Thesis

Computing Syzygies of Homogeneous Polynomials using Linear Algebra

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Abstract

Computing Syzygies of Homogeneous Polynomials using Linear Algebra

Given a ideal generated by polynomials $f_1, ..., f_n$ in $\mathbb{C}[x_1, ..., x_m]$ a syzygy is an n-tuple $(\alpha_1, .., \alpha_n)$, $\alpha_i \in \mathbb{C}[x_1, ..., x_m]$ such that $\sum_{i=1}^{n} \alpha_i \cdot f_i = 0$. Syzygies can be computed by Buchberger's algorithm for computing Gröbner Bases. However, Gröbner bases have been computationally impractical as the number of variables and number of polynomials increase. The aim of this thesis is to describe a way to compute syzygies without the need for Gröbner bases but still retrieve some of the same information as Gröbner bases. The approach is to use the monomial structure of the polynomials in our generating set to build syzygies using Nullspace computations.

Table of Contents

CHAPTER 1

Introduction

Given a polynomial ring R = $\mathbb{C}[x_1, ..., x_m]$, where $\mathbb{C}$ is the field of complex numbers, and a set of homogeneous polynomials that serve as a generating set for an ideal of the ring R, we can ask if there exist relationships between these generators. One possible relationship that we may want to find is the following.

Definition 1.0.1. *Given a ideal generated by polynomials $f_1, ..., f_n$ in $\mathbb{C}[x_1, ..., x_m]$ a **syzygy** is an n-tuple $(\alpha_1, .., \alpha_n)$, $\alpha_i \in \mathbb{C}[x_1, ..., x_m]$ such that $\sum_{i=1}^{n} \alpha_i \cdot f_i = 0$*

Currently, syzygies can be computed by Buchberger's algorithm for computing Gröbner Bases.

Definition 1.0.2. *Fix a monomial order $>$ on $K[x_1, ..., x_n]$ and let $I \subset K[x_1, ..., x_n]$ be an ideal. A **Gröbner basis** for I (with respect to $>$) is a finite collection of polynomials $G = \{g_1, ..., g_t\} \subset I$ with the property that for every nonzero $f \in I$, $LT(f)$ is divisible by $LT(g_i)$ for some i, where $LT(f)$ is the leading term of f with respect to $>$.*
[1]

From experience, Gröbner Basis algorithms do not scale well. Mayr and Meyer showed that these computations are doubly exponential in the number of variables, though that is the worst-case scenario see [2]. Generally speaking, unless there is some special structure to the problem, these methods are typically prohibitively slow for problems of moderate size, say 10-20 variables. Another limitation is the stability of Gröbner bases with floating point coefficients for polynomials see [3] and [4]. An aim of this research is to avoid these limitations by finding syzygies without Gröbner bases. Software packages include Macaulay2 [5], and Singular [6] (with a nice introduction in [7]).

An example of ideals and syzygies is as follows :

EXAMPLE 1.0.1. *Let* $I = \langle x^3y + z^4, xy^2, x^3 + y^3, z^5 \rangle \subseteq \mathbb{C}[x, y, z]$ *[8]*

Using software package Singular [6], we can compute the syzygies of this ideal. Using the *syz* command in Singular, one obtains the following syzygies:

$$(0, x^3 + y^3, -xy^2, 0), (yz, -x^2z, 0, -y), (xz, y^2z, -xyz, -x), (xy^2, y^4 - z^4, -xy^3, 0),$$
$$(x^3 + y^3, 0, -x^3y - z^4, 0), (y^4 + z^4, -x^2y^3, -yz^4, -z^3)$$

Theses syzygies are found using algorithms based on Gröbner basis techniques. The fundamental question is using only Linear Algebra is it possible to create syzygies without Gröbner bases? We will use this system as our running example to illustrate the two algorithms in this paper.

