RANGE AND DEFICIT ANALYSIS USING MARKOV CHAINS

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ABSTRACT

Two properties of the partial sums of random variables are investigated: the range and the maximum accumulated deficit. The relevance of this study follows from the fact that the range (or the adjusted range) is used in the design of storage capacities for full regulation of river discharges and the maximum accumulated deficit is used in the case of partial regulation.

A general approach to the distribution of the range of partial sums of independent random variables is developed. Starting with discrete random variables, the distribution of the range is shown to follow from the theory of Markov chains, when the state space is such that the boundary states are absorbing. By analogy, the distribution of the range of partial sums of continuous, independent random variables is obtained. Some results are given in closed form, and others are obtained numerically.

Similarly, a general approach to the distribution of the maximum accumulated deficit of partial sums of independent random variables is developed. Starting with discrete random variables, the distribution of the maximum accumulated deficit is shown to follow from the theory of Markov chains, when the state space is such that one boundary state is absorbing and the other is reflecting. By analogy, the distribution of the maximum accumulated deficit of partial sums of continuous, independent random variables is obtained. Some results are given in closed form, and others are obtained numerically. In particular, new asymptotic results are derived.

The similarities between range and deficit analysis and Moran's theory of reservoirs are pointed out and the theory exposed is extended to the case of serially correlated random variables.

Practical applications are discussed and a brief note on the so-called Hurst phenomenon is included.

Chapter I
INTRODUCTION

1. Preliminaries

The theory of stochastic processes applied to the design and operation of reservoirs has emerged in recent years as one of the most dynamic topics of statistical hydrology. It has attracted engineers simply because the inherently stochastic nature of hydrological phenomena could not be ignored. It has attracted statisticians not only due to the extremely interesting mathematics involved but also because of the obvious relationships between this problem and other areas of statistical interest such as the theory of provisioning and the queuing theory.

The growing world shortage of water resources, the increased competitions between water used, and the technological advances of society in general dramatized the importance of the study of the theory of reservoirs. However, the problem is extremely complex. In dealing with annual streamflows, for instance, the assumption of independence of events may be acceptable; however, in dealing with daily, weekly or monthly flows, the correlation structure is significant. Furthermore, such stochastic processes are apt to be nonstationary due to seasonality. To all these problems, the variabilty in water demand and the competition between water used have to be added.

A good way to illustrate the complexity of the problem is to approach it from a historical viewpoint. In order to do so, a few definitions are needed.

Let X_i be a sequence of random variables, and

$$S_i = X_1 + X_2 + \dots + X_i;\ i = 1,2,\dots, n$$

$$M_n = \max(0, S_1, S_2,\dots, S_n)$$

$$m_n = \min(0, S_1, S_2,\dots, S_n)$$

$$R_n = M_n - m_n \quad . \tag{1.1}$$

The random variable S_i is called the cumulative or partial sum, M_n the maximum partial sum, m_n the minimum partial sum and R_n the range of partial sums (see Fig. 1.1). In this paper, M_n and m_n are *not* called surplus and deficit, as in some other works on this subject, to avoid confusion with another concept which will be introduced later.

Another set of definitions follows when each component of the partial sum is adjusted for the sample mean $\overline{X}_n$:

$$S_i^* = S_i - \frac{i}{n} S_n$$

$$M_n^* = \max(0, S_1^*, S_2^*,\dots, S_n^*)$$

$$m_n^* = \min(0, S_1^*, S_2^*,\dots, S_n^*)$$

$$R_n^* = M_n^* - m_n^* \quad . \tag{1.2}$$

The random variable S_i^* is called the adjusted partial sum, M_n^* the adjusted maximum partial sum, m_n^* the adjusted minimum partial sum and R_n adjusted range (see Fig. 1.2).

In this paper, the underlying random variable X_i will be referred to as *net input*, or simply *input*.

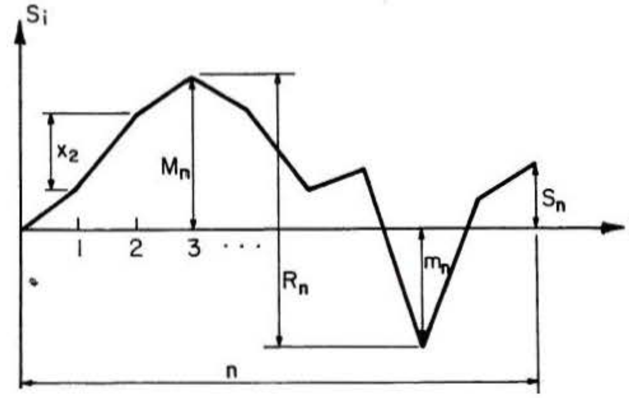

Fig. 1.1. Definition of the maximum partial sum (M_n), the minimum partial sum (m_n), and the range (R_n).

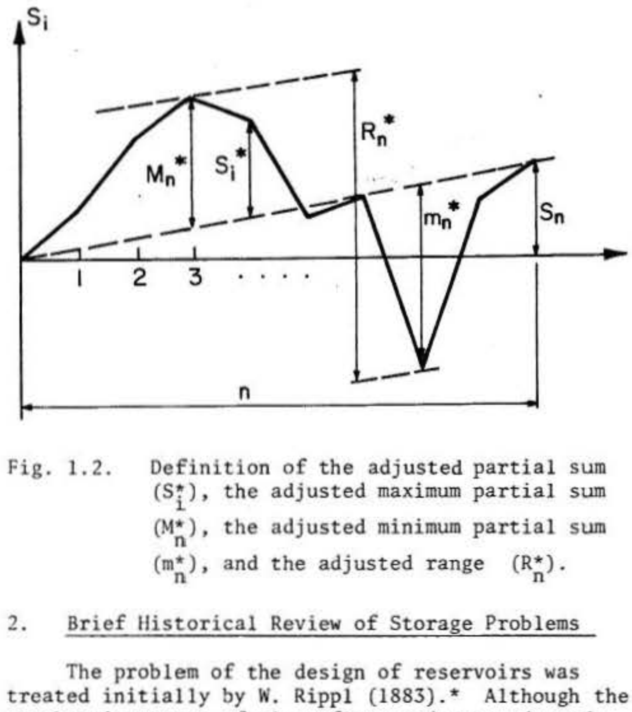

Fig. 1.2. Definition of the adjusted partial sum (S_i^*), the adjusted maximum partial sum (M_n^*), the adjusted minimum partial sum (m_n^*), and the adjusted range (R_n^*).

2. Brief Historical Review of Storage Problems

The problem of the design of reservoirs was treated initially by W. Rippl (1883).* Although the stochastic nature of river flows and water demands was ignored, Rippl's work is important because it introduced the concept of mass-curves as a tool to determine the storage capacity required.

*Name and/or date in parentheses refer to the author's name and date of publication given in the bibliography.

Later, A. Hazen (1914) published the first paper in which the problem was seen within the context of uncertainty. In this paper, data from several river stations were transformed to comparable values by "standardization." The result was the production of enough data to approach the problem from a probabilistic viewpoint. Incidentally, in this paper Hazen presented his invention of the "probability paper," a well-known graphical tool used by statisticians and engineers.

C. E. Sudler (1927) treated the problem of reservoir design by extending records of, say, 50 years, into artificial records of 1000 years. The method consisted of writing the observed annual runoff values on cards, which were shuffled and drawn one by one, without replacement, until all cards were used. Following this procedure twenty times, the artificial record was generated. The technique is obviously poor; for instance, the maximum and minimum values of the historical record are necessarily the maximum and minimum values of the sample of size 1000. However, the importance of Sudler's work derives from the fact that he was possible the first man in statistical hydrology to use simulation methods.

The next influential paper in this field was written by H.E. Hurst (1951). Using a modification of Rippl's idea of mass-curve, and an impressive quantity of long-term annual records, Hurst computed for each record the cumulative sums of the departures of the annual totals from the long-term mean. The storage required to yield the average flow, each year, was taken as the range from the maximum to the minimum of these cumulative totals (see definition of adjusted range in Fig. 1.2). Hurst showed that the storage computed in this manner, from long-term records of natural phenomena, was proportional to $n^{0.729}$, where n is the length of the period of time. In the same paper Hurst found that the mean range when the "variation from the mean is distributed normally" is proportional to $n^{0.5}$. He concluded that although "the frequency characteristics of river discharges (in the investigations of Messrs. Hazen and Sudler) are assumed to be like those of random events: this is only an approximation "in cases in which storage over long periods of time is concerned." The apparent departure from the square-root law found in this paper became later known as the "Hurst phenomenon."

Subsequently W. Feller (1951), whose attention had been called to Hurst's paper, attacked the problem using the theory of Brownian motion, in a sophisticated and much celebrated paper. He found the asymptotic distribution of the range and adjusted range of partial sums of independent random variables, and consequently the asymptotic moments. It was then made clear that because the partial sums of independent random variables X_i with finite variance are asymptotically normally distributed, the asymptotic distributions of the range and adjusted range are independent of the distribution of the random variable X_i. Feller mentioned that the Hurst phenomenon could conceivably be explained starting from the assumption that the variables X_i are not independent. Incidentally, Feller's results were derived under the additional assumption that the mean value of the random variable X_i was zero, which is relevant only in terms of the unadjusted range.

Later, P.A.P. Moran (1954) initiated a new line of research. Instead of studying simply the properties of partial sums in order to develop ideas about the convenient size of the reservoir, Moran studied the influence of the inflow and various operation policies in the distribution of the amount of water stored, given the size of the reservoir.

For a delightful reading of the more recent history of stochastic reservoir theory, the interested reader is referred to E. H. Lloyd (1974a).

It is the belief of this writer that the complexity of the problem has been well demonstrated by the fact that, in the pursuit of a solution, notable engineers like Rippl, Hazen, Sudler and Hurst were able, respectively, to introduce the concept of mass-curve, to invent such a useful device as the "probability paper," to pioneer methods of simulation, and to raise a question still unresolved.

3. Approaches to Storage Problems

Approaches commonly used in the design of storage capacities may be classified into three groups: empirical, experimental, and analytical. The empirical approach consists of the application of Rippl's mass-curve to the observed hydrological sequence. Input and output are both taken as known functions of the time. This approach is clearly inadequate, for the probability of repetition of the same flow sequence is zero. Unfortunately, the method is still quite widely used.

The experimental approach is simply the application of the so-called Monte-Carlo method or data generation method. It is also called the synthetic hydrology method, and it consists of the generation of a large number of flow sequences statistically indistinguishable from the historical record. Rippl's method, or a modification of it, is applied to each flow sequence, and the probability distribution of storage capacities is approached from a relative frequency viewpoint.

The analytical approach consists of the derivation of exact, asymptotic, or approximated distributions, and moments of statistics related to the design of storage capacities, and it is the subject of this paper. Within this approach, two lines of research are usually identified: the line initiated by Moran and expanded considerably in the last two decades, and the line initiated by Hurst and Feller, which consists of random variables. In this paper these lines of research will be referred to as Moran's analysis and range analysis, respectively.

Range analysis is sometimes referred to as the infinite reservoir theory. The reasoning behind this seems to be that, although the object of the study is simply the properties of the partial sums of random variables, one may conceive the existence of a reservoir capable of storing any water surplus and of supplying any deficit of water. An infinite reservoir clearly satisfies such conditions. On the other hand Moran's analysis is sometimes called the finite reservoir theory even though in some cases the top (or the bottom) of the reservoir is abolished in order to assure mathematical tractability and elegance. Perhaps the contributions from Moran's school which abolish the top (or the bottom) of the reservoir should be referred to as the semi-infinite reservoir theory.

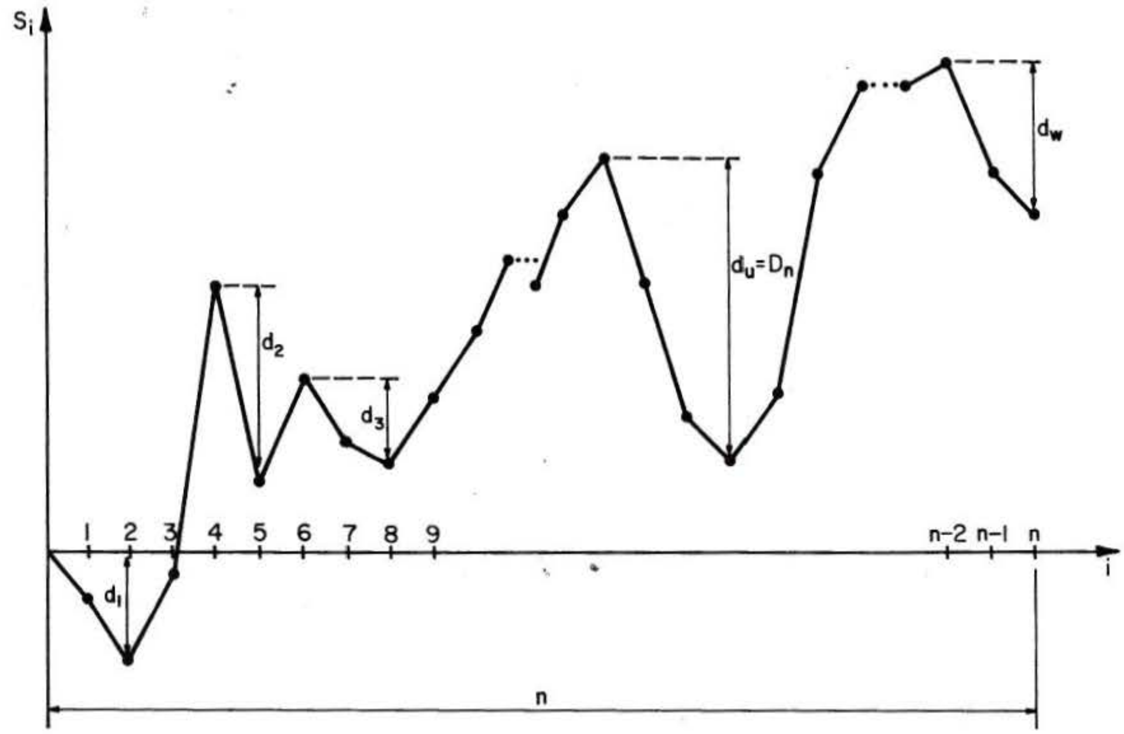

Fig. 1.3. Definition of the maximum accumulated deficit $D_n = \max\limits_{1 \leq i \leq W} \{d_i\}$.

One of the reasons to study the range is that since Hurst's initial work, the behavior of the range as n increases has acquired independent mathematical and scientific interest as an indicator of the structure of stochastic processes (Anis and Lloyd, 1975). The emphasis in range analysis has been on the determination of the expected value of the range for exchangeable (equally correlated) inputs simply because it follows directly from the expected value of the maximum (or minimum) of partial sums of exchangeable inputs, which is easier to study.

In terms of Moran's analysis, a very large number of papers deal with the time-dependent probability function of storage levels, their limiting distributions, probability of water overflow and probability of emptiness of the finite reservoir, for stationary independent inflows. The most significant contribution in this field was the extension of Moran's initial idea to seasonal and serially correlated inflows, which was given by E. H. Lloyd (1963,1964). After this contribution, several papers were published studying what became known as the Lloyd reservoir.

4. Objectives and General Approach in this Investigation

Some engineers interpret the range as the required storage capacity to avoid both overflows and emptiness of the reservoir. This is an interpretation valid only in the case of full regulation of discharges. Full regulation of river discharges is tantamount to assuming that the random variables X_i presented in Eq. (1.1) have zero expectation (i.e., the average net input is zero). When the average net input is positive, the regulation is only partial and overflows are implied in the design procedure. In such a case, the partial sums may appear as shown in Fig. 1.3 (recall that the X_i's in this case are still departures from the desired regulated discharge but their expectation is no longer zero). Clearly there can exist a random number of accumulated deficits, which are the random variables $\{d_i;\ i=1,2,\ldots,w\}$ shown in Fig. 1.3. The required storage capacity is the maximum of these accumulated deficits, say, D_n. The study of the random variable D_n will be called maximum accumulated deficit analysis, or simply deficit analysis, and will be one of the subjects of this paper.

Another objective of this paper will be to study the exact distribution of the range. In so doing, it will be shown that both range analysis and deficit analysis can be approached from a finite-reservoir viewpoint.

It is well known that Moran's analysis is a direct application of the theory of Markov chains, when the boundaries are reflecting. It will be shown that range analysis can be derived from the same theory, when the boundaries are absorbing. Furthermore, it will be shown that deficit analysis follows from the theory of Markov chains with one absorbing and one reflecting boundary. It is interesting to be able to derive all analytical approaches to storage problems by simply changing the character of the boundary in the theory of Markov chains.

As a consequence, the distribution of the range will be shown to be closely related to the probability of emptiness before overflow and to the probability of overflow before emptiness in the finite reservoir. Also, the obvious relationship between the maximum accumulated deficit and the probability of emptiness with or without overflow of a finite, initially full reservoir will be pointed out.

The basic approach in this investigation will be to work with discrete random variables as input (that is why the term Markov chains rather than Markov processes is used). Starting with independent identically distributed random variables, the distribution of the range and of the maximum accumulated deficit will be studied. The possibility of extension to the case of seasonal and serially correlated inputs will be indicated. The case of continuous random variables as input will be studied in some cases in which the integrals involved exist in closed form. When this is not the case, the distribution of the range and the distribution of the maximum accumulated deficit will be obtained numerically, simply by "discretization" of the input (i.e., by choosing an "analogue" discrete distribution to approximate the continuous input).

It is the hope of this writer that the reader will come to the conclusion that, at least in the single reservoir problem, the gap between theory and practical needs is not as wide as generally believed.

Chapter II
REVIEW OF LITERATURE

This chapter summarizes the main results in the study of the range of partial sums (range analysis), following J. D. Salas-La Cruz (1972) and briefly describes some of the contributions to the study of the finite reservoir (Moran's analysis). Furthermore, the lack of theoretical work on the maximum accumulated deficit (deficit analysis) is discussed and a note on the Hurst phenomenon is included.

1. Range Analysis

The asymptotic distribution of the maximum partial sum (M_n) of independent identically distributed random variables with mean zero and unit variance was given by P. Erdos and M. Kac (1946) as:

$$P[|\frac{M_n}{\sqrt{n}}| \leq x] \approx \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}u^2} du \quad . \qquad (2.1)$$

The asymptotic mean adjusted range of partial sums (R_n^*) of independent identically distributed random variables with unit variance was given by H. E. Hurst (1951) as:

$$E\{R_n^*\} \approx \sqrt{\frac{n\pi}{2}} \approx 1.2533n^{\frac{1}{2}} \quad . \qquad (2.2)$$

Hurst used a combinatorial lemma related to the maximum partial sum, in the particular case in which the last partial sum (S_n) equals zero. Multiplying the result by two, he obtained the asymptotic mean adjusted range. It is not obvious at first glance that one can approach the adjusted maximum partial sum by studying the unadjusted maximum partial sum conditioned to $S_n = 0$.

W. Feller (1951) found the asymptotic distribution of R_n as well as the asymptotic distribution of R_n^*, for independent identically distributed random variables with mean zero and unit variance, using the theory of Brownian motion. In particular, he obtained the asymptotic mean and variance in each case:

$$E\{R_n\} \approx \sqrt{\frac{8n}{\pi}} \approx 1.5958n^{\frac{1}{2}} \quad , \qquad (2.3)$$

$$\text{Var}\ \{R_n\} \approx 4n(\ln 2 - 2/\pi) \approx 0.2261n \quad , \qquad (2.4)$$

$$E\{R_n^*\} \approx \sqrt{\frac{n\pi}{2}} \approx 1.2533n^{\frac{1}{2}} \quad , \qquad (2.5)$$

and

$$\text{Var}\ \{R_n^*\} \approx \frac{\pi}{2}(\frac{\pi}{3} - 1)\ n \approx 0.0741n \quad . \qquad (2.6)$$

The exact expected value of the maximum of the partial sums $S_1, S_2, \ldots, S_n$ of independent standard normal variables was given by A. A. Anis and E. H. Lloyd (1953):

$$E[\max(S_1, S_2, \ldots, S_n)] = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n-1} i^{-\frac{1}{2}}. \qquad (2.7)$$

It can be easily shown from Eq. (2.7) that

$$E\{M_n\} = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} i^{-\frac{1}{2}} \quad . \qquad (2.8)$$

which leads to the expected value of the range

$$E\{R_n\} = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} i^{-\frac{1}{2}} \quad . \qquad (2.9)$$

A. A. Anis (1955) published the exact second moment of the maximum of the partial sums $S_1, S_2, \ldots, S_n$ and later (1956) presented a recursive relationship for numerical evaluation of all the moments of the maximum of the partial sums $S_1, S_2, \ldots, S_n$ of independent standard normal variables.

Results similar to the above can be obtained using F. Spitzer's (1956) identity, which is more general. Considering a sequence of independent and identically distributed random variables and $S_j = X_1 + X_2 + \ldots + X_j$, $M_j = \max(0, S_1, S_2, \ldots, S_j)$ and $S_j^+ = \max(0, S_j)$, Spitzer derived the identity

$$\sum_{j=0}^{\infty} \theta_j(t) \cdot z^j = \exp[\sum_{j=1}^{\infty} j^{-1}\ \Omega_j(t)\ z^j] \quad , \qquad (2.10)$$

where $\theta_j(t)$ and $\Omega_j(t)$ are the characteristic functions of M_j and S_j^+, respectively.

From Eq. (2.10), the moments of M_j can be written as a function of the moments of S_j^+, which are easier to compute. In particular, it can be shown that

$$E\{M_n\} = \sum_{i=1}^{n} i^{-1}\ E\{S_i^+\} \quad . \qquad (2.11)$$

and

$$E\{M_n^2\} = \sum_{i=1}^{n} i^{-1}\ E(S_i^{+2}) + \sum_{i=2}^{n}\sum_{j=1}^{i-1} j^{-1}\ (i-j)^{-1}\ E(S_j^+)\ E(S_{i-j}^+) \ . \qquad (2.12)$$

Equation (2.11), applied to the case of independent normal variables with mean zero and variance σ^2 leads to

$$E\{M_n\} = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} i^{-1}\ [\text{Var}\ \{S_i\}]^{\frac{1}{2}} \quad . \qquad (2.13)$$

and thus

$$E\{R_n\} = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} i^{-1}\ [\text{Var}\ \{S_i\}]^{\frac{1}{2}} \ , \qquad (2.14)$$

where $\text{var}(S_i) = i\ \sigma^2$. Notice that for $\sigma = 1$ Eq. (2.14) reduces to Eq. (2.9), as it should.

M. E. Solari and A. A. Anis (1957) derived the first two moments of the maximum adjusted partial sum for independent, standard normal variables:

$$E\{M_n^*\} = \frac{1}{2}\sqrt{\frac{n}{2\pi}}\sum_{i=1}^{n} i^{-\frac{1}{2}}(n-i)^{-\frac{1}{2}} , \qquad (2.15)$$

and

$$E\{M_n^{*2}\} = \frac{1}{6}\left[\frac{n^2-1}{n} + \frac{\sqrt{n}}{2\pi}\sum_{i=2}^{n-1}\sum_{j=1}^{i-1}\frac{i(2i-n)}{\sqrt{j^3(n-i)(i-j)^3}}\right] . \qquad (2.16)$$

P. A. P. Moran (1964), exploring Spitzer's result further, showed that $E(R_n)$ varies as $n^{1/\gamma}$, when considering the range of partial sums of independent random variables having the characteristic function $\exp(-|t|^\gamma)$ (i.e., symmetric stable random variables).

A procedure for obtaining the exact distribution of M_n, m_n and R_n was described by V. Yevjevich (1965), for the values of $n = 2$ and $n = 3$. For higher values of n, Yevjevich used the data generation method to investigate the properties of M_n, m_n, R_n, M_n^*, m_n^*, and R_n^*, for a first order autoregressive process. He also used the data generation method to assess the effects of nonnormality, in the case of independent random variables.

M. J. Melentijevich (1965), using the data generation method, found approximate equations for the expected value and variance of the range when the output is linearly dependent on storage.

V. Yevjevich (1967) suggested that the expected range of linearly dependent normal variables could be expressed by Eq. (2.14), which was derived for independent normal variables. Using the data generation method, he showed that for the case of the first and second order autoregressive models and the simple moving average scheme the results given by Eq. (2.14) closely approximate the exact (and unknown) values.

J. D. Salas-La Cruz (1972) found the exact expected value of M_n for the case of random variables with general covariance structure, for $n = 2$ and $n = 3$. Salas also proposed approximate expressions for the mean and variance of the range of periodic-stochastic series.

D. C. Boes and J. D. Salas-La Cruz (1973) summarized the existing expressions for the expected range and expected adjusted range, showing that they follow from a single expression, namely the expected range of the partial sums of exchangeable random variables. They also obtained a new asymptotic result:

$$E\{R_n^*\} \doteq \left[\frac{\pi n}{2}(1-\rho)\right]^{\frac{1}{2}} \qquad (2.17)$$

for exchangeable normally distributed random variables. Notice that when the coefficient of correlation ρ is equal to zero, Eq. (2.17) reduces to Eq. (2.2) as it should.

Subsequently, J. D. Salas-La Cruz and D. C. Boes (1974) elaborated on the previous study, and among other things, graphically illustrated the transient nature of the general formulas for the expected adjusted range.

A. A. Anis and E. H. Lloyd (1975), following Boes' and Salas-La Cruz' reasoning with exchangeable random variables (1973), showed that the exact expected value of the rescaled adjusted range (meaning the ratio between the adjusted range and the sample standard deviation) for independent normal summands is

$$E\{R_n^{**}\} = \frac{1}{\sqrt{2\pi}}\,\frac{\Gamma[(\frac{n-1}{2})]}{\Gamma(\frac{n}{2})}\sum_{i=1}^{n-1}\left(\frac{n-i}{i}\right)^{\frac{1}{2}} , \qquad (2.18)$$

which leads to the asymptotic value given by Eq. (2.2). The relevance of this result follows from the fact that Hurst's experimental study referred exactly to the rescaled adjusted range. Furthermore, Anis and Lloyd showed that the same result holds for exchangeable multivariate normal summands. Therefore, the asymptotic value of R_n^{**} departs drastically from Eq. (2.17), which indicates that in some cases the assumption that the behavior of the rescaled adjusted range can be inferred from the behavior of the adjusted range is not justifiable.

One should notice that the emphasis in these works has been on the expected value of the range, simply because it can be approached through the study of the maximum partial sum, which is a simpler problem. The exact distribution of the range and the determination of higher moments have been approached only by the data generation method with the exception of Yevjevich's exact solution for the distribution of R_n for the cases $n = 2$ and $n = 3$. The only other work which neither uses the data generation method nor approaches the range through the maximum partial sum is Feller's derivation of asymptotic results.

2. Moran's Analysis

A slight modification of P.A.P. Moran's (1954) initial work will be presented. The model is formulated in discrete time, by considering a finite reservoir of size k and the water net input (input minus output) as a sequence of independent, identically distributed discrete random variables such that $P(X_t = i) = p_i$.

The reservoir is such that when full, it continues full only if the next input is nonnegative (and thus an overflow may occur), and when empty, it continues emply only if the next net input is equal to zero. Then the amount of water stored follows a simple homogeneous Markov chain with state space $\{0, 1, 2, \ldots, k\}$ and one step transition matrix as follows:

	0	1	2	3	...	k-2	k-1	k
0	ℓ_0	ℓ_{-1}	ℓ_{-2}	ℓ_{-3}	...	ℓ_{-k+2}	ℓ_{-k+1}	ℓ_{-k}
1	p_{+1}	p_0	p_{-1}	p_{-2}	...	p_{-k+3}	p_{-k+2}	p_{-k+1}
2	p_{+2}	p_{+1}	p_0	p_{-1}	...	p_{-k+4}	p_{-k+3}	p_{-k+2}
3	p_{+3}	p_{+2}	p_{+1}	p_0	...	p_{-k+5}	p_{-k+4}	p_{-k+3}
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.	.	.	.	.	.			
k-2	p_{+k-2}	p_{+k-3}	p_{+k-4}	p_{+k-5}	...	p_0	p_{-1}	p_{-2}
k-1	p_{+k-1}	p_{+k-2}	p_{+k-3}	p_{+k-4}	...	p_{+1}	p_0	p_{-1}
k	u_{+k}	u_{+k-1}	u_{+k-2}	u_{+k-3}	...	u_{+2}	u_{+1}	u_0

where the elements in the first and last rows are to be interpreted as follows:

$$\ell_{-j} = p_{-j} + p_{-j-1} + p_{-j-2} + \dots \quad (j=0,1,2,\dots,k)$$

$$u_{+j} = p_{+j} + p_{+j+1} + p_{+j+2} + \dots \quad (j=0,1,2,\dots,k).$$

Once the amount of water stored follows a homogeneous Markov chain, Moran and others emphasize the problem of determining the "steady state" probabilities, which will be discussed in the next chapter.

The transient distribution of the amount of water stored was reported by N. U. Prabhu (1965), for the case of geometric inputs. For illustration purposes, this result will be presented.

Consider the case in which the net input has a geometric distribution:

$$p_i = ab^{i+1} \quad (= -1,\ 0,\ 1,\ 2,\ \dots) \tag{2.19}$$

where $0 < a < 1$ and $a + b = 1$. For a finite reservoir of size $k-1$, the transition matrix is

	0	1	2	. . . .	k-1
0	$a+ab$	a	0	. . . .	0
1	ab^2	ab	a	. . . .	0
2	ab^3	ab^2	ab	. . . .	0
.	.	.	.	.	.
.	.	.	.	.	.
k-2	ab^{k-1}	ab^{k-2}	ab^{k-3}	. . . .	a
k-1	b^k	b^{k-1}	b^{k-2}	. . . .	b

(2.20)

Let the n-step transition probabilities be denoted by

$$q'^{(n)}(j,\ i) = P[Y_n = j \,|\, Y_o = i]$$

where Y_t is the amount of water at time t.

Prabhu defined the generating function

$$G(j,i) = \sum_{n=2}^{\infty} q'^{(n)}(j,i)\, z^n \quad (|z| < 1) \tag{2.21}$$

and showed that

$$G(j,i) = \sum_{v=0}^{i-1} (az)^{i+1-v} b^{j+1-v} V_v(z)(1-z)^{-1}$$

$$+ \sum_{v=i}^{j+1} (az)^2 b^{j+1+2} V_v(z)(1-z)^{-1}, \tag{2.22}$$

where

$$V_0 = \frac{\lambda_1^{k-i} - \lambda_2^{k-i} - bz(\lambda_1^{k-i-1} - \lambda_2^{k-i-1})}{\lambda_1^{k+1} - \lambda_2^{k+1}},$$

$$V_v = \frac{[\lambda_1^{k-i} - \lambda_2^{k-i} - bz(\lambda_1^{k-i-1} - \lambda_2^{k-i-1})][\lambda_1^{v+2} - \lambda_2^{v+2} - az(\lambda_1^{v+1} - \lambda_2^{v+1})]}{(\lambda_1 - \lambda_2)(\lambda_1^{k+1} - \lambda_2^{k+1})},$$

$(1 \le v \le i-1)$

$$V_v = \frac{[\lambda_1^{i+1} - \lambda_2^{i+1} - az(\lambda_1^{i} - \lambda_2^{i})][\lambda_1^{k-v-1} - \lambda_2^{k-v-1} - bz(\lambda_1^{k-v-2} - \lambda_2^{k-v-2})]}{(\lambda_1 - \lambda_2)(\lambda_1^{k+1} - \lambda_2^{k+1})},$$

$(i \le v \le j+1)$,

and

$$\lambda_1 = \frac{1 + \sqrt{1-4abz}}{2}, \qquad \lambda_2 = \frac{1 - \sqrt{1-4abz}}{2}.$$

Notice that to "invert" this result, to obtain $q'^{(n)}(j,i)$ is not an easy task. For $n = 10$, say, one has to differentiate $G(j,i)$ ten times with respect to z. Clearly this result indicates that the usefulness of some results in closed forms can be questioned. It is more appealing to the engineer to solve the problem numerically, for $q'^{(n)}(j,i)$ is simply the (j,i) entry in the n·th power of the transition matrix shown in Eq. (2.20)

Other authors analyzed the problem of emptiness with overflow and before overflow. B. Weesakul's (1961) result, typical of the rest, refers to the case of geometric inputs. He analyzed the cases of first emptiness before overflow and first emptiness regardless of occurrence of overflows. This second result is transcribed below for illustration purposes and because of its relevance to the concepts exposed in Chapter V.

Using the same input shown in Eq. (2.19), Weesakul showed that for a finite reservoir of size $k-1$ which had an initial content $u > 0$, the probability of first emptiness occurring at time $t + u$, regardless of how many times overflow occurs, is given by

$$-4a^u(ab)^t \sum_{v=1}^{[\frac{1}{2}k]} (2\cos\alpha_v)^{2t+u-1} \sin\alpha_v$$

$$\times \frac{\{a \sin[(k-u+1)\alpha_v] - b \sin[(k-u+1)\alpha_v]\}}{\{a(k+1)\cos[(k+1)\alpha_v] - b(k-1)\cos[(k-1)\alpha_v]\}} \tag{2.23}$$

where α_v $(v = 1, 2, \dots, [\frac{1}{2}k])$ are the distinct roots of

$$a \sin[(k+1)\alpha] - b \sin[(k-1)\alpha] = 0 \tag{2.24}$$

which lie in the subintervals

$$\left(\frac{(v-1)\pi}{k-1}, \frac{v\pi}{k}\right) \quad (v = 1, 2, \dots, [\tfrac{k}{2}])$$

and where $[\frac{k}{2}]$ denotes the largest integer contained in $k/2$.

E. H. Lloyd (1963) extended Moran's model, to take into account serial correlation of the inputs. Assuming that the sequence of inputs can be described by a Markov chain, Lloyd redefined the "state" of the system in terms of the values of the pair of variables storage-input and introduced a bivariate transition probability, namely the conditional probability that the pair storage-input at time t assume specified values, given the values of the pair storage-input at time $(t-1)$. With this device, the Markov property is restored and methods similar to the ones used in the univariate problem are applicable. The size of the matrix involved increases drastically. Instead of $k+1$ states $\{0, 1, 2, \ldots, k\}$, one has to consider $m(k+1)$ redefined states with m being the number of different possible values assumed by the input.

E. H. Lloyd and S. Odoom (1964) extended Moran's model to take into account seasonality of inputs. This has been accomplished simply by considering a set of transition probability matrices, one for each season. A detailed analysis of the simple two seasons model was given for illustration purposes.

Only contributions directly related to this paper have been discussed. There is a large number of other interesting works, and for a comprehensive view the interested reader is referred to review papers by N. U. Prabhu (1964), J. Gani (1969) and E. H. Lloyd (1974a).

3. Deficit Analysis

Very little work has been done on deficit analysis. To the knowledge of this writer, only two papers deal specifically with the maximum accumulated deficit, and both are "practical" papers in the sense that one presents an empirical treatment of actual data and the other used the data generation method.

E. H. Hurst (1951), using his long term sequences of natural phenomena, made an attempt to find the relationship between the adjusted maximum accumulated deficit (i.e., maximum accumulated deficit when the draft is a percentage of the sample mean) and the adjusted range. His method of analysis consisted of plotting observed values of the pair adjusted range--adjusted maximum deficit, and fitting curves "of the exponential and square-root forms: by simple regression techniques. The empirical formulae proposed were:

$$\log(D_n^*/R_n^*) = -0.11 - 0.88\ (\bar{Z}-B)/S \qquad (2.25)$$

$$(D_n^*/R_n^*) = 0.91 - 0.89\ \sqrt{(\bar{Z}-B)/S} \qquad (2.26)$$

where $\bar{Z}$ is the (sample) mean discharge, B is the constant output, S is the sample standard deviation of the natural discharge, and R_n^* and D_n^* are the adjusted range and the adjusted maximum accumulated deficit, respectively. Hurst concluded that "as far as closeness of fit is concerned, over the range of observations, there is no significant difference between one type of curve and the other. At some future time, it may perhaps be possible to decide that one or the other has some theoretical justification, but this has not so far been possible."

M. B. Fiering (1965), using the data generation method, investigated a wide range of possible data combination characterized by several input populations with different coefficients of skewness and serial correlation, by several levels of regulation and by record lengths typical in hydrologic studies. For each combination, either R_n^* or D_n^* was taken as the storage capacity required, depending on whether $\bar{Z} = B$ or $\bar{Z} \neq B$, respectively.

It is important to stress that in both works, the mean adjusted range and the mean adjusted maximum accumulated deficit rather than the unadjusted ones were taken as the storage capacity required, and in opinion of this writer, it is not easy to justify this criterion.

4. A Note on the Hurst Phenomenon

Hurst (1951) derived the asymptotic value of the mean adjusted range of partial sums of independent random variables with unit variance (Eq. (2.2)).

In order to verify this result experimentally, Hurst generated 30 sequences of size 100 of independent random variables, computed for each sequence the statistic $R_n^*/S\sqrt{n}$, where S^2 is the (biased) sample variance, and obtained an average value for this statistic close to 1.25, thus indicating that the derivation of Eq. (2.2) was probably correct.

Subsequently, in the analysis of data relative to natural phenomena (meaning rainfall, discharge, temperature, pressure, growth of tree rings, thickness of layers of mud and sunspot numbers), Hurst came to the conclusion that the mean rescaled range (meaning R_n^*/S) varies as n^K, where K has mean 0.729 and variance 0.303, and consequently, that the square-root law found before does not prevail for natural phenomena.

The important feature to observe is that Hurst used two different methods of analysis: in dealing with generated random data, he assumed that

$$E(R_n^*/S) = a.n^{0.5} \qquad (2.27)$$

and estimated the value of the parameter a (thus, in effect, he *imposed* the square-root law); however, in dealing with data of natural phenomena, he assumed that

$$E(R_n^*/S) = (n/2)^K \qquad (2.28)$$

or, equivalently,

$$K = \log[(R_n^*/S)]/\log(n/2)\ . \qquad (2.29)$$

To illustrate that the two methods of analysis lead to different conclusions, it suffices to go back to Hurst's own generated data (used to show that the square-root law prevails for "random events") and apply Eq. (2.29). The conclusion is that the estimated mean value of K is 0.64 and thus the square-root law does not prevail!

However, the reasoning behind Eq. (2.28) is sound: one would like to have $R_n^* = S$ for $n = 2$, independently of the value of K (it can be easily verified that when the biased estimator for the variance is used,

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n}$$

then $R_n^* = S$ for $n = 2$). Another reason for Hurst's proposal of Eq. (2.29) is that it seems to fit his data well (certainly the indisputable fact is that his data depart from Eq. (2.2)). Probably for this reason the following estimator has not been used:

$$K' = \frac{\log(R_n^*/S) - \log(1.25)}{\log(n)} \quad . \qquad (2.30)$$

Interestingly enough, using Eq. (2.30), the mean value of K' for Hurst's 690 cases of natural phenomena turns out to be 0.57, still larger than 0.50, but much smaller than 0.73. Furthermore, when Eq. (2.30) is applied to Hurst's 30 sequences of generated data, the mean value of K' is 0.50.

Once Eq. (2.29) gives the right result for $n = 2$ but leads to inconsistencies when applied to $n = 100$ (in the case of Hurst's generated data, for instance), the obvious conclusion is that it is not reasonable to assume that the relationship between the logarithm of the rescaled range and log (n) is linear. The reader may find it illustrative to plot values of $\log[E(R_n^*/S)]$ given by Eq. (2.18) against log (n) to see that the relationship is not linear for small values of n, and that even though the square-root law holds for large values of n, the rescaled range behaves as higher powers of n, in a pre-asymptotic sense. This argument (Hurst phenomenon as a simple transient effect) was first presented by E. H. Lloyd (1967), based on the analysis of Eq. (2.15) rather than (2.18).

Exploring further the idea of transience, it was natural to follow Feller's (1951) suggestion and to study the mean adjusted range for dependent random variables. N. Matalas and C. S. Huzzen (1967) simulated 10,000 sequences of Gaussian-Markov processes for each of several combinations of n (record length) and ρ (lag one coefficient of correlation). For each sequence, the coefficient K as defined by Eq. (2.29) was computed. His conclusion was that in general the results were similar to Hurst's, with the mean value of K ranging from 0.58 to 0.87.

Since Hurst's basic argument was that the square-root law apparently does not hold for geophysical data, it was natural to look for possible explanations outside the Gaussian Markov framework. As mentioned before, P. A. P. Moran (1964) showed that the range of partial sums of independent stable random variables behaves as $n^{1/\gamma}$. D. C. Boes and J. D. Salas-La Cruz (1973) showed that this is also the case for the adjusted range. However, it is important to note that stably distributed random variables with parameter $1<\gamma<2$ have finite mean but infinite varia[nce,] some hydrologists find it difficult to acc[ept that] hydrologic processes have infinite varianc[e.] point to stress is that in some cases one can acc[ept] the idea of similarity of behavior between the adjusted range R_n^* and the rescaled range R_n^*/S. However, when the expectation of the sample variance does not exist, one may be tempted to conclude that the adjusted range and the rescaled range behave in different fashions. Therefore, it is this writer's opinion that the reasoning with stable distributions cannot be accepted as a candidate to explain the Hurst phenomenon before the behavior of the rescaled range is assessed.

Another attempt to explain the Hurst phenomenon outside the Gaussian-Markov framework was made by B. B. Mandelbrot and J. R. Wallis (1968, 1969a, 1969b). They proposed an alternative generator of Hurst-like sequences, called "fractional Gaussian noise," characterized by a property called "self-similarity." This model assumes that geophysical processes have "infinite memory" (meaning that the distant past exerts small but nonnegligible influence in the present), and some hydrologists find difficulties in accepting this assumption (A. E. Scheidegger, 1970, V. Klemes, 1974).

The other area explores as an alternative explanation for the Hurst phenomenon is that of possible nonstationarity of geophysical time series. Hurst (1957) proposed an interesting model, in which the mean input suffers random finite jumps, randomly in time. P. E. O'Connell (1971) claimed Hurst-like properties for particular autoregressive integrated moving-average (ARIMA) models. Recently, V. Klemes (1974) elaborated further on Hurst's idea of random jumps occurring randomly in time.

Later in this paper, departing from these possible explanations (infinite memory, infinite variance, or nonstationarity), the argument that "short memory" (meaning that the influence of the distant past in the present is negligible) models preserve the so-called Hurst phenomenon will be presented. The argument will be original, but the reader should note that the idea is old: it goes back to Feller's (1951) conjecture, and it has been verified by Matalas and Huzzen (1967), and by V. Klemes (1974), using the data-generation method.

Chapter III
BACKGROUND MATERIAL

1. Markov Chains

In this section a summary of the theory of Markov chains is presented. Although this topic is well-known and can be found in basic text books, it is convenient to present it here for quick reference. A very attractive presentation of the subject has been made by E. H. Lloyd (1974), and in this section his contribution is summarized.

1.1 Generalities. Considering the time structure of a univariate discrete process $\{Y_t\}$, two extreme cases may arise: the situation when the random variables $Y_0, Y_1, Y_2, \ldots$ are all independent of each other (Eq. (3.1)) and the situation in which the distribution of the variables is influenced by all earlier observations (Eq. (3.2)).

$$P[Y_t = i, Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_0 = w]$$
$$= P[Y_t = i] \cdot P[Y_{t-1} = j] \cdot P[Y_{t-2} = k] \cdot \ldots \cdot P[Y_0 = w] \tag{3.1}$$

$$P[Y_t = i, Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_1 = v, Y_0 = w]$$
$$= P[Y_t = i | Y_{t-1} = j, \ldots, Y_0 = w] \cdot P[Y_{t-1} = j | Y_{t-2} = k, \ldots, Y_0 = w] \cdot \ldots \cdot P[Y_1 = v | Y_0 = w] \cdot P[Y_0 = w] \tag{3.2}$$

In Eq. (3.2), the expression $P[Y_r = s|C]$ denotes the conditional probability the Y_r should take the value s, given the condition C.

A model intermediate between (3.1) and (3.2), in which the distribution of the value Y_t is influenced only by the previous k observations is called a k-step Markov chain (Eq. (3.3)).

$$P[Y_t = i | Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_{t-k} = m, Y_{t-k-1} = n, \ldots, Y_0 = w]$$
$$= P[Y_t = i | Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_{t-k} = m] \tag{3.3}$$

Each value of k calls for its own methods of analysis, which are similar to some extent to the methods used for the case $k = 1$, which is the 1-step Markov chain, or simple Markov chain.

1.2 Simple Markov Chains. For the simple Markov chain, Eq. (3.2) becomes

$$P[Y_t = i, Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_1 = v, Y_0 = w]$$
$$= P[Y_t = i | Y_{t-1} = j] \cdot P[Y_{t-1} = j | Y_{t-2} = k] \cdot \ldots \cdot P[Y_1 = v | Y_0 = w] \cdot P[Y_0 = w]. \tag{3.4}$$

The conditional probability $P[Y_r = s | Y_{r-1} = u]$ is called a transition probability, and is sometimes written in short notation as $p_{u,s}$, indicating a transition from the "state" u to the "state" s. Because it is a conditional probability, it follows that, for each fixed value of the conditioning variable Y_{r-1}, $\sum_s P[Y_r = s | Y_{r-1} = u] = \sum_s p_{u,s} = 1$, summed over the conditioned variable Y_r.

From (3.4), the marginal distribution of Y_t can be obtained as

$$P[Y_t = i] = \sum_j \ldots \sum_w P[Y_t = i, Y_{t-1} = j, Y_{t-2} = k, \ldots, Y_1 = v, Y_0 = w]$$
$$= \sum_j \ldots \sum_w q'_t(i,j) \cdot q'_{t-1}(j,k) \cdot \ldots \cdot q'_1(v,w) \cdot P[Y_0 = w] \tag{3.5}$$

where $q'_r(s,u) = p_{u,s} = P[Y_r = s | Y_{r-1} = u]$.

The symbol $q'_r(s,u)$ rather than $p_{u,s}$ is used to proportionate a more convenient matrix notation to Eq. (3.5). Let ε_t denote a column vector with elements $\varepsilon_t(s) = P[Y_t = s]$, $s = 0, 1, \ldots$, and Q'_t a square matrix in which the (i,j) entry is $q'_t(i,j)$. Then Eq. (3.5) can be written as

$$\varepsilon_t = Q'_t \cdot Q'_{t-1} \cdot Q'_{t-2} \cdot \ldots \cdot Q'_2 \cdot Q'_1 \cdot \varepsilon_0 \tag{3.6}$$

and, equivalently,

$$\varepsilon_t = Q'_t \cdot \varepsilon_{t-1} \quad t = 1,2,\ldots. \tag{3.7}$$

If the restriction of time homogeneity is imposed, so that $Q'_t = Q'$, for every t, Eqs. (3.6) and (3.7) become

$$\varepsilon_t = Q'^{\,t} \cdot \varepsilon_0 \tag{3.8}$$

and

$$\varepsilon_t = Q' \cdot \varepsilon_{t-1}. \tag{3.9}$$

Whilst the distribution vectors ε_0, $\varepsilon_1 = Q' \cdot \varepsilon_0$, $\varepsilon_2 = Q'^2 \cdot \varepsilon_0, \ldots$ are in general different from each other, there is a large and important subclass of so-called ergodic Markov chains for which ε_t converged to a unique limit ε which is a probability vector and which is independent of the initial state of the system. Let the elements in this vector be denoted by $\varepsilon(r) = P[Y_t = r]$.

In the ergodic case, $\varepsilon_t \to \varepsilon$, and thus it follows from (3.9) that

$$\varepsilon = Q' \cdot \varepsilon \tag{3.10}$$

or, equivalently,

$$(Q' - I) \cdot \varepsilon = \underline{0} \tag{3.11}$$

where I stands for the identity matrix and $\underline{0}$ is a vector with all elements equal to zero.

Equation (3.11) represents a system of linear equations which, in conjunction with the condition

$$\sum_r \varepsilon\,(r) = 1 \tag{3.12}$$

determines ε uniquely.

