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The CFAR Adaptive Subspace Detector is a Scale-Invariant GLRT

Shawn Kraut and Louis L. Scharf

***Abstract*—The constant false alarm rate (CFAR) matched subspace detector (CFAR MSD) is the uniformly most-powerful-invariant test and the generalized likelihood ratio test (GLRT) for detecting a target signal in noise whose covariance structure is known but whose level is unknown. Recently, the CFAR adaptive subspace detector (CFAR ASD), or adaptive coherence estimator (ACE), was proposed for detecting a target signal in noise whose covariance structure and level are both unknown and whose covariance structure is estimated with a sample covariance matrix based on training data. We show here that the CFAR ASD is GLRT when the test measurement is not constrained to have the same noise level as the training data. As a consequence, this GLRT is invariant to a more general scaling condition on the test and training data than the well-known GLRT of Kelly.**

***Index Terms*—Adaptive arrays, matched filters, maximum likelihood detection, multidimensional signal detection, radar detection.**

I. INTRODUCTION

Recently, we have suggested the constant false alarm rate (CFAR) adaptive subspace detector (CFAR ASD) [3] for detecting a target signal $\underline{\psi}$ in a complex multivariate measurement $\underline{y}$ whose distribution is complex normal $\underline{y} \sim CN[\mu e^{j\alpha}\underline{\psi}, \sigma^2 \boldsymbol{R}]$. The signal scaling μ determines the null hypothesis H_0: $\mu = 0$ and alternate hypothesis H_1: $\mu > 0$. We factor out a noise scaling σ^2 from the noise covariance structure $\boldsymbol{R}$: a step to be clarified in the subsequent discussion.

When the noise covariance structure and scaling $\boldsymbol{R}$ and σ^2 are both known, the appropriate noncoherent detection statistic is the matched filter magnitude-squared or the matched subspace detector (MSD). This uses the inner product of the whitened measurement $\underline{z} = \boldsymbol{R}^{-(1/2)}\underline{y}$ with the whitened signal template $\underline{\phi} = \boldsymbol{R}^{-(1/2)}\underline{\psi}$

$$\chi^2 = \frac{|\underline{\psi}^\dagger \boldsymbol{R}^{-1}\underline{y}|^2}{\underline{\psi}^\dagger \boldsymbol{R}^{-1}\underline{\psi}\sigma^2} = \frac{\underline{z}^\dagger \boldsymbol{P}_{\underline{\phi}}\underline{z}}{\sigma^2} \gtrless \eta \tag{1}$$

where $\boldsymbol{P}_{\underline{\phi}} = \underline{\phi}(\underline{\phi}^\dagger\underline{\phi})^{-1}\underline{\phi}^\dagger$ is the projection onto $\underline{\phi}$. This statistic is complex chi-squared (or gamma) distributed; the MSD compares it with the threshold η to decide on hypothesis H_0 or H_1.

When the covariance matrix $\boldsymbol{R}$ is known but the scaling σ^2 is unknown, the MSD may be normalized by the magnitude squared of the measurement weighted by $\boldsymbol{R}^{-1}$. This measures the direction-cosine squared of the angle that $\underline{z}$ makes with $\underline{\phi}$:

$$\cos^2 = \frac{|\underline{\psi}^\dagger \boldsymbol{R}^{-1}\underline{y}|^2}{(\underline{\psi}^\dagger \boldsymbol{R}^{-1}\underline{\psi})\,(\underline{y}^\dagger \boldsymbol{R}^{-1}\underline{y})} \gtrless \eta. \tag{2}$$

This statistic has a "beta" density under H_0; under H_1, it is most clearly described as a monotone function of a statistic with a scaled noncentral "F" distribution

$$\cos^2 = \frac{F}{F+1}; \quad F = \frac{\underline{z}^\dagger \boldsymbol{P}_{\underline{\phi}}\underline{z}}{\underline{z}^\dagger \boldsymbol{P}_{\underline{\phi}}^{\perp}\underline{z}} \tag{3}$$

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The authors are with the Department of Electrical and Computer Engineering, University of Colorado, Boulder, CO 80309-0425 USA.