CHAPTER 2

First Syzygies

Before we start with the running example, there are a few points that can be better explained with a smaller example. Consider the example below:

Example 2.0.2. $I = \left\langle x+4y+z, -\frac{1}{3}x-2y, \frac{1}{3}x+z \right\rangle$

Not multiplying by any monomials lets us consider if there exists a syzygy comprised of constant polynomials. To do this, we first write these polynomials in their coefficient vector form. We then concatenate these vectors to construct a matrix.

$$A = \begin{array}{cc} & \begin{array}{ccc} x+4y+z & -\frac{1}{3}x-2y & \frac{1}{3}x+z \end{array} \\ & \left(\begin{array}{ccc} 1 & -\frac{1}{3} & \frac{1}{3} \\ 4 & -2 & 0 \\ 1 & 0 & 1 \end{array}\right)\begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

We now wish to find a vector $\mathbf{x}$ such that A$\mathbf{x} = \mathbf{0}$. We can clearly solve this first problem by computing the Nullspace of our matrix, resulting in the following vector as a basis vector:

$$\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

We can pull off each coefficient as a component of our syzygy and obtain the syzygy (-1,-2,1).

Next, we will consider when a syzygy is of the form $(ax+by+cz, dx+ey+fz, gx+hy+iz)$, (i.e. syzygies found by multiplying each polynomial by degree one monomials). This syzygy can be represented as a Nullspace vector with a labeling of the monomials in each component.

EXAMPLE 2.0.3. *Let* $f = x + 4y + z,\ g = -\frac{1}{3}x - 2y,\ h = \frac{1}{3}x + z.$

$$\begin{aligned}
x \cdot f &= x^2 + 4xy + xz \\
y \cdot f &= xy + 4y^2 + yz \\
z \cdot f &= xz + 4yz + z^2 \\
x \cdot g &= -\tfrac{1}{3}x^2 - 2xy \\
y \cdot g &= -\tfrac{1}{3}xy - 2y^2 \\
z \cdot g &= -\tfrac{1}{3}xz - 2yz \\
x \cdot h &= \tfrac{1}{3}x^2 + xz \\
y \cdot h &= \tfrac{1}{3}xy + yz \\
z \cdot h &= \tfrac{1}{3}xz + z^2
\end{aligned}$$

These products can also be written in matrix form:

$$\begin{array}{c}
\begin{array}{ccccccccc} x\cdot f & y\cdot f & z\cdot f & x\cdot g & y\cdot g & z\cdot g & x\cdot h & y\cdot h & z\cdot h \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & -1/3 & 0 & 0 & 1/3 & 0 & 0 \\
4 & 1 & 0 & -2 & -1/3 & 0 & 0 & 1/3 & 0 \\
1 & 0 & 1 & 0 & 0 & -1/3 & 1 & 0 & 1/3 \\
0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{array}{l} x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{array}
\end{array}$$

Notice that by the restriction to homogeneous polynomials the number of monomials can guarantee that our coefficient vectors are all of the same length. Taking the Nullspace of the previous matrix we get a basis for the Nullspace using Maple.

$$\left\{ \begin{bmatrix} 2/3 \\ 4 \\ -1 \\ 2 \\ 8 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2 \\ 0 \\ 1 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

The Nullspace is four dimensional, and by pulling off coefficients in triples we can get components of each syzygy with the first coefficient being multiplied by x, second by y and lastly the third by z. Doing this through each vector we obtain the following syzygies:

$$(\tfrac{2}{3}x + 4y - z, 2x + 8y, z),$$
$$(-y, -2y, y), (-x, -2x, x), (\tfrac{1}{3}x + 2y, x + 4y + z, 0)$$

The two syzygies $(-y, -2y, y)$, $(-x, -2x, x)$ are both in the span of the lower degree syzygy $(-1, -2, 1)$, e.g., $(-y, -2y, y) = y \cdot (-1, -2, 1)$. To make sure we only obtain syzygies that are linearly independent to all other syzygies as coefficient vectors, we will only keep syzygyies in the degree n^{th} step if they are linearly independent to the span of the syzygies in the $(n-1)^{th}$ step in n^{th}. Removing such redundancy from the previous example, we get the set of syzygies:

$$(-1,-2,1),(\tfrac{1}{3}x+2y,x+4y+z,0)$$

Comparing this with Singular which had the following syzygies:

$$(0,x+3z,x+6y),(1,2,-1)$$

The syzygy consisting of constants is equivalent to Singular's since they differ by a negative sign. Also, the existence of syzygies from lower degree products must be accounted for when finding new linearly independent syzygies at higher degrees. Now that these points have been highlighted let us consider a larger example with polynomials of different degrees, our running example.

