \geq C_1.G$ by Lemma 4.11.

Assume now that E has rank $r > 1$, and that the statement is true for all subobjects of $\mathcal{O}_S$ of rank $< r$.

By Lemma 4.22, we can assume that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola. Morever, if we let $C = c_1(H^0(Q))$, where Q is the quotient of $\mathcal{O}_S$ by E in $\mathcal{A}$, we have that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also a Left Hyperbola, and that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ for $u >> 0$. By the rank 1 case, we know that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ for all $u \geq C_1.G$.

If the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$, we are done. Supposing not, we have by Lemma 4.11 that the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ for all $u \leq u(\sigma)$. Denote this statement by $(\star)$.

If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ instersects $s = \mu_H(\underline{K})$ (using our usual notation for the Harder-Narasihman filtration from Section 4.3.3), then it will be inside the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$, which will be inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ by induction, contradicting $(\star)$. Thus $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ does not instersect $s = \mu_H(\underline{K})$, and since $u(\sigma) \geq C_1.G$, $(\star)$ implies that the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ at $u = C_1.G$. Since $\mathcal{O}_S(-C_1)$ is Bridgeland stable at $u = C_1.G$, this can only happen if E is not a subobject of $\mathcal{O}_S(-C_1)$ in $\mathcal{A}$ at the points on the wall $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{C_1.G}$. This means that the wall $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ had to intersect $s = \mu_H(\underline{K})$ for some $C_1.G \leq u < u(\sigma)$.

Consider a value of u with $C_1.G \leq u < u(\sigma)$ where the wall $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(\underline{K})$. There, the wall $\mathcal{W}(E, \mathcal{O}_S)$ would have to be inside the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ by Lemma 4.14, and the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ would be inside $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ by induction hypothesis, contradicting $(\star)$. Thus the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$. $\qquad\square$

We now prove the second part of our main Theorem 4.1, i.e., that Conjecture 4 is true for surfaces of Picard Rank 2 that only have one irreducible curve of negative self-intersection. We first give a lemma, then prove a result which is stronger than the conjecture in this situation.

LEMMA 4.27. *Let S be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by C_1 and C_2, that $C_1.G > 0$, and that C_1 is the only irreducible curve in S of negative self-intersection. Let C be an effective curve. If $C.G < 0$, then $C^2 \geq 0$.*

PROOF. We saw above that we can write $H = eC_1 + fC_2$ with $e, f > 0$, and obtain $fC_2.G = -eC_1.G$. Since $(eC_1 + fC_2)^2 = H^2 > 0$, we have that $2ef C_1.C_2 > -e^2 C_1^2 - f^2 C_2^2$.

Let C be an effective curve. We have that $C = aC_1 + bC_2$ with $a, b \geq 0$. Assume that $C.G < 0$. Then, $0 > fC.G = afC_1.G + bfC_2.G = (af - be)C_1.G$, and therefore, $af - be < 0$.

Therefore, $efC^2 = a^2efC_1^2 + b^2efC_2^2 + 2abefC_1.C_2 > a^2efC_1^2 + b^2efC_2^2 - abe^2C_1^2 - abf^2C_2^2 = ae(af - be)C_1^2 + bf(be - af)C_2^2 \geq 0$, and $C^2 \geq 0$. □

PROPOSITION 4.28. *Let S be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by C_1 and C_2, that $C_1.G > 0$, and that C_1 is the only irreducible curve in S of negative self-intersection. Then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$.*

PROOF. Note that, since C_1 is the only irreducible curve of negative self-intersection, then $\mathcal{O}_S(-C_2)$ does not weakly destablize $\mathcal{O}_S$, and the wall $\mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S)$ is empty in the region where $\mathcal{O}_S(-C_2) \in \mathcal{A}$.

We prove the following statement by induction on the rank of E: If $E \subseteq \mathcal{O}_S$ for some stability condition σ, and $\beta(E) \geq \beta(\mathcal{O}_S)$, then the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$.

If $u(\sigma) \geq C_1.G$, we already know this statement to be true by the proof of Proposition 4.26. We therefore only need to prove the statement in the case when $u(\sigma) < C_1.G$.

If the rank of E is 1, and $\beta(E) \geq \beta(\mathcal{O}_S)$ at σ, then $E = \mathcal{I}_Z(-C)$ for some curve C of negative self-intersection and some zero-dimensional scheme Z. Suppose, by contradiction, that the wall $\mathcal{W}(E, \mathcal{O}_S)$ were not inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$. Choose a $\sigma' \in \mathcal{W}(E, \mathcal{O}) \cap \Pi_{u(\sigma)}$. Then at σ' we have $\beta(\mathcal{O}_S(-C)) \geq \beta(\mathcal{I}_Z(-C)) > \beta(\mathcal{O}_S(-C_1))$. Therefore, at $\mathcal{O}_S(C_1) \otimes \sigma'$ we have $\beta(\mathcal{O}_S(-C + C_1)) > \beta(\mathcal{O}_S)$. This means that $\mathcal{O}_S(-C + C_1)$ weakly destabilizes $\mathcal{O}_S$ at $\mathcal{O}_S(C_1) \otimes \sigma'$. By Proposition 4.17, we must have that $(C - C_1)^2 < 0$, and by Lemma 4.27, $(C - C_1).G > 0$. This implies that the wall $\mathcal{W}(\mathcal{O}_S(-C + C_1), \mathcal{O}_S)$ is

a Left Hyperbola that could only weakly destabilize $\mathcal{O}_S$ in the region $u > 0$, but we have

$$u(\mathcal{O}_S(C_1) \otimes \sigma') = u(\sigma) - C_1.G < 0.$$

Assume now that E has rank $r > 1$, and that the statement is true for all subobjects of $\mathcal{O}_S$ of rank $< r$.

By Lemma 4.22, we can assume that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola. Morever, if we let $C = c_1(H^0(Q))$, we have that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also a Left Hyperbola, and that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ for $u >> 0$. By the rank 1 case, we know that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ for all u.

If the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$, we are done. Supposing not, we have by Lemma 4.11 that the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ for all $u \leq u(\sigma)$. Denote this statement by $(\star)$.

If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ instersects $s = \mu_H(\underline{K})$ (using our usual notation for the Harder-Narasihman filtration from Section 4.3.3), then it will be inside the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$, which will be inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ by induction, contradicting $(\star)$.

Thus $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ does not instersect $s = \mu_H(\underline{K})$. Choose a $\sigma' \in \mathcal{W}(E, \mathcal{O}_S)\Pi_{u(\sigma)}$. Then $\beta(E) > \beta(\mathcal{O}_S(-C_1))$ at σ' implies $\beta(E(C_1)) > \beta(\mathcal{O}_S)$ at $\mathcal{O}_S(C_1) \otimes \sigma'$. Therefore, $E(C_1)$ weakly destabilizes $\mathcal{O}_S$ at $\mathcal{O}_S(C_1) \otimes \sigma'$.

Note that $u(\mathcal{O}_S(C_1) \otimes \sigma') = u(\sigma) - C_1.G < 0$. Consider the highest semi-circular wall at $u(\mathcal{O}_S(C_1) \otimes \sigma')$, corresponding to an object $E' \subseteq \mathcal{O}_S$. Since E' actually destabilizes $\mathcal{O}_S$, we have that the wall $\mathcal{W}(E', \mathcal{O}_S)$ must be a Left Hyperbola. However, the proof of Proposition 4.26 shows that the wall $\mathcal{W}(E', \mathcal{O}_S)$ would have to be inside the wall $\mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S)$ for all $u \leq C_2.G$. But this cannot be true, because the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola, and the wall $\mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S)$ is empty. $\qquad\square$

We now move on to studying the stability of $\mathcal{O}_S[1]$. This can be done via duality, except for the stability conditions $\sigma_{D,H}$ with $D = uG$, i.e. $s = 0$.

Note that, given a Bridgeland stability condition σ, $\mathcal{O}_S[1] \in \mathcal{A}$ iff $s \geq 0$.

4.5.1. Subobjects of $\mathcal{O}_S[1]$. Let σ be a Bridgeland stability condition, and let E be a proper subobject of $\mathcal{O}_S[1]$ in $\mathcal{A}$. We have a short exact sequence $0 \to E \to \mathcal{O}_S[1] \to Q' \to 0$ in $\mathcal{A}$ for some $Q' \in \mathcal{A}$. The long exact sequence in cohomology is

$$0 \longrightarrow H^{-1}(E) \longrightarrow \mathcal{O}_S \longrightarrow H^{-1}(Q) \longrightarrow H^0(E) \longrightarrow 0,$$

and therefore $Q' = H^{-1}(Q)[1]$ is the shift of a sheaf. We will denote $H^{-1}(Q)$ by Q. Also, since $H^{-1}(Q) \in \mathcal{F}$ is torsion-free, we have that either $H^{-1}(E) = \mathcal{O}_S$ or $H^{-1}(E) = 0$. However, if $H^{-1}(E) = \mathcal{O}_S$, then $H^{-1}(Q) = H^0(E)$, and this is not possible, since the first sheaf is in $\mathcal{F}$, and the second one is in $\mathcal{T}$. Therefore, $H^{-1}(E) = 0$, and $E = H^0(E)$ is a sheaf.

To summarize, if $E \subseteq \mathcal{O}_S[1]$ is a proper subobject in $\mathcal{A}$, then E is a sheaf in $\mathcal{T}$, and the quotient is of the form $Q[1]$ for some sheaf $Q \in \mathcal{F}$. We have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_S \longrightarrow Q \longrightarrow E \longrightarrow 0.$$

4.5.2. The $s = 0$ case. Let us start by proving that $\mathcal{O}_S[1]$ is Bridgeland stable when $s = 0$.

Lemma 4.29. *If $s = 0$, $\mathcal{O}[1]$ has no proper subobjects in $\mathcal{A}$, and is therefore Bridgeland stable.*

PROOF. Let $E \subseteq \mathcal{O}_S[1]$ be a proper subobject of $\mathcal{O}_S[1]$ in $\mathcal{A}$ with quotient $Q[1]$ as above. Since $s = 0$, we have that $\operatorname{Im} Z(\mathcal{O}_S[1]) = 0$. Since $\operatorname{Im} Z(\mathcal{O}_S[1]) = \operatorname{Im} Z(E) + \operatorname{Im} Z(Q[1])$, and they all have non-negative imaginary parts, we must have that $\operatorname{Im} Z(E) = \operatorname{Im} Z(Q[1]) = 0$. Therefore, all three objects have maximal phase. The only objects in $\mathcal{T}$ of maximal phase are torsion sheaves supported in dimension 0. We therefore have the short exact sequence of sheaves $0 \to \mathcal{O}_S \to Q \to E \to 0$, with E a torsion sheaf supported in dimension 0. But this cannot happen unless the sequence splits, in which case Q would have torsion, which is impossible. Therefore, if $s = 0$, $\mathcal{O}_S[1]$ cannot have proper subobjects, and is Bridgeland stable. $\qquad\square$

We can therefore assume that $s > 0$.

4.5.3. DUALITY. The following duality result allows us to apply results on the stability of $\mathcal{O}_S$ to that of $\mathcal{O}_S[1]$. It follows as in [25, Lemma 3.2] with a slightly different choice of functor. Specifically, we consider the functor $E \mapsto E^\vee := R\mathcal{H}om(E, \mathcal{O}_S)[1]$. Note that $\mathcal{O}_S^\vee = \mathcal{O}_S[1]$ and vice versa.

LEMMA 4.30. *Let D be a divisor with $D.H < 0$. Then $\mathcal{O}_S$ is $\sigma_{D,H}$-(semi)stable if and only if $\mathcal{O}_S[1]$ is $\sigma_{-D,H}$-(semi)stable.*

PROOF. This follows from [25, Lemma 3.2(d)]. $\qquad\square$

Note that if $\mathcal{O}_S(-C) \subseteq \mathcal{O}_S$ destabilizes $\mathcal{O}_S$ at $\sigma_{D,H}$, then applying $(\_)^\vee$ shows that the quotient $\mathcal{O}_S[1] \twoheadrightarrow \mathcal{O}_S(C)[1]$ destabilizes $\mathcal{O}_S[1]$ at $\sigma_{-D,H}$. The kernel of the map $\mathcal{O}_S[1] \twoheadrightarrow \mathcal{O}_S(C)[1]$ in $\mathcal{A}$ is $\mathcal{O}_S(C)|_C$. Thus Theorem 4.20, Proposition 4.28, and Lemma 4.29 yield the following result.

PROPOSITION 4.31.

(1) Let S be a surface of any Picard rank such that there are no curves $C \subset S$ with $C^2 < 0$.