1.3 Nonsimple Markov Chains. For the purposes of this paper, a presentation of the 2-step Markov chain will suffice. In this case, Eq. (3.2) becomes

$$\begin{aligned} P[Y_t &= i,\ Y_{t-1} = j,\ Y_{t-2} = k,\ Y_{t-3} = \ell, \ldots, Y_2 \\ &= u,\ Y_1 = v,\ Y_0 = w] \\ &= P[Y_t = i \mid Y_{t-1} = j,\ Y_{t-2} = k]\cdot P[Y_{t-1} \\ &= j \mid Y_{t-2} = k,\ Y_{t-3} = \ell] \\ &\ldots P[Y_2 = u \mid Y_1 = v,\ Y_0 = w]\cdot P[Y_1 = v,\ Y_0 = w]. \end{aligned} \tag{3.13}$$

To use a technique similar to the one presented for the simple Markov chain, the transition probability can be written as

$$\begin{aligned} P[Y_t = i \mid Y_{t-1} = j,\ Y_{t-2} = k] &= P[Y_t = i,\ Y_{t-1} \\ &= j \mid Y_{t-1} = j,\ Y_{t-2} = k] = a_t(ij \mid jk). \end{aligned}$$

The "marginal" distribution of the pair (Y_t, Y_{t-1}) can be obtained by

$$\begin{aligned} P[Y_t = i,\ Y_{t-1} = j] &= \sum_k \sum_\ell \ldots \sum_w P[Y_t = i,\ Y_{t-1} \\ &= j, \ldots, Y_0 = w] \\ &= \sum_k \sum_\ell \ldots \sum_w a_t(ij \mid jk)\cdot a_{t-1}(jk \mid k\ell)\ldots a_2(uv \mid vw)\cdot P[Y_1 \\ &= v,\ Y_0 = w]\ . \end{aligned} \tag{3.14}$$

Or, in matrix notation,

$$\delta_t = A_t \cdot A_{t-1} \cdot \ldots \cdot A_2 \cdot \delta_1 \tag{3.15}$$

or,

$$\delta_t = A_t \cdot \delta_{t-1} \tag{3.16}$$

where δ_t is the vector with elements $\delta_t(i,j) = P[Y_t = i,\ Y_{t-1} = j]$, ordered as

$$\delta_t^T = \{\delta_t(0,0),\ \delta_t(0,1), \ldots, \delta_t(1,0),$$

$$\delta_t(1,1), \ldots, \delta_t(2,0),\ \delta_t(2,1)\ldots\},$$

the symbol T denoting the transpose of the vector. The elements of the matrix A_t are $a_t(ij \mid jk)$, arranged as shown in the following example:

(i,j) \ (j,k)	00	01	02	10	11	12	20	21	22
00	*	*	*	0	0	0	0	0	0
01	0	0	0	*	*	*	0	0	0
02	0	0	0	0	0	0	*	*	*
10	*	*	*	0	0	0	0	0	0
11	0	0	0	***	*	*	0	0	0
12	0	0	0	0	0	0	*	*	*
20	*	*	*	0	0	0	0	0	0
21	0	0	0	*	*	*	0	0	0
22	0	0	0	0	0	0	*	*	*

The asterisks indicate nonidentically zero entries. For example, the entry *** is $a_t(11 \mid 10)$.

If the transition probabilities are not time dependent, Eqs. (3.15) and (3.16) become, respectively,

$$\delta_t = A^{t-1} \cdot \delta_1 \tag{3.17}$$

and

$$\delta_t = A \cdot \delta_{t-1} \qquad t = 2,3,\ldots. \tag{3.18}$$

As in the case of the simple Markov chains, if the transition matrix is ergodic, δ_t will converge, with increasing t, to a limit vector, which defines the joint equilibrium distribution of the consecutive pairs of variables. From this, the univariate limiting vector can be obtained.

2. The Symmetric Random Walk and the Method of Images

The purpose of this section is to present the classical method of images and to apply it in the derivation of expressions that will be used later in this paper.

2.1 Generalities. Consider the sequence of independent random variables $\{X_i;\ i = 1,2,\ldots\}$ such that $P[X_i = +1] = p$, $P[X_i = -1] = q$ and $p + q = 1$.

The distribution of the sum $S_m = X_1 + X_2 + \ldots + X_m (m = 1,2,\ldots)$ is given by

$$P[S_m = s] = {}_mC_{(m+s)/2} \cdot p^{(m+s)/2} \cdot q^{(m-s)/2} \tag{3.19}$$

where

$${}_mC_{(m+s)/2} = \frac{m!}{[(m+s)/2]!\ [(m-s)/2]!}$$

and

$$s = -m,\ -m+2,\ -m+4, \ldots, m-4,\ m-2,\ m.$$

This process is called a simple random walk.

For the symmetric random walk, $p = q = \frac{1}{2}$ and Eq. (3.19) reduces to

$$P[S_m = s] = {}_mC_{(m+s)/2} \cdot (1/2)^m. \tag{3.20}$$

The X_i's are independent and identically distributed and thus the following relationship holds:

$$P[S_{m+n} = s | S_m = u] = P[X_{m+1} + X_{m+2} + \ldots + X_{m+n} = s-u]$$

$$= P[S_n = s-u] = {}_nC_{(n+s-u)/2} \cdot (1/2)^n . \qquad (3.21)$$

2.2 The One-Boundary Problem. The ideal coin-tossing will be used to illustrate the different cases presented in this section, following W. Feller (1970).

Consider a player "betting against the house"; assume that the "house" is infinitely rich, but the player has a finite initial capital C. The game is "head and tails" and the player loses one dollar each time the outcome is, say, a head. Thus, a head stands for a -1 and a tail stands for a +1, from the player's viewpoint.

One of the questions that arises is what is the probability that the player will have a final capital S at the end of n coin tosses. To answer this question one has to have in mind that the player may very well go broke before the n·th coin toss, in which case the game would not continue. The solution is easily found using a geometric reasoning, which is the essence of the so-called method of images.

Referring to Fig. 3.1, U' is the point with coordinates (0,-u) and it is the image of point U(0,+u) with respect to the line y = 0. The geometric reasoning is as follows: the number of paths going from U(0,+u) to S(n,+s) which touch or cross the line y = 0 equals the number of *all* paths from U'(0,-u) to S(n,+s).

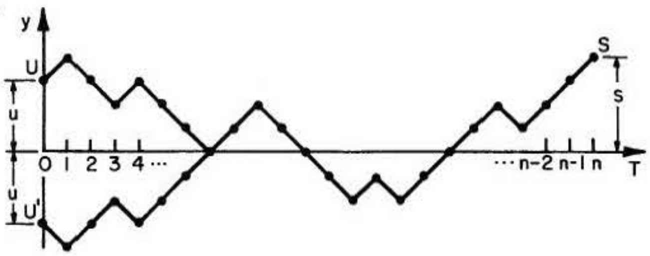

Fig. 3.1. The essence of the method of images.

Consequently, the number of paths from U to S which do not cross or touch the axis is the difference between the number of *all* paths from U to S and the number of *all* paths from U' to S. The probability of such an event is obtained by dividing this difference between the number of paths by 2^n (total number of n-step paths starting in U).

Applying Eq. (3.21), the probability that the player will have a final capital S = s at the end of the n·th coin toss, given that the initial capital U was u, is

$$P[S_n = s-u] - P[S_n = s+u] = {}_nC_{(n+s-u)/2} \cdot (1/2)^n$$

$$- {}_nC_{(n+s+u)/2} \cdot (1/2)^n \qquad (3.22)$$

Or, introducing an obvious notation:

$$P[S_n = s-u] - P[S_n = s+u] = v_n(u,s) - v_n(-u,s).$$

A boundary such as y = 0 in the above example is called an absorbing boundary, in the sense that, once it is reached, the "system" continues in this "state" with probability one (the player is broke and the game ends).

Now suppose that the house is generous enough not to collect the player's last dollar. Thus, when the player has only one dollar, the "system" continues in "state" 1 with probability ½ (when the outcome is a head) and goes to "state" 2 with probability ½ (when the outcome is a tail).

It can be shown that the probability that the player will have a final capital S = s at the end of the n·th coin toss, given the initial capital U = u and given that the house does not charge him for his last dollar is

$$P[S_n = s-u] + P[S_n = s+u-1] = v_n(u,s) + v_n(-u+1,s)$$

where $P[S_n = s-u]$ is given by Eq. (3.21).

This is tantamount to adding the number of *all* paths from U to S to the number of *all* paths from U" to S, where U" has coordinates (0,-u+1) and it is the image of point U with respect to the line y = ½ (see Fig. 3.2).

A boundary such as the axis y = 1 in the above example is called a reflecting boundary, in the sense that it does not allow the axis y = 0 or ("state" 0) to be reached.

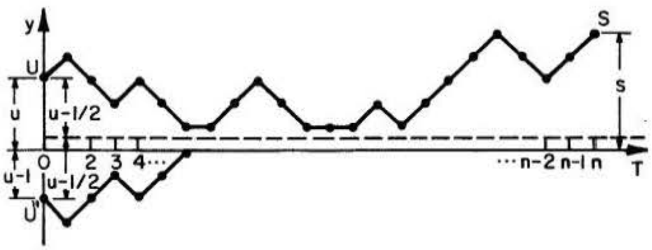

Fig. 3.2. Reflecting boundary for the endless game.

2.3 The Two-Boundary Problem. Now consider the case of two players with finite capitals. The game ends when one of them goes broke, and thus the problem involves two absorbing boundaries.

Referring to Fig. 3.3, one may ask what is the nubmer of paths from A to B which do not cross or touch either boundary.

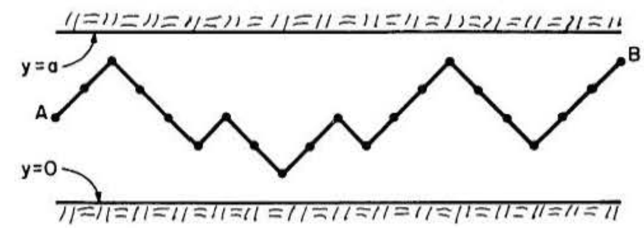

Fig. 3.3. Absorbing boundaries for the finite duration game between two players.

Denote by $A^{(1)}$ the image of point A with respect to the axis y = 0, by $A^{(2)}$ the image of point $A^{(1)}$ with respect to the axis y = a, by $A^{(3)}$

the image of point $A^{(2)}$ with respect to the axis $y = 0$, and so on. Similarly, denote by $A_*^{(1)}$ the image of point A with respect to the axis $y = a$, by $A_*^{(2)}$ the image of point $A_*^{(1)}$ with respect to the axis $y = 0$, by $A_*^{(3)}$ the image of point $A_*^{(2)}$ with respect to the axis $y = a$, and so on.

The solution to the problem is given by considering the number of *all* paths going from A to B, subtracting the number of *all* paths going from $A^{(1)}$ and $A_*^{(1)}$ to B, adding the number of *all* paths going from $A^{(2)}$ and $A_*^{(2)}$ to B, subtracting the number of *all* paths going from $A^{(3)}$ and $A_*^{(3)}$ to B, and so on.

Considering the case in which the players do not collect each other's last dollar, the problem involves two reflecting boundaries (in such a way that no player goes broke).

Referring to Fig. 3.4, the probability of being at B, after the n·th play, given that the process started at A, is found by considering the repeated images as before, now with respect to the axes $y = a - 1/2$ and $y = 1/2$, and by adding all probabilities involved (recall that in the previous case addition and subtraction were performed alternately).

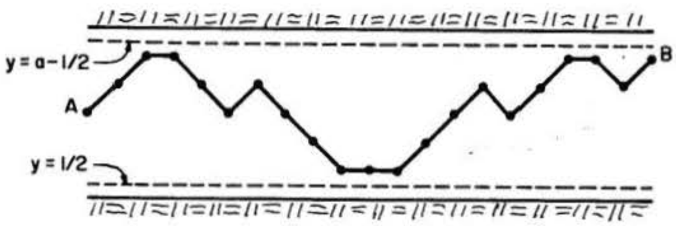

Fig. 3.4. Reflecting boundaries for the endless game between two players.

The case of one absorbing and one reflecting boundary can be treated similarly. A more careful analysis of the changes of sign is necessary in this case.

It is convenient to homogenize notation before giving explicit results for each case of the two-boundary problem. In Section 1 the symbol $q_t'(s,u)$ was introduced to denote the one-step transition probability from state u to state s. This would be better defined by $q_t^{(1)}(s,u)$, so that in general $q_t^{(n)}(s,u)$ will denote the n-step transition probability from state u to state s. The subscript t can be dropped whenever time-homogeneity is assumed, as in the problem in this section. For convenience, the state space considered will be $\{0,1,2,\ldots k,k+1\}$.

Two absorbing boundaries

The random walk in this case is a Markov chain with the following one-step transition matrix Q':

	0	1	2		k-1	k	k+1
0	1	1/2	0	......	0	0	0
1	0	0	1/2	......	0	0	0
2	0	1/2	0	......	0	0	0
3	0	0	1/2	......	0	0	0
	.	.	.	......	.	.	.
	.	.	.	......	.	.	.
k-2	0	0	0	......	1/2	0	0
k-1	0	0	0	......	0	1/2	0
k	0	0	0	......	1/2	0	0
k+1	0	0	0	......	0	1/2	1

The probability of going from state $u = 1, 2,\ldots, k$ to state $s = 1, 2,\ldots, k$ in n steps without touching or crossing the boundaries will be denoted by $q^{(n)}(s,u)$ and it is the (s,u) entry in the n·th power of the following matrix:

$$Q = \begin{bmatrix} 0 & 1/2 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 1/2 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots\cdots & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \cdots\cdots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \cdots\cdots & 0 & 1/2 & 0 \end{bmatrix}$$

which is obtained from matrix Q' simply by deleting the first and the last row and column.

Using the method of images as described, it is found that

$$q^{(n)}(s,u) = \sum_{j=-\infty}^{j=+\infty} [v_n(2j(k+1)+u,s) - v_n(2j(k+1)-u,s)] \tag{3.23}$$

where $v_n(t,r) = P[S_n = r-t]$ is given by Eq. (3.21), and where only finitely many nonzero terms exist.

Finally, the relationship between the n·th powers of the matrices Q and Q' is the following:

$$Q'^n = \begin{bmatrix} 1 & \ell_n^T & 0 \\ \underline{0} & Q^n & \underline{0} \\ 0 & u_n^T & 1 \end{bmatrix}$$

where $\underline{0}$ is a column vector with all elements equal to zero and u_n^T and ℓ_n^T are row vectors (T stands for transposed, u stands for absorption in the upper boundary and ℓ stands for the absorption in the lower boundary).

The vectors u_n and ℓ_n are given by

$$u_n = (I+Q+Q^2+Q^3+\ldots+Q^{n-1}) \cdot u_1,$$

where

$$u_1^T = [0 \ \ldots \ 0 \quad 0 \quad 1/2],$$

and

$$\ell_n = (I+Q+Q^2+Q^3+\ldots+Q^{n-1}) \cdot \ell_1,$$

where

$$\ell_1^T = [1/2 \quad 0 \quad 0 \ \ldots \ 0] \ .$$

Two reflecting boundaries

The matrix Q' in this case is

	0	1	2		k-1	k	k+1
0	1/2	1/2	0	......	0	0	0
1	1/2	0	1/2	......	0	0	0
2	0	1/2	0	......	0	0	0
	.	.	.	......	.	.	.
k-1	0	0	0	......	0	1/2	0
k	0	0	0	......	1/2	0	1/2
k+1	0	0	0	......	0	1/2	1/2

and there is no advantage in defining the matrix Q. $q'^{(n)}(s,u)$ is the (s,u) entry in the n·th power of the matrix Q', and applying the method of images, it is given by

$$q'^{(n)}(s,u) = \sum_{j=-\infty}^{j=+\infty} [v_n(2j(k+1)+u,s) + v_n(2j(k+1)-u+1,s)] \qquad (3.24)$$

where, as before, $v_n(t,r) = P[S_n = r-t]$ is given by Eq. (3.21), and where only finitely many nonzero terms exist.

One absorbing and one reflecting boundary

In this case, the matrix Q' is shown below, for the case when the absorbing state is state 0. The matrix Q can be obtained by deleting the first row and the first column (recall that in the case of the two absorbing boundaries, the first and the last rows and columns were deleted).

	0	1	2		k-1	k	K·
0	1	1/2	0	......	0	0	(
1	0	0	1/2	......	0	0	(
2	0	1/2	0	......	0	0	(
3	0	0	1/2	......	0	0	(
	.	.	.	......	.	.	.
	.	.	.	......	.	.	.
k-1	0	0	0	......	0	1/2	(
k	0	0	0	......	1/2	0	1,
k+1	0	0	0	......	0	1/2	1,

$q^{(n)}(s,u)$ is the (s,u) entry in the n·th power of the matrix Q and its value follows from tl application of the method of images:

$$q^{(n)}(s,u) = \sum_{j=-\infty}^{j=+\infty} (-1)^j \cdot [v_n(2j(k+3/2)+u,s)$$

$$- v_n(2j(k+3/s)-u,s)] \qquad (3.25)$$

where, once more, $v_n(t,r) = P[S_n = r-t]$ is given by Eq. (3.21) and where only finitely many nonzero terms exist.

The relationship between the n·th powers of the matrices Q and Q' is

$$Q'^n = \begin{bmatrix} 1 & \ell_n^T \\ \underline{0} & Q^n \end{bmatrix}$$

where $\underline{0}$ is a column vector with all elements equal to zero and

$$\ell_n = (I+Q+Q^2+Q^3+\ldots+Q^{n-1}) \cdot \ell_1$$

$$\ell_1^T = [1/2 \quad 0 \quad 0 \ \ldots \ 0]$$

where, as usual, T stands for "transpose" and the symbol ℓ is related to absorption in the lower boundary. Later in this paper, the matrices Q' an Q for the case of one absorbing and one reflecting boundary will be denoted by P' and P, to avoid confusion with the case of two absorbing boundaries.

Chapter IV
RANGE ANALYSIS FOR INDEPENDENT, IDENTICALLY DISTRIBUTED INPUTS

The concept of the range of partial sums of random variables is of great importance in hydrology. Surprisingly, only a few results are known, such as the asymptotic distribution (Feller, 1951), mean range (Spitzer, 1956, and others) and the exact distribution obtained numerically for very small values of n (Yevjevich, 1965).

In this chapter, a general approach to the exact distribution of the range is described. Starting with discrete random variables, the formulation is extended in the sequel to continuous random variables. Evaluation of moments follows immediately from the procedure to be described and applications are shown for the case of some well known probability distributions.

1. Discrete Net Inputs

Consider the sequence of independent, identically distributed discrete random variables $\{X_t;\ t = 1, 2, \ldots, n\}$ and

$$S_t = X_1 + X_2 + \ldots + X_t;\ t = 1, 2, \ldots, n$$

$$M_n = \max(0, S_1, S_2, \ldots, S_n)$$

$$m_n = \min(0, S_1, S_2, \ldots, S_n)$$

$$R_n = M_n - m_n = M_n + |m_n|$$

As defined previously, $\{S_t\}$ are called partial sums, M_n is their (nonnegative) maximum, m_n is their (nonpositive) minimum, and R_n is the range.

In this section, the joint distribution of M_n and m_n is initially discussed. From this discussion the distribution of the range follows directly.

It is convenient to approach the problem using a terminology similar to Moran's in the analysis of the finite reservoir. Let X_t denote the *net input* (i.e., input minus output) at discrete time t into a reservoir of size (k+1), such that $P(X_t = i) = p_i$. Clearly, X_t can assume negative vlaues. Furthermore, let this reservoir be such that when full, it continues full with probability one, and when empty, it continues empty with probability one.

Then the amount of water stored follows a simple homogeneous Markov chain with state space $\{0,1,2,\ldots, k+1\}$ and one-step transition matrix Q' as shown at the top of the next column.

The elements in the first and last rows are to be interpreted as

$$u_j = p_j + p_{j+1} + p_{j+2} + \ldots,\ (j = 1,2,\ldots,k)$$

	0	1	2	3	...	k-2	k-1	k	k+1
0	1	ℓ_{-1}	ℓ_{-2}	ℓ_{-3}	...	ℓ_{-k+2}	ℓ_{-k+1}	ℓ_{-k}	0
1	0	p_0	p_{-1}	p_{-2}	...	p_{-k+3}	p_{-k+2}	p_{-k+1}	0
2	0	p_{+1}	p_0	p_{-1}	...	p_{-k+4}	p_{-k+3}	p_{-k+2}	0
3	0	p_{+2}	p_{+1}	p_0	...	p_{-k+5}	p_{-k+4}	p_{-k+3}	0
.	.	.	.	.		.	.	.	.
.	.	.	.	.		.	.	.	.
.	.	.	.	.		.	.	.	.
k-2	0	p_{+k-3}	p_{+k-4}	p_{+k-5}	...	p_0	p_{-1}	p_{-2}	0
k-1	0	p_{+k-2}	p_{+k-3}	p_{+k-4}	...	p_{+1}	p_0	p_{-1}	0
k	0	p_{+k-1}	p_{+k-2}	p_{+k-3}	...	p_{+2}	p_{+1}	p_0	0
k+1	0	u_{+k}	u_{+k-1}	u_{+k-2}	...	u_{+3}	u_{+2}	u_{+1}	1

and

$$\ell_{-j} = p_{-j} + p_{-j-1} + p_{-j-2} + \ldots,\ (j = 1,2,\ldots,k).$$

The matrix Q' can be partitioned as

$$Q' = \begin{bmatrix} 1 & \ell^T & 0 \\ \underline{0} & Q & \underline{0} \\ 0 & u^T & 1 \end{bmatrix} \qquad (4.2)$$

where $\underline{0}$ is a column vector of size k with all elements equal to zero, the symbol T stands for transpose, and where u^T, ℓ^T and Q are as follows:

$$u^T = [u_k\ u_{k-1}\ u_{k-2} \cdots u_2\ u_1]\ , \qquad (4.3)$$

$$\ell^T = [\ell_{-1}\ \ell_{-2}\ \ell_{-3} \cdots \ell_{-k+1}\ \ell_{-k}]\ , \qquad (4.4)$$

and

$$Q = \begin{bmatrix} p_0 & p_{-1} & p_{-2} & \cdots & p_{-k+3} & p_{-k+2} & p_{-k+1} \\ p_{+1} & p_0 & p_{-1} & \cdots & p_{-k+4} & p_{-k+3} & p_{-k+2} \\ p_{+2} & p_{+1} & p_0 & \cdots & p_{-k+5} & p_{-k+4} & p_{-k+3} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ p_{+k-3} & p_{+k-4} & p_{+k-5} & \cdots & p_0 & p_{-1} & p_{-2} \\ p_{+k-2} & p_{+k-3} & p_{+k-4} & \cdots & p_{+1} & p_0 & p_{-1} \\ p_{+k-1} & p_{+k-2} & p_{+k-3} & \cdots & p_{+2} & p_{+1} & p_0 \end{bmatrix}$$

The n-step transition matrix is then

$$Q'^{n} = \begin{bmatrix} 1 & \ell^{T}\cdot(I+Q+Q^{2}+\dots+Q^{n-1}) & 0 \\ \underline{0} & Q^{n} & \underline{0} \\ 0 & u^{T}\cdot(I+Q+Q^{2}+\dots+Q^{n-1}) & 1 \end{bmatrix} \quad (4.6)$$

where I is the identity matrix.

The matrix Q^n will be called the n-step "restricted" transition matrix for obvious reasons.

1.1 Joint Distribution of M_n and m_n. Keeping the same notation presented in Chapter III, $q^{(n)}(s,u)$ is the (s,u) entry in the matrix Q^n, and it denotes the probability of a transition from the state $u = 1,2,\dots,k$ to the state $s = 1,2,\dots,k$, without passing through the states zero or $(k+1)$.

Then $\sum_{s=1}^{s=k} q^{(n)}(s,u)$ *denotes the probability that the system does not reach the boundaries (states zero and* $k+1$*) in the first* n *steps, given the initial state* u*. But this is clearly the joint probability* $P(M_n \leq k-u,\ |m_n| \leq u-1)$*, where the symbol* $|m_n|$ *stands for the absolute value of* m_n.

It is convenient to use the index k to emphasize that $\sum_{s=1}^{s=k} q_k^{(n)}(s,u)$ is the sum of all elements of the u·th column in the n·th power of the matrix Q, *of size* k.

The probability mass function $P(M_n = k-u,\ |m_n| = u-1)$ is given by

$$P(M_n = k-u,\ |m_n| = u-1) = P(M_n = k-u,\ |m_n| \leq u-1) - P(M_n = k-u,\ |m_n| \leq u-2)$$

but

$$P(M_n = k-u,\ |m_n| \leq u-1) = P(M_n \leq k-u,\ |m_n| \leq u-1) - P(M_n \leq k-u-1,\ |m_n| \leq u-1)$$

and

$$P(M_n = k-u,\ |m_n| \leq u-2) = P(M_n \leq k-u,\ |m_n| \leq u-2) - P(M_n \leq k-u-1,\ |m_n| \leq u-2).$$

and thus

$$P(M_n = k-u,\ |m_n| = u-1) = P(M_n \leq k-u,\ |m_n| \leq u-1) - P(M_n \leq k-u-1, |m_n| \leq u-1) - P(M_n \leq k-u, |m_n| \leq u-2) + P(M_n \leq k-u-1,\ |m_n| \leq u-2)$$

where all the terms in the right hand side can be written as sums of elements of particular columns of n-step "restricted" transition matrices of sizes $k, k-1$ and $k-2$:

$$P(M_n = k-u,\ |m_n| = u-1) = \sum_{s=1}^{s=k} q_k^{(n)}(s,u) - \sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u) - \sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u-1) + \sum_{s=1}^{s=k-2} q_{k-2}^{(n)}(s,u-1)\,. \quad (4.7)$$

Notice that $q_{k-1}^{(n)}(s,u-1)$ and $q_{k-2}^{(n)}(s,u-1)$ are different, because they are entries in matrices of different sizes. Notice also that once the joint distribution of M_n and m_n is known, their marginal distributions can be obtained easily.

Although the underlying concepts are very simple, their exposition may be obscured by the unfortunately complicated notation. To help clarify the procedure outlined, a simple example is given.

Example 4.1

The joint distribution of M_n and m_n will be found in the case $n = 3$, for the following binomially distributed net input:

$$P(X_t = i) = {}_4C_{(2+i)} \cdot (1/2)^4 \text{ for } i = -2,-1,0,1,2$$

$$P(X_t = i) = 0 \text{ otherwise.}$$

The symbol C stands for "combination." For instance, ${}_4C_2 = \frac{4!}{2!2!} = 6$. Notice that $E(X_t) = 0$ and $\mathrm{var}(X_t) = 1$.

Only two particular values are evaluated in detail. The procedure to find all other probability masses is exactly the same and the final results are shown in Table 4.1.

To find, say, $P(M_3 = 1,\ |m_3| = 2)$, Eq. (4.7) furnishes

$$P(M_3 = 1,\ |m_3| = 2) = \sum_{s=1}^{s=4} q_4^{(3)}(s,3) - \sum_{s=1}^{s=3} q_3^{(3)}(s,3) - \sum_{s=1}^{s=3} q_3^{(3)}(s,2) + \sum_{s=1}^{s=2} q_2^{(3)}(s,2)\,. \quad (4.8)$$

The one-step "restricted" transition matrix (Eq. (4.5)) in this cas^ is

$$\left[\begin{array}{cc|cccc} 6/16 & 4/16 & 1/16 & 0 & 0 & 0 \\ 4/16 & 6/16 & 4/16 & 1/16 & 0 & 0 \\ \hline 1/16 & 4/16 & 6/16 & 4/16 & 1/16 & 0 \\ 0 & 1/16 & 4/16 & 6/16 & 4/16 & 1/16 \\ 0 & 0 & 1/16 & 4/16 & 6/16 & 4/16 \\ 0 & 0 & 0 & 1/16 & 4/16 & 6/16 \end{array}\right]$$

A matrix of size 6 was shown for convenience. Actually only matrices of sizes 2, 3, and 4 are used at this point. The dotted line indicates how to obtain the matrix of size 2 from the given matrix. Similarly matrices of sizes 3 and 4 can be defined.

The terms in the right hand side of Eq. (4.8) are

$$\sum_{s=1}^{s=4} q_4^{(3)}(s,3) = [1111]\begin{bmatrix} 6/16 & 4/16 & 1/16 & 0 \\ 4/16 & 6/16 & 4/16 & 1/16 \\ 1/16 & 4/16 & 6/16 & 4/16 \\ 0 & 1/16 & 4/16 & 6/16 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 2755/4096$$

$$\sum_{s=1}^{s=3} q_3^{(3)}(s,3) = [111]\begin{bmatrix} 6/16 & 4/16 & 1/16 \\ 4/16 & 6/16 & 4/16 \\ 1/16 & 4/16 & 6/16 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1619/4096$$

$$\sum_{s=1}^{s=3} q_3^{(3)}(s,2) = [111]\begin{bmatrix} 6/16 & 4/16 & 1/16 \\ 4/16 & 6/16 & 4/16 \\ 1/16 & 4/16 & 6/16 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 2096/4096$$

$$\sum_{s=1}^{s=2} q_2^{(3)}(s,2) = [11]\begin{bmatrix} 6/16 & 4/16 \\ 4/16 & 6/16 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 1000/4096$$

and thus

$$P(M_3 = 1, |m_3| = 2) = (2755 - 1619 - 2096 + 1000)/4096 = 40/4096$$

For illustration, another probability is evaluated:

$$P(M_3 = 3, |m_2| = 2) = \sum_{s=1}^{s=6} q_6^{(3)}(s,3) - \sum_{s=1}^{s=5} q_5^{(3)}(s,3) - \sum_{s=1}^{s=5} q_5^{(3)}(s,2) + \sum_{s=1}^{s=4} q_4^{(3)}(s,2)$$

where

$$\sum_{s=1}^{s=6} q_6^{(3)}(s,3) = (\tfrac{1}{16})^3 \cdot [111111]\begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3672}{4096}$$

$$\sum_{s=1}^{s=5} q_5^{(3)}(s,3) = (\tfrac{1}{16})^3 \cdot [11111]\begin{bmatrix} 6 & 4 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{3416}{4096}$$

$$\sum_{s=1}^{s=5} q_5^{(3)}(s,2) = (\tfrac{1}{16})^3 \cdot [11111]\begin{bmatrix} 6 & 4 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3011}{4096}$$

$$\sum_{s=1}^{s=4} q_4^{(3)}(s,2) = (\tfrac{1}{16})^3 \cdot [1111]\begin{bmatrix} 6 & 4 & 1 & 0 \\ 4 & 6 & 4 & 1 \\ 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{2755}{4096}$$

and thus

$$P(M_3 = 3, |m_3| = 2) = (3672-3416-3011+2755)/4096 = 0.$$

In a similar fashion, all values shown in Table 4.1 can be easily evaluated.

TABLE 4.1 JOINT DISTRIBUTION OF M_3 AND m_3 FOR A PARTICULAR BINOMIAL NET INPUT*

$\lvert m_n\rvert$ \ M_n	0	1	2	3	4	5	6	7
0	216	784	619	252	71	12	1	0
1	784	312	40	4	0	0	0	0
2	619	40	2	0	0	0	0	0
3	252	4	0	0	0	0	0	0
4	71	0	0	0	0	0	0	0
5	12	0	0	0	0	0	0	0
6	1	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0

*Entries in the table must be divided by 4096

1.2 Distribution of the Range. Now that the joint distribution of M_n and $|m_n|$ has been found, the distribution of their sum follows directly:

$$P(R_n = k-1) = \sum_{u=1}^{u=k} P(R_n = k-1, |m_n| = u-1)$$

$$= \sum_{u=1}^{u=k} P(M_n = k-u, |m_n| = u-1)$$

because

$$P(M_n = k-u,\ |m_n| = u-1) = P(M_n + |m_n| = k-1,\ |m_n| = u-1)$$

$$= P(R_n = k-1,\ |m_n| = u-1)$$

Using Eq. (4.7),

$$P(R_n = k-1) = \sum_{u=1}^{u=k} \left(\sum_{s=1}^{s=k} q_k^{(n)}(s,u) - \sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u) - \sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u-1) + \sum_{s=1}^{s=k-2} q_{k-2}^{(n)}(s,u-1) \right)$$

or

$$P(R_n = k-1) = \sum_{u=1}^{u=k}\sum_{s=1}^{s=k} q_k^{(n)}(s,u) - \sum_{u=1}^{u=k-1}\sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u) - \sum_{u=2}^{u=k}\sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u-1) + \sum_{u=2}^{u=k-1}\sum_{s=1}^{s=k-2} q_{k-2}^{(n)}(s,u-1)$$

or, finally,

$$P(R_n = k-1) = \sum_{u=1}^{u=k}\sum_{s=1}^{s=k} q_k^{(n)}(s,u) - 2\sum_{u=1}^{u=k-1}\sum_{s=1}^{s=k-1} q_{k-1}^{(n)}(s,u) + \sum_{u=1}^{u=k-2}\sum_{s=1}^{s=k-2} q_{k-2}^{(n)}(s,u)$$

where special attention should be paid to the fact that the adjustment in the values of u in the above summations is valid.

Clearly $\sum_{u=1}^{u=j}\sum_{s=1}^{s=j} q_j^{(n)}(s,u)$ is the sum of all elements in an n-step "restricted" transition matrix of size j. Using an obvious notation,

$$P(R_n = k-1) = \lambda_k^{(n)} - 2\lambda_{k-1}^{(n)} + \lambda_{k-2}^{(n)}$$

or equivalently,

$$P(R_n = k) = \lambda_{k+1}^{(n)} - 2\lambda_k^{(n)} + \lambda_{k-1}^{(n)} \qquad (4.9)$$

where $\lambda_j^{(n)}$ is understood to be zero for $j \leq 0$.

A simple example is now given for illustration purposes.

Example 4.2

The distribution of the range of partial sums will be found in the case $n = 3$, for the same net input of Example 4.1:

From Table 4.1, it is obvious that

$P(R_3 = 0) = 216/4096$
$P(R_3 = 1) = (784 + 784)/4096 = 1568/4096$
$P(R_3 = 2) = (619 + 312 + 619)/4096 = 1550/4096$
$P(R_3 = 3) = (252 + 40 + 40 + 252)/4096 = 584/4096$
$P(R_3 = 4) = (71 + 4 + 2 + 4 + 71)/4096 = 152/4096$
$P(R_3 = 5) = (12 + 12)/4096 = 24/4096$
$P(R_3 = 6) = (1 + 1)/4096 = 2/4096$
$P(R_3 \geq 7) = 0$

Equation (4.9) is used to verify some of these results. For instance,

$$P(R_3 = 3) = \left(\tfrac{1}{16}\right)^3 [1\ 1\ 1\ 1] \begin{bmatrix} 6 & 4 & 1 & 0 \\ 4 & 6 & 4 & 1 \\ 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$- 2 \cdot \left(\tfrac{1}{16}\right)^3 [1\ 1\ 1] \begin{bmatrix} 6 & 4 & 1 \\ 4 & 6 & 4 \\ 1 & 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$+ \left(\tfrac{1}{16}\right)^3 [1\ 1] \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{9252}{4096} - 2 \cdot \frac{5334}{4096} + \frac{2000}{4096} = \frac{584}{4096}$$

and

$$P(R_3 = 1) = \left(\tfrac{1}{16}\right)^3 [1\ 1] \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-2 \cdot \left(\tfrac{6}{16}\right)^3 + 0 = \frac{2000}{4096} - 2 \cdot \frac{216}{4096} + 0 = \frac{1568}{4096}$$

and, of course,

$$P(R_3 = 0) = \left(\tfrac{6}{16}\right)^3 - 2 \cdot 0 + 0 = \frac{216}{4096}$$

Although the above example was given for independent and symmetric net input with zero expectation, Eq. (4.9) holds in general. As a matter of fact, it holds even for dependent inputs, but in this case the n-step transition matrix is not simply the n-th power of the one-step transition matrix (see Chapter III, Section 1.3).

Using Eq. (4.9), the cumulative distribution function (c.d.f.) of the range can be readily obtained as

$$P(R_n \le k) = P(R_n = 0) + P(R_n = 1) + P(R_n = 2)$$
$$+ \dots + P(R_n = k)$$
$$= \lambda_1^{(n)} + (\lambda_2^{(n)} - 2\lambda_1^{(n)}) + (\lambda_3^{(n)} - 2\lambda_2^{(n)}$$
$$+ \lambda_1^{(n)}) + \dots + (\lambda_{k+1}^{(n)} - 2\lambda_k^{(n)} + \lambda_{k-1}^{(n)})$$
$$= \lambda_{k+1}^{(n)} - \lambda_k^{(n)} . \tag{4.10}$$

Thus it is clear that there exists a value K sufficiently large so that $P(R_n = K) = 0$ and

$$P(R_n \le K) = \lambda_{K+1}^{(n)} - \lambda_K^{(n)} = 1$$

or equivalently,

$$\lambda_{K+1}^{(n)} = 1 + \lambda_K^{(n)} . \tag{4.11}$$

The m·th moment of the range is given by

$$E(R_n^m) = P(R_n = 1) + 2^m \cdot P(R_n = 2)$$
$$+ 3^m \cdot P(R_n = 3) + \dots$$

For $m = 1$, the expectation of the range is

$$E(R_n) = P(R_n = 1) + 2 \cdot P(R_n = 2) + 3 \cdot P(R_n = 3) + \dots$$

or, using a large value K such that $P(R_n = K) = 0$ and $P(R_n \le K) = 1$,

$$E(R_n) = P(R_n = 1) + 2 \cdot P(R_n = 2) + 3\, P(R_n = 3)$$
$$+ \dots + K \cdot P(R_n = K)$$

Using Eq. (4.9), the mean range simplifies to

$$E(R_n) = K\lambda_{K+1}^{(n)} - (K+1)\,\lambda_K^{(n)}$$

But, from Eq. (4.11), $\lambda_{K+1}^{(n)} = 1 + \lambda_K^{(n)}$ and thus

$$E(R_n) = K - \lambda_K^{(n)} . \tag{4.12}$$

Similarly, the second moment of the range is

$$E(R_n^2) = P(R_n = 1) + 4 \cdot P(R_n = 2)$$
$$+ 9 \cdot P(R_n = 3) + \dots + K^2 \cdot P(R_n = K)$$
$$= 2\lambda_1^{(n)} + 2\lambda_2^{(n)} + 2\lambda_3^{(n)}$$
$$+ \dots + 2\lambda_{K-1}^{(n)} + K^2 - (2K - 1)\,\lambda_K^{(n)}$$

where Eqs. (4.9) and (4.11) have been used.

After some elementary manipulations, the second moment of the range can be rewritten as

$$E(R_n^2) = E(R_n) + 2 \sum_{k=1}^{K-1} [E(R_n) - (k - \lambda_k^{(n)})] \tag{4.13}$$

Some useful relationships can now be derived. The relationship between Q'^n and Q^n has been shown in Eq. (4.6). But Q'^n is a transition matrix and thus the elements of each column add to unity. Consequently, the sum of all elements in Q'^n is equal to its size, namely, k+2, and the following relationship holds:

$$u^T (I + Q + Q^2 + \dots + Q^{n-1})\, \underline{1} + \underline{1}^T Q^n\, \underline{1}$$
$$+ \ell^T (I + Q + Q^2 + \dots + Q^{n-1})\, \underline{1} = k$$

where $\underline{1}$ is a column vector of size k with all elements equal to 1, and $\underline{1}^T Q^n \underline{1}$ is clearly $\lambda_k^{(n)}$.

Using the index k to emphasize that the vectors and matrices involved have size k, one has

$$\lambda_k^{(n)} = k - (u_k^T + \ell_k^T)(I_k + Q_k + Q_k^2 + \dots + Q_k^{n-1})\, \underline{1}_k . \tag{4.14}$$

Making $k = K$ (large) in Eq. (4.14), and using Eq. (4.12),

$$E(R_n) = K - \lambda_K^{(n)} = (u_K^T + \ell_K^T)(I_K + Q_K + \dots + Q_K^{n-1})\underline{1}_K$$

or, equivalently,

$$E(R_n) - E(R_{n-1}) = (u_K^T + \ell_K^T)\, Q_K^{n-1}\, \underline{1}_K . \tag{4.15}$$

Equation (4.15) indicates that it is easier to study the difference between consecutive values of the mean range than to study the mean range itself. This conclusion is apparent also from Spitzer's result (1956).

For symmetric inputs with zero expectation Eq. (4.15) simplifies to

$$E(R_n) - E(R_{n-1}') = 2\, u_K^T\, Q_K^{n-1}\, \underline{1}_K = 2\, \ell_K^T\, Q_K^{n-1}\, \underline{1}_K . \tag{4.16}$$

Using Eqs. (4.13) and (4.14), similar results can be found for the second moment:

$$E(R_n^2) - E(R_{n-1}^2) = (u_K^T + \ell_K^T) Q_K^{n-1}\, \underline{1}_K + 2 \sum_{k=1}^{K-1} [(u_K^T + \ell_K^T)\, Q_K^{n-1}\, \underline{1}_K - (u_k^T + \ell_k^T) Q_k^{n-1}\, \underline{1}_k \tag{4.17}$$

and for symmetric inputs with zero expectation

$$E(R_n^2) - E(R_{n-1}^2) = 2\ell_K^T\, Q_K^{n-1}\, \underline{1}_K$$
$$+ 4 \sum_{k=1}^{k=K-1} [\ell_K^T\, Q_K^{n-1}\, \underline{1}_K - \ell_k^T\, Q_k^{n-1}\, \underline{1}_k] \tag{4.18}$$

1.3 <u>Range Analysis for the Random Walk Process</u>. In this particular case, the net input X_t is such that

$$P(X_t = i) = 1/2 \qquad (i = -1, +1)$$

and thus,

$$E(X_t) = 0, \text{ and } \operatorname{var}(X_t) = 1 .$$

Some interesting results can be derived to be used later in this paper.

The one-step "restricted" transition matrix in this case is

$$Q = \begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 & 0 & 0 \\ 0 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \cdots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 & 0 \end{bmatrix}$$

and the n-step "restricted" transition matrix can be found by the method of images, as shown in Chapter III. Thus, the (s,u) entry in the n·th power of the k by k matrix Q is

$$q_k^{(n)}(s,u) = \sum_{j=-\infty}^{j=+\infty} \{v_n[2j(k+1) + u,s] - v_n[2j(k+1) - u,s]\} \qquad (4.19)$$

where

$$v_n(r,t) = {}_nC_{(n+r-t)/2} \cdot (1/2)^n = {}_nC_{(n+t-r)/2} \cdot (1/2)^n .$$

The probability distribution function of R_n is

$$P(R_n = k) = \lambda_{k+1}^{(n)} - 2\lambda_k^{(n)} + \lambda_{k-1}^{(n)}$$

where

$$\lambda_k^{(n)} = \sum_{u=1}^{u=k} \sum_{s=1}^{s=k} q_k^{(n)}(s,u)$$

$$= \sum_{u=1}^{u=k} \sum_{s=1}^{s=k} \sum_{j=-\infty}^{j=+\infty} \{v_n[2j(k+1) + u,s] - v_n[2j(k+1) - u,s]\}$$

with $\lambda_{k+1}^{(n)}$ and $\lambda_{k-1}^{(n)}$ similarly defined.

To find the mean value of the range, Eq. (4.16), slightly modified, can be used:

$$E(R_{n+1}) - E(R_n) = 2\, \ell_K^T\, Q_K^n\, \underline{1}_K .$$

Recalling that $\ell_K^T = [1/2\ 0\ 0\ \ldots\ 0\ 0]$, it follows that $[E(R_{n+1}) - E(R_n)]$ is simply the sum of all elements in the first row of the matrix Q_K^n. But the matrix is symmetric and thus the first row is equal to the first column and then

$$E(R_{n+1}) - E(R_n) = \sum_{s=1}^{s=K} q_K^{(n)}(s,1) \qquad (4.20)$$

For general k, $2\,\ell_k^T\, Q_k^n\, \underline{1}_k = \sum_{s=1}^{s=k} q_k^{(n)}(s,1)$ can be found using Eq. (4.19):

$$\sum_{s=1}^{s=k} q_k^{(n)}(s,1) = \sum_{s=1}^{s=k} \sum_{j=-\infty}^{j=+\infty} \{v_n[2j(k+1) + 1,s] - v_n[2j(k+1) - 1,s]\}$$

$$= \sum_{j=-\infty}^{j=+\infty} \{v_n[2j(k+1) + 1,1] + v_n[2j(k+1) + 1,2] - v_n[2j(k+1) - 1, k-1] - v_n[2j(k+1) - 1,k]\} \qquad (4.21)$$

In particular, when $k = K$ (recall that K is a very large number), it follows from the expression defining $v_n(r,t)$ that the only nonzero values in Eq. (4.21) are $v_n[2j(K+1) + 1,1]$ and $v_n[2j(K+1) + 1,2]$ for the particular value $j = 0$. Thus, Eq. (4.20) becomes

$$E(R_{n+1}) - E(R_n) = v_n(1,1) + v_n(1,2)$$

$$= {}_nC_{n/2} \cdot (1/2)^n + {}_nC_{(n-1)/2} \cdot (1/2)^n \qquad (4.22)$$

where only one of the terms at the right hand side is nonzero, depending upon n being odd or even.

Similarly, to find the second moment of the range, Eq. (4.18), slightly modified, can be used:

$$E(R_{n+1}^2) - E(R_n^2) = 2\ell_K^T\, Q_K^n\, \underline{1}_K + 2 \sum_{k=1}^{k=K-1} [2\, \ell_K^T\, Q_K^n\, \underline{1}_K - 2\ell_k^T Q_k^n\, \underline{1}_k]$$

where $2\ell_k^T\, Q_k^n\, \underline{1}_k$ is given by Eq. (4.21) and $2\ell_K^T\, Q_K^n\, \underline{1}_K$ has been shown to equal ${}_nC_{n/2} \cdot (1/2)^n + {}_nC_{(n-1)/2}(1/2)^n = v_n(1,1) + v_n(1,2)$.

Thus,

$$E(R_{n+1}^2) - E(R_n^2) = v_n(1,1) + v_n(1,2) + 2 \sum_{k=1}^{k=K-1} \{v_n(1,1) + v_n(1,2) - \sum_{j=-\infty}^{+\infty} [v_n(2j(k+1) + 1,1) + v_n(2j(k+1) + 1,2) - v_n(2j(k+1) - 1, k-1) - v_n(2j(k+1) - 1, k)]\} \qquad (4.23)$$

where $v_n(r,t) = {}_nC_{(n+r-t)/2}(1/2)^n = {}_nC_{(n+t-r)/2}(1/2)^n$. Equations (4.22) and (4.23), used recursively, furnish the mean range and the second moment of the range. Table 4.2 summarizes the values of $E(R_n)$, $E(R_n^2)$ and $var(R_n)$ for $n = 1,2,\ldots, 100$.

Taking into account that the random walk process is the simplest possible discrete input, it should be clear by now that to obtain results explicitly (especially the second moment of the range) is not an easy task. Even when such results are found, as it was just done for the random walk process, final equations may be so complicated that one would be better off using directly the more general results (Eq. (4.9) for the distribution of the range, and Eqs. (4.15) and (4.17) for the first two moments), solving the problem numerically.

1.4 Closing remarks. In this section, a general approach was described to obtained the joint distribution of the maximum and minimum of partial sums of discrete, identically distributed, independent random variables (Eq. (4.7)). With minor modifications, it will be shown that the approach holds for dependent random variables as well.

From the joint distribution of M_n and m_n, the distribution of the range (and consequently, its moments) was found (Eqs. (4.19), (4.15) and (4.17)).

The approach described was applied to the simplest possible input, to illustrate that the usefulness of some results in closed form may be questionable.