EXAMPLE 2.0.4. ***Ideal:*** $I \doteq \langle x^3y+z^4, xy^2, x^3+y^3, z^5\rangle$

We can disregard all polynomials up to degree 0, 1, and 2 as each polynomial in I has degree greater than all those possible degrees, so we start at degree 3. This suggests that we consider when we get all possible polynomials to degree 3. In this case we only have two polynomials: xy^2, x^3+y^3. Converting these vectors into coefficients using a pure lexicographic ordering we get the matrix:

$$\begin{array}{ccc} xy^2 & x^3+y^3 & \\ \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right) & & \begin{array}{c} x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{array} \end{array}$$

This matrix can easily be seen to have full rank. Given that the matrix above is full rank, we know that our Nullspace is trivial so we must go to the next degree.

Considering degree 4, we have a slightly larger polynomial system:

$$\langle x^3y + z^4, x^2y^2, xy^3, xy^2z, x^4 + xy^3, x^3y + y^4, x^3z + y^3z \rangle$$

This also gives a matrix as follows:

$$
\begin{array}{cc}
\begin{array}{ccccccc} x^3y+z^4 & x^2y^2 & xy^3 & xy^2z & x^4+xy^3 & x^3y+y^4 & x^3z+y^3z \end{array} & \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
&
\begin{array}{c} x^4 \\ x^3y \\ x^3z \\ x^2y^2 \\ x^2yz \\ x^2z^2 \\ xy^3 \\ xy^2z \\ xyz^2 \\ xz^3 \\ y^4 \\ y^3z \\ y^2z^2 \\ yz^3 \\ z^4 \end{array}
\end{array}
$$

Continuing this process up to degree 8, we get a set of syzygies/Nullspace vectors for each coefficient matrix. This set of syzygies may have higher degree syzygies, recall from 2.0.2 there are products of syzygies spanned by a lower degree so we will remove these from the final list below. This list has the linearly independent syzygies in each degree that are not spanned by any lower degree.

$$\big\langle(-yz, x^2z, 0, y), (-xz, , -y^2z, xyz, x), (0, -x^3 - y^3, xy^2, 0), (-x^3 - y^3, 0, x^3y + z^4, 0),$$
$$(-xy^2, x^3y + z^4, 0, 0), (x^3y - z^4, -x^5, 0, z^3)\big\rangle$$

We now have an algorithm for calculating syzygies of a system of homogeneous polynomials in a fixed choice of degrees. We require that we multiply by monomials needed to get all polynomials to a certain degree. If a polynomial is above the degree that we wish to go to then we will say the syzygy component corresponding to that polynomial must be zero as there may be no way to cancel the higher degree terms of this polynomial. We also have seen a reason to make sure we only have linearly independent Nullspace vectors at each step since some vectors may be the span of a lower degree Nullspace vector in a previous calculation. Taking a moment we can consider the advantages and disadvantages to this algorithm.

2.0.1. Advantages:

- No Gröbner bases were needed in the construction of the above syzygies.
- Since Linear Algebra is the main mathematical tool it is possible to use Numerical Linear Algebra to compute *numerical syzygies.*

2.0.2. Disadvantages:

- There is no way *a priori* to know if we can stop constructing matrices and taking Nullspaces to find syzygies at a certain degree. This is equivalent to saying we have no stopping criterion for the more general problem of finding *all* syzygies.
- The ability to find a minimal generating set is also of great concern for further research.
- Numerical computation will pose its own problems with stability of computations as well as correctness of rank/Nullspace calculations.

CHAPTER 3

Second Syzygies

Given a set of syzygies for a polynomial system one can pose the question, do there exist syzygies for the syzygies we have? This question will lead us to a new algorithm that will hopefully help us find “higher syzygies” that is, syzygies of the syzygies. First, let us consider what it would mean to a be a syzygy on the syzygies.