Then $\mathcal{O}_S[1]$ is σ-stable whenever in $\mathcal{A}$.

(2) Let S be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by C_1 and C_2, that $C_1.G > 0$, and that C_1 is the only irreducible curve in S of negative self-intersection. Then $\mathcal{O}_S[1]$ is only destabilized by $\mathcal{O}_S(C_1)|_{C_1}$.

CHAPTER 5

Partial Result when S has Two Irreducible

Negative Curves

Let S be a surface of Picard rank 2. If S has no curves of negative self-intersection, or just one, then we completely understand the stability of line bundles in $\text{Stab}_{div}(S)$ (see Theorem 4.20 and Proposition 4.28, respectively). Recall that an understanding of the stability of $\mathcal{O}_S$ suffices to understand that of line bundles in general.

For surfaces S with Picard rank 2 and two curves C_1, C_2 of negative self-intersection, we have only partial results. In Proposition 4.26 it is shown that in the 3-spaces $\mathcal{S}_{G,H} \subset \text{Stab}_{div}(S)$, if $\sigma_{D,H}$ is far enough away from the plane given by $D = sH$, then $\mathcal{O}_S$ can only be destabilized by $\mathcal{O}_S(-C_1)$ or $\mathcal{O}_S(-C_2)$.

The main result of this chapter, Theorem 5.1, extends Proposition 4.26 in a certain situation, but does not give a full characterization of the stability of $\mathcal{O}_S$. It states that if the ample divisor H is such that in $\mathcal{S}_{G,H}$, the curves C_1, C_2 have a particular arrangement, then in $\mathcal{S}_{G,H}$ the only walls for $\mathcal{O}_S$ are $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ and $\mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S)$. In the statement of the theorem, note that the regions $T1$ and $T2$ referenced are defined in Definition 5.7.

THEOREM 5.1. *Let S be a surface of Picard Rank 2, where C_1 and C_2 are the generators of the cone of effective curves on S. If there is an ample divisor H such that there are no points $-C$ in the regions $T1$ or $T2$ in $\mathcal{S}_{H,G}$ (where C is an integral effective curve), then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$ and $\mathcal{O}_S(-C_2)$ in $\mathcal{S}_{H,G}$. Note that the set $\mathcal{S}_{H,G}$ is determined by H since S has Picard rank 2.*

In this chapter we introduce xy-coordinates on the $t = 0$-plane of the 3-spaces $\mathcal{S}_{H,G}$ and also some terminology which does not appear in Chapter 4. While not strictly necessary, these can aid in the process of research. For example, in xy-coordinates, walls for $\mathcal{O}_S$ have axis along the x or y axes.

5.1. LEGIT/ACTIVE REGIONS AND ROTATED COORDINATES

Here we quickly introduce some useful definitions regarding when a triangle is a short exact sequence and when that sequence "weakly destabilizes" $\mathcal{O}$. Following that, we rotate our standard su-coordinates and obtain new equations for walls.

DEFINITION 5.2.

(1) For a triangle $E \to \mathcal{O} \to Q$ we set $\mathbf{legit}(E \hookrightarrow \mathcal{O}) := \{\sigma \mid E \to \mathcal{O} \to Q \text{ a short exact sequence in } \mathcal{A}\}$. We also call this $\mathbf{legit}(\mathcal{O} \twoheadrightarrow Q)$. In the following, we will often restrict these notions to only considering a specific $\mathcal{S}_{H,G}$.

(2) For a triangle $E \to \mathcal{O} \to Q$ we set $\mathbf{active}(E \hookrightarrow \mathcal{O}) := \{\sigma \in \mathbf{legit}(E \hookrightarrow \mathcal{O}) \mid \beta(E) \geq \beta(\mathcal{O})\}$.

(3) A triangle $E \to \mathcal{O} \to Q$ **weakly destabilizes** $\mathcal{O}$ if $\mathbf{active}(E \hookrightarrow \mathcal{O}) \neq \varnothing$.

Choose a $\mathcal{S}_{H,G}$ and let $\mathrm{ch}(E) = (r, d_h H + d_g G + \alpha, c)$. Note that we may choose G so that $d_g \geq 0$. We saw in Section 4.3.1 that the equation for the wall $\mathcal{W}(E, \mathcal{O}_S)$ inside $\mathcal{S}_{H,G}$ is

$$d_h s^2 - 2cs - 2d_g u s + d_h t^2 + d_h u^2 = 0$$

In order to separate variables and obtain nice axes for the wall, we apply the substitution $s = x + y$, $u = -x + y$ (which rotates the plane $-\pi/4$ radians and scales by a factor of $\sqrt{2}/2$) and this becomes

$$w(E) := 2((d_h + d_g)x^2 - cx) + 2((d_h - d_g)y^2 - cy) + d_h t^2 = 0$$

If $d_h + d_g \neq 0$ and $c \neq 0$ then completing the squares and simplifying yields

(5–1)
$$W(E) := \frac{(x - \frac{c}{2(d_h + d_g)})^2}{\frac{1}{2(d_h + d_g)}\frac{c^2 d_h}{d_h^2 - d_g^2}} + \frac{(y - \frac{c}{2(d_h - d_g)})^2}{\frac{1}{2(d_h - d_g)}\frac{c^2 d_h}{d_h^2 - d_g^2}} + \frac{t^2}{\frac{c^2}{d_h^2 - d_g^2}} = 1$$

The region $\mathbf{active}(E \hookrightarrow \mathcal{O})$ is given by $w(E) \geq 0$. It corresponds to $\mathcal{W}(E, \mathcal{O})$ as well as the region it encloses with the $t = 0$-plane. The point P of Section 4.3.1 is now

$$P = \left(\frac{c}{2(d_h + d_g)}, \frac{c}{2(d_h - d_g)} \right)$$

and we have $(0,0) = 0P, 2P \in \mathcal{W}(E, \mathcal{O})$. Further, note that if $c \neq 0$ then (in the $t = 0$-plane) we have $\frac{dW}{dx}(0,0) = -\frac{W_x}{W_y}(0,0) = -1$ and so $\mathcal{W}(E, \mathcal{O})$ is tangent to the line $s = 0$. The same slope is obtained at $2P$.

DEFINITION 5.3. For a sheaf E we denote by E_A the line $s = \mu_H(E)$ inside $\mathcal{S}_{H,G}$.

Note that in xy-coordinates we have $E_A = \{y = -x + \mu_H(E)\}$.

We now consider what form the walls take based on the values of $d_h + d_g$ and c. In the following, when we speak of $\mathcal{W}(E, \mathcal{O})$ we mean it's intersection with the $t = 0$-plane.

Case 1: $d_h + d_g = 0$:

- If $c = 0$ then $w(E) = 2(d_h - d_g)y^2 + d_h t^2 < 0$ and so $\mathbf{active}(E \hookrightarrow \mathcal{O}) = \varnothing$.

- $(\mathbf{Par^+})$ If $c < 0$ then $w(E) = 0$ simplifies to

$$2(d_h - d_g)\left(y - \frac{c}{2(d_h - d_g)} \right)^2 + d_h t^2 - \frac{c^2}{2(d_h - d_g)} = 2cx$$

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Then $\mathcal{W}(E, \mathcal{O})$ is a parabola with vertex in the second quadrant and opening in the positive x direction. Since $\mathcal{O}_\mathcal{A} \leq \mathcal{W}(E, \mathcal{O})$ then (in $\mathcal{S}_{H,G}$) we have **active**$(E \hookrightarrow \mathcal{O}) = \varnothing$.

- (**Par**) If $c > 0$ then $\mathcal{W}(E, \mathcal{O})$ is again a parabola but with vertex in the fourth quadrant and opening in the negative x direction.

Case 2: $d_h + d_g < 0$**:**

- If $c = 0$ then $w(E) = 2(d_h + d_g)x^2 + 2(d_h - d_g)y^2 + d_h t^2 < 0$ and so **active**$(E \hookrightarrow \mathcal{O}) = \varnothing$.

- (**Ell$^+$**) If $c < 0$ then $\mathcal{W}(E, \mathcal{O})$ is an ellipse with center ξ in the first quadrant. Since $\mathcal{O}_\mathcal{A} \leq \mathcal{W}(E, \mathcal{O})$ then (in $\mathcal{S}_{H,G}$) we have **active**$(E \hookrightarrow \mathcal{O}) = \varnothing$.

- (**Ell**) If $c > 0$ then $\mathcal{W}(E, \mathcal{O})$ is again an ellipse but with center ξ in the third quadrant.

Case 3: $d_h + d_g > 0$**:**

- (**Con**) If $c = 0$ then $w(E) = 0$ gives $2(d_h + d_g)x^2 + 2(d_h - d_g)y^2 + d_h t^2 = 0$. Then $\mathcal{W}(E, \mathcal{O})$ is a cone bounded by the two lines $y = \pm Mx$ where $M = \sqrt{\frac{d_h + d_g}{-(d_h - d_g)}}$. Note that $M < 1$. The region **active**$(E \hookrightarrow \mathcal{O})$ corresponds to the region bounded by these lines and the x-axis.

- (**Hyp$_L$**) If $c < 0$ then $\mathcal{W}(E, \mathcal{O})$ is a hyperbola with center ξ in the second quadrant and opening in the x directions.

- (**Hyp$_R$**) If $c > 0$ then $\mathcal{W}(E, \mathcal{O})$ is again hyperbola but with center ξ in the fourth quadrant.

COROLLARY 5.4. *If* **active**$(E \hookrightarrow \mathcal{O}) \neq \varnothing$ *then* $\mathcal{W}(E, \mathcal{O})$ *is of type* **Ell**, **Par**, **Con**, **Hyp$_R$** *or* **Hyp$_L$**.

REMARK 5.5. The walls mentioned above, those corresponding to an E with $d_g \geq 0$, are called horizontal "h" walls. Walls corrsponding to an E with $d_g < 0$ are called vertical "v" walls. If one defines $v(E) = (\mathrm{ch}_1(E) - \alpha)/r$ then the cases considered above are effected by the position of $v(E)$: the condition $d_g \geq 0$ corresponds to $u(v(E)) \geq 0$, and if $v(E)$ lies on the negative x-axis, then we are in Case 1. If $v(E)$ lies in the 3rd quadrant, we are in Case 2, and if $v(E)$ lies in the 2nd quadrant, we are in Case 3.

Before moving to the proof of Theorem 5.1, we state two definitions which speak to aspects of walls.

DEFINITION 5.6.

(1) Let $\mathcal{W}$ be a wall for $\mathcal{O}_S$ in a given $\mathcal{S}_{G,H}$. We denote by $\mathrm{in}(\mathcal{W})$ the convex region bounded by $\mathcal{W}$ in the $t = 0$-plane.

(2) Let $\mathcal{W}(E, \mathcal{O}_S)$ and $\mathcal{W}(F, \mathcal{O}_S)$ be two $\mathbf{Hyp}_L^h$ walls in a given $\mathcal{S}_{G,H}$. We say $M(E) > M(F)$ to mean that the slope of the line through the origin and $v(E)$ is less than the slope of the line through the origin and $v(F)$ (where $v(-)$ is as in Remark 5.5). Note that this implies that $\mathcal{W}(F, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ for all $u \gg 0$.

5.2. CHARACTERIZATION OF STABILITY

In this section we prove Theorem 5.1. We first define the regions $T1$ and $T2$, then identify three properties that an actually destabilizing object for $\mathcal{O}_S$ must satisfy, and in Proposition 5.10 show that no object can satisfy these. We assume, without loss of generality, that $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O})$ is $\mathbf{Hyp}_L^h$ and $\mathcal{W}(\mathcal{O}(-C_2), \mathcal{O})$ is $\mathbf{Hyp}_L^v$.

DEFINITION 5.7. Suppose that $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O}) = \mathbf{Hyp}_L^h$, then $T1$ is defined by the inequalities $y > 0$, $x(-C_1) < x$, and $y - y(-C_1) < -(x - x(-C_1))$. The region $T2$ is defined analogously. Note that $\mathcal{W}(\mathcal{O}(-C_2), \mathcal{O}) = \mathbf{Hyp}_L^v$.
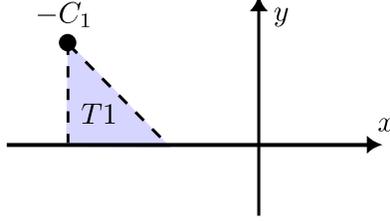
FIGURE 5.1. The region $T1$

Recall from Proposition 4.21 that if an object E actually destabilizes $\mathcal{O}_S$ at σ with quotient Q, then $\mathcal{W}(E, \mathcal{O}_S)$ is $\mathbf{Hyp}_L$ (we may assume $\mathbf{Hyp}_L^h$). Moreover, $C = c_1(H^0(Q))$ is a curve of negative self-intersection such that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also $\mathbf{Hyp}_L^h$, and we have $M(-C) > M(E)$. Note that we also must have $\beta(E) > \beta(\mathcal{O}_S(-C_1))$.