A final remark can be made, having to do with the interpretation of the i·th element ($i = 1,2,\ldots, k$)

TABLE 4.2 MOMENTS OF THE RANGE FOR THE RANDOM WALK PROCESS

n	E(R)	$E(R^2)$	VAR(R_n)	n	E(R)	$E(R^2)$	VAR(R_n)
1	1.0000	1.0000	0.0000	51	10.4521	120.5936	11.3479
2	1.5000	2.5000	0.2500	52	10.5622	123.1363	11.5766
3	2.0000	4.5000	0.5000	53	10.6723	125.6982	11.8002
4	2.3750	6.3750	0.7344	54	10.7804	128.2453	12.0288
5	2.7500	8.5000	0.9375	55	10.8885	130.8109	12.2524
6	3.0625	10.5625	1.1836	56	10.9946	133.3622	12.4810
7	3.3750	12.7812	1.3906	57	11.1007	135.9313	12.7047
8	3.6484	14.9453	1.6342	58	11.2051	138.4866	12.9331
9	3.9219	17.2266	1.8455	59	11.3094	141.0590	13.1570
10	4.1680	19.4570	2.0851	60	11.4120	143.6181	13.3853
11	4.4141	21.7832	2.2993	61	11.5145	146.1938	13.6092
12	4.6396	24.0625	2.5362	62	11.6155	148.7564	13.8375
13	4.8652	26.4229	2.7523	63	11.7164	151.3351	14.0615
14	5.0747	28.7402	2.9876	64	11.8157	153.9012	14.2897
15	5.2842	31.1277	3.2051	65	11.9151	156.4828	14.5137
16	5.4806	33.4758	3.4393	66	12.0129	159.0521	14.7419
17	5.6769	35.8854	3.6578	67	12.1108	161.6365	14.9660
18	5.8624	38.2590	3.8911	68	12.2072	164.2089	15.1941
19	6.0479	40.6872	4.1103	69	12.3036	166.7960	15.4182
20	6.2241	43.0822	4.3430	70	12.3986	169.3713	15.6463
21	6.4003	45.5264	4.5629	71	12.4936	171.9609	15.8705
22	6.5685	47.9397	4.7950	72	12.5873	174.5391	16.0985
23	6.7367	50.3978	5.0153	73	12.6810	177.1312	16.3227
24	6.8978	52.8271	5.2470	74	12.7735	179.7121	16.5506
25	7.0590	55.2974	5.4678	75	12.8659	182.3065	16.7749
26	7.2140	57.7407	5.6990	76	12.9571	184.8901	17.0028
27	7.3690	60.2219	5.9202	77	13.0484	187.4867	17.2272
28	7.5184	62.6777	6.1510	78	13.1384	190.0728	17.4550
29	7.6679	65.1687	6.3725	79	13.2285	192.6716	17.6794
30	7.8123	67.6356	6.6031	80	13.3174	195.2601	17.9072
31	7.9568	70.1355	6.8249	81	13.4063	197.8610	18.1317
32	8.0967	72.6125	7.0552	82	13.4942	200.4518	18.3594
33	8.2367	75.1204	7.2772	83	13.5820	203.0547	18.5839
34	8.3725	77.6065	7.5073	84	13.6688	205.6477	18.8116
35	8.5084	80.1218	7.7296	85	13.7556	208.2526	19.0361
36	8.6404	82.6163	7.9594	86	13.8414	210.8478	19.2638
37	8.7725	85.1384	8.1819	87	13.9272	213.4546	19.4884
38	8.9011	87.6406	8.4115	88	14.0120	216.0519	19.7160
39	9.0297	90.1689	8.6342	89	14.0968	218.6605	19.9406
40	9.1550	92.6782	8.8637	90	14.1807	221.2597	20.1682
41	9.2804	95.2123	9.0865	91	14.2645	223.8701	20.3929
42	9.4028	97.7281	9.3158	92	14.3475	226.4713	20.6204
43	9.5252	100.2676	9.5388	93	14.4305	229.0834	20.8451
44	9.6448	102.7896	9.7680	94	14.5125	231.6865	21.0726
45	9.7644	105.3341	9.9911	95	14.5946	234.3002	21.2973
46	9.8814	107.8618	10.2201	96	14.6758	236.9051	21.5249
47	9.9984	110.4110	10.4434	97	14.7571	239.5204	21.7496
48	10.1130	112.9440	10.6723	98	14.8375	242.1271	21.9771
49	10.2275	115.4977	10.8956	99	14.9178	244.7440	22.2018
50	10.3398	118.0357	11.1244	100	14.9974	247.3524	22.4293

in the vector $u_k^T Q_k^{n-1}$. From Eq. (4.6), it is clear that the i·th element in the vector $u_k^T (I_k + Q_k + Q_k^2 + \ldots + Q_k^{n-1})$ is the probability that the system is at state (k+1), at discrete time n, given that the initial state was i. The i·th element in the vector $u_k^T (I_k + Q_k + Q_k^2 + \ldots + Q_k^{n-2})$ can be similarly interpreted and consequently, the i·th element in the vector $u_k^T Q_k^{n-1}$ is the probability that the system, starting at state i, reaches state (k+1) in exactly n steps, for the first time, without ever passing through state zero. Using the jargon of the followers of Moran, this is the probability of first overflow occurring at time n, before emptiness, given the initial state i of a finite reservoir of size (k+1).

Similarly, the i·th element in the vector $\ell_k^T Q_k^{n-1}$ is the probability of first emptiness occurring at time n, before overflow, given the initial state i of a finite reservoir of size (k+1).

Equation (4.17) indicates that the second moment of the range can be written in terms of finite reservoir concepts like the probabilities of first emptiness before overflow and of first overflow before emptiness.

2. Continuous Net Inputs

For illustration purposes, a convenient approach to the range analysis for continuous inputs is to start with a convenient "discretization" of input, to find the solution for this discrete case, and to impose the conditions under which this solution tends to the solution of the continuous case.

Normally distributed net inputs are studied first. The distribution of R_1 and R_2 are derived and the distribution of R_3 is shown to depend on integrals that do not exist in closed form, thus suggesting that numerical evaluation is unavoidable. Actually, even the distribution of R_2 depends on an integral that does not exist in closed form, namely, the cumulative distribution function (c.d.f.) of the normal distribution, which is, of course, tabulated.

The second type of net input studied is the Laplace distributed input, because the integrals involved can be easily evaluated and results in closed form are obtainable.

Finally, exponentially distributed inputs are studied, to illustrate that for moderately large values of n, the type of input is relatively unimportant. The exponential distribution is chosen as a drastic departure from normality. The case of gamma distributions which are important in practice falls between the exponential case and the normal case.

2.1 Normally Distributed Net Inputs. Consider the following binomially distributed net input, for m even:

$$P(X_t = i) = P_i = {}_mC_{(\frac{m}{2}+i)} \ (1/2)^m \qquad (4.24)$$

for $i = -\frac{m}{2}, -\frac{m}{2}+1, \ldots, 0, \ldots, \frac{m}{2}-1, \frac{m}{2}$.

Notice that $E(X_t) = 0$ and $var(X_t) = m/4$. Furthermore, notice that the distribution is symmetric, i.e., $p_i = p_{-i}$. Thus the one-step "restricted" transition matrix can be written as

$$Q = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_{k-3} & p_{k-2} & p_{k-1} \\ p_1 & p_0 & p_1 & \cdots & p_{k-4} & p_{k-3} & p_{k-2} \\ p_2 & p_1 & p_0 & \cdots & p_{k-5} & p_{k-4} & p_{k-3} \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ p_{k-3} & p_{k-4} & p_{k-5} & \cdots & p_0 & p_1 & p_2 \\ p_{k-2} & p_{k-3} & p_{k-4} & \cdots & p_1 & p_0 & p_1 \\ p_{k-1} & p_{k-2} & p_{k-3} & \cdots & p_2 & p_1 & p_0 \end{bmatrix}$$

Using Eq. (4.9), for n = 1, and using the symbol ' to denote the discrete range,

$$P(R_1' = k) = \lambda_{k+1} - 2\lambda_k + \lambda_{k-1} ,$$

Recalling that λ_{k+1}, λ_k and λ_{k-1} are the sums of all elements in the one-step "restricted" transition matrices of sizes k+1, k and k - 1, respectively, one has

$$P(R_1' = k) = (\lambda_{k+1} - \lambda_k) - (\lambda_k - \lambda_{k-1}) = 2\ p_k$$

or

$$P(R_1' = k) = 2\ {}_mC_{(\frac{m}{2}+k)} \ (1/2)^m$$

For large m, the normal approximation to the binomial distribution can be used and

$$P(R_1' = k) \approx \frac{4}{\sqrt{m}} \phi\left(\frac{2k}{\sqrt{m}}\right)$$

where $\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ is the density function of the normal distribution.

It is convenient to express the range in units of the standard deviation of the net input, and thus

$$P\left(\frac{2R_1'}{\sqrt{m}} = \frac{2k}{\sqrt{m}}\right) \approx \frac{4}{\sqrt{m}} \phi\left(\frac{2k}{\sqrt{m}}\right) .$$

Changing variables $y = \frac{2k}{\sqrt{m}}$ and $R_1 = \frac{2R_1'}{\sqrt{m}}$ and then taking the limit as $m \to \infty$,

$$f_{R_1}(y) = \frac{4}{\sqrt{m}} \cdot \phi(y) \cdot \frac{\sqrt{m}}{2} = 2\phi\ (y) \qquad (4.25)$$

Now moments can be easily evaluated:

$$E(R_1) = \int_0^\infty 2y\phi(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty y\ e^{-1/2\ y^2} dy = \sqrt{\frac{2}{\pi}}$$

and

$$E(R_1^2) = \int_0^\infty 2\, y^2 \phi(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty y^2 e^{-1/2\, y^2} dy = 1$$

Equation (4.25) is an already known result: the probability density function (p.d.f.) of the range for $n = 1$ is equal to the p.d.f. of the absolute value of the net input.

To find the distribution of R_2', Eq. (4.9) is used again:

$$P(R_2' = k) = \lambda_{k+1}^{(2)} - 2\lambda_k^{(2)} + \lambda_{k-1}^{(2)} .$$

$\lambda_k^{(2)}$ can be written as $\underline{1}^T Q_k^2 \underline{1}$ where $\underline{1}$ is a column vector of size k with all elements equal to unity, and $\underline{1}^T$ is its transpose:

$$\lambda_k^{(2)} = \underline{1}^T Q_k^2 \underline{1} = (\underline{1}^T Q_k)(\underline{1}^T Q_k)^T$$

and

$$\underline{1}^T Q_k = [s_1\ s_2 \ldots\ldots s_{k-1}\ s_k]$$

where s_i is the sum of elements in the i·th column of Q_k.

Then,

$$\lambda_k^{(2)} = \sum_{i=1}^{i=k} s_i^2 .$$

Similarly,

$$\lambda_{k+1}^{(2)} = \underline{1}^T Q_{k+1}^2 \underline{1} = (\underline{1}^T Q_{k+1})(\underline{1}^T Q_{k+1})^T$$

and

$$\underline{1}^T Q_{k+1} = [s_1+p_k\ s_2+p_{k-1} \ldots s_{k-1}+p_2\ s_k+p_1\ s_{k+1}]$$

where s_i $(i = 1,2,\ldots, k)$ is the sum of elements in the i·th column of Q_k and s_{k+1} is the sum of elements in the last column of Q_{k+1}.

Then,

$$\lambda_{k+1}^{(2)} = \sum_{i=1}^{i=k} (s_i + p_{k+1-i})^2 + s_{k+1}^2 .$$

Similarly,

$$\lambda_{k-1}^{(2)} = \underline{1}^T Q_{k-1}^2 \underline{1} = (\underline{1}^T Q_{k-1})(\underline{1}^T Q_{k-1})^T$$

and

$$\underline{1}^T Q_{k-1} = [s_1 - p_{k-1}\ s_2 - p_{k-2} \ldots\ldots s_{k-1} - p_1]$$

where the s_i's have been defined previously.

Then

$$\lambda_{k-1}^{(2)} = \sum_{i=1}^{i=k-1} (s_i - p_{k-i})^2$$

and finally

$$\begin{aligned} P(R_2' = k) &= \lambda_{k+1}^{(2)} - 2\cdot\lambda_k^{(2)} + \lambda_{k-1}^{(2)} \\ &= s_{k+1}^2 + \sum_{i=1}^{i=k} (s_i + p_{k+1-i})^2 \\ &\quad - 2 \sum_{i=1}^{i=k} s_i^2 + \sum_{i=1}^{i=k-1} (s_i - p_{k-i})^2 . \end{aligned}$$

After some elementary transformations, this equation can be rewritten as

$$P(R_2' = k) = 4\, p_k \left(\frac{p_0}{2} + p_1 + \ldots + p_{k-1} + \frac{p_k}{2}\right) + 2\left(\frac{p_0 p_k}{2} + p_1 p_{k-1} + \ldots + p_{k-1} p_1 + \frac{p_k p_0}{2}\right) \quad (4.26)$$

Proceeding as before, the range is expressed in units of the standard deviation of the input, the normal approximation to the binomial distribution is introduced, a convenient change of variables is made and the limit as $m \to \infty$ is considered. The result is, then,

$$f_{R_2}(y) = 4\phi(y) \int_0^y \phi(u)\, du + 2 \int_0^y \phi(u)\phi(y-u)\, du$$

where

$$\phi(y) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2} y^2},$$

$$\int_0^y \phi(u) du = \int_{-\infty}^y \phi(u)\, du - \int_{-\infty}^0 \phi(u)\, du = \Phi(y) - 1/2,$$

and

$$\begin{aligned} \int_0^y \phi(u)\phi(y-u) du &= \int_0^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-u)^2} du \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4}y^2} \int_0^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{2}u - y/\sqrt{2})^2} du \\ &= \frac{1}{\sqrt{2\pi}}\, \phi(y/\sqrt{2}) \int_{-y/\sqrt{2}}^{+y/\sqrt{2}} \phi(w)\, dw \\ &= \frac{1}{\sqrt{2}}\, \phi(y/\sqrt{2})\, [\Phi(y/\sqrt{2}) - \Phi(-y/\sqrt{2})] \\ &= \sqrt{2}\, \phi(y/\sqrt{2})\, [\Phi(y/\sqrt{2}) - 1/2]. \end{aligned}$$

Finally, one has

$$f_{R_2}(y) = 4\phi(y)\, [\Phi(y) - 1/2] + 2\sqrt{2}\, \phi(y/\sqrt{2})\, [\Phi(y/\sqrt{2}) - 1/2]. \quad (4.27)$$

For completeness, the first two moments of R_2 will be derived, using Eq. (4.27):

$$E(R_2) = \int_0^\infty 4y\phi(y)\Phi(y)\,dy - \int_0^\infty 2y\phi(y)\,dy$$

$$+ \int_0^\infty 2\sqrt{2}\,y\phi(y/\sqrt{2})\Phi(y/\sqrt{2})\,dy - \int_0^\infty \sqrt{2}\,y\phi(y/\sqrt{2})\,dy$$

$$E(R_2) = (4 + 4\sqrt{2}) \int_0^\infty y\phi(y)\Phi(y)\,dy - (2 + 2\sqrt{2}) \int_0^\infty y\phi(y)dy$$

$$= (4 + 4\sqrt{2}) \cdot \frac{1}{2\sqrt{2\pi}} (1 + \frac{1}{\sqrt{2}}) - (2 + 2\sqrt{2}) \cdot \frac{1}{\sqrt{2\pi}}$$

or

$$E(R_2) = \sqrt{\frac{2}{\pi}}\ (1 + \frac{1}{\sqrt{2}}),$$

which agrees with Anis and Lloyd (1953).

Similarly,

$$E(R_2^2) = \int_0^\infty 4\ y^2\phi(y)\Phi(y)\ dy - \int_0^\infty 2\ y^2\phi(y)\ dy$$

$$+ \int_0^\infty 2\sqrt{2}\ y^2\ \phi(y/\sqrt{2})\Phi(y/\sqrt{2})\ dy - \int_0^\infty \sqrt{2}\ y^2\phi(y/\sqrt{2}\ )dy$$

$$= (4+8) \int_0^\infty y^2\phi(y)\Phi(y)\ dy - (2+4)\int_0^\infty y^2\phi(y)\ dy$$

where

$$\int_0^\infty y^2\ \phi(y)\ dy = 1/2$$

and

$$\int_0^\infty y^2\ \phi(y)\Phi(y)\ dy = \frac{3}{8} + \frac{1}{4\pi},$$

so that

$$E(R_2^2) = \frac{9}{2} + \frac{3}{\pi} - 3 = \frac{3}{2} + \frac{3}{\pi}.$$

The purpose of the derivation as presented here is to emphasize similarities between the discrete and the continuous cases, because the former can be used as a numerical integration algorithm to solve the latter.

It is obvious from Eq. (4.27) that to evaluate $f_{R_2}(y)$ for a particular $y = a$ one has to pick the values of $\Phi(a)$ and $\Phi(a/\sqrt{2})$ from tables, because the integral $\int_{-\infty}^{a} \phi(y)\ dy$ does not exist in closed form. Consequently, in this sense, closed form solutions for the range of partial sums of normal variables do not exist, except for $n = 1$. This fact can be once more illustrated by studying the range for $n = 3$.

$$P(R_3 = k) = \lambda_{k+1}^{(3)} - 2\lambda_k^{(3)} + \lambda_{k-1}^{(3)}.$$

As before, the solution consists in writing:

$$\lambda_{k+1}^{(3)} = (\underline{1}^T\ Q_{k+1}) \cdot Q_{k+1}^{\cdot}(\underline{1}^T\ Q_{k+1})^T,$$

$$\lambda_k^{(3)} = (\underline{1}^T\ Q_k) \cdot Q_k^{\cdot}\ (\underline{1}^T\ Q_k)^T,$$

and
$$\lambda_{k-1}^{(3)} = (\underline{1}^T\ Q_{k-1}) \cdot Q_{k-1}\cdot(\underline{1}^T\ Q_{k-1})^T$$

There is a considerable amount of algebraic work involved. After simplifications, the range is expressed in units of the standard deviation of the input, the normal approximation is introduced, a change of variables is performed and the limit as $m \to \infty$ is taken. The final result is

$$f_{R_3}(y) = \phi(y) + 4\phi(y)[\Phi(y) - 1/2]^2$$

$$+ 4\phi(y)\ \Phi(y/\sqrt{2})[\Phi(y/\sqrt{2}) - 1]$$

$$+ 4\sqrt{2}\phi(y/\sqrt{2})[\Phi(y) - 1/2][\Phi(y/\sqrt{2}) - 1/2]$$

$$+ \sqrt{2}\phi(y/\sqrt{3})T(y)$$

where

$$\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}$$

$$\Phi(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\ du$$

$$T(v) = \int_0^v \phi(\sqrt{\frac{3}{2}}\,u - \sqrt{\frac{2}{3}}\,v)[\Phi(\sqrt{2}\ v-u/\sqrt{2}) - \Phi(-u/\sqrt{2})]\ du.$$

Clearly both $\Phi(v)$ and $T(v)$ do not exist in closed forms. Furthermore, to the knowledge of this writer, $T(y)$ has not been previously tabulated. Thus, the next step would be to tabulate such a function, solving the integral by numerical methods. But this would solve only the case $n = 3$. For $n = 4$, a new integral would be introduced, which would also have to be tabulated, and so on.

An obvious alternative approach is to solve the whole function $f_{R_n}(y)$ numerically, rather than solving numerically only parts of it, such as $T(y)$ in $f_{R_3}(y)$. This can be done by using a binomial input such as the one given by Eq. (4.24), for a large value of m.

The selection of m is tantamount to the selection of the increment Δy in a conventional numerical integration algorithm. It can be shown that in order to obtain a specified accuracy over a wide range of values of n, m can be chosen inversely proportional to n. Then, it is clear that there exists a value of n sufficiently large so that m can be very small. Thus, for large n, even the simple random walk is a good approximation, and that was essentially Feller's approach (1951) to the asymptotic distribution of the range of partial sums of independent random variables.

In this paper, the value $m = 100$ was selected. The numerical results for $n = 2$ were compared with the exact results from Eq. (4.27), and the accuracy was considered satisfactory. The value $m = 100$ was kept constant and n varied from 2 to 50. Consequently, the accuracy, which was satisfactory for $n = 2$, increased with the increase of n.

The program for this numerical evaluation is extremely simple, because the matrix Q is obviously patterned, and thus it can be represented simply by a vector. Instead of powers of Q the program computes vectors $\underline{1}^T Q^n$ and thus computer memory requirements are minimal.

Figures 4.1, 4.2 and 4.3 shown the probability density function of the range for $n = 2$ and 3, $n = 4$ and 5, and $n = 6$ and 7, respectively. Figure 4.4 shows the density of the range for $n = 8$ and compares some of the density functions shown in previous figures.

Figure 4.5 shows the probability density function of the range for normal net inputs and $n = 2$, $n = 8$ and $n = 50$, as compared with the asymptotic density found by Feller (1951). A change of variables was necessary to make such comparison: the range is expressed in units of $\sqrt{n}$.

It is interesting to study the distribution of the standardized range, i.e., the distribution of

$$\frac{R_n - E(R_n)}{[\mathrm{Var}(R_n)]^{1/2}} .$$

This was done for $n = 2,3,\ldots, 6$ and for the asymptotic range as well. In Figs. 4.6, 4.7, 4.8, 4.9, and 4.10, the standardized asymptotic density function is compared to the standardized density function for $n = 2$, 3, 4, 5, and 6, respectively. The conclusion is that the asymptotic result is a remarkably good approximation even for n as low as 2, when the influence of the first and second moments has been removed. For $n = 6$ the standardized exact density and the standardized asymptotic density are practically identical.

Consequently, if one desires to have a result in closed form for the density of the range, in the case of normal net inputs, it suffices to correct the asymptotic result for the exact mean and variance. The exact mean is known (Anis and Lloyd, 1953) to be

$$E[R_n] = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{i=n} i^{-1/2}$$

and the exact second moment obtained numerically is shown in Fig. 4.11, for $n = 1,2,3,\ldots, 50$.

2.2 Laplace Distributed Net Inputs. The Laplace distribution is also called the double exponential distribution, or the first law of errors. Its density function is:

$$f_X(x) = \frac{\sqrt{2}}{2\sigma} e^{-\frac{\sqrt{2}}{\sigma}|x-\mu|} \qquad (-\infty<x<+\infty)$$

where $E[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$ and $|x-\mu|$ denotes the absolute value of $(x-\mu)$.

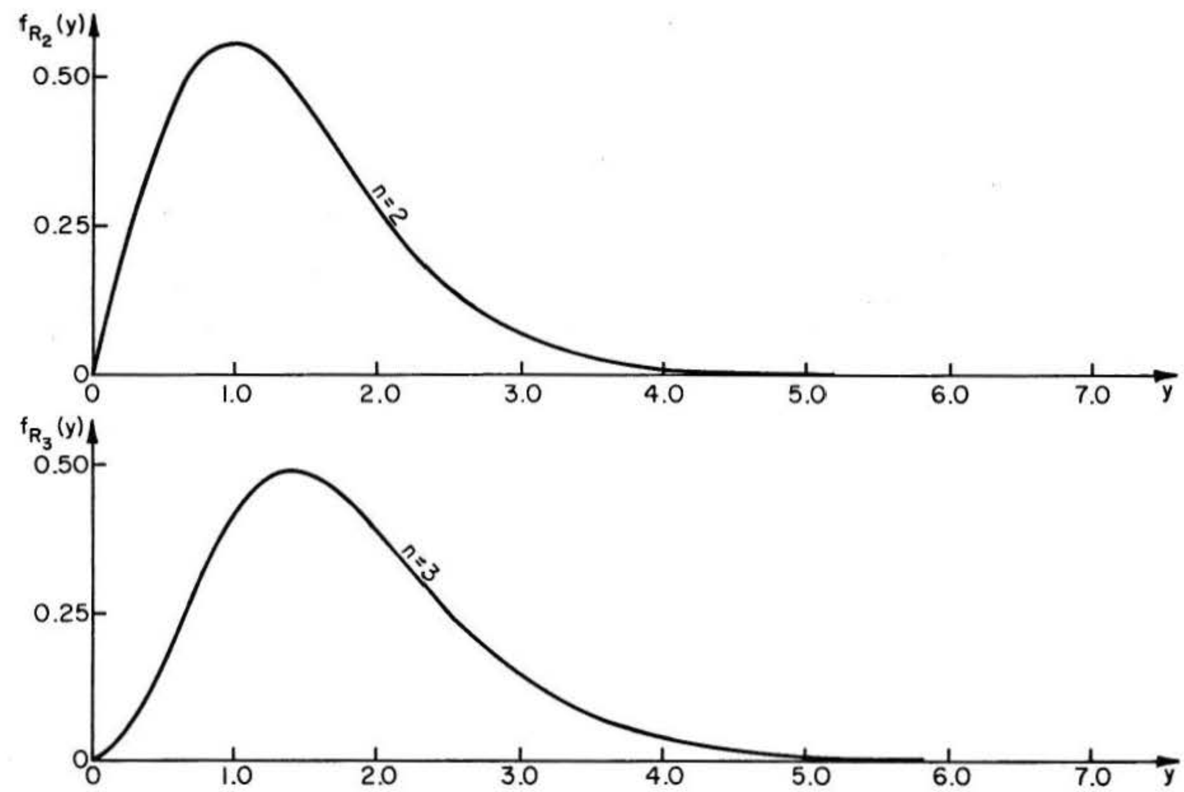

Fig. 4.1. Distribution of R_n for independent normal net inputs (n=2 and n=3).

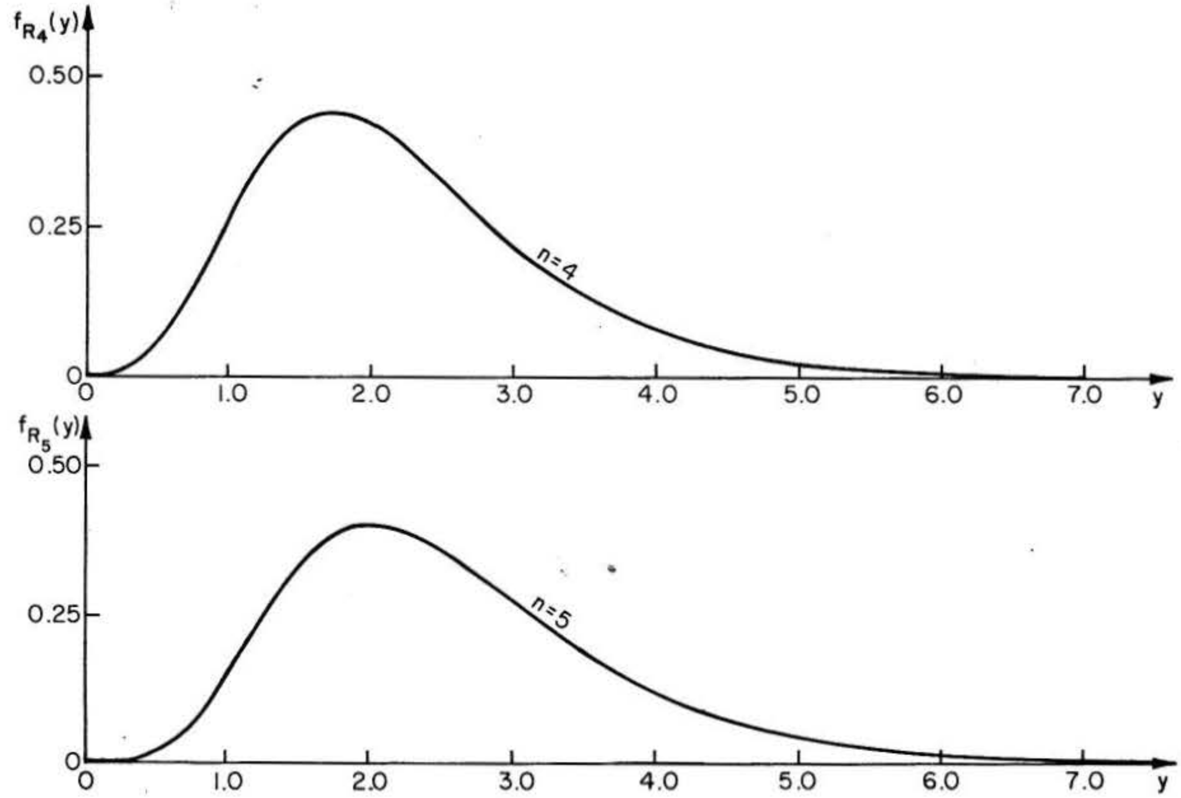

Fig. 4.2. Distribution of R_n for independent normal net inputs (n=4 and n=5).

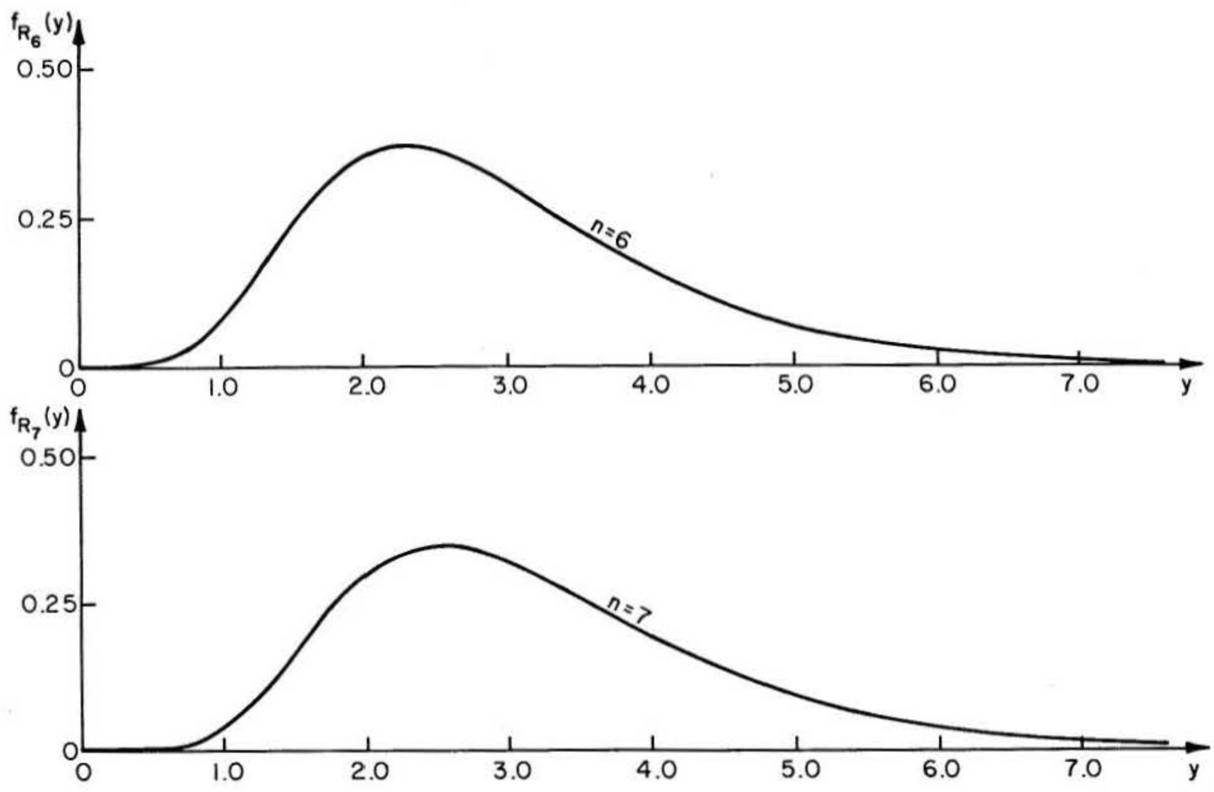

Fig. 4.3. Distribution of R_n for independent normal net inputs (n=6 and n=7).

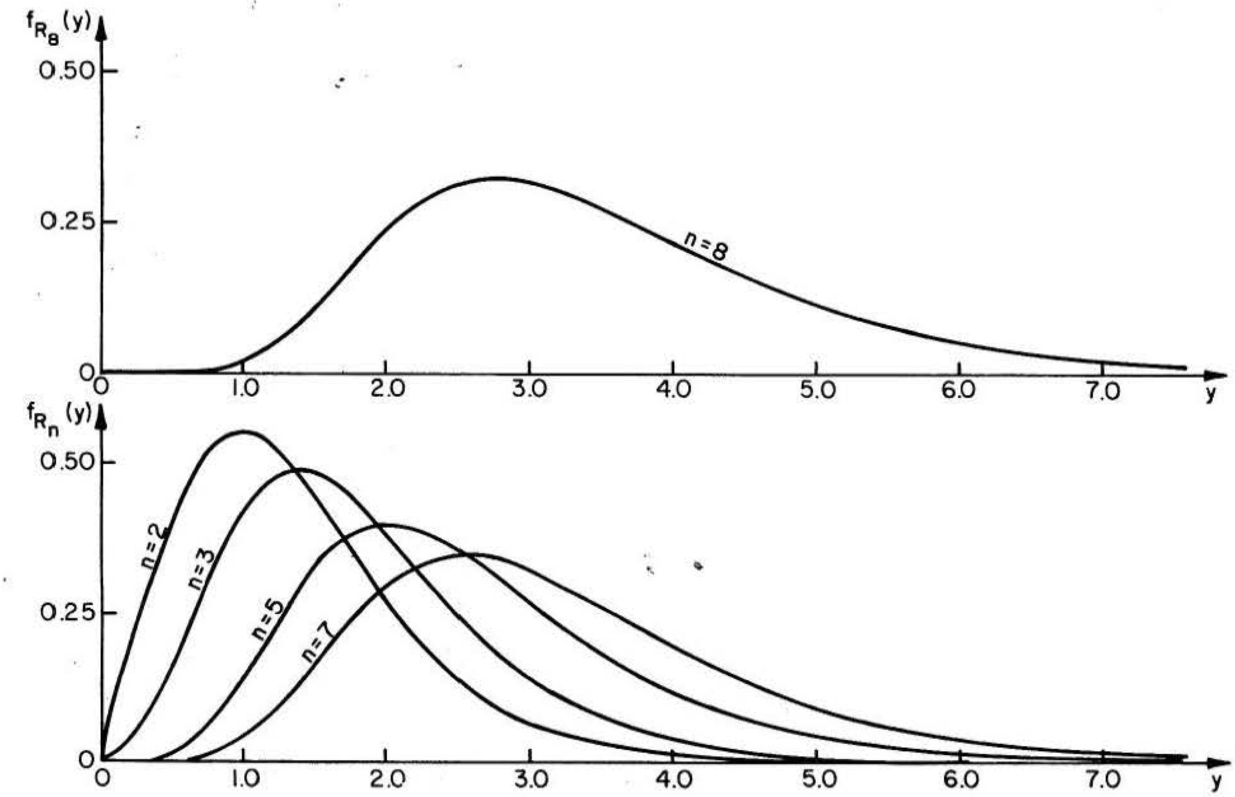

Fig. 4.4. Distribution of R_n for independent normal net inputs.

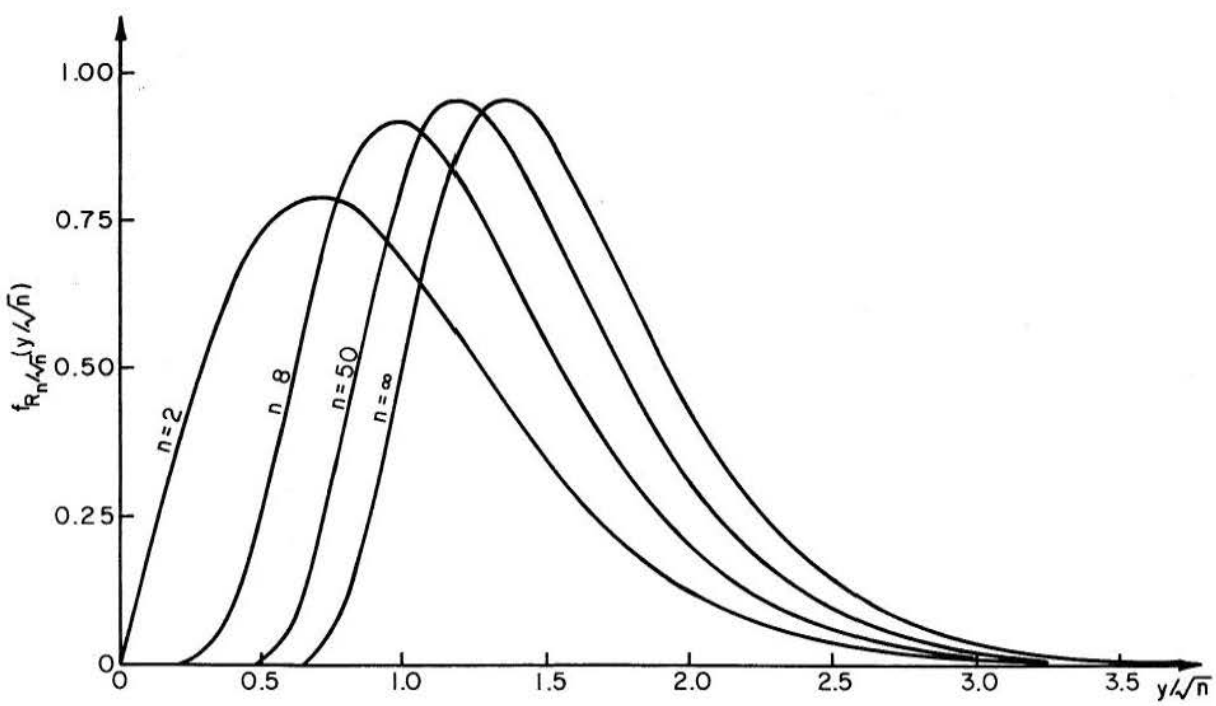

Fig. 4.5. Distribution of $R_n/\sqrt{n}$ for independent normal net inputs ($n=2,8,50,\infty$).

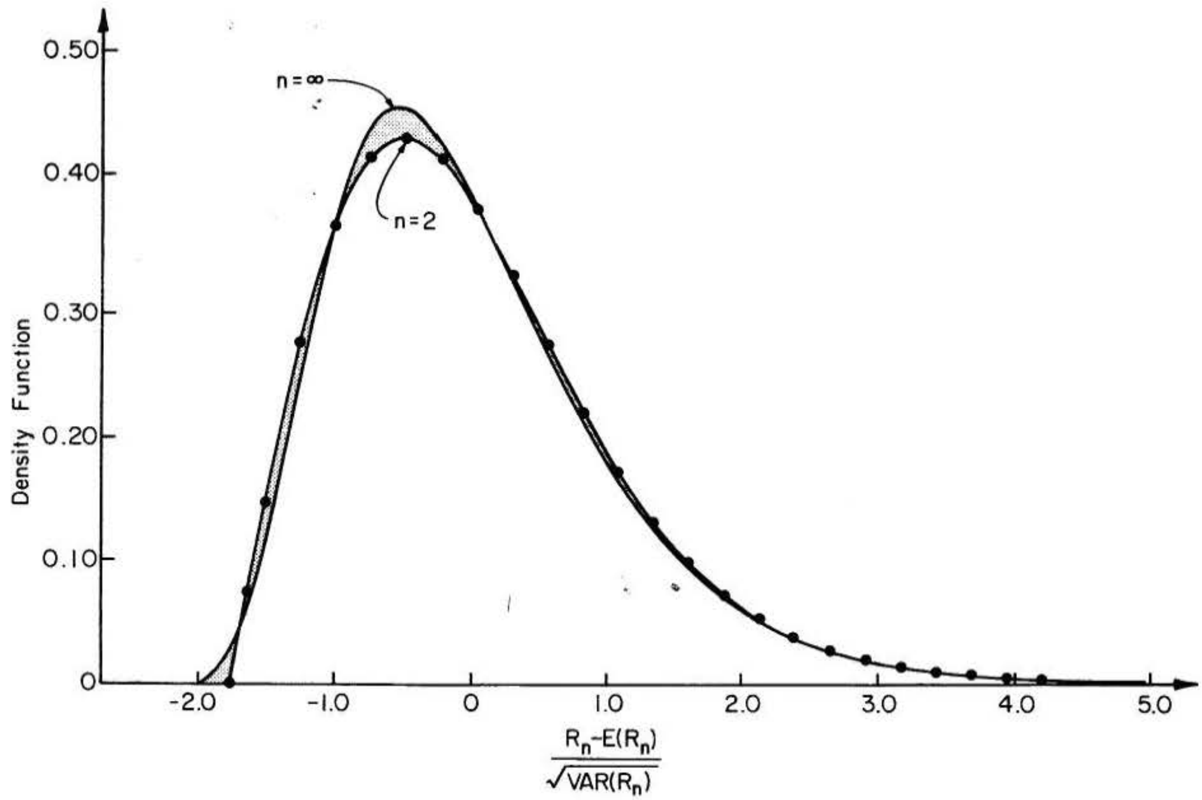

Fig. 4.6. Distribution of $[R_n-E(R_n)]/\sqrt{var(R_n)}$ for independent normal net inputs (n=2 and n=∞).

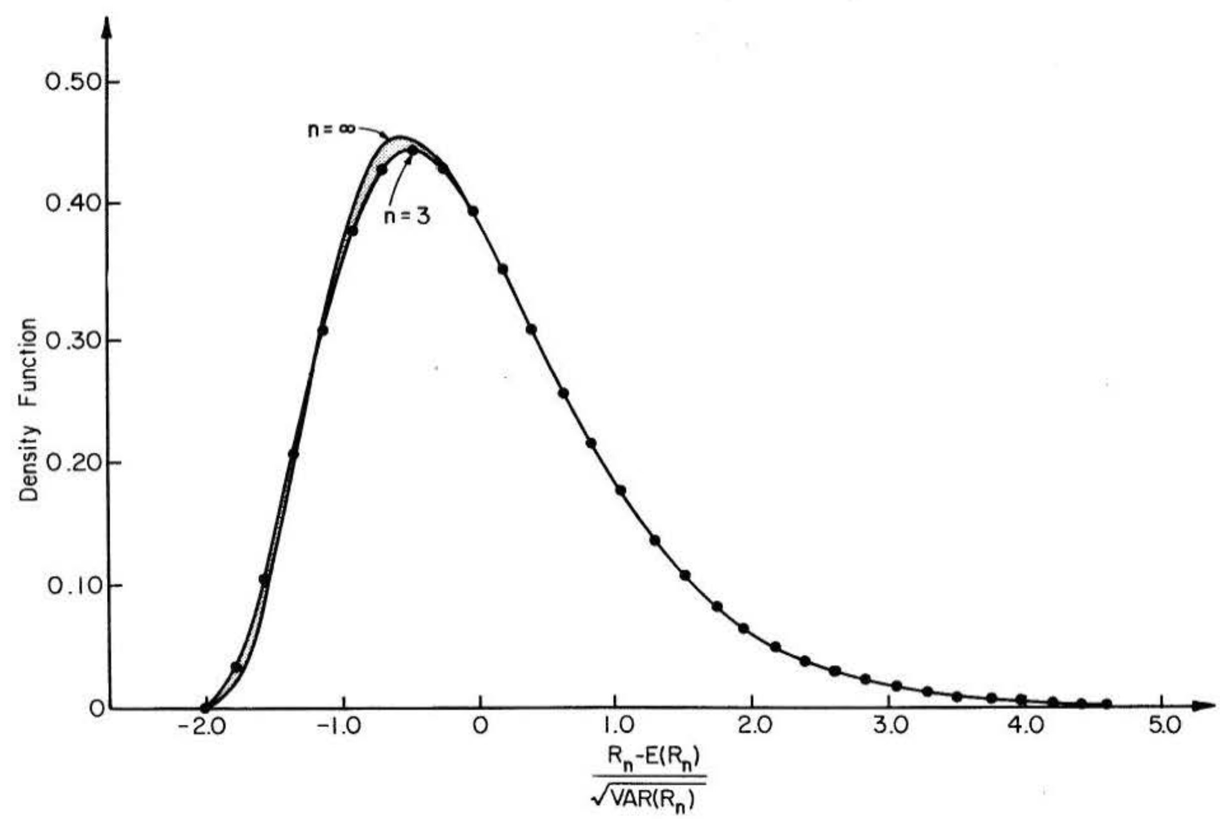

Fig. 4.7. Distribution of $[R_n-E(R_n)]/\sqrt{var(R_n)}$ for independent normal net inputs (n=3 and n=∞).

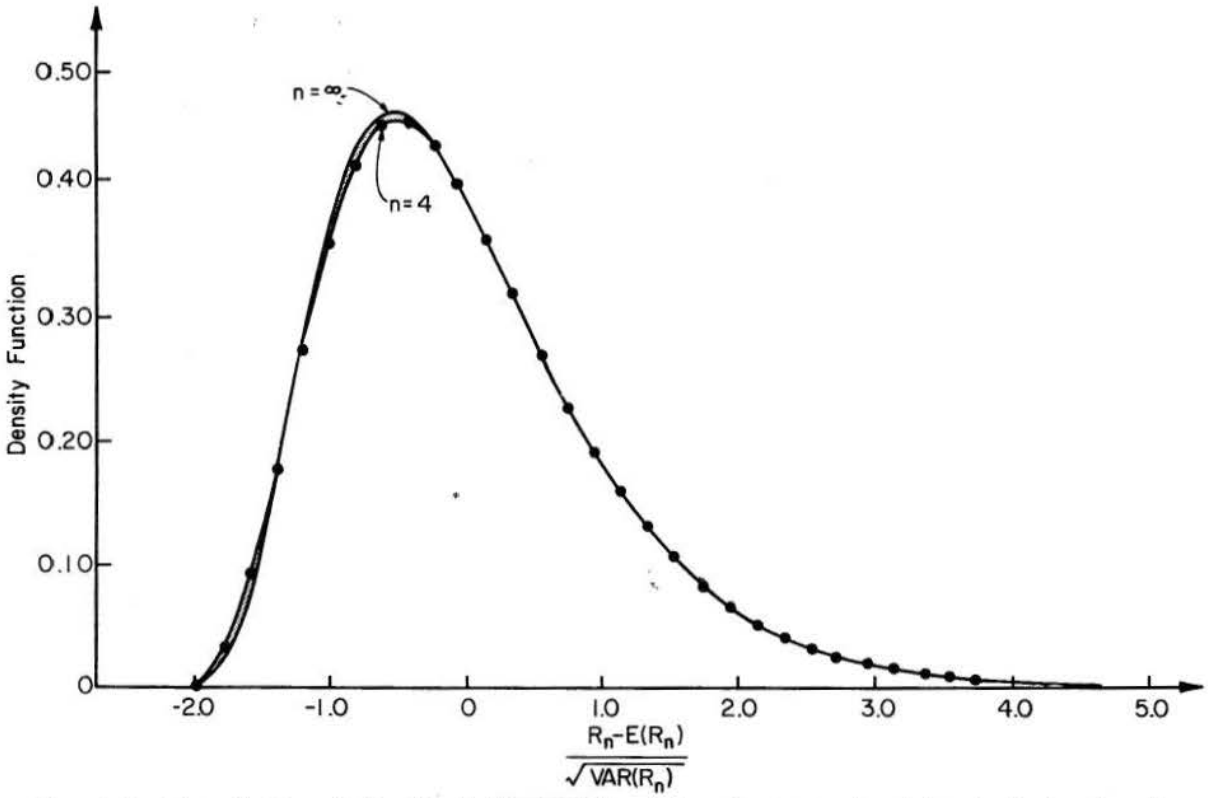

Fig. 4.8. Distribution of $[R_n - E(R_n)]/\sqrt{var(R_n)}$ for independent normal net inputs (n=4 and n=∞).