From Example 2.0.4 the components of the first syzygies are:

$$\begin{bmatrix} -yz & -xz & 0 & -x^3-y^3 & -xy^2 & x^3y-z^4 \\ x^2z & -y^2z & -x^3-y^3 & 0 & x^3y+z^4 & -x^5 \\ 0 & xyz & xy^2 & x^3y+z^4 & 0 & 0 \\ y & x & 0 & 0 & 0 & z^3 \end{bmatrix}$$

Notice that now that each of these components are polynomial systems (i.e the first polynomial system we can write is $\langle -yz, -xz, 0, -x^3-y^3, -xy^2, x^3y-z^4\rangle$). This system differs from any original system we may have been given by the fact that we have a zero as a polynomial in our system. There are two issues that this zero poses for us to find syzygies for this system.

First, this zero may or may not be the constant 0. Given the way we constructed the first syzygies this zero could in fact be the polynomial $0x+0y+0z$ or possibly any other homogenous zero polynomial of some other degree d. However, we can find the degree of this zero by keeping track of how the syzygy $\langle 0, -x^3-y^3, xy^2, 0\rangle$ was produced and recording the degree of each zero in that syzygy; this is a matter of book-keeping. Doing this for all the components even the non-zero ones we obtain the matrix of polynomial degrees:

$$\begin{bmatrix} 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 3 & 3 & 4 & 4 & 5 \\ 3 & 3 & 3 & 4 & 4 & 5 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{bmatrix}$$

Given this matrix, we can remove the zeroes without losing information of what degree the zeroes are. With the zeroes removed we can solve for syzygies using the algorithm outlined in Section 2.

In addition, the other issue that a zero can pose is that it can be multiplied by any polynomial of any degree and still give zero. This allows for an infinite number of choices for the element in our syzygy that is to be multiplied by the zero. We will take the approach that after we have found syzygies for the system without zeroes we can mark with a "*" where the zeroes were and find out what possibly could be in the component of the syzygy to multiply with zero. We will illustrate what is to be done to solve for this "*" shortly.

Getting rid of zeroes in the components of first syzygies for example 2.0.4:

$$\begin{bmatrix} -yz & -xz & & -x^3-y^3 & -xy^2 & x^3y-z^4 \\ x^2z & -y^2z & -x^3-y^3 & & x^3y+z^4 & -x^5 \\ & xyz & xy^2 & x^3y+z^4 & & \\ y & x & & & & z^3 \end{bmatrix}$$

Now that all components have had zeroes removed, we can calculate syzygies for each of the components using the algorithm outlined in Section 2. The systems and their corresponding syzygies calculated with products up to degree eight are given below.

First Components (First Row):

$$\text{Ideal: } \langle -yz, -xz, -x^3-y^3, -xy^2, x^3y-z^4 \rangle$$

Syzygies:$\langle (-x, y, 0, 0, 0), (-xy, 0, 0, z, 0), (-y^2, -x^2, z, 0, 0), (-z^3, 0, 0, x^2, y), (0, -z^3, xy, -y^2, x) \rangle$

Second Components:

$$\text{Ideal: } \langle x^2z, -y^2z, -x^3 - y^3, x^3y + z^4, -x^5 \rangle$$

$$\text{Syzygies:} \langle (x, -y, z, 0, 0), (y^2, x^2, 0, 0, 0), (0, -z^3, -x^3, -y^2, x), (x^3, 0, 0, 0, z), (-z^3, 0, 0, x^2, y) \rangle$$

Third Components:

$$\text{Ideal: } \langle xyz, xy^2, x^3y + z^4 \rangle$$

$$\text{Syzygies: } \langle (-y, z, 0), (-z^3, -x^3, xy) \rangle$$

Fourth Components:

$$\text{Ideal: } \langle y, x, z^3 \rangle$$

$$\text{Syzygies: } \langle (-x, y, 0), (-z^3, 0, y), (0, -z^3, x) \rangle$$

We now have all potential syzygies calculated, with the algorithm in Section 2, where the systems are the components with zeroes removed. These syzygies are missing components corresponding to the removed zeroes before calculation. We will put a "*" in a the i^{th} component of a syzygy if the system before removing zeroes had a zero in the i^{th} position. This will allow us to have the entire syzygy with unknowns denoted by ""*". The above syzygies after this process are below:

Table 3.0.1.

