DISSERTATION

THE GROUP OF THE MONDELLO BLT-SETS

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ABSTRACT

THE GROUP OF THE MONDELLO BLT-SETS

A BLT-set is a set of $(q+1)$ points of a $Q(4,q)$ parabolic quadric with a collinearity condition. There are many infinite families of BLT-sets, all of which have had their stabilizers computed except for the Mondello BLT-sets of Penttila [42]. Following an introduction to, and survey of BLT-sets and their related geometries, we compute the group stabilizing the Mondello BLT-sets.

TABLE OF CONTENTS

1. INTRODUCTION

Finite geometry is the study of finite configurations with an incidence structure. For this thesis, the focus is objects called BLT-sets. Preceding and following the construction of BLT-sets in 1990 by Bader, Lunardon and Thas [1], there has been much activity in the geometrically related areas of elation generalized quadrangles, flocks of the quadratic cone, translation planes, q-clans, and hyperbolic fibrations. Most recently, BLT-sets have been related to hemisystems as well as association schemes [5]. Therefore, despite this paper's focus on BLT-sets, the work contained is related to many other areas within finite geometry.

For many years, the main focus was on construction of new examples. In the post-BLT era, many constructions of infinite families of elation generalized quadrangles associated with flocks, and hence BLT-sets, have been given: the Subiaco family of Cherowitzo et al. in 1996, the Mondello family of Penttila in 1998, the family of Law and Penttila in 2001, the Adelaide family of Cherowitzo et al. in 2003. These added to the pre-BLT era infinite families of Fisher, Fisher-Thas/Walker, Kantor semifield, Kantor likeable, Kantor-Payne monomial, and Ganley. Further information, including references to original papers, can be found in Payne's survey paper of the area [38].

Today, there are many currently sporadic examples of BLT-sets. Hence, some focus has shifted from construction to categorization. One way to do this is by parameterizing examples into infinite families. All infinite families have had their groups computed, with the single exception of the Mondello family (introduced in Section 4.1.1). In most cases (and all recent ones), the approach to the calculation of the groups has been via the fundamental theorem of q-clan geometry of Payne [37]. This explains the omission of the Mondello examples, as Payne "worked out a q-clan representation of this BLT-set, but it seems too involved to be helpful" [38]. For this reason, this paper's approach will not be in terms of q-clans, but instead will use BLT-sets.

BLT-sets are subsets of the points of the P$\Omega(5, q)$ polar space. Therefore, the stabilizer of a BLT-set, which after this point will be called the group, will be a subgroup of the stabilizer of the whole space, PΓO$(5, q)$. Subgroups of PΓO$(5, q)$ are known due to work by H.H. Mitchell [34] as well as W. Kantor and R. Liebler [27]. Thus, we have a list of all possible groups of BLT-sets. This knowledge provides a superior approach towards the group computations. We start the computation of the group of a Mondello BLT-set using this knowledge (Chapter 5).

The groups of the associated generalized quadrangles, flocks, Thas-Walker planes and hyperbolic fibrations can be computed from those of the BLT-sets. This is a result of Payne and Thas [40] and others. How to transfer the group information between configurations is explained in Section 2.4.1. Thus, doing the group computation for the BLT-set is just as useful as the group computation for any of the related objects. But, working with the objects as BLT-sets, we gain a subgroup list.

We begin by providing necessary background knowledge in Chapter 2. This includes information on the projective and polar spaces and quadrics we will encounter. It also has a brief survey of the currently known infinite families of BLT-sets. In Chapter 3, we provide a nice construction of a family of BLT-sets from twisted cubics. In Chapter 4, we begin studying the Mondello BLT-sets in earnest, by presenting the model in which they were first given. This chapter includes what turns out the be the group of the Mondello BLT-set. The remaining chapters are needed to prove this fact.

From the original paper on the Mondello BLT-sets [42], the group is known to be transitive. Starting with transitivity, in Chapter 5, we characterize the Fisher-Thas/Walker BLT-sets as the only BLT-sets admitting a transitive group acting irreducibly on the underlying five dimensional vector space (Theorem 5.2.1). It follows directly that the group of the Mondello BLT-sets act reducibly (Corollary 5.2.2). In Chapter 6 we prove that for $q \neq 11$

the group of a Mondello BLT-set fixes a hyperplane (Lemma 6.1.2). Knowing that this hyperplane is fixed, we continue on in that chapter by looking at the group's action on this fixed hyperplane. We prove that for $q \neq 11$ the group fixes a pair of perp external lines (Lemma 6.2.3). Finally, in this chapter, we prove for $q \neq 9$ or 11, that the pair of external lines the group fixes are within the model of the Mondello BLT-sets (Lemma 6.2.5). In Chapter 7 we put together the results of previous chapters to prove the main theorem of this paper. This is the first proof that the Mondello BLT-sets (and related objects) are in fact distinct.

Theorem 1.0.1. *The stabilizer in $P\Gamma O(5, q)$ of a Mondello BLT-set $\mathcal{P}$, for $q = p^h > 11$, is isomorphic to $C_{q+1} \rtimes C_{2h}$. The group is generated by the permutations ϕ and ψ where ϕ is the map $(x, y, a) \mapsto (\eta^2 x, \eta^3 y, a)$ for a fixed $\eta \in GF(q^2)$ with $|\eta| = q + 1$ and ψ is the map $(x, y, a) \mapsto (\epsilon x^p, \epsilon y^p, a^p)$ for $\epsilon = 1$ if $\sqrt{5} \in GF(p)$ and $\epsilon = -1$ if $\sqrt{5} \notin GF(p)$.*

2. BACKGROUND

To study BLT-sets, one needs an understanding of the underlying spaces, namely projective and polar spaces. As this thesis is interested in the group of a Mondello BLT-set, we also need to know the automorphisms of the ambient space. This section is intended to give a foundation in these areas, as well as to provide knowledge that will be useful in the computation of the group of a Mondello BLT-set.

2.1 Finite Projective and Polar Spaces

In this thesis, we are concerned with geometric spaces containing a finite number of points. These spaces arise from vector spaces of dimension $n+1$ over a finite field, $\mathrm{GF}(q)$, of order q. We define a **projective space**, $\mathrm{PG}(n, q)$, to be the subspaces of this vector space where incidence is symmetric inclusion. We also define non-zero scalar multiples of vectors to be equivalent. Points of a projective space are elements of dimension 1 of the vector space, dimension 0 of the projective space. Similar definition can be given for lines and higher dimensional spaces. Hyperplanes are defined to be subspaces of codimension 1.

In 1916, Veblen and Young [49] gave axioms for finite projective spaces in terms of points and lines.

1. Any two points lie on a unique line.
2. Any line contains at least three points.
3. Any line which meets two sides of a triangle, not at a vertex, also meets the third side.

Any finite geometry of dimension greater than 2 is PG(n, q) for some integer n and some power of a prime q. For dimension 2, there exist finite projective planes not isomorphic to PG$(2, q)$. For a thorough treatment of projective spaces, the author recommends Beutelspacher and Rosenbaum's text [9].

Projective geometries satisfy a duality principle. That is, any statement that is true of a projective geometry is also true if inclusion is inverted and dimension is replaced by codimension. This duality will be revisited after looking at automorphism of projective spaces.

An automorphism of a projective space needs to preserve the underlying structure, namely the incidence structure or collinearity. For this reason, the group of those actions on the space that preserve collinearity are called the **collineation group**. These actions on the underlying vector space, V, come from semilinear transformations. A transformation $\varphi : V \to V$ is called **semilinear** if there exists a field automorphism of α of GF(q) such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(cv) = \alpha(c)\varphi(v)$ for all $v, w \in V$ and $c \in$ GF(q). All invertible semilinear transformations are collineations of the corresponding projective space PG(n, q). The group of all invertible semilinear transformations is denoted P$\Gamma(n + 1, q)$.

When defining the projective space we had an equivalence relation on vectors, so here we will need to mod out elements of P$\Gamma(n + 1, q)$ which fix points of PG(n, q) via scalar multiplication. These elements are the center of the P$\Gamma(n + 1, q)$, $Z = \{v \mapsto cv : c \in GF(q), c \neq 0\}$. Thus, the collineation group of PG(n, q) is the **projective semilinear group** PΓL$(n + 1, q) \cong \Gamma$L$(n + 1, q)/Z$.

Theorem 2.1.1 (Fundamental Theorem of Projective Geometry)**.** *Assuming that n is at least two, the automorphism group of PG(n, q) is PΓL$(n + 1, q)$.*

Instead of automorphisms, collineations, of a projective geometry, let us look at actions that use the duality principle. Elements that are inclusion-reversing bijections of the space. The group that is generated by all such inclusion-reversing bijections is called the **correlation group**. For the purpose of this paper, we wish to look at those correlations of a space that have order 2, called **polarities**. These polarities will give rise to geometries called polar spaces. Polar spaces are where BLT-sets live.

Let ρ be a polarity of a projective space that arises from a finite field. Those subspaces, W of PG(n, q), with $W \subseteq \rho(W)$ are called **totally isotropic**. The set of all totally isotropic subspaces for a given polarity, where incidence is inherited from the projective space, is called a **polar space**. Over the next few paragraphs we will explain another way to view these polar spaces.

Semilinear transformations lead to collineations of PG(n, q). What types of transformations lead to polarities of PG(n, q)? This question was answered by Birkhoff and von Neumann in 1936 [10].

Theorem 2.1.2. *Every duality of PG(n, q), for $n \geq 2$, arises from a non-degenerate sesquilinear form on the underlying vector space.*

Lemma 2.1.3. *The duality arising from a non-degenerate sesquilinear form is a polarity if and only if that form is reflexive.*

A **sesquilinear form** on a vector space V is a function f from (V, V) to GF(q) that is linear in the first variable and semilinear in the second variable. If the function is linear in the second variable, the form is called **bilinear**. The **radical** of a form is Rad$(f) = \{v \in V : f(v, w) = 0$ for all $w \in V\}$. A form is called **nondegenerate**, if the radical is $\{0\}$. The form is called **reflexive**, if for all $v, w \in V$, $f(v, w) = 0$ implies that $f(w, v) = 0$.

As stated earlier, polar spaces come from the totally isotropic subspaces of a projective space. But, we can define them in terms of their point set, rather than by their subspaces.

A **quadratic form** on V is a function Q that takes V to GF(q) such that $Q(cv) = c^2Q(v)$ for all $v \in V$ and all $c \in$ GF(q) such that $f(v, w) = Q(v + w) - Q(v) - Q(w)$ is bilinear. f is a **symmetric bilinear form**, i.e. $f(v, w) = f(w, v)$. A vector $v \in V$ is **singular** if $Q(v) = 0$. The singular radical of Q is $\{v \in$ Rad$(f) : Q(v) = 0\}$. The quadratic form Q is called **non-degenerate**, if the singular radical is $\{0\}$.

The set $\mathcal{Q}$ of points $\langle v \rangle$ of PG(n, q) satisfying $Q(v) = 0$ form a **quadric**. When this set of points inherits the subspaces of PG(n, q) that satisfy $Q(v) = 0$ for all v in a given subspace, then $\mathcal{Q}$ is an orthogonal polar space arising from the quadratic form.

Therefore, one can think of polar spaces as either coming from a reflexive non-degenerate sesquilinear form or from a non-degenerate quadratic form. In 1974 Buekenhout and Shult [8] gave an axiomatization of non-degenerate polar spaces similar to what Veblen and Young gave for projective spaces. These are the **classical polar spaces**.

1. Every line contains at least three points.
2. No point is collinear with all other points.
3. Two points are on at most one line and every point is on at least three lines.
4. If a point P is not on a line l, then either P is collinear with exactly one point of l or P is collinear with all points of l.

A polar space has rank 2 if "all" never arises in the fourth axiom; otherwise the rank is greater than 2. Just as with projective spaces, there is a classification of polar spaces.

Theorem 2.1.4 (Tits 1974 [48]). *A finite polar space of polar rank greater than 2 is a classical polar space.*

There exists non-classical polar spaces of rank 2 (generalized quadrangles. The author recommends Payne and Thas' book on finite GQs [41]. Other non-classical generalized quadrangles are mentioned in Section 2.4.

Specifically for this thesis, there are two polar spaces of importance. The first polar space (where a BLT-set lies) is $Q(4, q)$, a parabolic quadric. There is a unique non-degenerate quadratic form on GF$(q)^5$, up to similarity. Also in this paper, the polar space $Q^+(3, q)$, a hyperbolic quadric, will arise as the intersection of a hyperplane of PG$(4, q)$ and $Q(4, q)$. Both of these polar spaces are orthogonal spaces.

The parabolic quadric $Q(4, q)$ arises from a non-degenerate quadratic form with associated symmetric bilinear polar form. For example the sum of squares:

$$Q(x, y, z, w, v) = x^2 + y^2 + z^2 + w^2 + v^2$$

and

$$f(u, v) = Q(u + v) - Q(u) - Q(v).$$

The automorphism group of $Q(4, q)$ is $\mathrm{P\Gamma O}(5, q)$, the projective semilinear group. Indeed, the automorphism group of any non-degenerate classical polar space of rank at least two (apart from $\mathrm{P}\Omega^+(4, q)$ spaces) is the corresponding classical projective semisimilarity group.We will need some facts about $Q(4, q)$ later in the thesis. First, we will look at an invariant of orthogonal spaces over fields of odd characteristic. Second ,we will view $Q(4, q)$ as a generalized quadrangle.

The invariant of orthogonal spaces we will use is called the **discriminant**. Some introduction is necessary. As q is odd, then $\mathrm{GF}(q)^*$ is a (multiplicative) group with subgroup the squares $\Box = \{f^2 : f \in GF(q)^*\}$ and the other coset, the non-squares $\boxtimes = \{f : f \neq g^2 \text{ for any } g \in \mathrm{GF}(q)^*\}$. For a non-degenerate orthogonal space V over $\mathrm{GF}(q)$, q odd, there is a corresponding quadratic form Q and its related polar form f. Let B be the nonsingular matrix, with respect to some basis for V, associated with f: $f(u, v) = uBv^T$.

We define the discriminant of Q to be $\mathrm{disc}(Q) = \det(B)\Box$. Now we must show that this is an invariant. Suppose that we had two isometric orthogonal spaces V_1 and V_2 with corresponding quadratic forms Q_1 and Q_2. There exists an isometry φ taking V_1 to V_2. To each quadratic form there is a nonsingular matrix with respect to some basis for V_1 and V_2, call them B_1 and B_2 respectively. Then there exists a matrix A that corresponds to φ that takes B_1 to B_2, namely $B_2 = A^T B_1 A$. As $\det B_2 = \det(A)^2 \cdot \det B_1$, we see that up to a square, the determinants are equal. Therefore, $\mathrm{disc}(Q)$ is an invariant. In fact, for non-degenerate spaces, it is a complete invariant. In other words, two non-degenerate orthogonal spaces over GF(q), q odd, are isometric if and only if their quadratic forms have the same discriminant.

$Q(4,q)$ is a polar space, but it is also a generalized quadrangle. This fact can be used to prove that a BLT-set is of maximal size given the BLT-set property. We will take a short diversion into the axioms of a generalized quadrangle.

A **generalized quadrangle**, **GQ**, is an incidence structure with points and lines and symmetric incidence satisfying:

1. Each point is incident with a constant number $(t+1)$ of lines, and two distinct points are incident with at most one line.

2. Each line is incident with a constant number $(s+1)$ of points, and two distinct lines are incident with at most one points.

3. For a non-incident point-line pair (P,l), there is a unique point Q and a unique line m such that P is incident with m which is incident with Q which is incident with l.

We call (s,t) the order of the GQ. The number of points of a GQ is $(s+1)(st+1)$ and the number of lines is $(t+1)(st+1)$. As $Q(4,q)$ is a generalized quadrangle of order (q,q), it has $(q+1)(q^2+1)$ points and lines.

The other polar space we will encounter in this paper is $Q^+(3,q)$, a hyperbolic quadric. It has collineation group PΓO$^+(4,q)$. A hyperbolic quadric is also a GQ and has order $(q,1)$. Therefore, it has $(q+1)^2$ points and $2(q+1)$ lines. The lines split exactly in half into two reguli - 2 sets of skew lines such that each line of the opposite regulus is a transversal. In Section 4.3 and Section 6.2.2 we will need the number of lines of $PG(3,q)$ that are external to $Q^+(3,q)$ (i.e. they do not intersect $Q^+(3,q)$).