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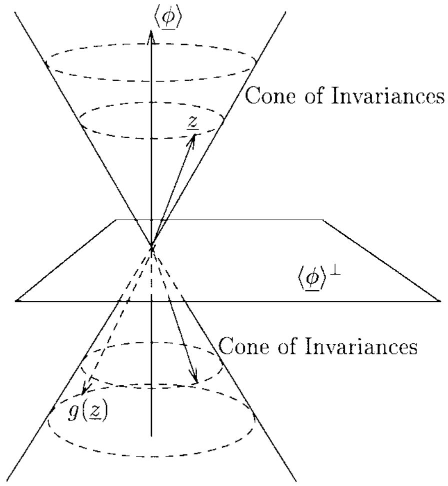

Fig. 1. Geometry and invariances of the CFAR MSD. It measures the cosine-squared of the angle the whitened measurement $\underline{z} = \boldsymbol{R}^{-(1/2)}\underline{y}$ makes with the whitened signal $\underline{\phi} = \boldsymbol{R}^{-(1/2)}\underline{\psi}$. It is invariant to scaling of the measurement and to rotations of $\underline{z}$ in the subspaces $\langle\underline{\phi}\rangle$ and $\langle\underline{\phi}\rangle^{\perp}$.

where $\boldsymbol{P}_{\underline{\phi}}^{\perp} = \boldsymbol{I} - \boldsymbol{P}_{\underline{\phi}}$ is the projection onto the subspace perpendicular to $\underline{\phi}$. In its F version, this detector has a similar form as χ^2 but is normalized by a scaled estimate of σ^2, namely, $(N-1)\overline{\sigma^2} = \underline{z}^{\dagger}\boldsymbol{P}_{\underline{\phi}}^{\perp}\underline{z}$. This makes it have a CFAR with respect to the unknown noise scaling σ^2; thus, we term it the CFAR MSD.

The MSD and the CFAR MSD have interesting invariances with respect to transformations of the whitened measurement $\underline{z}$. The MSD is invariant to translations of $\underline{z}$ in the subspace $\langle\underline{\phi}\rangle^{\perp}$ and to rotations in the subspace $\langle\underline{\phi}\rangle$The CFAR MSD is invariant to rotations in the subspaces $\langle\underline{\phi}\rangle$ and $\langle\underline{\phi}\rangle^{\perp}$; it is also invariant to scaling of the measurement, as shown in Fig. 1. Both the MSD and CFAR MSD have been shown to uniformly most powerful within the class of detectors that share their respective invariances (UMP invariant) [1]. They are also generalized likelihood ratio tests (GLRT's) [2], which are obtained by inserting maximum-likelihood (ML) estimates for unknown parameters into the likelihood ratio, which is the ratio of the probability density function (pdf) of $\underline{y}$ under H_1 to that under H_0.

When the noise structure is not known *a priori* but is estimated from training data, then a reasonable, though seemingly *ad hoc*, procedure for generalizing the CFAR MSD is to simply replace the known $\boldsymbol{R}$ with the sample covariance matrix estimate $\boldsymbol{S} = \hat{\boldsymbol{R}}$ based on training data. This procedure produces the CFAR ASD statistic $\widehat{\cos^2}$, which is often referred to as the adaptive coherence estimator (ACE) [3], [5]:

$$\widehat{\cos^2} = \frac{|\underline{\psi}^{\dagger}\boldsymbol{S}^{-1}\underline{y}|^2}{(\underline{\psi}^{\dagger}\boldsymbol{S}^{-1}\underline{\psi})\,(\underline{y}^{\dagger}\boldsymbol{S}^{-1}\underline{y})} \gtrless \eta. \tag{4}$$

This adaptive version was independently proposed by Conte *et al.* [6], [7] for use with the related scenario of compound-Gaussian noise, multivariate Gaussian with random amplitude scaling. The adaptive statistic retains the invariances of its nonadaptive counterpart in (2).