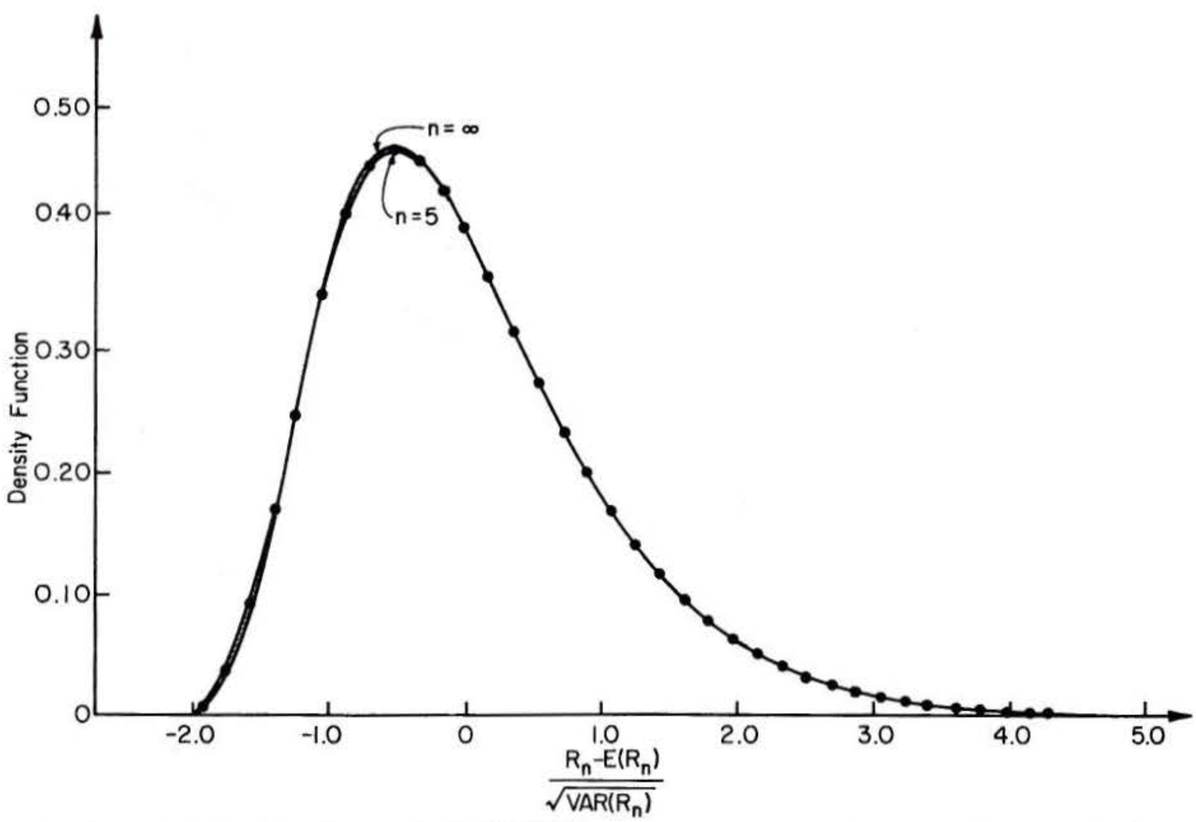

Fig. 4.9. Distribution of $[R_n - E(R_n)]/\sqrt{var(R_n)}$ for independent normal net inputs (n=5 and n=∞).

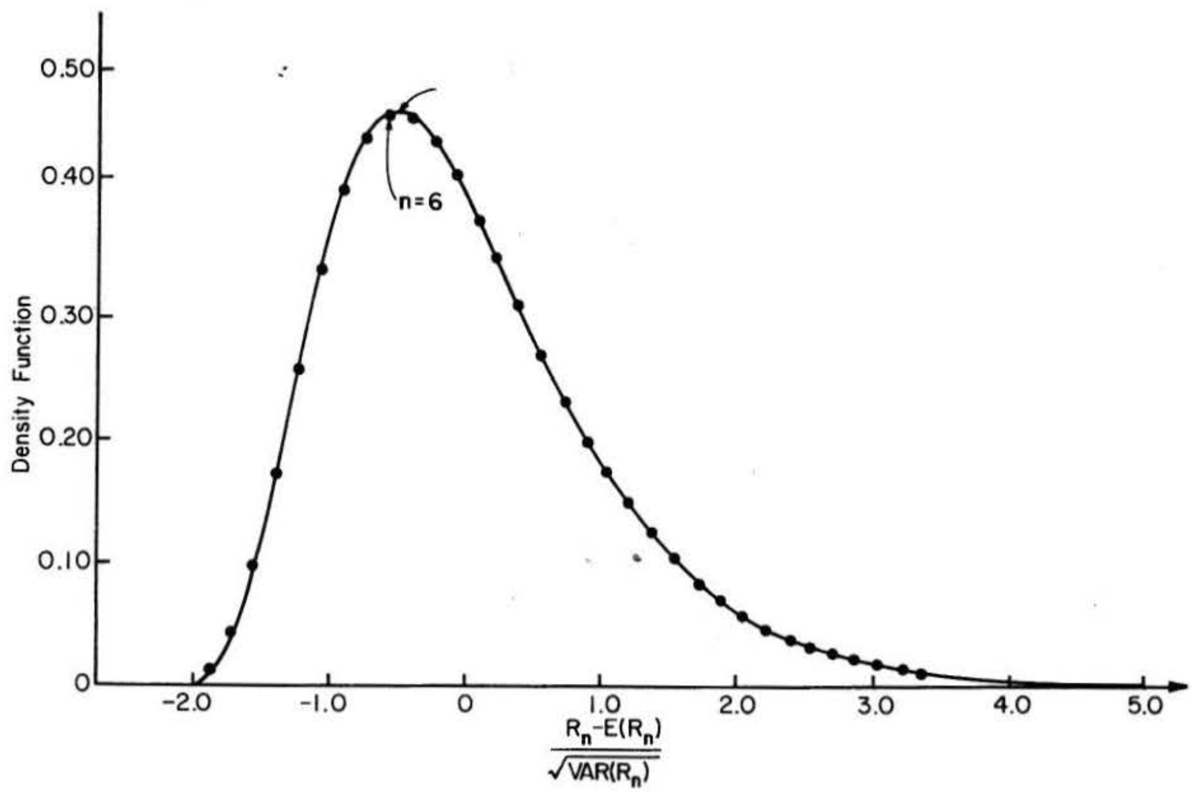

Fig. 4.10. Distribution of $[R_n - E(R_n)]/\sqrt{var(R_n)}$ for independent normal net inputs (n=6 and n=∞).

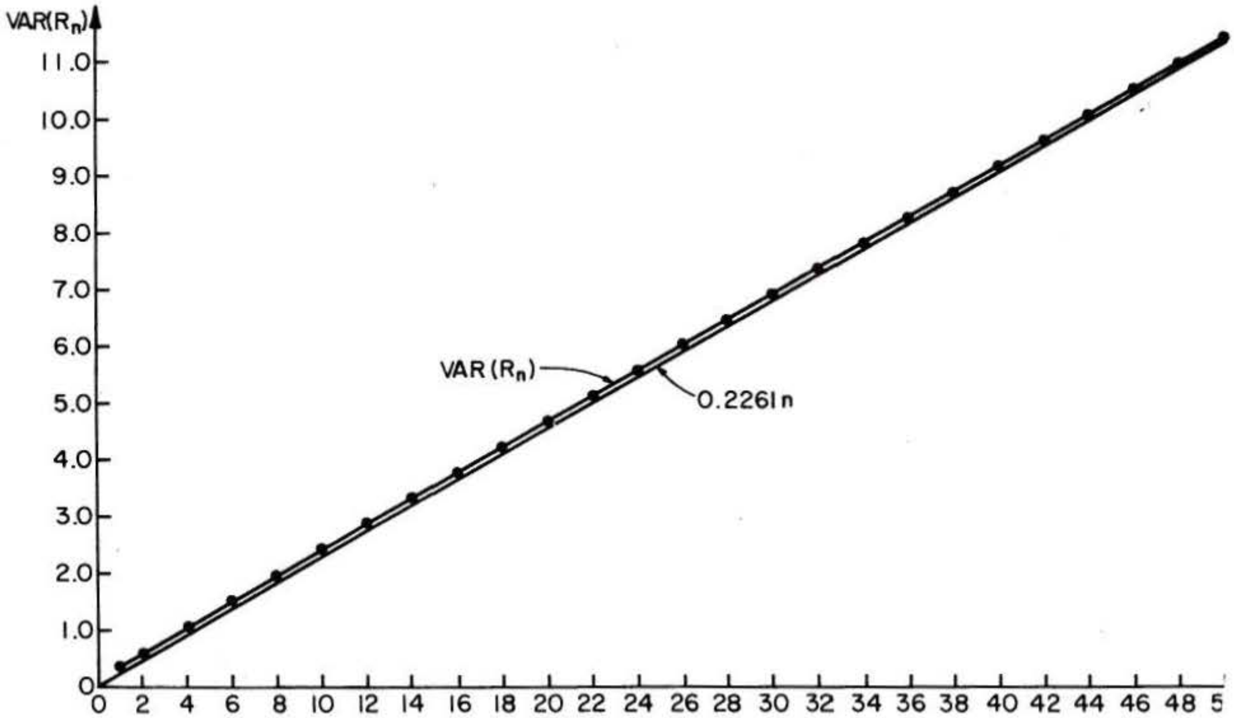

Fig. 4.11. Variance of the range of the partial sums of independent normal variates as compared to the asymptotic result $var(R_n)$=0.226 n.

For the standard Laplace distribution, $\mu = 0$ and $\sigma = 1$, and thus,

$$f_X(x) = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,|x|}.$$

For $n = 1$, the distribution of the range of partial sums of Laplace distributed random variables is, of course, the distribution of the absolute value of the net input (see Eq. (4.25) for normal net inputs),

$$f_{R_1}(y) = \sqrt{2}\; e^{-\sqrt{2}\,y} \qquad (y>0) \qquad (4.28)$$

and thus

$$E(R_1) = \sqrt{2} \int_0^\infty y \; e^{-\sqrt{2}\,y} = \frac{\sqrt{2}}{2}$$

$$E(R_1^2) = \sqrt{2} \int_0^\infty y^2 \, e^{-\sqrt{2}\,y} = 1.$$

For $n = 2$, using Eq. (4.26),

$$P(R_2' = k) = 4p_k \left(\frac{p_0}{2} + p_1 + \dots + p_{k-1} + \frac{p_k}{2}\right)$$

$$+ 2 \left(\frac{p_o p_k}{2} + p_1 p_{k-1} + \dots + p_{k-1}\, p_1 + \frac{p_k p_0}{2}\right)$$

where now $p_i = P(X_t = i)$ refers to some convenient discrete approximation to the double exponential distribution. Recall that Eq. (4.26) was derived for symmetric inputs, and thus it is applicable here. A similar expression can be easily found for nonsymmetric inputs.

Changing variables as appropriate in Eq. (4.26) and imposing the conditions under which the discrete approximation tends to the actual continuous net input, in the limit the result is:

$$f_{R_2}(y) = 4 \cdot \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\,y} \; \left(\int_0^y \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\,u}\, du\right)$$

$$+ 2 \int_0^y \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\,u} \cdot \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\,(y-u)}\, du$$

$$= [\sqrt{2} + y]\; e^{-\sqrt{2}\,y} - \sqrt{2}\; e^{-2\sqrt{2}\,y} \qquad (4.29)$$

and thus

$$E(R_2) = \sqrt{2} \int_0^\infty y\, e^{-\sqrt{2}\,y}\, dy + \int_0^\infty y^2\, e^{-\sqrt{2}\,y}\, dy$$

$$- \sqrt{2} \int_0^\infty y\, e^{-2\sqrt{2}\,y}\, dy$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} = \frac{7\sqrt{2}}{8}$$

$$E(R_2^2) = \sqrt{2} \int_0^\infty y^2\, e^{-\sqrt{2}\,y}\, dy + \int_0^\infty y^3\, e^{-\sqrt{2}\,y}\, dy$$

$$- \sqrt{2} \int_0^\infty y^2\, e^{-2\sqrt{2}\,y}\, dy$$

$$= \frac{3}{2} + 1 - \frac{1}{8} = \frac{19}{8}$$

Before continuing, it should be noticed that Eq. (4.9) is actually a second-order difference equation, which is the discrete analogue of a second derivative:

$$P(R_n = k) = \lambda_{k+1}^{(n)} - 2\lambda_k^{(n)} + \lambda_{k-1}^{(n)}$$

$$= (\lambda_{k+1}^{(n)} - \lambda_k^{(n)}) - (\lambda_k^{(n)} - \lambda_{k-1}^{(n)}) .$$

Consequently, an obvious approach consists in writing the continuous analogue of $\lambda_k^{(n)}$ and differentiating it twice to obtain $f_{R_n}(y)$.

The continuous analogue of $Q_k \underline{1}_k$ is

$$\psi(u_1,y) = \int_0^y f(u_1-u_0)\; du_0, \qquad (0 \le u_1 \le y)$$

where y is the analogue of k, after a convenient change of variables (see reasoning leading to Eq. (4.25)), and where $f(\cdot)$ is the density function of the net input.

Similarly, the continuous analogue of $Q_k^2 \underline{1}_k$ is

$$\psi(u_2,y) = \int_0^y f(u_2-u_1)\; \psi(u_1,y)\; du_1.$$

Following the same reasoning, the continuous analogue of $Q_k^n \underline{1}_k$ is

$$\psi(u_n,y) = \int_0^y f(u_n-u_{n-1})\; \psi(u_{n-1},y)\; du_{n-1} \qquad (4.30)$$

which can also be written as

$$\psi(u_n,y) = \int_0^y \int_0^y \dots \int_0^y \int_0^y f(u_n-u_{n-1})\; f(u_{n-1}-u_{n-2}) \dots$$

$$f(u_2-u_1)\; f(u_1-u_0) \quad du_0\, du_1 \dots du_{n-2}\, du_{n-1}. \qquad (4.31)$$

In view of Eq. (4.30), the continuous analogue of $\underline{1}_k^T Q_k^n \underline{1}_k$ is simply

$$\gamma_n(y) = \int_0^y \psi(u_n,y) du_n \qquad (4.32)$$

which can also be written as

$$\gamma_n(y) = \int_0^y \int_0^y \int_0^y \cdots \int_0^y \int_0^y f(u_n-u_{n-1}) \quad f(u_{n-1}-u_{n-2}) \cdots$$

$$\cdots f(u_2-u_1)\, f(u_1-u_0)\, du_0\, du_1 \cdots du_{n-2}\, du_{n-1}\, du_n. \tag{4.33}$$

Now the first derivative of $\gamma_n(y)$ with respect to y is the cumulative distribution function (c.d.f.) of the range and the second derivative of $\gamma_n(y)$ is its probability density function.

Furthermore, the analogue of Eq. (4.12) is:

$$E(R_n) = \lim_{y\to\infty} \; [y - \gamma_n(y)], \tag{4.34}$$

and the analogue of Eq. (4.13) is

$$E(R_n^2) = 2 \int_0^\infty \{E(R_n) - [y - \gamma_n(y)]\}\, dy. \tag{4.35}$$

The above results are general. In the sequel, they are applied to the particular case of Laplace distributed net inputs.

For $n = 1$ and Laplacian net inputs, Eq. (4.31) gives

$$\psi(u_1,y) = \int_0^y \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\;|u_1-u_0|}\, du_0$$

$$= \int_0^{u_1} \frac{\sqrt{2}}{2} e^{\sqrt{2}\;(u_0-u_1)}\, du_0 + \int_{u_1}^y \frac{\sqrt{2}}{2} e^{-\sqrt{2}(u_0-u_1)}\, du_0$$

$$= 1 - \frac{1}{2} e^{-\sqrt{2}\, u_1} - \frac{1}{2} e^{-\sqrt{2}\;(y-u_1)}.$$

Equation (4.32) furnishes

$$\gamma_1(y) = \int_0^y \psi(u_1,y)\, du_1 = y - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{-\sqrt{2}\, y}.$$

Using Eqs. (4.34) and (4.35),

$$E(R_1) = \lim_{y\to\infty} \; [\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}\, y}] = \frac{\sqrt{2}}{2}$$

and

$$E(R_1^2) = 2 \int_0^\infty \frac{\sqrt{2}}{2} \cdot e^{-\sqrt{2}\, y}\, dy = 1, \text{ as expected.}$$

Finally,

$$f_{R_1}(y) = \frac{d^2\gamma_1(y)}{dy^2} = \frac{d}{dy}\,[1 - e^{-\sqrt{2}\, y}\,] = \sqrt{2}\, e^{-\sqrt{2}\, y},$$

as in Eq. (4.28).

Similarly, for $n = 2$

$$\psi(u_2,y) = \int_0^y \frac{\sqrt{2}}{2} \; e^{-\sqrt{2}|u_2-u_1|} \; [1 - \frac{1}{2} \; e^{-\sqrt{2}\, u_1}$$

$$- \frac{1}{2} \; e^{-\sqrt{2}(y-u_1)}]\, du_1$$

$$= 1 - \frac{1}{2} e^{-\sqrt{2}\, u_2} \; [\frac{5}{4} - \frac{1}{4} e^{-\sqrt{2}\, y} + \frac{\sqrt{2}}{2} u_2]$$

$$- \frac{1}{2} e^{-\sqrt{2}(y-u_2)} \; [\frac{5}{4} - \frac{1}{4} e^{-\sqrt{2}\, y} + \frac{\sqrt{2}}{2} (y-u_2)]$$

and

$$\gamma_2(y) = \int_0^y \psi(u_2,y)\, du_2 = y - \frac{7\sqrt{2}}{8}$$

$$+ [\sqrt{2} + \frac{1}{2}y]\; e^{-\sqrt{2}\, y} - \frac{\sqrt{2}}{8} e^{-\,2\sqrt{2}\, y}$$

then

$$E(R_2) = \lim_{y\to\infty} \; \{\frac{7\sqrt{2}}{8} - [\sqrt{2} + \frac{1}{2}\, y]\; e^{-\sqrt{2}\, y}$$

$$+ \frac{\sqrt{2}}{8} e^{-2\sqrt{2}\, y}\} = \frac{7\sqrt{2}}{8},$$

$$E(R_2^2) = 2 \int_0^\infty [\sqrt{2} + \frac{1}{2}\, y]\; e^{-\sqrt{2}\, y}\, dy$$

$$- 2 \int_0^\infty \frac{\sqrt{2}}{8} e^{-2\sqrt{2}\, y}\, dy = \frac{19}{8},$$

and finally,

$$f_{R_2}(y) = \frac{d^2\gamma_2(y)}{dy^2} = [\sqrt{2} + y]\; e^{-\sqrt{2}\, y} - \sqrt{2}\, e^{-2\sqrt{2}\, y},$$

as in Eq. (4.29).

Similarly, for $n = 3$,

$$\psi(u_3,y) = 1 - \frac{1}{2} e^{-\sqrt{2}u_3} \Big\{ [\frac{11}{8} - \frac{7}{16} e^{-\sqrt{2}\, y} - \frac{\sqrt{2}}{8} y\, e^{-\sqrt{2}y}$$

$$+ \frac{1}{16} e^{-2\sqrt{2}y}]$$

$$+ [\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{8} e^{-\sqrt{2}\, y}]\; u_3 + \frac{1}{4} u_3^2 \Big\}$$

$$- \frac{1}{2} e^{-\sqrt{2}\;(y-u_3)}$$

$$\cdot \Big\{ [\frac{11}{8} - \frac{7}{16} e^{-\sqrt{2}\, y} - \frac{\sqrt{2}}{8} y\, e^{-\sqrt{2}\, y} + \frac{1}{16} \; e^{-2\sqrt{2}\, y}]$$

$$+ [\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{8} e^{-\sqrt{2}\, y}] \cdot (y-u_3) + \frac{1}{4} (y-u_3)^2 \Big\}$$

and

$$\gamma_3\,(y) = y - \frac{19\sqrt{2}}{16} + [\frac{47\sqrt{2}}{32} + \frac{9}{8} y + \frac{\sqrt{2}}{8} y^2]\; e^{-\sqrt{2}y}$$

$$- [\frac{10\sqrt{2}}{32} + \frac{1}{4}\, y]\; e^{-2\sqrt{2}\, y} + \frac{\sqrt{2}}{32} e^{-3\sqrt{2}\, y}$$

Then

$$E(R_3) = \lim_{y\to\infty} \{\frac{19\sqrt{2}}{16} - [\frac{47\sqrt{2}}{32} + \frac{9}{8} y + \frac{\sqrt{2}}{8} y^2]\, e^{-\sqrt{2}\, y} + [\frac{10\sqrt{2}}{32} + \frac{1}{4} y]\, e^{-2\sqrt{2}\, y} - \frac{\sqrt{2}}{32}\, e^{-3\sqrt{2}\, y}\} = \frac{19\sqrt{2}}{16},$$

$$E(R_3^2) = 2 \cdot \int_0^\infty [\frac{47\sqrt{2}}{32} + \frac{9}{8} y + \frac{\sqrt{2}}{8} y^2]\, e^{-\sqrt{2}\, y}\, dy - 2 \cdot \int_0^\infty [\frac{10\sqrt{2}}{32} + \frac{1}{4} y]\, e^{-2\sqrt{2}\, y}\, dy + 2 \cdot \int_0^\infty \frac{\sqrt{2}}{32}\, e^{-3\sqrt{2}\, y}\, dy = \frac{95}{24},$$

and finally

$$f_{R_3}(y) = \frac{d^2\gamma_3(y)}{d\,y^2} = [\frac{15\sqrt{2}}{16} + \frac{5}{4} y + \frac{\sqrt{2}}{4} y^2]\, e^{-\sqrt{2}\, y} - [\frac{3\sqrt{2}}{2} + 2y]\, e^{-2\sqrt{2}} + \frac{9\sqrt{2}}{16}\, e^{-3\sqrt{2}\, y}.$$

Similarly, for $n = 4$,

$$\begin{aligned}\psi(u_4,y) = 1 &- \frac{1}{2} e^{-\sqrt{2}\, u_4} \Big\{ [\frac{93}{64} - \frac{37}{64} e^{-\sqrt{2}\, y} - \frac{\sqrt{2}}{4} y\, e^{-\sqrt{2}\, y} - \frac{1}{16} y^2 e^{-\sqrt{2}\, y} + \frac{9}{64} e^{-2\sqrt{2}\, y} + \frac{\sqrt{2}}{16} y\, e^{-2\sqrt{2}\, y} - \frac{1}{64} e^{-3\sqrt{2}\, y}] \\ &+ [\frac{29\sqrt{2}}{32} - \frac{\sqrt{2}}{4} e^{-\sqrt{2}\, y} - \frac{1}{8} y\, e^{-\sqrt{2}\, y} + \frac{\sqrt{2}}{32} e^{-2\sqrt{2}\, y}]\, u_4 + [\frac{7}{16} - \frac{1}{16} e^{-\sqrt{2}\, y}]\, u_4^2 + \frac{\sqrt{2}}{24} u_4^3 \Big\} \\ &- \frac{1}{2} e^{-\sqrt{2}\,(y-u_4)} \Big\{ [\frac{93}{64} - \frac{37}{64} e^{-\sqrt{2}\, y} - \frac{\sqrt{2}}{4} y\, e^{-\sqrt{2}\, y} - \frac{1}{16} y^2 e^{-\sqrt{2}\, y} + \frac{9}{64} e^{-2\sqrt{2}\, y} + \frac{\sqrt{2}}{16} y\, e^{-2\sqrt{2}\, y} - \frac{1}{64} e^{-3\sqrt{2}\, y}] \\ &+ [\frac{29\sqrt{2}}{32} - \frac{\sqrt{2}}{4} e^{-\sqrt{2}\, y} - \frac{1}{8} y\, e^{-\sqrt{2}\, y} + \frac{\sqrt{2}}{32} e^{-2\sqrt{2}\, y}] \cdot (y-u_4) + [\frac{7}{16} - \frac{1}{16} e^{-\sqrt{2}\, y}]\, (y-u_4)^2 + \frac{\sqrt{2}}{24} (y-u_4)^3 \Big\}\end{aligned}$$

and

$$\gamma_4(y) = y - \frac{187\sqrt{2}}{128} + [\frac{61\sqrt{2}}{32} + \frac{57}{32} y + \frac{10\sqrt{2}}{32} y^2 + \frac{1}{24} y^3]\, e^{-\sqrt{2}\, y} - [\frac{17\sqrt{2}}{32} + \frac{11}{16} y + \frac{\sqrt{2}}{8} y^2]\, e^{-2\sqrt{2} y} + [\frac{3\sqrt{2}}{32} + \frac{3}{32} y]\, e^{-3\sqrt{2}\, y} - \frac{\sqrt{2}}{128} e^{-4\sqrt{2}\, y}$$

then

$$E(R_4) = \frac{187}{128}\sqrt{2},$$

$$E(R_4^2) = 2\int_0^\infty [\frac{61\sqrt{2}}{32} + \frac{57}{32} y + \frac{10\sqrt{2}}{32} y^2 + \frac{1}{24} y^3]\, e^{-\sqrt{2} y}\, dy - 2\int_0^\infty [\frac{17\sqrt{2}}{32} + \frac{11}{16} y + \frac{\sqrt{2}}{8} y^2]\, e^{-2\sqrt{2}\, y}\, dy + 2\int_0^\infty [\frac{3\sqrt{2}}{32} + \frac{3}{32} y]\, e^{-3\sqrt{2}\, y}\, dy - 2\int_0^\infty \frac{\sqrt{2}}{128} e^{-4\sqrt{2}\, y} = \frac{8722}{1536},$$

and finally,

$$f_{R_4}(y) = \frac{d^2\gamma_{4(y)}}{dy^2} = [\frac{7\sqrt{2}}{8} + \frac{21}{16} y + \frac{3\sqrt{2}}{8} y^2 + \frac{1}{12} y^3]\, e^{-\sqrt{2}\, y} - [\frac{7\sqrt{2}}{4} + \frac{7}{2} y + \sqrt{2}\, y^2]\, e^{-2\sqrt{2}\, y} + [\frac{9\sqrt{2}}{8} + \frac{27}{16} y]\, e^{-3\sqrt{2}\, y} - \frac{\sqrt{2}}{4} e^{-4\sqrt{2}\, y}.$$

Figures 4.12 and 4.13 show the density function of the range for Laplacian net inputs compared to the density function of the range for normal net inputs, for $n = 1,2,3,$ and 4.

For general n and Laplacian net inputs, $\psi(u_n,y)$ can be written as

$$1 - \frac{1}{2} e^{-\sqrt{2}\, u_n} [a_0 + a_1 u_n + a_2 u_n^2 + \ldots + a_{n-1} u_n^{n-1}] - \frac{1}{2} e^{-\sqrt{2}(y-u_n)} [a_0 + a_1 (y-u_n) + a_2 (y-u_n)^2 + \ldots + a_{n-1} (y-u_n)^{n-1}]$$

where the a_i's are functions of y.

Writing $\psi(u_{n+1},y)$ as

$$1 - \frac{1}{2} e^{-\sqrt{2}\, u_{n+1}} [\alpha_0 + \alpha_1 u_{n+1} + \alpha_2 u_{n+1}^2 + \ldots + \alpha_{n-1} u_{n+1}^{n-1} + \alpha_n u_{n+1}^n] - \frac{1}{2} e^{-\sqrt{2}\,(y-u_{n+1})} [\alpha_0 + \alpha_1 (y - u_{n+1}) + \ldots + \alpha_n (y - u_{n+1})^n],$$

the recursive relation between a_i's and α_i's is

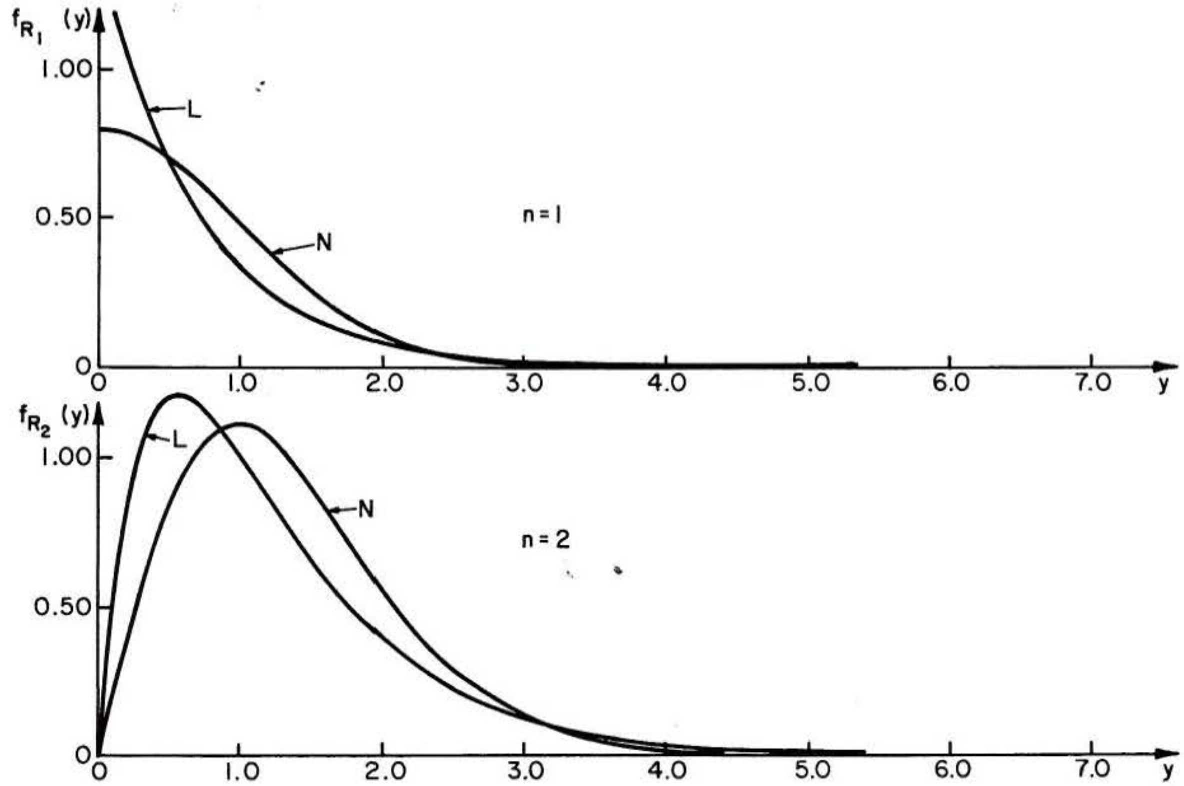

Fig. 4.12. Distribution of R_1 and R_2 for independent Laplacian (L) and normal (N) net inputs.

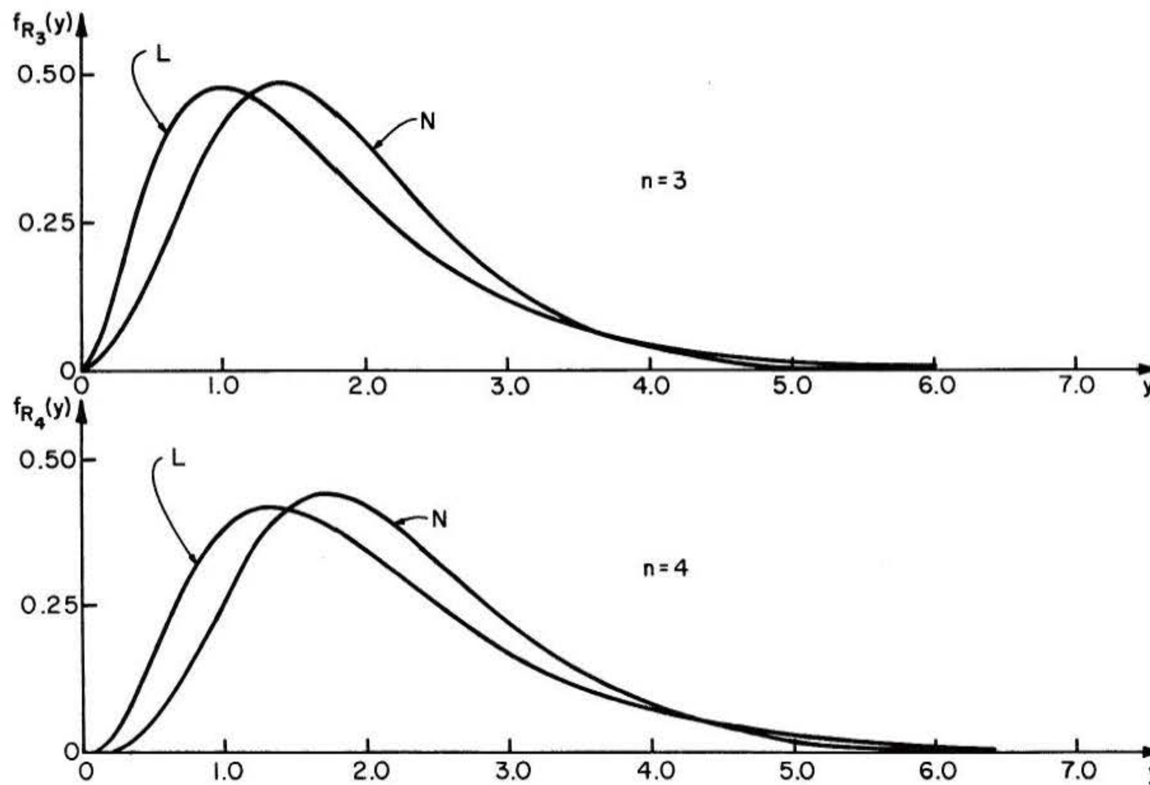

Fig. 4.13. Distribution of R_3 and R_4 for independent Laplacian (L) and normal (N) net inputs.

$$\begin{bmatrix}\alpha_o^*\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \cdot\\ \cdot\\ \cdot\\ \alpha_{n-1}\\ \alpha_n\end{bmatrix} = \frac{\sqrt{2}}{2}\cdot\begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{(2\sqrt{2})^2} & 2!\frac{1}{(2\sqrt{2})^3} & 3!\frac{1}{(2\sqrt{2})^4} & \cdots & (n-2)!\frac{1}{(2\sqrt{2})^{n-1}} & (n-1)!\frac{1}{(2\sqrt{2})^{n}}\\ 1 & \frac{1}{2\sqrt{2}} & 2!\frac{1}{(2\sqrt{2})^2} & 3!\frac{1}{(2\sqrt{2})^3} & \cdots & (n-2)!\frac{1}{(2\sqrt{2})^{n-2}} & (n-1)!\frac{1}{(2\sqrt{2})^{n-1}}\\ 0 & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{3!}{2!}\frac{1}{(2\sqrt{2})^2} & \cdots & \frac{(n-2)!}{2!}\frac{1}{(2\sqrt{2})^{n-3}} & \frac{(n-1)!}{2!}\frac{1}{(2\sqrt{2})^{n-2}}\\ 0 & 0 & \frac{1}{3} & \frac{1}{2\sqrt{2}} & \cdots & \frac{(n-2)!}{3!}\frac{1}{(2\sqrt{2})^{n-4}} & \frac{(n-1)!}{3!}\frac{1}{(2\sqrt{2})^{n-3}}\\ 0 & 0 & 0 & \frac{1}{4} & \cdots & \frac{(n-2)!}{4!}\frac{1}{(2\sqrt{2})^{n-5}} & \frac{(n-1)!}{4!}\frac{1}{(2\sqrt{2})^{n-4}}\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n-1} & \frac{1}{2\sqrt{2}}\\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{n}\end{bmatrix}\begin{bmatrix}a_o\\ a_1\\ a_2\\ a_3\\ a_4\\ \\ \\ \\ a_{n-1}\\ \\ \end{bmatrix} \quad (4.36)$$

and

$$\alpha_0 = \alpha_0^* + 1 - \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,y}\Big[\sum_{i=0}^{n-1} a_i \sum_{r=0}^{i} \frac{i!}{(i-r)!}\frac{y^{(i-r)}}{(2\sqrt{2})^{r+1}}\Big]. \quad (4.37)$$

There is no recursive relation for $\gamma_n(y)$, which would be, of course more appealing. Consequently, one has to use the recursive relation for the functions a_i's and to get $\gamma_n(y)$ in the following way:

$$\gamma_n(y) = y - \int_0^y e^{-\sqrt{2}\,u_n}\,[a_0 + a_1 u_n + \dots + a_{n-1} u_n^{n-1}]\,du_n$$

$$= y + \sum_{i=0}^{n-1} [\int_0^y e^{-\sqrt{2}\,u_n}\cdot a_i \cdot u_n^i\,du_n]$$

$$= y + \sum_{i=0}^{n-1} [a_i\, e^{-\sqrt{2}\,y} \sum_{r=0}^{i} \frac{i!}{(i-r)!}\frac{y^{i-r}}{(\sqrt{2})^{r+1}}] - \sum_{i=0}^{n-1} \frac{i!}{(\sqrt{2})^{i+1}}\cdot a_i. \quad (4.38)$$

Recalling that the a_i's are functions of y, $\gamma_n(y)$ can be rewritten as

$$\gamma_n(y) = y - E(R_n) +$$

$$+ [b_{1\cdot 0}+b_{1\cdot 1}y+b_{1\cdot 2}y^2+\dots+b_{1\cdot (n-1)}y^{n-1}]e^{-\sqrt{2}\,y}$$

$$+ [b_{2\cdot 0}+b_{2\cdot 1}y+b_{2\cdot 2}y^2+\dots+b_{2\cdot (n-2)}y^{n-2}]e^{-2\sqrt{2}\,y}$$

$$+ [b_{3\cdot 0}+b_{3\cdot 1}y+b_{3\cdot 2}y^3+\dots+b_{3\cdot (n-3)}y^{n-3}]e^{-3\sqrt{2}\,y}$$

$$+\dots+[b_{(n-1)\cdot 0}+b_{(n-1)\cdot 1}y]e^{(n-1)\sqrt{2}\,y} + b_{n\cdot 0}e^{-n\sqrt{2}y}$$

$$= y - E(R_n) + \sum_{i=1}^{i=n}\sum_{j=0}^{j=n-i} b_{i\cdot j}\, y^j\, e^{-i\sqrt{2}\,y}. \quad (4.39)$$

Consequently, ther are $n(n+1)/2$ parameters in the density $f_{R_n}(y)$. The second moment of the range follows from Eq. (4.35) and (4.39):

$$E(R_n^2) = 2\int_0^\infty \sum_{i=1}^{i=n}\sum_{j=0}^{j=n-i} b_{i\cdot j}\, y^j\, e^{-i\sqrt{2}\,y}$$

$$= 2\cdot\sum_{i=1}^{i=n}\sum_{j=0}^{j=n-i}\frac{j!}{(i\sqrt{2})^{j+1}}\, b_{i\cdot j}. \quad (4.40)$$

The first moment of the range is more easily seen from the following expressions:

$$E(R_1) = \frac{\sqrt{2}}{2} = \sqrt{2}\ {}_2C_1\ (\tfrac{1}{2})^2$$

$$E(R_2) - E(R_1) = \frac{7\sqrt{2}}{8} - \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{8} = \sqrt{2}\ {}_4C_2\ (\tfrac{1}{2})^4$$

$$E(R_3) - E(R_2) = \frac{19\sqrt{2}}{16} - \frac{7\sqrt{2}}{8} = \frac{5\sqrt{2}}{16} = \sqrt{2}\ {}_6C_3\ (\tfrac{1}{2})^6$$

$$E(R_4) - E(R_3) = \frac{187\sqrt{2}}{128} - \frac{19\sqrt{2}}{16} = \frac{35}{128}\sqrt{2} = \sqrt{2}\ {}_8C_4\ (\tfrac{1}{2})^8$$

or in general,

$$E(R_n) - E(R_{n-1}) = \sqrt{2}\ {}_{2n}C_n\ (\tfrac{1}{2})^{2n} \quad (4.41)$$

or equivalently,

$$E(R_n) = \sqrt{2}\sum_{i=1}^{i=n} {}_{2i}C_i (\tfrac{1}{2})^{2i} \quad (4.42)$$

This result could be derived from Spitzer's Lemma (1956).

It is interesting to note that applying Stirling's approximation to the factorials involved in Eq. (4.41), one has

$$E(R_n) - E(R_{n-1}) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$$

which is an exact result for normal inputs (Anis and Lloyd, 1953).

Table 4.3 shows the values of $E(R_n)$ from Eq. (4.42) and the values of $E(R_n^2)$ from Eqs. (4.36), (4.37), (4.38), (4.39) and (4.40), for several values of n. Figure 4.14 illustrates the convergence of the exact density of the range for Laplacian net inputs to the asymptotic density function.

TABLE 4.3 MOMENTS OF THE RANGE FOR LAPLACE DISTRIBUTED NET INPUTS

n	$E(R_n)$	$E(R_n^2)$	$VAR(R_n)$
1	0.7071	1.0000	0.5000
2	1.2374	2.3750	0.8438
3	1.6794	3.9583	1.1380
4	2.0661	5.6784	1.4097
5	2.4141	7.4949	1.6670
6	2.7331	9.3884	1.9184
7	3.0294	11.3407	2.1636
8	3.3071	13.3434	2.4065
9	3.5694	15.3867	2.6462
10	3.8186	17.4661	2.8846
11	4.0564	19.5757	3.1211
12	4.2844	21.7126	3.3568
13	4.5035	23.8730	3.5911
14	4.7149	26.0551	3.8249
15	4.9192	28.2563	4.0578
16	5.1171	30.4751	4.2902
17	5.3092	32.7097	4.5219
18	5.4960	34.9591	4.7533
19	5.6778	37.2219	4.9842
20	5.8551	39.4973	5.2148
21	6.0282	41.7843	5.4451
22	6.1973	44.0822	5.6750
23	6.3628	46.3902	5.9048
24	6.5248	48.7078	6.1342
25	6.6836	51.0343	6.3635
26	6.8393	53.3693	6.5926
27	6.9922	55.7123	6.8215
28	7.1423	58.0629	7.0503
29	7.2898	60.4206	7.2790
30	7.4349	62.7852	7.5075

2.3 Exponentially Distributed Net Inputs. The distribution of the range of partial sums of exponential random variables is studied here to illustrate the influence of departures from normality in general and of the coefficient of skewness in particular. The exponential distribution is chosen as a drastic example of departure from normality.

The distribution of the range for $n = 2$ was obtained analytically, and for the cases $n = 8$, $n = 50$, the solution was numerical. In Fig. 4.15 these distributions are compared to Feller's asymptotic result is different from the cases of normal and Laplace inputs, in the sense that the mode of the exact distributions is larger than the asymptotic mode.

In the particular case $n = 2$, the result is

$$f_{R_2}(y) = e^{-2}\,[2(e^{y}-e^{-y}) + y(e^{y}+ e^{-y})] \quad \text{for } 0\leq y<1$$

$$f_{R_2}(y) = e^{-2}\,(2-y)(e^{y}-e^{-y}) + 2e^{-(1+y)} \quad \text{for } 1<y\leq 2$$

$$f_{R_2}(y) = e^{-y}\,(2\,e^{-1} - 2e^{-2} + ye^{-2}) \quad \text{for } 2\leq y.$$

A final remark can be made, having to do with the fact that finite jumps, as the one that occurs in the case $n = 2$, exist also for higher values of n, and the numerical integration algorithm, being a discretization procedure, may not detect them. Of course, this does not invalidate the conclusions regarding convergence of the exact results to the asymptotic one.

2.4 Closing Remarks. In this section, the probability density function of the range of partial sums of independent, identically distributed, continuous random variables was shown to be given by

$$f_{R_n}(y) = \frac{d^2}{dy^2}\left[\int_0^y\int_0^y\int_0^y \cdots \int_0^y\int_0^y\int_0^y f(u_n-u_{n-1})\, f(u_{n-1}-u_{n-2}) \cdots f(u_2-u_1)\, f(u_1-u_0) \cdot du_0\, du_1\, du_2 \cdots du_{n-2}\, du_{n-1}\, du_n\right]$$

or, in short notation,

$$f_{R_n}(y) = \frac{d^2\gamma_n(y)}{dy^2}.$$

Recall that $f(\cdot)$ is the density function of the input.

An example of application of these results was given for the case of Laplace distributed net inputs. Although algebraically complicated, the solution is conceptually very simple and this writer certainly disagrees with Feller's assertion (1951) that "it is practically impossible to calculate the exact distribution of the ranges even for $n = 3$ and simple forms of the underlying distribution (input)."

In the case of normal net imputs, the solution was shown to be necessarily numerical and the range for binomial net inputs was used as an alternative algorithm for numerical integration.

Finally, the range for exponential net inputs was studied to investigate the influence of nonnormality in range analysis.

In all cases, comparisons were made between exact and asymptotic density function.

3. A Note on Existing Asymptotic Results

In this section, the asymptotic distribution of m_n is derived in a very simple manner, using the method of images. This result was first obtained by Erdos and Kac (1946) and it is presented here to illustrate that, as often happens in science, it was known before the works of Hurst (1951) and Feller (1951) that the square-root law holds asymptotically for any independent summands with zero expectation and finite variance. Another objective of the

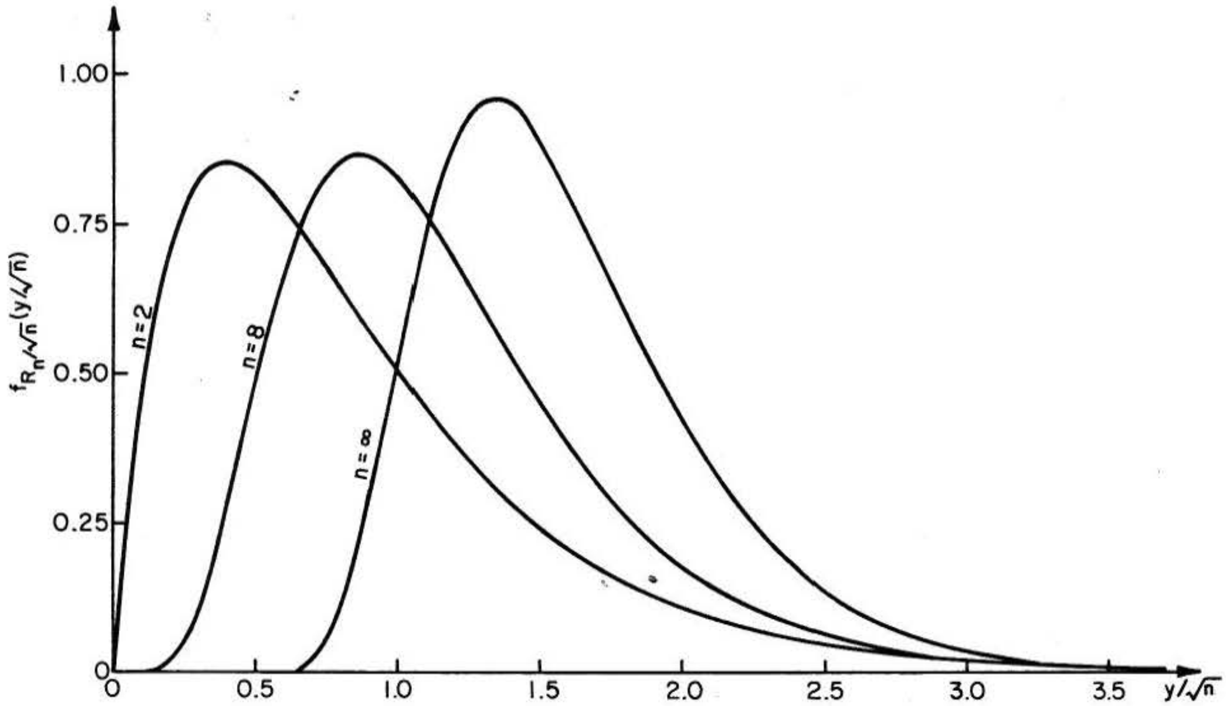

Fig. 4.14. Distribution of $R_n/\sqrt{n}$ for independent Laplacian net inputs (n=2,8,∞).