Lemma 2.1.5. *There are $q^2(q-1)^2/2$ external lines to $Q^+(3,q)$.*

Proof. Lines of PG$(3,q)$ are incident with $Q^+(3,q)$ in $0,1,2$, or $(q+1)$ points. Let l_0, l_1, l_2, and l_{q+1} denote the number of these line types. There are $(q^2+1)(q^2+q+1)$ lines of

PG$(3, q)$ so

$$l_0 + l_1 + l_2 + l_{q+1} = (q^2 + 1)(q^2 + q + 1).$$

The lines l_{q+1} are the lines of $Q^+(3, q)$, so $l_{q+1} = 2(q+1)$. The l_2 lines meet the hyperbolic quadric in two points. They can be counted by first choosing two points of $Q^+(3, q)$ and then removing those pairs of points that lie on a line of the quadric. So $l_2 = \binom{(q+1)^2}{2} - 2(q+1)\binom{q+1}{2}$. The l_1 lines are tangent lines to $Q^+(3, q)$. On each point there are $q^2 + q + 1$ lines. For a point on the quadric, 2 of these lines are of type l_{q+1} and $2q$ are of type l_2. Thus, on each point of the quadric there are $q^2 + q + 1 - 2 - 2q = q - 2$ tangent lines. Thus, in total $l_1 = (q - 2)(q + 1)^2$. Knowing l_1, l_2, and l_{q+1} we can subtract to find that $l_0 = q^2(q - 1)^2/2$. □

2.2 BLT-Sets

BLT-sets were introduced in a paper by Bader, Lunardon, and Thas [1], although the nomenclature is due to Kantor [26]. They arose as a way to create new flocks from old. But, the intermediary step, a BLT-set, was a worthwhile object to study as it is a more general object than a flock. A BLT-set is a set B of $q+1$ points of the generalized quadrangle $Q(4, q)$ for odd q with an incidence condition. As $Q(4, q)$ is the P$\Omega(5, q)$ polar space, BLT-sets can be thought of in either setting.

Definition 2.2.1. A **BLT-set** $\mathcal{B}$ is a set of $q + 1$ points of $Q(4, q)$, such that for any three points of $\mathcal{B}$, there is no point of $Q(4, q)$ that is collinear (in a line of $Q(4, q)$) with all three of the points.

Let us now explore further the size of a BLT-set, as well as the condition that q is odd. Let B be a set of points of $Q(4, q)$ with the BLT-set property: no point of $Q(4, q)$ is collinear (in $Q(4, q)$) with more than two points of B. We will prove in two different ways that the maximum size B can take on is $q + 1$.

First, we give a proof via double counting. Let $\mathcal{B}$ be a subset of size b of points of $Q(4,q)$. Remember that $Q(4,q)$ is a GQ of order (q,q). Let t_1 be the number of points of the GQ without $\mathcal{B}$ that are collinear with a point of $\mathcal{B}$. Let t_2 be the number of points of the GQ without $\mathcal{B}$ that are collinear with two points of $\mathcal{B}$. Let (P,T) be a pair of collinear points where P is in $\mathcal{B}$ and T is in the GQ without $\mathcal{B}$. Then, by double counting, $t_1 + 2t_2 = 2bq$. Let (P,Q,T) be a set of unordered triples where P and Q are distinct points of $\mathcal{B}$ and T is a point of the GQ without $\mathcal{B}$ and each of P and Q are collinear with T. Then, by double counting, $2t_2 = 2b(b-1)$. Solving for t_1, we get $t_1 = 2bq - 2b(b-1) = 2b(q+1-b) \geq 0$ and thus, $b \leq q+1$. Therefore, a set with the BLT-set property is of maximal size if $b = q+1$.

Next, we present a proof using the properties of a GQ. Assume two points P and Q of $\mathcal{B}$ were collinear. A third point of $\mathcal{B}$ would lie on a unique line intersecting the line PQ at a point R. But then the point R would be collinear with more than two points of a BLT-set, a contradiction. Thus, no two points of a BLT-set are collinear (another property of BLT-sets). Now consider a line intersecting $\mathcal{B}$ at a point P. By the GQ axioms, every other point of $\mathcal{B}$ must be on a unique line meeting the original line in a point. These points must be unique, as if they were not, they would provide a point that is collinear with 3 points of the BLT-set. A line of $Q(4,q)$ has size $q+1$, and thus, the maximum size of $\mathcal{B}$ is $q+1$.

Now we will prove that for a BLT-set to exist, q needs to be odd. Let l be a line not meeting a BLT-set $\mathcal{B}$, and let P be a point on l. Assume P is contained in a line m which meets $\mathcal{B}$ in the point R. To each point of m we can associate a distinct point of $\mathcal{B}$. As R is on m and R will be associated with R, P must be associated with another point of $\mathcal{B}$. Hence, each point on a line not intersecting $\mathcal{B}$ must be collinear with either 0 or 2 points of $\mathcal{B}$. These points of intersection cannot overlap as that would run counter to the axioms of a GQ. Therefore, the points of $\mathcal{B}$ can be partitioned into disjoint pairs by the points of an external line to $\mathcal{B}$. Thus, $q+1$ must be even which forces q to be odd.

As stated above, no point of $Q(4,q)$ can be collinear with three points of $\mathcal{B}$. This is equivalently stated as the perp of the span of three distinct points of $\mathcal{B}$ is an external line to $Q(4, q)$. For a set $\{x, y, z\}$ of points to be a subset of a BLT-set, Bader, O'Keefe and Penttila [2], using that the discriminant is a complete invariant of a non-degenerate quadratic form in odd characteristic, gave an extremely useful criterion for testing whether or not a set is a BLT-set.

Theorem 2.2.1. *If x, y, and z are pairwise linearly independent vectors of $Q(4, q)$ then $\langle x, y, z\rangle^{\perp}$ is an external line to the quadric (i.e. $\{x, y, z\}$ form a partial BLT-set) if and only if*

$$\frac{-2f(x, y)f(x, z)f(y, z)}{disc(Q)} = \boxtimes,$$

where Q is the quadratic form defining $Q(4, q)$ and f is the associated bilinear form.

The triple condition, when combined with a result of Johnson [22], reduced the test from all possible triples, to all triples containing a fixed point. This quicker test was implemented by Law and Penttila when searching for BLT-sets [32]. Also, as will be seen in Chapter 5, essentially the same computer search was put into use in this thesis.

Lemma 2.2.2 ([2])**.** *Let $\mathcal{B}$ be a set of at least 3 points of $Q(4, q)$. If there exists an $x \in \mathcal{B}$ such that $\{x, y, z\}$ is a partial BLT-set for all $\{y, z\} \in \mathcal{B}\backslash\{x\}$, then $\mathcal{B}$ is a partial BLT-set.*

2.3 Known BLT-Sets

There are nine infinite families of BLT-sets known. Other, presently sporadic, examples are known for small field orders. It is possible that these examples may one day be included in an infinite family. One can see an up to date list, with explicitly given data, of all known BLT-sets with field order less than a certain number (currently classified up to $q = 67$) on Betten's webpage [6].

As stated in the introduction, many of the following BLT-set families are known to have nice presentations in terms of their related q-clans. Hence, those families with known

q-clan presentations, will be given as q-clans. **The triples (a_t, b_t, c_t) give the q-clan**

$$C = \left\{ \begin{pmatrix} a_t & \frac{1}{2}b_t \\ \frac{1}{2}b_t & c_t \end{pmatrix} : t \in \mathrm{GF}(q) \right\}.$$

These q-clans, for the quadratic form $x_1x_5 + x_2x_4 - x_3^2$ give the following BLT-set

$$B = \left\{ \left(1, c_t, \frac{-b_t}{2}, a_t, \left(\frac{b_t}{2}\right)^2 - a_tc_t \right) : t \in \mathrm{GF}(q) \right\} \cup (0,0,0,0,1).$$

Along with the q-clan, the order of the stabilizer of the BLT-set in PΓO$(5, q)$ will be given for large enough $q = p^h$. If a characterization of the family is known, the characterization will be given in terms of both the flock and the BLT-set.

The following theorem is not used in any computations in this paper, but explains the number of corresponding flocks of an infinite family of BLT-sets. Bader, Lundardon, and Thas [1] proved the following, in their original paper on derivation of flocks.

Theorem 2.3.1. *The number of distinct, non-isomorphic flocks arising from a BLT-set is equal to the number of orbits of the stabilizer of the BLT-set in $P\Gamma O(5, q)$.*

Thus, if the group is transitive, there is only one corresponding flock. If the group is not transitive, the number of corresponding non-isomorphic flocks will be given. For a more detailed working of the known infinite families, see Law's thesis [30] or Payne's survey [38].

<u>The Classical BLT-sets</u>

$(t, 0, -nt)$ for n a non-square.

They give the linear flocks, the classical GQs, namely $H(3, q^2)$, the Desarguesian Thas-Walker planes and the André planes via hyperbolic fibrations. The group is known to be transitive with order $2h(q-1)q(q+1)^2$.

The Fisher-Thas/Walker (FTW) BLT-sets

$$(t, 3t^2, 3t^3), \;\; q \equiv 2 \text{ (mod 3)}$$

These BLT-sets were first found by Walker [50] as a flock and a plane, by Fisher-Thas [16] as a flock and a plane, and by Kantor [23] as a GQ. The group, as will be shown in this paper, is transitive and irreducible. For $q = 5$ the group is S_6. For $q > 5$ the group is PΓL$(2, q)$ which has order $h(q-1)q(q+1)$.

Thas in 1993 proved a characterization of the Fisher-Thas-Walker flocks using q-arcs. In general, a k-arc of PG(d, q) is a set of k points such that any $(d+1)$ points of the set span the whole space. In this paper, he proved that any q-arc of PG$(3, q)$, q odd and $q > 83$, is extendable to a unique $(q+1)$-arc. He then used this lemma to prove the following FTW flock characterization.

Theorem 2.3.2 (Thas [46])**.**

- *Let $\mathcal{F} = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone K of PG$(3, q)$, $q \geq 4$. Also, assume that in PG$(3, q)$ each q-arc is extendable to a $(q+1)$-arc ($q > 83$). Then $\mathcal{F}$ is the flock of FTW if no four of the planes π_i, with $C_i \subset \pi_i$, have a point in common.*

- *With the same conditions on q, a set of $q+1$ points of $Q(4, q)$ is a FTW BLT-set if and only if it is a $(q+1)$-arc.*

As a corollary of the results of Chapter 5, we have a new (complete) characterization of the FTW BLT-sets.

Theorem 2.3.3. *The only infinite family of BLT-sets whose group acts transitively on the BLT-set and irreducibly on GF$(q)^5$ is the Fisher-Thas-Walker family. There are two other transitive irreducible BLT-sets (both due to Law-Penttila): one in $Q(4, 29)$ with group S_6 and one in $Q(4, 59)$ with group S_5.*

The Fisher BLT-sets

They were first given by Fisher [16] as a flock and a plane as well as a GQ by Thas [45]. Their q-clan presentation is complex, but as we will see in Section 4.2, the Fisher BLT-sets when viewed in a different model are simple. The representation given here is from [36]. For all odd q, let ζ be a primitive element of GF(q^2), so $\omega = \zeta^{q+1}$ is a primitive element of GF(q) and hence a nonsquare of GF(q). Let $i = \zeta^{(q+1)/2}$, so $i^2 = \omega$, and $i^q = -i$. Let $z = \zeta^{q-1} = a + bi$, so z has order $q+1$ in the multiplicative group of GF(q^2). Then the triples are $(t, 0, -\omega t)$ for those $t \in$ GF(q) with $t^2 - 2(1+a)^{-1}$ a square in GF(q) and $(-a_{2j}, 2b_{2j}, -\omega a_{2j})$ for $0 \le j \le (q-1)/2$ and

$$a_k = \frac{z^{k+1} + z^{-k}}{z+1}$$

$$b_k = \frac{i(z^{k+1} - z^{-k})}{z+1}.$$

The group acts transitively and has order $2h(q+1)^2$ for $q > 7$.

In a work from 1991, Payne and Thas proved a characterization of the Fisher flocks by using intersections of the associated planes.

Theorem 2.3.4 (Payne-Thas [39])**.**

- *For π a plane of $P = PG(3,q)$, q odd, and x be a point of $P - \pi$, consider a nonsingular conic C in π, and the cone K with vertex x and containing C. The Fisher flock is the unique nonlinear flock $\mathcal{F}$ of K for which at least (i.e. exactly) $(q-1)/2$ of the planes associated with the flock contain a common line.*

- *The Fisher BLT-set is the only BLT-set which meets a classical BLT-set in $(q+1)/2$ points.*

The Kantor Likeable BLT-sets

$$(t, t^2, t^3/3 - nt^5 - t/n),\ \ q = 5^h \text{ for } n \text{ a non-square}$$

They were first constructed as a plane by Kantor [24] and recognized in the flock setting by Gevaert-Johnson [17]. This group's action is not transitive. Each BLT-set gives rise to 2 non-isomorphic flocks. The group has order $2hq$ for $q > 5$.

The Ganley BLT-sets

$$(t, t^3, -(nt + t^9/n)),\ \ q = 3^h \text{ for } n \text{ a non-square}$$

They were first constructed as a plane by Ganley [14] and recognized as a flock by Gevaert-Johnson [17]. This group action is also not transitive and derives to 2 non-isomorphic flocks (one semifield and one non-semifield). The group has order $2hq$ for $q > 27$.

The Kantor Semifield BLT-sets

$$(t, 0, -nt^{\sigma}) \text{ for non-prime } q, n \text{ a non-square, and } \sigma \text{ a nontrivial automorphism of GF}(q)$$

They were constructed first as a GQ by Kantor [25] and later were recognized as a flock by Thas [45]. The group of these BLT-sets are transitive. If $\sigma^2 \neq 1$ the group has order $2hq(q-1)q(q+1)$ and if $\sigma^2 = 1 \neq \sigma$ the group has order $4h(q-1)q(q+1)$.

In a 1987 work, Thas characterized the Kantor semifield flocks again via intersections of the associated planes.

Theorem 2.3.5 (Thas [45])**.**

- *If all planes of a nonlinear flock $\mathcal{F}$ intersect in a common point, then $\mathcal{F}$ is a Kantor semifield flock.*

- *The points of the Kantor semifield BLT-set span a hyperplane of PG$(4, q)$ which meets $Q(4, q)$ in a hyperbolic quadric.*

The Kantor Monomial BLT-sets

$$(t, 5t^3, 5t^5),\ \ q \equiv \pm 2 \text{ (mod 5)}$$

They were first given as a GQ by Kantor [25] and were later related to BLT-sets by Thas [45]. They have a non-transitive group and derive to 2 non-isomorphic flocks. The group has order $2h(q-1)$ for $q > 13$.

The Mondello BLT-sets

They were given and named by Penttila [42] and are a BLT-set family with no useful q-clan representation that arises for $q \equiv \pm 1 \pmod{10}$. A full presentation of the Mondello BLT-sets is given in Chapter 4. Their groups are transitive and we will show their groups have order $2h(q+1)$ for $q > 11$.

The Law-Penttila BLT-sets

$$(t, -t^4 - nt^2, -n^{-1}t^9 + t^7 + n^2t^3 - n^3t),\ q = 3^h \text{ for } n \text{ a non-square}$$

They were first given by Law and Penttila [31]. Payne [38] has a paper with the exact number of non-isomorphic derived flocks, in general there are many. The group has order $2h$ for $q > 7$.

There are two currently sporadic examples that are transitive and irreducible. Both examples are due to Law and Pentilla: one in $Q(4, 29)$ with group S_6 and another in $Q(4, 59)$ with group S_5 [32].

There are also known transitive but reducible BLT-sets that are currently sporadic. Three examples are due to Royle and Penttila [43]: one in $Q(4, 19)$ with a group of order 20 acting regularly, one in $Q(4, 23)$ with a group of order 1152, and in $Q(4, 23)$ with a group of order 24 acting regularly. Another example is due to De Clerck and Penttila in $Q(4, 47)$ with a group of order 2304 (unpublished [13]). Lastly, there is an example in $Q(4, 41)$ due to Betten with a group of order 84 [7].