Syzygies for first components:

$$\Big\langle(-x, y, *, 0, 0, 0), (-xy, 0, *, 0, z, 0), (-y^2, -x^2, *, z, 0, 0), (-z^3, 0, *, 0, x^2, y), (0, -z^3, *, xy, -y^2, x)\Big\rangle$$

Syzygies for second components:

$$\Big\langle(x, -y, z, *, 0, 0), (y^2, x^2, 0, *, 0, 0), (0, -z^3, -x^3, *, -y^2, x), (x^3, 0, 0, *, 0, z), (-z^3, 0, 0, *, x^2, y)\Big\rangle$$

Syzygies for third components:

$$\langle(*, -y, z, 0, *, *), (*, -z^3, -x^3, xy, *, *)\rangle$$

Syzygies fourth components:

$$\langle(-x, y, *, *, *, 0), (-z^3, 0, *, *, *, y), (0, -z^3, *, *, *, x)\rangle$$

Since "*" denotes a zero in our polynomial system, that means in each syzygy we could multiply by any polynomial in that spot. Depending on what degree of products we are at for creating syzygies we will know the degree of polynomials that could replace "*". The following definition will allow us to narrow the possibilities for "*" to a finite list.

Definition 3.0.3. *Two potential syzygies* $(f_1, f_2, ..., f_n) \neq 0$ *and* $(g_1, g_2, ..., g_n) \neq 0$ *are* ***similar*** *if* $deg(f_i) = deg(g_i)$ *for* $f_i \neq$ *"*" and* $g_i \neq$ *"*".*

This definition allows us to gather what type of syzygies come from multiplying by the same degree of monomials. If we can find syzygies that satisfy this property, then we can allow a "*" in the i^{th} spot to be any of the possible polynomials in the i^{th} spot of the similar syzygies. In searching for similar syzygies in Table 3.0.1, we find that there are two sets of similar syzygies. Consider the first set:

$$(-x, y, ``*", 0, 0, 0), (x, -y, z, ``*", 0, 0), (``*", -y, z, 0, ``*", ``*"), (-x, y, ``*", ``*", ``*", 0)$$

These syzygies are all similar to each other. Consider the "*" in the third component of the first syzygy. Using all the similar syzygies we have that the only possible substitution for that "*" is z since z is the only non-star in the 3^{rd} position. This allows us to have the following possible syzygy $(-x, y, z, 0, 0, 0)$.

Let us consider a little more involved situation. The syzygy $(``*", -y, z, 0, ``*", ``*")$ has three unknowns so we can find all possible combinations of three polynomials in the corresponding spots to put into our unknowns. The "*" in the first slot has possibilities $\{-x, x\}$, the fifth slot has possibility $\{0\}$, and the last slot has possibility $\{0\}$. Using this information we can find all possible syzygies with substitutions they are below:

$$(x, -y, z, 0, 0, 0), (-x, -y, z, 0, 0, 0)$$

Continuing this process we get all possible combinations for each "*" by using all possibilities from similar syzygies. Recall that each syzygy actually corresponded to a Nullspace vector of a matrix comprised of coefficient vectors of products of the polynomial system. To find a syzygy or Nullspace vector that is common between all four systems in our example we can consider the intersection of their corresponding Nullspaces. We must find what subspace of the Nullspaces for each system correspond to each other. Luckily, definition 3.0.3 has shown us which syzygies/Nullspace vectors correspond to each other. The definition 3.0.3

also guarantees that our Nullspace vectors are of the same size. Given this information we can now intersect all corresponding Nullspace vectors that are from similar syzygies. Doing this we obtain the following list of syzygies.

$$(x, -y, z, 0, 0, 0), (0, -z^3, -x^3, xy, -y^2, x), (-z^3, 0, 0, 0, x^2, y)$$

The minimal generating set for the second syzygies of this system has three syzygies. This is found using the *betti* command in Singular [6]. Thus, we seem to have found all second syzygies for this set.