We will show in this correspondence that when the measurement $\underline{y}$ is distributed $CN[\mu e^{j\alpha}\underline{\psi}, \sigma^2\boldsymbol{R}]$ while the training data is distributed $CN[\underline{0}, \boldsymbol{R}]$ (i.e., the noise of the test data $\underline{y}$ may be scaled by σ^2 relative to the training data), then the CFAR ASD is, in fact, a GLRT detector statistic. As far as we know, this is the only formal derivation of the ACE statistic. This claim lends credence to the heuristics of [3] and [5]–[7]. This statistic may be contrasted with the well-known GLRT statistic of Kelly [4], which assumes the same noise scaling in both test and training data ($\sigma^2 = 1$). In the following derivation of the scale-invariant GLRT, we will adhere closely to the notation and procedure of Kelly's original paper, with the exception that the likelihoods will be maximized over the additional unknown parameter σ^2.

II. Deriving the Scale-Invariant GLRT

We first consider what the unknown noise-scaling factor σ^2 means in the adaptive case, where the assumption of known covariance is relaxed, and it is instead assumed that one has access to training data vectors $\{\underline{x}_i\}$ that share the same noise covariance as the test data $\underline{y}$. A consistent interpretation is that σ^2 is a *relative* scaling of the noise power in the test data, with respect to that in the training data, that is, $\underline{y} \sim CN[\mu e^{j\alpha}\underline{\psi}, \sigma^2\boldsymbol{R}]$, whereas $\underline{x}_i \sim CN[\underline{0}, \boldsymbol{R}]$. We allow for the possibility of additional scaling that the training data does not account for, by leaving σ^2 as a free parameter. Thus, in the derivation of the GLRT, the unknown parameters are the noise structure $\boldsymbol{R}$, noise scaling σ^2, and signal scaling and phase $\mu e^{j\alpha}$; only the signal template $\underline{\psi}$ is known. (Again, Kelly's problem [4] differs in that the scaling σ^2 is a known parameter, which is assumed to be unity.)

A. Densities

We assume that there are K training vectors $\underline{x}_i$ that are independently distributed, and we construct the data matrix $\boldsymbol{X}$ containing these vectors as its columns $\boldsymbol{X} = [\underline{x}_1 \cdots \underline{x}_K]$. The GLRT is obtained by considering the joint pdf of the measurement $\underline{y}$ *and* the training data $\boldsymbol{X}$, which, under the alternate hypothesis H_1, is

$$\begin{aligned} f_1(\boldsymbol{X}, \underline{y}) &= f_1(\underline{y}) \prod_{i=1}^{K} f(\underline{x}_i) \\ &= \frac{1}{\pi^N \|\sigma^2\boldsymbol{R}\|} \exp\left\{-\frac{1}{\sigma^2}(\underline{y} - \mu e^{j\alpha}\underline{\psi})^{\dagger}\boldsymbol{R}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})\right\} \\ &\quad \cdot \prod_{i=1}^{K} \frac{1}{\pi^N \|\boldsymbol{R}\|} \exp\{-\underline{x}_i^{\dagger}\boldsymbol{R}^{-1}\underline{x}_i\} \end{aligned} \tag{5}$$

where $\|\;\|$ denotes determinant. The density under H_0 is f_1 evaluated at $\mu = 0$, i.e., $f_0(\boldsymbol{X}, \underline{y}) = f_1(\boldsymbol{X}, \underline{y})|_{\mu=0}$, which is a notation we will use throughout. The densities under the hypotheses H_0, H_1 may be rewritten as

$$f_{0,1} = \left\{\frac{1}{\pi^N \|\boldsymbol{R}\| \sigma^{2N/(K+1)}} \exp\{-tr(\boldsymbol{R}^{-1}\boldsymbol{T}_{0,1})\}\right\}^{K+1} \tag{6}$$

where $\boldsymbol{T}_1$ and $\boldsymbol{T}_0$ are the matrices

$$\begin{aligned} \boldsymbol{T}_1 &= \frac{1}{K+1}\left\{\frac{1}{\sigma^2}\,(\underline{y} - \mu e^{j\alpha}\underline{\psi})\,(\underline{y} - \mu e^{j\alpha}\underline{\psi})^{\dagger} + \sum_{i=1}^{K}\underline{x}_i\,\underline{x}_i^{\dagger}\right\} \\ \boldsymbol{T}_0 &= \boldsymbol{T}_1|_{\mu=0}. \end{aligned} \tag{7}$$