This list of sporadic examples, along with the above families, is believed to be a complete list of all known transitive BLT-sets.

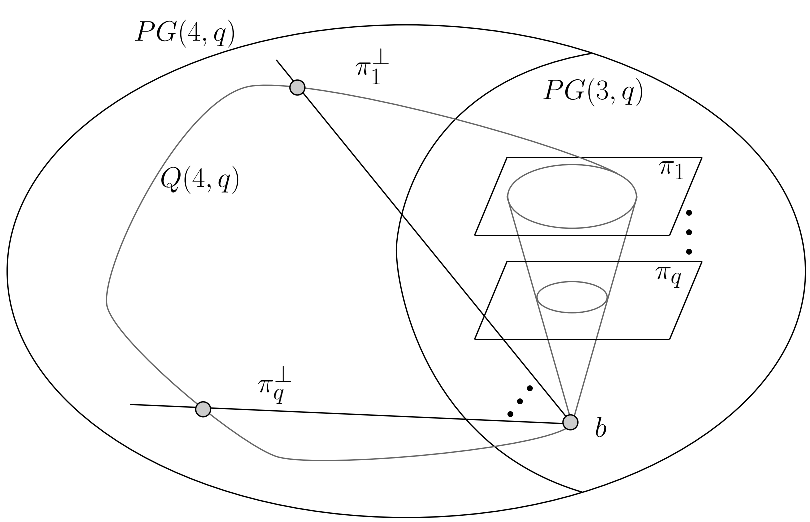
2.4 Related Configurations

The history for BLT-sets is more abundant than one might assume for an object named as recently as 1991. This richness is due to the geometric correspondences between BLT-sets and objects that have been studied as early as the 1950's. What follows is a brief history of the objects that are directly related to BLT-sets.

What follows is how Bader, Lunardon, and Thas [1] first described the objects now called BLT-sets. They were searching for new constructions of flocks. They started with a flock in PG$(3, q)$. They embeded that projective space in PG$(4, q)$ and relate the flock to a set of $(q + 1)$ points of a parabolic quadric of PG$(4, q)$. From each of those $(q + 1)$ points, a new flock arises. The flocks found in this way are called derived flocks. Flocks of quadratic cones are also related to certain generalized quadrangles and translation planes.

Let $\mathcal{B}$ be a BLT-set and let b be a point of $\mathcal{B}$. The set of points of $Q(4, q)$ collinear with b form a quadratic cone in the polar hyperplane of b. The polar hyperplanes coming from the points of $\mathcal{B}$, other than b, form the flock of the quadratic cone given by b. The converse construction also holds (i.e. going from a flock of a quadratic cone to a BLT-set). Choosing a different starting point has the possibility of leading to a new non-isomorphic flock.

The following diagram is a pictorial representation of derivation of flocks. Remember that, although this picture is continuous, the objects being described are discrete.



In 1976, Walker [50], and independently, at about the same time, Thas (unpublished), constructed an ovoid of $Q^+(5, q)$ from a flock of a quadratic cone. Via the Klein correspondence this ovoid is equivalent to a spread of lines of $\mathrm{PG}(3, q)$. This spread of lines gives rise to a translation plane via the André/Bruck-Bose construction. This construction involves embedding the flock in the Klein quadric as a hyperplane, taking the union of the perps of the planes of the flock, and then applying the Klein correspondence.

Between 1998-2005, Baker, Dover, Ebert, Wantz [3] and Baker, Ebert, and Penttila [4] found another connection between flocks, hence BLT-sets, and translation planes. A hyperbolic fibration is a partition of $\mathrm{PG}(3, q)$ into $q - 1$ hyperbolic quadrics and 2 lines. Choosing one of the two reguli on each of the hyperbolic quadrics gives 2^{q-1} spreads and so 2^{q-1} translation planes.

In this case, isomorphism of hyperbolic fibrations corresponds to orbits of the stabilizer of the BLT-set B arising from the flock on ordered pairs of points of B. These spreads of lines are in general different that those created via the Thas-Walker construction.

Between 1980 and 1987, Kantor, Payne and Thas [23, 35, 25, 45] laid the groundwork for and constructed elation generalized quadrangles, EGQs, of order (q^2, q) from flocks of Miquelian Laguerre planes. Their work used the notation of q-clans and showed that to every flock there is a corresponding EGQ, and conversely. Knarr [29] gives a geometric construction of the EGQ directly from the BLT-set.

The most recent connection of BLT-sets to other geometric objects is from Bamberg, Giudici, and Royle [5]. They proved that every flock generalized quadrangle contains a hemisystem. They also showed that a hemisystem gives rise to a cometric 4-class association scheme.

2.4.1 Groups of Related Configurations

The motivations for the study of BLT-sets are manifold: many projective planes (in two essentially different ways), generalized quadrangles, hemisystems, and association schemes all arise from BLT-sets. Our computation of the group of a Mondello BLT-set is of use to each of these different areas, as there are known conversions between groups of these configurations.

Each of these related configurations have automorphism groups. Using the constructions going back and forth between these objects, we can transfer the group information. Therefore, computing the group of a Mondello BLT-set is equivalent to computing the group of any of the related objects.

The group of a flock is the stabilizer of the disjoint conics (and hence the vertex/ distinguished point) and it must preserve the original quadratic cone. We can think of this group as the subgroup of the stabilizer of the degenerate quadric (the cone) fixing these conics (or, equivalently, the planes subtending them). The group of the (degenerate) polar space arising from a conic turns out to be the stabilizer of the cone in $\mathrm{P\Gamma L}(4, q)$. For a given BLT-set, there is a flock corresponding to each point of the BLT-set. This distinguished point is the vertex of the cone. Because of this, the subgroup of the group of the BLT-set which fixes the distinguished point is the group of the corresponding flock. Thus, the order of the group of the flock depends upon the number of orbits on points of the BLT-set. These group computations follow directly from the original BLT paper [1].

In 1991, Payne and Thas [40] related the group of a BLT-set and the group of the related generalized quadrangle. To each BLT-set there corresponds only on GQ. Each collineation of the GQ gives an element of $\mathrm{P\Gamma O}(5, q)$ which fixes the BLT-set. The points of the BLT-set correspond to the lines through (∞) of the GQ. Therefore, the group of the BLT-set corresponds to the subgroup of the automorphism group of the GQ that acts on the lines through (∞) modulo the kernel. The kernel has q^5 elations and $q - 1$ collineations. Therefore, the group of the GQ is $q^6 - q^5$ times larger than the group of the BLT-set. The preceding statements do not hold true if the BLT-sets are Classical or Kantor-semifield. In these two families, there exists elements of the kernel which act non-trivially on the BLT-set.

The groups of BLT-sets and hyperbolic fibrations were related by Baker, Ebert, and

Penttila [4]. BLT-sets with a distinguished point give a flock, and each plane of the flock determines a hyperbolic fibration. As each point different than the distinguished point corresponds to a plane, equivalence of hyperbolic fibrations corresponds to orbits on ordered pairs of points of a BLT-set under the group of the BLT-set. The order of the group of a hyperbolic fibration is $2(q+1)$ times larger than the group of the related BLT-set. The group of the hyperbolic fibration is known to be a subgroup of the group of the corresponding spreads of lines of PG$(3, q)$, although it is not known whether in general they are equal.

In the simultaneous papers [17, 18], Gevaert, Johnson, and Thas proved the relationship between the groups of corresponding translation planes, spreads of PG$(3, q)$, and flocks of the quadratic cone. An important result is: a spread of PG$(3, q)$ that consists of the union of q reguli arises from the Thas-Walker construction of a flock of the quadratic cone of PG$(3, q)$. Also, there are no further reguli, if it is not a regular spread. Equivalently, if the spread comes from a non-classical flock, then there are no further reguli. The spread acts on the set of q reguli, and the group fixing each reguli has order q. Therefore, the group of the spread is q times larger than the group of the flock. The group of the corresponding translation plane is $q^5 - q$ times larger than the group of the spread.

2.5 Group Theoretic Background

The group of PSL$(2, q)$ will frequently show up during computations in the following chapters. Thankfully, the subgroups of PSL$(2, q)$ are known from books by Dickson and Huppert [15, 20]. The following list is from Cameron et.al. [12].

Theorem 2.5.1. *The subgroups of PSL$(2, q) =$ PSL$(2, p^h)$, p odd, are as follows:*

i) $q(q \mp 1)/2$ *cyclic subgroups C_d of order d where $d \mid (q \pm 1)/2$.*

ii) $q(q^2 - 1)/(4d)$ *dihedral subgroups D_{2d} of order $2d$ where $d \mid (q \pm 1)/2$ and $d > 2$ and $q(q^2 - 1)/24$ subgroups D_4.*

iii) $q(q^2 - 1)/24$ *subgroups A_4.*

iv) $q(q^2-1)/24$ *subgroups* S_4 *when* $q = 7 \pmod 8$.

v) $q(q^2-1)/60$ *subgroups* A_5 *when* $q = \pm 1 \pmod{10}$.

vi) $p^h(p^{2h}-1)/(p^m(p^{2m}-1))$ *subgroups PSL*$(2, p^m)$ *where* $m \mid h$.

vii) The elementary Abelian group of order p^m *for* $m \leq h$.

viii) A semidirect product of the elementary Abelian group of order p^m *and the cyclic group of order* d *where* $d \mid (q-1)/2$ *and* $d \mid p^m - 1$.

Specifically when proving that the FTW-BLT-sets are equivalent to the BLT-set construction in Chapter 3, we need some basic group theory lemmas.

Lemma 2.5.2. *Let* G *act transitively on a set* $\mathcal{B}$ *of size* $q+1$. *Let* $A = G^{(\infty)}$ *and* $B = A_b$ *for* $b \in \mathcal{B}$. *Then* $[A : B]$ *divides* $q+1$.

Proof. As A is normal in G, then A is also normal in AG_b. An isomorphism theorem states that $[AG_b : G_b] = [A : B] = m$ for some integer m. As G acts transitively on $\mathcal{B}$, we know the index of G_b in G is $q+1$. Therefore, as $[AG_b : G_b] \cdot [G : AG_b] = m \cdot [G : AG_b] = q+1$ then $m = [A : B]$ divides $q+1$. □

Lemma 2.5.3. *The centralizer of PSL*$(2, q)$ *in P*Γ*L*$(2, q)$ *is trivial*.

Proof. PSL$(2, q)$ acts transitively on PG$(1, q)$. Therefore, $G = C_{\mathrm{P\Gamma L}(2,q)}(\mathrm{PSL}(2, q))$ acts semiregularly on PG$(1, q)$. Hence, the order of G must divide $q+1$.

Suppose the order of G is neither 1 nor $q+1$. Then the orbits of G form a non-trivial system of imprimitivity for PSL$(2, q)$ on PG$(1, q)$. This is a contradiction as PSL$(2, q)$ is 2-transitive and thus, primitive.

Next, assume that $|G| = q+1$. Then G is transitive, so $C_{\mathrm{PGL}(2,q)}(G)$ is semiregular. It follows that the order of $C_{\mathrm{PGL}(2,q)}(G)$ must divide $q+1$. This leads to a contradiction as PSL$(2, q) \leq C_{\mathrm{PGL}(2,q)}(G)$ and $|\mathrm{PSL}(2, q)| > q+1$.

Therefore, $G = C_{\mathrm{P\Gamma L}(2,q)}(\mathrm{PSL}(2, q)) = 1$. □

Lemma 2.5.4. *G is absolutely irreducible if and only if $C_{GL(d,q)}(G) \cong C_{q-1}$.*

Proof. This proof, albeit in slightly different forms, can be found in Isaacs [21] Theorem 9.2 as well as in Kleidman and Liebeck [28] Lemma 2.10.1. $\square$

Lemma 2.5.5. *If H_1 and H_2 are irreducible subgroups of PGO$(5,q)$ and conjugate in PGL$(5,q)$ then they are conjugate in PGO$(5,q)$.*

Proof. As $H_1 \sim H_2$ in PGL$(5,q)$ then there exists a $g \in$ PGL$(5,q)$ such that $H_2 = gH_1g^{-1}$. Let π be the polarity coming from the quadratic form giving PGO$(5,q)$. As $g^{-1}\pi gH_1(g^{-1}\pi g)^{-1} = g^{-1}\pi H_2\pi^{-1}g = g^{-1}H_2g = H_1$ then $g^{-1}\pi g$ also centralizes H_1. By Kleidman and Liebeck [28] Lemma 2.10.3, $\pi = g\pi g^{-1}$ which implies that $g \in C_{\mathrm{PGL}(5,q)}(\pi) =$ PGO$(5,q)$. $\square$

We will further use the fact that the stabilizer of a point of a BLT-set inside of PSL$(2,q)$ forces PSL$(2,q)$ to act transitively on the set. Using Lemma 2.5.2 with $A =$ PSL$(2,q)$ and $B = A_b$ for b a point of the BLT-set $\mathcal{B}$, it follows that $[A : B]$ must divide $q+1$. Thus, we need to search for subgroups of PSL$(2,q)$ with index dividing $q+1$. What follows is a case by case look at the subgroups of PSL$(2,q)$.

i) C_d of order d has index $q(q \mp 1)$ which doesn't divide $q+1$.

ii) D_{2d} of order $2d$ has index $q(q \mp 1)$ which doesn't divide $q+1$.

   D_4 has index $(q+1)q(q-1)/8$ which doesn't divide $q+1$.

iii) A_4 doesn't have index dividing $q+1$ for $q > 3$.

iv) S_4 when $q = 7 (\bmod 8)$ doesn't have index dividing $q+1$.

v) A_5 when $q = \pm 1 (\bmod 10)$ doesn't have index dividing $q+1$.

vi) PSL$(2,p^m)$ doesn't have index dividing $q+1$ unless $m = h$.

vii) The elementary Abelian group of order p^m has index $(q+1)p^{h-m}(q-1)/2$ which doesn't divide $q+1$.

viii) A semidirect product of the elementary Abelian group of order p^m and the cyclic group of order d where $d \mid (q-1)/2$ and $d \mid p^m - 1$.

   a) If $m \neq h$ then the index doesn't divide $q+1$.

   b) If $m = h$ and $d \neq (q-1)/2$ then the index is divisible by $(q+1)(q-1)/2d$ so doesn't divide $q+1$.

   c) If $m = h$ and $d = (q-1)/2$ then the index is $q+1$.

Thus, the point stabilizer $B = A_b$ must be a semidirect product of the elementary Abelian group of order q and $C_{(q-1)/2}$ or $\mathrm{PSL}(2,q)$. Either way $q' = q$ and the index of B in A is $q+1$, so A must be transitive on $\mathcal{B}$.

2.6 A Matrix Model of PGO$^+(4,q)$

Later in this thesis, we will need to know about the action of $\mathrm{PGL}(2,q) \times \mathrm{PGL}(2,q)$ on $\mathrm{PG}(3,q)$. One way of viewing this action is by imposing a matrix model on the space. Take the points of $\mathrm{PG}(3,q)$, homogeneous vectors of length 4 over $\mathrm{GF}(q)$, and turn them into a two by two matrix by

$$(a,b,c,d) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let this set of two by two matrices be denoted by $\mathrm{PM}(2,q)$. For A and B in $\mathrm{GL}(2,q)$ define the map from $\mathrm{PM}(2,q)$ to $\mathrm{PM}(2,q)$ by

$$\begin{aligned} \phi(A,M): &\quad \mathrm{PM}(2,q) \mapsto \mathrm{PM}(2,q) \\ \text{by} &\quad\quad X \mapsto AXB^T. \end{aligned}$$

The determinant is a quadratic form, $ad - bc$, so those points with determinant zero define a hyperbolic quadric, a $Q^+(3, q)$ space. $\phi(A, B)$ fixes $Q^+(3, q)$ for all A and B in GL$(2, q)$. If $\det(X) = 0$ then $\det(AXB^T) = \det(A)\det(X)\det(B) = 0$. The transpose of X in the previous map also preserves elements of $Q^+(3, q)$, in fact, it switches the reguli.