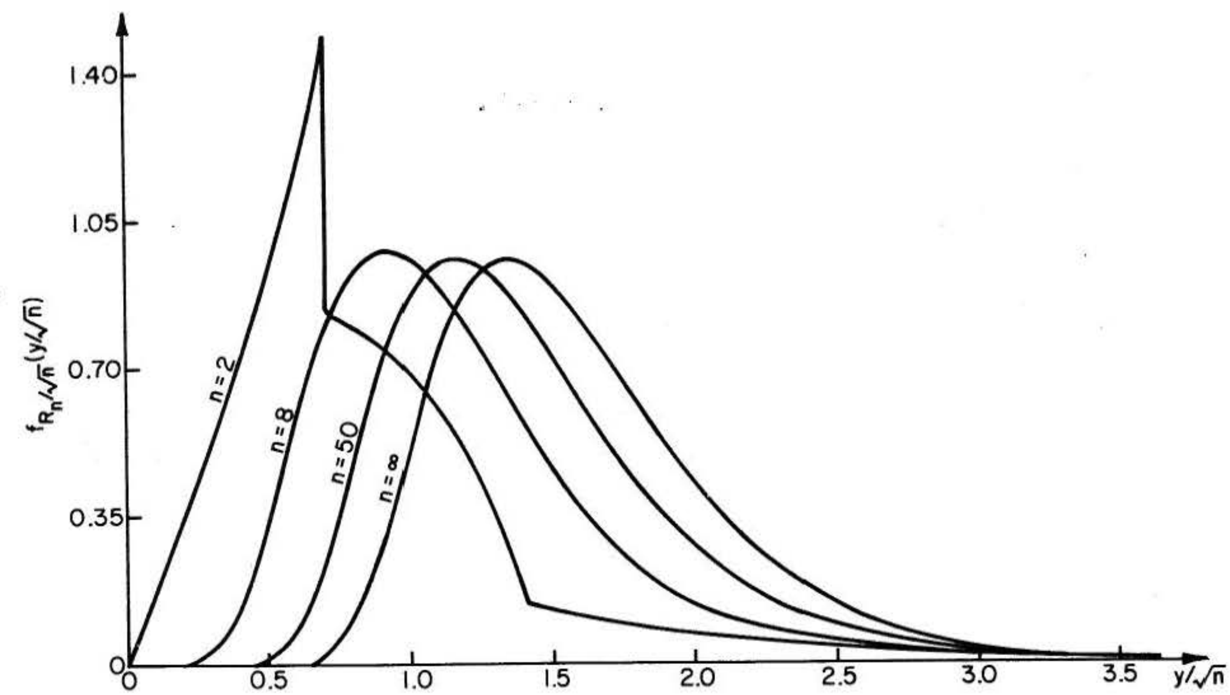

Fig. 4.15. Distribution of $R_n/\sqrt{n}$ for independent exponential net inputs (n=2,8,50,∞).

presentation of this result is to propose an approximation to the distribution of m_n for small n, when the inputs are normally distributed.

The asymptotic distribution of R_n, derived initially by Feller (1951) is also discussed and an alternative format is proposed.

3.1 The Asymptotic Distribution of m_n. The partial sums of any independent sequence of random variables which have finite variance are asymptotically normal, and thus the asymptotic distribution of m_n does not depend on the type of input. The argument is made with reference to the simplest input, namely the symmetric random walk process, in the presence of one absorbing state.

The probability that the system moves from the initial state j to any state $i = 1,2,3,\ldots$ without passing through (absorbing) state zero is obtained by summing Eq. (3.22) over all values of $i = 1,2,3,\ldots$.

$$\sum_{i=1}^{i=\infty} [{}_nC\left(\frac{n+i-j}{2}\right)(1/2)^n - {}_nC\left(\frac{n+i+j}{2}\right)(1/2)^n],$$

and only finitely many nonzero terms exist in this expression.

But when the system moves from state j to state $i = 1,2,3,\ldots$ without passing through state zero, then $|m_n|<j$.

For large n, the normal approximation to the binomial distribution can be used:

$$\sum_{i=1}^{i=\infty} {}_nC\left(\frac{n+i-j}{2}\right)(1/2)^n =$$

$$[{}_nC\left(\frac{n+1-j}{2}\right)+ {}_nC\left(\frac{n+2-j}{2}\right)+ \cdots] (1/2)^n \approx \Phi(j/\sqrt{n})$$

$$\sum_{i=1}^{i=\infty} {}_nC\left(\frac{n+i+j}{2}\right)(1/2)^n =$$

$$[{}_nC\left(\frac{n+1+j}{2}\right) + {}_nC\left(\frac{n+2+j}{2}\right) + \cdots] (1/2)^n \approx 1 - \Phi(j/\sqrt{n})$$

and thus

$$P[|m_n| \le j] = 2\Phi(j/\sqrt{n}) - 1 \tag{4.43}$$

or changing variables $j = x\sqrt{n}$

$$P[|\frac{m_n}{\sqrt{n}}| \le x] = 2\Phi(x) - 1 = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}u^2} du$$

which is the Erdos and Kac (1946) result. (Recall that $|m_n|$ and $|M_n|$ have the same distribution).

Consequently,

$$E(|\frac{m_n}{\sqrt{n}}|) = \int_0^\infty \{1 - [2\Phi(x) - 1]\}dx = 2\int_0^\infty [1 - \Phi(x)] = \sqrt{\frac{2}{\pi}}$$

or, equivalently,

$$E(|m_n|) \approx \sqrt{\frac{2}{\pi}}\sqrt{n} \approx 0.7979 \sqrt{n} .$$

Recall that for inputs having zero expectation, $E(R_n) = 2 \cdot E(M_n)$ and thus $E(R_n) = \sqrt{\frac{8}{\pi}}\sqrt{n} \approx 1.5958\sqrt{n}$ which is Feller's (1951) result. For $n = 1$ and normal inputs, it is clear that

$$P[|m_1| \le y] = \Phi(y) = 1 - \Phi(-y). \tag{4.44}$$

Rewriting Eq. (4.43),

$$P[|m_n| \le y] = 2\Phi(y/\sqrt{n}) - 1 = 1 - 2\Phi(-y/\sqrt{n}). \tag{4.45}$$

Equations (4.44) and (4.45) can be written as

$$P[|m_n| \le y] = 1 - g(n,y)\Phi(-y/\sqrt{n})$$

where $g(1,y) = 1$ and $\lim_{n\to\infty} g(n,y) = 2$.

An approximation to the distribution of $|m_n|$ for normal inputs and small n follows when $g(n,y)$ is considered a function of n only, neglecting the influence of y:

$$P[|m_n| \le y] = F_{|m_n|}(y) = 1 - g(n)\Phi(-y/\sqrt{n}). \tag{4.46}$$

Now a result due to Anis and Lloyd (1953) can be used, namely

$$P[|m_n| \le 0] = {}_{2n}C_n (1/2)^{2n}$$

so that, from Eq. (4.46),

$${}_{2n}C_n(1/2)^{2n} = 1 - \frac{1}{2}g(n)$$

or equivalently,

$$g(n) = 2[1 - {}_{2n}C_n (1/2)^{2n}]$$

where, obviously, $g(1) = 1$

$$\lim_{n\to\infty} g(n) = 2 .$$

The expression

$$P[|m_n| \le y] = F_{|m_n|}(y)$$

$$= 1 - 2[1 - {}_{2n}C_n (1/2)^{2n}]\Phi(-y/\sqrt{n}) \tag{4.47}$$

can be used as an approximation to the exact distribution of m_n for the case of normal inputs. Figure 4.16 shows the reasonable agreement between Eq. (4.47) and the Monte Carlo results presented by Yevjevich (1965).

From Eq. (4.47),

$$E[|m_n|] = \int_0^\infty [1 - F_{m_n}(y)] dy$$

$$= 2[1 - {}_{2n}C_n(1/2)^{2n}] \int_0^\infty [1 - \Phi(y/\sqrt{n})] dy$$

$$= 2[1 - {}_{2n}C_n(1/2)^{2n}] \frac{\sqrt{n}}{\sqrt{2\pi}}.$$

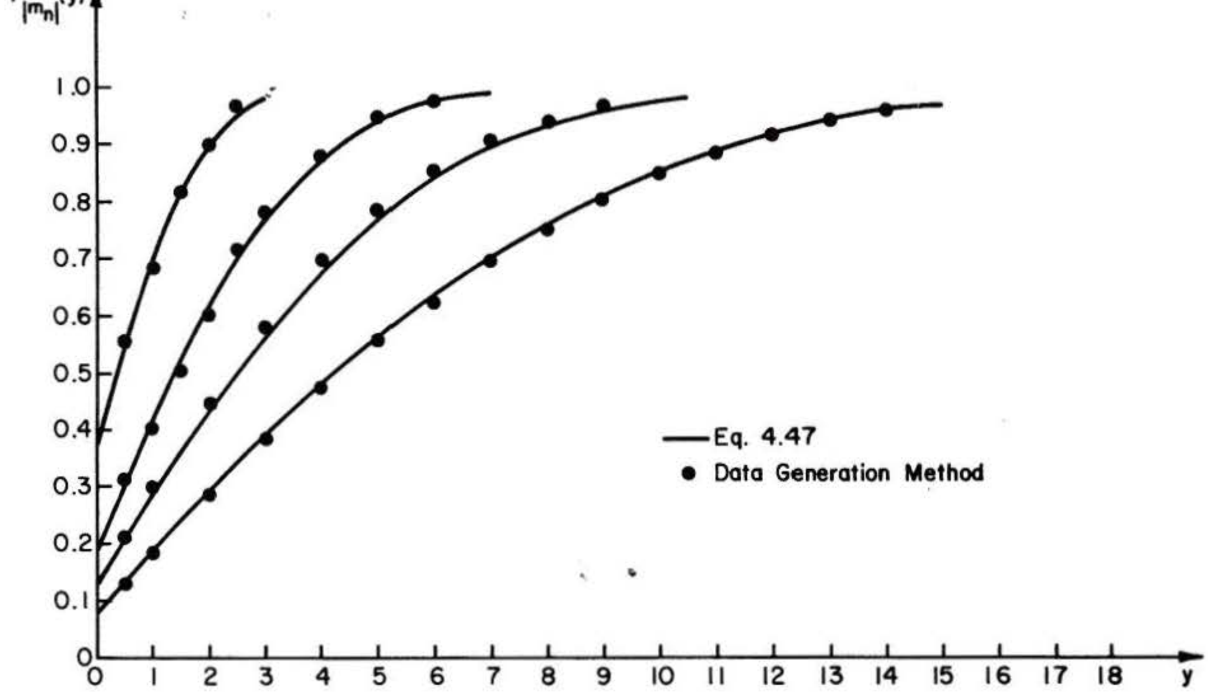

Fig. 4.16. Approximate cumulative distribution function of the minimum partial sum of independent normal variates.

Using Stirling's approximation:

$$_{2n}C_n\ (1/2)^{2n} \doteq \frac{1}{\sqrt{\pi n}}$$

and thus

$$E[|m_n|] = \sqrt{\frac{2}{\pi}}\sqrt{n} - \sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{\sqrt{\pi n}} \approx 0.7979\sqrt{n} - 0.4502$$

and $E(R_n) = 2 \cdot E[|m_n|] = 1.5958\sqrt{n} - 0.9003$ which is a better approximation than simply $1.5958\sqrt{n}$.

Notice that to get Eq. (4.47) from Eq. (4.46), the determination of $g(n)$ was arbitrary. One could very well determine $g(n)$ by imposing another condition, such as

$$E[|m_n|] = \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{n} i^{-\frac{1}{2}}$$

which is the Anis and Lloyd (1953) exact result. In this case, another approximation to the distribution of m_n would follow. Furthermore, this approximation would be as good as the one shown in Fig. 4.16.

3.2 The Asymptotic Distribution of R_n. Feller's (1951) result follows immediately from Section IV.1.3, namely, the range analysis for the random walk process, when the normal approximation to the binomial distribution is used. Note that this is not an alternative derivation of the asymptotic density function. It is essentially the same derivation.

The results found by this writer are

$$f_{R_n/\sqrt{n}}(x) = 2\sum_{j=1}^{j=\infty} j\,(-1)^j\{(j-1)\,\phi\,[(j-1)\,x] - 2j\,\phi\,(jx) + (j+1)\,\phi\,[(j+1)x]\} \quad (4.48)$$

for the probability density function, and

$$F_{R_n/\sqrt{n}}(x) = 2\sum_{j=1}^{j=\infty} j\,(-1)^j\{\Phi\,[(j-1)\,x] - 2\,\Phi\,(jx) + \Phi\,[(j+1)\,x]\} \quad (4.49)$$

for the cumulative distribution function, where

$$\phi\,(v) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}$$

and

$$\Phi\,(u) = \int_{-\infty}^{u}\phi(v)\,dv.$$

Note that

$$f_{R_n/\sqrt{n}}(0) = 0,$$

$$F_{R_n/\sqrt{n}}(0) = 0,$$

and

$$F_{R_n/\sqrt{n}}(\infty) = 1,$$

as they should.

It can be easily shown that Eq. (4.48) can be simplified to

$$f_{R_n/\sqrt{n}}(x) = 8 \sum_{j=1}^{j=\infty} (-1)^{j+1} j^2 \phi(jx) \qquad (4.50)$$

which is the original Feller (1951) result. It is not obvious from Eq. (4.50) that $f_{R_n/\sqrt{n}}(0) = 0$. Furthermore, it is not obvious that this function is non-negative.

The main advantage of Eq. (4.48) over Eq. (4.50) is that the mean can be obtained by termwise integration. In his paper, Feller found the mean by analogy with an existing result in the Kolmogorov-Smirnov theorem on empirical distribution functions. Using Eq. (4.48) this analogy is no longer necessary, and the mean is obtained by straight forward integration as follows.

$$E[R_n/\sqrt{n}] = 2 \sum_{j=2}^{j=\infty} j (-1)^j \int_0^\infty (j-1) x \phi[(j-1)x] dx$$

$$- 4 \sum_{j=1}^{j=\infty} j (-1)^j \int_0^\infty j x \phi(jx) dx$$

$$+ 2 \sum_{j=1}^{j=\infty} j (-1)^j \int_0^\infty (j+1) x \phi[(j+1)x] dx.$$

Recalling that $\int_0^\infty kx \phi(kx) dx = \frac{1}{k}\int_0^\infty v \phi(v) dv = \frac{1}{k\sqrt{2\pi}}$, one has

$$E[R_n/\sqrt{n}] = 4 \frac{1}{\sqrt{2\pi}} - 2 \frac{1}{2\sqrt{2\pi}} + 2 \frac{1}{\sqrt{2\pi}}$$

$$\cdot \sum_{j=2}^{j=\infty} j (-1)^j \left[\frac{1}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[3 + 4 \sum_{j=2}^{\infty} \frac{(-1)^j}{j^2-1}\right]$$

$$= \frac{4}{\sqrt{2\pi}} \approx 1.5958 \ldots.$$

A final remark can be made, having to do with the fact that the asymptotic moments follow very easily from the moments of the range of the random walk process. For instance, using Stirling's approximation in Eq. (4.22),

$$E(R_{n+1}) - E(R_n) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$$

and consequently,

$$E(R_{n+1}) \approx \sqrt{\frac{2}{\pi}} \left(1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}\right)$$

$$\approx \sqrt{\frac{2}{\pi}}\, 2\sqrt{n} \approx 1.5958 \sqrt{n}$$

or equivalently, $E(R_n/\sqrt{n}) \approx 1.5958\ldots.$

Similarly, using the normal approximation to the binomial distribution in Eq. (4.23), the asymptotic second moment of the range can be obtained.

The second term in the right hand side of Eq. (4.23) can be written as

$$2 \sum_{k=1}^{k=K-1} \left\{ \sum_{j=-\infty}^{j=\infty} [v_n(2j(k+1) - 1, k-1) - v_n(2j(k+1)-1, k)] \right.$$

$$\left. - \sum_{\substack{j=-\infty \\ (j\neq 0)}}^{j=+\infty} [v_n(2j(k+1) + 1, 1) - v_n(2j(k+1) + 1, 2)] \right\}$$

and it can be approximated by

$$2 \sum_{k=1}^{k=K-1} \left\{ \sum_{j=-\infty}^{j=+\infty} \frac{2}{\sqrt{n}} \phi[(2j-1)(k+1)/n] \right.$$

$$\left. - \sum_{\substack{j=-\infty \\ (j\neq 0)}}^{j=+\infty} \frac{2}{\sqrt{n}} \phi[2j(k+1)/\sqrt{n}] \right\}$$

$$= 2 \sum_{k=1}^{k=K-1} \sum_{i=1}^{\infty} (-1)^{i+1} \frac{4}{\sqrt{n}} \phi\left[\frac{i(k+1)}{\sqrt{n}}\right]$$

$$\approx 8 \sum_{i=1}^{i=\infty} \frac{(-1)^{i+1}}{i} \int_{2i/\sqrt{n}}^{\infty} \phi(v) dv$$

where $\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}$

and thus

$$\lim_{n\to\infty} [E(R_{n+1}^2) - E(R_n^2)] = \lim_{n\to\infty} \left\{ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \right.$$

$$\left. + 8 \sum_{i=1}^{i=\infty} \frac{(-1)^{i+1}}{i} \left[1 - \Phi\left(\frac{2i}{\sqrt{n}}\right)\right] \right\}$$

$$= 4 \sum_{i=1}^{i=\infty} \frac{(-1)^{i+1}}{i} = 4 \ln 2$$

and finally $\mathrm{var}[R_n/\sqrt{n}] = 4 \ln 2 - \frac{8}{\pi} \approx 0.2261\ldots.$

3.3 Closing Remarks. In this section, an approximation to the distribution of the minimum (or maximum) of partial sums of a finite number of independent normal variates was proposed (Eq. (4.47) and Fig. 4.20).

An alternative format for the asymptotic density function of the range was also proposed (Eq. (4.48)), and the asymptotic moments of the range were shown to follow easily from the concepts outlined in Section IV.1.3.

4. Summary

The main items investigated in this chapter are summarized as:

i) General approach to the distribution of the range for independent discrete inputs (Eq. (4.9)) and is first two moments (Eq. (4.12) and (4.13)).

ii) General approach to the distribution of the range for independent continuous inputs (Eq. (4.33)) and its first two moments (Eq. (4.34) and (4.35)).

iii) Illustration of the convergence of exact distributions to the asymptotic distribution, emphasizing that for moderately large values of n the first exact two moments are the only information needed in practice.

Several examples were given to illustrate the concepts outlined and a section of comments on existing asymptotic results was included.

Chapter V

DEFICIT ANALYSIS FOR INDEPENDENT, IDENTICALLY DISTRIBUTED INPUTS

As stated previously, range analysis may be relevant to the design of storage capacities when the regulation of flows is complete (alternative expressions are "full regulation" and "regulation on the mean"). This implies that the net input (input minus output) has zero expectation.

When the mean net input is positive (i.e., the regulated mean discharge is smaller than the mean natural discharge), overflows are unavoidable, and are implied in the design. Negative mean net inputs do not need to be studied because it is impossible to regulate a discharge larger than the mean natural discharge for long periods of time. Nevertheless, the approach described in this chapter can be applied for both positive and negative mean net inputs.

The study of storage problems involving partial regulation (i.e., cases when the mean net input is positive) will be called the maximum accumulated deficit analysis, or simply deficit analysis. One may argue that deficit analysis as described in this chapter should be applied even in the case of full regulation of discharges, and that the criterion of "designing for the range" may thus be questioned.

The procedure used in the design of storage capacities, say 30 years ago, consisted in the application of Rippl's mass curve to the observed hydrologic sequence (Hurst, 1951), as shown in Fig. 5.1 (this procedure has been sometimes referred to as the "sequent-peak method").

In Fig. 5.1, the cumulative sum of departures from an arbitrary (and convenient) base value B is plotted. To study the cumulative sum of departures from other base values (say, B_1), the summation curve is referred to inclined axes (say, OB_1). It is obvious that as this inclination changes, different points on the summation curve may become maxima, or minima. For instance, AA_1 is the range and A_1A_2 the maximum accumulated deficit with respect to the base value B, CC_1 is the maximum accumulated deficit with respect to the base B_1 and C_2C_1 is the maximum accumulated deficit with respect to the base B_2. Clearly the concept of range is meaningless for small base values.

The maximum accumulated deficit with respect to whatever discharge is to be regulated (base value) would be considered the storage capacity required. Often times this procedure would be used to find what would be the regulated discharge, given the storage capacity. The conventional way to answer this question would be to consider several base values, to plot the relation between storage capacity and regulated discharge and to fit a smooth curve, usually called the "storage-yield relationship." Clearly, the only purpose of considering inclined axes such as OB_1 and OB_2 in Fig. 5.1 is to avoid drawing a new graphic for each base value.

The modern procedure is not very different: it is essentially the application of the same old procedure, in the framework of the Monte Carlo method. In other words, sequences statistically indistinguishable from the actual record are generated, the old procedure is applied to each realization, and the

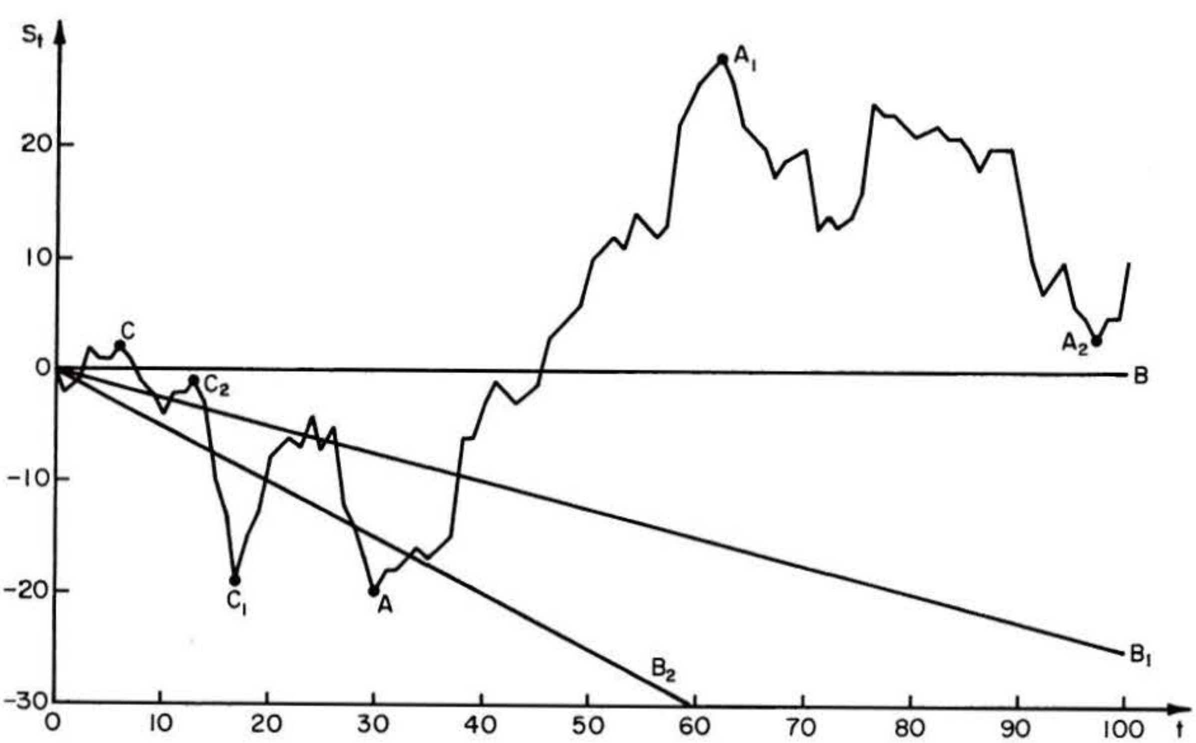

Fig. 5.1. Example of application of a Rippl's mass curve.

distribution of the maximum accumulated deficit is approached from a relative-frequency standpoint. Usually the sample mean value is taken as the storage required (Fiering, 1965).

Surprisingly, virtually no theoretical work has been done on this topic. The main reason seems to be that figures such as Fig. 5.1 are misleading, in the sense that the engineer may reach the conclusion that no simple connection exists between the storages required to guarantee different discharges.

The objective of this chapter is to study the properties of the maximum accumulated deficit. As in Chapter IV, the case of discrete inputs will be treated first. In the sequel, asymptotic results will be derived and the case of continuous inputs will be considered.

1. Discrete Net Inputs

Initially, some concepts related to the theory of Markov chains will be outlined, the relevance of which will become apparent later.

Consider the sequence of independent, identically distributed discrete random variables X_t $(t=1,2,\ldots,n)$ such that $P[X_t=i] = P_i$. Using a terminology similar to Moran's analysis of the finite reservoir, let X_t denote the net input at discrete time t into a reservoir of size $k+1$. Furthermore, let this reservoir be such that when empty, it continues empty with probability one, and when full, it continues full only if the net input in the next discrete time is nonnegative. Then the amount of water stored follows a simple homogeneous Markov chain with state space $\{0,1,2,\ldots,k+1\}$ and one-step transition matrix P' as shown here

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & & k-2 & k-1 & k & k+1 \\
\hline
0 & 1 & \ell_{-1} & \ell_{-2} & \ell_{-3} & \cdots & \ell_{-k+2} & \ell_{-k+1} & \ell_{-k} & \ell_{-k-1} \\
1 & 0 & p_0 & p_{-1} & p_{-2} & \cdots & p_{-k+3} & p_{-k+2} & p_{-k+1} & p_{-k} \\
2 & 0 & p_{+1} & p_0 & p_{-1} & \cdots & p_{-k+4} & p_{-k+3} & p_{-k+2} & p_{-k+1} \\
3 & 0 & p_{+2} & p_{+1} & p_0 & \cdots & p_{-k+5} & p_{-k+4} & p_{-k+3} & p_{-k+2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
k-2 & 0 & p_{+k-3} & p_{+k-4} & p_{+k-5} & \cdots & p_0 & p_{-1} & p_{-2} & p_{-3} \\
k-1 & 0 & p_{+k-2} & p_{+k-3} & p_{+k-4} & \cdots & p_{+1} & p_0 & p_{-1} & p_{-2} \\
k & 0 & p_{+k-1} & p_{+k-2} & p_{+k-3} & \cdots & p_{+2} & p_{+1} & p_0 & p_{-1} \\
k+1 & 0 & u_{+k} & u_{+k-1} & u_{+k-2} & \cdots & u_{+3} & u_{+2} & u_{+1} & u_0
\end{array} \tag{5.1}$$

where

$$u_j = p_j + p_{j+1} + p_{j+2} + \ldots \quad (j = 0,1,2,\ldots k)$$

and

$$\ell_{-j} = p_{-j} + p_{-j-1} + p_{-j-2} + \ldots \quad (j = 1,2,3,\ldots k+1).$$

Clearly the matrix P' can be partitioned as

$$P'_{k+2} = \begin{bmatrix} 1 & \ell^T_{k+1} \\ \underline{0}_{k+1} & P_{k+1} \end{bmatrix} \tag{5.2}$$

where $\underline{0}_{k+1}$ is a column vector of size $(k+1)$, with all elements equal to zero; the symbol T stands for transpose, and the subindexes denote the size of the vector or matrix. Furthermore,

$$\ell^T_{k+1} = [\ell_{-1}\ \ell_{-2}\ \ldots\ \ell_{-k}\ \ell_{-k-1}].$$

The n-step transition matrix is then

$$P'^{\,n}_{k+2} = \begin{bmatrix} 1 & \ell^T_{k+1}\,(I_{k+1} + P_{k+1} + P^2_{k+1} + \ldots + P^{n-1}_{k+1}) \\ \underline{0}_{k+1} & P^n_{k+1} \end{bmatrix} \tag{5.3}$$

The matrix P^n_{k+1} will be called the n-step "restricted" transition matrix, for obvious reasons. Notice that this matrix is different from the "restricted" transition matrix used in Chapter IV.

1.1 Formulation of the Problem.

Given the sequence of independent, identically distributed discrete random variables

$$X_t;\ t = 1,2,\ldots,n$$

consider the partial sums

$$S_{-1} = 0,\ S_0 = 0,$$

and

$$S_t = X_1 + X_2 + \ldots + X_t;\ t = 1,2,\ldots,n$$

and in particular the partial sums which are local maxima, that is,

$$\Big\{S_{t_i} : S_{t_i} \geq \max(S_{t_i-1}, S_{t_i+1});$$
$$i = 1,2,\ldots, J;\ t_i < t_k \text{ for } i < k\Big\},$$

where J is the number of local maxima among $\{S_t;\ t = 0,1,\ldots,n-1\}$.

For each local maxima S_{t_i} consider the partial sums immediately following it that are equal to or smaller than S_{t_i}. Let u_i be the largest integer such that

$$\{S_{t_i+v} \leq S_{t_i};\ v = 1,2,\ldots,u_i\}.$$

Let

$$s_{t_i} = \min_{1 \leq v \leq u_i} S_{t_i} + v$$

and define the deficit d_{t_i} by

$$d_{t_i} = S_{t_i} - s_{t_i};\ i = 1,2,\ldots,J.$$

The maximum accumulated deficit D_n can be defined by

$$D_n = \max_{1 \leq i \leq J} d_{t_i}.$$

Notice that some of the sets $\{S_{t_i+v}:\ S_{t_i+v} < S_{t_i};\ v = 1,2,\ldots,u_i\}$ may be subsets of a large one and thus the corresponding deficit need not be considered, because it could not be the maximum deficit.

1.2 Distribution of the Maximum Deficit D_n. The (s,u) entry in the matrix P_{k+1}^n, represented by $p_{k+1}^{(n)}(s,u)$, denotes the probability of a transition from state $u = 1,2,\ldots,k+1$ to state $s = 1,2,\ldots,k+1$ without passing through state zero.

Then $\sum_{s=1}^{s=k+1} p_{k+1}^{(n)}(s,k+1)$ denotes the probability that the system does not reach state zero in the first n steps, given the initial state $u = k+1$. But this is simply $P[D_n \le k]$ and Fig. 5.2 illustrates this fact. A realization of the process $\{S_t;\ t = 1,2,\ldots,n\}$ is shown in the upper part of the graphic and the same process, as routed through a hypothetical, initially full reservoir of size k+1, is shown in the lower part. Clearly the filtered process preserves only the deficits d_{t_i} and the distribution of the maximum deficit follows immediately.

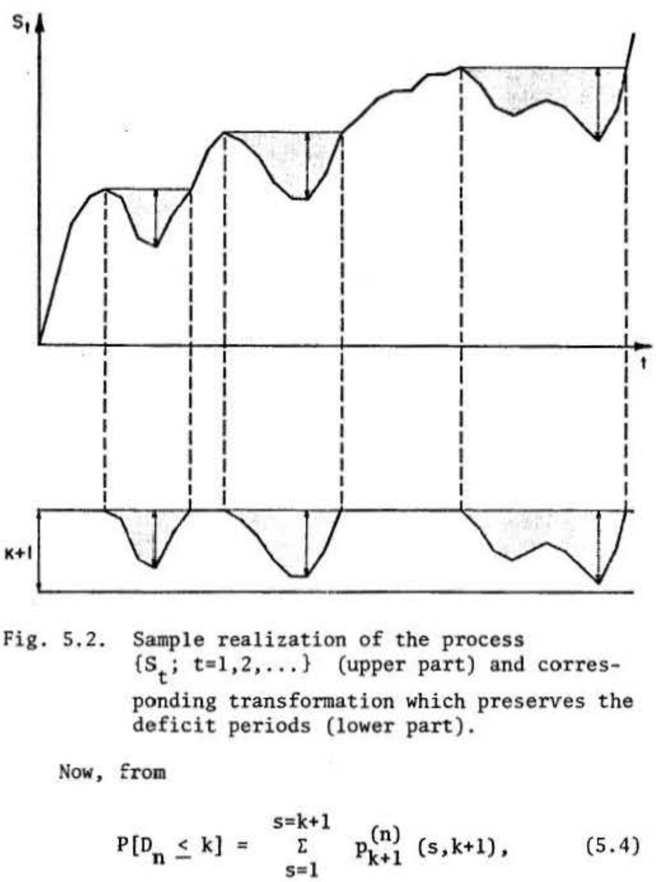

Fig. 5.2. Sample realization of the process $\{S_t;\ t=1,2,\ldots\}$ (upper part) and corresponding transformation which preserves the deficit periods (lower part).

Now, from

$$P[D_n \le k] = \sum_{s=1}^{s=k+1} p_{k+1}^{(n)}(s,k+1), \qquad (5.4)$$

it follows that

$$P[D_n = k] = \sum_{s=1}^{s=k+1} p_{k+1}^{(n)}(s,k+1) - \sum_{s=1}^{s=k} p_{k}^{(n)}(s,k). \qquad (5.5)$$

Equation (5.4), in matrix notation, becomes

$$P[D_n \le k] = \underline{1}_{k+1}^T\ P_{k+1}^n\ \underline{\theta}_{k+1} \qquad (5.6)$$

where T stands for transpose, the subindex k+1 denotes the size of the vectors and matrix involved, P_{k+1}^n is given by Eq. (5.3) and

$$\underline{1}_{k+1}^T = [1\ 1\ \ldots\ 1\ 1],$$

and

$$\underline{\theta}_{k+1}^T = [0\ 0\ \ldots\ 0\ 1].$$

Similarly, Eq. (5.5) can be written as

$$P[D_n = k] = \underline{1}_{k+1}^T\ P_{k+1}^n\ \underline{\theta}_{k+1} - \underline{1}_k^T\ P_k^n\ \underline{\theta}_k. \qquad (5.7)$$

For practical applications, it may be convenient to denote the level of regulation α by

$$\alpha = [1 - \frac{E(X_t)}{E(Y_t)}] \cdot 100\% \qquad (5.8)$$

where $E(X_t)$ is the mean net input and $E(Y_t)$ is the mean natural discharge (gross input). Clearly, for $E(X_t) = 0$, $\alpha = 100\%$.

For the particular case of constant output (constant regulated discharge), X_t and Y_t have the same variance σ^2 and their mean values can be written as

$$E(X_t) = \mu\sigma$$

$$E(Y_t) = c\sigma$$

where c is the inverse of the coefficient of variation C_V and μ is a number between zero and c. In this case, Eq. (5.8) simplifies to

$$\alpha = [1 - \frac{\mu}{c}] \cdot 100\%$$

or

$$\alpha = [1 - \mu C_V] \cdot 100\%. \qquad (5.9)$$

An example will now be given to help clarify the concepts exposited.

Example 5.1

The distribution of the maximum accumulated deficit will be found, in the case n = 3, for the following binomially distributed net inputs:

I) $P(X_t = i) = {}_4C_{2+i}\ (1/2)^4 \quad (i = -2,-1,0,1,2)$

(notice that $E(X_t) = 0$ and $\mathrm{var}(X_t) = 1$)

II) $P(X_t' = i) = {}_4C_{1+i}\ (1/2)^4 \quad (i = -1,0,+1,+2,+3)$

(notice that $E(X_t') = 1$ and $\mathrm{var}(X_t') = 1$)

For the first net input, to find, say, $P(D_3 \le 2)$, Eq. (5.6) can be used:

$$P(D_3 \le 2) = (1/16)^3 \cdot [1\ 1\ 1] \cdot \begin{bmatrix} 6 & 4 & 1 \\ 4 & 6 & 4 \\ 1 & 5 & 11 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3715}{4096}.$$

Similarly,

$$P(D_3 \le 1) = (1/16)^3 \cdot [11] \cdot \begin{bmatrix} 6 & 4 \\ 5 & 11 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{2863}{4096}$$

and

$$P(D_3 \le 0) = P(D_3 = 0) = (11/16)^3 = \frac{1331}{4096}.$$

Consequently,

$$P(D_3 = 2) = P(D_3 \le 2) - P(D_3 \le 1) = 852/4096$$

$$P(D_3 = 1) = P(D_3 \le 1) - P(D_3 = 0) = 1532/4096.$$

For other values of k in Eq. (5.7), the results are summarized in the following tabulation

k	0	1	2	3	4	5	6
$P(D_3 = k)$	$\frac{1331}{4096}$	$\frac{1532}{4096}$	$\frac{852}{4096}$	$\frac{292}{4096}$	$\frac{76}{4096}$	$\frac{12}{4096}$	$\frac{1}{4096}$

Notice that $E(D_3) = 1.0942 \ldots.$ Compare this result with $E(R_3) = 1.7480 \ldots$, from Example 4.2.

For the second net input, one has

$$P(D_3 \le 3) = (1/16)^3 [1111] \begin{bmatrix} 4 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 4 & 6 & 4 & 1 \\ 1 & 5 & 11 & 15 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{4096}{4096}$$

$$P(D_3 \le 2) = (1/16)^3\ [111] \begin{bmatrix} 4 & 1 & 0 \\ 6 & 4 & 1 \\ 5 & 11 & 15 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{4095}{4096}$$

$$P(D_3 \le 1) = (1/16)^3\ [1\ 1] \begin{bmatrix} 4 & 1 \\ 11 & 15 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{4061}{4096}$$

$$P(D_3 \le 0) = P(D_3 = 0) = (15/16)^3 = \frac{3375}{4096}$$

and consequently:

$$P(D_3{=}0) = \frac{3375}{4096},\ P(D_3{=}1) = \frac{686}{4096},\ P(D_3{=}2)$$

$$= \frac{34}{4096} \text{ and } P(D_3{=}3) = \frac{1}{4096}.$$

Finally, $E(D_3) = \frac{755}{4096} = 0.1843 \ldots.$

An interesting feature is that the analysis of net inputs can be used for several different gross inputs (natural discharge). In other words, the result $E(D_3) = 0.1843 \ldots$ found above holds for any value of $E(Y_t)$. If $E(Y_t)$ is, say 4, Eq. (5.8) gives $\alpha = 75\%$. If $E(Y_t)$ is, say, 8, Eq. (5.8) gives $\alpha = 87.5\%$. This illustrates the obvious fact that less variable natural discharges require smaller storage capacities for a given level of regulation.

It is also interesting to notice that the output (regulated discharge or water demand) can be random. This simply increases the variance of the net input (and consequently increases the storage capacity required), but the procedure is not altered.

A final remark can be made, having to do with the fact that a reasonable coefficient of variation for annual flows of American rivers is 0.25. In this case, considering the output constant, the level of regulation correspondent to $E(D_3) = 0.1843 \ldots$ would be 75% (Eq. (5.9) for $\mu = 1.0$ and $C_v = 0.25$). Notice the drastic reduction in $E(D_3)$ as compared with the case of full regulation, in which $E(D_3) = 1.0942$.

1.3 An Alternative Expression for the Distribution of $\underline{D}_n$. Another way of approaching the problem, which will be advantageous later to derive the analogue result for continuous inputs, is to partition P'_{k+2} as follows:

$$P'_{k+2} = \begin{bmatrix} 1 & \ell_k^T & \ell_{-k-1} \\ \underline{0}_k & Q_k & p_k \\ 0 & u_k^T & u_0 \end{bmatrix} \tag{5.10}$$

where $p_k^T = [p_{-k}\ p_{-k+1} \cdots p_{-2}\ p_{-1}]$, and all other terms are obvious from Eq. (5.1). Notice that now the matrix Q_k, used in Chapter IV, appears explicitly.

Taking into account Eq. (5.3) and (5.6) and recalling that the sum of elements of each column of the matrix $P'^{\,n}_{k+2}$ is unity, if follows that the element in the first row and last column of $P'^{\,n}_{k+2}$ is $1 - P(D_n \le k) = P(D_n > k)$.

It is convenient to work only with the first row of $P'^{\,n}_{k+2}$, keeping in mind that only the last element in this row is of interest. Let this first row be written as

$$[1\ d_k^{(n)^T}\ d_0^{(n)}]$$

where $d_k^{(n)}$ is a vector of size k, T stands for transpose, $d_0^{(n)}$ is a number (equal to $P(D_n > k)$) and the preservation of the scalar 1 is apparent from Eq. (5.3).

Clearly, for $n = 1$:

$$d_k^{(1)^T} = \ell_k^T \tag{5.11}$$

and

$$d_0^{(1)} = \ell_{-k-1}, \tag{5.12}$$

and in general, the following recursive relationship is obvious:

$$[1\ d_k^{(n)^T}\ d_0^{(n)}] = [1\ d_k^{(n-1)^T}\ d_0^{(n-1)}] \begin{bmatrix} 1 & \ell_k^T & \ell_{-k-1} \\ \underline{0}_k & Q_k & p_k \\ 0 & u_k^T & u_0 \end{bmatrix}$$

or equivalently,

$$\begin{aligned} d_k^{(n)^T} &= \ell_k^T + d_k^{(n-1)^T} Q_k + d_0^{(n-1)} u_k^T \\ d_0^{(n)} &= \ell_{-k-1} + d_k^{(n-1)^T} p_k + d_0^{(n-1)} u_0. \end{aligned} \tag{5.13}$$

Equations (5.13) will be useful later in this chapter.

1.4 Closing Remarks. In this section, a general approach was described to obtain the distribution of the maximum deficit of partial sums of independent, identically distributed discrete random variables. In a later chapter this approach will be extended to dependent random variables.

It is apparent that the solution to the problem is simpler than its formulation.

An example was given to illustrate, among other things, the drastic reduction in storage capacity required in the case of partial regulation, as compared to the case of full regulation.

A final remark can be made, having to do with the obvious relationship between deficit analysis and Moran's analysis of the finite reservoir: the probability that a reservoir of size (k+1), initially full, is empty for the first time at discrete time n, regardless of the occurrence of overflows is simply

$$P[D_n > k] - P[D_{n-1} > k].$$

Consequently, results like Weesakul's probability of first emptiness with or without overflows for the case of geometric inputs (1961) can be used directly to obtain the distribution of D_n for geometric inputs.

2. Asymptotic Results

The maximum accumulated deficit is a function of the partial sums S_t (recall definition in Section V 1.1). These partial sums are asymptotically normally distributed for all independent inputs which have finite variance, and consequently, the asymptotic distribution of the maximum deficit is independent of the underlying random variable (input).

In this section, the asymptotic result will be derived based on the deficit analysis for the random walk process.

2.1 Maximum Accumulated Deficit for the Case of Full Regulation. Consider the following probability distribution for the input:

$$p(X_t = i) = 1/2 \quad (i = -1, +1).$$

Clearly, $E(X_t) = 0$ and $\mathrm{var}(X_t) = 1$.

In this case, the one-step "restricted" transition matrix P_{k+1} is

$$\begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 & 0 & 0 \\ 0 & 1/2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \cdots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 & 1/2 \end{bmatrix}$$

and its n·th power can be found using the method of images.

It follows from Section III 2.3 that the (s,u) entry in the matrix P_{k+1}^n is given by Eq. (3.25):

$$p_{k+1}^{(n)}(s,u) = \sum_{j=-\infty}^{j=+\infty} (-1)^j \left\{ v_n\,[2j(k+3/2) + u,s] - v_n\,[2j(k+3/2) - u,s] \right\}$$

where $s = 1,2,\ldots,k+1$; $u = 1,2,\ldots,k+1$, and where $v_n(r,t)$ is given by

$$v_n(r,t) = {}_nC_{\left(\frac{n+r-t}{2}\right)} (1/2)^n = {}_nC_{\left(\frac{n+t-r}{2}\right)} (1/2)^n.$$

Now Eq. (5.4) can be used as

$$P(D_n \le k) = \sum_{s=1}^{s=k+1} p_{k+1}^{(n)}(s,k+1) = \sum_{s=1}^{s=k+1} \sum_{j=-\infty}^{j=+\infty} (-1)^j \cdot \left\{ v_n[j\,(2k+3) + (k+1),\ s] - v_n[j(2k+3) - (k+1),\ s] \right\}. \tag{5.14}$$

For simplicity, rewrite Eq. (5.14) as

$$P(D_n \le k) = V_0 + V_1 + V_2 + V_3 + V_4$$

where

$$V_0 = \sum_{s=1}^{s=k+1} v_n[(k+1),s] - \sum_{s=1}^{s=k+1} v_n[-(k+1),s]$$

$$\begin{aligned} V_1 &= \sum_{j/2=+1}^{j/2=+\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s] \\ &\quad - \sum_{j/2=-1}^{j/2=-\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s] \\ &= \sum_{j/2=+1}^{j/2=+\infty} \left\{ \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s] \right. \end{aligned}$$

$$- \sum_{s=1}^{s=k+1} v_n[-j(2k+3) - (k+1),s]\Bigg\}$$

$$V_2 = \sum_{j/2=-1}^{j/2=-\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s]$$

$$- \sum_{j/2=+1}^{j/2=+\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s]$$

$$= \sum_{j/2=+1}^{j/2=+\infty} \Bigg\{ \sum_{s=1}^{s=k+1} v_n[-j(2k+3) + (k+1),s]$$

$$- \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s]\Bigg\}$$

$$V_3 = \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s]$$

$$- \sum_{(j-1)/2=-1}^{(j-1)/2=-\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s]$$

$$= \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Bigg\{ \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s]$$

$$- \sum_{s=1}^{s=k+1} v_n[-j(2k+3) + (k+1),s]\Bigg\}$$

$$V_4 = \sum_{(j-1)/2=-1}^{(j-1)/2=-\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) - (k+1),s]$$

$$- \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s]$$

$$= \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Bigg\{ \sum_{s=1}^{s=k+1} v_n[-j(2k+3)-(k+1),s]$$

$$- \sum_{s=1}^{s=k+1} v_n[j(2k+3) + (k+1),s]\Bigg\} .$$

In particular, the terms in V_0 are

$$\sum_{s=1}^{s=k+1} v_n[(k+1),s] = [{}_nC_{\frac{(n+k)}{2}} + {}_nC_{\frac{(n+k-1)}{2}} + \dots + {}_nC_{\frac{n}{2}}] \cdot (1/2)^n$$

and

$$\sum_{s=1}^{s=k+1} v_n[-(k+1),s] = [{}_nC_{\frac{(n-k-2)}{2}} + {}_nC_{\frac{(n-k-3)}{2}} + \dots + {}_nC_{\frac{(n-2k-2)}{2}}] \cdot (1/2)^n .$$

For large n, the normal approximation to the binomial distribution can be used as

$$\sum_{s=1}^{s=k+1} v_n[(k+1),s] \approx \Phi(k/\sqrt{n}) - 1/2$$

and

$$\sum_{s=1}^{s=k+1} v_n[-(k+1),s] \approx \Phi(-k/\sqrt{n}) - \Phi(-2k/\sqrt{n})$$

$$= \Phi(2k/\sqrt{n}) - \Phi(k/\sqrt{n})$$

and thus

$$V_0 \approx 2\,\Phi(k/\sqrt{n}) - \Phi(2k/\sqrt{n}) - 1/2. \tag{5.15}$$

Similarly, one has

$$V_1 \approx \sum_{j/2=+1}^{j/2=+\infty} \Big\{\Phi[(j(2k+3) + k)/\sqrt{n}] - \Phi[j(2k+3)/\sqrt{n}]$$

$$- \Phi[(-j(2k+3) - k)/\sqrt{n}] + \Phi[(-j(2k+3) - 2k)/\sqrt{n}]\Big\}$$

or

$$V_1 \approx \sum_{j/2=+1}^{j/2=+\infty} \Big\{2\,\Phi[(j(2k+3) + k)/\sqrt{n}]$$

$$- \Phi[(j(2k+3) + 2k)/\sqrt{n}] - \Phi[j(2k+3)/\sqrt{n}]\Big\}, \tag{5.16}$$

$$V_2 \approx \sum_{j/2=+1}^{j/2=+\infty} \Big\{\Phi[(-j(2k+3) + k)/\sqrt{n}] - \Phi[-j(2k+3)/\sqrt{n}]$$

$$- \Phi[(j(2k+3) - k)/\sqrt{n}] + \Phi[(j(2k+3) - 2k)/\sqrt{n}]\Big\}$$

or

$$V_2 \approx \sum_{j/2=+1}^{j/2=+\infty} \Big\{- 2\,\Phi[(j(2k+3) - k)/\sqrt{n}]$$

$$+ \Phi[(j(2k+3) - 2k)/\sqrt{n}] + \Phi[j(2k+3)/\sqrt{n}]\Big\} \tag{5.17}$$

$$V_3 \approx \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Big\{\Phi[(j(2k+3) - k)/\sqrt{n}]$$

$$- \Phi[(j(2k+3) - 2k)/\sqrt{n}]$$

$$- \Phi[(-j(2k+3) + k)/\sqrt{n}] + \Phi[-j(2k+3)/\sqrt{n}]\Big\}$$

or

$$V_3 \approx \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Big\{2\,\Phi[(j(2k+3) - k)/\sqrt{n}]$$

$$- \Phi[(j(2k+3) - 2k)/\sqrt{n}] - \Phi[j(2k+3)/\sqrt{n}]\Big\}, \tag{5.18}$$

$$V_4 \approx \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Big\{\Phi[(-j(2k+3) - k)/\sqrt{n}] - \Phi[(-j(2k+3)$$

$$- 2k)/\sqrt{n}] - \Phi[(j(2k+3) + k)/\sqrt{n}] + \Phi[j(2k+3)/\sqrt{n}]\Big\}$$

or

$$V_4 \approx \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Big\{ -2\,\Phi\,[(j(2k+3)\;+k)/\sqrt{n}] + \Phi\,[(j(2k+3)\;+\;2k)/\sqrt{n}] + \Phi\,[j(2k+3)/\sqrt{n}] \Big\},$$

and finally the asymptotic cumulative distribution function of the maximum deficit is

$$F_{D_n/\sqrt{n}}\,(k/\sqrt{n}) = [V_0 + V_1 + V_2 + V_3 + V_4]$$

$$\approx 2\,\Phi\,(k/\sqrt{n}) - \Phi\,(2k/\sqrt{n}) - 1/2$$

$$+ \sum_{j/2=+1}^{j/2=+\infty} \Big\{ 2\,\Phi\,[(j(2k+3)\;+k)/\sqrt{n}] - 2\,\Phi\,[(j(2k+3)\;-k)/\sqrt{n}] + \Phi[(j(2k+3)\;-2k)/\sqrt{n}] - \Phi[(j(2k+3)\;+2k)/\sqrt{n}] \Big\}$$

$$+ \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \Big\{ 2\,\Phi\,[(j(2k+3)\;-k)/\sqrt{n}] - 2\Phi[(j(2k+3)+k)/\sqrt{n}] + \Phi\,[(j(2k+3)\;+2k)/\sqrt{n}] - \Phi[(j(2k+3)\;-2k)/\sqrt{n}] \Big\}$$

$$= 2\,\Phi\,(k/\sqrt{n}) - \Phi(2k/\sqrt{n}) - 1/2$$

$$+ \sum_{j=1}^{j=+\infty} (-1)^j \Big\{ 2\Phi[(j(2k+3)\;+k)/\sqrt{n}] - 2\Phi[(j2k+3)-k)/\sqrt{n}] + \Phi[(j(2k+3)\;-2k)/\sqrt{n}] - \Phi[(j(2k+3)\;+\;2k)/\sqrt{n}] \Big\}$$

$$\approx 2\,\Phi\,(k/\sqrt{n}) - \Phi\,(2k/\sqrt{n}) - 1/2$$

$$+ \sum_{j=1}^{j=+\infty} (-1)^j \Big\{ 2\,\Phi\,[(2j+1)k/\sqrt{n}] - 2\Phi[(2j-1)k/\sqrt{n}] + \Phi\,[(2j-2)k/\sqrt{n}] - \Phi[(2j+2)k/\sqrt{n}] \Big\}$$

$$\approx 4[\Phi(k/\sqrt{n}) - \Phi(3k/\sqrt{n}) + \Phi(5k/\sqrt{n}) - \Phi(7k/\sqrt{n}) + \ldots\;] - 1$$

or

$$F_{D_n/\sqrt{n}}\,(x) = 4 \sum_{(j+1)/2=1}^{(j+1)/2=+\infty} (-1)^{\frac{j-1}{2}}\;\Phi\,(jx) - 1. \qquad (5.19)$$

Consequently, the asymptotic density function of the maximum deficit is

$$f_{D_n/\sqrt{n}}\,(x) = 4 \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} (-1)^{\frac{j-1}{2}}\; j\,\phi\,(jx). \qquad (5.20)$$

The moments are easily obtained by termwise integration:

$$E(D_n/\sqrt{n}) = 4 \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} (-1)^{\frac{j-1}{2}} \int_0^\infty jx\;\phi\;(jx)\;dx$$

$$= \frac{4}{\sqrt{2\pi}} \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \frac{(-1)^{\frac{j-1}{2}}}{j}$$

$$= \frac{4}{\sqrt{2\pi}}\;(1 - 1/3 + 1/5 - 1/7 + \ldots) = \frac{4}{\sqrt{2\pi}} \cdot \frac{\pi}{4}$$

$$= \sqrt{\frac{\pi}{2}}$$

or, equivalently for inputs with variance σ^2:

$$E(D_n) \approx \sqrt{\frac{\pi}{2}} \cdot \sqrt{n} \cdot \sigma \approx 1.2533\;\sqrt{n} \cdot \sigma \qquad (5.21)$$

$$E(D_n^2/n) = 4 \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} (-1)^{\frac{j-1}{2}} \int_0^\infty j\;x^2\,\phi\;(jx)\;dx$$

$$= 2 \sum_{(j+1)/2=+1}^{(j+1)/2=+\infty} \frac{(-1)^{\frac{j-1}{2}}}{j^2}$$

$$= 2\;(1 - 1/9 + 1/25 - 1/49 + \ldots) = 1.8319\;\ldots$$

or, equivalently, for inputs with variance σ^2:

$$E(D_n^2) \approx 1.8319\;n \cdot \sigma^2 \qquad (5.22)$$

and consequently

$$\mathrm{var}(D_n) = (1.8319 - \tfrac{\pi}{2}) \cdot n \cdot \sigma^2 = 0.2611 \cdot n \cdot \sigma^2 \qquad (5.23)$$

and

$$C_V\;(D_n) = \frac{\sqrt{0.2611}}{\sqrt{\pi/2}} = 0.4077$$

where C_V stands for the coefficient of variation.