B. Maximum-Likelihood Estimates

We now find maximum-likelihood (ML) estimates for the noise structure $\boldsymbol{R}$, noise scaling σ^2, and signal scaling and phase $\mu e^{j\alpha}$, and we insert them into the densities to obtain the generalized likelihood ratio (GLR). The ML estimates of the structure $\boldsymbol{R}$ under the two hypotheses are given by $\hat{\boldsymbol{R}}_0 = \boldsymbol{T}_0$ and $\hat{\boldsymbol{R}}_1 = \boldsymbol{T}_1$. Inserting these estimates into the densities yields

$$\max_{\boldsymbol{R}} f_{0,1} = \left\{\frac{1}{(e\pi)^N \|\boldsymbol{T}_{0,1}\| \sigma^{2N/(K+1)}}\right\}^{K+1}. \tag{8}$$

Following the procedure of Kelly [4], the determinants $\|\boldsymbol{T}_{0,1}\|$ may be written in terms of $\|\boldsymbol{S}\|$, where $\boldsymbol{S}$ is the sample-covariance estimate based solely on the training data $\boldsymbol{S} = (1/K)\,\Sigma_i \underline{x}_i \underline{x}_i^\dagger$. That is

$$\begin{aligned}\|\boldsymbol{T}_1\| &= \left(\frac{K}{K+1}\right)^N \|\boldsymbol{S}\| \\ &\quad \cdot \left(1 + \frac{1}{K\sigma^2}\,(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})\right) \\ \|\boldsymbol{T}_0\| &= \|\boldsymbol{T}_1\||_{\mu=0}. \end{aligned} \tag{9}$$

To maximize the likelihood functions under σ^2, we minimize the expression $\{1+(1/K\sigma^2)(\underline{y}-\mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y}-\mu e^{j\alpha}\underline{\psi})\}(\sigma^2)^{N/K+1}$ with respect to σ^2. This yields ML estimates for the noise scaling

$$\begin{aligned}\widehat{\sigma^2}_1 &= \frac{K-N+1}{KN}\,(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi}) \\ \widehat{\sigma^2}_0 &= \widehat{\sigma^2}_1|_{\mu=0}. \end{aligned} \tag{10}$$

Inserting these estimates into the densities yields

$$\begin{aligned}\max_{\boldsymbol{R},\,\sigma^2} f_1 &= \frac{(K+1)^{(N-1)(K+1)}(K-N+1)^{K-N+1}(KN)^N}{(e\pi K)^{N(K+1)}\|\boldsymbol{S}\|^{K+1}} \\ &\quad \cdot \frac{1}{[(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})]^N} \\ \max_{\boldsymbol{R},\,\sigma^2} f_0 &= \max_{\boldsymbol{R},\,\sigma^2} f_1|_{\mu=0}. \end{aligned} \tag{11}$$

At this stage, we can write the intermediate GLR, after substituting in the ML estimates $\hat{\boldsymbol{R}}_{0,1}$ and $\widehat{\sigma^2}_{0,1}$, as

$$\begin{aligned}\hat{\lambda}(\boldsymbol{X},\underline{y}) &= \frac{\max_{\boldsymbol{R},\sigma^2} f_1}{\max_{\boldsymbol{R},\sigma^2} f_0} \\ &= \left(\frac{\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y}}{(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})}\right)^N. \end{aligned} \tag{12}$$

Finally, we need to evaluate the ML estimate of the signal scaling and phase $\mu e^{j\alpha}$. As in Kelly's paper, this can be found by completing the square to minimize the quadratic form $(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})$:

$$\begin{aligned}&(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi}) \\ &\quad = \underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y} + \underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}\left|\mu e^{j\alpha} - \frac{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}}\right|^2 \\ &\qquad - \frac{|\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}|^2}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}}. \end{aligned} \tag{13}$$