We show a stronger relationship between the walls $\mathcal{W}(E, \mathcal{O}_S)$ and $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$, and control the region where E weakly destabilizes $\mathcal{O}_S$ before a $\mathcal{O}(-C_1)$ does. But we first give this region a name.

DEFINITION 5.8. Suppose $E \hookrightarrow \mathcal{O}$ at σ with $\mathrm{ch}_1(Q_0) = C$ for some negative curve C. Suppose also that E weakly destabilizes $\mathcal{O}$ and the walls $\mathcal{W}(E, \mathcal{O})$ and $\mathcal{W}(\mathcal{O}(-C), E)$ are both $\mathbf{Hyp}_L^h$ in a given $\mathcal{S}_{G,H}$. Then we define $\mathbf{bR}(E, \mathcal{O}; -C_1)$ as the set of points (x, y) such that there is a $\sigma_{x,y,t} \in \mathbf{legit}(E \hookrightarrow \mathcal{O})$ with $\beta(\mathcal{O}) = \beta(E) \geq \beta(\mathcal{O}(-C_1))$.

Note that $E \hookrightarrow \mathcal{O}$ implies that $E \hookrightarrow O(-C) \hookrightarrow \mathcal{O}(-C_1)$, where the last inclusion follows since $\mathcal{W}(\mathcal{O}(-C), \mathcal{O})$ and $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O})$ are both $\mathbf{Hyp}_L^h$, which gives that $C = aC_1 + bC_2$ with $a \geq 1$. Thus E weakly destabilizes $\mathcal{O}(-C_1)$ over $\mathbf{bR}(E, \mathcal{O}; -C_1)$.

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LEMMA 5.9. *Suppose E (with $rk(E) \geq 2$) actually destabilizes $\mathcal{O}$ with $ch_1(Q_0) = C$ a negative curve with $\mathcal{W}(\mathcal{O}(-C), \mathcal{O}), \mathcal{W}(E, \mathcal{O}_S) = \mathbf{Hyp}_L^h$. Then the following are satisfied.*

(1) There exists a Π_{u_i} in which $\mathcal{W}(E, \mathcal{O}) = \mathcal{W}(\mathcal{O}(-C), \mathcal{O})$ and we have $u(-C) > u_i$.

(2) We have $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$, i.e. that $u(-C_1) > u(p)$ for all points $p \in \mathbf{bR}(E, \mathcal{O}; -C_1)$.

PROOF. (1): Note that the existence of u_i follows from Proposition 4.21. The following shows a picture of the situation we shall preclude for an actually destabilizing E.
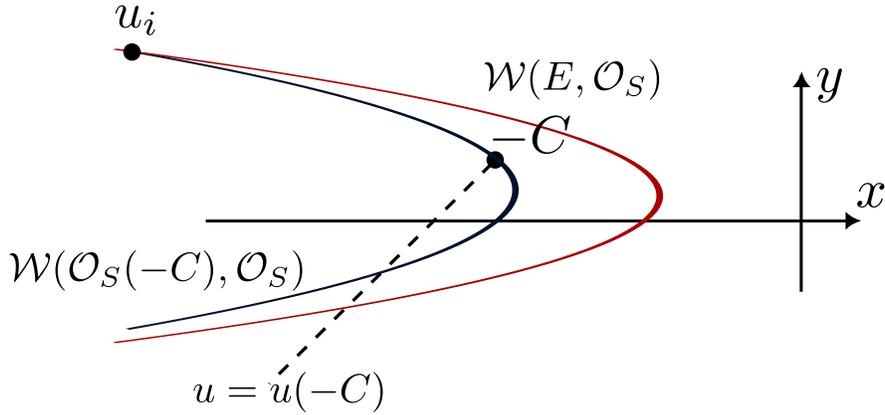
FIGURE 5.2. The situation of $u_i > u(-C)$

We prove that if E weakly destabilizes $\mathcal{O}_S$ as in Figure 5.2, then there is an E' such that $\mathbf{active}(E' \hookrightarrow \mathcal{O}_S) \supset \mathbf{active}(E \hookrightarrow \mathcal{O}_S)$. To that end, suppose that E weakly destabilizes $\mathcal{O}$ with $ch_1(Q_0) = C$ a negative curve with $\mathcal{W}(\mathcal{O}(-C), \mathcal{O}), \mathcal{W}(E, \mathcal{O}_S) = \mathbf{Hyp}_L^h$. Also, suppose that $u_i > u(-C)$ where u_i is such that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_i} = \mathcal{W}(\mathcal{O}(-C), \mathcal{O}) \cap \Pi_{u_i}$.

Since $\mu_H(K_i) < \mu_H(\mathcal{O}_S(-C))$ for each Mumford H-semistable factor of E, we have that $\underline{K}_{\mathcal{A}}$ cuts through $\mathcal{W}(E, \mathcal{O}_S)$. By approaching $\underline{K}_{\mathcal{A}}$ from lower s values we see that $\mathcal{W}(E, \mathcal{O}_S)$ lies inside $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ for all Π which intersect the portion of $\underline{K}_{\mathcal{A}}$ lying inside

in($\mathcal{W}(E, \mathcal{O}_S)$). Denote by p the point on $\underline{K}_\mathcal{A} \cap \mathcal{W}(E, \mathcal{O}_S)$ with higher u value, and q the point with lower u value.

Suppose that $\mathcal{W}(E_{n-1}, \mathcal{O}_S) = \mathbf{Hyp}_L^h$ (note that it must be a horizontal wall). If there is a u_0 with $0 < u_0 < u(q)$ with $\mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$, then $\mathbf{active}(E_{n-1} \hookrightarrow \mathcal{O}_S) \supset \mathbf{active}(E \hookrightarrow \mathcal{O}_S)$ and E does not actually destabilize $\mathcal{O}_S$.

If instead $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \preceq \mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_u$ for all $0 < u < u(p)$ (we know this holds at least for $u(q) \leq u \leq u(p)$), then we have that $K_{n-1,\mathcal{A}}$ cuts through $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$.

This is also the case if $\mathcal{W}(E_{n-1}, \mathcal{O}_S) = \mathbf{Hyp}_R, \mathbf{Ell}, \mathbf{Par}$, or $\mathbf{Con}$. And furthermore, by Lemma 4.22, by continuing this process of "cutting down" E we obtain an E_i with $\mathcal{W}(E_i, \mathcal{O}_S) = \mathbf{Hyp}_L$ and $\mathbf{active}(E_i \hookrightarrow \mathcal{O}_S) \supset \mathbf{active}(E_{n-1} \hookrightarrow \mathcal{O}_S)$. Thus we may as well assume that $\mathcal{W}(E_{n-1}, \mathcal{O}_S) = \mathbf{Hyp}_L$ as above.

Thus, with each "cut" we obtain a $\mathbf{Hyp}_L$ wall which either covers $\mathcal{W}(E, \mathcal{O}_S)$ for low u or has an active region which contains $\mathbf{active}(E \hookrightarrow \mathcal{O}_S)$. If this process continues until $E = E_2$, then $\mathbf{active}(E_2 \hookrightarrow \mathcal{O}_S) \supset \mathbf{active}(E \hookrightarrow \mathcal{O}_S)$ is forced, because $K_1 = E_1$ is Bogomolov and thus $E_{1,\mathcal{A}}$ cannot cut through $\mathcal{W}(E_1, \mathcal{O}_S)$.

We conclude that an actually destabilizing E behaves as in the following figure. Note that the above proof holds for *any surface S*.
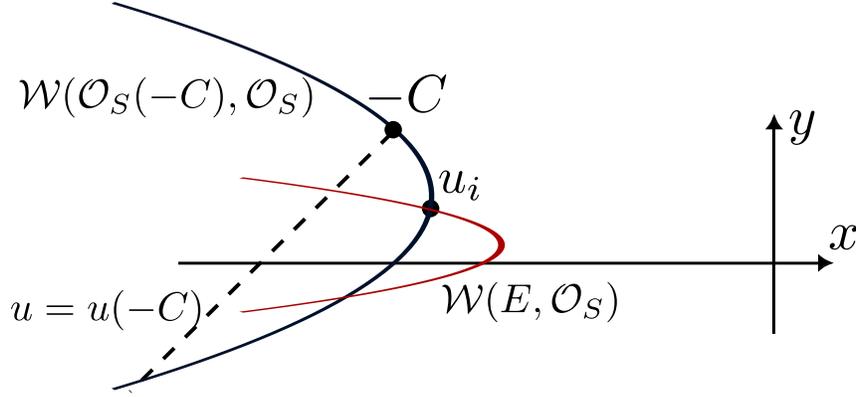
FIGURE 5.3. The situation if E actually destabilizes $\mathcal{O}$

(2): Suppose that E satisfies **AD**. For E to actually destabilize $\mathcal{O}$ we must have $\mathbf{bR}(E, \mathcal{O}; -C_1) \neq \varnothing$. Note that $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O}) = \mathbf{Hyp}_L^h$ and $M(-C_1) \geq M(-C) > M(E)$ and finally $x(-C) \leq x(-C_1)$. We claim that $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$, i.e. that $u(-C_1) > u(p)$ for all points $p \in \mathbf{bR}(E, \mathcal{O}; -C_1)$. For E to violate this, $\mathcal{W}(E, \mathcal{O})$ would have to intersect the wall for $\mathcal{O}(-C_1)$ at a u higher than $u(-C_1)$. Note that this would imply that there is part of $\mathrm{in}(\mathcal{W}(E, \mathcal{O}))$ with y value larger than $y(-C_1)$. But by our assumptions on S we know that $-C_1$ must live somewhere in this shaded region:

Now, if there is part of $\mathrm{in}(\mathcal{W}(E, \mathcal{O}))$ with y value larger than $y(-C_1)$, then this would imply that $\mathrm{in}(\mathcal{W}(E, \mathcal{O}))$ contains $-C$ (since if $p \in \mathrm{in}(\mathcal{W}(E, \mathcal{O}))$ then $R_p^0 \subset \mathrm{in}(\mathcal{W}(E, \mathcal{O}))$, where R_p^0 is the infinite ray pointing out from the origin and starting at p). This contradicts Lemma 5.9(1) and thus we have $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$.

$\square$

We now single out three properties which an actually destabilizing object E (with $\mathrm{rk}(E) \geq 2$) must satisfy. Our strategy is to only consider objects which satisfy these properties, and in fact obtain a contradiction and show that no such objects exist (see Proposition 5.10). It

follows that no E with $\mathrm{rk}(E) \geq 2$ actually destabilizes $\mathcal{O}$ and we obtain our characterization of $\mathcal{O}$'s stability.

We say that E satisfies properties **B1** if properties (a1),(b1), and (c1) are satisfied:

(a1) $E \hookrightarrow \mathcal{O}$ with $\mathrm{ch}_1(Q_0) = C$ for some negative curve C

(b1) E weakly destabilizes $\mathcal{O}$ and the walls $\mathcal{W}(E, \mathcal{O})$ and $\mathcal{W}(\mathcal{O}(-C), E)$ are both $\mathbf{Hyp}_L^h$

(c1) $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$

Note that we have abused notation in (c1) and have used $-C_1$ to mean the point v_{-C_1} - we will often make this substitution. There is an analogous set of properties **B2** concerning $-C_2$ and $\mathbf{Hyp}_L^v$ walls. Note that if an E satisfies **B1** or **B2** then there is such an E with minimal rank. Also, because of our hypothesis on S, all rank 1 walls are contained in either $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O})$ or $\mathcal{W}(\mathcal{O}(-C_2), \mathcal{O})$ and thus the minimal rank of such E's is 2 or more.

Note that if $\mathbf{bR}(E, \mathcal{O}; -C_1) \neq \varnothing$ then either $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$ or $u(-C_1) < u(\mathbf{bR}(E, \mathcal{O}; -C_1))$. This is because the wall $\mathcal{W}(E, \mathcal{O}(-C_1))$ is either a horizontal or vertical wall. If it is horizontal, then $u(\mathbf{active}(E \hookrightarrow \mathcal{O}(-C_1))) > u(-C_1)$, and if it is vertical the inequality is flipped. Then note that $\mathbf{active}(E \hookrightarrow \mathcal{O}(-C_1)) \supset \mathbf{bR}(E, \mathcal{O}; -C_1)$.

