TREES FOR A 3-VALUED LOGIC

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IN [3] Slater uses trees to restrict the classical propositional logic to avoid the paradoxes of material implication and other odd features of the classical logic such as these:

(1) Something (a tautology) follow from nothing

and

(2) Something follows from a contradiction.

The paradoxes of material implication are avoided by restricting the synthetic tree rule[1] that would permit one to write 'p v q' in a path that contains 'p'. (1) and (2) are avoided by requiring that every deduction tree have, respectively, a non-empty and a non-contradictory initial list. Though Slater's project is a good one there are some problems with his discussion. For example, as noted above, he says that (2) is false in his system, but he also says that his system could be presented as a natural deduction system in which there is the Rule of Simplification. With this rule 'p' follows from 'p & ~ p', and thus (2) is not false. Another problem is that he provides no semantic account of validity. I will present a modified tree account of validity and a semantic account of validity with the same extension. We will see that with this account of validity we can avoid the paradoxes and oddities that Slater wished to avoid.

[1] Throughout the discussion I will follow the terminology of [1], and I will mime some of the arguments of [1], though only classical logics are developed in [1].

Consider a language in which statements are constructed in the standard way from the letters A_1, A_2, A_3, ..., parentheses, and the connectives '&' and '~'. To construct trees we will use the following (standard) analytic rules:

$$\frac{\begin{array}{c} p \\ \sim p \end{array}}{x} \qquad \frac{\sim\sim p}{p} \qquad \frac{p \,\&\, q}{\begin{array}{c} p \\ q \end{array}} \qquad \frac{\sim(p \,\&\, q)}{\sim p \mid \sim q}$$

Moreover, we will use one (non-standard) synthetic rule:

$$\frac{p}{q \mid \sim q}$$

To use the synthetic rule the initial list has to contain the letters that occur in q. To construct a *deduction tree* we will require that the synthetic rule is not used before all uses of the analytic rules have been exhausted in the tree under construction. Let us call the initial part of the deduction tree that cannot be extended without the use of the synthetic rule the *initial analytic part* (IAP) *of the deduction tree*. Any part of a path P of a deduction tree D that occurs in the IAP of D will be in the *IAP of P of D*. A path P of D will be *open* if 'x' (the 'x' of the analytic rule mentioned first) does not appear in the IAP of P of D; otherwise, it is *closed*. We will say that $P_1, \ldots, P_n$ *syntactically entails* C ($P \vdash C$) if and only if there is a deduction tree with initial list $P_1, \ldots, P_n$ such that (1) there is an open path in the deduction tree, and (2) C occurs in every open path.

To define semantic entailment we will first define valuations as functions which map statements into the set {0, 1, 2}. (0, 1 and 2 can be thought of as false, odd (or meaningless) and true, respectively.) A valuation V meets these conditions: ($V(\sim p) = 2 - V(p)$; $V(p \,\&\, q) = 1$ if either $V(p) = 1$ or $V(q) = 1$, and, otherwise, $V(p \,\&\, q) =$ the minimum of $V(p)$ and $V(q)$. (This definition of V is attributed by Rescher to Bochvar. See p. 29 of [2].) We will say that $P_1, \ldots, P_n$ *semantically entails* C ($P \models C$) if there is a valuation V such that $V(P_1 \,\&\, \ldots \,\&\, P_n) = 2$, and if for any valuation V $V(C) = 2$ if $V(P_1 \,\&\, \ldots \,\&\, P_n) = 2$.

That the notions of syntactic and semantic entailment have the same extension is a corollary of the following four claims.

1. If $P \vdash C$ then there is a valuation V such that $V(P) = 2$. Proof: assume the antecedent. Then there is an open path OP in a deduction tree D with P as its initial list. Let V be a valuation that assigns 2 to those statement letters that are full lines of the IAP of OP and let V assign 0 to all of the other statement letters that occur in the IAP of OP. Any line in the IAP of OP is either part of the initial list or was placed by one of the analytic rules. Since for each

of the analytic rules a 2 below the line guarantees a 2 above the line, it follows that $V(P) = 2$.

2. If $P \vdash C$ then $V(C) = 2$ if $V(P) = 2$. Proof: assume that $V(P) = 2$. If the upper part of an analytic rule is assigned the value 2 then at least one branch below the line is assigned the value 2. The same holds for the synthetic rule. For if $V(P) = 2$ then any statement composed of letters in P must have the value 0 or 2. (For suppose there were a letter in P that had the value 1. Then P would have the value 1. But if each letter in a statement has the value 0 or the value 2 then so does the entire statement.) But then either the 'q' or the '$\sim$q' of the synthetic rule has to have the value 2. So if $V(P) = 2$ then in any deduction tree with P as initial list there is a path in which each full line is assigned the value 2. Such a path must be open. So if we assume that $P \vdash C$ it follows that C is in this path, and thus $V(C) = 2$.

3. If $P \models C$ then there is an open path in any deduction tree that has P as its initial list. Proof: assume the antecedent. Then there is a valuation V such that $V(P) = 2$. By an argument in the preceding paragraph it follows that there is a path in the tree in which every full line is assigned 2 by V. But this path has to be an open path.

4. If $P \models C$ then there is a deduction tree with initial list P and with C in every open path. Proof: first note that if $P \models C$ then any deduction tree with P and $\sim$C as the initial list has no open paths in its IAP. For suppose there were such a path. Then by the argument for the first of the four claims under consideration there would be a valuation V such that $V(P) = 2$ and $V(\sim C) = 2$. But then it would be false that $P \models C$. Keeping this in mind, we use the following recipe to construct a tree:

(a) Construct the IAP of a deduction tree with P as its initial list.
(b) Apply the synthetic rule to every open path, putting 'C' to the left and '$\sim$C' to the right. (Any '$\sim$C' and anything that extends from it will be said to be in a 'right branch'.) Note that there is no difficulty in meeting the qualification on the synthetic rule since if C contained a letter S that did not occur in P then a valuation V that assigns 2 to P but 1 to S would assign 1 to C. Thus, it would be false that $P \models C$.
(c) Apply analytic rules to the 'right branches' until no more applications are possible.

Since each of the 'right branches' is closed (by the argument we are keeping in mind) it follows that C occurs in every open path.

That syntactic and semantic entailment have the same extension, for arguments with one premise, follows directly from statements 1–4. There is no difficulty in generalizing the argument to cover arguments with a greater number of premises.

Let us complete our discussion by looking at some of the puzzles Slater mentioned in [3]. To separate the notion of entailment discussed above from classical entailment, let us call it superentailment. Since if V is any valuation $V(A_1 \& \sim A_1) = 0$ or $V(A_1 \& \sim A_1) = 1$, '$A_1 \& \sim A_1$' does not superentail anything (though it entails everything). Since there is a valuation V such that $V(A_1) = 2$, $V(A_2) = 1$ (and thus $V(\sim (A_2 \& \sim A_1) = 1)$, it follows that it is false that $A_1 \models \sim (A_2 \& \sim A_1)$. Thus, not everything superentails that everything 'materially implies' it (though everything entails that everything materially implies it).

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REFERENCES

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[2] Nicholas Rescher, *Many-Valued Logic*, New York: McGraw-Hill, 1969.
[3] B. H. Slater, 'Direct Tableaux Proofs', *Analysis*, 41.4 (October 1981), 192-4.