This yields the ML estimate of $\mu e^{j\alpha}$

$$\widehat{\mu e^{j\alpha}} = \frac{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}}. \tag{14}$$

C. Generalized Likelihood Ratio and the CFAR ASD

With the substitution of the signal's estimated scaling and phase $\widehat{\mu e^{j\alpha}}$, the final GLR is given by

$$\begin{aligned}\hat{\lambda}(\boldsymbol{X},\underline{y}) &= \frac{\max_{\boldsymbol{R},\,\sigma^2,\,\mu e^{j\alpha}} f_1}{\max_{\boldsymbol{R},\,\sigma^2} f_0} = \left(\frac{\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y}}{\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y} - \dfrac{|\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}|^2}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}}}\right)^N \\ &= \left(\frac{1}{1-\widehat{\cos^2}}\right)^N = (1+\hat{F})^N \end{aligned} \tag{15}$$

where $\widehat{\cos^2}$ is the statistic defined in (4), and $\hat{F}$ is the "F" version of $\widehat{\cos^2}$, which is the adaptive version of the F statistic of (3) (see also [1], [2], and [8]). Since $\hat{\lambda}$ is a monotone function of $\widehat{\cos^2}$, the GLRT is the CFAR ASD test of (4). This is our key result.

This detector is CFAR with respect to the noise structure $\boldsymbol{R}$, due to the sample-covariance inverse $\boldsymbol{S}^{-1}$. It is also CFAR with respect to the noise scaling σ^2, due to the normalization by $\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y}$, which is an observation that is more easily understood with the "$\hat{F}$" form:

$$\widehat{\cos^2} = \frac{\hat{F}}{\hat{F}+1}; \quad \hat{F} = \frac{\underline{\hat{z}}^\dagger \boldsymbol{P}_{\underline{\hat{\phi}}}\underline{\hat{z}}}{\underline{\hat{z}}^\dagger \boldsymbol{P}_{\underline{\hat{\phi}}}^\perp \underline{\hat{z}}} \tag{16}$$

where $\boldsymbol{P}_{\underline{\hat{\phi}}} = \underline{\hat{\phi}}(\underline{\hat{\phi}}^\dagger \underline{\hat{\phi}})^{-1}\underline{\hat{\phi}}^\dagger$ projects the approximately whitened measurement $\underline{\hat{z}} = \boldsymbol{S}^{-(1/2)}\underline{y}$ onto the approximately whitened signal $\underline{\hat{\phi}} = \boldsymbol{S}^{-(1/2)}\underline{\psi}$, and $\boldsymbol{P}_{\underline{\hat{\phi}}}^\perp = \boldsymbol{I} - \boldsymbol{P}_{\underline{\hat{\phi}}}$. This F form is explicitly normalized by the ML estimate of the noise scaling, $\widehat{\sigma^2}$ [obtained by substituting the results of (13) and (14) into (10)]

$$\widehat{\sigma^2} = \frac{K-N+1}{KN}\underline{\hat{z}}^\dagger \boldsymbol{P}_{\underline{\hat{\phi}}}^\perp \underline{\hat{z}} \tag{17}$$

making it CFAR with respect to σ^2.

D. Related Detectors: The Coherent and Multirank CFAR ASD

There are two detectors related to the CFAR ASD, which can also be shown to be GLRT with very little modification of the derivation given in Section II-B.

The Coherent CFAR ASD: The *coherent* CFAR ASD is used when the phase $e^{j\alpha}$ of the signal is known:

$$\begin{aligned}&\max[0, \mathrm{Re}\{\widehat{\cos}\}] \gtrless \eta, \\ &\qquad \widehat{\cos} = \frac{e^{-j\alpha}\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}}{(\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi})^{1/2}(\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y})^{1/2}}. \end{aligned} \tag{18}$$

To show this is GLRT, the minimization of the the quadratic form of (13) $(\underline{y} - \mu e^{j\alpha}\underline{\psi})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu e^{j\alpha}\underline{\psi})$ is performed only over the scaling parameter μ, constrained to be real and positive:

$$\hat{\mu} = \max\left[0, \mathrm{Re}\left\{\frac{e^{-j\alpha}\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}}\right\}\right]. \tag{19}$$

Inserting this ML estimate into the quadratic form yields the GLR

$$\hat{\lambda}(\boldsymbol{X},\underline{y}) = \frac{\max_{\boldsymbol{R},\,\sigma^2,\,\mu} f_1}{\max_{\boldsymbol{R},\,\sigma^2} f_0} = \left(\frac{1}{1-\max[0,\mathrm{Re}\{\widehat{\cos}\}]^2}\right)^N \tag{20}$$

with $\widehat{\cos}$ given by (18).