Finally, if $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$ then we also have $u(\mathbf{bR}(E, \mathcal{O}; -C_1)) > 0$ and $s(\mathbf{bR}(E, \mathcal{O}; -C_1)) < s(-C_1)$. The first inequality follows from $\mathbf{bR}(E, \mathcal{O}; -C_1) \subset \mathbf{active}(E \hookrightarrow \mathcal{O})$ and the second follows from $\mathbf{bR}(E, \mathcal{O}; -C_1) \subset \mathbf{active}(E \hookrightarrow \mathcal{O}(-C_1))$.

PROPOSITION 5.10. *Let S be a smooth projective surface of Picard rank 2 with its effective cone generated by two negative curves Eff$= \langle C_1, C_2 \rangle$. Suppose also that there is no curve $-C$ in T_1 or T_2. Then there is no E which satisfies **B1** or **B2**.*

PROOF. Let E be a sheaf of minimal rank among those which satisfy **B1** or **B2**. Then $\mathrm{rk}(E) \geq 2$ and Q_{-1} is non-zero. Our first goal is to show that there exists a σ with $E \hookrightarrow$

$\mathcal{O}(-C_1) \hookrightarrow \mathcal{O}$ and $\beta(\mathcal{O}) = \beta(\mathcal{O}(-C_1)) = \beta(E)$. We begin by showing that the arrangement of $\mathcal{W}(E, \mathcal{O})$ and $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O})$ is as in Figure 5.4.

Note that since $\mathbf{bR}(E, \mathcal{O}; -C_1) \neq \varnothing$ there must be u values with $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O}) \cap \Pi_u \preceq \mathcal{W}(E, \mathcal{O}) \cap \Pi_u$. Let $2P$ denote the (x, y) point on $\mathcal{W}(E, \mathcal{O})$ with tangent slope -1 and $u > 0$ (c.f. Section 5.1).

If $x(2P) \geq x(-C_1)$ then we must have $M(E) > M(-C_1)$. But this together with $u(-C_1) > u(\mathbf{bR}(E, \mathcal{O}; -C_1))$ implies the existence of σ with $u(\sigma) = u(-C_1)$ and E weakly destabilizing $\mathcal{O}(-C_1)$ at σ. This cannot be, as $\mathcal{O}(-C_1)$ is stable in $u = u(-C_1)$. Thus $x(-C_1) < x(2P)$.

If $M(E) \geq M(-C_1)$ then again we obtain a σ with $u(\sigma) = u(-C_1)$ and E weakly destabilizing $\mathcal{O}(-C_1)$ at σ. Thus $M(-C_1) > M(E)$ and the two $\mathbf{Hyp}_L^h$ walls intersect at some $u = u_i$. If $u_i \geq u(-C_1)$ we also obtain a σ with $u(\sigma) = u(-C_1)$ and E weakly destabilizing $\mathcal{O}(-C_1)$ at σ. Thus we have $u(-C_1) > u_i$ and our situation is as shown in Figure 5.4 below.
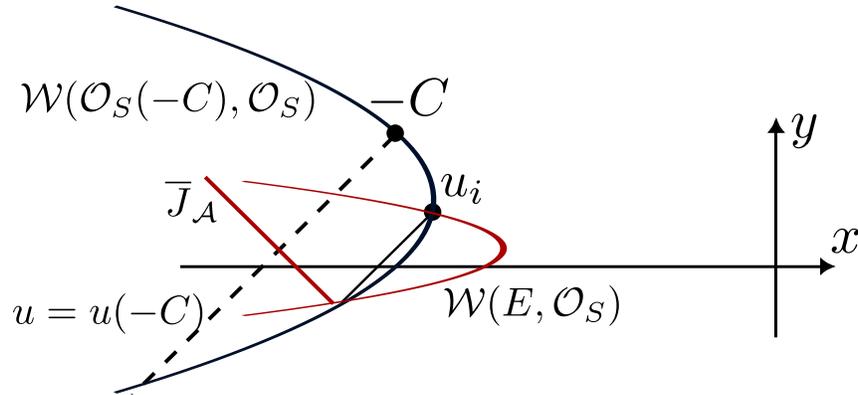
FIGURE 5.4. Illustration of the situation where a minimal E satisfies **B1**

If we denote by q the point of $\overline{J}_{\mathcal{A}} \cap \mathcal{W}(E, \mathcal{O})$ with lower u value, then we must have $u(q) \geq u_i$. This is because there can be no $u_\star$ where $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O}) \cap \Pi_u \preceq \mathcal{W}(E, \mathcal{O}) \cap$

Π_u and $\overline{J}_{\mathcal{A}}$ cuts through $\mathcal{W}(E, \mathcal{O}) \cap \Pi_u$. To see this, suppose such a $u_\star$ existed. Then $\mathcal{W}(E, \mathcal{O}) \cap \Pi_{u_\star} \preceq \mathcal{W}(E^{2,m}, \mathcal{O}) \cap \Pi_{u_\star}$ and there would exist stability conditions on $\mathcal{W}(E^{2,m}, \mathcal{O})$ inside $\mathbf{active}(E^{2,m} \hookrightarrow E \hookrightarrow \mathcal{O})$. Thus $\sigma \in \mathbf{bR}(E^{2,m}, \mathcal{O}; -C_1)$, and since $u(-C_1) > u_\star$ by hypothesis, $E^{2,m}$ satisfies B1. But then $\mathrm{rk}(E^{2,m}) < \mathrm{rk}(E)$ contradicts the minimality of E.

Since $\overline{J}_{\mathcal{A}} < \underline{K}_{\mathcal{A}}$ there exists a σ with $u(-C_1) > u(\sigma) = u_i$ and $E \hookrightarrow \mathcal{O}(-C_1) \hookrightarrow \mathcal{O}$ and $\beta(\mathcal{O}) = \beta(\mathcal{O}(-C_1) = \beta(E)$, as we desired.
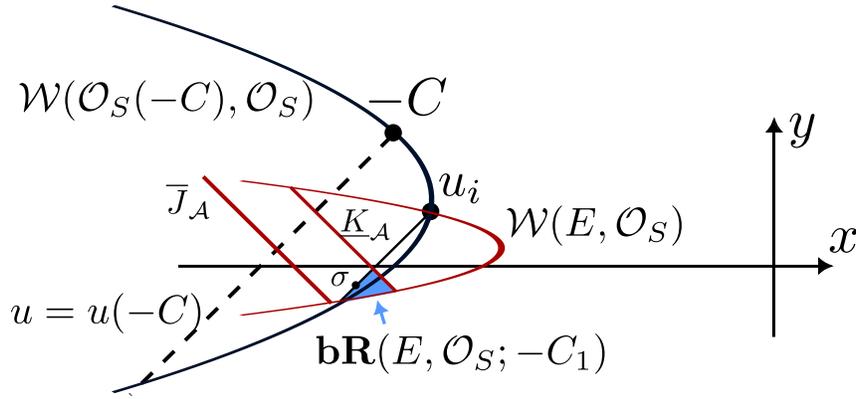
FIGURE 5.5. The Bad Region for E and $-C_1$ goes right up to u_i

For any such σ, at $\mathcal{O}(C_1) \otimes \sigma$ we have $E(C_1)$ weakly destabilizes $\mathcal{O}$. Note that $0 > u(\sigma \otimes \mathcal{O}(C_1))$. We may now "cut down" the quotient of $E(C_1) \hookrightarrow \mathcal{O}$ until we obtain a wall that is not a $\mathbf{Hyp}_L$, then "cut down" the associated subobject until we obtain a $\mathbf{Hyp}_L$ wall, and continue this process until forced to stop with one of the two cases listed below occurring at a stability condition directly above $\mathcal{O}(C_1) \otimes \sigma$ (i.e. after increasing t):

Case (1): $\mathcal{O}(-\widetilde{C}) \hookrightarrow \mathcal{O}$ with $\beta(\mathcal{O}(-\widetilde{C})) = \beta(\mathcal{O})$ and $\beta(\mathcal{O}(C_1)) > \beta(\mathcal{O}) > \beta(E(C_1))$

Case (2): $\widetilde{E}(C_1) \hookrightarrow \mathcal{O}(-\widetilde{C}) \hookrightarrow \mathcal{O}$ with $\beta(\widetilde{E}(C_1)) = \beta(\mathcal{O}) > \beta(\mathcal{O}(-\widetilde{C}))$ and $\beta(\mathcal{O}(C_1)) > \beta(\mathcal{O}) > \beta(E(C_1))$

Here $\widetilde{C}$ is a negative curve with $\mathcal{W}(\mathcal{O}(-\widetilde{C}), \mathcal{O}) = \mathbf{Hyp}_L^v$ and $\widetilde{E}(C_1) =: F$ is a subobject of $\mathcal{O}$ with $\mathrm{rk}(F) > \mathrm{rk}(E)$ and such that if $F' = \mathrm{coker}(F \hookrightarrow \mathcal{O})$ then $\mathrm{ch}_1(F_0') = \widetilde{C}$. Note that

$\mathcal{W}(\mathcal{O}(-\widetilde{C}), \mathcal{O}) = \mathbf{Hyp}_L^v$ implies that $\widetilde{C} = aC_1 + bC_2$ with $b \geq 1$ and thus $\mathcal{O}(-\widetilde{C}) \hookrightarrow \mathcal{O}(-C_2)$. We consider each situation in turn: in Case (1), we find that we violate our assumption on curves in S, and in Case (2) we violate the minimality of E.

Case (1): Lowering t we have at $\mathcal{O}(C_1) \otimes \sigma$ that $\beta(\mathcal{O}(-\widetilde{C})) > \beta(\mathcal{O}) = \beta(\mathcal{O}(C_1))$. Tensoring by $\mathcal{O}(-C_1)$ yields that at σ we have $\beta(\mathcal{O}(-C_1 - \widetilde{C})) > \beta(\mathcal{O})$. Thus $-C_1 - \widetilde{C}$ is a negative curve with $\mathcal{W}(\mathcal{O}(-C_1 - \widetilde{C}), \mathcal{O}) = \mathbf{Hyp}_L^h$. But then $\mathcal{W}(\mathcal{O}(\widetilde{C}), \mathcal{O}) = \mathbf{Hyp}_L^v$ implies that $-C_1 - \widetilde{C}$ is in $T1$ contradicting our assumption on S. Thus Case (1) cannot happen.

Case (2): Let σ' be the stability condition above $\mathcal{O}(C_1) \otimes \sigma$ from the statement of Case (2) above. If at σ' we have $\beta(\mathcal{O}(-C_2)) \geq \beta(\mathcal{O})$ then we are done as in Case (1), but now with C_2 instead of $\widetilde{C}$. So assume that at σ' we have $\beta(\mathcal{O}) = \beta(F) > \beta(\mathcal{O}(-C_2))$. We show that F satisfies (a2),(b2), and (c2), which contradicts the minimality of E since $\mathrm{rk}(F) < \mathrm{rk}(E)$.

The fact that F satisfies (a2) and (b2) comes from the stopping criteria for the process of successively "cutting down" quotients and subobjects described above, when we obtained the statements of Case (1) and (2). To finish, we must show (c2), i.e. that $u(\mathbf{bR}(E, \mathcal{O}; -C_2)) > u(-C_2)$. We have $\mathbf{bR}(E, \mathcal{O}; -C_2) \neq \varnothing$ and thus either $u(\mathbf{bR}(E, \mathcal{O}; -C_2)) > u(-C_2)$ or $u(\mathbf{bR}(E, \mathcal{O}; -C_2)) < u(-C_2)$. Let us suppose the latter and show we contradict our assumption on S.

Then $\sigma' \in \mathbf{bR}(E, \mathcal{O}; -C_2)$ and thus $u(-C_2) > u(\sigma')$. But $u(\sigma' \otimes \mathcal{O}(-C_1)) = u_i$ which intersects the line $x = x(-C_1)$ above $y = 0$. This implies that $-C_1 - C_2$ is in T_1, contradicting our assumption on S. Thus Case (2) cannot happen either.

FIGURE 5.6. Case (2) causes $-C_1 - C_2$ to be in T_1

Since we obtain a contraction in either case, our assumption that there exists an E satisfying **B1** or **B2** yields a contradiction. Thus no such E exists, as we desired to show. $\square$

Combining Lemma 5.9 and Proposition 5.10 yield a proof of Theorem 5.1 - we need only recall that our assumption on S implies that each rank 1 wall is contained completely inside $\mathcal{W}(\mathcal{O}(-C_1), \mathcal{O})$ or $\mathcal{W}(\mathcal{O}(-C_2), \mathcal{O})$.