$\phi(A, B)$ for A and B in PGL$(2, q)$ is isomorphic to PGL$(2, q) \times$ PGL$(2, q)$. We can show this by looking at the reguli of the hyperbolic quadric. The reguli are given by

$$\mathcal{R}_1 = \{l^1_{a,b} : (a, b) \in \text{GF}(q)^2 \backslash (0, 0)\} \text{ for } l^1_{a,b} = \{(a, b, \lambda a, \lambda b) : \lambda \in \text{GF}(q)\} \cup \{(0, 0, a, b)\}$$

$$\mathcal{R}_2 = \{l^2_{a,b} : (a, b) \in \text{GF}(q)^2 \backslash (0, 0)\} \text{ for } l^2_{a,b} = \{(a, \lambda a, b, \lambda b) : \lambda \in \text{GF}(q)\} \cup \{(0, a, 0, b)\}$$

The set $\{\phi(A, I) : A \in \text{PGL}(2, q)\} \cong \text{PGL}(2, q)\}$ is the linewise stabilizer of $\mathcal{R}_1$. The set $\{\phi(I, B) : B \in \text{PGL}(2, q)\} \cong \text{PGL}(2, q)\}$ is the linewise stabilizer of $\mathcal{R}_2$. It can now be checked that PGL$(2, q) \times$ PGL$(2, q)$ is isomorphic to the direct product of these two groups.

For a given $X \in$ PM$(2, q)$ not in the hyperbolic quadric $Q^+(3, q)$, $\det(X) = \det(cX) = c^2 \det(X)$. Thus, points have a well defined non-zero determinant modulo the squares in GF(q) for odd q.

By a similar argument to above, $\phi(A, B)$ for A and B in PSL$(2, q)$ is isomorphic to PSL$(2, q) \times$ PSL$(2, q)$. $\phi(A, B)$ for A and B in PSL$(2, q)$ will preserve the determinant of the point/matrix being acted on. Therefore, PSL$(2, q) \times$ PSL$(2, q)$ will have two orbits on points not of the hyperbolic quadric, namely those points with $\det = \square$ and those with $\det = \boxtimes$. Knowing that there are $q^3 + q^2 + q + 1$ points of PG$(3, q)$ and $(q + 1)^2$ points of $Q^+(3, q)$ then these orbits are of size $(q^3 - q)/2$.

The same argument holds for $\phi(A, I)$ for A in PSL$(2, q)$ and $\phi(I, B)$ for B in PSL$(2, q)$. They both have orbits of length $(q^3 - q)/2$ on points not of $Q^+(3, q)$. These orbit lengths will be useful in Section 6.2.1.

3. FISHER, THAS, AND WALKER REVISITED

This chapter can stand alone as an interesting construction of an infinite family of BLT-sets, although, the construction is not the sole purpose of this chapter. It will be shown in Chapter 5 that the BLT-sets constructed here are equivalent to the FTW BLT-sets.

The last chapter of Lüneburg's [33] text on translation planes has a section devoted to twisted cubics. Here proofs can be found for the following statements and lemmas.

A set of points of PG$(3, q)$ that can be mapped by an element of PΓL$(4, q)$ onto the set

$$C = \{(s^3, s^2t, st^2, t^3) : \ s, t \in \text{GF}(q), \ (s,t) \neq (0,0)\}$$

is called a **twisted cubic**. This is slightly non-standard. An alternate definition would be a non-singular cubic curve of PG$(3, q)$ not lying in a plane.

Lemma 3.0.1. *The stabilizer G of a twisted cubic $\mathcal{C}$ of PG$(3, q)$ is isomorphic to PΓL$(2, q)$ and acts triply transitivly, assuming $q > 3$.*

Triple transitivity will be useful later to reduce the computations needed to prove the construction of a BLT-set from a twisted cubic.

Lemma 3.0.2. *There exists a symplectic polarity π of PG$(3, q)$, $\ q > 2$ and characteristic not 3, such that for all $P \in C$, P^π is the osculating plane. There is exactly one such polarity.*

The osculating plane, is the tangent plane to a point. The symmetric form

$$f(x, y) = x_1y_4 - x_4y_1 - 3x_2y_3 + 3x_3y_2$$

defines a symplectic polarity with the desired properties.

3.1 Converting a Twisted Cubic into a BLT-Set

To each point on the twisted cubic we can define a unique tangent line using a theorem due to Segre [44].

Theorem 3.1.1. *A q-arc of PG$(2,q)$ ($q \geq 5$, q-odd) completes uniquely to an oval (a $(q+1)$-arc).*

Given the twisted cubic lying in PG$(3,q)$, take one point P of the twisted cubic and look at PG$(3,q)/P \cong$ PG$(2,q)$. The twisted cubic will become a set of q points of PG$(2,q)$, no three collinear, so a q-arc. Using Segre's theorem, let Q denote the unique completion of the q-arc to an oval. We define the line PQ in PG$(3,q)$ to be *the* tangent line. These tangent lines are disjoint.

As PΓL$(2,q)$ acts triply transitively on points of the twisted cubic, it also acts triply transitively on the tangent lines to C. Thus, we have a transitive action on a set of $q+1$ objects. Using the Klein correspondence, we can turn this set of $q+1$ lines into a set of $q+1$ points of the dual space. This, it turns out, will result in a BLT-set of the dual space, $Q(4,q)$.

For any three points of a BLT-set $\mathcal{B}$, there are no points of $Q(4,q)$ collinear (in a line of $Q(4,q)$) with all three points. We now rephrase this under our polarity; for any three lines of the set, there does not exist a transversal. This test will decide whether or not the set of $q+1$ tangent lines to the twisted cubic is a BLT-set after dualizing.

Once we find one tangent line, we can use the group, as it acts transitively on the tangent lines, to find all of the other tangent lines. Let $P = (1,0,0,0)$. The twisted cubic modulo P in PG$(2,q)$ lies within the oval given by $y^2 - xz$ for points (x,y,z). The point that completes the q-arc to an oval is the point $Q = (1,0,0)$. The line connecting P and Q in PG$(3,q)$ is $t_1 = \{(a,b,0,0) : a,b \in \text{GF}(q)\}$. Therefore, t_1 is the tangent line to C at

$(1,0,0,0)$. Using the group, we find that $t_2 = \{(0,0,c,d) : c,d \in \text{GF}(q)\}$ is the tangent line at $(0,0,0,1)$, and $t_3 = \{(3e, 2e+f, e+2f, 3f) : e,f \in \text{GF}(q)\}$ is the tangent line at $(1,1,1,1)$.

We need to test whether or not there exists a transversal through any three tangent lines. Without loss of generality, we have chosen $t_1,\ t_2$, and t_3. There exists a transversal if and only if there exists a,b,c,d,e,f such that the following three equations, which comes from the symmetric form above, equal zero simultaneously. A solution will relate to the non-existence of BLT-sets.

$$\begin{aligned} f(t_1,t_2) &= ad - 3bc = 0 \\ f(t_1,t_3) &= 3af - 3be - 6bf = 0 \\ f(t_2,t_3) &= -3de + 3cf + 6ce = 0 \end{aligned}$$

After homogenizing coordinates and noting that no variable can be zero, we arrive at

$$\begin{aligned} f(t_1,t_2) &= d - 3b = 0 \\ f(t_1,t_3) &= 3f - b - 6bf = 0 \\ f(t_2,t_3) &= -d + 3f + 2 = 0 \end{aligned}$$

The solutions to this system of equations are $b = \frac{3\pm\sqrt{-3}}{6}$. If -3 is a square then there exists a transversal, so there is not a BLT-set. Thus, a BLT-set exists if -3 is a non-square in the field.

Lemma 3.1.2. *In a finite field (not of characteristic 3), -3 is a square when the order of the field is congruent to 1 modulo 3.*

Proof. -3 is a square if and only if $x^2+x+1=0$ has solutions. There are solutions if and only if $x^3 = 1$ has solutions for $x \neq 1$. This occurs if and only if there exists a primitive cube root of unity in GF$(q)^*$. There is a primitive cube root of unity in GF$(q)^*$ if and only if 3 divides $q-1$. $q-1$ being divisible by 3 is equivalent to q being congruent to 1 modulo 3. $\square$

Therefore, we have created an infinite family of BLT-sets which only exists for $q \equiv 2 \mod 3$.

From Lüneburg [33], we know that the group stabilizing the twisted cubic is PΓL$(2, q)$ for $q > 3$, q odd. PΓL$(2, q)$ has a unique orbit of length $q + 1$, namely the points of the twisted cubic. Each point has a unique tangent line, so therefore, PΓL$(2, q)$ has a unique orbit of length $q + 1$ on the tangent lines to the twisted cubic. Stated in another way, there is a unique orbit of length $q + 1$ on the totally isotropic points of the Sp$(4, q)$ polar space, and hence, a unique orbit of length $q + 1$ on the points arising from the tangent lines in the dual space (via the Klein correspondence).

The last term of the derived series of PΓL$(2, q)$ is PSL$(2, q)$. Using the subgroups of PSL$(2, q)$ and arguments from Section 2.5, PSL$(2, q)$ must act transitively on a set of $q + 1$ points, namely the tangent lines (points of dual space). Thus, we have a subgroup $H =$ PSL$(2, q)$ of P$\Omega(5, q)$, which, after dualizing, stabilizes the set of tangent lines to the twisted cubic. We have shown that these objects being stabilized are BLT-sets. All twisted cubics are equivalent, thus, their groups are conjugate in PGL$(3, q)$, and hence conjugate in PGL$(5, q)$.

Consider the following centralizer $C = C_{\mathrm{GL}(5,q)}H$. Let $Z = Z(\mathrm{GL}(5, q)) \cong C_{q-1}$. Then C/Z dualized is a subgroup of PΓL$(2, q)$. But, by Lemma 2.5.3, $C_{\mathrm{P\Gamma L}(2,q)}(H) = 1$. As elements of C/Z must both centralize H and be members of PΓL$(2, q)$, $C/Z = 1$, giving $C = Z$. Thus, by Lemma 2.5.4, since $C = Z = C_{q-1}$, then $H =$ PSL$(2, q)$ (in $GL(5, q)$) is absolutely irreducible.

In Section 5.1.1, we will prove that the BLT-sets just created are the FTW BLT-sets. We have just proven that the group stabilizing these BLT-sets, PΓL$(2, q)$, has a subgroup, $PSL(2, q)$, which acts irreducibly on GF$(q)^5$. This, it turns out, is enough to prove the equivalence of the BLT-sets just constructed and the FTW BLT-sets.

4. THE $(\mathrm{GF}(q^2), \mathrm{GF}(q^2), \mathrm{GF}(q))$ MODEL

A BLT-set is a subset of the points of $Q(4, q)$. $Q(4, q)$ is itself a subset of the points of $\mathrm{PG}(5, q)$. Usually the points of a BLT-set are given by a set of $(q+1)$ homogenized vectors of length five over $\mathrm{GF}(q)$. But, viewing a BLT-set in this manner does not display, or make visible, the symmetries in specific infinite families. By changing the way a BLT-set is specified, certain symmetries become obvious. Forcing the symmetries into the model has the added benefit of making the presentation far simpler.

In this chapter, the model first described by Penttila [42] will be explained in Section 4.1. Originally the model was devised as a way to give a cleaner description of the Fisher BLT-sets, but it also provided the path toward finding the Mondello BLT-sets. In Section 4.1.1, we will prove the infinite family of Mondello BLT-sets are in fact BLT-sets using this model. As well, the infinite families which fit nicely into this model and their groups, in the model, will be given in Section 4.2. Lastly, in Section 4.3, the group of the model will be investigated. This will result in Lemma 4.3.1, which will be useful when computing the full group of the Mondello BLT-sets.

4.1 The Model

The model given by Penttila [42] uses the isomorphism between $\mathrm{GF}(q)^2$ and $\mathrm{GF}(q^2)$ as $\mathrm{GF}(q)$ vector spaces to force a cyclic group of order $(q+1)$ to become visible in the description of a BLT-set (as will be described later). Instead of thinking of the points of $\mathrm{PG}(5, q)$ being of the form $\{(x, y, z, w, v) : x, y, z, w, v \in \mathrm{GF}(q)\}$, we can think of them as being in

$$V = \{(x, y, a) : x, y \in \text{GF}(q^2),\ a \in \text{GF}(q)\}.$$

For a description of a BLT-set to be given, we need to attach a quadratic form to define our polar space $Q(4, q)$. Let

$$Q(x, y, a) = x^{q+1} + y^{q+1} - a^2 = N(x) + N(y) - a^2$$

be the quadratic form with its associated bilinear form

$$f((x, y, a), (z, w, b)) = T(xz^q) + T(yw^q) - 2ab$$

where $T(x) = x + x^q$ and $N(x) = x^{q+1}$. Note that this quadratic form varies slightly from that in the original paper: $Q(x, y, a) = x^{q+1} + y^{q+1} + a^2$.

V is the orthogonal direct sum of $\{(x, 0, 0) : x \in GF(q^2)\}$, $\{(0, y, 0) : y \in GF(q^2)\}$, and $\{(0, 0, a) : a \in GF(q)\}$. Therefore, the discriminant of Q equals the product of the discriminants of Q restricted to these subspaces. The first two subspaces are isometric and thus have the same discriminant. The third subspace has discriminant -2. Therefore, $disc(Q) = -2\,\square$. Using this, the theorem of Bader, O'Keefe, and Penttila [2] (from Section 2.2) becomes:

Theorem 4.1.1. *Let B be a set of at least 3 points of $Q(4, q)$ with the quadratic form: $Q(x, y, a) = x^{q+1} + y^{q+1} - a^2$. Then B is a partial BLT-set if and only if for all $x, y, z \in B$*

$$f(x, y) \cdot f(x, z) \cdot f(y, z) = \boxtimes,$$

4.1.1 The Mondello BLT-sets

Theorem 4.1.2. *Let $\eta \in GF(q^2)$ be an element of order $q+1$. Then*

$$\mathcal{B} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq q\}$$

is a BLT-set in (V, Q) for $q \equiv \pm 1(mod\ 10)$.

The proof of this theorem will come in four sections. First, we will prove the necessary congruence condition on the characteristic of the finite field. Then, we will show that $\mathcal{B}$ has the correct size and the points of $\mathcal{B}$ are points of $Q(4, q)$. Finally, we will show that the points of $\mathcal{B}$ satisfy the triple/discriminant condition given in Theorem 4.1.1.

For $\mathcal{B}$ to be a BLT-set, $\sqrt{5}$ needs to exist. If $\sqrt{5} = 0$, (i.e. $GF(q)$ has characteristic 5) the BLT-set will lie in a hyperplane and Thas proved if a BLT-set lies in a hyperplane it is either Classical or Kantor Semifield [45]. So, we will rule out GF(q) having characteristic 5. As the following lemma proves, 5 is a square if the characteristic is congruent to 1 or 9 modulo 10.

Lemma 4.1.3. *For q odd and $q \not\equiv 0\ (mod\ 5)$, $5 = \square$ if and only if $q \equiv \pm 1\ (mod\ 10)$.*

Proof. We will split this proof into three cases: q is prime and q is an odd or even power of a prime.

Assuming that q is a prime, we can use quadratic reciprocity. $5 = \square(\text{mod } q)$ if and only if $q = \square(\text{mod } 5)$. The squares modulo 5 are 1 and 4. But, q is odd so $q \equiv \pm 1(\text{mod } 10)$.

Now assume that $q = p^h$ for p an odd prime and h odd. We will prove that $5 = \square$ (in $GF(p)$) if and only if $5 = \square$ (in $GF(p^h)$). The forward direction is obvious. For the other directions assume that $x^2 - 5$ factorizes over $GF(p^h)$ but is irreducible over $GF(p)$. This forces an even extension, i.e. h even, which is a contradiction.

Next, assume that $q = p^h$ for p an odd prime and h even. An odd number to an even power is always equivalent to 1,5, or 9 modulo 10. As q is not of characteristic 5, then

$q = p^h \equiv \pm 1 (\text{mod} 10)$.

In conclusion, assuming q is not characteristic 5, 5 is a square iff q is equivalent to 1 or 9 modulo 10. $\square$

For $\mathcal{B}$ to be a BLT-set, $\mathcal{B}$ must have size $q+1$.