CHAPTER 4

Motivation for Second Syzygy Algorithm

The second syzygy algorithm has lacked motivation so we will attempt to remedy this. Consider the following example:

Example 4.0.5. ***Ideal:*** $\langle yz - xw, y^3 - x^2z, z^3 - yw^2, xz^2 - y^2w\rangle$

This example comes from the online documentation for the *betti* command on Singular's webpage [6]. Using the algorithm outlined in Section 2 we can obtain the first syzygies for this system. They are as follows:

$$\langle (wy, 0, -x, z), (-xz, w, 0, y), (-y^2, z, 0, x), (z^2, 0, -y, w)\rangle$$

Before the second syzygy algorithm outlined in section three there were many ideas of how to resolve the issue of having zeroes in a polynomial system. An interesting issue was brought to light by example 4.0.5. Let us consider this issue now.

Taking the syzygies of the first components of the above syzygies we obtain a list of syzygies. These syzygies are given below:

$$\langle (0, z, 0, x), (y, 0, w, 0), (0, 0, z^2, y^2), (-z^2, 0, 0, wy), (0, -y^2, xz, 0), (xz, wy, 0, 0)\rangle$$

At first these syzygies seem very mundane. Let us give the first components to Singular and compute the syzygies. The result turns out to be $(y, -z, w, -x)$. This syzygy is a linear combination of the first two syzygies that we computed above. If we take the first two syzygies we computed they are in fact not syzygies of all components, but this linear combination does end up being a syzygy of all components.

In Linear Algebra terms, we have computed a Nullspace for the first set of polynomials. However, this Nullspace may contain vectors which are not in the Nullspaces for the other sets of polynomials. Therefore, we calculate the Nullspace for each set of polynomials and intersect them to find only vectors which are in the Nullspace of all the sets of polynomials.

This possibility for the intersection of each component's Nullspace to reduce the dimension of the Nullspace for all components is what led to the final computation in Section 3. The problem when this was first discovered is there was no good way of knowing which Nullspaces held information that could be compared. Definition 3.0.3 was created to allow for a uniform way of finding Nullspaces that contained compatible information.

Following the algorithm for second syzygies in Section 3 we obtain the syzygy $(-y, z, -w, x)$.

CHAPTER 5

Formal Algorithms

Algorithm 1: Algorithm for first syzygies

Data: Polynomial System, Degree_of_products_requested

Result: A table of syzygies where entry d is an array of syzygies from products of degree d

```
for i 1 to Degree_of_products_requested do
    Build Products of degree i ;
    Convert products to vectors;
    Build Coefficient matrix;
    Compute Nullspace of coefficient matrix ;
    if Nullspace is trivial then
        break;
    else
        Check lower degree for redundancies;
        Keep linearly independent syzygies;
    end
end
```

The algorithm in Section 2 is described above. Some minor details described in Section 2 are omitted for ease of reading.

Algorithm 2: Algorithm for second syzygies

Data: Syzygy_Table, Polynomial_System_For_Syzygy_Table

Result: Table_Of_Second_Syzygies

```
Construct components from Syzygy_Table;
Compute degrees of components;
Remove zeroes from components;
for i 1 to number of components without zeroes do
|   Compute syzygies up to specified degree;
|   Store syzygy tables components;
end
for i 1 to number_of_new_syzygy_tables do
|   for j 1 to number of syzygies in table_i do
|   |   if Components of syzygy table_i has zeroes then
|   |   |   Place "*";
|   |   else
|   |   |   Don't place "*";
|   |   end
|   end
end
Find which syzygies are similar;
for All similar syzygies do
|   for Syzygies of similarity s do
|   |   Compute all possibilities for "*";
|   end
end
for Each set of similar syzygies that have "*" solved for do
|   Intersect coefficient vectors;
end
Record what degree of products gave the intersection syzygy;
Store intersection syzygy;
```

The above algorithm is for second syzygies described in Section 3. For more information about each step refer back to Section 3.

CHAPTER 6

Numerical Computation

Currently, there does not exist a way to compute numerical syzygies. Work is being done in this area using numerical Gröbner Bases see [4] and [3]. In particular, there has been work to use interval arithmetic at points in Gröbner Basis calculation to help with numerical inaccuracy. Also, trying to estimate the ill-conditionedness of floating-point Gröbner Basis calculations.