CHAPTER 6

PROJECTIVITY OF BRIDGELAND MODULI SPACES ON $\mathbb{P}^1 \times \mathbb{P}^1$

Bridgeland stability conditions are not a priori tied to Geometric Invariant Theory (GIT), and consequently the structure of Bridgeland moduli spaces $\mathcal{M}_\sigma(v)$ is not fully understood in general. However, in some cases, the structure is known. In [6] nef divisors associated to stability conditions are constructed on Bridgeland moduli spaces - for K3 surfaces, these divisors are ample and Bayer and Macrì show that (generic) Bridgeland moduli spaces are irreducible and projective.

Another method to deduce structure for these moduli spaces is to find particular stability conditions that do have ties to GIT, and relate other stability conditions to these special ones. This was done in [3] for $S = \mathbb{P}^2$. There, certain stability conditions are found which have a heart whose objects can be seen as finite-dimensional representations of a quiver. There, Bridgeland stability is shown to agree with the stability of King [21]. Finally it is shown that after choosing invariants v of interest, any stability condition can be moved into a "quiver region" without crossing a wall for v. Then projectivity of the Brigeland moduli spaces follows from the result of King for representations of a quiver.

We now carry out the program from [3] for $S = \mathbb{P}^1 \times \mathbb{P}^1$. In each $\mathcal{S}_{H,G}$ we find regions governed by quivers by considering the position via the central charge of certain line bundles. Tensoring with line bundles then tiles the entire $t = 0$-plane with these regions. Then Bertram's Nested Wall Theorem [22, Theorem 3.1] allows us to "slide down the wall" and obtain projectivity of moduli spaces for any $\sigma \in \text{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$.

6.1. Quiver Hearts

In [14] it is shown that, on Del Pezzo surfaces, to certain collections of exceptional objects one can associate a heart that is equivalent to representations of a quiver (with relations) determined by the irreducible maps between the exceptional objects.

We will be interested in the collection $\mathbb{E} = (\mathcal{O}(0,0), \mathcal{E}, \mathcal{O}(1,0), \mathcal{O}(0,1))$, where $\mathcal{E}$ is *the* exceptional object with $(r, d_1, d_2) = (3,1,1)$ and $\mathcal{O}(a,b)$ is the line bundle $p_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b)$ with p_1, p_2 the natural projections from $\mathbb{P}^1 \times \mathbb{P}^1$. This exceptional collection generates a geometric helix and the heart $\mathcal{A}_{\mathbb{F}}$ associated to it is the extension closed category generated by

$$\mathbb{F} = (F_4, F_3, F_2, F_1) = (\mathcal{O}(-2,-1)[2], \mathcal{O}(-1,-2)[2], \mathcal{O}(-1,-1)[1], \mathcal{O}(0,0))$$

Note that $\mathbb{F}$ is an "Ext" exceptional collection in the sense of [23, Definition 3.10]. The heart $\mathcal{A}_{\mathbb{F}}$ is naturally equivalent to finite-dimensional contravariant representations of the quiver (with relations)
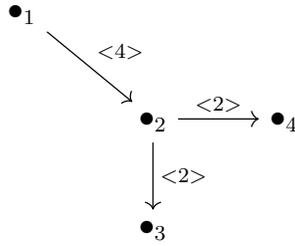
Where the labels on the arrows represent the number of arrows, and the labels on the vertices are chosen so that $F_i \leftrightarrow S_i$ under the equivalence, where S_i is the simple representation at the ith vertex. This picture also holds for any tensor of $\mathbb{F}$ by line bundles - we denote these by $\mathcal{O}(p,q) \otimes \mathbb{F}$.

6.2. Locating the Quiver Regions

Here we introduce a convenient reparameterizing of the spaces $\mathcal{S}_{G,H}$ in $\mathrm{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$,

and then find certain regions within these slices whose stability conditions have heart $\mathcal{A}_{\mathbb{F}}$.

6.2.1. New Coordinates for $\mathcal{S}_{G,H}$. The Picard rank of $\mathbb{P}^1 \times \mathbb{P}^1$ is 2, and any divisor in

$\mathbb{P}^1 \times \mathbb{P}^1$ is in some linear system $|\mathcal{O}(x, y)|$. An $\mathbb{R}$-divisor H is ample if and only if it is in some

$|\mathcal{O}(a, b)|$ with $a, b > 0$. For such an H, we write $\mathcal{S}_{G,H} = \mathcal{S}_{a,b} = \{\sigma_{x,y,ta,tb} \mid x, y \in \mathbb{R}, t > 0\}$.

Here $\sigma_{x,y,a,b} = \sigma_{D,H}$ where $D \in |\mathcal{O}(x, y)|$ and $H \in |\mathcal{O}(a, b)|$. We will associate $\sigma_{x,y,ta,tb}$ with

the tuple (x, y, t).

If $\mathrm{ch}(E) = (r, (d_1, d_2), c)$, then then in $\mathcal{S}_{a,b}$ the equation $\mu_{a,b}(E) = \mu_{a,b}(\mathcal{O}(x, y))$ simplifies

to $y = -\frac{b}{a}(x - \frac{d_1}{r}) + \frac{d_2}{r}$. Therefore, the line $E_{\mathcal{A}}$ is these coordinates is the line of slope $-b/a$

through the point $v_E = (\frac{d_1}{r}, \frac{d_2}{r})$.

Since the lines $E_{\mathcal{A}}$ have negative slope, we may speak of a point (x, y) lying "to the left"

or "to the right" of $E_{\mathcal{A}}$. Specifically, if P is a set of points in the $t = 0$ plane, we say P **lies**

to the left of $E_{\mathcal{A}}$ and write $P < E_{\mathcal{A}}$ if $y < -\frac{b}{a}(x - \frac{d_1}{r}) + \frac{d_2}{r}$ for each $(x, y) \in P$. We define

lying to the right ($E_{\mathcal{A}} < P$) similarly. If $P \leq E_{\mathcal{A}}$ we say P **lies weakly to the left** of $E_{\mathcal{A}}$,

and similarly for $E_{\mathcal{A}} \leq P$.

6.2.2. Quiver Regions. Note that the objects in the heart of any $\sigma_{x,y,a,b} = \sigma$ can be

non-zero only in positions 0 and -1. Thus, the objects of $\mathbb{F}$ cannot all be in the heart of σ.

However, by aligning the phases of different shifts of these objects, we can then act on σ by

a rotation to get a stability condition $\sigma' \in \mathrm{Stab}(\mathbb{P}^1 \times \mathbb{P}^1) \backslash \mathrm{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$, such that σ' has

heart $\widetilde{\mathcal{A}_{\mathbb{F}}}$. "Rotation" is a restriction of the action of $\widetilde{\mathrm{GL}^+(2, \mathbb{R})}$ given in [12, Lemma 8.2]. It

is essentially adjusting the slicing of σ by a constant (see [12, Definition 1.1 and 3.3]). Note

that stability of an object is unchanged by a rotation of σ, and thus the moduli spaces of stable objects are also unchanged.
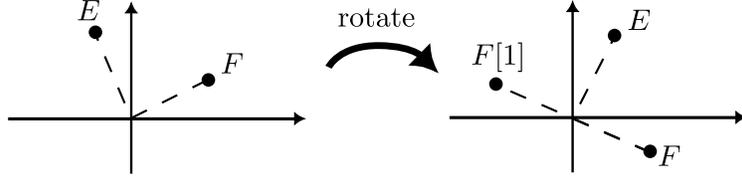
FIGURE 6.1. A clockwise rotation

We define a **quiver region** to be a subset of $\text{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$, where after rotation its heart is given by $\mathcal{A}_{\mathbb{F}}$ (or a tensor of $\mathbb{F}$ by a line bundle). We call a stability condition a **quiver** stability condition if it is in a quiver region. Since all line bundles (and their shifts) are stable for any $\sigma \in \text{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$, [23, Lemma 3.16] implies that whenever the objects of $\mathbb{F}$ have phases in $(0, 1]$, the heart of the stability condition is $\mathcal{A}_{\mathbb{F}}$.

We will show that, near the $t = 0$ plane in $\mathcal{S}_{ab}$, all stability conditions are quiver stability conditions. Then, starting from any stability condition $\sigma_{x,y,a,b}$ Bertram's Nested Wall Theorem [22, Theorem 3.1] allows us to "slide down the wall" and enter a quiver region without crossing any walls.

PROPOSITION 6.1. *For any $\mathcal{S}_{a,b} =: \mathcal{S} \subset \text{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$, there is a $t_0 > 0$ such that any (x, y, t) with $t < t_0$ is in a quiver region. Specifically, we show that there is an open set $\mathfrak{U}$ over (i.e. "whose projection onto the xy-plane contains") $\{(x, y) \mid x, y \in [-1, 0]\} \backslash \{(-1, 0), (0, -1), (0, 0)\}$ cut by a finite number of inequalities such that each $\sigma \in \mathfrak{U}$ is a quiver stability condition (with heart $\mathcal{A}_{\mathbb{F}}$ after a rotation).*

PROOF. Choose $H \in |\mathcal{O}(a, b)|$. We will always be rotating clockwise (i.e. decreasing the phase of our objects), which implies that each object stays in the heart or gets replaced by a *positive* shift of itself. Thus, to have the heart $\mathcal{A}_{\mathbb{F}}$ after rotating, we must

have $\mathcal{O}(0,0), \mathcal{O}(-2,-1)[1]$ and $\mathcal{O}(-1,-2)[1]$ in our original heart. This gives the following inequalities:

- for $\mathcal{O}(0,0) \in \mathcal{A}$, we need $(x,y) < \mathcal{O}_{\mathcal{A}}$

- for $\mathcal{O}(-2,-1)[1] \in \mathcal{A}$, we need $\mathcal{O}(-2,-1)_{\mathcal{A}} \leq (x,y)$

- for $\mathcal{O}(-1,-2)[1] \in \mathcal{A}$, we need $\mathcal{O}(-1,-2)_{\mathcal{A}} \leq (x,y)$

Let $\mathbb{U} := \{(x,y) \mid x,y \in [-1,0]\}$ which we think of as a subset of $\mathcal{S}$. Note that the region $\mathfrak{R}_\heartsuit \subset \mathcal{S}$ given by the above inequalities sits over $\mathbb{U} \backslash \{(0,0)\}$. We now must control the arrangement of these objects via the central charge for $\sigma \in \mathfrak{R}_\heartsuit$. Specifically, we need both $\beta(\mathcal{O}(0,0)) > \beta(\mathcal{O}(-2,-1)[1])$ and $\beta(\mathcal{O}(0,0)) > \beta(\mathcal{O}(-1,-2)[1])$ so that rotating will shift $\mathcal{O}(-2,-1)[1]$ and $\mathcal{O}(-1,-2)[1]$, but not $\mathcal{O}(0,0)$.

In the xy-plane of $\mathcal{S}_{a,b}$, the wall $\mathcal{W}(\mathcal{O}(0,0),\mathcal{O}(-2,-1)[1])$ is an ellipse passing through $(0,0)$ and $(-2,-1)$ with slope $-b/a$. The region enclosed by the ellipsoidal wall in $\mathcal{S}_{a,b}$ corresponds to the inequality $\beta(\mathcal{O}(0,0)) > \beta(\mathcal{O}(-2,-1)[1])$. We call this region $\mathfrak{R}_{-2-1}$. Note that it sits over $\mathbb{U} \backslash \{(0,0),(0,-1)\}$.

Similarly, the region enclosed by the ellipsoidal wall $\mathcal{W}(\mathcal{O}(0,0),\mathcal{O}(-1,-2)[1])$ in $\mathcal{S}_{a,b}$ corresponds to the inequality $\beta(\mathcal{O}(0,0)) > \beta(\mathcal{O}(-1,-2)[1])$ and we call this region $\mathfrak{R}_{-2,-1}$. This region sits over $\mathbb{U} \backslash \{(0,0),(0,-1)\}$.

The line $\mathcal{O}(-1-1)_{\mathcal{A}}$ lies to the left of $\mathcal{O}(0,0)_{\mathcal{A}}$ and to the right of $\mathcal{O}(-2,-1)_{\mathcal{A}}$ and $\mathcal{O}(-1,-2)_{\mathcal{A}}$. Thus, $\mathcal{O}(-1,-1)$ or its shift can be in $\mathcal{A}$ and we must consider the arrangement of objects in each case.

If $(x,y) < \mathcal{O}(-1,-1)_{\mathcal{A}}$ so that $\mathcal{O}(-1,-1) \in \mathcal{A}$ then $\mathcal{O}$ is σ-stable and $\mathcal{O}(-1,-1) \hookrightarrow \mathcal{O}(0,0)$ in $\mathcal{A}$ implies we have $\beta(\mathcal{O}(0,0)) > \beta(\mathcal{O}(-1,-1))$. Thus rotating will shift $\mathcal{O}(-1,-1)$ but not $\mathcal{O}(0,0)$, as desired.