Lemma 4.1.4. $|\mathcal{B}| = q + 1$

Proof. As η has order $q+1$, $2\eta^{2j}$ has order $(q+1)/2$ and η^{3j} has order $(q+1)$ or $(q+1)/3$ (depending upon whether of not $(q+1)$ is congruent to three). As the least common multiple of $(q+1)/2$ and either $(q+1)$ or $(q+1)/3$ is $(q+1)$, $\mathcal{B}$ has size $q+1$. $\square$

For $\mathcal{B}$ to be a BLT-sets, all points of $\mathcal{B}$ need to be points of $Q(4, q)$.

Lemma 4.1.5. *For all $b \in \mathcal{B}$, $b \in Q(4, q)$.*

Proof.

$$Q(2\eta^{2j}, \eta^{3j}, \sqrt{5}) = (2\eta^{2j})^{q+1} + (\eta^{3j})^{q+1} - 5 = 2^{q+1} + 1 - 5 = 4 + 1 - 5 = 0$$

$\square$

For $\mathcal{B}$ to be a BLT-set, all triples of points need to satisfy the discriminant condition. First, we will prove a lemma to simplify this computation.

Lemma 4.1.6. *Let η be an element of $GF(q^2)$ of order $q+1$ and α be an element of $GF(q^2)$ with $\alpha^2 = \eta$. If l is odd, $4T(\eta^{2l}) + T(\eta^{3l}) - 10 = T(\alpha^l)^2 \cdot (T(\alpha^l)^2 + 5)^2$ which is a square. If l is even (and non divisible by $q+1$), $4T(\eta^{2l}) + T(\eta^{3l}) - 10 = (T(\alpha^l)^2 - 4) \cdot (T(\alpha^l)^2 + 1)^2$ which is a non-square in $GF(q)$.*

Proof. We will repeatedly use the following facts:

$$T(x^2) = T(x)^2 - 2N(x) \text{ for all } x \in GF(q^2)$$

$$T(x^3) = T(x)^3 - 3N(x)T(x) \text{ for all } x \in GF(q^2)$$

$$N(\alpha) = -1$$

Suppose l is odd. Then

$$\begin{aligned} 4T(\eta^{2l}) + T(\eta^{3l}) - 10 = &\quad 4(T(\eta^l)^2 - 2) + T(\eta^l)^3 - 3T(\eta^l) - 10 \\ = &\quad T(\eta^l)^3 + 4T(\eta^l)^2 - 3T(\eta^l) - 18 \\ = &\quad (T(\alpha^l)^2 + 2)^3 + 4(T(\alpha^l)^2 + 2)^2 - 3(T(\alpha^l)^2 + 2) - 18 \\ = &\quad T(\alpha^l)^6 + 10T(\alpha^l)^4 + 25T(\alpha^l)^2 \\ = &\quad T(\alpha^l)^2 \cdot (T(\alpha^l)^2 + 5)^2 \end{aligned}$$

Suppose l is even and not divisible by $q + 1$. Then

$$\begin{aligned} 4T(\eta^{2l}) + T(\eta^{3l}) - 10 = &\quad 4(T(\eta^l)^2 - 2) + T(\eta^l)^3 - 3T(\eta^l) - 10 \\ = &\quad T(\eta^l)^3 + 4T(\eta^l)^2 - 3T(\eta^l) - 18 \\ = &\quad (T(\alpha^l)^2 - 2)^3 + 4(T(\alpha^l)^2 - 2)^2 - 3(T(\alpha^l)^2 - 2) - 18 \\ = &\quad T(\alpha^l)^6 - 2T(\alpha^l)^4 - 7T(\alpha^l)^2 - 4 \\ = &\quad (T(\alpha^l)^2 - 4) \cdot (T(\alpha^l)^2 + 1)^2 \end{aligned}$$

The minimal polynomial of α^l over $GF(q)$ is $x^2 - T(\alpha^l)x + N(\alpha^l) = x^2 - T(\alpha^l)x + 1$. Since α^l is not in $GF(q)$, it follows that the discriminant of this polynomial, $T(\alpha^l)^2 - 4$, is a nonsquare in $GF(q)$. $\square$

Lemma 4.1.7. *The points of $\mathcal{B}$ satisfy the triple/discriminant condition, Theorem 4.1.1, for a partial BLT-set.*

Proof. We need to prove that

$$\begin{aligned} f(x,y) \cdot f(x,z) \cdot f(y,z) = &\quad f((2\eta^{2i}, \eta^{3i}, \sqrt{5}), (2\eta^{2j}, \eta^{3j}, \sqrt{5}) \\ &\cdot f((2\eta^{2i}, \eta^{3i}, \sqrt{5}), (2\eta^{2k}, \eta^{3k}, \sqrt{5}) \\ &\cdot f((2\eta^{2j}, \eta^{3j}, \sqrt{5}), (2\eta^{2k}, \eta^{3k}, \sqrt{5}) \end{aligned}$$

is a non-square. Using

$$\begin{aligned} f((2\eta^{2i},\eta^{3i},\sqrt{5}),(2\eta^{2j},\eta^{3j},\sqrt{5}) &= T(4\eta^{2(i+qj)})+T(\eta^{3(i+qj)})-10 \\ &= 4T(\eta^{2(i-j)})+T(\eta^{3(i-j)})-10, \end{aligned}$$

we get that

$$\begin{aligned} f(x,y)\cdot f(x,z)\cdot f(y,z) = \quad & 4T(\eta^{2(i-j)})+T(\eta^{3(i-j)})-10 \\ & \cdot 4T(\eta^{2(i-k)})+T(\eta^{3(i-k)})-10 \\ & \cdot 4T(\eta^{2(j-k)})+T(\eta^{3(j-k)})-10. \end{aligned}$$

The set $\{i-j, i-k, j-k\}$ is either a set of two odd numbers and one even number, or a set of three even numbers. Thus, the proof splits into two final cases.

If the set has two odds and one even, then, by using Lemma 4.1.6,

$$f(x,y)\cdot f(x,z)\cdot f(y,z) = \underbrace{(T(\alpha^l)^2)-4)}_{\boxtimes}\cdot\square = \boxtimes.$$

If the set consists of three evens, then, by using the product of three non-squares is a non-square and Lemma 4.1.6,

$$f(x,y)\cdot f(x,z)\cdot f(y,z) = \underbrace{(T(\alpha^l)^2)-4)}_{\boxtimes}\cdot\underbrace{(T(\alpha^m)^2)-4)}_{\boxtimes}\cdot\underbrace{(T(\alpha^n)^2)-4)}_{\boxtimes}\cdot\square = \boxtimes.$$

Thus, we have proved that $f(x,y)\cdot f(x,z)\cdot f(y,z) = \boxtimes$. $\square$

Having proven that $\mathcal{B}$ exists for $q \equiv \pm 1 \pmod{10}$, is of the right size, consists of points of $Q(4,q)$, and satisfies the triple/discriminant condition, we have proven Theorem 4.1.2.

4.2 Stabilizer in the Group of the Model

In this section we will look at the group M of the given model,

$$V = \{(x, y, a) : x, y \in \mathrm{GF}(q^2),\ a \in \mathrm{GF}(q)\}.$$

The stabilizer in M of a BLT-set is not guaranteed to be the full group of the BLT-set. The full group could possibly, and will in general, be larger than the stabilizer in M.

The group of the model is

$$M = ((C_{q+1} \rtimes C_2) \wr C_2) \curlywedge_{C_2} C_{2h}.$$

The cyclic group of order $(q+1)$ maps $(x, y, a) \mapsto (\eta x, y, a)$ for an $\eta \in \mathrm{GF}(q^2)$ with order $q+1$. The cyclic group of order 2 paired with C_{q+1} maps (x, y, a) to (x^q, y, a). The wreath product with a cyclic group of order two swaps the first two coordinates, allowing those actions on the first coordinate to also be done on the second. The cyclic group of order $2h$ maps (x, y, a) to (x^p, y^p, a^p). Finally, the subdirect product glues two maps $(x, y, a) \mapsto (x^q, y^q, a^q)$ together. Combining these groups, we arrive at the group of the model which has order $8h(q+1)^2$.

As stated in the introduction to this chapter, there are infinite families, other than the Mondello family, that have nice descriptions within this model. Those families which fall into this category are: the Classical BLT-sets, the Fisher BLT-sets, and The FTW BLT-sets. For ease of computing the group within the model of these BLT-sets, the possible elements of the stabilizer in M are the following (as well as powers and products of these elements):

$$\begin{aligned} g_1: &\quad (x,y,a) \mapsto (y,x,a) \\ g_{2x}: &\quad (x,y,a) \mapsto (x^q,y,a) \\ g_{2y}: &\quad (x,y,z) \mapsto (x,y^q,a) \\ g_3: &\quad (x,y,a) \mapsto (\zeta^\alpha x, \zeta^\beta y, a) \\ g_4: &\quad (x,y,a) \mapsto (\zeta^\alpha x^p, \zeta^\beta y^p, a^p) \end{aligned}$$

for $\zeta \in \mathrm{GF}(q^2)$ with order of $\zeta = q+1$.

The Classical BLT-sets

The Classical BLT-sets are given in (V,Q) by

$$\mathcal{B} = \{(x,0,1) : x \in GF(q^2), N(x) = 1\}.$$

The element g_1 does not stabilize $\mathcal{B}$.

The element g_{2x} does stabilize the set, but it will generate a subgroup of the cyclic group generated by g_4. The element g_{2y} trivially stabilizes the set.

For the Classical BLT-sets, an element of type g_3 acts on $\mathcal{B}$ as $g_3 : (x,0,1) \mapsto (\zeta^\alpha x, 0, 1)$. As

$$N(\zeta^\alpha x) = N(\zeta^\alpha) \cdot N(x) = N(\zeta)^\alpha \cdot N(x) = 1^\alpha \cdot (1) = 1,$$

any ζ and all α stabilize the BLT-set. Let $g_{3'} : (x,0,1) \mapsto (\zeta x, 0, 1)$, then $g_{3'}$ is a generator of a cyclic group of order $q+1$.

The possible group element type g_4 acts on $\mathcal{B}$ as $g_4 : (x,0,1) \mapsto (\zeta^\alpha x^p, 0, 1)$. This element is a product of an element of type g_3 and the to map $(x,y,a) \mapsto (x^p, y^p, a^p)$. Thus, without loss of generality, we can assume $\alpha = 0$. As $N(x^p) = 1$ and $x \in \mathrm{GF}(q^2)$ this group element has order $2h$ ($x^{p^{2h}} = x^{p^{h^2}} = x^{q^2} = x$). Therefore, g_4 with $\alpha = 0$ will generate a cyclic group of order $2h$.

In conclusion, the stabilizer of a Classical BLT-set in M has order $2h(q+1)$ and has two cyclic subgroups; one with order $2h$ and the other with order $q+1$.

The Fisher BLT-sets

For a fixed $b \neq 1$ with $N(b) = 1$, the Fisher BLT-sets are given in the model by

$$\mathcal{B} = \left\{(b^{2i}, 0, 1) : 0 \leq i \leq \frac{q+1}{2}\right\} \cup \left\{(0, b^{2i}, 1) : 0 \leq i \leq \frac{q+1}{2}\right\}.$$

The element g_1 preserves the BLT-set and contributes an involution to the stabilizer of $\mathcal{B}$ in M.

The elements g_{2x} and g_{2y} will be subgroups of the cyclic group generated by an element of type g_4.

The element g_3 splits into two possible elements types: $g_{3_1} : (b^{2i}, 0, 1) \mapsto (\zeta^{\alpha} b^{2i}, 0, 1)$ and $g_{3_2} : (0, b^{2i}, 1) \mapsto (0, \zeta^{\beta} b^{2i}, 1)$. By symmetry, the proofs that they both lead to cyclic groups of order $(q+1)/2$ are identical. Therefore, we will focus on g_{3_1}. b is fixed and both ζ and b are of norm 1, so without loss of generality, $\zeta = b$. We can now rewrite an element of type g_{3_1} as $g_{3'_1} : (b^{2i}, 0, 1) \mapsto (b^{2i+\alpha}, 0, 1)$. As long as α is even, we generate stabilize $\mathcal{B}$. All possible α can be generated by $\alpha = 2$. So, both $g_{3'_1}$ and $g_{3'_2}$, with $\alpha = 2,\ \beta = 0$ and $\alpha = 0,\ \beta = 2$ respectively, generate cyclic groups of order $(q+1)/2$.

The elements of type g_4 will lead to a cyclic group of order $2h$. Elements of type g_4 act on $\mathcal{B}$ as $(b^{2i}, 0, 1) \mapsto (\zeta^{\alpha}(b^{2i})^p, 0, 1)$ and $(0, b^{2i}, 1) \mapsto (0, \zeta^{\alpha}(b^{2i})^p, 1)$. Once again, without loss of generality, we can let $\alpha = 0$ and $\beta = 0$. Let $g_{4'}$ map (x, y, a) to (x^p, y^p, a^p). Then $g_{4'}$ in its action on the BLT-set, generates a cyclic subgroup of order $2h$.

In conclusion, the stabilizer in M of a Fisher BLT-set is of order $h(q+1)^2$ with four cyclic subgroups; one has order 2, one has order $2h$, and the other two have order $(q+1)/2$.

The FTW BLT-sets

The FTW BLT-sets are given in the model by

$$\mathcal{B} = \{(ax, bx^2, 1) : N(x) = 1\}$$

where $N(a) = 4/3$, $N(b) = -1/3$, and $q \equiv 2 \pmod 3$.

Switching of the first two coordinates is not an element of the stabilizer in M.

The elements g_{2x} and g_{2y} do not stabilize the set.

Without loss of generality let $\zeta = x$. Then g_3 acts on $\mathcal{B}$ as $(ax, bx^2, 1) \mapsto (ax^{1+\alpha}, bx^{2+\beta}, 1)$. To stabilize the set, the exponent of x in the second coordinate needs to equal twice the exponent of x in the first coordinate. So, $2(1+\alpha) = 2+\beta$, and $2\alpha = \beta$. Once again, we can get all possible elements of this type by letting $\alpha = 1$ and taking powers of this element of type g_2. There are $q+1$ such elements. Let $g_{2'} : (ax, bx^2, 1) \mapsto (ax^2, bx^4, 1)$, then $g_{2'}$ generates a cyclic group of order $q+1$.

The map $g_{4'}$ on $\mathcal{B}$ takes $(ax, bx^2, 1)$ to $(\zeta^\alpha a^p x^p, \zeta^\beta b^p x^{2p}, 1)$. Once again, without loss of generality, we can let $\alpha = 0$ and $\beta = 0$. As $4/3$ and $-1/3$ are both necessarily in GF(p), a^p and b^p remain elements of norm $4/3$ and $-1/3$ respectively. Thus, the map of type g_4 with $\alpha, \beta = 0$ preserves the set and generates a cyclic group of order $2h$.

In conclusion, the stabilizer in M of a FTW BLT-sets is of order $2h(q+1)$ with two cyclic subgroups; one has order $2h$, and the other has order $(q+1)$.

The Mondello BLT-sets

The Mondello BLT-sets are given by

$$\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq q\}$$

for fixed $\eta \in$ GF(q^2) with order $q+1$ and $q \equiv \pm 1 (\bmod 10)$.

From here until the end of the paper, we will use $\mathcal{P}$ to denote the Mondello BLT-sets.

The element g_1 does not stabilize $\mathcal{P}$.

Both of the elements g_{2x} and g_{2y} stabilize the set, but will be subgroups of the cyclic group arising from elements of type g_4.

For an element of type g_3, without loss of generality, let $\zeta = \eta$ as $|\eta| = q+1$. Then this element of type g_3 acts on $\mathcal{P}$ as

$$(2\eta^{2j}, \eta^{3j}, \sqrt{5}) \mapsto (\eta^{\alpha} \cdot 2\eta^{2j}, \eta^{\beta} \cdot \eta^{3j}, \sqrt{5}) = (2\eta^{2j+\alpha}, \eta^{3j+\beta}, \sqrt{5}).$$

We can get all solution by taking powers of an element of this type with $\alpha = 2$ and $\beta = 3$. Let $q_{3'} : (x, y, a) \mapsto (\eta^2 x, \eta^3 y, a)$. Then $g_{3'}$ generates a regular cyclic subgroup of order $q+1$, and is an element of the stabilizer in M.