In the current work, our main tool has been Linear Algebra, allowing for a natural extension to Numerical Linear Algebra. However, numerical computation can cause problems since there is an truncation of coefficients causing possible round-off error. In our case, allowing floating point precision makes finding the Nullspace of a matrix less stable. Given the rank of the matrix, we know how many Nullspace vectors we expect to find in the computation. The singular value decomposition of the matrix is a more stable way of computing the rank of a matrix numerically.

Computing Nullspace vectors numerically allows each syzygy to have floating point coefficients. Due to this, the syzygies we compute may have more (perhaps all) monomials than in the exact coefficient case. This difference in monomial structure in the computation of first syzygies leads to another possible difference in second syzygies. If a floating point matrix (i.e., the coefficient matrix computation in second syzygies) has different rank than in the exact coefficients case we may get lower degree syzygies that did not exist in the exact computation. These lower degree syzygies limit the number of syzygies in higher degrees and may make our intersection empty. Increased precision and accuracy of computation may help with all of these issues; this is the subject of future work on this project.

Aside from the numerical differences above we can compute the same number of first and second syzygies for Example 4.0.5. For the running example, Example 2.0.4, we can compute the correct number of first syzygies without issue. However, the second syzygies have lower degree syzygies that did not exist in the exact coefficients case, as described above. This may be remedied by increasing accuracy and precision while computing the rank of the corresponding coefficient matrices.

CHAPTER 7

Conclusion

Given a set of homogenous polynomials that serve as a generating set for an ideal of a polynomial ring R, we have asked if there exist a certain relationship between these generators. Knowing that there exists a way using Gröbner Bases, we wish to see if there exists a way using only Linear Algebra. Rephrasing our problem as a Nullspace computation, we were able to compute syzygies of this set of generators if they have exact coefficients.

Successful with first syzygies, we endeavored to compute relationships on the syzygies that we had just computed. In this case, we have the issue of zeroes being in our generating set. We were able to resolve this issue by ignoring zeroes until we had computed potential syzygies of the system. After computing these potential syzygies we found suitable polynomials to multiply with each zero using Definition 3.0.3. From this we found a set of syzygies that have the syzygy property on each of the components of our first syzygies.

Numerically there are issues with stability and possible syzygies that are artifacts of numerical inaccuracy. These issues may be resolved with higher precision and accuracy. Future work includes finding heuristics for how much precision and accuracy is needed. Another future direction is the computation of Betti numbers.

Bibliography

[1] D. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry.* Springer Verlag, first ed., 1998.

[2] E. W. Mayr and A. R. Meyer, *The Complexity of the Word Problems for Commutative Semigroups and Polynomial Ideals.* Massachusetts Institute of Technology, Laboratory for Computer Science, first ed., 1981.

[3] T. Sasaki and F. Kako, "Floating-point grbner basis computation with ill-conditionedness estimation," *Computer Mathematics,* 8^{th} *Asian Symposium, ASCM 2007, Singapore. Revised and Invited Papers.*, 2007.

[4] M. Bodrato and A. Zanoni, "Intervals, syzygies, numerical grbner bases: A mixed study," *Computer Algebra in Scientific Computing* 9^{th} *International Workshop*, 2006.

[5] D. R. Grayson and M. E. Stillman, "Macaulay2, a software system for research in algebraic geometry." Available at `http://www.math.uiuc.edu/Macaulay2/`http://www.math.uiuc.edu/Macaulay2/.

[6] W. Decker, G. G.-M., P. G., and S. H., "Singular 3-1-6 — A computer algebra system for polynomial computations," 2012. http://www.singular.uni-kl.de.

[7] G.-M. Greuel and G. Pfister, *A Singular Introduction to Commutative Algebra.* Springer Verlag Berlin Heidelberg, second ed., 2008.

[8] R. Wiegand, "What is a syzygy?," *American Mathematical Society Notice*, 2006.

[9] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry.* Spinger Science+Buisness Media, LLC, 2004.