It is interesting to notice that the asymptotic mean maximum accumulated deficit for the case of full regulation is equal to the asymptotic mean adjusted range (Feller 1951).

Feller's results are compared with the results of this section in the following tabulation:

	$R_n/\sqrt{n}$	$R_n^*/\sqrt{n}$	$D_n/\sqrt{n}$
$E(\cdot)$	1.5958...	1.2533...	1.2533...
$Var(\cdot)$	0.2261...	0.0741...	0.2611...
$C_V(\cdot)$	29.80%	21.72%	40.77%

2.2 Maximum Accumulated Deficit for the Case of Partial Regulation. Consider the following probability distribution for the net input:

$$P[X_t = +1] = p$$

$$P[X_t = -1] = q$$

Clearly, $E(X_t) = p-q$ and $\mathrm{var}\;(X_t) = 4pq$.

In this case, the one-step "restricted" transition matrix P_{k+1} is

$$\begin{bmatrix} 0 & q & 0 & \dots & 0 & 0 & 0 \\ p & 0 & q & \dots & 0 & 0 & 0 \\ 0 & p & 0 & \ddots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & q & 0 \\ 0 & 0 & 0 & \dots & p & 0 & q \\ 0 & 0 & 0 & \dots & 0 & p & p \end{bmatrix}$$

The problem is that the $n\cdot$th power of this matrix cannot be obtained as easily as in the case $p=q=1/2$, where the method of images was applicable. In the present case, a procedure similar to Kac's (1947) could be used, but this writer decided simply to illustrate the fact that

$$\lim_{n\to\infty} \frac{E(D_n|\text{partial regulation})}{E(D_n|\text{full regulation})} = 0. \qquad (5.24)$$

Figure 5.3 shows the exact mean maximum deficit for the random walk process, for the cases

$$\text{(I)} \quad \mu = \frac{p-q}{2\sqrt{pq}} = 0, \qquad (p=0.5)$$

$$\text{(II)} \quad \mu = \frac{p-q}{2\sqrt{pq}} = 1/2, \qquad (p=0.723606798)$$

$$\text{(III)} \quad \mu = \frac{p-q}{2\sqrt{pq}} = 1, \qquad (p=0.853553391)$$

and for n ranging from 1 to 1000.

Figure 5.3 indicates that Eq. (5.24) is true.

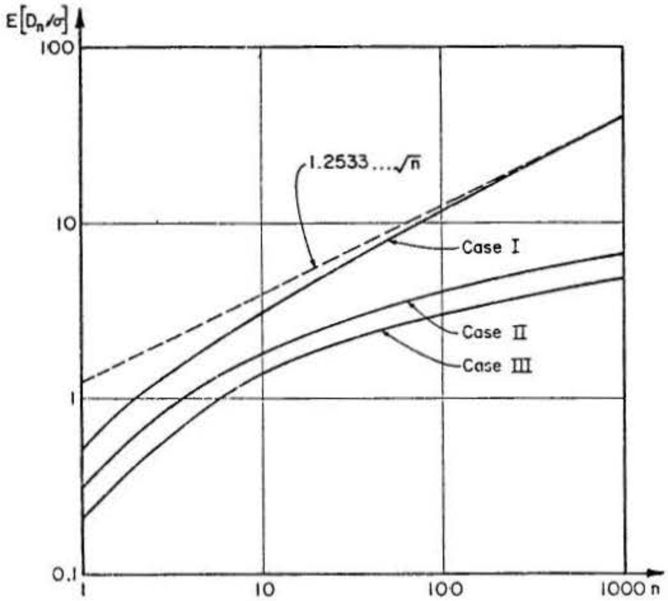

Fig. 5.3. The mean maximum accumulated deficit for the simple random walk process.

3. Continuous Net Inputs

Starting with the results obtained in the discrete case, their continuous analogues are discussed, and applications for some particular continuous distributions are made.

It is clear from Eq. (5.13),

$$d_k^{(n)^T} = \ell_k^T + d_k^{(n-1)^T} Q_k + d_0^{(n-1)} u_k^T,$$

and

$$d_0^{(n)} = \ell_{-k-1} + d_k^{(n-1)^T} p_k + d_0^{(n-1)} u_0,$$

that their continuous analogues are

$$d_n(v_n,y) = \ell(v_n,y) + \int_0^y d_{n-1}(v_{n-1},y)\, f(v_n-v_{n-1})\, dv_{n-1}$$
$$+ d_{n-1}(0,y)\cdot u(v_n), \qquad (5.25)$$

where $0 \le v_n \le y$,

and

$$d_n(0,y) = \ell(0,y) + \int_0^y d_{n-1}(v_{n-1},y)\, f(-v_{n-1})\, dv_{n-1}$$
$$+ d_{n-1}(0,y)\cdot u(0), \qquad (5.26)$$

respectively. In the above expressions $f(\cdot)$ is the density function of the net input, y is the analogue of k and $d_n(0,y)$ is $P(D_n>y) = 1 - P(D_n\le y) = 1 - F_{D_n}(y)$.

Furthermore,

$$\ell(v_n,y) = \int_{-\infty}^{-y+v_n} f(x)\, dx \qquad (5.27)$$

and

$$u(v_n) = \int_{v_n}^{+\infty} f(x)\, dx. \qquad (5.28)$$

Using the recursive relation implied by Eqs. (5.25) and (5.26),

$$d_n(0,y) = u(0)\, d_{n-1}(0,y) + \ell(0,y)$$
$$+ \int_0^y \Big\{ u(v_{n-1})\, d_{n-2}(0,y) + \ell(v_{n-1},y)$$
$$+ \int_0^y d_{n-2}(v_{n-2},y)\, f(v_{n-1}-v_{n-2})\, dv_{n-2} \Big\} f(-v_{n-1})\, dv_{n-1}$$
$$= d_{n-1}(0,y)\, u(0) + d_{n-2}(0,y) \int_0^y u(v_{n-1})\, f(-v_{n-1})\, dv_{n-1}$$
$$+ \ell(0,y) + \int_0^y \ell(v_{n-1},y)\, f(-v_{n-1})\, dv_{n-1}$$

$$+ \int_0^y \int_0^y d_{n-2}(v_{n-2},y)\ f(v_{n-1} - v_{n-2})\ f(-v_{n-1})dv_{n-2}dv_{n-1}$$

and continuing in this fashion,

$$\begin{aligned}
d_n(0,y) = d_{n-1}(0,y)\ u(0) & \\
& + d_{n-2}(0,y) \int_0^y u(v_{n-1})\ f(-v_{n-1})\ dv_{n-1} \\
& + d_{n-3}(0,y) \int_0^y \int_0^y u(v_{n-2})\ f(v_{n-1} \\
& - v_{n-2})\ f(-v_{n-1})\ dv_{n-2}\ dv_{n-1} \\
& + d_{n-4}(0,y) \int_0^y \int_0^y \int_0^y u(v_{n-3})\ f(v_{n-2} \\
& - v_{n-3})\ f(v_{n-1} - v_{n-2})\ f(-v_{n-1}) \\
& \cdot dv_{n-3}\ dv_{n-2}\ dv_{n-1} \\
& \cdot \\
& \cdot \\
& \cdot \\
& \cdot \\
& + d_1(0,y) \int_0^y \ldots (n-2) \ldots \int_0^y u(v_2)\ f(v_3 \\
& - v_2) \ldots f(v_{n-1} - v_{n-2})\ f(-v_{n-1}). \\
& \cdot \quad \cdot dv_2\ dv_3 \ldots dv_{n-1} \\
& + \ell(0,y) \\
& + \int_0^y \ell(v_{n-1},y)\ f(-v_{n-1})\ dv_{n-1} \\
& + \int_0^y \int_0^y \ell(v_{n-2},y)\ f(v_{n-1} \\
& - v_{n-2})\ f(-v_{n-1})\ dv_{n-2}\ dv_{n-1} \\
& + \int_0^y \int_0^y \int_0^y \ell(v_{n-3},y)\ f(v_{n-2}-v_{n-3})\ f(v_{n-1} \\
& - v_{n-2})\ f(-v_{n-1})dv_{n-3}dv_{n-2}dv_{n-1} \\
& \cdot \\
& \cdot \\
& \cdot \\
& \cdot \\
& + \int_0^y \ldots (n-1) \ldots \int_0^y \ell(v_1,y)\ f(v_2 \\
& - v_1) \ldots f(v_{n-1} - v_{n-2})\ f(-v_{n-1}). \\
& \cdot dv_1\ dv_2 \ldots dv_{n-1} \qquad (5.29)
\end{aligned}$$

where $d_1(0,y) = \int_{-\infty}^{-y} f(x)dx = 1 - F_{D_1}(y)$.

3.1 Normally Distributed Net Inputs. In this case, the density of the net input is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} = \phi(x-\mu).$$

Notice that the mean net input is μ and that the variance is unity, without loss of generality. Clearly, for the case of full regulation, $\mu = 0$.

For $n = 1$, the results are obvious but they will be presented here for illustration purposes. In this case, one has

$$d_1(0,y) = \int_{-\infty}^{-y} \phi(x-\mu)\ dx = \Phi(-y-\mu) = 1 - \Phi(y+\mu) \qquad (5.30)$$

and thus $F_{D_1}(y) = \Phi(y+\mu)$ for $y \geq 0$

is the cumulative distribution function of D_1. The density function is then

$$f_{D_1}(y) = \phi(y+\mu) \quad \text{for } y \geq 0$$

and the mean value of the maximum deficit is

$$\begin{aligned}
E(D_1) &= \int_0^\infty y\ \phi(y+\mu)\ dy = \int_\mu^\infty (w-\mu)\ \phi(w)\ dw \\
&= \int_\mu^\infty w\ \phi(w)\ dw - \int_\mu^\infty \mu\ \phi(w)\ dw = \phi(\mu) - \mu[1 - \Phi(\mu)].
\end{aligned}$$

In the case of full regulation, $\mu = 0$, then

$$E(D_1) = \phi(0) = \frac{1}{\sqrt{2\pi}}$$

The second moment of D_1 is

$$\begin{aligned}
E(D_1^2) &= \int_0^\infty y^2\ \phi(y+\mu)\ dy = \int_\mu^\infty (w-\mu)^2\ \phi(w)\ dw \\
&= \int_\mu^\infty w^2\ \phi(w)\ dw - 2\mu \int_\mu^\infty w\,\phi(w)\ dw + \mu^2 \int_\mu^\infty \phi(w)\ dw
\end{aligned}$$

where $\int_\mu^\infty w^2 \phi(w)\ dw = w\,\phi(w)\big|_\infty^\mu + \int_\mu^\infty \phi(w)\ dw$

and thus

$$\begin{aligned}
E(D_1^2) &= \mu\ \phi(\mu) + (1 + \mu^2)\ [1 - \Phi(\mu)] - 2\mu\ \phi(\mu) \\
&= (1 + \mu^2)\ [1 - \Phi(\mu)] - \mu\ \phi(\mu).
\end{aligned}$$

In the case of full regulation, $\mu = 0$, then

$$E(D_1^2) = 1 - \Phi(0) = 1/2.$$

Some values of $E(D_1)$ and $E(D_1^2)$ are given in the following tabulation.

μ	0	1/2	1	3/2	2
$E(D_1)$	0.3989	0.1979	0.0833	0.0293	0.0084
$E(D_1^2)$	0.5000	0.2096	0.0754	0.0229	0.0060

Assuming that the gross input (natural discharge) has the coefficient of variation equal to 0.25 and that the output is constant, Eq. (5.9) says that the level of regulation is 100%, 87.5%, 75%, 62.5% and 50%, respectively for $\mu = 0$, 0.5, 1, 1.5, and 2. Figure 5.4 emphasizes the fact that the mean maximum deficit decreases very fast as the level of regulation decreases.

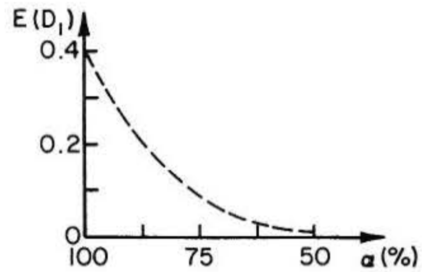

Fig. 5.4. Expected value of D_1 as a function of the level of regulation for independent normal net inputs.

For the case $n = 2$, Eq. (5.29) gives

$$d_2(0,y) = d_1(0,y)\,u(0) + \ell(0,y) + \int_0^y \ell(v_1,y)\,f(-v_1)\,dv_1 .$$

Recalling that $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} = \phi(x-\mu)$, Eqs. (5.27), (5.28) and (5.30) can be used as follows:

$$\ell(v_1,y) = \int_{-\infty}^{-y+v_1} f(x)dx = \int_{-\infty}^{-y+v_1} \phi(x-\mu)dx = \Phi(v_1-y-\mu)$$

$$\ell(0,y) = \Phi(-y-\mu) = 1 - \Phi(y+\mu)$$

$$u(0) = \int_0^\infty f(x)dx = \int_0^\infty \phi(x-\mu)\,dx = \Phi(\mu)$$

$$d_1(0,y) = 1 - \Phi(y+\mu)$$

and thus

$$d_2(0,y) = \Phi(\mu)\cdot[1 - \Phi(\mu+y)] + [1 - \Phi(\mu+y)] + \int_0^y \Phi(v_1-y-\mu)\,\phi(-v_1-\mu)\,dv_1$$

$$= [1 + \Phi(\mu)][1 - \Phi(\mu+y)] + \int_0^y \Phi(v_1-y-\mu)\phi(v_1+\mu)\,dv ,$$

and consequently

$$F_{D_2}(y) = 1 - d_2(0,y)$$

$$= \Phi(\mu+y) - \Phi(\mu) + \Phi(\mu)\,\Phi(\mu+y) - \int_0^y \Phi(v_1-y-\mu)\,\phi(v_1+\mu)\,dv_1 \tag{5.31}$$

and

$$f_{D_2}(y) = \phi(\mu+y) + \Phi(\mu)\,\phi(\mu+y) + \int_0^y \phi(v_1-y-\mu)\phi(v_1+\mu)dv_1 - \Phi(-\mu)\,\phi(\mu+y)$$

$$= 2\,\Phi(\mu)\,\phi(\mu+y) + \int_0^y \phi(v_1-y-\mu)\,\phi(v_1+\mu)\,dv_1.$$

Noticing that

$$\int_0^y \phi(v_1-y-\mu)\,\phi(v_1+\mu)\,dv_1 = \int_{-\mu}^{y-\mu} \phi(w-y)\,\phi(w+2\mu)\,dw$$

$$= \int_{-\mu}^{y-\mu} \phi(\sqrt{2}w - \frac{y-2\mu}{\sqrt{2}})\,\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)\,dw$$

$$= \frac{1}{\sqrt{2}}\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)\int_{-\frac{y}{\sqrt{2}}}^{\frac{y}{\sqrt{2}}} \phi(z)\,dz = \frac{1}{\sqrt{2}}\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)[2\Phi(\frac{y}{\sqrt{2}}) - 1]$$

one has, finally,

$$f_{D_2}(y) = 2\Phi(\mu)\,\phi(\mu+y) + \frac{1}{\sqrt{2}}\,\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)\,[2\Phi(\frac{y}{\sqrt{2}}) - 1]. \tag{5.32}$$

Notice that a probability mass exists in the point $D_2 = 0$. Its value is given by Eq. (5.31), making $y = 0$:

$$P(D_2=0) = \Phi^2(\mu)$$

and in the case of full regulation, $\mu = 0$, then

$P(D_2=0) = 1/4$, as it should.

The mean value of D_2 can now be found

$$E(D_2) = \int_0^\infty y\,f_{D_2}(y)\,dy$$

$$= 2\Phi(\mu)\int_0^\infty y\,\phi(\mu+y)\,dy$$

$$+ 2\int_0^\infty \frac{1}{\sqrt{2}}\cdot y\cdot\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)\,\Phi(\frac{y}{\sqrt{2}})\,dy$$

$$- \int_0^\infty \frac{1}{\sqrt{2}}\cdot y\,\phi(\frac{y}{\sqrt{2}} + \sqrt{2}\mu)dy$$

$$= 2\Phi(\mu)\int_0^\infty (\mu+y)\,\phi(\mu+y)dy - 2\Phi(\mu)\int_0^\infty \mu\,\phi(\mu+y)dy$$

$$+ 2 \int_0^\infty \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) \phi\left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) \Phi \left(\frac{y}{\sqrt{2}}\right) dy$$

$$- 2 \int_0^\infty \sqrt{2}\mu \, \phi \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) \Phi \left(\frac{y}{\sqrt{2}}\right) dy$$

$$- \int_0^\infty \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) \phi \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) dy + \int_0^\infty \sqrt{2}\mu \, \phi \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) dy$$

$$= 2 \Phi (\mu) \int_\mu^\infty w \, \phi(w) dw - 2\mu \, \Phi (\mu) \int_\mu^\infty \phi(w) dw$$

$$+ 2\sqrt{2} \int_{\sqrt{2}\mu}^\infty w \, \phi (w) \, \Phi (w-\sqrt{2}\mu) dw - 4\mu \int_{\sqrt{2}\mu}^\infty \phi (w) \, \Phi (w-\sqrt{2}\mu) dw$$

$$- \sqrt{2} \int_{\sqrt{2}\mu}^\infty w \, \phi (w) dw + 2\mu \int_{\sqrt{2}\mu}^\infty \phi (w) dw$$

$$= 2\Phi (\mu) \, \phi (\mu) - 2\mu \, \Phi (\mu) \, [1-\Phi(\mu)]$$

$$+ 2\sqrt{2} \int_{\sqrt{2}\mu}^\infty w \, \phi(w) \, \Phi (w-\sqrt{2}\mu) dw - 4\mu \int_{\sqrt{2}\mu}^\infty \phi(w) \, \Phi (w-\sqrt{2}\mu) dw$$

$$- \sqrt{2} \, \phi (\sqrt{2}\mu) + 2\mu \, [1-\Phi (\sqrt{2}\mu)]. \tag{5.33}$$

To evaluate the integral

$$\int_{\sqrt{2}\mu}^\infty \phi(w) \, \Phi (w-\sqrt{2}\mu) dw$$

one may use the fact that

$$\int_0^\infty f_{D_2} (y) \, dy = 1 - \Phi^2(\mu)$$

or using Eq. (5.32)

$$2\Phi(\mu) \int_0^\infty \phi (y+\mu) \, dy + 2 \int_0^\infty \frac{1}{\sqrt{2}} \phi \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) \Phi \left(\frac{y}{\sqrt{2}}\right) dy$$

$$- \int_0^\infty \frac{1}{\sqrt{2}} \phi \left(\frac{y}{\sqrt{2}} + \sqrt{2}\mu\right) dy = 1 - \Phi^2(\mu)$$

or

$$2 \, \Phi (\mu) \int_\mu^\infty \phi (w) \, dw + 2 \int_{\sqrt{2}\mu}^\infty \phi(w) \Phi (w-\sqrt{2}\mu) \, dw$$

$$- \int_{\sqrt{2}\mu}^\infty \phi (w) \, dw = 1 - \Phi^2(\mu)$$

or

$$2 \, \Phi(\mu) \, [1-\Phi(\mu)] + 2 \int_{\sqrt{2}\mu}^\infty \phi(w) \Phi (w-\sqrt{2}\mu) \, dw$$

$$- [1-\Phi(\sqrt{2}\mu)] = 1 - \Phi^2 (\mu)$$

or finally,

$$2 \int_{\sqrt{2}\mu}^\infty \phi(w) \, \Phi (w-\sqrt{2}\mu) dw = 2 + \Phi^2(\mu) - 2\Phi(\mu) - \Phi (\sqrt{2}\mu). \tag{5.34}$$

The integral

$$\int_{\sqrt{2}\mu}^\infty w \, \phi (w) \, \Phi (w - \sqrt{2}\mu) dw$$

can be obtained as follows:

$$\int_{\sqrt{2}\mu}^\infty w \, \phi (w) \, \Phi (w-\sqrt{2}\mu) dw = - \phi (w) \, \Phi (w-\sqrt{2}\mu) \Big|_{\sqrt{2}\mu}^\infty$$

$$+ \int_{\sqrt{2}\mu}^\infty \phi (w) \, \phi (w-\sqrt{2}\mu) dw$$

$$= \frac{1}{2} \phi (\sqrt{2}\mu) + \int_{\sqrt{2}\mu}^\infty \phi (w) \, \phi (w-\sqrt{2}\mu) dw$$

$$= \frac{1}{2} \phi (\sqrt{2}\mu) + \int_{\sqrt{2}\mu}^\infty \phi (\sqrt{2}w-\mu) \, \phi (\mu) dw$$

$$= \frac{1}{2} \phi (\sqrt{2}\mu) + \frac{1}{\sqrt{2}} \phi(\mu) \int_\mu^\infty \phi (z) dz$$

$$= \frac{1}{2} \phi (\sqrt{2}\mu) + \frac{1}{\sqrt{2}} \phi (\mu) \, [1 - \Phi (\mu)] . \tag{5.35}$$

Substituting Eqs. (5.34) and (5.35) in Eq. (5.33) after simplifications one has

$$E(D_2) = 2 \, \phi (\mu) - 2\mu \, [1 - \Phi (\mu)] \tag{5.36}$$

and in particular, for full regulation, $\mu = 0$ and $E(D_2) = \sqrt{2/\pi}$. Notice that $E(D_2) = 2 \, E(D_1)$ and thus Fig. 5.4 holds, when the values of $E(D_1)$ are multiplied by two.

Recalling that the asymptotic mean maximum deficit is proportional to $\sqrt{n}$, it is clear that the fact that $E(D_2) = 2 \, E(D_1)$ is simply a transient effect.

For higher values of n $(n = 3,4,\ldots)$, the problem is similar to the one encountered in the study of the distribution of the range of partial sums: integrals which do not exist in closed formula will appear and will have to be evaluated numerically. Following the same reasoning exposed in Section IV 2.1, this writer decided to approach numerically the entire distribution of the maximum deficit, rather than to solve numerically only parts of it.

The most convenient algorithm to this particular numerical integration is to choose a binomial input such as the one given by Eq. (4.24), for a large value of m, and to apply Eq. (5.13) recursively. Notice that the selection of m in the analogue discrete distribution is tantamount to the selection of the increment Δy in a conventional numerical integration algorithm.

The numerically obtained density function of D_n is shown in Fig. 5.5 for small values of n and for the case of full regulation ($\mu = 0$). In this figure, as well as in the next ones, the probability mass at $D_n = 0$ is not shown, and it is given in all cases by $[\Phi(\mu)]^n$. Figure 5.6 compares the exact density function of $(D_n/\sqrt{n})$ for $n = 8$ and $n = 30$ with the asymptotic density function, for the case of full regulation. Notice that D_n has been expressed in terms

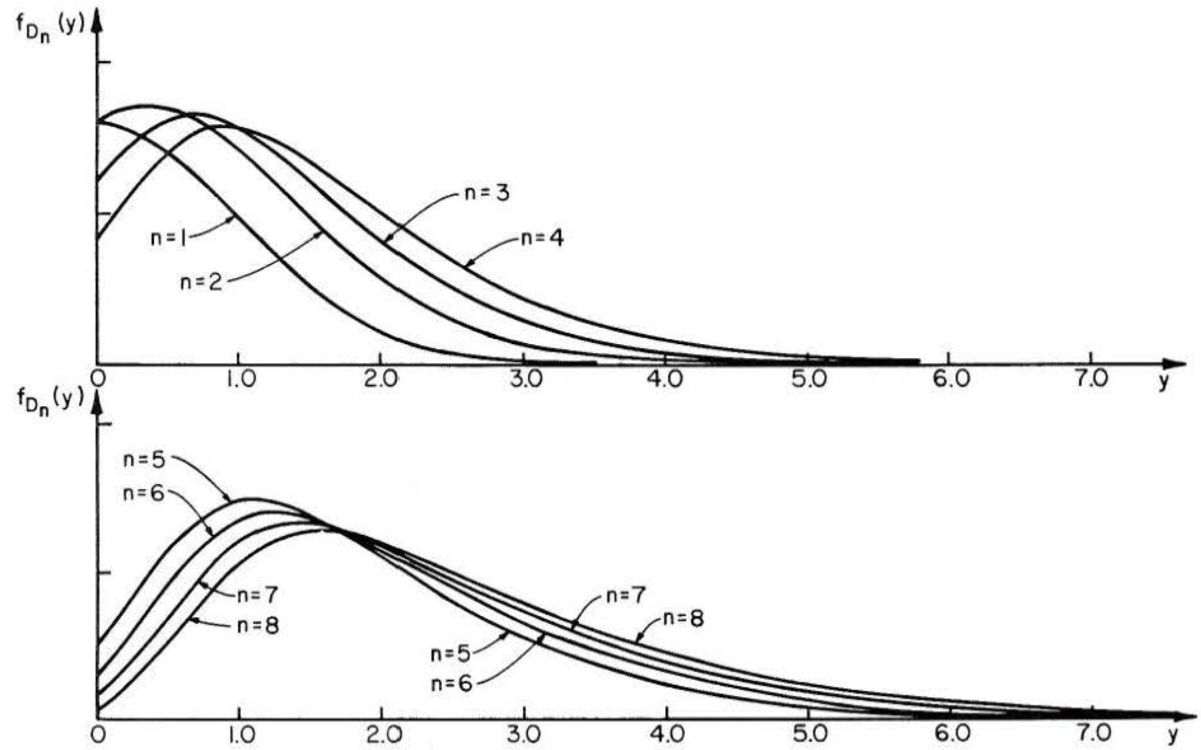

Fig. 5.5. Distribution of D_n for independent normal net inputs (μ=0; n=1,2,...,7,8).

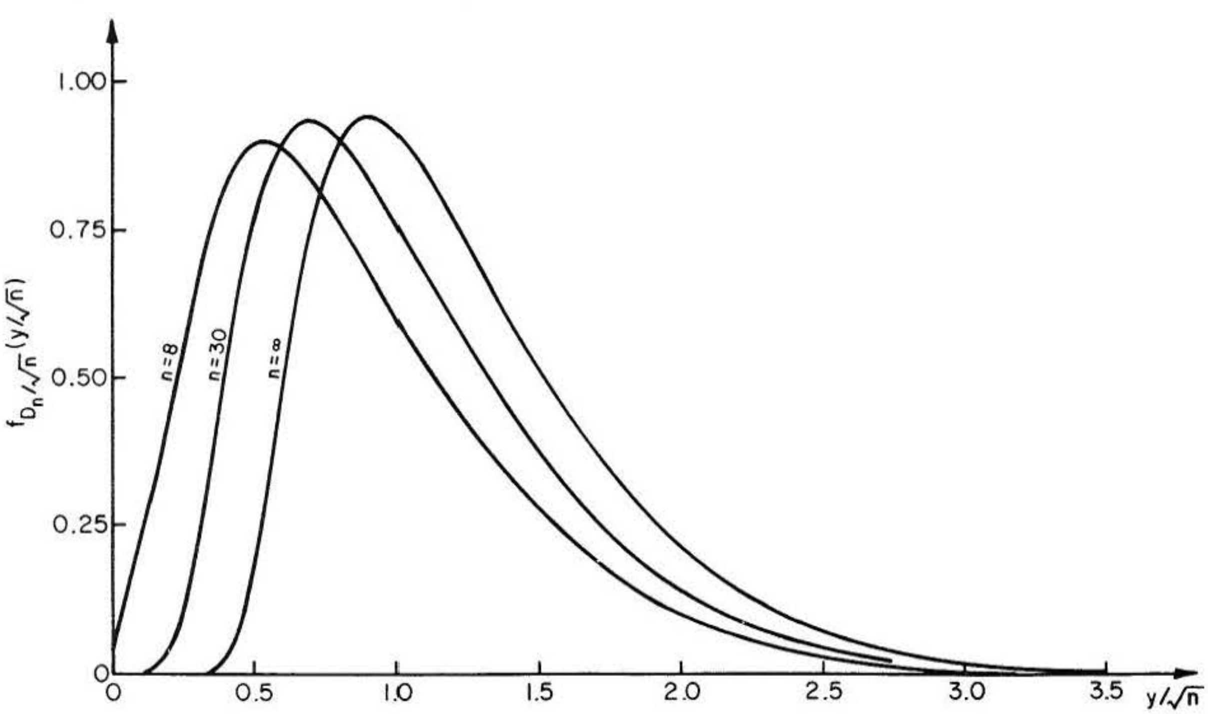

Fig. 5.6. Distribution of $D_n/\sqrt{n}$ for independent normal net inputs (μ=0; n=8,30,∞).

of $\sqrt{n}$ to allow such comparison. Figure 5.7 compares the standardized exact density of D_n (for $n = 15$ and $\mu = 0$) with the standardized asymptotic density. Recall that the same comparison has been made in Chapter IV, for the standardized distribution of the range. In the present case the convergence of the standardized exact density to the standardized asymptotic density is slower because of the influence of the probability mass at $D_n = 0$.

The case of partial regulation is illustrated by Fig. 5.8 for $\mu = 1$. Notice that D_n has *not* been expressed in units of $\sqrt{n}$, indicating that the mean maximum deficit does *not* increase as $\sqrt{n}$.

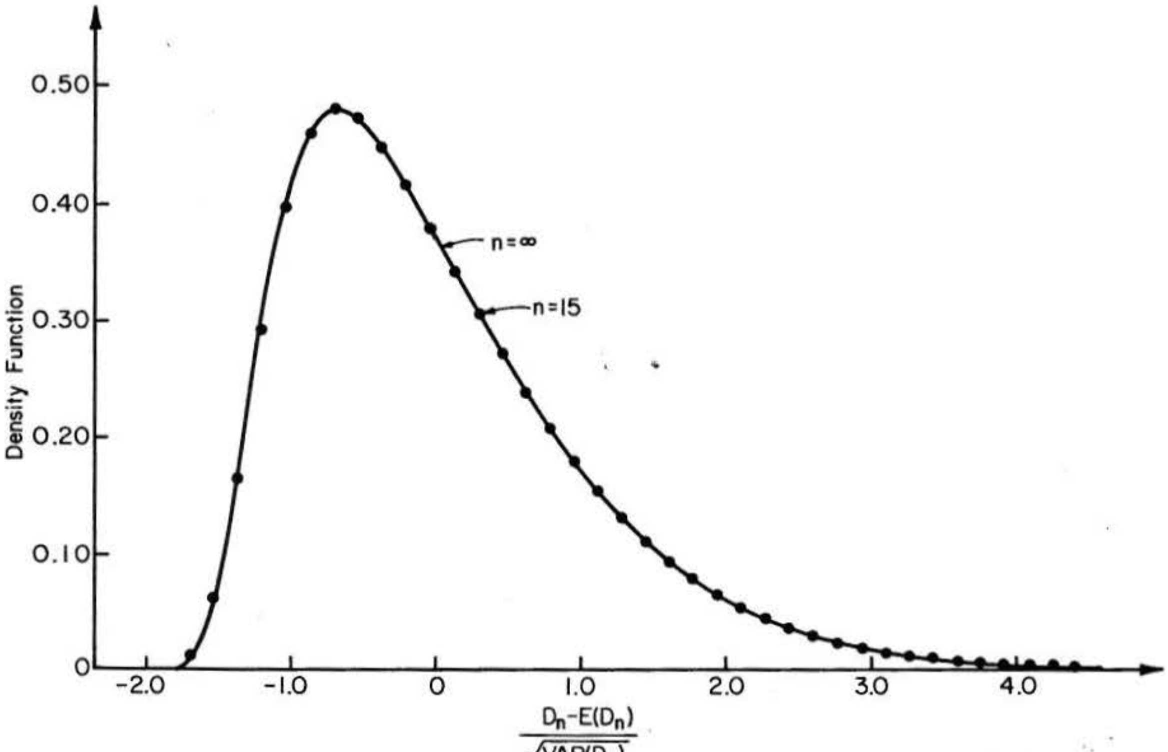

Fig. 5.7. Distribution of $[D_n-E(D_n)]/\sqrt{Var(D_n)}$ for independent normal net inputs (μ=0; n=15 and n=∞).

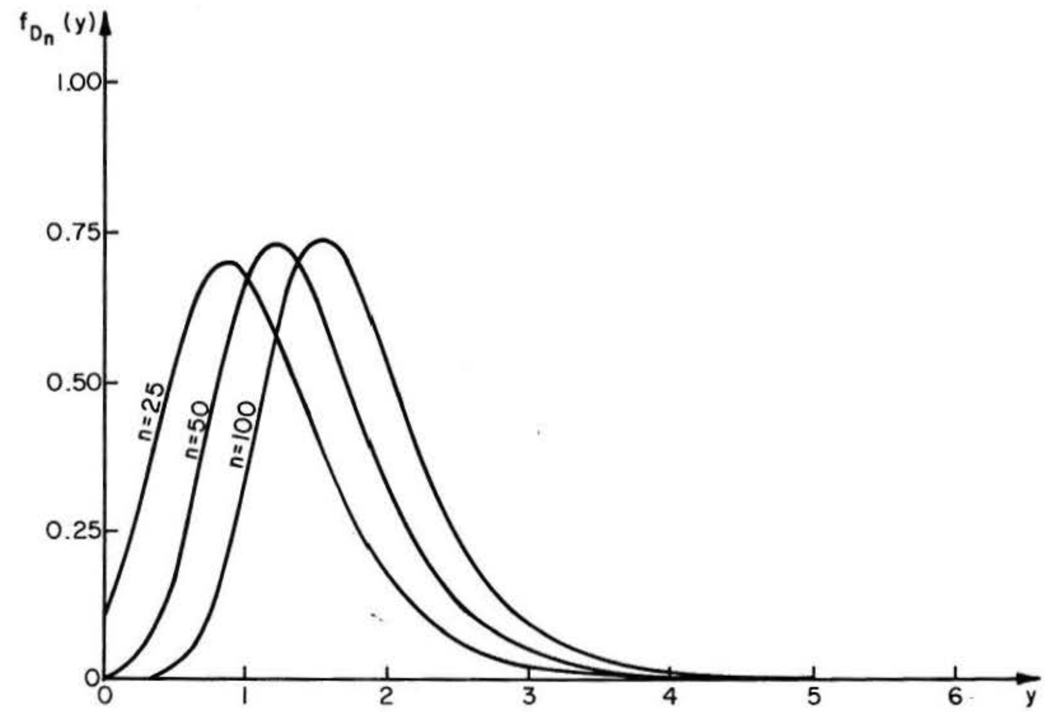

Fig. 5.8. Distribution of D_n for independent normal net inputs (μ=1; n=25,50,100).

Finally, the mean and the variance of D_n are shown for various values of n and μ, in Figs. 5.9 and 5.10. Recall that the level of regulation can be found by using Eqs. (5.8) and (5.9).

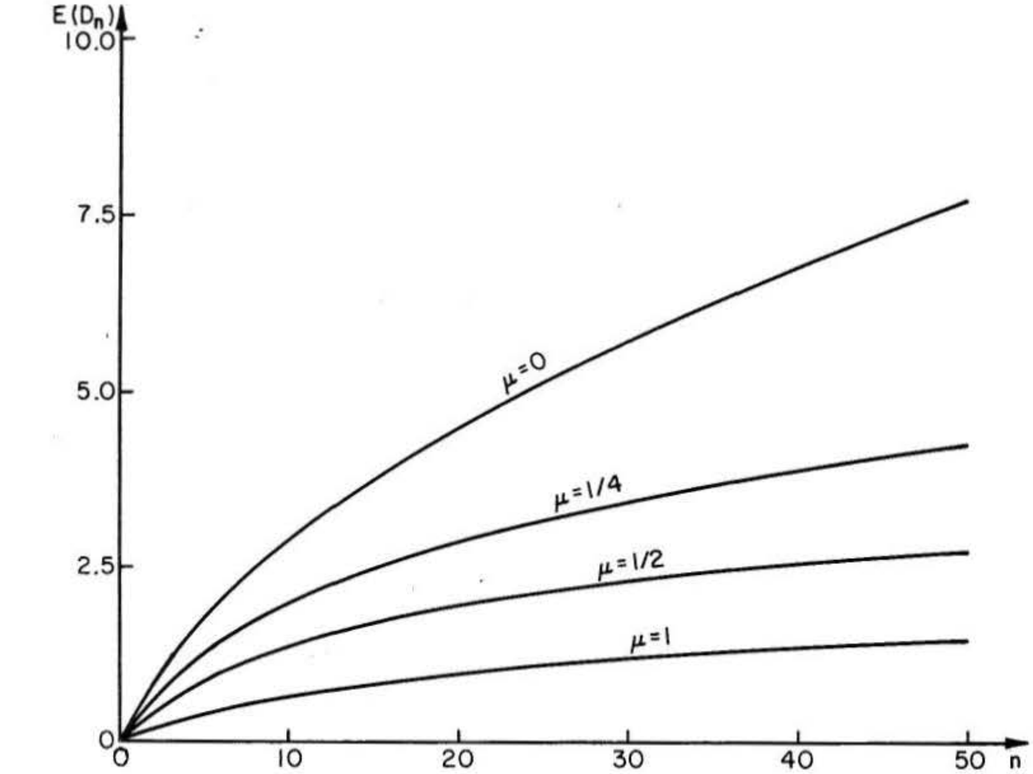

Fig. 5.9. Expected value of D_n for independent normal net inputs (μ=0,1/4,1/2,1).

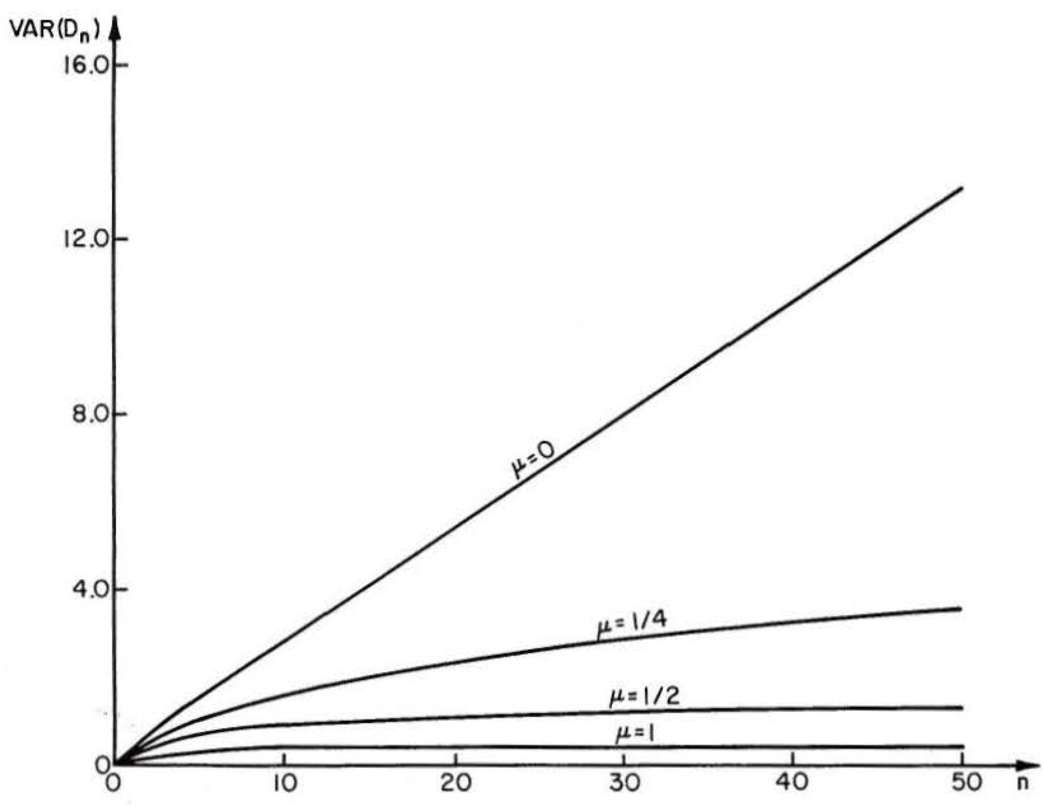

Fig. 5.10. Variance of D_n for independent normal net inputs (μ=0,0.25,0.50,1.00).

3.2 Laplace Distributed Net Inputs. Laplace distributed net inputs are studied here because solutions in closed form always exist.

In this case, the density function of the net input is

$$f(x) = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,|x-\mu|}. \qquad (5.37)$$

Notice that the mean net input is μ and that its variance is unity. In considering unit variance, there is no loss of generality.

Using Eq. (5.29) for $n = 1$,

$$d_1(0,y) = \int_{-\infty}^{-y} f(x)dx$$

$$= \frac{\sqrt{2}}{2} \int_{-\infty}^{-y} e^{-\sqrt{2}|x-\mu|}dx = \frac{\sqrt{2}}{2} \int_{-\infty}^{-y} e^{+\sqrt{2}(x-\mu)}dx$$

$$= \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu} \int_{-\infty}^{-y} e^{\sqrt{2}x}dx = \frac{1}{2} e^{-\sqrt{2}(\mu+y)} \qquad (5.38)$$

and thus the cumulative distribution function of the maximum deficit is

$$F_{D_1}(y) = 1 - \frac{1}{2} e^{-\sqrt{2}(\mu+y)},$$

and the probability density function is

$$f_{D_1}(y) = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,(\mu+y)}, \qquad (5.39)$$

and the moments of D_1 can be obtained as

$$E(D_1) = \int_0^{\infty} y\, f_{D_1}(y)\, dy = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu} \int_0^{\infty} y\, e^{-\sqrt{2}\,y}\, dy$$

$$= \frac{\sqrt{2}}{4} e^{-\sqrt{2}\mu}, \qquad (5.40)$$

and

$$E(D_1^2) = \int_0^{\infty} y^2 f_{D_1}(y)\, dy = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu} \int_0^{\infty} y^2\, e^{-\sqrt{2}\,y}\, dy$$

$$= \frac{1}{2} e^{-\sqrt{2}\mu}. \qquad (5.41)$$

Some values of $E(D_1)$ and $E(D_1^2)$ are given in the following tabulation:

μ	0	1/2	1	3/2	2
$E(D_1)$	0.3536	0.1743	0.0860	0.0424	0.0209
$E(D_1^2)$	0.5000	0.2465	0.1216	0.0599	0.0296

Using Eq. (5.9) and assuming that the output is constant and that the gross input (natural discharge) has the coefficient of variation 0.25, the level of regulation is 100% for $\mu = 0$, 87.5% for $\mu = 0.5$, 75% for $\mu = 1$, 62.5% for $\mu = 1.5$, and 50% for $\mu = 2$. In this case, the relationship between the maximum deficit and the level of regulation is shown in Fig. 5.11.

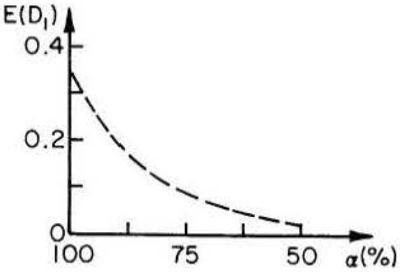

Fig. 5.11. Expected value of D_1 as a function of the level of regulation for independent Laplacian net inputs.