If $\mathcal{O}(-1,-1)_{\mathcal{A}} \leq (x,y)$ so that $\mathcal{O}(-1,-1)[1] \in \mathcal{A}$ then we must have $\beta(\mathcal{O}(-1,-1)[1]) >$ $\beta(\mathcal{O}(-2,-1)[1])$ and $\beta(\mathcal{O}(-1,-1)[1]) > \beta(\mathcal{O}(-1,-2)[1])$ so that we may shift $\mathcal{O}(-2,-1)[1]$ and $\mathcal{O}(-1,-2)[1]$ but not $\mathcal{O}(-1,-1)[1]$. But inequalities hold since $\mathcal{O}(-2,-1)[1]$ and $\mathcal{O}(-1,-2)[1]$ are σ-stable and have $\mathcal{O}(-1,-1)$ as a quotient in $\mathcal{A}$. Thus $\mathcal{O}(-1,-1)$ or $\mathcal{O}(-1,-1)[1]$ is positioned correctly for each $\sigma \in \mathfrak{R}_{\heartsuit}$.

Collecting these results, we see the open set we sought is $\mathfrak{U} = \mathfrak{R}_{\heartsuit} \cap \mathfrak{R}_{-2-1} \cap \mathfrak{R}_{-1-2} = \mathfrak{R}_{-2-1} \cap \mathfrak{R}_{-1-2}$. Each $\sigma \in \mathfrak{U}$ can be rotated to a stability condition with heart $\mathcal{A}_{\mathbb{F}}$.

We may tensor with line bundles to obtain other quiver regions. Namely, σ is a quiver stability condition with heart (up to rotation) $\mathcal{A}_{\mathbb{F}}$ iff $\mathcal{O}(p,q) \otimes \sigma$ is a quiver stability condition with heart (up to rotation) $\mathcal{A}_{\mathcal{O}(p,q) \otimes \mathbb{F}}$. Thus we obtain quiver regions $\mathfrak{U}_{\mathcal{O}(p,q) \otimes \mathbb{F}} = \mathfrak{U} + (p,q,0)$, and the union of these quivery regions lies over the entire $t = 0$ plane.

To find the t_0 from the statement of the proposition, let $M_{\mathbb{F}}(x,y) = t$, where (x,y,t) is the unique point in $\mathcal{S}$ on the boudary of $\mathfrak{U}$ (if such a point exists), and similarly define $M_{\mathbb{F} \otimes \mathcal{O}(p,q)}(x,y)$. Note that each M is continuous where it is defined.

Now, for $(x,y) \in \mathbb{U}$ define $m(x,y)$ as the maximum of $M_{\mathbb{F}}(x,y)$, $M_{\mathcal{O}(1,0) \otimes \mathbb{F}}(x,y)$, $M_{\mathcal{O}(0,1) \otimes \mathbb{F}}(x,y)$, and $M_{\mathcal{O}(1,1) \otimes \mathbb{F}}(x,y)\}$. Note that m is continuous and positive, and since it is defined on the compact set $\mathbb{U}$ it has a minimum value t_0 there. Finally, the action of line bundles on $\mathrm{Stab}_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$ restricts to an action on quiver regions, and hence this t_0 serves as the value desired in the statement of the proposition. $\square$

COROLLARY 6.2. *For any* $\sigma_{x,y,a,b} \in Stab_{div}(\mathbb{P}^1 \times \mathbb{P}^1)$ *and choice of Bogomolov chern class, the moduli space of stable (semi-stable) objects with those invariants is quasi-projective (projective).*

PROOF. By [22, Corollary 3.2] and Proposition 6.1 above, our moduli space is isomorphic to a moduli space for a quiver σ', which by [14] is isomorphic to a moduli space of quiver representations for a given dimension vector. Following the proof of [3, Proposition 8.1] we see that our stability condition is equivalent to the definition of [21] and thus the moduli space is quasi-projective if only stable objects are considered, and projective when semi-stable objects are considered. $\square$

CHAPTER 7

HELICES AND TILTING

As seen in Chapter 6, quiver stability conditions (or "algebraic stability conditions" as they are also known) can be useful in determining structure for Bridgeland moduli spaces. They have also been used in [7, Sections 6 and 7] to deduce the topology of (a connected component of) the space of stability conditions for local $\mathbb{P}^2$.

Besides these applications, considerations regarding the exceptional collections which give rise to algebraic stability conditions are interesting in their own right. In particular, in [14] Bridgeland and Stern (BS) exhibit a tight connection between tilting on quiver algebras and mutations via a height function on associated helices (which are infinite collections of exceptional objects).

After stating some preliminaries, we collect a number of results surrounding the operation of tilting. More specifically, in Section 7.2 we show that the information necessary to perform a height function mutation is contained in the associated quiver. In Section 7.3 we realize d-block mutations as repeated height function mutations, and in Section 7.4 we characterize the geometric helices on $\mathbb{P}^1 \times \mathbb{P}^1$. While not all results contained in this Chapter are new to the mathematical community, they have all been obtained independently by the author.

7.1. TILTING PRELIMINARIES

Let $D = D^b(\operatorname{Coh} X)$ for X smooth, projective variety over $\mathbb{C}$. An **exceptional object** $E \in D$ is one such that $\operatorname{Hom}^k(E, E) = 0$ for $k \neq 0$ and $\operatorname{Hom}^0(E, E) = \mathbb{C}$. An ordered sequence of exceptional objects $\mathbb{E} = (E_1, \ldots, E_n)$ is called an **exceptional collection** if $1 \leq j < i \leq n$ implies $\operatorname{Hom}^\bullet(E_i, E_j) = 0$.

Given an exceptional collection $\mathbb{E}$ we define the **right orthogonal category** as the full subcategory $\mathbb{E}^\perp = \{F \in D \mid \mathrm{Hom}^\bullet(E, F) = 0 \text{ for } E \in \mathbb{E}\}$. The **left orthogonal category** $^\perp\mathbb{E}$ is defined similarly.

An infinite collection of exceptional objects $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ is called a **helix** of type (n, d) if each **thread** $(E_{i+1}, \ldots, E_{i+n})$ is a full exceptional collection, and if we have $E_{i-n} = S_D(E_i)[1 - d]$ for each i. Here S_D is the Serre functor of D. For us, this is $S_D(E) = (E \otimes \omega_X)[\dim X]$.

To a **geometric helix** $\mathbb{H}$ (meaning $\mathrm{Hom}^\bullet(E_i, E_j) = \mathrm{Hom}^0(E_i, E_j)$ for all $i < j$) one obtains an equivalence between the bounded derived category of local X and the bounded derived category of modules over the "rolled-up helix algebra" of $\mathbb{H}$ (see [14, Section 3.2 and 3.3]).

A crucial operation to this theory is that of **mutation**. For E an exceptional object and $F \in {}^\perp E$, the **left mutation** of F through E, $L_E(F)$, is defined by the canonical evaluation triangle $\mathrm{Hom}^\bullet(E, F) \otimes E \to F \to L_E(F)$. Note that $L_E(F) \in E^\perp$. Dually, we may define a **right mutation**.

To an exceptional collection $\mathbb{E} = (E_1, \ldots, E_n)$ we associate the **dual exceptional collection** $\mathbb{F} = (F_n, \ldots, F_1)$ defined by $F_j = L_{E_1} \cdots L_{E_{j-1}}(E_j)$. These objects (resp. their pushforwards) generate the quiver heart associated to a full, strong exceptional collection (resp. a geometric helix).

The quiver associated to a geometric helix has a vertex for each object in a thread of the helix, and we associate a vertex to its respective object. One can then place arrows either by considering irreducible maps between objects in the helix, or extensions between dual objects.

We say a map $E_i \rightarrow E_j$ is **irreducible** if it is *not* in the image of

$$c_{i,j} : \bigoplus_{i<k<j} \mathrm{Hom}(E_i, E_k) \otimes \mathrm{Hom}(E_k, E_j) \longrightarrow \mathrm{Hom}(E_i, E_j)$$

For $j - i < n$, the number of arrows from vertex i to vertex j, $n_{i,j}$, is $\dim \mathrm{coker}\, c_{i,j}$. Alternatively, $n_{i,j} = \dim \mathrm{Hom}^1(F_j, F_i)$.

The description of $n_{i,j}$ in terms of Ext's between dual objects suggests the following definition, which is central to performing a height function mutation. If $i < j$ we say that E_i and E_j are p-**related** if $\mathrm{Hom}^k(F_j, F_i) = 0$ for $k \neq p$. We say that E_i and E_j are ***strictly p-related*** if they are p-related and $\mathrm{Hom}^p(F_j, F_i) \neq 0$. If $\mathrm{Hom}^k(F_j, F_i) = 0$ for all k we say that E_i and E_j are **all-related**.

We now build towards the description of a height function mutation. A levelling on a helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ of type (n, d) is a function $\phi : \mathbb{H} \rightarrow \mathbb{Z}$ such that $i \leq j$ implies $\phi(E_i) \leq \phi(E_j)$, and such that $\phi(E_{i+n}) = \phi(E_i) + d$ for each i.

Let $\mathbb{E} = (E_1, \ldots, E_n)$ be a full, strong exceptional collection. A **height function for an object** $E \in \mathbb{E}$ is a levelling $\phi : \mathbb{E} \rightarrow \mathbb{Z}$ such that $\phi^{-1}(0) = \{E\}$, and $\phi(E_j) = p \neq 0$ implies that E and E_j are p-related (if $p > 0$), or if $p < 0$ then E_j and E are $-p$-related. One defines height functions for helices be asking the above to hold for any thread in the helix.

Given a helix $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ with a height function for E. A **height function mutation** at E, henceforth called a **tilt** at E, constructs a related helix and levelling. To perform a tilt at E, choose a thread of $\mathbb{H}$ which contains both E and $\phi^{-1}(-1) =: \mathbb{E}_{-1} :$ $(\ldots, \mathbb{E}_{-1}, E, \ldots)$. Now left mutate (and shift) E through $\mathbb{E}_{-1}$ and keep this new positioning:

$(\ldots, L_{\mathbb{E}_{-1}}E[-1], \mathbb{E}_{-1}, \ldots)$. Now, using this updated thread, generate a helix $\mathbb{H}'$ and a level-ling ϕ' such that $\phi'^{-1}(-1) = L_{\mathbb{E}_{-1}}E[-1]$ and $\phi'^{-1}(0) = \mathbb{E}_{-1}$. This new helix and levelling is the result of the tilt at E.

7.2. TILTING VIA QUIVERS

To perform a tilt at an object E of a helix $\mathbb{H}$, one must know $\phi^{-1}(-1) =: \mathbb{E}_{-1}$. Here we add to a result of [14] (BS) to clarify how knowledge of the quiver associated to $\mathbb{H}$ is enough to deduce $\mathbb{E}_{-1}$. We first state the result of BS.

PROPOSITION 7.1 (BS Proposition 7.6). *Let Z be a smooth Fano variety of dimension $d-1$ and suppose $\mathbb{H}$ is a geometric helix in $D(Z)$ of type (n, d). Suppose there exists a height function $\phi : \mathbb{H} \to \mathbb{Z}$ for an object $\mathbb{E}_0 \in \mathbb{H}$ and write $\sigma_0(\mathbb{H}, \phi) = (\mathbb{H}', \phi')$. Then the algebra $B' = B(\mathbb{H}')$ is the left tilt of the algebra $B = B(\mathbb{H})$ at the vertex corresponding to E_0.*

In the following, we identify two helices if they differ by a rearrangement of mutually orthogonal, adjacent objects. Since A mutually orthogonal to B implies that $L_A B = B$, it follows that the objects of the dual collection $\mathcal{F}$ do not change under shuffling mutually orthogonal objects of $\mathbb{E}$.

PROPOSITION 7.2. *Moreover, taking the thread $\mathbb{E}$ such that E_0 is the last object in it, let $\mathbb{E}'_{-1} := \{objects\ in\ \mathbb{E}\ whose\ vertex\ has\ arrows\ to\ E_0\text{'s}\ vertex\}$. Then $\mathbb{H}'$ is obtained by taking E_0, moving it behind the last object of $\mathbb{E}'_{-1}$ and then replacing E_0 with $L_{\mathbb{E}'_{-1}}(E_0)[-1]$ (and generating the helix).*

PROOF. To tilt $\mathbb{H}$ at E_0, one takes E_0, moves it past $\mathbb{E}_{-1}$ and then replaces E_0 with $L_{\mathbb{E}_{-1}}(E_0)[-1]$. We show that our process produces the same result.