Whether or not $g_4 : (x, y, a) \mapsto (\zeta^{\alpha} x^p, \zeta^{\beta} y^p, a^p)$ stabilizes $\mathcal{P}$ depends upon whether or not $\sqrt{5}$ is in $GF(p)$. Independent of where $\sqrt{5}$ lives, the element coming from g_4 will always generate a cyclic group of order $2h$ for $q = p^h$. Once again, without loss of generality, let $\zeta = \eta$. This element of type g_4 acts on $\mathcal{P}$ as

$$(2\eta^{2j}, \eta^{3j}, \sqrt{5}) \mapsto (\eta^{\alpha} \cdot (2\eta^{2j})^p, \eta^{\beta} \cdot \eta^{3jp}, \sqrt{5}^{\,p}) = (2\eta^{2jp+\alpha}, \eta^{3jp+\beta}, \sqrt{5}^{\,p}).$$

If $\sqrt{5}$ is in $GF(p)$, then $\sqrt{5}^{\,p} = \sqrt{5}$ and

$$g_4(2\eta^{2j}, \eta^{3j}, \sqrt{5}) = (2\eta^{2jp+\alpha}, \eta^{3jp+\beta}, \sqrt{5}).$$

The powers on η need to be respectively, 2 and 3 times the same parameter. Let $\alpha = 0$ and $\beta = 0$. Elements without $\alpha = \beta = 0$ can be gotten from a product of an element of type g_4 with an element of type g_3. The element of type g_4 preserves $\mathcal{P}$ and becomes $(x, y, a) \mapsto (x^p, y^p, a^p)$. For $q = p^h$, the $2h^{\text{th}}$ power of g_4 is the identity. Thus, this element of type g_4 generates a cyclic group of order $2h$.

If $\sqrt{5}$ is not in $GF(p)$, then $\sqrt{5}^{\,p} = -\sqrt{5}$.

$$g_4(2\eta^{2j}, \eta^{3j}, \sqrt{5}) = (2\eta^{2jp+\alpha}, \eta^{3jp+\beta}, -\sqrt{5}) = (2\eta^{2jp+\alpha+(q+1)/2}, \eta^{3jp+\beta+(q+1)/2}, \sqrt{5}).$$

The last equality holds as scalar multiples are equal and $\eta^{(q+1)/2} = -1$. Once again, without loss of generality, let $\alpha = \beta = 0$. Thus, in this case, let $g_{4'} : (x, y, a) \mapsto (\eta^{(q+1)/2}x^p, \eta^{(q+1)/2}y^p, -a^p) = (-x^p, -y^p, -a^p)$. The element $g_{4'}$ restricted to its action on $\mathcal{P}$ has order $2h$.

In conclusion, the stabilizer in M of a Mondello BLT-set has order $2h(q+1)$ with two cyclic groups: one has order $q+1$ and the other has order $2h$. This group fixes the point $(0,0,1)$ and the two lines $\{(x,0,0) : x \in \text{GF}(q^2)\}$ and $\{(0,y,0) : y \in \text{GF}(q^2)\}$.

Letting $A =< g_{3'} >\cong C_{q+1}$ and $B =< g_{4'} >\cong C_{2h}$, then the stabilizer in M of a Mondello BLT-set is generated by A and B. Let G be this stabilizer in M of $\mathcal{B}$. The intersection of A and B is empty as A is a regular cyclic group and B fixes the element $(2, 1, \sqrt{5})$. Also, A is normal in G as $g_{4'}g_{3'}g_{4'}^{-1} : (x, y, a) \mapsto (\zeta^{2p}x, \zeta^{3p}y, a)$. Therefore, the stabilizer in M is the semidirect product of A and B.

$$G = A \rtimes B = \langle g_{3'},\ g_{4'} : g_{3'}^{q+1} = g_{4'}^{2h} = 1,\ g_{4'}g_{3'}g_{4'}^{-1} = g_{3'}^{p}\rangle \cong C_{q+1} \rtimes C_{2h}.$$

Theorem 4.2.1. *The stabilizer in M of a Mondello BLT-set is isomorphic to $C_{q+1} \rtimes C_{2h}$. The group is generated by two permutations ϕ and ψ where ϕ maps (x, y, a) to $(\eta^2 x, \eta^3 y, a)$ for a fixed $\eta \in GF(q^2)$ with $|\eta| = q+1$ and ψ maps (x, y, a) to $(\epsilon x^p, \epsilon y^p, a^p)$ for $\epsilon = 1$ if $\sqrt{5} \in GF(p)$ and $\epsilon = -1$ if $\sqrt{5} \notin GF(p)$.*

4.3 Geometry of the Model

In the previous section, it was shown that the model contains two cyclic group of order $q+1$. What will be shown in this section is a further characterization of the model in its action on a hyperplane. The group M of the $(\text{GF}(q^2), \text{GF}(q^2), \text{GF}(q))$ model is the stabilizer of two perp external lines and a point: $\{\{(x,0,0) : x \in \text{GF}(q^2)\}, \{(0,y,0) : y \in \text{GF}(q^2)\}, (0,0,1)\}$. M clearly is a subgroup of the stabilizer of this set, but as we will show in the next few paragraphs, in fact it is the whole stabilizer.

M fixes the point $(0,0,1)$. Fixing a point fixes the perp of the point, which in this case is the hyperplane $H = \{(x, y, 0) : x, y \in \text{GF}(q^2)\}$. This hyperplane meets $Q(4, q)$ in a hyperbolic quadric. As we are studying BLT-sets, we are working within the stabilizer of $Q(4, q)$ which is PΓO$(5, q)$. As $H \cap Q(4, q)$ is the hyperbolic quadric, it must be fixed. Therefore, the stabilizer of the point, in PΓO$(5, q)$, contains the stabilizer of the hyperbolic quadric which is PΓO$^+(4, q)$. The order of PΓO$^+(4, q)$ is $h(q + 1)^2 q^2 (q - 1)^2$ for $q = p^h$. The reflection in H, $(x, y, z, w, v) \mapsto (-x, -y, -z, -w, v)$, is not contained in P$\Gamma$O$^+(4, q)$ but acts trivially on the hyperplane. Adding on this element of order two, the order of the stabilizer of $(0, 0, 1)$ in PΓO$(5, q)$ becomes $2h(q + 1)^2 q^2 (q - 1)^2$.

Next, we need to add the lines $l = \{(x, 0, 0) : x \in \text{GF}(q^2)\}$ and $m = \{(0, y, 0) : y \in \text{GF}(q^2)\}$. Both lines are in H and are external to the hyperbolic quadric $Q^+(3, q)$. By Witt's theorem, as all external lines are isometric, the isometry group GO$^+(4, q)$ acts transitively on external lines. Therefore, the subgroup fixing $(0, 0, 1)$ and l will have index equal to the number of external lines to $Q^+(3, q)$ in the stabilizer of $(0, 0, 1)$ in PΓO$(5, q)$.

From Lemma 2.1.5, the orbit of external lines to $Q^+(3, q)$ has size $q^2(q-1)^2/2$. Therefore, the subgroup stabilizing $\{(0, 0, 1), l\}$ has order $4h(q + 1)^2$. Lastly, to stabilize m, we add on an element of order two that switches l and m. Thus, the stabilizer of $\{(0, 0, 1), l, m\}$ in PΓO$(5, q)$ has order $8h(q + 1)^2$. The group of the model also has order $8h(q + 1)^2$ and stabilizes the same point and lines, therefore we have the following lemma.

Lemma 4.3.1. *M, the group of the model, is the stabilizer of the set*

$$\left\{\{(x, 0, 0) : x \in \textit{GF}(q^2)\}, \{(0, y, 0) : y \in \textit{GF}(q^2)\}, (0, 0, 1)\right\}.$$

5. REDUCIBILITY

The main theorem of this chapter is that the group of a Mondello BLT-set acts irreducibly on $\mathrm{GF}(q)^5$. The way we will prove this result is to show that the only infinite family of BLT-sets whose group acts irreducibly on $\mathrm{GF}(q)^5$ and transitively on the set, is the FTW family. This, by itself, is a new complete characterization of the FTW family.

The way we get to this characterization of the Mondello BLT-sets is by using the collineation group of $Q(4,q)$: $\mathrm{P\Gamma O}(5,q)$. As this group is a classical group, much is known about its subgroups. Due to Mitchell [34], and Kantor and Liebler [27], we know all maximal subgroups of $\mathrm{P\Gamma O}(5,q)$. Knowing all subgroups is equivalent to knowing all possible groups of a BLT-set. Thus, for this chapter, we will parse this list to see which subgroups could act transitively on a BLT-set and act irreducibly on $\mathrm{GF}(q)^5$.

5.1 $\mathrm{P\Gamma O}(5,q)$

Define $\Omega(5,q)$ to be $\mathrm{O}(5,q)'$, so that $\mathrm{P}\Omega(5,q)$ is simple for $q \geq 2$. Using the exceptional isomorphism $\mathrm{P}\Omega(5,q) \cong \mathrm{PSp}(4,q)$ and Kantor and Liebler's [27] paper as well as H.H. Mitchell's [34] paper, listing the maximal subgroups of $\Gamma\mathrm{Sp}(4,q)$ and $\mathrm{PSp}(4,q)$ respectively, we get the following list of subgroups for $\Gamma\mathrm{O}(5,q)$. Note that $K^{(\infty)}$ denotes the last term of the derived series.

Theorem 5.1.1. *Let $K \leq \Gamma O(5,q)$, q odd, then one of the following holds.*

1. $K^{(\infty)} = \Omega(5,q')$ *with* $GF(q') \subseteq GF(q)$.

2. *K fixes a t.s. line or a t.s. point or pair of non-collinear t.s. points, or a non-*

degenerate point with perp a hyperbolic quadric.

3. *K fixes a non-degenerate point with perp an elliptic quadric.*

4. *K fixes a normal rational curve over $GF(q') \subseteq GF(q)$, the characteristic is greater than 3 and $K^{(\infty)} = SL(2, q')$.*

5. *K fixes an anisotropic (i.e. external) line.*

6. *$K \cap GO(5, q) \leq C_{q-1} \wr S_5$.*

7. *$K^{(\infty)}$ is $2.A_5$ or $2.A_6$.*

8. *$K^{(\infty)}$ is $2.A_7$ in characteristic 7.*

The search in this chapter is for BLT-sets with groups that acts transitively on the set and irreducibly on $\mathrm{GF}(q)^5$. Thus, this list can be pared down for our purposes. By the orbit-stabilizer theorem, for a group to act transitively on a BLT-set, the order of the group must be divisible by $q+1$, the size of the BLT-set. More specifically, it must have an orbit of length $q+1$. Using this divisibility condition and irreducibility of action, the following groups are removed as not being able to admit the BLT-sets we are searching for.

- Case 1: Assume $q = p^h$ then $q' = p^k$ for some m such that $mk = h$. Then for $q+1 = p^h + 1 = p^{mk} + 1$ to divide $\mid \Omega(5, q') \mid = \frac{1}{2} p^{2k}(p^{4k} - 1)(p^{4k} - p^{2k})$ either $q' = q$ or $q' = \sqrt{q}$. All orbits of $\Omega(5, q)$ and $\Omega(5, \sqrt{q})$ have length greater than $q+1$, and therefore, cannot admit a transitive BLT-set.

- Case 2: The group action is reducible on $\mathrm{GF}(q)^5$ as it fixes a line or a point. Therefore, it can be ignored for our search.

- Case 3: The group action is reducible on $\mathrm{GF}(q)^5$ as it fixes a point. Therefore, it can be ignored for our search.

- Case 5: The group action is reducible on GF$(q)^5$ as it fixes a line. Therefore, it can be ignored for our search.

The cases that remain after these reductions are cases $4, 6, 7$, and 8. They will be dealt with in depth in the following sections. Due to cases 6,7, and 8 being restricted by group size, they will not be able to admit an infinite family of BLT-sets. Hence, case 4 is the only case that can possibly admit an infinite family.

5.1.1 Special Linear Group over a Subfield

Lemma 5.1.2. *For $q > 7$ and q odd, there is a unique conjugacy class of subgroups of P$\Omega(5, q)$ which act irreducibly on GF$(q)^5$ whose elements are isomorphic to PSL$(2, q)$.*

Proof. PSL$(2, 5)$ is a subgroup of A_5 and A_6 and PSL$(2, 7)$ is a subgroup of A_7. Thus, we restrict the cases further with $q > 7$. That restriction along with irreducibility forces Case 4 from the above list of subgroups of $\Omega(5, q)$. □

From this lemma, we know that any copy of $PSL(2, q)$ that acts irreducibly on GF$(q)^5$ will produce equivalent BLT-sets. It is known that a $PSL(2, q)$ subgroup of P$\Omega(5, q)$ stabilizes the FTW BLT-sets.

Lemma 5.1.3. *The $PSL(2, q)$ that stabilizes a FTW BLT-set acts irreducibly on GF$(q)^5$.*

Proof. Let H be the $PSL(2, q)$ stabilizing a FTW BLT-set. Let $C = C_{\mathrm{GL}(5,q)}(H)$ and $Z = Z(\mathrm{GL}(5, q)) \cong C_{q-1}$, H has a unique orbit of length $q + 1$ on totally singular points of $O(5, q)$ polar space (the FTW BLT-set B), so its normalizer stabilizes B. Known results say that the stabilizer B in PΓO$(5, q)$ is PΓL$(2, q)$ (for $q > 5$, for $q = 5$, it is S_6). Since $C_{\mathrm{P\Gamma L}}(H) = 1$, then $C = Z = C_{q-1}$ resulting in H (in GL$(5, q)$) being absolutely irreducible. □

We then arrive at the following corollary.

Corollary 5.1.4. *The Fisher-Thas/Walker BLT-sets, but no others, admit*

$$K^{(\infty)} = SL(2, q')$$

Going back to Chapter 3, this also shows that the BLT-sets created from the twisted cubic are equivalent to the FTW BLT-sets.

5.1.2 Imprimitive Case

For a BLT-set to be transitive, $q+1$ needs to divide $|C_{q-1} \wr S_5|$. Also, as q is odd, the greatest common divisor of $q-1$ and $q+1$ is 2. Therefore, the most that the base group, C_{q-1}, can contribute, for large enough q, is a C_2. So, $q+1$ must divide $2^5 \cdot 5!$. This restricts possible field orders to be from the set

$$q \in \{3, 5, 7, 9, 11, 19, 23, 31, 47, 59, 79, 127, 191, 239, 383, 479, 1279\}.$$

We can think of the C_2 as acting by negation on the coordinate entries of a point and the S_5 action permutes the entries of the point. Let the quadratic form be the sum of squares, $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$, with polar form $f(x, y) = Q(x + y) - Q(x) - Q(y)$. The group action of $C_2 \wr H$ (for H a subgroup of S_5) preserves the resulting parabolic quadric.

The details provided here will be for points of the form $(1, a, b, c, d)$, a point with no zero entries. The points $(0, 1, a, b, c)$, $(0, 0, 1, a, b)$, $(0, 0, 0, 1, a)$, and $(0, 0, 0, 0, 1)$, $a, b,$ or c possibly zero, can be dealt with in a similar fashion. These are the only points that need to be tested, as to force irreducibility, we must have at least C_5 acting on the entries.

For the search, we will us the discriminant/triples condition of Bader, O'Keefe, and Penttila for BLT-sets [2]. For $(1, a, b, c, d)$ a general point with no zero entries, let the C_2

act on the first two entries to get two more points:

$$\begin{aligned} x &= (1, a, b, c, d) \\ y &= (-1, a, b, c, d) \\ z &= (1, -a, b, c, d). \end{aligned}$$

Then

$$\begin{aligned} f(x,y) &= 4a^2 + 4b^2 + 4c^2 + 4d^2 \\ f(x,z) &= 4 + 4b^2 + 4c^2 + 4d^2 \\ f(y,z) &= 4b^2 + 4c^2 + 4d^2 \end{aligned}$$

and

$$\begin{aligned} -2f(x,y)f(x,z)f(y,z) &= -2(4a^2 + 4b^2 + 4c^2 + 4d^2)(4 + 4b^2 + 4c^2 + 4d^2) \\ &\quad (4b^2 + 4c^2 + 4d^2) \\ &= -2 * 64(-1)(-a^2)(-1 - a^2) \\ &= 128a^2(1 + a^2). \end{aligned}$$

For this set of three to be contained in a BLT-set, we need $128a^2(1 + a^2)$ to be a non-square. As both 128 and a^2 are squares in the possible list of field orders, we need $1 + a^2$ to be a non-square.