For higher values of n, Eqs. (5.27) and (5.28) will be used:

$$\ell(v_n,y) = \int_{-\infty}^{-y+v_n} f(x)\, dx = \int_{-\infty}^{-y+v_n} \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,|x-\mu|}dx.$$

Recalling that $v_n \leq y$

$$\ell(v_n,y) = \int_{-\infty}^{-y+v_n} \frac{\sqrt{2}}{2} e^{+\sqrt{2}\,(x-\mu)}\, dx = \frac{1}{2} e^{-\sqrt{2}\,(\mu+y-v_n)}. \qquad (5.42)$$

Similarly,

$$u(v_n) = \int_{v_n}^{+\infty} \frac{\sqrt{2}}{2} e^{-\sqrt{2}\,|x-\mu|}\, dx$$

$$= \frac{\sqrt{2}}{2} \int_{v_n}^{\mu} e^{\sqrt{2}\,(x-\mu)}\, dx + \frac{\sqrt{2}}{2} \int_{\mu}^{+\infty} e^{-\sqrt{2}\,(x-\mu)}\, dx$$

$$= \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu} \int_{v_n}^{\mu} e^{\sqrt{2}\,x}\, dx + \frac{\sqrt{2}}{2} e^{\sqrt{2}\mu} \int_{\mu}^{\infty} e^{-\sqrt{2}\,x}\, dx$$

$$= \frac{1}{2} e^{-\sqrt{2}\mu}[e^{\sqrt{2}\mu} - e^{\sqrt{2}\,v_n}] + \frac{1}{2} e^{\sqrt{2}\mu} \cdot e^{-\sqrt{2}\mu}$$

$$= 1 - \frac{1}{2} e^{-\sqrt{2}(\mu-v_n)}. \qquad (5.43)$$

Eq. (5.43) is valid for $0 \leq v_n \leq \mu$.
For the case $v_n \geq \mu$,

$$u(v_n) = \frac{\sqrt{2}}{2} \int_{v_n}^{+\infty} e^{-\sqrt{2}\,(x-\mu)}\, dx = \frac{\sqrt{2}}{2} e^{\sqrt{2}\mu} \int_{v_n}^{+\infty} e^{-\sqrt{2}\,x}\, dx$$

$$= \frac{1}{2} e^{\sqrt{2}\mu} e^{-\sqrt{2}v_n} = \frac{1}{2} e^{-\sqrt{2}\,(v_n-\mu)}. \qquad (5.44)$$

For $n = 2$, Eq. (5.29) reduces to

$$d_2(0,y) = d_1(0,y)\, u_0 + \ell(0,y) + \int_0^y \ell(v_1,y)\, f(-v_1)\, dv_1,$$

and now Eqs. (5.38), (5.42) and (5.43) can be used:

$$d_2(0,y) = \frac{1}{2} e^{-\sqrt{2}(\mu+y)} [1 - \frac{1}{2} e^{-\sqrt{2}\mu}] + \frac{1}{2} e^{-\sqrt{2}(\mu+y)}$$

$$+ \int_0^y \frac{1}{2} e^{-\sqrt{2}(\mu+y-v_1)} \frac{\sqrt{2}}{2} \cdot e^{-\sqrt{2}|-v_1-\mu|} dv_1,$$

and thus

$$d_2(0,y) = e^{-\sqrt{2}(\mu+y)} - \frac{1}{4} e^{-\sqrt{2}(2\mu+y)}$$

$$+ \frac{\sqrt{2}}{4} \int_0^y e^{-\sqrt{2}(2\mu+y)} dv_1,$$

or

$$d_2(0,y) = e^{-\sqrt{2}(\mu+y)} - \frac{1}{4} e^{-\sqrt{2}(2\mu+y)} + \frac{\sqrt{2}}{4} y\, e^{-\sqrt{2}(2\mu+y)}. \tag{5.45}$$

Consequently, the cumulative distribution function of the maximum deficit is

$$F_{D_2}(y) = 1 - e^{-\sqrt{2}(\mu+y)} + \frac{1}{4} e^{-\sqrt{2}(2\mu+y)} - \frac{\sqrt{2}}{4} y e^{-\sqrt{2}(2\mu+y)}$$

and the density function is

$$f_{D_2}(y) = \sqrt{2}\, e^{-\sqrt{2}(\mu+y)} - \frac{\sqrt{2}}{2} e^{-\sqrt{2}(2\mu+y)} + \frac{1}{2} y e^{-\sqrt{2}(2\mu+y)}.$$

Furthermore, the mean value of the maximum deficit is

$$E(D_2) = \int_0^\infty [1 - F_{D_2}(y)]dy = \int_0^\infty d_2(0,y)dy$$

$$= e^{-\sqrt{2}\mu} \int_0^\infty e^{-\sqrt{2}\, y} dy - \frac{1}{4} e^{-2\sqrt{2}\mu} \int_0^\infty e^{-\sqrt{2}\, y} dy$$

$$+ \frac{\sqrt{2}}{4} e^{-2\sqrt{2}\mu} \int_0^\infty y\, e^{-\sqrt{2}\, y} dy$$

$$= \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu} - \frac{\sqrt{2}}{8} e^{-2\sqrt{2}\mu} + \frac{\sqrt{2}}{8} e^{-2\sqrt{2}\mu} = \frac{\sqrt{2}}{2} e^{-\sqrt{2}\mu}. \tag{5.46}$$

Notice that $E(D_2) = 2E(D_1)$, which was also found to be true in the case of normal inputs.

The second moment of the maximum deficit is

$$E(D_2^2) = \int_0^\infty 2y[1 - F_{D_2}(y)]dy = \int_0^\infty 2y\, d_2(0,y)dy$$

$$= 2\, e^{-\sqrt{2}\mu} \int_0^\infty y\, e^{-\sqrt{2}\, y} dy - \frac{1}{2} e^{-2\sqrt{2}\mu} \int_0^\infty y\, e^{-\sqrt{2}\, y} dy$$

$$+ \frac{\sqrt{2}}{2} e^{-2\sqrt{2}\mu} \int_0^\infty y^2\, e^{-\sqrt{2}\, y} dy$$

$$= e^{-\sqrt{2}\mu} - \frac{1}{4} e^{-2\sqrt{2}\mu} + \frac{1}{2} e^{-2\sqrt{2}\mu} = e^{-\sqrt{2}\mu} + \frac{1}{4} e^{-2\sqrt{2}\mu}.$$

In the case of Laplacian inputs, Eq. (5.29) can always be solved, leading to the exact distribution of the maximum deficit, in closed form, for any value of n. The integrals involved can be easily, although tediously, performed.

The case $n = 3$ will be studied now to show how tedious the procedure is, and also to emphasize that $E(D_3)$ is not equal to $3 \cdot E(D_1)$, as the reader may be led to believe due to the fact that $E(D_2) = 2 \cdot E(D_1)$. Of course, $E(D_n)$ could not be equal to $n \cdot E(D_1)$, for the asymptotic mean deficit was shown to vary as $\sqrt{n}$.

Using Eq. (5.29) for $n = 3$,

$$d_3(0,y) = d_2(0,y)\, u(o) + d_1(0,y) \int_0^y u(v_2)\, f(-v_2)\, dv_2$$

$$+ \ell(0,y) + \int_0^y \ell(v_2,y)\, f(-v_2)\, dv_2$$

$$+ \int_0^y \int_0^y \ell(v_1,y)\, f(v_2-v_1)\, f(-v_2)\, dv_1\, dv_2 \tag{5.47}$$

where $f(\cdot)$ is the density function of the net input and where the following expressions are known from Eqs. (5.38), (5.42), (5.43), and (5.45):

$$d_1(0,y) = \frac{1}{2} e^{-\sqrt{2}(\mu+y)}$$

$$\ell(v_2,y) = \frac{1}{2} e^{-\sqrt{2}(\mu+y-v_2)}$$

$$\ell(0,y) = \frac{1}{2} e^{-\sqrt{2}(\mu+y)}$$

$$u(0) = 1 - \frac{1}{2} e^{-\sqrt{2}\mu}$$

$$d_2(0,y) = e^{-\sqrt{2}(\mu+y)} - \frac{1}{4} e^{-\sqrt{2}(2\mu+y)} + \frac{\sqrt{2}}{4}\, y\, e^{-\sqrt{2}(2\mu+y)}$$

and the following expressions have to be evaluated:

i) $$\int_0^y \ell(v_2,y)\, f(-v_2)dv_2$$

$$= \frac{\sqrt{2}}{4} \int_0^y e^{-\sqrt{2}(\mu+y-v_2)}\, e^{-\sqrt{2}|-v_2-\mu|}\, dv_2$$

$$= \frac{\sqrt{2}}{4} \int_0^y e^{-\sqrt{2}(2\mu+y)}\, dv_2 = \frac{\sqrt{2}}{4} y\, e^{-\sqrt{2}(2\mu+y)}$$

ii) $$\int_0^y u(v_2)\, f(-v_2)dv_2 = J_1.$$

Using Eqs. (5.43) and (5.44), for $y \geq \mu$

$$J_1 = \int_0^\mu [1 - \frac{1}{2} e^{-\sqrt{2}(\mu-v_2)}] \frac{\sqrt{2}}{2} e^{-\sqrt{2}|-v_2-\mu|} dv_2$$

$$+ \int_\mu^y \frac{1}{2} e^{-\sqrt{2}(v_2-\mu)} \cdot \frac{\sqrt{2}}{2} e^{-\sqrt{2}|-v_2-\mu|} dv_2$$

$$= \frac{1}{2}(e^{-\sqrt{2}\mu} - e^{-2\sqrt{2}\mu}) - \frac{\sqrt{2}\mu}{4} e^{-2\sqrt{2}\mu} + \frac{1}{8}(e^{-2\sqrt{2}\mu} - e^{-2\sqrt{2}\ y})$$

$$= \frac{1}{2} e^{-\sqrt{2}\mu} - \frac{3}{8} e^{-2\sqrt{2}\mu} - \frac{\sqrt{2}\mu}{4} e^{-2\sqrt{2}\mu} - \frac{1}{8} e^{-2\sqrt{2}\ y}$$

and for $y \le \mu$,

$$J_1 = \int_0^y [1 - \frac{1}{2} e^{-\sqrt{2}(\mu-v_2)}] \cdot \frac{\sqrt{2}}{2} \cdot e^{-\sqrt{2}|-v_2-\mu|} dv_2$$

$$= \frac{1}{2} [e^{-\sqrt{2}\mu} - e^{-\sqrt{2}(\mu+y)}] - \frac{\sqrt{2}}{4} y\, e^{-2\sqrt{2}\mu}$$

$$= \frac{1}{2} e^{-\sqrt{2}\mu} - \frac{1}{2} e^{-\sqrt{2}(\mu+y)} - \frac{\sqrt{2}}{4} y\, e^{-2\sqrt{2}\mu}$$

iii) $\int_0^y \int_0^y \ell(v_1,y)\, f(v_2-v_1)\, f(-v_2)\, dv_1\, dv_2 = J_2$.

Using Eq. (5.42),

$$J_2 = \int_0^y \frac{1}{2} e^{-\sqrt{2}(\mu+y-v_1)} [\int_0^y \frac{1}{2} e^{-\sqrt{2}|v_2-v_1-\mu|} e^{-\sqrt{2}|-v_2-\mu|} dv_2]\, dv_1$$

where, for $v_1 + \mu \le y$,

$$\int_0^y e^{-\sqrt{2}|v_2 - v_1 - \mu|} e^{-\sqrt{2}|-v_2-\mu|} dv_2$$

$$= \int_0^{v_1+\mu} e^{\sqrt{2}(v_2-v_1-\mu)} e^{-\sqrt{2}(v_2+\mu)} dv_2$$

$$+ \int_{v_1+\mu}^y e^{-\sqrt{2}(v_2-v_1-\mu)} e^{-\sqrt{2}(v_2+\mu)} dv_2$$

$$= (\mu+v_1) e^{-\sqrt{2}(2\mu+v_1)} + \int_{v_1+\mu}^y e^{-\sqrt{2}(2v_2-v_1)} dv_2$$

$$= (\mu+v_1+ \frac{\sqrt{2}}{4}) e^{-\sqrt{2}(2\mu+v_1)} - \frac{\sqrt{2}}{4} e^{-\sqrt{2}(2y-v_1)}$$

and for $v_1 + \mu \ge y$,

$$\int_0^y e^{-\sqrt{2}|v_2-v_1-\mu|} e^{-\sqrt{2}|-v_2-\mu|} dv_2 = \int_0^y e^{-\sqrt{2}(v_1+2\mu)} dv_2 = y\, e^{-\sqrt{2}(v_1+2\mu)}$$

and thus, for $y \ge \mu$,

$$J_2 = \int_0^{y-\mu} \frac{1}{4} e^{-\sqrt{2}(\mu+y-v_1)} [(\mu+v_1+\frac{\sqrt{2}}{4}) e^{-\sqrt{2}(2\mu+v_1)} - \frac{\sqrt{2}}{4} e^{-\sqrt{2}(2y-v_1)}] + \int_{y-\mu}^y \frac{1}{4} e^{-\sqrt{2}(\mu+y-v_1)} \cdot y\, e^{-\sqrt{2}(v_1+2\mu)}\, dv_1$$

$$= \frac{1}{4} e^{-\sqrt{2}(3\mu+y)} \int_0^{y-\mu} (\mu+v_1+ \frac{\sqrt{2}}{4})\, dv_1 - \frac{\sqrt{2}}{16} e^{-\sqrt{2}(\mu+3y)} \int_0^{y-\mu} e^{2\sqrt{2}\ v_1}\, dv_1 + \frac{y}{4} e^{-\sqrt{2}(3\mu+y)} \int_{y-\mu}^y dv_1$$

$$= \frac{1}{8} e^{-\sqrt{2}(3\mu+y)} (y^2 + \frac{\sqrt{2}}{2} y+2\mu y-\mu^2 - \frac{\sqrt{2}}{2}\mu - \frac{1}{4}) + \frac{1}{32} e^{-\sqrt{2}(\mu+3y)}$$

and, for $y \le \mu$

$$J_2 = \int_0^y \frac{1}{4} e^{-\sqrt{2}\ (\mu+y-v_1)} \cdot y \cdot e^{-\sqrt{2}\ (v_1+2\mu)}\, dv_1$$

$$= \frac{y}{4} e^{-\sqrt{2}(3\mu+y)} \int_0^y dv_1 = \frac{y^2}{4} e^{-\sqrt{2}\ (3\mu+y)}.$$

Now, going back to Eq. (5.47), in the case $y \ge \mu$,

$$d_3(0,y) = [e^{-\sqrt{2}\ (\mu+y)} - \frac{1}{4} e^{-\sqrt{2}(2\mu+y)} + \frac{\sqrt{2}}{4} y e^{-\sqrt{2}(2\mu+y)}] \cdot [1- \frac{1}{2} e^{-\sqrt{2}\mu}] + \frac{1}{2} e^{-\sqrt{2}(\mu+y)} \cdot [\frac{1}{2} e^{-\sqrt{2}\mu} - \frac{3}{8} e^{-2\sqrt{2}\mu} - \frac{\sqrt{2}\mu}{4} e^{-2\sqrt{2}\mu} - \frac{1}{8} e^{-2\sqrt{2}\ y}] + \frac{1}{2} e^{-\sqrt{2}(\mu+y)} + \frac{\sqrt{2}}{4} y\, e^{-\sqrt{2}(2\mu+y)} + \frac{1}{8} e^{-\sqrt{2}(3\mu+y)} [y^2 + \frac{\sqrt{2}}{2} y + 2\mu y - \mu^2 - \frac{\sqrt{2}}{2}\mu - \frac{1}{4}] + \frac{1}{32} e^{-\sqrt{2}(\mu+3y)}$$

$$= [\frac{3}{2} e^{-\sqrt{2}\mu} + e^{-2\sqrt{2}\mu} (\frac{\sqrt{2}}{2} y - \frac{1}{2}) + e^{-3\sqrt{2}\mu}(\frac{y^2}{8} - \frac{\mu^2}{8} - \frac{\sqrt{2}}{16} y - \frac{3\sqrt{2}\mu}{16} + \frac{\mu y}{4} - \frac{3}{32})] \cdot e^{-\sqrt{2}\ y} - \frac{1}{32} e^{-\sqrt{2}\ (\mu+3y)} \qquad (5.48)$$

and in the case of $y \le \mu$, similarly, one has

$$d_3(0,y) = [\frac{3}{2} e^{-\sqrt{2}\mu} + e^{-2\sqrt{2}\mu} (\frac{\sqrt{2}}{2} y - \frac{1}{2}) + e^{-3\sqrt{2}\mu} (\frac{y^2}{4} - \frac{\sqrt{2}}{4} y + \frac{1}{8})] \cdot e^{-\sqrt{2}\ y} - \frac{1}{4} e^{-2\sqrt{2}(\mu+y)} \qquad (5.49)$$

and now the cumulative distribution function and the probability density function can be obtained by

$$F_{D_3}(y) = 1 - d_3(0,y)$$

$$f_{D_3}(y) = \frac{d}{dy}[F_{D_3}(y)].$$

For instance, in the case of full regulation ($\mu = 0$), Eq. (5.48) results

$$d_3(0,y) = (\frac{29}{32} + \frac{7\sqrt{2}}{16} y + \frac{1}{8} y^2)\, e^{-\sqrt{2}\, y} - \frac{1}{32} e^{-3\sqrt{2}\, y}$$

and thus

$$F_{D_3}(y) = 1 - (\frac{29}{32} + \frac{7\sqrt{2}}{16} y + \frac{1}{8} y^2)\, e^{-\sqrt{2}\, y} + \frac{1}{32} e^{-3\sqrt{2}\, y}$$

$$f_{D_3}(y) = (\frac{15\sqrt{2}}{32} + \frac{5}{8} y + \frac{\sqrt{2}}{8} y^2)\, e^{-\sqrt{2}\, y} - \frac{3\sqrt{2}}{32}\, e^{-3\sqrt{2}\, y}. \tag{5.50}$$

Notice that $F_{D_3}(0) = \frac{1}{8} = (\frac{1}{2})^3$, as it should. For a more general verification, consider Eq. (5.49) for $y = 0$.

$$d_3(0,0) = \frac{3}{2} e^{-\sqrt{2}\mu} - \frac{3}{4} e^{-2\sqrt{2}\mu} + \frac{1}{8} e^{-3\sqrt{2}\mu} = 1 - F_{D_3}(0)$$

Thus

$$F_{D_3}(0) = 1 - \frac{3}{2} e^{-\sqrt{2}\mu} + \frac{3}{4} e^{-2\sqrt{2}\mu} - \frac{1}{8} e^{-3\sqrt{2}\mu}$$

$$= (1 - \frac{1}{2} e^{-\sqrt{2}\mu})^3$$

$$= \left(\int_0^\infty \frac{\sqrt{2}}{2} e^{-\sqrt{2}|x-\mu|}\, dx \right)^3 = \left(\int_0^\infty f(x)dx \right)^3$$

$= [P(X > 0)]^3$, as it should.

The mean value of D_3 is, in general,

$$E(D_3) = \int_0^\infty [1 - F_{D_3}(y)]dy = \int_0^\infty d_3(0,y)dy.$$

Using Eqs. (5.48) and (5.49), one has

$$E(D_3) = \int_0^\infty [\frac{3}{2} e^{-\sqrt{2}\mu} + (\frac{\sqrt{2}}{2} y - \frac{1}{2})\, e^{-2\sqrt{2}\mu}] \cdot e^{-\sqrt{2}\, y}\, dy$$

$$+ \int_0^\mu e^{-3\sqrt{2}\mu} \cdot (\frac{1}{4} y^2 - \frac{\sqrt{2}}{4} y + \frac{1}{8})\, e^{-\sqrt{2}\, y}\, dy$$

$$+ \int_\mu^\infty e^{-3\sqrt{2}\mu} \cdot (\frac{1}{8} y^2 - \frac{1}{8}\mu^2 - \frac{\sqrt{2}}{16} y - \frac{3\sqrt{2}}{16}\mu$$

$$+ \frac{1}{4}\mu y - \frac{3}{32}) \cdot e^{-\sqrt{2}\, y}\, dy$$

$$- \int_0^\mu \frac{1}{4} \cdot e^{-2\sqrt{2}\mu} \cdot e^{-2\sqrt{2}\, y}\, dy - \int_\mu^\infty \frac{1}{32} \cdot e^{-\sqrt{2}\mu}\, e^{-3\sqrt{2}\, y}\, dy$$

$$= (\frac{3}{2} e^{-\sqrt{2}\mu} - \frac{1}{2} e^{-2\sqrt{2}\mu}) \cdot \int_0^\infty e^{-\sqrt{2}\, y}\, dy$$

$$+ \frac{\sqrt{2}}{2} e^{-2\sqrt{2}\mu} \int_0^\infty y\, e^{-\sqrt{2}\, y}\, dy$$

$$+ \frac{1}{8} \cdot e^{-3\sqrt{2}\mu} \cdot \int_0^\mu e^{-\sqrt{2}\, y} dy - \frac{\sqrt{2}}{4} e^{-3\sqrt{2}\mu} \int_0^\mu y\, e^{-\sqrt{2}\, y}\, dy$$

$$+ \frac{1}{4} e^{-3\sqrt{2}\mu} \int_0^\mu y^2 e^{-\sqrt{2}y} dy - (\frac{1}{8}\mu^2 + \frac{3\sqrt{2}}{16}\mu + \frac{3}{32}) \cdot e^{-3\sqrt{2}\mu}$$

$$\cdot \int_\mu^\infty e^{-\sqrt{2}y}\, dy + (\frac{1}{4}\mu - \frac{\sqrt{2}}{16}) \cdot e^{-3\sqrt{2}\mu} \int_\mu^\infty y\, e^{-\sqrt{2}y}\, dy$$

$$+ \frac{1}{8} \cdot e^{-3\sqrt{2}\mu} \int_\mu^\infty y^2 e^{-\sqrt{2}y} dy - \frac{1}{4} \cdot e^{-2\sqrt{2}\mu} \int_0^\mu e^{-2\sqrt{2}y}\, dy$$

$$- \frac{1}{32} e^{-\sqrt{2}\mu} \int_\mu^\infty e^{-3\sqrt{2}\, y}\, dy$$

$$= \frac{3\sqrt{2}}{4} e^{-\sqrt{2}\mu} - \frac{\sqrt{2}}{4} e^{-2\sqrt{2}\mu} + \frac{\sqrt{2}}{4} e^{-2\sqrt{2}\mu} + \frac{\sqrt{2}}{16} e^{-3\sqrt{2}\mu}$$

$$- \frac{\sqrt{2}}{16} e^{-4\sqrt{2}\mu}$$

$$- \frac{\sqrt{2}}{8} \cdot e^{-3\sqrt{2}\mu} + (\frac{1}{4} \cdot \mu + \frac{\sqrt{2}}{8})\, e^{-4\sqrt{2}\mu} + \frac{\sqrt{2}}{8} e^{-3\sqrt{2}\mu}$$

$$- (\frac{\sqrt{2}}{8}\mu^2 + \frac{1}{4} \cdot \mu + \frac{\sqrt{2}}{8})\, e^{-4\sqrt{2}\mu} - (\frac{\sqrt{2}}{16}\mu^2$$

$$+ \frac{3}{16}\mu + \frac{3\sqrt{2}}{64})\, e^{-4\sqrt{2}\mu}$$

$$+ (\frac{\sqrt{2}}{8}\mu^2 + \frac{1}{16} \cdot \mu - \frac{\sqrt{2}}{32})\, e^{-4\sqrt{2}\mu}$$

$$- (\frac{\sqrt{2}}{16}\mu^2 + \frac{1}{8}\mu + \frac{\sqrt{2}}{16})\, e^{-4\sqrt{2}\mu}$$

$$+ \frac{\sqrt{2}}{16} e^{-4\sqrt{2}\mu} - \frac{\sqrt{2}}{16} e^{-2\sqrt{2}\mu} - \frac{\sqrt{2}}{192} e^{-4\sqrt{2}\mu}.$$

After simplifications, one has

$$E(D_3) = \frac{3\sqrt{2}}{4} e^{-\sqrt{2}\mu} - \frac{\sqrt{2}}{16} e^{-2\sqrt{2}\mu} + \frac{\sqrt{2}}{16} e^{-3\sqrt{2}\mu} - \frac{\sqrt{2}}{48} e^{-4\sqrt{2}\mu} \tag{5.51}$$

and in particular, for $\mu = 0$,

$$E(D_3) = \frac{35\sqrt{2}}{48} \neq 3 \cdot E(D_1) = \frac{3\sqrt{2}}{4}.$$

Some values of $E(D_3)$, from Eq. (5.51), are given in the following tabulation:

μ	0	1/2	1	3/2	2
$E(D_3)$	1.0312	0.5103	0.2538	0.1260	0.0624

The change in the mean value of D_n as μ increases can be better appreciated by computing

$$\frac{E(D_n)}{E(D_n|\mu=0)}$$

where $E(D_n|\mu=0)$ is the mean value of D_n for the case of full regulation.

In the following tabulation the expressions

$$\text{(I)} \quad \frac{E(D_1)}{E(D_1|\mu=0)} \quad \text{and} \quad \text{(II)} \quad \frac{E(D_3)}{E(D_3|\mu=0)}$$

are compared, using Eq. (5.40) and (5.51)

μ	0	1/2	1	3/2	2
Expression I	1.0000	0.4929	0.2432	0.1199	0.0591
Expression II	1.0000	0.4949	0.2461	0.1222	0.0605

Recall that for constant output and for a gross input with the coefficient of variation equal to 0.25, the values of μ = 0, 0.5, 1.0, 1.5 and 2.0 correspond to α = 100.0, 87.5, 75.0, 62.5 and 50.0 percent. Notice the drastic reduction in the mean maximum deficit (and in practical terms, reduction in storage capacity required) when the level of regulation decreases.

The objective of this section was to emphasize that even when the net input is such that all integrals involved can be easily performed, the analytical solution to the problem still involves long and tedious algebraic transformations. Nevertheless deficit analysis, as range analysis, is conceptually very simple and numerical solutions can be very easily found in all cases.

3.3 Closing Remarks. In this section, the distribution of the maximum deficit for continuous inputs was derived, by analogy with the discrete inputs problem.

Examples of application were given for normal and Laplace inputs. Some results are in closed forms, and others were obtained numerically. In this sense, similarities between deficit analysis and range analysis were stressed.

The asymptotic distribution of the maximum deficit was derived in the case of full regulation and some exact densities were compared to the asymptotic result, by standardization of variables.

In a later chapter, examples will be given in order to assess the influence of skewness in deficit analysis.

4. Summary

The main items discussed in this chapter can be summarized as follows:

(i) General approach to the distribution of the maximum deficit for independent discrete inputs (Eq. (5.5)),

(ii) Derivation of asymptotic results (Section 2),

(iii) General approach to the distribution of the maximum deficit for independent continuous inputs (Eq. (5.29)), and

(iv) Illustration of convergence of exact results to asymptotic ones.

The influence of skewness will be illustrated in a later chapter. It is important to notice that the effects of nonnormal inputs are not only due to skewness. For low levels of regulation and low values of n, the effect of nonnormality of symmetric inputs can be substantial. For instance, for μ = 2 (say, 50% regulation) and n = 2, the mean maximum accumulated deficit for Laplace inputs (Eq. (5.46)) is almost 150 percent larger than the mean maximum deficit for normal inputs (Eq. (5.36)), and the skewness coefficient is zero in both cases.

Finally, it is important to stress that the graph obtained by plotting the mean net input (which is related to the level of regulation) against the mean maximum deficit (say, storage capacity required) for a given value of n, is simply the storage-yield relationship. Although the storage-yield curve is one of the oldest concepts in water resources, this is the first time in which this relationship was determined exactly. Of course, depending upon the designer's criterion, the storage capacity could be some quantile in the cumulative distribution function of D_n rather than the mean maximum deficit.

Chapter VI
RANGE AND DEFICIT ANALYSIS FOR CORRELATED INPUTS

In this chapter the theory exposed before is extended for the case of correlated inputs. Clearly, once range and deficit analysis have been shown to follow directly from the theory of Markov chains, the generalization to correlated inputs is similar to Lloyd's (1963) extension of Moran's work and consequently the same limitations (drastic increase in the size of matrices involved, for instance) are found here. The theory can also be extended for seasonal inputs. In this case, a different transition matrix is considered for each season and the basic approach remains unchanged. As a matter of fact, seasonality is so easily taken into account in range and deficit analysis that this writer will not elaborate on it at this time.

1. Range Analysis for Correlated Inputs

The assumption of independence of inputs has not been made in the derivation of Eq. (4.9), and thus it holds in general. Recall that $\lambda_k^{(n)}$ is the sum of all elements in the n-step "restricted" transition matrix of size k. Also, recall that only in the case of independent inputs this matrix is the n-th power of the one-step "restricted" transition matrix.

In this section, the case of Markovian inputs is considered and the procedure to be outlined can be regarded as a numerical integration algorithm to obtain the distribution of the range for the continuous case of first order autoregressive inputs.

When the inputs follow a Markov chain, the distribution of the state of the system Y_t depends both on the previous state Y_{t-1} and the previous net input X_{t-1}. But X_{t-1} can be written as $(Y_{t-1} - Y_{t-2})$ and thus the distribution of Y_t depends on Y_{t-1} and Y_{t-2}. The important point to observe in this reasoning is that the net input is not given by simple subtraction when the boundaries are reached, but this does not affect the problem because once one of the boundaries is reached, the system remains at the corresponding absorbing state with probability one.

Consequently the solution of the problem involves the consideration of two-step Markov chain, as described in Section III-1.3, in the case when the boundaries are absorbing. Referring to Eq. (3.17) and (3.18), the elements of the matrix A are $a(ij|jk)$ and when neither j nor k are absorbing states, the elements in the column (j,k) constitute simply the distribution of the net input X_t given that $X_{t-1} = j-k$.

For illustration purposes, the simple dependent (-1, +1) process will be studied, and it will become apparent that even a process as simple as this one can lead to important and relevant practical conclusions.

Consider the process characterized by the following marginal distribution:

$$P(X_t = +1) = P(X_t = -1) = 1/2 \tag{6.1}$$

and by the following conditional distribution:

$$P(X_t = +1|X_{t-1} = +1) = P(X_t = -1|X_{t-1} = -1) = p$$

$$P(X_t = +1|X_{t-1} = -1) = P(X_t = -1|X_{t-1} = +1) = q \tag{6.2}$$

where

$$p + q = 1.$$

Notice that $E(X_t) = 0$, var $(X_t) = 1$ and

$$\text{cov}(X_t, X_{t-1}) = \rho = p-q = 2p-1 \tag{6.3}$$

where ρ stands for the lag-one coefficient of correlation.

Now consider a system with state space {0, 1, 2, 3, 4} where 0 and 4 are absorbing states. Using the notation introduced in Section III-1.3, it is clear that if the initial state is, say, $Y_0 = 2$, then the joint distribution of the pair Y_1 and Y_0 is

$$\delta_1^T = \{\delta_1(0,0)\ldots\ \delta_1(0,4)\ \delta_1(1,0)\ldots\delta_1(1,4)$$

$$\ldots\ldots\ldots\delta_1(4,0)\ldots\delta_1(4,4)\}$$

where $\delta_1(1,2)$ and $\delta_1(3,2)$ are equal to 1/2 and all other $\delta_1(i,j)$ are equal to zero.

The joint distribution of Y_2 and Y_1 is given by Eq. (3.18):

$$\delta_2 = A\ \delta_1 \tag{6.4}$$

where A is a square matrix of size 25 such that

$$a(0,0|0,0) = a(0,0|0,1) = a(4,4|4,3) = a(4,4|4,4) = 1$$

$$a(0,1|1,2) = a(1,2|2,3) = a(3,2|2,1) = a(4,3|3,2) = p$$

$$a(1,2|2,1) = a(2,1|1,2) = a(2,3|3,2) = a(3,2|2,3) = q$$

and all other $a(ij|jk)$ are equal to zero. The reader may find it necessary to write down the entire matrix as explained in Section III-1.3, to fully understand the reasoning.

Fortunately the vector δ_t and the matrix A can be simplified by elimination of the impossible transitions, and Eq. (6.4) can be rewritten as

$$
\delta_2 = \begin{bmatrix} \delta_2(0,0) \\ \delta_2(0,1) \\ \delta_2(1,2) \\ \delta_2(2,1) \\ \delta_2(2,3) \\ \delta_2(3,2) \\ \delta_2(4,3) \\ \delta_2(4,4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ p/2 \\ 0 \\ q/2 \\ q/2 \\ 0 \\ p/2 \\ 0 \end{bmatrix} \qquad (6.5)
$$

Now the distribution of Y_3 given $Y_0 = 2$ can be obtained using Eqs. (6.2) and (6.5). Notice that this result is simply the three-step transition probabilities $q^{(3)}(i,2)$ for $i = 0, 1, 2, 3, 4$:

$$
\begin{aligned}
q^{(3)}(0,2) &= P(Y_3 = 0 | Y_0 = 2) = \delta_2(0,0) + \delta_2(0,1) \\
&\quad + p \cdot \delta_2(1,2) = p/2 \\
q^{(3)}(1,2) &= P(Y_3 = 1 | Y_0 = 2) = q \cdot \delta_2(2,1) \\
&\quad + p \cdot \delta_2(2,3) = q/2 \\
q^{(3)}(2,2) &= P(Y_3 = 2 | Y_0 = 2) = q \cdot \delta_2(1,2) \\
&\quad + q \cdot \delta_2(3,2) = 0 \qquad (6.6) \\
q^{(3)}(3,2) &= P(Y_3 = 3 | Y_0 = 2) = p \cdot \delta_2(2,1) \\
&\quad + q \cdot \delta_2(2,3) = q/2 \\
q^{(3)}(4,2) &= P(Y_3 = 4 | Y_0 = 2) = p \cdot \delta_2(3,2) \\
&\quad + \delta_2(4,3) + \delta_2(4,4) = p/2
\end{aligned}
$$

Eqs. (6.6) can be written in matrix notation, as follows

$$
\begin{bmatrix} q^{(3)}(0,2) \\ q^{(3)}(1,2) \\ q^{(3)}(2,2) \\ q^{(3)}(3,2) \\ q^{(3)}(4,2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_2(0,0) \\ \delta_2(0,1) \\ \delta_2(1,2) \\ \delta_2(2,1) \\ \delta_2(2,3) \\ \delta_2(3,2) \\ \delta_2(4,3) \\ \delta_2(4,4) \end{bmatrix}
$$

or, by using Eq. (6.5):

$$
\begin{bmatrix} q^{(3)}(0,2) \\ q^{(3)}(1,2) \\ q^{(3)}(2,2) \\ q^{(3)}(3,2) \\ q^{(3)}(4,2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \qquad (6.7)
$$

Equation (6.7) represents one column (the third) in the three-step transition matrix $Q'^{(3)}$. A similar reasoning leads to the other four columns of this matrix:

$$
Q'^{(3)} = \begin{bmatrix} 1 & 1 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6.8)
$$

The three-step "restricted" matrix $Q^{(3)}$ can be obtained by deleting the first and last rows and columns of $Q'^{(3)}$:

$$
Q^{(3)} = \begin{bmatrix} 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad (6.9)
$$

For general $n \geq 2$ the n-step "restricted" transition matrix is given also by Eq. (6.9), provided A is substituted by A^{n-2}.

Recall that Eq. (6.9) corresponds to the case when the state space is $\{0,1,2,3,4\}$. For the general case of state space $\{0,1,2,\ldots,k,k+1\}$, the matrix $Q^{(n)}$ would have size k and the matrix A would have size $2k+2$. For instance, for $k = 5$,

$$
Q^{(n)} = \begin{bmatrix} 0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 \end{bmatrix} \cdot A^{n-2} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

(6.10)

and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad (6.11)$$

where lines have been drawn to emphasize that the matrix is obviously patterned.

To use Eq. (4.9), the quantity $\lambda_k^{(n)}$ is needed. Recalling that $\lambda_k^{(n)} = \underline{1}^T Q_k^{(n)} \underline{1}$ and using Eq. (6.10), one has, for $k = 5$:

$$\lambda_5^{(n)} = [0\ 0\ q\ 1\ 1\ 1\ 1\ 1\ 1\ q\ 0\ 0] \cdot A^{n-2} \cdot \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \qquad (6.12)$$

Using the fact that the elements in the columns of the matrix $Q'^{(n)}$ add to unity, it can be shown that Eq. (6.12) can be rewritten as

$$\lambda_5^{(n)} = 5 - [1\ 1\ p\ 0\ 0\ 0\ 0\ 0\ 0\ p\ 1\ 1] \cdot A^{n-2} \cdot \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \qquad (6.13)$$

or, equivalently, for general k,

$$\lambda_k^{(n)} = k - [1\ 1\ p\ 0\ \ldots\ 0\ p\ 1\ 1] \cdot A^{n-2} \cdot \begin{bmatrix} 0 \\ 1/2 \\ \cdot \\ \cdot \\ \cdot \\ 1/2 \\ 0 \end{bmatrix} \qquad (6.14)$$

where A is a square matrix of size $2k+2$.

Using Eq. (4.9) and (6.14), the probability density function of the range was computed for $n = 50$ and $n = 100$ and for $p = 0.50$ $(\rho=0)$, $p = 0.60$ $(\rho=0.20)$, and $p = 0.75$ $(\rho=0.50)$. The results are shown in Fig. 6.1.

Figure 6.2 indicates that for large values of n the exact distribution of the standardized range of partial sums of Markovian inputs tends to the asymptotic distribution of the standardized range, found by Feller (1951).

Obviously, the moments of the range can also be obtained numerically. In particular, the mean is given by Eq. (4.12), where K is a large number. Applying Eqs. (4.12) and (6.14), it is readily seen that

$$E(R_n) = [1\ 1\ p\ 0\ \ldots\ 0\ p\ 1\ 1] \cdot A^{n-2} \cdot \begin{bmatrix} 0 \\ 1/2 \\ \cdot \\ \cdot \\ \cdot \\ 1/2 \\ 0 \end{bmatrix} \qquad (6.15)$$

where A is a square matrix of size $2K+2$.

Figure 6.3 shows the values of $E(R_n)$ given by Eq. (6.15) for various values of ρ and n. For large n, these results are approximations of the continuous case of the first order autoregressive process. This figure illustrates the known fact that the square-root law prevails asymptotically for summands of any sequence of random variables subjected to the central limit theorem, and confirms a conjecture of Yevjevich (1967), namely, that the following relationship holds asymptotically:

$$E(R_n) = \sqrt{\frac{8n}{\pi}} \cdot \sqrt{\frac{1+\rho}{1-\rho}} \approx 1.5958 \sqrt{n} \sqrt{\frac{1+\rho}{1-\rho}}. \qquad (6.16)$$

Equation (6.16) is formally derived in Appendix A.

The main contribution of Fig. 6.5, however, is to illustrate the drastic increase with ρ in the size of the transient region.

2. Adjusted Range Analysis for Correlated Inputs

In this section Hurst's (1951) idea of studying the mean unadjusted range conditioned to the last partial sum being equal to zero is extended for correlated inputs.

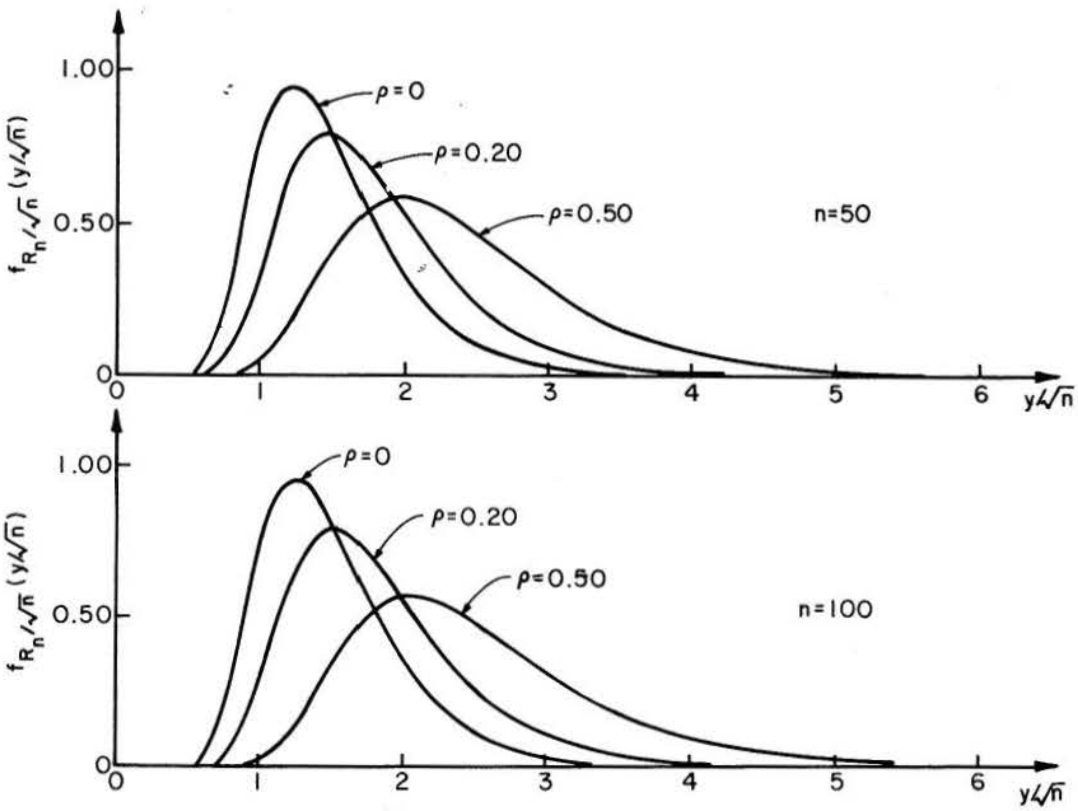

Fig. 6.1. Distribution of $R_n/\sqrt{n}$ for Markovian inputs (ρ=0,0.20,0.50; n=50 and n=100).

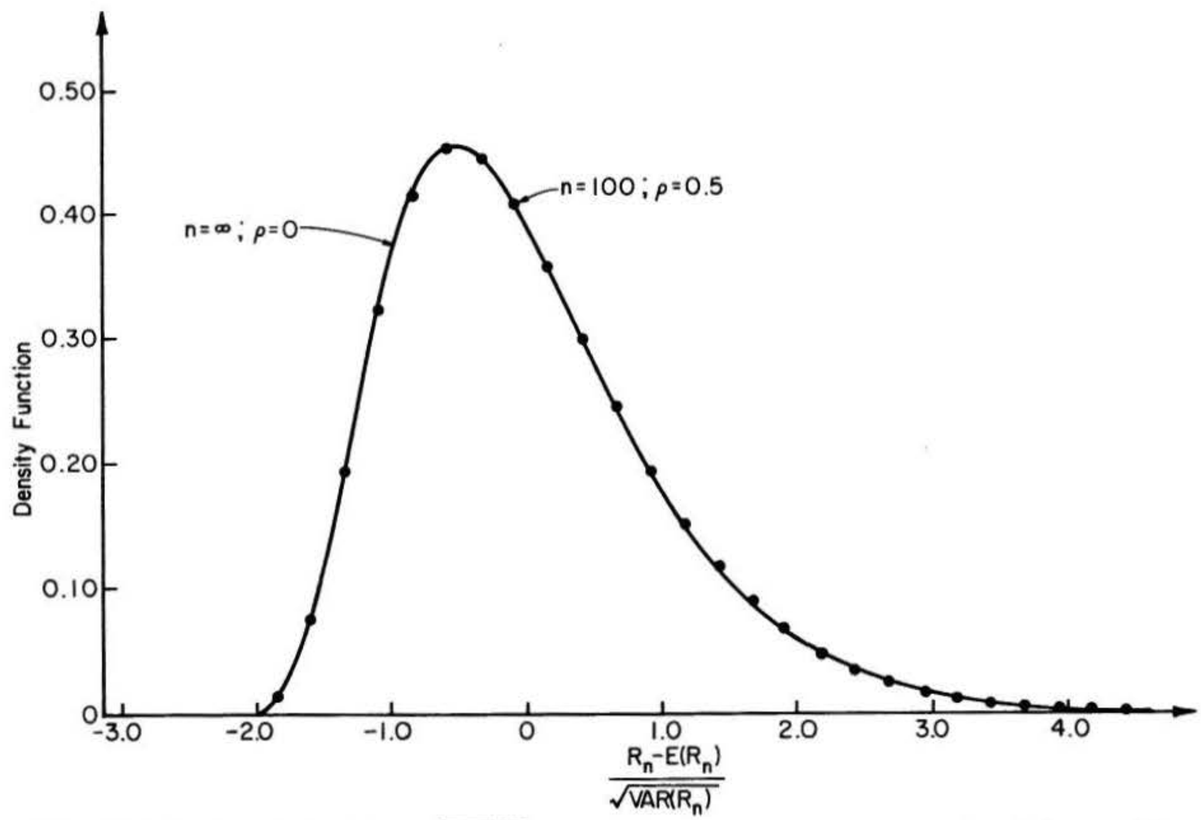

Fig. 6.2. Distribution of $[R_n-E(R_n)]/\sqrt{Var(R_n)}$ for Markovian inputs [(n=100; ρ=0.50) and (n=∞; ρ=0)].

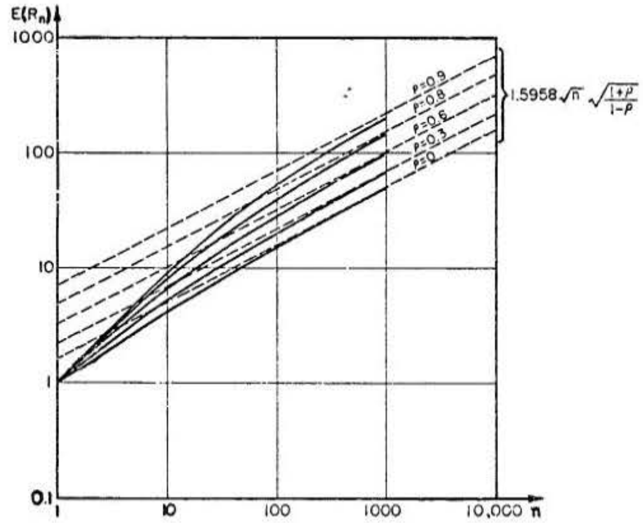

Fig. 6.3. Expected value of R_n for the dependent (-1,+1) process.

Consider a system with state space $\{0,1,\ldots,k+1\}$ such that states 0 and $k+1$ are absorbing states. Let $q_k^{(n)}(u,u)$ denote the probability of a transition from state $u = 1,2,\ldots,k$ to the same state u. Obviously this implies that the boundaries (absorbing states) have not been reached. But this is simply the joint probability $P(M_n \leq k - u, |m_n| \leq u - 1, S_n = 0)$. Using the same reasoning that led to Eq. (4.7), one has

$$P(M_n = k-u, |m_n| = u-1, S_n = 0) = q_k^{(n)}(u,u)$$

$$- q_{k-1}^{(n)}(u,u) - q_{k-1}^{(n)}(u-1,u-1) + q_{k-2}^{(n)}(u-1,u-1)$$

and then

$$P(R_n = k-1, S_n=0) = \sum_{u=1}^{k} P(M_n = k-u, |m_n| = u-1, S_n=0)$$

$$= \sum_{u=1}^{k} q_k^{(n)}(u,u) - \sum_{u=1}^{k-1} q_{k-1}^{(n)}(u,u) - \sum_{u=2}^{k} q_{k-1}^{(n)}(u-1,u-1)$$

$$+ \sum_{u=2}^{k-1} q_{k-2}^{(n)}(u-1,u-1)$$

$$= \sum_{u=1}^{k} q_k^{(n)}(u,u) - 2\sum_{u=1}^{k-1} q_{k-1}^{(n)}(u,u) + \sum_{u=1}^{k-2} q_{k-2}^{(n)}(u,u)$$

where special attention should be paid to the fact that the adjustment in the values of u in the above summations is valid. Furthermore, notice that $\sum_{u=1}^{j} q_j^{(n)}(u,u)$ is simply the trace (sum of elements of the principal diagonal) of the n-step transition matrix. Using an obvious notation, one has

$$P(R_n = k-1, S_n = 0) = \nu_k^{(n)} - 2\,\nu_{k-1}^{(n)} + \nu_{k-2}^{(n)}$$

or, equivalently,

$$P(R_n = k, S_n = 0) = \nu_{k+1}^{(n)} - 2\,\nu_k^{(n)} + \nu_{k-1}^{(n)}$$

and finally,

$$P(R_n = k|S_n = 0) = (\nu_{k+1}^{(n)} - 2\,\nu_k^{(n)} + \nu_{k-1}^{(n)})/P(S_n = 0) \tag{6.17}$$

Notice that independence of inputs has not been assumed and thus Eq. (6.17) holds in general. Following the reasoning that led to Eq. (4.12), it can be shown that

$$E(R_n|S_n = 0) = K - [\nu_K^{(n)}/P(S_n = 0)] \tag{6.18}$$

where K is a sufficiently large number so that

$$P(R_n \leq K|S_n = 0) = 1$$

In the case of the dependent (-1,+1) process defined by Eqs. (6.1) and (6.2), the n-step "restricted" transition matrix (and thus, its trace) can be obtained as shown in the previous section. To use Eq. (6.18), $P(S_n=0)$ has to be evaluated for the process defined by Eqs. (6.1) and (6.2). Following Gabriel (1959), it can be shown that for n even,

$$P(S_n=0) = p^{n-1} \sum_{j=1}^{n-1} \left(\frac{n-2}{2}\right)^{C}_{[\frac{j}{2}]} \cdot \left(\frac{n-2}{2}\right)^{C}_{[\frac{j-1}{2}]} \cdot \left(\frac{1-p}{p}\right)^{j} \tag{6.19}$$

where $[\cdot]$ denotes the integer part of the argument.