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First we show the new positions of our objects match BS: Let $\mathcal{F} = \{F_n, \ldots, F_1\}$ be the dual collection corresponding to $\mathbb{E}$. Since a height function exists by assumption, if $\mathrm{Hom}^p(F_0, F_i) \neq 0$ for some p, then p is the only such degree. Now, suppose $E_i \in \mathbb{E}_{-1}$. Then either $(n_{i,0} =) \mathrm{Hom}^1(F_0, F_i) \neq 0$ (and thus E_i's vertex has an arrow to E_0's vertex) or $\mathrm{Hom}^p(F_0, F_i) = 0$ for all p (in which case there are no arrows from E_i's vertex to E_0's vertex).

By definition, $\mathbb{E}'_{-1}$ is the subset of $\mathbb{E}$ with $(n_{i,0} =)\mathrm{Hom}^1(F_0, F_i) \neq 0$ and thus $\mathbb{E}'_{-1} \subset \mathbb{E}_{-1}$. It follows that any object $E_\star$ of $\mathbb{E}_{-1}$ that is to the left of $\mathbb{E}'_{-1}$ (in the ordering of $\mathbb{E}$) has $\mathrm{Hom}^p(F_0, F_\star) = 0$ for all p and so can be moved into $\mathbb{E}_{-2}$. Doing this gives an slightly different levelling $\widetilde{\phi}$. Thus moving E_0 just past $\mathbb{E}'_{-1}$ gives the same position as BS using $\widetilde{\phi}$.

Next, we must show that (using $\widetilde{\phi}$), we have $L_{\mathbb{E}_{-1}}(E_0) \cong L_{\mathbb{E}'_{-1}}(E_0)$, so that E_0 is replaced by the object specified in BS. Proposition 7.1 of [14] shows that $\Phi_\mathbb{E}(P_0^{(1)}) = L_{\mathbb{E}_{-1}}(E_0)$, where

$$P_0^{(1)} = [0 \longrightarrow \bigoplus_{j \in Q_0} P_j^{\oplus d_{j0}^1} \longrightarrow P_0 \longrightarrow 0]$$

and we use $d_{j0}^1 = \dim \mathrm{Hom}^1(S_0, S_j)$.

The same proof (using $m = 0, k = 1, i = 0$) shows that $\Phi_\mathbb{E}(P_0^{(1)}) = L_{\mathbb{E}'_{-1}}(E_0)$ and gives our result: Let $R := \Phi_\mathbb{E}(P_0^{(1)}) = [\bigoplus_j E_j^{\oplus d_{j0}^1} \longrightarrow E_0]$. We have the triangle $E_0 \xrightarrow{0,\mathrm{id}} R \longrightarrow A$, where $A = [\bigoplus_j E_j^{\oplus d_{j0}^1} \longrightarrow 0] \in \langle \mathbb{E}'_{-1} \rangle$ since no arrows to vertex 0 imply $d_{j0}^1 = \dim \mathrm{Hom}^1(F_0, F_\star) = n_{j0} = 0$. As in BS, by breaking the projective resolution of S_0 into three pieces and applying $\Phi_\mathbb{E}$, we have a triangle $B' \longrightarrow R \longrightarrow F_0$ with $B' \in \langle E_j \in \mathbb{E} \mid \varphi(E_j) < -1 \rangle$. Then, since $i < j$ implies $\mathrm{Hom}^\bullet(E_j, E_i) = 0$ (in an exceptional collection) and $\mathrm{Hom}^k(E_i, F_j) \neq 0$ iff $i = j$ and $k = 0$, we have that $R \in \mathbb{E}^\perp_{-1}$ and hence $R \in (\mathbb{E}'_{-1})^\perp$ since $\mathbb{E}'_{-1} \subset \mathbb{E}_{-1}$.

Now, consider the triangle $A[-1] \longrightarrow E_0 \longrightarrow R$. Since $A[-1] \in \langle \mathbb{E}'_{-1} \rangle$, and $E_0 \in {}^\perp(\mathbb{E}'_{-1})$, and $R \in (\mathbb{E}'_{-1})^\perp$, we have by [14, Proposition 2.2.b] that $R = L_{\mathbb{E}'_{-1}}(E_0)$.

Lastly, the fact that our updated height function, $\widetilde{\phi}$ gives a helix that differs from the original $\mathbb{H}'$ by shuffling mutually orthogonal objects follows from our proof above: Specifically, we have shown that $L_{\mathbb{E}_{-1}}(E_0) \cong L_{\mathbb{E}'_{-1}}(E_0)$ for any choice of height function for E_0. We also showed that, we can choose to have any object $\widetilde{E}$ of $\mathbb{E}_{-1}$ which is to the left of $\mathbb{E}'_{-1}$ be the start of $\mathbb{E}_{-2}$ (denote by $\widetilde{\mathbb{E}}_{-1}$ the set of all such $\widetilde{E}$). Thus, placing $L_{\mathbb{E}'_{-1}}(E_0)[-1]$ to the left or right of any object of $\widetilde{\mathbb{E}}_{-1}$ gives a geometric helix. But X being to the left Y in an exceptional collection implies that $\mathrm{Hom}^\bullet(Y, X) = 0$. So the fact that we still obtain a helix with $L_{\mathbb{E}'_{-1}}(E_0)[-1]$ to the left or to the right of any object of $\bar{\mathbb{E}}_{-1}$ gives that $L_{\mathbb{E}'_{-1}}(E_0)[-1]$ is mutually orthogonal to all objects in $\widetilde{\mathbb{E}}_{-1}$. $\hfill\square$

7.3. Tilting and d-Block Mutations

d-block mutations are helix mutations which preserve collections of orthogonal elements. Proposition 7.4 shows that d-block mutations for Del Pezzo surfaces are a coarser notion than tilting. We first prove a lemma which restricts $\mathbb{E}_{-1}$ when using an appropriate thread.

LEMMA 7.3. *Let Z be a (connected) Fano variety and $\mathbb{E} = (E_1, \ldots, E_n)$ a thread of $\mathbb{H}$, a geometric helix for Z of type (n, d) with $d \geq 3$. There are no height functions for E_n that allow E_1 to be in $\mathbb{E}_{-1}$.*

PROOF. We show that mutating E_n down through E_1 (which is the required operation of tilting) will never result in a geometric helix. Suppose ϕ a height function for E_n and $\mathbb{E}_{-1} = \{E_1, \ldots, E_{n-1}\}$. Then, following the tilting operation, we replace E_n with $L := L_{\mathbb{E}_{-1}}(E_n)[-1] = L_{E_1, \ldots, E_{n-1}}(E_n)[-1]$. But by [14, Remark 3.2.b] and our assumption that

$d \geq 3$ we have that $L = E_0[k]$ for $k = -1 - (1 - d) \geq 1$. So our new generating thread of $\mathbb{H}'$ is $\mathbb{E}' = (E_0[k], E_1, \ldots, E_{n-1})$. Now, $\mathbb{H}$ is a geometric helix and so $\mathrm{Hom}^\bullet(E_0, E_i) = \mathrm{Hom}^0(E_0, E_i)$. But any nonzero element of $\mathrm{Hom}^0(E_0, E_i)$ for $i \in \{1, \ldots, n-1\}$ gives a nonzero element of $\mathrm{Hom}^k(E_0[k], E_i)$, which cannot be, since $k \geq 1$ and, by the construction of BS, our mutated helix $\mathbb{H}'$ is geometric. Thus, for $i \in \{1, \ldots, n-1\}$ we have $\mathrm{Hom}^0(E_0, E_i)$ ($= \mathrm{Hom}^\bullet(E_0, E_i)) = 0$. But then $\langle E_0 \rangle \perp \langle E_1, \ldots, E_{n-1} \rangle$ and so $\mathrm{D}(Z) = \langle \langle E_0 \rangle, \langle E_1, \ldots, E_{n-1} \rangle \rangle$ implies that Z is disconnected, a contradiction. $\square$

Here we introduce our notation for d-block mutations. For a height function ϕ, let the ith level be denoted $\overline{\overline{\mathbb{E}}}_i$. Let the d-block structure of an exceptional collection be denoted by $\mathbb{E} = (\mathbb{E}_1, \ldots, \mathbb{E}_d)$, where the objects in each $\mathbb{E}_i$ are mutually orthogonal. A left d-block mutation at $\mathbb{E}_d$ takes $\mathbb{E}$ and gives $\tau_d \mathbb{E} = (\mathbb{E}_1, \ldots, L_{\mathbb{E}_{d-1}} \mathbb{E}_d[-1], \mathbb{E}_{d-1})$.

For example, the exceptional collection $\mathbb{E} = (\mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1))$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has a 3-block decomposition as $\mathbb{E}_1 = \mathcal{O}(0,0), \mathbb{E}_2 = \{\mathcal{O}(1,0), \mathcal{O}(0,1)\}$, and $\mathbb{E}_3 = \mathcal{O}(1,1)$.

PROPOSITION 7.4. *Let Z be a Del Pezzo Surface and $\mathbb{H}$ a geometric helix of type (n, d), $d \geq 3$ on Z with a d-block decomposition. Any d-block mutation can be realized as a sequence of tilts.*

PROOF. Without loss of generality, we may consider just the mutation τ_d. Since the objects within each $\mathbb{E}_i$ are mutually orthogonal, we can bring E_0 to the far left of $\mathbb{E}_d$ without obtaining an inequivalent helix. From the proof of Theorem 5.3 in [14], we have the triangle

$$(7\text{–}1) \qquad L_{\mathbb{E}_{d-1}} E_0[-1] \longrightarrow \bigoplus_{E \in \mathbb{E}_{d-1}} \mathrm{Hom}(E, E_0) \otimes E \longrightarrow E_0$$

We show first that $\overline{\overline{\mathbb{E}}}_{-1}$ is "effectively contained" in $\mathbb{E}_{d-1}$, i.e. that any object of $\overline{\overline{\mathbb{E}}}_{-1}$ to the left of $\mathbb{E}_{d-1}$ is all-related to E_0. Let $E_i \in \overline{\overline{\mathbb{E}}}_{-1}$ lie to the left of $\mathbb{E}_{d-1}$ and suppose it has an irreducible map to E_0, i.e. that E_i and E_0 are strictly 1-related. We now apply $\mathrm{Hom}(E_i, \_)$ to Equation 7–1 and use the fact that $\mathbb{H}$ is geometric and for A, B sheaves and V a vector space we have $\mathrm{Hom}(A, V \otimes B) \cong \mathrm{Hom}(A, B) \otimes V$ ($\cong \mathrm{Hom}(A \otimes V, B)$). The result is the following long exact sequence.

$$0 \longrightarrow \mathrm{Hom}(E_i, L_{\mathbb{E}_{d-1}} E_0[-1]) \longrightarrow \bigoplus_{E \in \mathbb{E}_{d-1}} \mathrm{Hom}(E_i, E) \otimes \mathrm{Hom}(E, E_0) \overset{\mathrm{comp}}{\longrightarrow}$$

$$\overset{\mathrm{comp}}{\longrightarrow} \mathrm{Hom}(E_i, E_0) \overset{g}{\longrightarrow} \mathrm{Hom}(E_i, L_{\mathbb{E}_{d-1}} E_0) \longrightarrow 0.$$

where "comp" is the natural composition map.

Since $E_i \to E_0$ is irreducible, the map "comp" is not surjective. Thus $\ker(g) = \mathrm{im}(\mathrm{comp}) \neq \mathrm{Hom}(E_i, E_0)$. Hence $g \neq 0$ and so $\mathrm{Hom}(E_i, L_{\mathbb{E}_{d-1}} E_0) \neq 0$. But $\mathrm{Hom}(E_i, L_{\mathbb{E}_{d-1}} E_0) = \mathrm{Hom}^1(E_i, L_{\mathbb{E}_{d-1}} E_0[-1])$ and thus the helix generated by $\tau_d \mathbb{E}$ is not geometric, contradicting [14, Theorem 5.3]. Thus, any object of $\overline{\overline{\mathbb{E}}}_{-1}$ that is to the left of $\mathbb{E}_{d-1}$ is all-related to E_0 and so can be moved into $\overline{\overline{\mathbb{E}}}_{-2}$, making $\overline{\overline{\mathbb{E}}}_{-1} \subset \mathbb{E}_{d-1}$.