Using the same x as above, but now negating the remaining possible pairs of entries we find that $1 + b^2, 1 + c^2, 1 + d^2, a^2 + b^2, a^2 + c^2, a^2 + d^2, b^2 + c^2, b^2 + d^2$, and $c^2 + d^2$ all need to be non-square. These conditions narrow the search field far enough to make a computer search feasible. What follows is the code that was run in MAGMA [11] to show that no such BLT-sets exists admitting this group.

```
q:=p^h;
d:=(q-1)/2;
F:=GF(q);
V:=VectorSpace(F,5);
Q:=func<x|x[1]^2+x[2]^2+x[3]^2+x[4]^2+x[5]^2>;
f:=func<x,y|Q(x+y)-Q(x)-Q(y)>;
```

The following creates N as the set of elements f such that $1 + f^2$ is a non-square.

```
N:=[];
for f in F do
if not IsSquare(F!(1+f^2)) then N:=Include(N,f);
end if; end for;
```

The following function builds all possible C_2 actions on a fixed point, i.e. it forms the orbit of a point under the C_2 action.

```
c2:=function(X);
Y:=[];
for a,b,c,d,e in {1,-1} do
if a*b*c*d*e eq 1 then
y:=V![a*X[1],b*X[2],c*X[3],d*X[4],e*X[5]];
Y:=Include(Y,y);
end if; end for;
return Y;
end function;
```

The following is a function to test the discriminant/triple condition. It is code modified from the code run by Law and Penttila [32].

```
BLT:=function(B);
if #B ge 3 then b:=B[1];
for i in {2..#B-1} do c:=B[i];
flag:=true;
for j in {3..#B} do d:=B[j];
flag:=(F!(-4*f(b,c)*f(b,d)*f(c,d)))^d eq F!(-1);
if not flag then break;
end if; end for;
if not flag then break;
end if; end for;
return flag;
else return true;
end if; end function;
```

The actual search.

```
for a,b,c in N do
if IsSquare(F!(-(1^2+a^2+b^2+c^2))) then
for d in {SquareRoot(F!(-(1^2+a^2+b^2+c^2))),
              -SquareRoot(F!(-(1^2+a^2+b^2+c^2)))} do
if not IsSquare(F!(a^2+b^2)) then
if not IsSquare(F!(a^2+c^2)) then
if not IsSquare(F!(a^2+d^2)) then
if not IsSquare(F!(b^2+c^2)) then
if not IsSquare(F!(b^2+d^2)) then
if not IsSquare(F!(c^2+d^2)) then
x:=V![1,a,b,c,d];
B:=c2(x);
if BLT(B) then b;
```

```
end if; end if; end if; end if; end if; end if;
end if; end for; end if; end for;
```

As nothing results after running this code in MAGMA [11] for all possible q, we know that no BLT-sets exists under these specific conditions. Points of the form $(0, 1, a, b, c)$, $(0, 0, 1, a, b)$, $(0, 0, 0, 1, a)$, and $(0, 0, 0, 0, 1)$ can be dealt with in a similar fashion with no BLT-sets being returned.

Lemma 5.1.5. *No transitive irreducible BLT-sets exist that admit*

$$K \cap GO(5, q) \leq C_{q-1} \wr S_5$$

.

5.1.3 Alternating Groups

$K^{(\infty)}$ is $2.A_5$

As $q + 1$ needs to divide $|2.A_5| = 120$, then $q \in \{3, 5, 7, 9, 11, 19, 23, 29, 59\}$. These field orders are all covered by Betten's BLT-set list [6]. The only example exists in $Q(4, 59)$ and is due to Law and Penttila [32] with group S_5.

Lemma 5.1.6. *For $q = 59$ an example due to Law-Penttila exists with group S_5 which acts transitive and irreducibly. For $q \neq 59$, no transitive irreducible BLT-sets exist that admit $K^{(\infty)}$ to be $2.A_5$.*

$K^{(\infty)}$ is $2.A_6$

As $q + 1$ needs to divide $|2.A_6| = 720$, then

$$q \in \{3, 5, 7, 9, 11, 17, 19, 23, 29, 47, 59, 71, 79, 89, 179, 239, 359, 719\}.$$

Since we are looking for transitive BLT-sets, $q+1$ must also be the size of an orbit. We will look at elements of GF$(q)^6$ with $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ that are in the perp of $(1,1,1,1,1,1)$. As $Q(1,1,1,1,1,1) \neq 0$ these elements are the elements of $Q(4,q)$. Possible orbit lengths are as listed. These orbit lengths come from elements of the shape (distinct letters represent distinct elements of GF(q)).

Shape	Stabilizer in S_6	Orbit Length
aaaaaa	S_6	1
aaaaab	S_5	6
aaaabb	$S_4 \times S_2$	15
aaabbb	$S_3 \times S_3$	20
aaaabc	S_4	30
aaabbc	$S_3 \times S_2$	60
aaabcd	S_3	90
aabbcc	$S_2 \times S_2 \times S_2$	120
aabbcd	$S_2 \times S_2$	180
aabcde	S_2	360
abcdef	1	720

Therefore, we are left with the field orders $q \in \{5, 19, 29, 59, 89, 179, 359, 719\}$. For $q \leq 67$ Betten's list [6] is complete. The known examples for $q \leq 67$ are in $Q(4,5)$ where the FTW BLT-set has group S_6 and in $Q(4,29)$ where an example by Law and Penttila has group S_6. Therefore, we need to test $q \in \{89, 179, 359, 719\}$.

In MAGMA [11], we first created elements of GF$(q)^6$, such that they were in the space defined above and fit the proper shape for the field/orbit/stabilizer size. Then we applied the results of discriminants/triples, where elements of the possible BLT-sets were elements in the orbit (under A_6 or S_6) of the starting point. No BLT-sets resulted from the search.

The following is the code that was run for the $q = 359$ case.

```
Q:=func<x|x[1]^2+x[2]^2+x[3]^2+x[4]^2+x[5]^2+x[6]^2>;
```

```
f:=func<x,y|Q(x+y)-Q(x)-Q(y)>;


F:=GaloisField(359);

V:=VectorSpace(F,6);


for b,c,d in GF(359) do

if 2+b^2+c^2+d^2+(-2-b-c-d)^2 eq 0 then

x:=V![1,1,b,c,d,-2-b-c-d];

for y in Orbit(AlternatingGroup(6),x) diff {x} do

for z in Orbit(AlternatingGroup(6),x) diff {x,y} do

if not BLT({x,y,z}) then break g; end if;

end for;

x; end for;

end if;

end for;


for c,d in GF(359) do

if 1+c^2+d^2+(-1-c-d)^2 eq 0 then

x:=V![0,0,1,c,d,-1-c-d];

for y in Orbit(AlternatingGroup(6),x) diff {x} do

for z in Orbit(AlternatingGroup(6),x) diff {x,y} do

if not BLT({x,y,z}) then break g; end if;

end for;

x; end for;

end if;

end for;
```

Simple changes can be made to run the other cases, all resulting in no BLT-sets.

Lemma 5.1.7. *For $q = 5$ the FTW BLT-set has group S_6 and for $q = 29$ an example by Law and Penttila also has this group. For $q \neq 29$ no transitive irreducible BLT-sets exist that admit $K^{(\infty)}$ to be $2.A_6$.*

$K^{(\infty)}$ *is* $2.A_7$ *with* $q \equiv 0 \pmod 7$

The condition that $q+1$ needs to divide $|2.A_7| = 5040$ is much more restrictive in this case. The only possible field order admitted is $q = 7$, where no examples exists with this group.

Lemma 5.1.8. *No transitive irreducible BLT-sets exist that admit $K^{(\infty)}$ to be $2.A_7$.*

5.2 Conclusion

By looking at BLT-sets in their natural context, having subgroups of PΓO$(5, q)$ as their stabilizers, we are able to do a case by case search of BLT-sets by their structure. Here the focus was on transitive groups with an irreducible action on GF$(q)^5$. This led to the following theorem and corollary.

Theorem 5.2.1. *The only infinite family of BLT-sets that act transitively and admit an absolutely irreducible group (on GF$(q)^5$) is the Fisher-Thas/Walker BLT family. The only other transitive irreducible BLT-sets are both due to Law-Penttila: $q = 29$ with group S_6 and $q = 59$ with group S_5.*

Proof. Most important to the theorem is the subgroups list of PΓO$(5, q)$ derived from Mitchell's [34], and Kantor and Liebler's [27] papers. Using this, we first narrowed our search by seeing which subgroups could admit a transitive group with irreducible action on GF$(q)^5$. That list had one subgroup admitting an infinite family of BLT-sets, namely the FTW BLT-sets. There can only be one such family, as there is a unique orbit of length $q + 1$. The other possible subgroups were done on a case by case basis. The only other BLT-sets with the given property are those listed in the theorem. □

Because there is only one infinite family admitting an absolutely irreducible group that acts transitively on the BLT-set, we arrive at the following corollary.

Corollary 5.2.2. *The group of the Mondello BLT-sets is reducible in its action on* $GF(q)^5$.

That a Mondello BLT-set has a reducible group is far too weak to completely determine the group. In the following chapter, Chapter 6, we will look at the group's action on a distinguished hyperplane. Is this hyperplane fixed? If it is fixed, is the groups action on the hyperplane reducible or irreducible? These questions will be answered there. The answers will lead to the computation of the group of a Mondello BLT-set.

6. ACTION ON THE DISTINGUISHED HYPERPLANE

From the results in the previous chapter, we know that the full group of a Mondello BLT-seta acts reducibly on the underlying vector space. We also know from the results in Chapter 4, that the stabilizer in M only fixes the following subspaces: a point $(0,0,1)$ and two lines $\{(x,0,0) : x \in \mathrm{GF}(q^2)\}$ and $\{(0,y,0) : y \in \mathrm{GF}(q^2)\}$. This forces restrictions upon the action by the full group. As it acts reducibly on $\mathrm{GF}(q)^5$, at least one of the subspaces that is fixed by the stabilizer in M must remain fixed.

The first section will deal with the case where only one of the lines is fixed. If this is the case, then $q = 11$. The second section is the case where the point is fixed. This will force the full group to fix an unordered pair of perp external lines. If these two lines are not the lines fixed by the stabilizer in M, then $q = 9$. Otherwise, we have forced the full group to be be a subgroup of the group of the model.

There originally seems to be more cases, but they all reduce to the two listed. If both lines are fixed, their span and its perp are fixed, but the perp of their span is the point, so all three are fixed. If the point and one of the lines were fixed, then the intersection of their perps (the other line) is fixed. So, if more than one line or the point is fixed, all three are fixed, and we are within the group of the model.

6.1 Hyperplane not fixed by Group

In this section, the point $(0,0,1)$ is not fixed, only one of the lines, either $\{(x,0,0) : x \in \mathrm{GF}(q^2)\}$ or $\{(0,y,0) : y \in \mathrm{GF}(q^2)\}$, is fixed. We will call the fixed line l and the non-fixed line m. Let π be the plane which is the span of the point and the non-fixed line, $\pi = \langle p, m\rangle$.

By looking at the action of the group on this plane, we will be able to prove that this case does not occur unless $q = 11$.

Let X be the projection of the Mondello BLT-set from l to π. If $l = \{(x, 0, 0) : x \in \mathrm{GF}(q^2)\}$ then $X = \{(0, \eta^{3i}, \sqrt{5})\}$ and has size $q + 1$ if $(q + 1) \not\equiv 0 \pmod 3$ and size $(q + 1)/3$ if $(q + 1) \equiv 0 \pmod 3$. If $l = \{(0, y, 0) : y \in \mathrm{GF}(q^2)\}$ then $X = \{2\eta^{2i}, 0, \sqrt{5})\}$ and has size $(q + 1)/2$.

For $q \neq 11$ the size of X will be greater than 4. A set of 5 points, no 3 collinear, lie on a unique conic. Therefore, if no three points of X are collinear, then X is contained in a unique conic.

Lemma 6.1.1. *No three points of X are collinear in π.*

Proof. We will split this proof into two cases. The first case is when $\{(x, 0, 0) : x \in \mathrm{GF}(q^2)\}$ is fixed and $X = \{(0, \eta^{3i}, \sqrt{5})\}$. The second case will be when $l = \{(0, y, 0) : y \in \mathrm{GF}(q^2)\}$ then $X = \{2\eta^{2i}, 0, \sqrt{5})\}$.

If three points of $X = \{(0, \eta^{3i}, \sqrt{5})\}$ were collinear, there would exists a GF(q) linear combination of two points of X to get a third point of X. Equivalently, this can be stated as, does $a(0, \eta^{3i}, \sqrt{5}) + b(0, \eta^{3j}, \sqrt{5}) = (0, \eta^{3k}, \sqrt{5})$ have solutions for a and b in GF(q) and $3i$ not equivalent to $3j$ not equivalent to $3k$ modulo $(q + 1)$? By looking at the third component, $b = 1 - a$. Then, using the first component we can solve for a. The solution,

$$a = \frac{\eta^{3k} - \eta^{3j}}{\eta^{3i} - \eta^{3j}}$$

does not lie in GF(q), therefore no three points are collinear.

Similarly, if three points of $X = \{2\eta^{2i}, 0, \sqrt{5})\}$ were collinear, then

$$a(2\eta^{2i}, 0, \sqrt{5}) + b(2\eta^{2j}, 0, \sqrt{5}) = (2\eta^{2k}, 0, \sqrt{5})$$

has a solution for $a, b \in \mathrm{GF}(q)$ where $2i \not\equiv 2j \not\equiv 2k \pmod{(q + 1)}$. Once again, $b = 1 - a$.

In a similar fashion to the preceding case,

$$a = \frac{\eta^{2k} - \eta^{2j}}{\eta^{2i} - \eta^{2j}}$$

is not in GF(q), so no three points of X are collinear. $\square$

By the immediately preceding two statements, if $q \neq 11$, X lies on a unique conic, C_X. Therefore, both the projection of the stabilizer in M and the full group must stabilize this conic. They must also stabilize the conic C that is the intersection of $Q(4, q)$ and π. As these groups stabilize these two conics, they will also stabilize the pencil of conics with basis $\{C_X, C\}$.

The two conics are disjoint, as no point of X lies on C and the group acting on C is transitive. We also know that the projection of the stabilizer in M fixes the line m and the point $(0, 0, 1)$. By the classification of pencils of quadrics (listed in Hirschfeld's text [19]), the pencil of conics arising from C and C_X consists of $(q-1)$ conics, a point (corresponding to a line pair in GF$(q)^2$), and a repeated line. The point and line from the pencil must be the point and line that are stabilized by the projection of the stabilizer in M. As stated, the projection of the full group must also stabilize this pencil. Therefore, the projection of the full group stabilizes the point $(0, 0, 1)$ and the line m, contradicting the assumption that only l was fixed.

For $q \neq 11$, the point $(0, 0, 1)$ is now known to be fixed.

Lemma 6.1.2. *The group of a Mondello BLT-set, for $q \neq 11$, fixes the point $(0, 0, 1)$ and the points perp, a distinguished hyperplane.*

We will explore what that means for the group in the next section.

For $q = 11$ it can be shown that the full group only fixes the line $\{(x, 0, 0) : x \in$ GF$(q^2)\}$. Both the line $\{(0, y, 0) : y \in$ GF$(q^2)\}$ and the point $(0, 0, 1)$ have orbits of length 3 under the full group of this BLT-set. This BLT-set was first given by De Clerck and Herssens [47].

Theorem 6.1.3. *The stabilizer in* $P\Gamma O(5,q)$ *of the Mondello BLT-set for* $q = 11$ *has order 144. The group is generated by the permutations* $(0,1,2,3,4,5,6,7,8,9,10,11)$, $(1,11)(2,10)(3,9)(4,8)(5,7)$, *and* $(2,11)(3,6)(7,10)$. *The labeling corresponds directly to the indexing in* $\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq 11\}$.