Now Eq. (6.18) can be used. Values of $E(R_n|S_n = 0)$ were determined for various combinations of n and $\rho = 2p-1$ and the results are shown in Fig. 6.4. This figure indicates that the following relationship holds asymptotically:

$$E(R_n|S_n = 0) \doteq \sqrt{\frac{\pi n}{2}}\ \sqrt{\frac{1+\rho}{1-\rho}} \simeq 1.2533\,\sqrt{n}\ \sqrt{\frac{1+\rho}{1-\rho}} \tag{6.20}$$

and illustrates the drastic increase with ρ in the size of the transient region. Eq. (6.20) is formally derived in Appendix A.

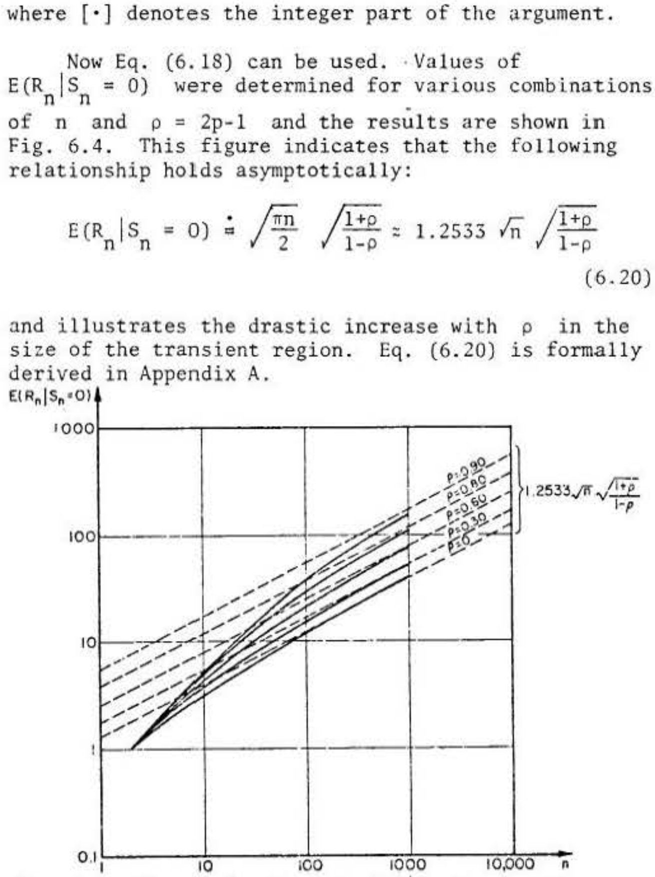

Fig. 6.4. Expected value of $(R_n|S_n=0)$ for the dependent (-1,+1) process.

The argument that variables that ultimately follow the square-root law may behave as higher powers of n in a pre-asymptotic sense was initially made by Lloyd (1967), reasoning with independent random variables. For dependent random variables, Fig. 6.4 indicates that this argument is much stronger than initially thought.

The results obtained from Eq. (6.18) for $p = 1/2$ $(\rho = 0)$ agree exactly with Hurst's original result. This can be verified analytically: for independent inputs, the n-step "restricted" transition matrix is simply the n·th power of the following matrix

$$\begin{bmatrix} 0 & 1/2 & 0 & . & . & . & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & . & . & . & 0 & 0 & 0 \\ 0 & 1/2 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & & & . & & \\ . & . & . & . & & & . & & \\ . & . & . & . & & & . & & \\ 0 & 0 & 0 & . & . & . & 0 & 1/2 & 0 \\ 0 & 0 & 0 & . & . & . & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & . & . & & 0 & 1/2 & 0 \end{bmatrix}$$

which can be found using the method of images. In particular, the elements in the principal diagonal of the n-step "restricted" transition matrix are given by Eq. (3.23), for $s = u$:

$$q_K^{(n)}(u,u) = \sum_{j=-\infty}^{+\infty} [v_n(2j(K+1)+u,u) - v_n(2j(K+1) - u,u)].$$

Using the fact that $v_n(t,r) = {}_nC_{(n-r+t)/2}(1/2)^n$, one has

$$q_K^{(n)}(u,u) = \sum_{j=-\infty}^{+\infty} (1/2)^n \, [{}_nC_{(n+2j(K+1))/2} - {}_nC_{(n+2j(K+1)-2u)/2}] \,. \qquad (6.21)$$

But K is a very large number and thus the first term in the right hand side (RHS) of Eq. (6.21) is nonzero only when $j = 0$ and the second term in the RHS of this equation is nonzero only when $j = 0$ and $j = 1$. Consequently, Eq. (6.21) can be rewritten as

$$q_K^{(n)}(u,u) = (1/2)^n[{}_nC_{n/2} - {}_nC_{(K+1-u+n/2)} - {}_nC_{(-u+n/2)}]$$

and thus

$$\begin{aligned} \nu_K^{(n)} &= \sum_{u=1}^{K} q_K^{(n)}(u,u) \\ &= (1/2)^n \cdot K \cdot {}_nC_{n/2} \\ &\quad - (1/2)^n \sum_{u=1}^{K} [{}_nC_{(K+1-u+n/2)} + {}_nC_{(-u+n/2)}] \\ &= (1/2)^n \cdot K \cdot {}_nC_{n/2} - [1 - (1/2)^n \cdot {}_nC_{n/2}] \end{aligned}$$

or

$$\nu_K^{(n)} = (K+1)\, {}_nC_{n/2}\, (1/2)^n - 1. \qquad (6.22)$$

For $p = 1/2$, Eq. (6.19) reduces to the known result

$$P(S_n = 0) = {}_nC_{n/2}(1/2)^n. \qquad (6.23)$$

Substituting Eqs. (6.22) and (6.23) in Eq. (6.18), one has

$$E(R_n|S_n = 0) = K - \frac{(K+1)\,{}_nC_{n/2}(1/2)^n - 1}{{}_nC_{n/2}\,(1/2)^n} = \frac{2^n}{{}_nC_{n/2}} - 1$$

which is Hurst's original result. Notice that n has to be even, so that S_n can be equal to zero. Using Sterling approximation, it is easily seen that

$$E(R_n|S_n = 0) \approx \sqrt{\frac{\pi n}{2}} - 1.$$

3. Deficit Analysis for Correlated Inputs

In Chapter V it was shown that $P(D_n > k)$ is simply the probability that the system is at (absorbing) state 0, at time n, given that the system was at (reflecting) state k+1 at time zero. In this section this reasoning is applied to the dependent (-1,+1) process defined by Eq. (6.1) and (6.2). As in the previous sections, the problem can be solved by considering a two-step Markov chain, now with one absorbing and one reflecting boundary.

Consider a system with state space $\{0,1,\ldots,k, k+1\}$ such that state 0 is absorbing and state k+1 is reflecting. For simplicity, consider the case k=5. If the initial state Y_0 is k+1, clearly the joint distribution of the pair Y_1 and Y_0 is

$$\underline{\delta}_1^T = \{\delta_1(0,0)\ \delta_1(0,1)\ \delta_1(1,2)\ \delta_1(2,1)\ \delta_1(2,3)\ \delta_1(3,2)\ \delta_1(3,4)\ \delta_1(4,3)\ \delta_1(4,5)\ \delta_1(5,4)\ \delta_1(5,6)\ \delta_1(6,5)\ \delta_1(6,6)\}$$

where the impossible transitions have been deleted and where $\delta_1(6,6) = 1/2$, $\delta_1(5,6) = 1/2$ and all other $\delta_1(i,j)$ are equal to zero.

Using Eq. (3.17), the joint distribution of the pair Y_{n-2} and Y_{n-1} is

$$\delta_{n-1} = A^{n-2}\,\delta_1 \qquad (6.24)$$

where the matrix A is shown by Eq. (6.25). Notice that lines have been drawn to emphasize the obvious pattern of this matrix.

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & p
\end{bmatrix}
\tag{6.25}
$$

Using Eqs. (6.24) and (6.2), it is readily seen that the probability that the system is at state 0 at time n given that it was at state $k+1$ at time zero (i.e., the probability that D_n is larger than k) is

$$P(D_n > k) = \delta_{n-1}(0,0) + \delta_{n-1}(0,1) + \delta_{n-1}(1,2)$$

or, equivalently,

$$P(D_n > k) = [1 \;\; 1 \;\; p \;\; 0 \;\; \ldots. \;\; 0] \cdot A^{n-2} \cdot \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix} \tag{6.26}$$

where A is a square matrix of size $2k+3$.

Using Eq. (6.26), the distribution of D_n was obtained for the cases $n = 50$ ($p = 0.50$, $p = 0.60$, $p = 0.75$) and $n = 100$ ($p = 0.50$, $p = 0.60$, $p = 0.75$). The results are shown in Fig. 6.5, and they can be considered numerically obtained solution for the distribution of the maximum accumulated deficit of normal autoregressive processes (first order) when the lag one coefficient of correlation is $\rho = 0$ ($p = 0.5$), $\rho = 0.20$ ($p = 0.60$) and $\rho = 0.50$ ($p = 0.75$). Figure 6.6 indicates that for large values of n the standardized distribution of the maximum deficit for Markovian inputs tends to the asymptotic result derived in Section V. 2.1. Finally, Fig. 6.7 shows the expected value of D_n for various combinations of n and ρ and for mean net input $\mu = 0$ (full regulation). Notice the similarity between the results for $\mu = 0$ and $E(R_n | S_n = 0)$ (Fig. 6.4). Similar results can be obtained for the case of partial regulation.

4. Summary

In this chapter, range and deficit analysis were extended to the case of correlated inputs. Even

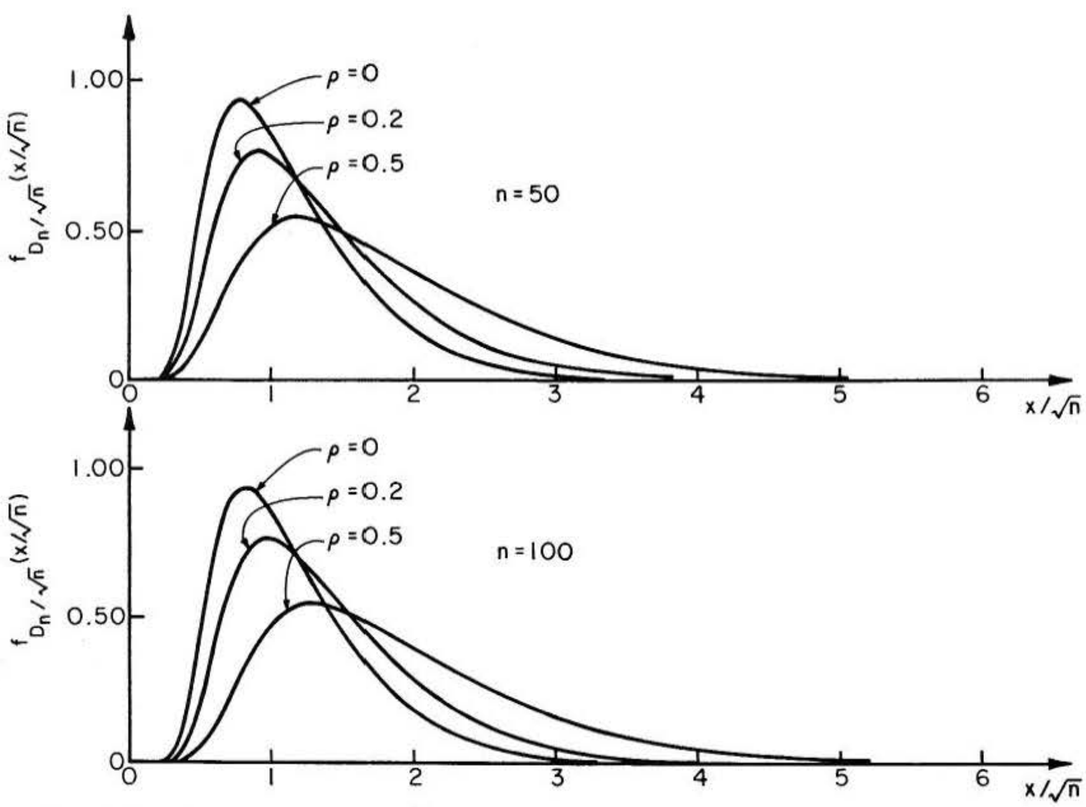

Fig. 6.5. Distribution of $D_n/\sqrt{n}$ for Markovian inputs (ρ=0,0.20,0.50; n=50 and n=100).

though the simplest possible case (the dependent (-1,+1) process) was used for illustration, the results led to important practical conclusions.

A final remark is in order, having to do with deficit analysis when the input can assume more than two positive values. In this case, when the reflecting state is reached and the system continues at this state in the next unit of time, the net input is unknown and a bivariate Markov chain rather than a two-step Markov chain has to be considered. The solution is practically the same, the only difference being that for bivariate Markov chains none of the entries in the transition matrix A are identically equal to zero.

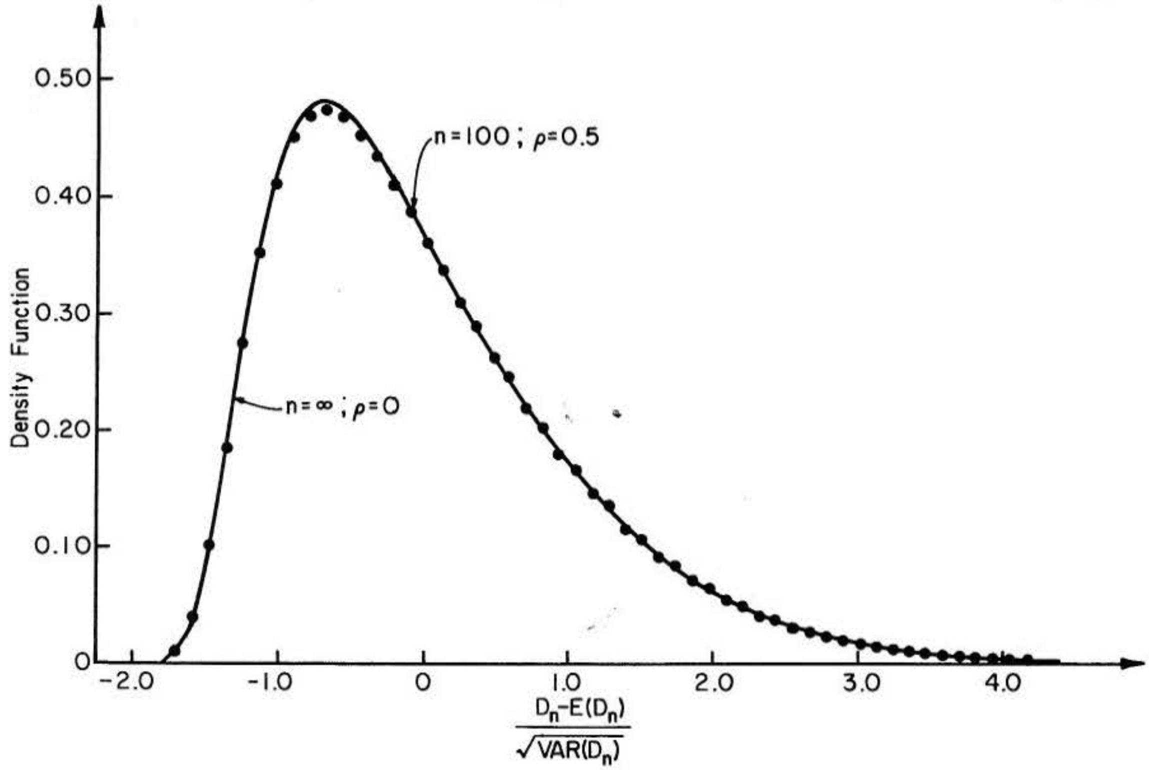

Fig. 6.6. Distribution of $[D_n-E(D_n)]/Var(D_n)$ for Markovian inputs [(n=100; ρ=0.50) and (n=∞; ρ=0)].

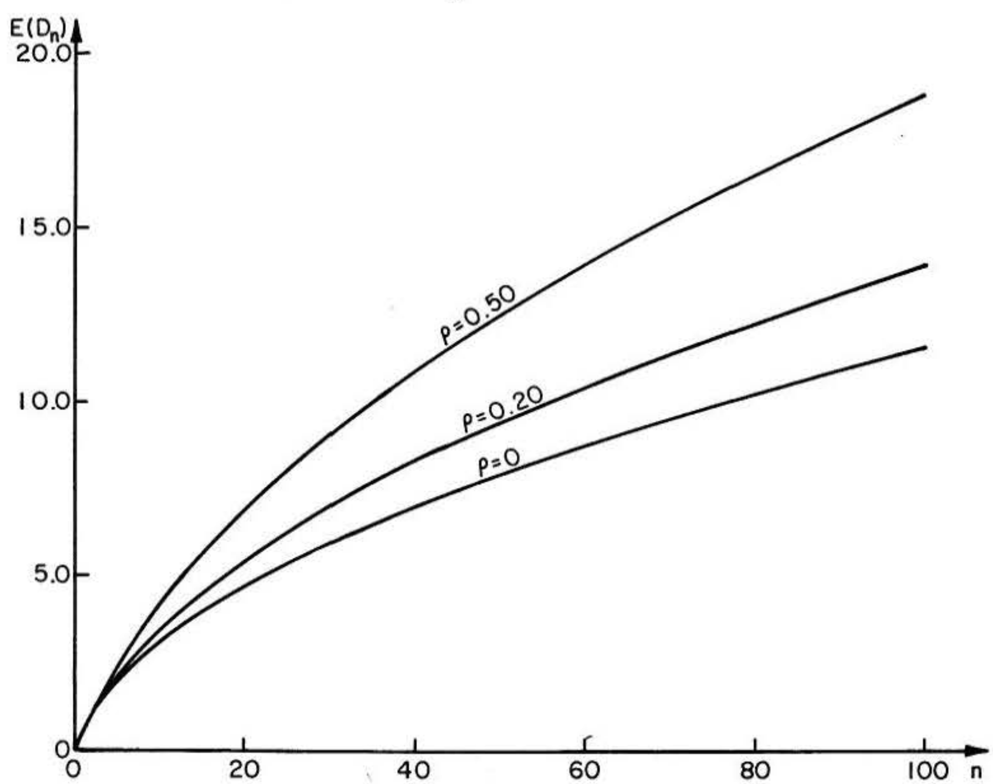

Fig. 6.7. Expected value of D_n for the dependent (-1,+1) process (μ=0).

Chapter VII
APPLICATIONS TO PRACTICAL HYDROLOGY

1. Range Analysis

Using the procedure described in previous chapters, the distribution of the range can be found, at least numerically, for a wide variety of cases of practical importance. Engineers who use the range as design criterion for the case of full regulation can now design for quantiles (say, the value r such that $P(R_n \leq r) = q$ where q is a probability level chosen by the designer) rather than design only for the expected value.

For large values of n the asymptotic distribution of R_n (Feller, 1951), corrected for the first two moments, can be used as an approximation to the exact distribution of R_n even when the inputs are Markovian. This conclusion emphasizes the relevance of previous studies in range analysis, which concentrated on approximate expressions of the first two moments of R_n for correlated inputs (Yevjevich, 1967; Salas La-Cruz, 1972).

2. Adjusted Range Analysis

In this section, $E(R_n | S_n = 0)$ is assumed to be an approximation of $E(R_n^*/S)$, following Hurst (1951).

Suppose that one has a large number of time series relative to Gaussian-Markovian models with various degrees of correlation. In the case of first order Markov processes, the observed values of (R_n^*/S) will fall around their expected values, approximated by the family of curves shown in Fig. 7.1, and a similar behavior may be anticipated for higher order models. When one considers the totality of observed values (R_n^*/S), regardless of the degree of correlation of the underlying process, a straight line passing through the point A $(R_n^*/S = 1,\ n = 2)$ with slope 0.75 will apparently fit well all the points, simply because this straight line fits reasonably well the family of curves of expected values. In particular notice that the two lines connecting point A to points B and C have slopes equal to 1.00 and 0.50, respectively, and the whole region where the pairs $(R_n^*/S,\ n)$, are expected to fall is bounded by those lines. Thus, it can be expected that for this hypothetical large number of Gaussian-Markovian time series one would find a frequency distribution of the slope K similar to the one found by Hurst, which is reproduced in Fig. 7.2.

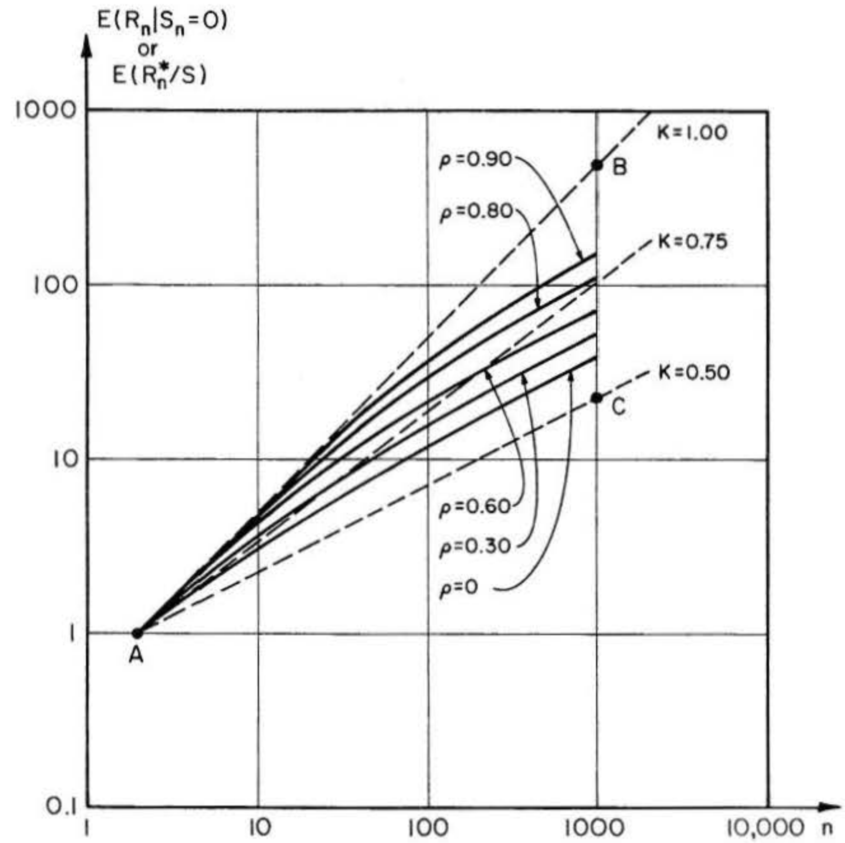

Fig. 7.1. Expected values of $(R_n | S_n = 0)$ for the dependent $(-1,+1)$ process (curves) as compared to Hurst's empirical law $E(R_n^*/S) = (n/2)^K$ for $K = 0.50, 0.75, 1.00$ (straight lines).

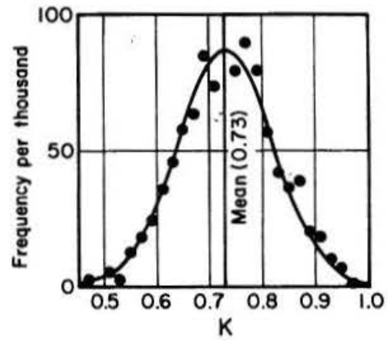

Fig. 7.2 Frequency distribution of the index K

In Fig. 7.3., the family of curves $E(R_n|S_n = 0)$ is compared with Hurst's data (Table 7 in his paper). Notice that the objective here is not to claim that Hurst's data corresponds specifically to first order autoregressive processes, but rather to emphasize that due to the drastic increase with ρ in the size of the transient region, one may easily confound an apparent, pre-asymptotic departure from the square root law with a definitive, asymptotic departure.

Another example of possible confusion between pre-asymptotic behavior and actual behavior will be given. Mandelbrot and Wallis (1969b) used, among other time series, the data from the St. Lawrence River (Yevjevich, 1963) to argue that the rescaled range R_n^*/S increases faster than $n^{0.5}$ and thus that "for practical purposes, geophysical records must be considered to have an 'infinite' span of statistical interdependence." The interesting point is that the same data have been studied by Yevjevich (1963), who concluded that a simple first-order autoregressive model fitted this particular data well. Depending on the estimation procedure the lag one coefficient of correlation could be estimated by 0.705 or 0.785. In Fig. 7.4 the exact values of $E(R_n|S_n=0)$ for the dependent (-1,+1) process with $\rho = 0.75$ are compared to a straight line with slope equal to 0.90 for $n \leq 100$. Clearly, the fact that the data from the St. Lawrence River ($n < 100$) shows a slope close to 0.90, cannot be regarded as conclusive evidence of departure from the square root law in an asymptotic sense.

3. Deficit Analysis

The immediate application of deficit analysis is in the determination of the exact storage-yield relationship. This relationship is obtained by plotting values of the storage required against the corresponding level of regulation. Using the procedure outlined in Chapter V, the distribution of the maximum deficit can be found, at least numerically, for a variety of cases of practical importance. The extension to the case of correlated inputs, presented in Chapter VI, has limitations, but at least indications of the effect of correlation on the maximum deficit can be found.

For completeness, a simple example will be given, having to do with the influence of skewness on the expected value of the maximum deficit D_n. Assume that the natural discharge can be approximated by a negative binomial distribution:

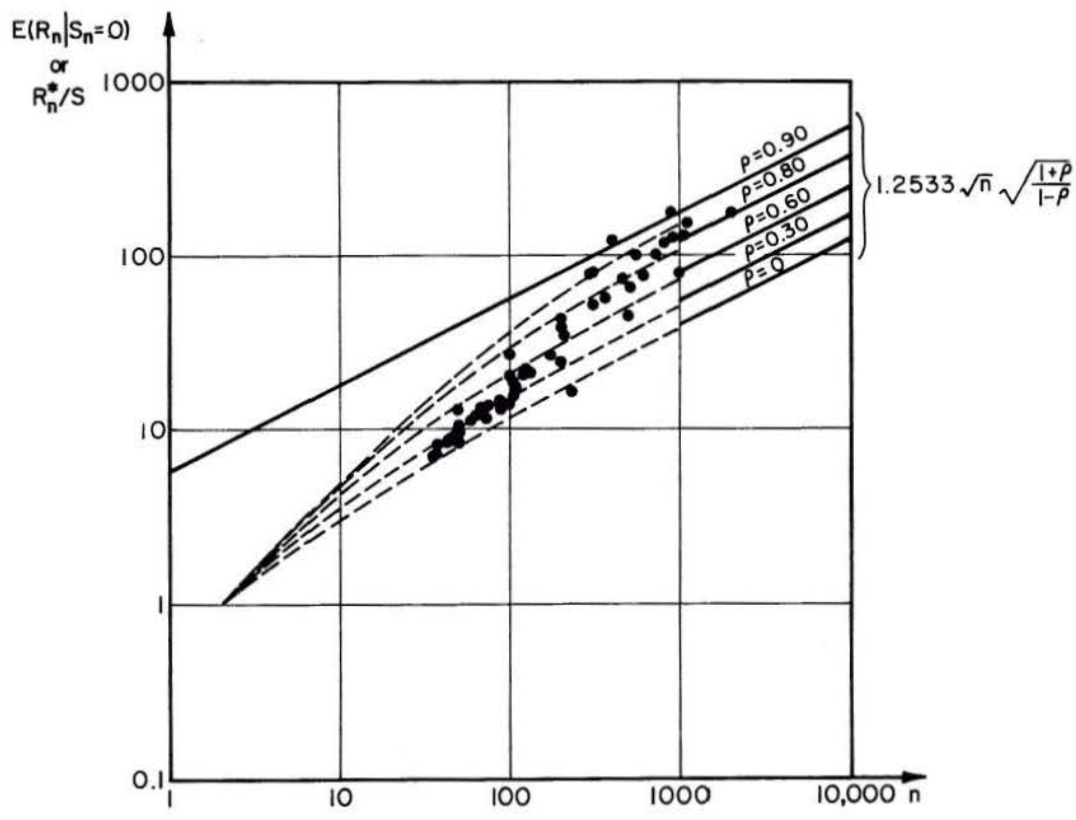

Fig. 7.3. Expected values of $(R_n|S_n=0)$ for the dependent (-1,+1) process (curves) as compared to Hurst's sample values of R_n^*/S (points).

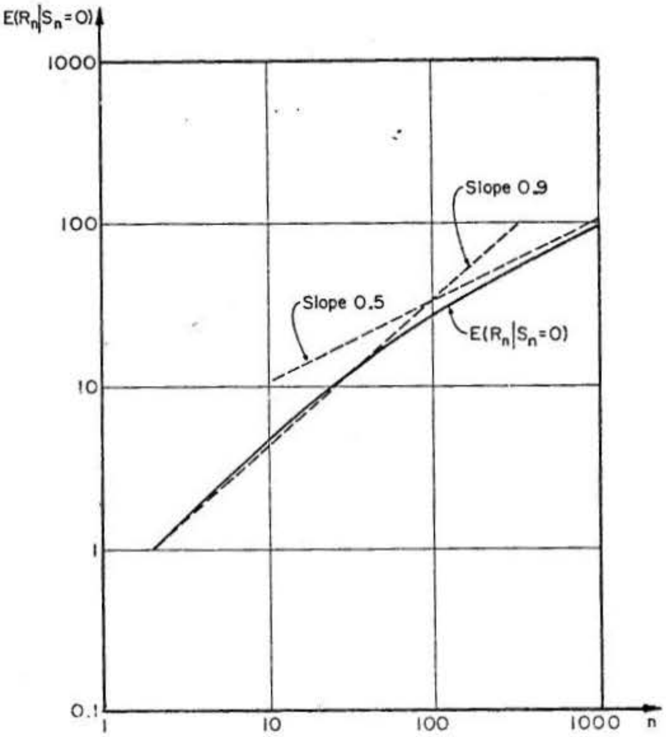

Fig. 7.4 Apparent slope of $E(R_n|S_n=0)$ for $\rho=0.75$ and $n<100$.

$$P(Z_t = i) = {}_{(r+i-1)}C_i \cdot p^r q^i \quad (i = 0,1,2,\ldots).$$

It is well known that for this distribution

$$E(Z_t) = rq/p,$$

$$\text{Var}(Z_t) = rq/p^2,$$

and

$$E[(Z_t - E(Z_t))^3] = rq(1+q)/p^3 .$$

and thus

$$C_V = \frac{E[(Z_t - E(Z_t))^3]}{[\text{var}(Z_t)]^{3/2}} = \frac{1+q}{\sqrt{rq}} . \tag{7.1}$$

For $p=q=1/2$ and $r=2$, one has

$$E(Z_t)=2 , \quad \text{Var}(Z_t)=4 \quad \text{and} \quad C_V=1.5 ,$$

and the following cases of net input can be considered:

i) $X_t=(Z_t-2)/2$; thus, $E(X_t)=0$, $\text{Var}(X_t)=1$ and $C_V=1.5$.

ii) $X_t=(Z_t-1)/2$; thus, $E(X_t)=1/2$, $\text{Var}(X_t)=1$ and $C_V=1.5$.

The values of $E(D_n)$ were obtained for both cases, for $n = 1,2,\ldots,10$ and the results were compared with $E(D_n)$ for normal inputs. This is shown in Fig. 7.5. As expected, the influence of skewness for the case of full regulation is less strong than in the case of partial regulation. Furthermore, one

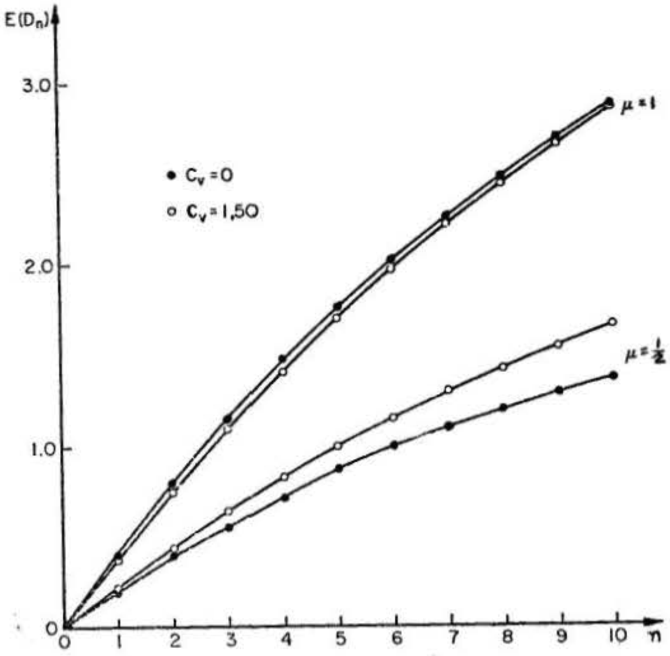

Fig. 7.5. Influence of skewness on the expected maximum accumulated deficit.

cannot "a priori" state whether skewed inputs will lead to larger or smaller values of $E(D_n)$. Notice that $C_V = 1.5$ is relatively high for river discharges.

In the case of partial regulation, some doubt can be cast on comparisons like the one shown in Fig. 7.5, for even though the new input has mean 1/2 and unit variance for both the negative binomial and the normal input, they may or may not correspond to different levels of regulation. In the case of the above example, the coefficient of variation of Z_t was equal to one and thus, for $\mu = 1$, D_n is identically equal to zero. The same difficulty arises when studying exponentially distributed inputs, which was done in terms of range analysis only. In summary, the reader should keep in mind that comparisons of this type are dangerous, and that for low levels of regulation, each case is a special case.

4. Summary

In this chapter the application of range and deficit analysis to the design of storage capacities was discussed. Of course, range analysis applies to both full and partial regulation cases. One may argue that deficit analysis should be used in all cases, for the sake of consistency and uniformity of criteria.

The main section of this chapter dealt exclusively with the so-called Hurst phenomenon. The exact values of $E(R_n|S_n = 0)$ for the dependent (-1, +1) process were compared with actual data, to argue once more that short memeory models do produce "Hurst-like sequences." This same conclusion was reached by Matalas and Huzzen (1967), who used the Monte-Carlo method and generated a large number of sequences following a first order autoregressive model.

Chapter VIII
SUMMARY AND CONCLUSIONS

The main objective of this study was to investigate two properties of the partial sums of random variables: the range and the maximum accumulated deficit. The range R_n or the adjusted range R_n^* are used by some engineers in the design of storage capacities for full regulation of river discharges (Salas La-Cruz, 1972; Hurst, 1951; Fiering, 1965). The maximum accumulated deficit D_n is used in the case of partial regulation (Hurst, 1951; Fiering, 1965).

In this paper a general approach to the distribution of the range of partial sums of independent random variables was developed. Starting with discrete random variables, the distribution of the range was shown to follow from the theory of Markov chains, when the state space contains two and only two absorbing states (the boundary states). By analogy, the distribution of the range for the case of independent continuous random variables was easily obtained. Several examples of application of the general formulae to particular probability distributions were given. Some results were obtained in closed form, and others were obtained numerically. In the case of continuous random variables such that the solution is necessarily numerical, it was argued that the most efficient approach consists of two steps: i) discretization of the input and ii) application of the general procedure for discrete random variables.

For each type of input considered, the exact distribution for finite values of n were compared to the asymptotic result found by Feller (1951), either by standardization or by considering the variable $R_n/\sqrt{n}$. The conclusion was that the asymptotic result, when corrected for the first two moments, is a good approximation of the exact distribution, even for low values of n. Other conclusions, such as the relative lack of importance of skewness and other departures from normality were previously known from simulation studies (Yevjevich, 1965).

The specific contribution of this paper to range analysis is that by using the approach described, the distribution of the range, and consequently its moments, can be obtained (at least numerically) for a wide variety of cases of practical interest. This is important because it allows engineers to use criteria other than the mean range (quantiles, for instance) in the design of storage capacities.

In another chapter of this paper, a general approach to the distribution of the maximum accumulated deficit D_n was described. Starting with independent discrete random variables, the distribution of D_n was shown to follow from the theory of Markov chains, when the state space is such that one boundary is absorbing and the other is reflecting. By analogy, the distribution of the maximum deficit D_n for the case of independent continuous random variables was obtained. Some examples of application were also given and again the solution of the continuous case by discretization was argued to be the most efficient.

The asymptotic distribution of D_n for the case of mean net input with expectation zero was derived and compared to results for finite values of n. The conclusion was that the asymptotic result, when corrected for the first two moments, is a good approximation of the exact distribution, for moderately large values of n is the existence of probability masses at the point $D_n = 0$, for finite values of n. Some conclusions, like the drastic reduction in storage capacity required when the level of regulation decreases, and the fact that this reduction depends on the value of n (indicating that Hurst's empirical formulae (Eq. 2.25) and (2.26)) are not adequate), were previously known from simulation studies (Fiering, 1965). A new conclusion is that departures from normality in general and skewness in particular may have strong influence, for low values of n and low levels of regulation.

The specific contribution of this paper to deficit analysis is that, by applying the approach developed, the distribution of D_n, and consequently its moments, can be obtained (at least numerically) for a wide variety of cases of practical interest. This is important because it allows the exact determination of the storage-yield relationship, one of the oldest concepts in water resources engineering.

The analogies between deficit and range analysis and Moran's theory of reservoirs were also pointed out. In so doing, the extension of the theory exposed to the case of seasonal inputs was merely mentioned and the extension to correlated inputs was made for very simple cases. The same limitation found by Lloyd, namely, the drastic increase in the size of the matrices involved, was present here. However, even though only a very simple discrete correlated input was considered, the analysis led to important practical conclusions: i) the asymptotic distributions of R_n and D_n, when corrected for the first two moments, are good approximations of the exact distributions of R_n and D_n even for Markovian inputs, ii) the square root law for the mean range holds asymptotically (this was previously known) but the size of the transient region (the region where the mean range behaves as higher powers of n) increases drastically with the degree of serial correlation, and iii) for the case of inputs following a simple Markov chain, the effect of correlation is to increase the storage capacity required by a factor smaller than $\sqrt{(1+\rho)/(1-\rho)}$.

In particular, Hurst's idea of approximating the rescaled range $R_n^{**} = (R_n^*/S)$ by the unadjusted range conditioned to the last partial sum being equal to zero $(R_n | S_n = 0)$ was extended to the case of inputs following a Markov chain, and the drastic increase in the size of the transient region (found before for the unadjusted unconditioned range) was noted. These results were compared to Hurst's results and the conclusion was that one can question the statement made by some authors (Mandelbrot and Wallis, 1969, and others) to the effect that short memory models fail to reproduce some drought characteristics.

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APPENDIX A

In this Appendix, an alternative expression for the mean conditioned range of partial sums of Markovian inputs (see Eq. (6.1) and (6.2) in the text) is derived, and some asymptotic results are formally obtained (see Eqs. (6.16) and (6.19) in the text).

A.1. The Mean Conditioned Range.

The mean conditioned range is twice the mean conditioned maximum partial sum, which can be obtained by using the tail of its cumulative distribution function:

$$E(R_n|S_n=0) = 2E(M_n|S_n=0) = 2\sum_{h=0}^{\infty} P[M_n>h|S_n=0]$$
$$= 2\sum_{h=0}^{\infty} P[M_n>h;\ S_n=0]/P[S_n=0]. \qquad (A.1)$$

To state the M_n is larger than h and S_n is equal to zero is equivalent to say that there exists an epoch m such that the sum of the inputs $X_1, X_2, \ldots, X_m$ equals $h+1$ for the first time (let us denote such an event by $S_m^{(1)}$) and that the sum of the remaining inputs $X_{m+1}, X_{m+2}, \ldots, X_n$ equals $-h-1$. Note that $S_m^{(1)}=h+1$ implies that $X_m=+1$, and thus,

$$P[M_n>h;S_n=0] = \sum_m P[(S_m^{(1)}=h+1;\ \sum_{t=m+1}^{t=n} X_t=-h-1)|\ X_m=+1]. \qquad (A.2)$$

Using Eq. (6.2), one has

$$P[\sum_{t=m+1}^{t=n} X_t=-h-1|X_m=+1] = q\cdot P[\sum_{t=m+2}^{t=n} X_t=-h|X_{m+1}=-1]$$
$$+ p\cdot P[\sum_{t=m+2}^{t=n} X_t=-h-2|X_{m+1}=+1]$$
$$= q\cdot P[\sum_{t=m+2}^{t=n} X_t=+h|X_{m+1}=+1]$$
$$+ p\cdot P[\sum_{t=m+2}^{t=n} X_t=-h-2|X_{m+1}=+1].$$

Using this result recursively, one has

$$P[\sum_{t=m+1}^{t=n} X_t=-h-1|X_m=+1] = q\cdot P[\sum_{t=m+2}^{t=n} X_t=+h|X_{m+1}=+1]$$
$$+ p\cdot q\cdot P[\sum_{t=m+3}^{t=n} X_t=+h+1|X_{m+2}=+1]$$
$$+ p^2q\cdot P[\sum_{t=m+4}^{t=n} X_t=+h+2|X_{m+3}=+1]$$
$$+ p^3q\cdot P[\sum_{t=m+5}^{t=n} X_t=+h+3|X_{m+4}=+1]+\ldots \qquad (A.3)$$

where only finitely many nonzero terms exist.

From Eqs. (A.2) and (A.3), one has

$$P[M_n>h;S_n=0] = \sum_m \Big\{ q\cdot P[(S_m^{(1)}=h+1;\ \sum_{t=m+2}^{t=n} X_t=+h)|X_{m+1}=+1]$$
$$+ p\cdot q\cdot P[(S_m^{(1)}=h+1;\ \sum_{t=m+3}^{t=n} X_t=h+1)|X_{m+2}=+1]$$
$$+ p^2q\cdot P[(S_m^{(1)}=h+1;\ \sum_{t=m+4}^{t=n} X_t=h+2)|X_{m+3}=+1]$$
$$+ \ldots \Big\}. \qquad (A.4)$$

The general term in Eq. (A.4) (i.e., the term $\sum_m p^i q\ P[\cdot]$) involves the probability of the event that the first m inputs add to $h+1$ for the first time and that the last $n-m-i-1$ inputs add to $h+i$ (given that X_{m+1+i} is equal to $+1$) for all possible values of m. But this is simply the probability of the event that $n-i-1$ inputs add to $2h+i+1$, and thus the general term in Eq. (A.4) can be written as

$$p^i\cdot q\cdot P[S_{n-i-1} = 2h+1+i].$$

Now Eq. (A.4) can be simplified to

$$P[M_n>h;S_n=0] = q\sum_{i=1}^{\infty} p^{i-1}P[S_{n-i}= 2h+i]. \qquad (A.5)$$

Using Eqs. (A.1) and (A.5), one has

$$E(R_n|S_n=0) = 2q\sum_{i=1}^{\infty} p^{i-1}P[S_{n-i}\geq i]/P[S_n=0]. \qquad (A.6)$$

Equation (A.6) is an alternative expression for the mean conditioned range, given before by the more general result from Eq. (6.18). In order to use Eq. (A.6), the value of $P[S_t=j]$ is needed, for general t and j. This result is known and due to Gabriel (1959). In particular, $P[S_n=0]$ is given by Eq. (6.19).

A.2. The Asymptotic Mean Conditioned Range.

To derive asymptotic results, the fact that $S_t = \sum_{i=1}^{t} X_i$ is asymptotically normally distributed with mean zero and variance tp/q can be used to rewrite Eq. (A.6) as

$$E(R_n|S_n=0) \simeq 2q\sum_{i=1}^{\infty} p^{i-1}\Phi\{[-(i-1)]/[(n-i)p/q]^{0.5}\}/P[S_n=0]. \qquad (A.7)$$

Let i^* be a large number such that $p^{i^*} \approx 0$ and let n be still larger, so that $i^*/(n-i^*)^{0.5} \approx 0$. Then $\Phi\{[-(i-1)]/[(n-i)p/q]^{0.5}\} \simeq \Phi(0) = \frac{1}{2}$ for all values of $i<i^*$, and Eq. (A.7) reduces to

$$E(R_n|S_n=0) \approx q\sum_{i=1}^{i^*} p^{i-1}/P[S_n=0] \approx 1/P[S_n=0] \qquad (A.8)$$

where $P[S_n=0]$ is given by Eq. (6.19); for large values of n, it can be shown that Eq. (6.19) can be approximately written as

$$P[S_n=0] \simeq (\pi n/2)^{-0.5}(p/q)^{-0.5}. \qquad (A.9)$$

Using Eqs. (A.8) and (A.9) and recalling that $\rho=2p-1$, one gets finally Eq. (6.20):

$$E(R_n|S_n=0) \approx \sqrt{\frac{\pi n}{2}}\sqrt{\frac{1+\rho}{1-\rho}} \approx 1.2533\sqrt{n}\sqrt{\frac{1+\rho}{1-\rho}}$$

A.3. The Asymptotic Mean Range

Let Eq. (A.6) be rewritten as

$$E(R_n|S_n=0) = 2q \sum_{i=1}^{\infty} p^{i-1}P[S_{n-i}>i-1]/P[S_n=0].$$

It can be shown that this is a particular case of the following more general result.

$$E(R_n|S_n=s) = 2s + 2q \sum_{i=1}^{\infty} p^{i-1}P[S_{n-i}>s+i-1]/P[S_n=s]; \quad s\geq 0. \qquad (A.10)$$

$$E(R_n|S_n=s) = 2q \sum_{i=1}^{\infty} p^{i-1}P[S_{n-i} > -s+i-1]/P[S_n=s]; \quad s\leq 0. \qquad (A.11)$$

For very large values of n, one can use the following approximation:

$$P[S_n>s]\approx P[S_{n-1}>s]\approx P[S_{n-2}>s+1]\approx P[S_{n-3}>s+2]$$

$$\approx \ldots \approx P[S_{n-i^*}>s+i^*-1]$$

where i^* is a value large enough so that $p^{i^*} \simeq 0$. Then Eqs. (A.10) and (A.11) simplify to

$$E(R_n|S_n=0)\approx 2s + 2\,P[S_n>s]/P[S_n=s]; \quad s\geq 0, \qquad (A.12)$$

$$E(R_n|S_n=0)\approx 2P[S_n\geq -s]/P[S_n=s]; \quad s\leq 0. \qquad (A.13)$$

The mean range is given by

$$E(R_n) = E[E(R_n|S_n)] = \sum_s E(R_n|S_n=s)\cdot P[S_n=s].$$

Using Eqs. (A.12) and (A.13), after routine transformations, one has

$$E(R_n) \approx 2E(|S_n|) - 2\,P[S_n>0] \qquad (A.14)$$

where $|S_n|$ denotes the absolute value of S_n. Recalling that n is large, the normal approximation can be used again:

$$E(R_n) \approx 2\int_0^{\infty} \sqrt{\frac{2}{\pi np/q}}\cdot x \cdot e^{-[x^2/(2np/q)]}\cdot dx - 1$$

$$= 2\sqrt{\frac{2}{\pi}}\sqrt{\frac{np}{q}} - 1.$$

Recalling that $\rho=2p-1$, one gets finally Eq. (6.16):

$$E(R_n) \approx \sqrt{\frac{8n}{\pi}}\sqrt{\frac{1+\rho}{1-\rho}} - 1 \approx 1.5958\sqrt{n}\sqrt{\frac{1+\rho}{1-\rho}}.$$

Key Words: Range Analysis, Storage Deficit Analysis, Water Storage, Storage Design.

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