We now show that any element of $\mathbb{E}_{d-1}$ that is not in $\overline{\overline{\mathbb{E}}}_{-1}$ is all-related to E_0 and thus can be put into $\overline{\overline{\mathbb{E}}}_{-1}$ by adjusting ϕ. Let $E_\star$ be in $\mathbb{E}_{d-1}$ but not in $\overline{\overline{\mathbb{E}}}_{-1}$. Since $d \geq 3$, Lemma 7.3 shows that $\mathbb{E}_{d-1}$ will never contain the first object in the thread. By [14, Lemma 6.3.a], we may consider the thread of $\mathbb{H}$ that starts with $E_\star$, as the maps between the corresponding dual objects are the same as those from our original thread. In other words, we may consider the thread $\mathbb{E} = (E_\star, \widehat{\mathbb{E}}, E_0, \widetilde{\mathbb{E}})$, where $\widehat{\mathbb{E}} = \{\widehat{E_1}, \dots, \widehat{E_k}\}$ (the objects of $\mathbb{E}_{d-1}$ beween $E_\star$ and E_0).

We wish to describe $\mathrm{Hom}^\bullet(L_{E_\star\widehat{\mathbb{E}}}E_0, E_\star)$. But, since shuffling mutually orthogonal elements does not change the dual objects, to obtain our result we may bring $E_\star$ right next to E_0 and then (as before) rotate our thread so that it becomes $\mathbb{E} = (E_\star, E_0, \widetilde{\mathbb{E}}')$, i.e. we have $\mathrm{Hom}^\bullet(L_{E_\star\widehat{\mathbb{E}}}E_0, E_\star) = \mathrm{Hom}^\bullet(L_{E_\star}E_0, E_\star)$.

Note that $\mathrm{Hom}(E_\star, E_0) = 0$, or else (since they are adjacent in the thread) these maps would be irreducible, which would imply that $E_\star$ and E_0 are strictly 1-related and hence that $E_\star \in \overline{\mathbb{E}}_{-1}$. But we have assumed that this is not the case. This, together with $\mathbb{H}$ geometric imply that $\mathrm{Hom}^\bullet(E_\star, E_0) = 0$. Thus, by considering the standard mutation triangle

$$L_{E_\star}E_0[-1] \longrightarrow \mathrm{Hom}^\bullet(E_\star, E_0) \otimes E_\star \longrightarrow E_0$$

we see that $L_{E_\star}E_0 = E_0$ and thus $\mathrm{Hom}^\bullet(L_{E_\star}E_0, E_\star) = \mathrm{Hom}^\bullet(E_0, E_\star) = 0$, i.e. $E_\star$ and E_0 are all-related. Since this is true for *any* object of $\mathbb{E}_{d-1}$ to the left of $\overline{\mathbb{E}}_{-1}$, we may adjust φ and choose these objects to be in $\overline{\mathbb{E}}_{-1}$.

Let $\mathbb{E}_d = \{E_0, E_1, \ldots, E_k\}$. We have shown that we can tilt at E_0 so that $\overline{\mathbb{E}}_{-1} = \mathbb{E}_{d-1}$. This takes our original thread $\mathbb{E} = (\ldots, \mathbb{E}_{d-1}, E_0, \ldots, E_k)$ and replaces it with $\mathbb{E}' = (\ldots, L_{\mathbb{E}_{d-1}}E_0[-1], \mathbb{E}_{d-1}, E_1, \ldots, E_k)$. Since Z is a Del Pezzo Surface, we know a height function (again, call it ϕ) exists for E_1 in this helix. We now show that we can also choose $\overline{\mathbb{E}}_{-1} = \mathbb{E}_{d-1}$ here:

Now, in the first part of the proof the choice of $E_0 \in \mathbb{E}_d$ was arbitrary since we can shuffle mutually orthogonal elements. Thus, for any $E_i \in \mathbb{E}_d$, the objects in $\mathbb{E}_{d-1}$ either have irreducible maps to E_i or are all-related to E_i. Thus, to show that for E_1 we may now choose $\overline{\mathbb{E}}_{-1} = \mathbb{E}_{d-1}$, we need only show that $L_{\mathbb{E}_{d-1}}E_0[-1]$ and E_1 are all-related.

Since $\mathbb{E}_{d-1}$ does not contain the first object of $\mathbb{E}$, $L_{\mathbb{E}_{d-1}}E_0[-1]$ is not the first object of $\mathbb{E}'$, and so (again, since we only care about maps between dual objects) we may move to the thread starting at $A := L_{\mathbb{E}_{d-1}}E_0[-1]$. Now A is its own dual object and, letting $B := L_{\mathbb{E}_{d-1}}E_1$, the dual object to E_1 is $L_A B$. Since $L_{\mathbb{E}_{d-1}} : {}^{\perp}\mathbb{E}_{d-1} \to \mathbb{E}_{d-1}^{\perp}$ is an equivalence, E_0 and E_1 mutually orthogonal imply that A and B are also. Thus $L_A B = B$. But $L_A B$ is the dual object corresponding to E_1. Thus A and B mutually orthogonal gives that $L_{\mathbb{E}_{d-1}}E_0[-1]$ and E_1 are all-related, as desired.

Tilting at E_1, the helix we have now is generated by

$$\mathbb{E}'' = (\ldots, L_{\mathbb{E}_{d-1}}E_0[-1], L_{\mathbb{E}_{d-1}}E_1[-1], \mathbb{E}_{d-1}, E_2, \ldots, E_k)$$

Continuing in a similar manner to above, and using the fact that $L_{\mathbb{E}_{d-1}} : {}^{\perp}\mathbb{E}_{d-1} \to \mathbb{E}_{d-1}^{\perp}$ is an equivalence, we see that we can continue to tilt at the remaining objects of $\mathbb{E}_d$ (with $\overline{\mathbb{E}}_{-1} = \mathbb{E}_{d-1}$ at each iteration) until the helix we end up with is

$$\mathbb{E}'^{\cdots\prime} = (\ldots, L_{\mathbb{E}_{d-1}}E_0[-1], L_{\mathbb{E}_{d-1}}E_1[-1], \ldots, L_{\mathbb{E}_{d-1}}E_k[-1], \mathbb{E}_{d-1}) = (\ldots, L_{\mathbb{E}_{d-1}}\mathbb{E}_d[-1], \mathbb{E}_{d-1})$$

which is $\tau_d \mathbb{E}$. $\qquad\square$

7.4. Geometric Helices on $\mathbb{P}^1 \times \mathbb{P}^1$

Varieties Z satisfying $\text{rank} K(Z) = \dim(Z) + 1$ have a well-behaved theory of mutations and geometric helices. Bondal and Polishchuk [**?**] show that, on such a Z, a mutation of a geometric helix produces a geometric helix, and Bridgeland [**?**] describes these mutations as an action of a certain braid group. In fact, on $\mathbb{P}^2$, every helix is geometric.

On surfaces which do not satisfy $\mathrm{rank}K(Z) = \dim(Z) + 1$, the theory is not quite so rigid. For instance, mutations of geometric helices on $\mathbb{P}^1 \times \mathbb{P}^1$ can result in non-geometric helices. (In fact, the content of [14] is to construct an operation (tilting) which extends mutation, and preserves the class of geometric helices.) An example is the geometric helix $\mathbb{H} = (\dots, \mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1), \dots)$. Left mutating at $\mathcal{O}(1,1)$ produces the helix $\mathbb{H}' = (\dots, \mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(-1,1), \mathcal{O}(0,1), \dots)$ which is not geometric, since $\mathrm{Hom}^1(\mathcal{O}(1,0), \mathcal{O}(-1,1)) \neq 0$.

Note that in $\mathbb{H}$, the slopes of the objects (with respect to the Mumford $(1,1)$-slope, μ) are increasing, whereas in $\mathbb{H}'$ they are not. We show that this property characterizes geometric helices on $\mathbb{P}^1 \times \mathbb{P}^1$. We begin with a lemma which precludes the possibility of forward maps of degree 2 in a helix on $\mathbb{P}^1 \times \mathbb{P}^1$.

Recall that by [19], exceptional sheaves on a quadric are locally free and μ-stable. Also, a helix of type $(4,3)$ satisfies $E_{i-n} = E_i \otimes \omega$ for all i.

LEMMA 7.5. *For $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ a helix of type $(4,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, there are no forward maps of degree 2, i.e. $\mathrm{Hom}^2(E_i, E_{i+k}) = 0$ for all i and all $k > 0$.*

PROOF. Suppose that $\mathrm{Hom}^2(E_0, E_k) \neq 0$. Then Serre duality gives $\mathrm{Hom}^2(E_k, E_{-n}) \neq 0$ and thus $\mu(E_k) < \mu(E_{-n})$ since each E_i is μ-stable. Define the positive integer q by the property $-n < k - qn < 0$. Note that $-n \neq k - qn$ for any q otherwise $E_{-n} = E_{k-qn}$ which is nonsense since $\mu(E_{k-qn}) < \mu(E_k) < \mu(E_{-n})$. Thus (E_{-n}, E_{k-qn}) is an exceptional pair.

Since $E_{k-qn} = E_k \otimes \omega^q \hookrightarrow E_k$ and $\mathrm{Hom}^2(E_k, E_{-n}) \neq 0$ we must have $\mathrm{Hom}(E_{k-qn}, E_{-n}) \neq 0$, for otherwise we would have a nonzero map from a torsion sheaf to E_{-n}. But this contradicts the fact that (E_{-n}, E_{k-qn}) is an exceptional pair. $\square$

Using Serre Duality, we quickly obtain the following.

COROLLARY 7.6. *For $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ a helix of type $(4,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, there are no backwards maps of degree 0.*

Note that, by Serre duality, there is a nonzero forward map of degree 1 iff there is a nonzero backwards map of degree 1.

THEOREM 7.7. *For $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ a helix on $\mathbb{P}^1 \times \mathbb{P}^1$, we have $\mathbb{H}$ is geometric iff $\mu(E_i) \leq \mu(E_{i+1})$ for all i. Here μ is the Mumford $(1,1)$-slope.*

PROOF. We first prove the forward direction via the contrapositive. Suppose we have $\mu(E_0) > \mu(E_1)$ (after renumbering, if necessary). Then by [19, Prop. 5.3.3] the mutation of E_1 past E_0 will be an L-ext mutation (notation from [26]), i.e. $\mathrm{Ext}^1(E_0, E_1) \neq 0$. Thus $\mathbb{H}$ is non-geometric.

We now prove the backwards direction via contradiction. Suppose $\mu_H(E_i) \leq \mu(E_{i+1})$ for all i. Suppose further that $\mathrm{Ext}^1(E_0, E_k) \neq 0$ for some $k > 0$ (Lemma 7.5 precludes degree 2 maps). If $1 \leq k \leq n-1$ then (E_0, E_k) is an exceptional pair and $\mu(E_k) < \mu(E_0)$ by [19]. This contradicts our assumption on slopes and so we may assume that $k \geq n$ and hence $\mu(E_k) > \mu(E_0)$.

We may choose k so that $\mathrm{Ext}^1(E_0, E_{k-qn}) = 0$ for all $0 \leq k - qn < k$. Choose an irreducible smooth (elliptic) curve $C \in |(2,2)|$. It is known that if E is an exceptional bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, then $E|_C$ is $\overline{\mu}$-stable where here $\overline{\mu} = \deg/\mathrm{rk}$. We also have $\mu(E) = \overline{\mu}(E|_C)$.

From the short exact sequence $0 \to E_0^* \otimes E_k \otimes \omega \to E_0^* \otimes E_k \to (E_0^* \otimes E_k)|_C \to 0$ we obtain the long exact sequence described in part below.

$$\cdots \to \mathrm{Ext}^1(E_0, E_k \otimes \omega) \to \mathrm{Ext}^1(E_0, E_k) \to \mathrm{Ext}^1(E_0|_C, E_k|_C) \to \cdots$$

If $\text{Ext}^1(E_0|_C, E_k|_C) \neq 0$ then by Serre duality on C (where $\omega_C = \mathcal{O}_C$) we have $\text{Ext}^1_C(E_0|_C, E_k|_C) = \text{Hom}_C(E_k|_C, E_0|_C) \neq 0$. But this is a contradiction, since $E_0|_C$ and $E_k|_C$ are $\overline{\mu}$-stable with $\overline{\mu}(E_k|_C) = \mu(E_k) > \mu(E_0) = \overline{\mu}(E_0|_C)$. Thus $\text{Ext}^1(E_0|_C, E_k|_C) = 0$ and hence $\text{Ext}^1(E_0, E_k \otimes \omega) \twoheadrightarrow \text{Ext}^1(E_0, E_k) \neq 0$. It follows that $\text{Ext}^1(E_0, E_k \otimes \omega) = \text{Ext}^1(E_0, E_{k-n}) \neq 0$, contradicting our choice of k. Thus $\mathbb{H}$ is geometric. $\qquad\square$

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