6.2 Hyperplane fixed by Group

From Lemma 6.1.2, we know the full group of a Mondello BLT-set ($q \neq 11$) fixes the point $P = (0,0,1)$ and therefore stabilizes its perp, the hyperplane $P^{\perp}$. As we are working inside of PΓO$(5,q)$, the stabilizer of $Q(4,q)$, the intersection of $P^{\perp}$ and $Q(4,q)$, a hyperbolic quadric H, must also be stabilized. The reguli of this hyperbolic quadric are given by $\mathcal{R}_1 = \{(x, \beta x, 0) : x \in \text{GF}(q^2)^*\}$ and $\mathcal{R}_2 = \{(x, -\beta^q x^q, 0) : x \in \text{GF}(q^2)^*\}$ where $N(\beta) = -1$.

Over the course of this argument, we will need to know a property of the projection of a Mondello BLT-set onto $P^{\perp}$. What follows are two different approaches to proving the result we need: the projection is a set of points external to $P^{\perp}$.

The line connecting P with a point of a Mondello BLT-set will be of the form $(2a\eta^2 i, a\eta^3 i, \sqrt{5} + b)$ for a fixed $i \in \{0, \ldots, q\}$ and $a, b \in \text{GF}(q)$. This lines intersection with $P^{\perp}$ (the projection) will then be $(2a\eta^2 i, a\eta^3 i, 0)$. These points do not satisfy $x^{q+1} + y^{q+1} = 0$, and thus, are not on the hyperbolic quadric.

The next proof has a more geometric flavor and begins with a lemma about tangent lines to $Q(4,q)$ through P.

Lemma 6.2.1. *A line* l *on* $P = (0,0,1)$, *is tangent to the parabolic quadric* $Q(4,q)$, *(given by* $Q(x,y,a) = x^{q+1} + y^{q+1} - a^2$*) if and only if the intersection of* l *and* $P^{\perp}$ *is a point of the intersection of* $P^{\perp}$ *and* $Q(4,q)$, *a hyperbolic quadric* H.

Proof. Assume $l \cap H = R$ is a point of H. Then R is perp to R, and R is perp to P, so R is perp to the span of R and P which is l. Therefore, l is singular. But, l is not contained in

$Q(4, q)$ as it contains P which is not contained in $Q(4, q)$. Thus, l is tangent to $Q(4, q)$.

Assume l is on P and tangent to $Q(4, q)$ at a point R. As l is tangent at R, l is contained in $R^{\perp}$. So P is perp to R as it is contained in l. But, l is not contained in $P^{\perp}$. If it were then l would be contained in both P and R perp, so it would be contained in the perp of their span. l is the perp of their span so l would be contained in l perp, but l cannot be in $Q(4, q)$ as P is contained in l and P is not singular. So, l is not contained in $P^{\perp}$. Thus, the intersection of $P^{\perp}$ and l is the point R, so R is on the hyperbolic quadric. $\square$

Using this lemma, consider a point B of a Mondello BLT-set and the line l connecting P and B. Then B is not on the hyperbolic quadric, but B is on $Q(4, q)$, so l is not tangent to $Q(4, q)$. But, l contains B a point of $Q(4, q)$, so l must be secant to $Q(4, q)$. P is non-degenerate, so $P^{\perp}$ intersected with l is also non-degenerate. Therefore, the projection of B, from P onto $P^{\perp}$, is not on H.

6.2.1 Group is Non-Solvable

We begin by looking at the group induced on the hyperplane and ask whether that group is solvable or non-solvable. In this section we will assume the full group G of a Mondello BLT-set is non-solvable. Let G_1 be G intersected with $\mathrm{PGL}(5, q)$. G_1 remains non-solvable. After this intersection we have lost a possible cyclic group of order h off the top of the group. Let G_2 be G_1 induced on the hyperbolic quadric H. G_2, will also remain non-solvable. Here we have lost a possible 2 on the bottom of the group, the reflection in the hyperplane. Let G_3 be the intersection of G_2 with the stabilizer of the reguli of H. Once again losing a possible 2 on the bottom interchanging the reguli but remaining non-solvable. Now we have a group G_3 which must be a non-solvable subgroup of the stabilizer of the reguli, $\mathrm{PGL}(2, q) \times \mathrm{PGL}(2, q)$. The reguli of this hyperbolic quadric are given by $\mathcal{R}_1 = \{(x, \beta x, 0) : x \in \mathrm{GF}(q^2)^*\}$ and $\mathcal{R}_2 = \{(x, -\beta^q x^q, 0) : x \in \mathrm{GF}(q^2)^*\}$ where $N(\beta) = -1$.

The maps $(x, y, a) \mapsto (\eta^2 x, \eta^3, y, a)$ and $(x, y, a) \mapsto (x^q, y^q, a)$, which come from the stabilizer in M, preserve the reguli of the hyperbolic quadric. The first mapping has order $(q+1)/2$ as $(x, \beta x, 0) \mapsto (\eta^2 x, \eta\beta(\eta^2 x), 0)$, $(x, -\beta^q x^q, 0) \mapsto (\eta^2 x, -\beta^q \eta^{3-2q}(\eta^2 x)^q, 0)$ and η^2 has order $(q+1)/2$. The second mapping has order 2 as expected. Therefore, the intersection of the stabilizer in M with PGL$(2,q) \times$PGL$(2,q)$ forces each of the projections to contain a dihedral group of order $(q+1)$.

For the group to be non-solvable, at least one of the projections needs to be non-solvable. Using the subgroups of PGL$(2,q)$, which can be derived from the subgroups of PSL$(2,q)$ in Section 2.5.1, at least one of the projections is either A_5, PSL$(2,q)$, or all of PGL$(2,q)$. A_5 could occur for $q = 9$, but cannot occur for any other finite field, as it does not have dihedral subgroups of order $(q+1)$ for any other q permissible by a Mondello BLT-set.

Thus, at least one of the projections must contain PSL$(2,q)$. Using results from the matrix model of PGL$(2,q) \times$ PGL$(2,q)$ introduced in Section 2.6, none of these groups have orbits short enough on external points. Thus, the group cannot be non-solvable.

Lemma 6.2.2. *The group of a Mondello BLT-set is solvable.*

6.2.2 Group is Solvable and Fixes a Pair of Perp External Lines

We now know that the full group of a Mondello BLT-set is solvable. Once again, let G_1 be G intersected with PGL$(5,q)$. G_1 will remain solvable. Let G_2 be the group induced on the hyperbolic quadrics H. G_2 will remain solvable. Finally, let G_3 be the intersection of G_2 and P$\Omega^+(4,q) \cong$ PSL$(2,q) \times$ PSL$(2,q)$. G_3 will remain solvable, and the projection onto each factor must also be solvable. We can use the list of subgroups of PSL$(2,q)$, listed in Section 2.5, to list all admissible subgroups.

The stabilizer in M once again enforces a divisibility condition. As in the non-solvable case, the projections must contain a dihedral group of order $(q+1)$. The dihedral group of order $(q+1)$ is maximal in PSL$(2,q)$. Therefore, the projections are both D_{q+1}, and the group G_3 must be $D_{q+1} \times D_{q+1}$.

The group $D_{q+1} \times D_{q+1}$ is the stabilizer in PGL$(2, q) \times$ PGL$(2, q)$ of an unordered pair of perp lines that are external to the hyperbolic quadric. From Lemma 2.1.5, we know that there are $q^2(q-1)^2/2$ external lines to the hyperbolic quadric. Also, by Witt's theorem, all external lines are isometric and are therefore acted on transitively by the stabilizer of the hyperbolic quadric. Thus, the stabilizer in PGO$(4, q)$ of a line l has order $4(q+1)^2$, by orbit stabilizer. The stabilizer of $\{l, l^\perp\}$ is twice as big as the stabilizer of l (there exists an element switching l with $l^\perp$). The stabilizer of l has index 2 in $D_{2(q+1)} \wr C_2$, so $D_{2(q+1)} \wr C_2$ is the stabilizer of the pair of unordered perp external lines $\{l, l^\perp\}$. Restricting to PSL$(2, q) \times$ PSL$(2, q)$, we know that $G_3 = D_{q+1} \times D_{q+1}$ stabilizes an unordered pair of perp external lines, and hence the full group must also stabilize this pair of lines.

Lemma 6.2.3. *The group of a Mondello BLT-set fixes an unordered pair of perp external lines to the hyperbolic quadric.*

Now we know that the group is solvable and fixes a pair of perp external lines. If this pair of lines is the pair fixed by the model, $\{l = \{(x, 0, 0) : x \in \text{GF}(q^2)\}, m = \{(0, y, 0) : y \in \text{GF}(q^2)\}\}$, then we are inside the group of the model, and our computations are completed. If not, we know from the model that l and m are not interchanged by the full group of Mondello as that switch is visible but does not stabilize the set. Thus, the question becomes: can there exist two other lines that are interchanged by the full group of Mondello? Equivalently, does the full group of Mondello act irreducibly on the hyperplane $P^\perp$?

Assuming that the full group of Mondello acts irreducibly on $P^\perp$, then there exists a subgroup of index two that swaps the two lines. Intersecting with the stabilizer in M, we still have a subgroup of index two. This subgroup will map $(x, y, a) \mapsto (\eta^4 x, \eta^6 y, a) \mapsto (\eta^8 x, \eta^{12} y, a)$. As we know that $P^\perp$ is fixed, these lines must lie in $\{(x, y, 0) : x, y \in \text{GF}(q^2)\}$. If this map were to fix a line then $(\eta^8 x, \eta^{12} y, 0)$ is a GF(q)-linear combination of $(x, y, 0)$ and $(\eta^4 x, \eta^6 y, 0)$.

$$a(x, y, 0) + b(\eta^4 x, \eta^6 y, 0) = (\eta^8 x, \eta^{12} y, 0)$$

So, $ax + b\eta^4 x = \eta^8 x$ and $ay + b\eta^6 y = \eta^{12} y$. We know these lines are different from l and m, so $x \neq 0$ and $y \neq 0$, and we can divide by x and y respectively to get: $a + b\eta^4 = \eta^8$ and $a + b\eta^6 = \eta^{12}$. Thus, η^4 and η^6 both satisfy the quadratic equation $X^2 = bX + a$ over GF(q). Therefore, η^4 and η^6 are conjugate over GF(q), so $\eta^{4q} = \eta^6$. Thus, $q + 1$ divides $4q - 6$, so $q + 1$ divides $4(q + 1) - (4q - 6) = 10$. The only possible q, given the constraint on a Mondello BLT-set's field order, is $q = 9$.

If $q = 9$ the Mondello BLT-set is also a Fisher BLT-set. The two halves of classical BLT-sets that lie within are given by letting the parameter in the Mondello BLT-set definition be even or odd.

Theorem 6.2.4. *The stabilizer in P*Γ*O*$(5, q)$ *of the Mondello BLT-set for* $q = 9$ *has order 400. The group is generated by the permutations* $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, $(3, 9)(5, 7)$, $(1, 9)(2, 8)(3, 7)(4, 6)$, *and* $(2, 6, 8, 4)(3, 7, 9, 5)$. *The labeling corresponds directly to the indexing in* $\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq 9\}$.

Lemma 6.2.5. *The group of a Mondello BLT-set, for* $q \notin \{9, 11\}$*, is a subgroup of* M*, the group of the model.*

7. CONCLUSIONS

Piecing together the results stated in the previous chapters, we are able to compute the group of a Mondello BLT-set and arrive at the main theorem of this thesis.

Theorem 7.0.6. *The stabilizer in P*Γ*O*$(5, q)$ *of a Mondello BLT-set* $\mathcal{P}$*, for* $q = p^h > 11$*, is isomorphic to* $C_{q+1} \rtimes C_{2h}$*. The group is generated by the permutations* ϕ *and* ψ *where* ϕ *is the map* $(x, y, a) \mapsto (\eta^2 x, \eta^3 y, a)$ *for a fixed* $\eta \in$ *GF*(q^2) *with* $|\eta| = q + 1$ *and* ψ *is the map* $(x, y, a) \mapsto (\epsilon x^p, \epsilon y^p, a^p)$ *for* $\epsilon = 1$ *if* $\sqrt{5} \in$ *GF*(p) *and* $\epsilon = -1$ *if* $\sqrt{5} \notin$ *GF*(p).

Proof. The computation of a Mondello BLT-sets group began with the knowledge, from the original paper [42], that the group acts transitively on the BLT-set. The group of a BLT-set is also forced to be a subgroup of PΓO$(5, q)$. Combining these two results, we began to look at how the group acts on the underlying vector space. In Chapter 5 it was shown that the group must act irreducibly on GF$(q)^5$ (Corollary 5.2.2). The visible group of a Mondello BLT-set (Section 4.2) fixes three subspaces: a point $P = (0, 0, 1)$ and two lines $l = \{(x, 0, 0) : x \in \text{GF}(q^2)\}$ and $m = \{(0, y, 0) : y \in \text{GF}(q^2)\}$. We then to looked at all possible cases where a subset of the point and lines are fixed. If only one line is fixed, we proved that $q = 11$ (Theorem 6.1.3). From this, we proved that the point and its perp, a hyperplane, must be fixed (Lemma 6.1.2).

Now we are able to restrict the groups action to this hyperplane. The hyperplane intersects the parabolic quadric of the BLT-set in a hyperbolic quadric. As we are working within the stabilizer of the parabolic quadric, this hyperbolic quadric, and its reguli, must also be fixed. We are able to prove that the group must be solvable (Lemma 6.2.2), and its action must fix a pair of perp external lines to the hyperbolic quadric (Lemma 6.2.3). If this pair of lines is not the pair of lines visible in the model, then $q = 9$ (Theorem 6.2.4). The group of the model is the stabilizer of $\{P, l, m\}$ (Lemma 4.3.1). Thus, we have forced the group of a Mondello BLT-set to be a subgroup of the group of the model. In Section 4.2, generators were given for the stabilizer of a Mondello BLT-set in the group of the model. The group generated by these permutations is the full group. $\square$

Using the information contained in Section 2.4.1, we can compute the group orders of the corresponding configurations. For $q = p^h > 11$:

- the group of the Mondello flock of the quadratic cone has order $2h$,
- the group of the Mondello GQ has order $(q^6 - q^5) \cdot (2h(q+1))$,
- the group of the Mondello hyperbolic fibration has order $2(q+1) \cdot 2h(q+1)$,
- the group of the spread arising from the Thas-Walker construction from the Mondello flock has order $2hq$, and
- the group of the translation plane arising from the Thas-Walker construction from the Mondello flock has order $2hq(q^5 - q^4)$.

It is an immediate consequence of these group calculations that the Mondello objects are inequivalent to objects arising from other infinite families for $q > 9$.

We reiterate that the groups of a Mondello BLT-set over GF(9) and GF(11) have extra symmetries. They are given in the following theorems.

Theorem 7.0.7. *The stabilizer in $P\Gamma O(5, q)$ of the Mondello BLT-set for $q = 9$ has order 400. The group is generated by the permutations* $(0,1,2,3,4,5,6,7,8,9)$, $(3,9)(5,7)$, $(1,9)(2,8)(3,7)(4,6)$, *and* $(2,6,8,4)(3,7,9,5)$. *The labeling corresponds directly to the indexing in* $\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq 9\}$.

Theorem 7.0.8. *The stabilizer in $P\Gamma O(5, q)$ of the Mondello BLT-set for $q = 11$ has order 144. The group is generated by the permutations* $(0,1,2,3,4,5,6,7,8,9,10,11)$,

$(1,11)(2,10)(3,9)(4,8)(5,7)$, *and* $(2,11)(3,6)(7,10)$. *The labeling corresponds directly to the indexing in* $\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) : 0 \leq j \leq 11\}$.

There remain many open problems in the area of BLT-sets and their related configurations. Two questions that fall in line with this thesis are:

- Can transitive BLT-sets with full stabilizer not regular be classified?
- Can a geometric argument be given to show that the stabilizer of a Mondello BLT-set ($q > 11$) fixes a line? This would give a geometrically more satisfying proof of the main result of this thesis, and perhaps also a geometric characterization of the Mondello BLT-sets.

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