Multirank Noncoherent CFAR ASD: The second detector is a generalization of the noncoherent CFAR ASD $\widehat{\cos^2}$, where the signal $\underline{\psi} = \boldsymbol{\Psi}\underline{\theta}$ is not completely specified but is only parametrized to be a superposition of known modes (the columns of $\boldsymbol{\Psi}$) with weights given by a vector of unknown coefficients $\underline{\theta}$ (the multidimensional generalization of the phase $e^{j\alpha}$). The resulting detector, for multirank signal subspaces, is given by a normalized projection

$$\widehat{\cos^2} = \frac{\underline{y}^\dagger \boldsymbol{S}^{-1}\boldsymbol{\Psi}(\boldsymbol{\Psi}^\dagger S^{-1}\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^\dagger \boldsymbol{S}^{-1}\underline{y}}{\underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y}} = \frac{\underline{\hat{z}}^\dagger \boldsymbol{P}_{\hat{\boldsymbol{\Phi}}}\underline{\hat{z}}}{\underline{\hat{z}}^\dagger \underline{\hat{z}}} \gtrless \eta \tag{21}$$

where $\underline{\hat{z}} = \boldsymbol{S}^{-(1/2)}\underline{y}$ is the approximately whitened measurement, and $\hat{\boldsymbol{\Phi}} = \boldsymbol{S}^{-(1/2)}\boldsymbol{\Psi}$ is the approximately whitened signal-mode matrix.

To show that this is GLRT, we again reconsider the intermediate GLR that contains only estimates of the noise structure and scaling

$\hat{\boldsymbol{R}}_{0,1}, \widehat{\sigma^2}_{0,1}$. This has the same form as (12), with $\boldsymbol{\Psi}$ replacing $\underline{\psi}$ and $\underline{\theta}$ replacing $e^{j\alpha}$:

$$\hat{\lambda}(\boldsymbol{X}, \underline{y}) = \frac{\max\limits_{\boldsymbol{R}, \sigma^2} f_1}{\max\limits_{\boldsymbol{R}, \sigma^2} f_0} = \left(\frac{\underline{y}^\dagger \boldsymbol{S}^{-1} \underline{y}}{(\underline{y} - \mu \boldsymbol{\Psi} \underline{\theta})^\dagger \boldsymbol{S}^{-1} (\underline{y} - \mu \boldsymbol{\Psi} \underline{\theta})} \right)^N. \tag{22}$$

We then evaluate the ML estimate of the signal scaling and parameter vector $\mu\underline{\theta}$ by completing the square to minimize the quadratic form $(\underline{y} - \mu\boldsymbol{\Psi}\underline{\theta})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu\boldsymbol{\Psi}\underline{\theta})$, yielding

$$\begin{aligned}
&(\underline{y} - \mu\boldsymbol{\Psi}\underline{\theta})^\dagger \boldsymbol{S}^{-1}(\underline{y} - \mu\boldsymbol{\Psi}\underline{\theta}) \\
&\quad = \underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y} - \mu\underline{\theta}^\dagger\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\underline{y} - \mu\underline{y}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi}\underline{\theta} + \mu^2\underline{\theta}^\dagger\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi}\underline{\theta} \\
&\quad = \|(\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi})^{1/2}[(\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\underline{y} - \mu\underline{\theta}]\|^2 \\
&\qquad + \underline{y}^\dagger\boldsymbol{S}^{-1}\underline{y} - \underline{y}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi}(\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\underline{y} \qquad (23)\\
&\quad \rightarrow \widehat{\mu\underline{\theta}} = (\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^\dagger\boldsymbol{S}^{-1}\underline{y} = \hat{\boldsymbol{\Phi}}^{\#}\,\underline{\hat{z}} \qquad (24)
\end{aligned}$$

where $\hat{\boldsymbol{\Phi}}^{\#} = (\hat{\boldsymbol{\Phi}}^\dagger\hat{\boldsymbol{\Phi}})^{-1}\hat{\boldsymbol{\Phi}}^\dagger$ is the pseudo-inverse of $\hat{\boldsymbol{\Phi}}$. With the substitution of $\widehat{\mu\theta}$, the GLR finally becomes

$$\hat{\lambda}(\boldsymbol{X}, \underline{y}) = \frac{\max\limits_{\boldsymbol{R}, \sigma^2, \mu\underline{\theta}} f_1}{\max\limits_{\boldsymbol{R}, \sigma^2} f_0} = \left(\frac{1}{1 - \widehat{\cos^2}} \right)^N \tag{25}$$

where $\widehat{\cos^2}$ is now given by (21).

III. Rapprochement with the Kelly and AMF Detectors

A. Matched Subspace Detector, Adaptive Matched Filter, and Kelly GLRT

A straightforward, although *ad hoc*, procedure for adapting the MSD of (1) is to simply replace $\boldsymbol{R}$ with the sample covariance $\boldsymbol{S}$. Assuming there is no scaling between test and training data $(\sigma^2 = 1)$, this approach yields the adaptive matched filter (AMF) of [9] and [10]:

$$\widehat{r^2} = \frac{|\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}|^2}{\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi}} \gtrless \eta. \tag{26}$$

However, the AMF is *not* GLRT. Rather, the GLRT is the well-known statistic obtained by Kelly [4]:

$$\widehat{\chi^2} = \frac{|\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{y}|^2}{(\underline{\psi}^\dagger \boldsymbol{S}^{-1}\underline{\psi})\,(K + \underline{y}^\dagger \boldsymbol{S}^{-1}\underline{y})} \gtrless \eta \tag{27}$$

where K is the sample support, or number of training vectors. As in [9] and [10], Kelly assumes that the training and test data share the same noise structure and noise level $\sigma^2 = 1$. This detector does not enjoy the scale invariances of the CFAR ASD to different scalings of $\boldsymbol{S}$ and $\underline{y}$.

IV. Conclusions

By characterizing a slightly different hypothesis testing problem than that of Kelly, we have shown the CFAR adaptive subspace detector to be a GLRT detector. Allowing for the possibility that the noise of the measurement is scaled relative to the noise of the training data, we introduce a scaling parameter σ^2. Maximizing the likelihoods over this additional parameter results in the CFAR ASD of (4), rather than the Kelly GLRT of (27).

In the CFAR ASD, the noise structure is estimated by the training data, and the noise scaling σ^2 is compensated using the test data (σ^2 is explicitly estimated with the test data in the "F" version of the CFAR ASD). The CFAR ASD also enjoys some attractive invariances, such as invariance to arbitrary scaling of the training data $\boldsymbol{X}$ and the measurement $\underline{y}$, that is, $(\boldsymbol{X}, \underline{y}) \rightarrow (g_1\boldsymbol{X}, g_2\underline{y})$. In contrast, the Kelly GLRT and the AMF are invariant only to *uniform* scaling of $\boldsymbol{X}$ and $\underline{y}$, that is, $(\boldsymbol{X}, \underline{y}) \rightarrow (g\boldsymbol{X}, g\underline{y})$. A complete taxonomy of the the coherent, rank-1, and multirank versions of all these detectors, and their statistical behavior and distributions, will be presented in a companion paper [11].

It is interesting that the CFAR matched subspace detector remains GLRT when the unknown covariance is simply replaced by its sample estimate $\boldsymbol{S}$. This is not true of the matched subspace detector [1], [2]. That is, substituting the sample covariance into the matched subspace detector produces the adaptive matched filter (AMF) of [9] and [10] and not the Kelly GLRT.

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